Stabilization of the Acrobot via sampled-data passivity-based control

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***Abstract*— The paper deals with the sampled-data asymptotic stabilization of the Acrobot at its upward equilibrium. The proposed controller results from the action of an Input-Hamiltonian-Matching (IHM) strategy that shapes the closed-loop energy combined with a Damping Injection (DI) feedback designed on the sampled-data equivalent model. Simulations show the effectiveness of the proposed controller.**

***Index Terms*— Sampled data control; Lyapunov methods; Energy systems**

I. Introduction

The Acrobot is a planar two-link robotic arm in the vertical plane actuated at the elbow. It provides an interesting case study to simulated or experimental tests for nonlinear control methods. The upward stabilization of this underactuated mechanical system has been proposed for the first time in [1] and then several control strategies have been developed in the continuous-time control literature such as partial feedback linearization [2], trajectory tracking [3], Lyapunov based control [4], optimal control [5], energy-based feedback [6].

One of the most celebrated control strategies for underactuated mechanical systems relies on Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) [7]. In this context, asymptotic stabilization is achieved by injecting damping into the passive system resulting from a suitable shaping of the total energy of the system according to the desired control objective. In [8] such an approach has been applied to design a controller stabilizing the Acrobot at the upward position.

Following these lines and motivated by recent results on the topic [9], [10], [11], we propose a control strategy for solving the problem in the digital context; namely, when the control signal is piecewise constant and implemented from synchronized sampled-data measures of the state. In this context, standard strategies relying on an approximate discrete-time IDA-PBC approach beyond emulation do not apply as the discrete-gradient function involved in the sampled-data model is highly nonlinear and not separable. The solution we propose combines two control components: an energy shaping feedback, designed to ensure matching at the sampling instants of the target continuous-time Hamiltonian [8]; a damping injection negative output feedback, set on the passivating average-output of the sampled-data equivalent model after the action of the energy shaping component. Simulations highlight the performances in a comparative perspective.

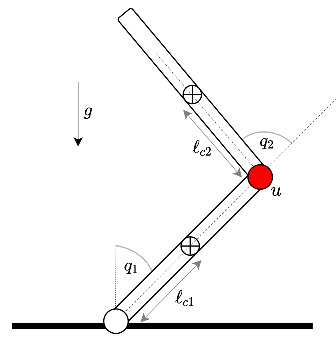

Fig. 1: The acrobot system

The paper is organized as follows. In Section II the model and preliminaries on the continuous-time stabilizer [8] are given and the problem is formally set. The main result is proposed in Section III for the case-study. Simulations that outperform the continuous-time solution are in Section IV. Section V concludes the paper.

Notations and definitions: Functions and vector fields are assumed smooth and complete over the respective definition spaces. $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers including 0. 0 denotes the zero matrix of suitable dimension, depending on the context. For any vector $z \in \mathbb{R}^n$, $\|z\|$ and $z^\top$ define respectively the norm and transpose of z. $x = \mathrm{col}(a_1, \dots, a_n) \in \mathbb{R}^{n_1+\cdots+n_n}$ denotes the column vector with entries provided by $a_i \in \mathbb{R}^{n_i}$ of suitable dimensions. I and I_d denote respectively the identity matrix (of suitable dimension) and identity operator. $\mathrm{L}_f = \sum_{i=1}^n f(\cdot)\frac{\partial}{\partial x_i}$ denotes the Lie derivative and $e^{\mathrm{L}_f} = I_d + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$ the exponential Lie series operator, associated with the vector field f. Given two vector fields $f(x)$, $g(x)$, $ad_f g(x) = (\mathrm{L}_f \mathrm{L}_g - \mathrm{L}_g \mathrm{L}_f)(x)$ denotes their Lie bracket. Given a twice continuously differentiable function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}$, ∇S represents its gradient (column) vector and $\nabla^2 S$ its Hessian matrix. Given a smooth real-valued function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}$, the corresponding discrete gradient is a vector-valued function, $\bar\nabla S|_x^z : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying, for all $x, z \in \mathbb{R}^n$, the variational equality

$$S(z) - S(x) = (z - x)^\top \bar\nabla S|_x^z, \quad \bar\nabla S_x^x = \nabla S(x). \qquad (1)$$

