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Stochastic Processes and their Applications 149 (2022) 248–277

stochastic processes and their applications

www.elsevier.com/locate/spa

Continuum and thermodynamic limits for a simple random-exchange model

Bertram Düring[a,1], Nicos Georgiou[b,2], Sara Merino-Aceituno[c,3], Enrico Scalas[b,*,4]

[a] *Mathematics Institute, University of Warwick, Zeeman Building, Coventry, CV4 7AL, UK*
[b] *University of Sussex, Department of Mathematics, Falmer Campus, Brighton BN1 9QH, UK*
[c] *University of Vienna, Faculty of Mathematics, Oscar-Morgenstern-Platz 1, 1090 Vienna, Austria*

Received 14 September 2020; received in revised form 18 October 2021; accepted 25 March 2022
Available online 4 April 2022

Abstract

We discuss various limits of a simple random exchange model that can be used for the distribution of wealth. We start from a discrete state space — discrete time version of this model and, under suitable scaling, we show its functional convergence to a continuous space — discrete time model. Then, we show a thermodynamic limit of the empirical distribution to the solution of a kinetic equation of Boltzmann type. We solve this equation and we show that the solutions coincide with the appropriate limits of the invariant measure for the Markov chain. In this way we complete Boltzmann's program of deriving kinetic equations from random dynamics for this simple model. Three families of invariant measures for the mean field limit are discovered and we show that only two of those families can be obtained as limits of the discrete system while the third is extraneous.

* Corresponding author.
E-mail addresses: bertram.during@warwick.ac.uk (B. Düring), N.Georgiou@sussex.ac.uk (N. Georgiou), sara.merino@univie.ac.at (S. Merino-Aceituno), E.Scalas@sussex.ac.uk (E. Scalas).
URLs: https://homepages.warwick.ac.uk/staff/Bertram.During/ (B. Düring), http://www.sussex.ac.uk/profiles/329373 (N. Georgiou), https://saramerinoaceituno.wordpress.com (S. Merino-Aceituno), http://www.sussex.ac.uk/profiles/330303 (E. Scalas).

1 B. Düring has been supported by the Leverhulme Trust research project grant: Novel discretisations for higher-order nonlinear PDE (RPG-2015-69).
2 N. Georgiou was partially supported by the EPSRC First Grant EP/P021409/1: The flat edge in last passage percolation.
3 S. Merino-Aceituno is supported by the Vienna Science and Technology Fund (WWTF) with a Vienna Research Groups for Young Investigators, grant VRG17-014.
4 E. Scalas has been supported by the Japan Society for the Promotion of Science Invitational Fellowship S18099.

https://doi.org/10.1016/j.spa.2022.03.015

MSC: 60J05; 60F17; 35Q91; 35Q20; 60J10; 60J20; 82B31; 82B40

Keywords: Wealth distribution; Mean-field limits; Functional limits; Markov chains; Kinetic equations; Partitions of integers

1. Introduction

This study was originally motivated by a new approach to macroeconomics modelling based on (1) continuous-time Markov chains to model stochastic dynamics interactions among agents and (2) combinations of stochastic processes and combinatorial analysis, called combinatorial stochastic processes [35]. Such an approach was extensively presented in [1]. Those authors argue that, in case (1), the master equation describes how states of the models evolve stochastically in time and, in case (2), combinatorial stochastic processes are applied to describe the random formation of clusters of agents as well as the distribution of cluster sizes. Mathematically, the two approaches are so strictly related that it is not necessary to distinguish between them. This point was already implicitly made in Chapter 10 of [24]. Moreover, both approaches are related to kinetic equations of Boltzmann type used in statistical physics [34].

We previously worked on the class of Markov-chain models described below in [13] where we focused on the existence and uniqueness of the invariant measures and on the stability of the Markov chains. Some results in this article can be found in the expository chapter [12], written with an eye for economists and with all the proofs omitted.

We explore the connection between combinatorial stochastic processes and kinetic equations of Boltzmann type via functional limit theorems of properly scaled processes in the spirit of [3,22,27].