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II. Preliminaries and problem statement

The Acrobot is an underactuated planar two-link robotic arm in the vertical plane [1] as depicted in Fig. 1. More in detail, for the i^{th} joint ($i=1,2$), we denote by $q_i \in \mathbb{R}$ the angle, $m_i > 0$ the mass, $I_i > 0$ the link inertia moment around the vertical axis passing through the center of mass, l_i the length, l_{c_i} the distance from the i^{th} link to the center of mass, $u \in \mathbb{R}$ the input torque acting on the elbow joint and g the gravitational constant. Accordingly, the dynamics admits the canonical port-controlled Hamiltonian (pcH) representation of the form

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\nabla_q H\\ \nabla_p H\end{pmatrix} + \begin{pmatrix}0\\ G\end{pmatrix} u \tag{2}$$

with $p = M(q)\dot q$ the momenta vector, $G = \begin{pmatrix}0 & 1\end{pmatrix}^\top$, Hamiltonian function

$$H(q,p) = \frac{1}{2}p^\top M^{-1}(q)p + V(q) \tag{3}$$

when setting the generalized inertia matrix as

$$M(q) = \begin{pmatrix} c_1 + c_2 + 2c_3\cos(q_2) & c_2 + c_3\cos(q_2)\\ c_2 + c_3\cos(q_2) & c_2\end{pmatrix}$$

and the potential energy as

$$V(q) = g\left(c_4\cos(q_1) + c_5\cos(q_1+q_2)\right)$$

with

$$\begin{aligned} c_1 &= m_1 l_{c_1}^2 + m_2 l_1^2 + I_1, \quad c_4 = m_1 l_{c_1} + m_2 l_1\\ c_2 &= m_2 l_{c_2}^2 + I_2, \quad c_3 = m_2 l_1 l_{c_2}, \quad c_5 = m_2 l_{c_2}.\end{aligned}$$

Setting

$$y = G^\top M^{-1}(q)p$$

the so defined input-output link $u \mapsto y$ is passive with output corresponds to the velocity at the second joint. We refer to [1] for further details on the mathematical model of the Acrobot system.

The problem we consider consists in defining a digital control law to swing-up the Acrobot from some configuration positions in the lower half plane, or equivalently to stabilize the upright equilibrium $q_\star = (0,0)^\top$, via Passivity-Based Control (PBC). As usual under sampling, we assume the measures of the state (q,p) available at the sampling instants $t = k\delta$ only and the control piecewise constant over sampling intervals of length $\delta > 0$; i.e. $u(t) = u_k$ for $t \in [k\delta, (k+1)\delta[$, for all $k \geq 0$.

A. Swing-up in continuous time

In continuous time, stabilization of the upright equilibrium relies upon Interconnection and Damping Assignment PBC (IDA-PBC) [7] by setting the control

$$u(q,p) = u_{\text{es}}(q,p) + u_{\text{di}}(q,p). \tag{4}$$

with: $u_{\text{es}}(q,p)$ the so-called energy shaping component designed to assign a suitably defined desired energy function $H_d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ of the form

$$H_d(q,p) = \frac{1}{2}p^\top M_d^{-1}p + V_d(q) \tag{5}$$

for a constant $M_d \succ 0$ and possessing its minimum at $x_\star = (q_\star^\top\, 0^\top)^\top$ with $q_\star = (0\,0)^\top$; $u_{\text{es}}(q,p)$ is the so-called damping injection component designed to guarantee asymptotic convergence to the swing-up configuration while assigning a suitably defined dissipation to the closed loop.

In detail, as proposed in [8, Proposition 6.1], one first sets

$$u = u_{\text{es}}(q,p) + v$$

with the energy-shaping component

$$u_{\text{es}}(q,p) = G^\top\Big(\nabla_q H(q,p) - M_d M^{-1}(q)\nabla_q V_d(q)\Big) \tag{6}$$

assigning the conservative pcH structure

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} = J(q)\begin{pmatrix}\nabla_q H_d\\ \nabla_p H_d\end{pmatrix} + \begin{pmatrix}0\\ G\end{pmatrix} v \tag{7}$$

with new interconnection matrix

$$J(q) = \begin{pmatrix}0 & M^{-1}(q)M_d\\ -M_d M^{-1}(q) & 0\end{pmatrix} \tag{8}$$

virtual generalized inertia matrix

$$M_d = \begin{pmatrix} k_2 - k_2\sqrt{\frac{c_1}{c_2}} & k_2\\ k_2 & k_3\end{pmatrix}, \quad k_2 < k_3\left(1 - \sqrt{\frac{c_1}{c_2}}\right)$$

and potential energy (see the Appendix for further details)

$$\begin{aligned} V_d(q) = {} & N(q_1)e^{A_{q_2}}\Gamma + b_1\cos(q_1) + b_2\cos(q_1+q_2) \qquad (9)\\ & + b_3\cos(q_1+2q_2) + b_4\cos(q_1-q_2) + \frac{k_p}{2}(q_1-\mu q_2)^2.\end{aligned}$$

The energy-shaping control makes the closed-loop link

$$v \mapsto y_d = G^\top M_d^{-1}p \tag{10}$$

passive, or more properly lossless, with dissipation rate $\dot H_d(q,p) = p^\top M_d^{-1}Gv$. Based on this, asymptotic stabilization at $q_\star = (0\,0)^\top$ is ensured by the output damping injection

$$u_{\text{di}}(p) = -k_v G^\top M_d^{-1}p, \quad k_v > 0 \tag{11}$$

assigning the desired closed-loop dynamics

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} = \begin{pmatrix}0 & M^{-1}(q)M_d\\ -M_d M^{-1}(q) & -k_v G G^\top\end{pmatrix}\begin{pmatrix}\nabla_q H_d\\ \nabla_p H_d\end{pmatrix} \tag{12}$$

with $\dot H_d(q,p) = -k_v p^\top G^\top G p \leq 0$.

B. Sampled-data systems and problem statement

Consider (2) under piecewise constant control over the sampling period $\delta > 0$. Setting $x = \text{col}(q,p)$, the sampled-data equivalent model is given in the form of a map [12]

$$x^+(u) = x + \delta F^\delta(x,u) = x + \delta\big(F_0^\delta(x) + g^\delta(x,u)u\big) \tag{13}$$

when denoting at time $t = k\delta$, $x = x_k$, $u = u_k$, and at time $t = (k+1)\delta$, $x^+(u) = x^+(u_k) = x_{k+1}$. From (2), setting

$$f(x) = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\nabla H, \quad B = \begin{pmatrix}0\\ G\end{pmatrix}$$

one computes

$$\delta F^{\delta}(x,u) = e^{\delta(\mathrm{L}_f + u\mathrm{L}_B)}x - x = \sum_{i\geq 1} \frac{\delta^i}{i!}(\mathrm{L}_f + u\mathrm{L}_B)^i x$$
$$F_0^{\delta}(x) = F^{\delta}(x,0), \quad g^{\delta}(x,u) = F^{\delta}(x,u) - F_0^{\delta}(x).$$

The problem stands in designing a sampled-data feedback $u = u^{\delta}(q,p)$ to swing-up the robot at the upward equilibrium $q_\star = (0\,0)^\top$ or equivalently making $x_\star = (0^\top, 0^\top)^\top$ asymptotically stable for the sampled-data equivalent model (13). Analogously to the continuous-time solution, we look for a feedback law of the form

$$u^{\delta}(x) = u^{\delta}_{\mathrm{es}}(x) + u^{\delta}_{\mathrm{di}}(x) \tag{14}$$

with $u^{\delta}_{\mathrm{es}}(x)$ and $u^{\delta}_{\mathrm{di}}(x)$ the energy-shaping and damping injection components respectively. The energy-shaping component is designed so that the feedback

$$u = u^{\delta}_{\mathrm{es}}(x) + v \tag{15}$$

assigns, at the sampling instants, the same energy function $H_d(x) = H_d(q,p)$ as in continuous time (5) with stable equilibrium at the desired configuration. As a consequence, the dissipation equality below holds

$$H_d(x^+(u)) - H_d(x) = vY_d^{\delta}(x,v).$$

with respect to a suitably defined passifying output $Y_d^{\delta}(x,v)$. According, setting v solution to the implicit equality $v = -k_v Y_d^{\delta}(x,v)$ with $k_v > 0$ achieves asymptotic stabilization of $x_\star = (0^\top 0^\top)^\top$ under damping feedback $v = u^{\delta}_{\mathrm{di}}(q,p)$.