In this article we study a simple discrete model for wealth dynamics using a coagulation — fragmentation process. This is the same as the one in [12,13].

The *discrete space, discrete time (DS-DT)* model is a Markov Chain on the integers partitions of n that have size N. In other words, the state space is comprised of all non-negative integer vectors $\mathbf{x}^{n,N} = (x_1, \dots, x_N) \in \mathbb{Z}_+^N$ so that $\sum_{i=1}^N x_i = n$. Note that in here the word "partition" does not mean that the x_i are ordered in a non-decreasing order but rather corresponds to what is called"compositions" (for example see [35]).

The x_i's represent the wealth of the ith individual and the superscripts are there to remind us of the total wealth and number of agents. We denote the state space by $S_{N-1}^{(n)} = n\Delta_{N-1} \cap \mathbb{Z}^N$, where

$$\Delta_{N-1} = \left\{ \mathbf{x} = (x_1, \dots, x_N) : x_i \geq 0 \text{ for all } i = 1, \dots, N \text{ and } \sum_{i=1}^N x_i = 1 \right\}, \tag{1.1}$$

is the N-dimensional unit simplex.

At every discrete time step, we choose an ordered pair of indices from 1 to N uniformly at random (say (i, j)) and add the individual wealths $x_i + x_j$ of the agents. After that, the first chosen agent i receives a uniform portion of the total wealth between 0 and $x_i + x_j$ and the rest goes to the second agent j. Let $\mathbf{X}_t^{n,N}$ denote the wealth distribution at time t. The transition probabilities for this chain are given by

$$\mathbb{P}\{\mathbf{X}_{t+1}^{n,N} = \mathbf{x}' | \mathbf{X}_t^{n,N} = \mathbf{x}\} = \sum_{(i,j): i \neq j} \left\{ \frac{1}{N} \frac{1}{N-1} \frac{1}{x_i + x_j + 1} \delta_{x_i + x_j, x_i' + x_j'} \prod_{k \neq i,j} \delta_{x_k, x_k'} \right\}. \tag{1.2}$$

By symmetry, the transition matrix for the chain is doubly stochastic, therefore the invariant distribution is uniform on $S^{(n)}_{N-1}$ which is also obtained as $t \to \infty$ because of irreducibility and aperiodicity.

After studying the discrete chain, it would be more realistic to allow the total wealth n to increase, but in general that would only alter the state space. However, there is way to converge to a *continuous space, discrete time (CS-DT) model*, if we alter the discrete model slightly. In particular, instead of looking at the distribution of wealth, we look at the distribution of the **proportion** of wealth, namely the process $\mathbf{Y}^{n,N} = n^{-1}\mathbf{X}^{n,N}$ which is a rescaling of the original discrete process by the total wealth. The state space for the $\mathbf{Y}^{n,N}$ process is the meshed simplex

$$\Delta_{N-1}(n) = \Big\{(q_1, \dots, q_N) : 0 \le q_i \le 1, \sum_{i=1}^{N} q_i = 1, nq_i \in \mathbb{N}_0\Big\} \subset \Delta_{N-1}. \tag{1.3}$$

Then in [13], it was shown (Proposition 3) that as $n \to \infty$ one had the weak convergence of one-dimensional marginals

$$\mathbf{Y}^{n,N}_t \Longrightarrow \mathbf{X}^{\infty,N}_t \text{ as } n \to \infty, \tag{1.4}$$

under the mild assumption that the initial distributions of $\mathbf{Y}^{n,N}$, $\mu^{n,N}_0$ converge weakly to some distribution $\mu^{\infty,N}_0$ on Δ_{N-1}. The process $\mathbf{X}^{\infty,N}_t$ is identified as a continuous space, discrete time Markov chain on Δ_{N-1}. At each discrete time step t, an ordered pair of agents, say (i, j) is selected uniformly at random, with total proportion of wealth $x_i + x_j$. Then an independent uniform random variable $u_{t,(i,j)} \sim \text{Unif}[0, 1]$ is drawn and the new proportion of wealth for agent i is $u_{t,(i,j)}(x_i + x_j)$ while for agent j is $(1 - u_{t,(i,j)})(x_i + x_j)$. Note that the agents' wealth is an exchangeable random variable; while the description above needs ordered pairs of agents, it has no bearing on the distribution of the eventual wealth, as both $u_{t,(i,j)}$ and $1 - u_{t,(i,j)}$ are uniformly distributed on $[0, 1]$.