III. Main result

The first result is based on the preservation of passivity under sampling with respect to a new output suitably defined based on discrete-time average passivity [12]. In this case, one recovers the result in [13], [9] deduced over suitably defined approximate discrete-time models of the lossless continuous-time system (2) [14].

Proposition 3.1: Consider the Acrobot dynamics (2) being lossless with respect to the output $y = B^\top M^{-1}(q)p$

$$\dot{H}(q,p) = p^\top M^{-1}(q)Bu. \tag{16}$$

Then the sampled-data equivalent model (13) with is lossless with respect to the modified output

$$Y^{\delta}(q,p,u) = \frac{1}{\delta}B^\top(q^+(u) - q) \tag{17}$$

so verifying the dissipation equality

$$\begin{aligned} H(q^+(u), p^+(u)) - H(q,p) =& \delta Y^{\delta}(q,p,u)u \\ =& (q^+(u) - q)^\top Bu. \end{aligned} \tag{18}$$

Proof: In the lossless case, the sampled-data passivating output reduces to the time average of the continuous-time output. By integrating the continuous-time dissipation equality (16) over the sampling interval, one gets

$$\int_{k\delta}^{(k+1)\delta} \dot{H}(q(s),p(s))\mathrm{d}s = uB^\top \int_{k\delta}^{(k+1)\delta} M^{-1}(q(s))p(s)\mathrm{d}s.$$

Substituting in the equality above $\dot{q} = M^{-1}(q)p$ and recalling that $q^+(u) = q_{k+1} = q((k+1)\delta)$ and $p^+(u) = p_{k+1} = p((k+1)\delta)$, one gets (18) and thus the result. ■

A. Sampled-data energy-shaping

The idea is to design a feedback so to make the sampled-data dynamics (13) conservative with respect to the desired energy function $H_d(\cdot)$ in (5). Namely, we design $u^{\delta}_{\mathrm{es}}(x) : \mathbb{R}^4 \to \mathbb{R}$ so to guarantee Input-Hamiltonian Matching (IHM) of the target Hamiltonian $H_d(x)$ along the continuous-time trajectories (7) when $v = 0$; i.e. for all $x \in \mathbb{R}^4$

$$H_d(x^+(u^{\delta}_{\mathrm{es}}(x))) - H_d(x) = 0.$$

Proposition 3.2: Consider the Acrobot dynamics (2) under the energy-shaping feedback (6) assigning the pcH structure (7) and let (13) be its sampled-data equivalent model. Then, the IHM equality

$$H_d(F^{\delta}(x,u)) - H(x) = \int_{k\delta}^{(k+1)\delta} \nabla^\top H_d(x(s)) f_d(x(s))\mathrm{d}s. \tag{19}$$

with $f_d(x) = J(q)\nabla H_d(q,p)$ and $J(q)$ in (8) admits a unique solution $u = u^{\delta}_{\mathrm{es}}(x)$ as a series expansion in powers of δ around the continuous-time feedback

$$u^{\delta}_{\mathrm{es}}(x) = u_{\mathrm{es}}(x) + \sum_{i>0} \frac{\delta^i}{(i+1)!} u^i_{\mathrm{es}}(x). \tag{20}$$

Proof: Setting $u = u^{\delta}_{\mathrm{es}}(x)$ in both sides of the IHM equality (19) and comparing the terms with the same power in δ, one gets the result from the Implicit Function Theorem because the rank condition $B^\top \nabla H_d(x) = B^\top M_d^{-1} p \neq 0$ holds for $p \neq 0$. It is a matter of computations to verify that, as $\delta \to 0$, (19) is solved by the continuous-time solution (6). More details can be found in [14]. ■

The next result describes the passivating output that can be associated to the sampled-data dynamics in closed loop with the energy-shaping feedback computed as the solution to the matching equality (19).

Theorem 3.1: Consider the Acrobot dynamics (2) with sampled-data equivalent (13). Let the control (15) with $u^{\delta}_{\mathrm{es}}(q,p)$ solution to (19). Then, the closed-loop dynamics

$$x^+(u^{\delta}_{\mathrm{es}}(x) + v) = x + \delta\big(F_d^{\delta}(x) + g_d^{\delta}(x,v)v\big) \tag{21}$$

with

$$\begin{aligned} F_d^{\delta}(x) =& F^{\delta}(x, u^{\delta}_{\mathrm{es}}(x)) \\ g_d^{\delta}(x,v)v =& g^{\delta}(x, u^{\delta}_{\mathrm{es}}(x) + v)v \\ &+ \big(g^{\delta}(x, u^{\delta}_{\mathrm{es}}(x) + v) - g^{\delta}(x, u^{\delta}_{\mathrm{es}}(x))\big)u^{\delta}_{\mathrm{es}}(x) \end{aligned}$$

is stable at the upward equilibrium $x_\star = (q_\star^\top, 0^\top)^\top$ and conservative for $v = 0$, i.e.

$$H_d(x^+(u^{\delta}_{\mathrm{es}}(x))) = H_d(x).$$

Moreover, for $v \neq 0$, it is lossless

$$H_d(x^+(u^{\delta}_{\mathrm{es}}(x) + v)) - H_d(x) = Y_d^{\delta}(x,v)v. \tag{22}$$

with respect to the output

$$Y_d^\delta(x,v) = \bar\nabla^\top H_d|_{x^+(u_{\rm es}^\delta(x))}^{x^+(u_{\rm es}^\delta(x)+v)} g_d^\delta(x,v) \tag{23}$$

with discrete-gradient function defined as

$$\bar\nabla H_d|_x^{x^+} = \begin{pmatrix} \bar\nabla V_d|_q^{q^+} \\ \frac{1}{2}M_d^{-1}(p^+ + p) \end{pmatrix}.$$

Proof: Because the energy-shaping component is solution to the IHM equality (19) and $\nabla^\top H_d(x) f_d(x) = \nabla^\top H_d(x) J(x) \nabla H_d(x) = 0$, one gets

$$\begin{aligned}\Delta_k H_d(x) =& H_d(x^+(u_{\rm es}^\delta(x)+v)) - H_d(x)\\ =& H_d(x^+(u_{\rm es}^\delta(x)+v)) - H_d(x^+(u_{\rm es}^\delta(x))).\end{aligned}$$

Exploiting the discrete gradient (1), the equality above reads

$$\Delta_k H_d(x) = \bar\nabla^\top H_d|_{x^+(u_{\rm es}^\delta(x))}^{x^+(u_{\rm es}^\delta(x)+v)} g_d^\delta(x,v)v = Y_d^\delta(x,v)v.$$

Thus, because $H_d(x_\star)=0$ one has $\bar\nabla H_d|_{x_\star}^{x_\star}\nabla H_d(x_\star)=0$ and by the matching equality (19), the upright equilibrium is stable for the sampled-data dynamics (21). ■

Remark 3.1: By Proposition 3.1 and the matching equality (19), the passivating output (23) gets the form

$$\begin{aligned}Y_d^\delta(x,v) =& \frac{1}{\delta}\int_{k\delta}^{(k+1)\delta} \nabla^\top H_d(x(s))B{\rm d}s\\ =& \frac{1}{\delta}G^\top M_d^{-1}\int_{k\delta}^{(k+1)\delta} p(s){\rm d}s\\ =& \frac{1}{\delta}G^\top M_d^{-1}\Big(M(q^+(u_{\rm es}^\delta+v))q^+(u_{\rm es}^\delta+v) - M(q)q\Big)\\ &-\frac{1}{\delta}G^\top M_d^{-1}\int_{k\delta}^{(k+1)\delta}\dot M(q(s))q(s){\rm d}s\end{aligned}$$

Again, by losslessness of the continuous-time the passivating output is the time average of the continuous-time one in (10) but over the sampled-data trajectories under (15). Accordingly, such an output is not the same as the one deduced in [9] which is based on an approximate discrete-time model of the dynamics.

B. Sampled-data damping injection

At this point, the main result can be stated.