For the CS-DT chain $\mathbf{X}^{\infty,N}_t$, it was further shown that the invariant distribution of wealth proportions as $t \to \infty$ is uniform on Δ_{N-1}.

Here, we go a few steps further. First, we show the process level convergence

$$\mathbf{Y}^{n,N} \Longrightarrow \mathbf{X}^{\infty,N} \text{ as } n \to \infty, \tag{1.5}$$

by showing convergence of the finite dimensional marginals of the process. Then, using the Poissonization trick [36], we will change time and consider a continuous-time version of our continuous-space Markov chain. In an other appropriate scaling limit, this will lead to one-dimensional kinetic equations of Boltzmann type as studied e.g. in [2]. Stochastic mean-field dynamics for interacting particle systems are well-studied; for example see [9] for models where components are exchangeable, as in our model here.

1.1. Kinetic equations for wealth models

A one-dimensional caricature of the three-dimensional Boltzmann equation for Maxwell molecules is the Kac model. While simpler, it retains key properties of the original Boltzmann equation, such as energy conservation in binary collisions. The Kac equation has been deeply analysed using Fourier analysis techniques, e.g. in [4,5]. This model also allows for a rigorous passage from the kinetic model with binary interactions to a Fokker–Planck equation in the grazing collisions limit [39,43].

In the last two decades, the mechanism of the binary interaction, originally developed for the Boltzmann equation, has been fruitfully adapted to describe collective dynamics in many-agent socio-economic systems. The basic idea is to describe the behaviour of a sufficiently large number of interacting agents in the socio-economic system by pairwise, microscopic interactions, similar to the physical models of rarefied gas dynamics, where molecules collide inside a container. One can then study the long-time dynamics of the system and observe the formation of macroscopic distributions, depending on the details of the microscopic interactions. This approach has been successfully followed to model wealth distribution in simple market economies [8,16,19,20], wealth distribution under taxation [17,41], opinion formation [15,21,40], asset pricing [14], continuous models for ratings [18,26], and others.

1.2. Kinetic equations as limits of discrete particle models

The derivation of kinetic equations from discrete particle models is classical in kinetic theory. This is generally a hard problem that involves proving that 'propagation of chaos' holds for the system. This corresponds to showing that the particles become statistically independent when their number grows large. Typically, proving propagation of chaos allows to close the BBGKY hierarchy, i.e., the hierarchy of equations giving the evolution of the marginal distributions associated to the system [6,37]. In the case of the classical Boltzmann equation, which describes hard-sphere collision dynamics, the kinetic limit was shown in [30], though there is still a proof missing for long times [23].

In this work we will use a probabilistic approach in order to obtain the kinetic equation of the system under consideration. On this account, Sznitman [38] showed the kinetic limit for McKean–Vlasov systems of Stochastic Differential equations using a coupling argument. This argument has been further extended recently in [11] to a piece-wise deterministic Markov process. Previous works also investigate the speed of convergence to the kinetic equation in terms of the number of particles.

For our results, we must use a different approach, since we consider a pure jump process. Particularly, the methodology used is based on computing the limit of the martingale formulation associated to the jump (Markov) process. The methodology used here has been applied with great success to the investigation of coagulation models and the Smoluchowski equation in [32,33] and later to a system of instantaneous coagulation–fragmentation processes in [31].