Theorem 3.2: Consider the Acrobot dynamics (2) with sampled-data equivalent (13). Let the control (15) with $u_{\rm es}^\delta(q,p)$ solution to (19). Consider the corresponding closed-loop dynamics provided by (21) passive with the output (23). Then, the damping injection equality

$$v = -k_v Y_d^\delta(x,v), \quad k_v>0 \tag{24}$$

admits a unique solution $v = u_{\rm di}^\delta(x)$ in the form of a series expansion in powers of δ; namely, one gets

$$u_{\rm di}^\delta(x) = u_{\rm di}(x) + \sum_{i>0}\frac{\delta^i}{(i+1)!}u_{\rm di}^i(x) \tag{25}$$

and $u_{\rm di}(x)$ the continuous-time damping feedback (11). Consequently, the piecewise constant feedback law (14), with energy shaping and damping injection components solutions to (19) and (24) respectively, makes the upright configuration $x_\star = (q_\star^\top\ 0)^\top$ asymoptotically stable for (2).

Proof: Existence of a solution to (24) in the form (25) follows from [12] by virtue of the Implicit Function Theorem. Accordingly, substituting (24) into the dissipation equality (22) one gets

$$\Delta_k H_d(x) = -k_v\|Y_d^\delta(x,u_{\rm di}^\delta(x))\|^2 \le 0, \quad k_v>0$$

and asymptotic stability of $x_\star = (q_\star^\top\ 0)^\top$ with $q_\star = (0\,0)^\top$ follows by zero-state detectability of (21) with output (23). ■

C. Computational aspects

Similarly to the continuous-time counterpart, the sampled-data stabilizer of the upright position of the Acrobot is a PBC feedback of the form (14) with both components computed as the implicit solutions to the corresponding series equalities (19) and (24) respectively. In particular, both of them admit a power expansion in powers of δ, the sampling period, and around the continuous time solutions (6) and (11) which are, then, naturally recovered as $\delta\to 0$. Despite exact forms cannot be computed in practice, each term of the series expansions (20)-(25) can be computed via an iterative and constructive procedure solving, at each step, a linear equality in the corresponding unknown. Substituting (20)-(25) into (19)-(24) and equating the terms with the same power of δ, one gets for the first terms

$$\begin{aligned}u_{\rm es}^1(x) =& (\nabla_q u_{es}(q,p))M^{-1}(q)p\\ &-(\nabla_p u_{es}(q,p))M_d M^{-1}(q)\nabla_q V_d(q)\\ u_{\rm di}^1(x) =& k_v^2(u_{di}(q,p)G^\top M_d^{-1}G - G^\top M^{-1}(q)\nabla_q V_d(q)\\ &-G^\top M_d^{-1}G k_v G^\top M_d^{-1}p) - k_v(\nabla_q^\top V_d(q)M^{-1}(q)\\ &-p^\top M_d^{-1}\nabla_p^\top\nabla_q H(q,p))G.\end{aligned}$$

Accordingly, the feedback law (14) rewrites in the form

$$u^\delta(x) = u(x) + \sum_{i>0}\frac{\delta^i}{(i+1)!}u^i(x),\ u^i(x) = u_{\rm es}^i(x) + u_{\rm di}^i(x). \tag{26}$$

As typical under sampling, only feedback laws deduced by truncating (26) at any fixed and arbitrary order $r\ge 0$ in δ can be implemented in practice. For, $r^{\rm th}$-order approximate feedback laws are formally defined as

$$u_{[r]}^\delta(x) = u(x) + \sum_{i=1}^r \frac{\delta^i}{(i+1)!}u^i(x) \tag{27}$$

recovering, for $r=0$, the well-known emulation-based control [15], typically implemented in practice. For $r>0$, performances are improved by including the so-called *correcting terms* $u^i(x)$, for $i=1,\dots,r$, ensuring practical asymptotic stability in closed loop. As a matter of fact, approximate solutions guarantee convergence of the closed-loop trajectories to a ball of radius $O(\delta^{r+1})$ centered the equlibrium to stabilize [16].

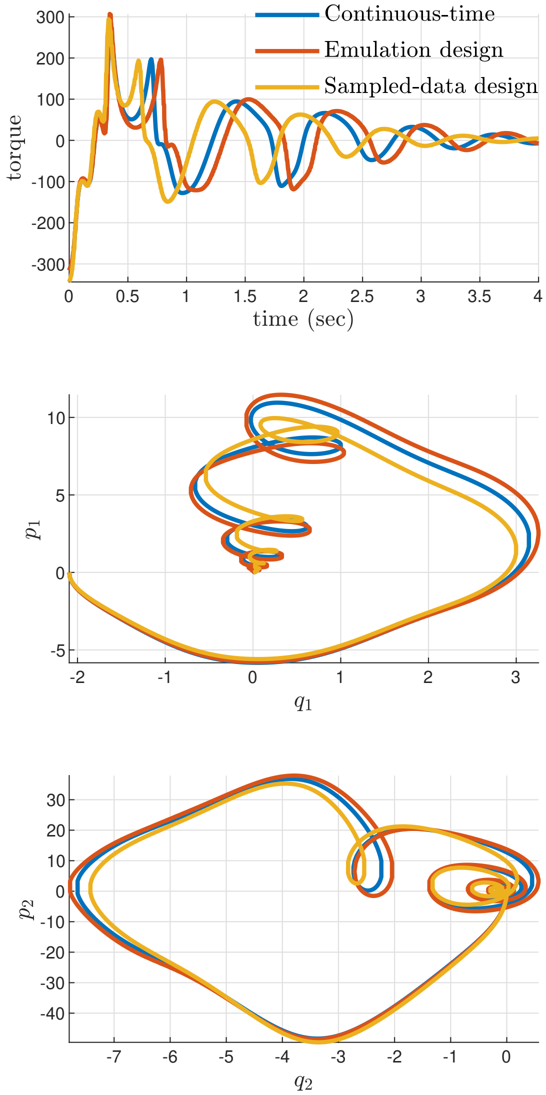

Fig. 2: Comparison for $\delta = 5 \cdot 10^{-3}$

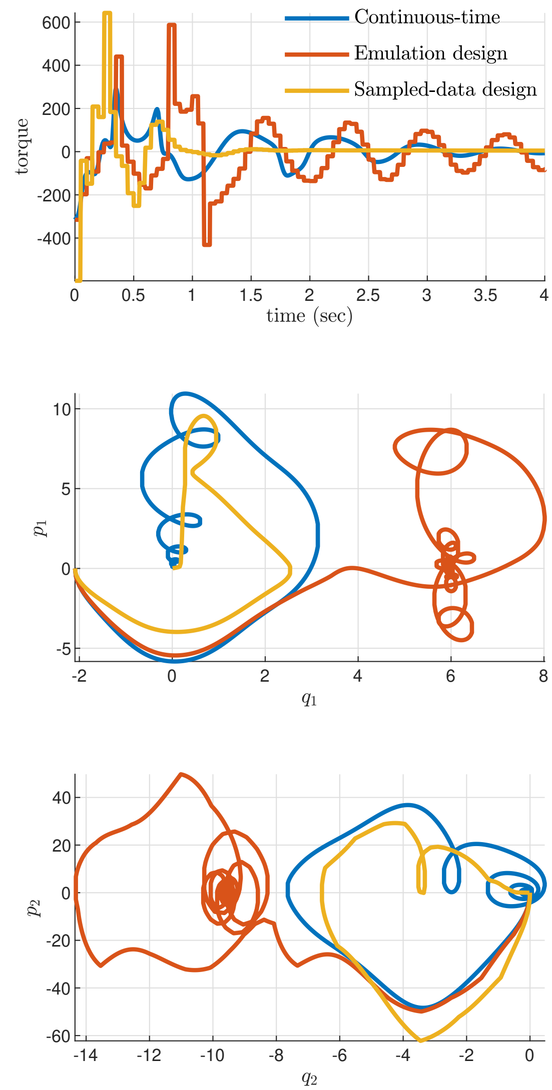

Fig. 3: Comparison for $\delta = 5 \cdot 10^{-2}$

IV. Simulations

Simulations reported in Fig. 2 and Fig. 3 are performed comparing the effect of the continuous-time feedback (4) with the emulation design, that is the controller (27) when setting $r = 0$, and the proposed approximate second-order sampled-data control, that is (27) with $r = 2$. The parameters considered in the simulations are displayed in Table I with $k_2 = 1$, $k_3 = 5.9073$, $\Gamma_2 = 10$, $k_p = 280$, and $k_v = 12$ set as in [8] and initial condition $q_0 = (-\frac{2\pi}{3}, 0)^\top$, $p_0 = (0, 0)^\top$.