1.3. Content and structure

In Section 2, we introduce the three connected models of the evolution of wealth, and present our results. The first one is an alternative formulation of the discrete model (discrete space, discrete time) with conserved wealth. The state space of the process is a discrete finite dimensional simplex. The dimension is the number of agents, and at each time step two agents interact (or collide). The Markovian evolution of the process is that of a discrete coagulation–fragmentation process.

The second model is obtained as a scaling spatial limit of the first one and is effectively the continuous space, discrete time analogue. Section 3 is dedicated to show process level convergence from the discrete to the continuous space model. Finally, the third model is the mean-field continuous limit for the empirical distribution of wealth. Agents are viewed as particles with binary interactions. In Section 4, by assuming the coagulation–fragmentation process jumps at the times of a Poisson process and letting the number of agents tend to infinity while appropriately scaling time, we obtain the relevant kinetic equations.

Section 5 is concerned with invariant distributions for the kinetic equation. While we find at least three potential invariant measures for the limiting empirical wealth (a delta, an exponential and a family of truncated exponential distributions) we show that from the particle system description only two of these are acceptable limits (the delta and the exponential). This highlights the power of the probabilistic approach, as a purely analytical one would not be able to *a priori* exclude that family. Similar laws of large numbers for empirical measures of particle systems can be found for a huge class of processes in the literature, e.g. [25].

To make the paper as self-consistent as possible, we have included an appendix on functional limit theorems for stochastic processes.

2. The models and results

We briefly describe the various models we are using, and collect the main results for an organised reference.

We consider N agents (originally N is fixed) and wealth W_N (originally fixed to be and integer denoted by n).

2.1. Equivalent construction of the DS-DT process.

For any $n \in \mathbb{N}$ the process $\mathbf{Y}^{(n)}$ is defined on $\Delta_{N-1}(n)$ given by (1.3), and we emphasise that for every n, $\Delta_{N-1}(n) \subset \Delta_{N-1}$, given by (1.1). $\Delta_{N-1}(n)$ is treated as the meshed simplex Δ_{N-1}; the mesh size is n^{-1}, which is precisely the reciprocal of the total wealth $W_n = n$.

Let $\mathcal{P}^n$ denote the law of the process $\mathbf{Y}^{(n)} = (\mathbf{Y}_0^{(n)}, \mathbf{Y}_1^{(n)}, \dots, \mathbf{Y}_k^{(n)}, \dots) \in (\Delta_{N-1}(n))^{\mathbb{N}_0} \subset (\Delta_{N-1})^{\mathbb{N}_0}$. The measure for $k+1$-th dimensional marginal $(\mathbf{Y}_0^{(n)}, \mathbf{Y}_1^{(n)}, \dots, \mathbf{Y}_k^{(n)})$ is denoted by

$$\mathcal{P}_k^n\{\cdot\} = \mathcal{P}^n\big\{(\mathbf{Y}_0^{(n)}, \mathbf{Y}_1^{(n)}, \dots, \mathbf{Y}_k^{(n)}) \in \cdot\big\}. \tag{2.1}$$

Similarly, denote by $\mathcal{P}^\infty$ and $\mathcal{P}_k^\infty$ the corresponding quantities for $\mathbf{X}^\infty$. The law of $\mathbf{Y}_0^{(n)}, \mathbf{X}_0^{(\infty)}$ are denoted by $\mu_0^{(n)} = \mathcal{P}_0^{(n)}$ and $\mu_0^{(\infty)} = \mathcal{P}_0^{(\infty)}$ respectively.

Starting from an initial distribution $\mu_0^{(n)}$ we construct the process $\mathbf{Y}^{(n)}$ using an i.i.d. sequence of uniform random variables

$$U_{i,j}^{(n)}(k) \sim \text{Unif}[0,1], \qquad 1 \le i, j \le N, i \ne j, k \in \mathbb{N}_0, n \in \mathbb{N}. \tag{2.2}$$

These random variables from (2.2) suffice to construct the whole process. The variable k plays the role of time index, and (i, j) is the ordered pair of agents that are selected. We assume – and use without a particular mention – that random variables (2.2) are independent of the initial distribution $\mu_0^{(n)}$.