We notice in Fig. 2 that for small sampling periods, such as $\delta = 5 \cdot 10^{-3}$, both the emulation and the proposed sampled-data design achieve stabilization of the Acrobot system in the upright configuration. However, the benefit of the proposed approximate controller with respect to the more standard emulation approaches of the continuous-time design (4) is emphasized in Fig. 3 for $\delta = 5 \cdot 10^{-2}$. In fact, despite the larger sampling period, the proposed sampled-data controller stabilizes the Acrobot at its upright position, with a piecewise constant torque vanishing after few

Symbol	Value	Unit	Symbol	Value	Unit
m_1	4	Kg	m_2	4	Kg
l_1	1	m	l_2	2	m
l_{c1}	1/2	m	l_{c2}	2	m
I_1	1/3	Kg·m^2	I_2	4/3	Kg·m^2

TABLE I: System parameters of the Acrobot

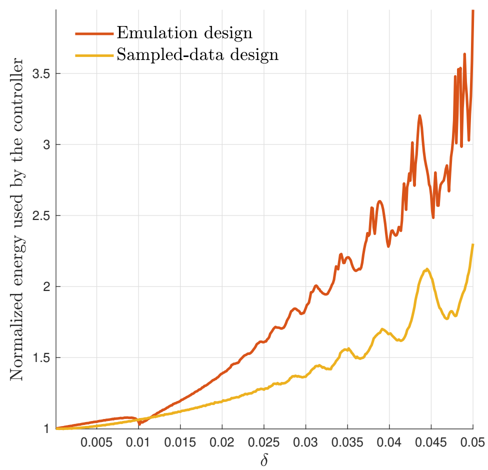

Fig. 4: Energy used by the digital controllers (normalized over the one of the continuous-time controller) for $\delta \in [0, 0.05]$.

seconds, while the emulation design fails in the stabilization objective as also underlined by further simulations available at `shorturl.at/fqAG1`. Finally, the energy used by both digital controllers (normalized with respect to the continuous-time one) are depicted in Fig. 4 when defined as,

$$u_{[r]}^{\max}(\delta) = \frac{\delta \sum_{k=0}^{k_{\text{fin}}-1} ||u_r^\delta(x_k)||^2}{\int_0^{t_{\text{fin}}} ||u(x(\tau))||^2 \mathrm{d}\tau}, \quad t_{\text{fin}} = k_{\text{fin}}\delta$$

for $r = 0, 2$, and $t_{\text{fin}} = 40$. The result highlights that the effort required by proposed sampled-data controller, and thus the control effort, is generally the half then the one used by the emulation design as δ increases.

V. Conclusions

In this paper, a new control scheme for stabilization of the Acrobot system at the upward equilibrium has been proposed via sampled-data PBC. The control consists of two components: the first one, whose design is based on IHM [17], assigns the energy of the system so to possess a minimum at the desired equilibrium; then, damping injection is performed to guarantee asymptotic swing up. Future works concern PBC of underactuated mechanical structures at large.

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Appendix

The desired potential energy given in (9), and formally computed in [8], comes with $N(q_1) = \text{col}(\sin(q_1), \cos(q_2))^\top$, $\Gamma = \text{col}(0, \Gamma_2)$ for arbitrary Γ_2, $\mu = -(1 + \sqrt{\frac{c_1}{c_2}})^{-1}$, matrix

$$A_{q_2} = \begin{pmatrix} 0 & \mu q_2 \\ -\mu q_2 & 0 \end{pmatrix},$$

and constants

$$k_p > \Gamma_2 - \frac{g(c_1c_2 - c_3^2)(c_4 + c_5)^2}{k_2(c_5c_1 \pm c_4\sqrt{c_1c_2} - c_4c_3 \mp c_5c_3\sqrt{\frac{c_1}{c_2}})}$$

$$b_1 = \frac{g}{2k_2}(c_3c_5 \pm 2\sqrt{c_1c_2}c_4), \quad b_3 = \frac{gc_3c_5\mu}{k_2(2\mu + 4)},$$

$$b_2 = \frac{g\mu}{2k_2(\mu + 1)}(c_3c_4 - 2\sqrt{c_1c_2}c_5), \quad b_4 = \frac{\mu(gc_3c_4)}{2k_2(\mu - 1)}.$$