For any $x \in \mathbb{R}_+$ we define

$$[x]_n = \frac{a}{n}, \qquad \text{so that } \frac{a}{n} \le x < \frac{a+1}{n}, \quad a \in \mathbb{N}_0,$$

and use this symbol for notational convenience when we define the evolution of the process directly on $\Delta_{N-1}(n)$.

Let $\mathbf{Y}_k^{(n)} = (y_1(k), \dots, y_N(k)) \in \Delta_{N-1}(n)$ be the vector of discrete wealths, normalised so that the total wealth is 1. Then, if indices $i.j$ were chosen to interact at time step k, the total

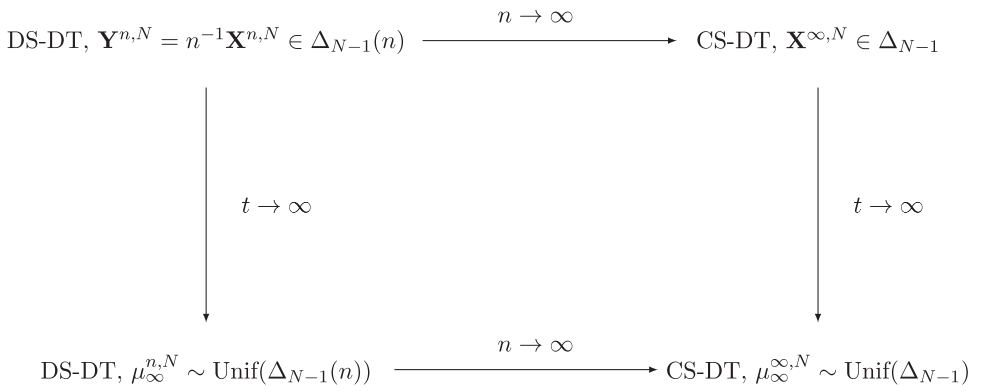

Fig. 1. Commutative diagram demonstrating the various limiting measures, depending on the order limits are taken, when the total wealth remains constant. Measures $\mu_\infty^{n,N}$ and $\mu_\infty^{\infty,N}$ denote the invariant distributions for the two Markov chains respectively.

wealth at time $k+1$ would become

$$\mathbf{Y}^{(n)}_{k+1} = (y_1(k), \dots, \underbrace{[U^{(n)}_{i,j}(k)(y_i(k)+y_j(k))]_n}_{y_i(k+1)}, \dots, \underbrace{y_i(k)+y_j(k)-y_i(k+1)}_{y_j(k+1)}, \dots, y_N(k))$$
$$= g_{i,j}(\mathbf{y}_k, U^{(n)}_{i,j}(k)).$$

Check to see that the coordinate $[U^{(n)}_{i,j}(k)(y_i(k)+y_j(k))]_n$ is uniformly distributed on the set $\{0, n^{-1}, \dots, (y_i(k)+y_j(k)-n^{-1}) \vee 0\}$, and therefore this procedure gives the same process as described in [13]. The function $g_{i,j}$ is a measurable function that depends on the value of the current state and the new uniform random variable, and the last display acts as the definition of $g_{i,j}$.

We prove the following theorem, which guarantees process-level convergence.

Theorem 2.1. *Assume the weak convergence of measures*

$$\mu_0^{(n)} \Longrightarrow \mu_0^{(\infty)}, \qquad \text{as } n \to \infty. \tag{2.3}$$

Furthermore, assume the weak convergence (as $n \to \infty$) of the i.i.d. sequence

$$\{U^{(n)}_{i,j}(k)\}_{i,j,k} \Longrightarrow \{U^{(\infty)}_{i,j}(k)\}_{i,j,k}, \tag{2.4}$$

so that the limiting sequence $\{U^{(\infty)}_{i,j}(k)\}_{i,j,k}$ is a sequence of i.i.d. uniform $[0,1]$ random variables, chosen in such a way that are also independent from $\mu_0^{(\infty)}$.

Then

$$\mathcal{P}^n \Longrightarrow \mathcal{P}^\infty, \qquad \text{as } n \to \infty.$$

The theorem gives that the order in which we take limits in the diagram of Fig. 1 is immaterial and the diagram is commutative. This will be proven in Section 3. Horizontal arrows in the diagram of Fig. 1 denote weak convergence, but the top one can be upgraded to almost sure convergence if we are concerned with finite sample paths (see also Remark 3.1). Note that assumption (2.4) is merely an implicit definition of the limit sequence and it does not impose any restriction in the theorem. When we only seek weak convergence (as stated), the limit sequence can be any i.i.d. uniform sequence, and the condition is automatically satisfied.

Moreover, we will investigate the mean field limit of the CS-DT process, as $N \to \infty$. In order to do this using kinetic theory, it is useful to switch to a continuous time Markov chain, where jump times coincide with those of a rate 1 Poisson process, which is why it is called a "Poissonisation trick". It is standard to argue that the long time behaviour of the discrete time process is the same as that of the Poissonised one when N is fixed, irrespective of the rate of the Poisson process. The finite time distribution of the proportions of wealth for the CS-CT Poissonised process, which we momentarily denote by $\mathbf{X}_t^{\text{Pois}}$, can also be rigorously obtained by standard conditioning on the number of Poisson events up to time t, using the following equation

$$\mathbb{P}\{\mathbf{X}_t^{\text{Pois}} \in A\} = \sum_{\ell=0}^{\infty} \mathbf{P}\{\mathbf{X}_\ell^{\infty,N} \in A\} P\{N_t = \ell\} = \sum_{\ell=0}^{\infty} \mathbf{P}\{\mathbf{X}_\ell^{\infty,N} \in A\} \frac{e^{-t/N} t^\ell}{\ell!\, N^\ell}. \tag{2.5}$$

N_t is the background Poisson process with rate $1/N$ and A is any Borel subset of the simplex. We omit the argument that the limiting distribution is still uniform on the simplex.

Remark 2.2. The coagulation–fragmentation process is very versatile and it is therefore well-studied in other contexts. For example it can be viewed also as a process on integer partitions of integers. To be precise, for any fixed $N \in \mathbb{N}$ we have that $\sum_i X_t^{i,N} = W_N$. If we assume W_N is an integer, we can interpret the vector $\mathbf{X}_t^N$ as a random (real) partition of the integer W_N and the process $\{\mathbf{X}_t^N\}_{t \geq 0}$ can be viewed as a Markov chain on these partitions. Most recently, a version of the process (with deterministic binary interactions at discrete time steps) has been studied in [7] in terms of its rate of convergence to the equilibrium.

2.2. *Martingale formulation for the CS-DT model.*

In general, it is not necessary to restrict to a case where the total wealth is 1 for all N, the same models can be studied when the total wealth is a function of N; here we do so for the kinetic model. Let us first introduce some notation. The total wealth in a system of N agents is denoted by a value $W_N \in \mathbb{R}_+$ (which we also allow to be 0). The state of the process at time t is a vector of non-negative real numbers

$$\mathbf{X}_t^N = (X_t^{1,N}, \dots, X_t^{N,N})$$

with state space

$$\Delta_{W_N} := \Big\{(x_1, \dots, x_N) : x_i \geq 0 \text{ for all } 1 \leq i \leq N \ \text{ and } \sum_{i=1}^N x_i = W_N\Big\}.$$

The dynamics on Δ_{W_N} are given by binary interactions, where an ordered pair of two agents (i, j) is chosen uniformly at random. The interactions are assumed to happen at constant rate $1/N$, at the events of a background Poisson process. After the interaction, the wealth of the pair $(X^{i,N}, X^{j,N})$ is changed to $((X^{i,N})', (X^{j,N})')$ with

$$(X^{i,N})' = r(X^{i,N} + X^{j,N}),$$
$$(X^{j,N})' = (1-r)(X^{i,N} + X^{j,N}),$$

where r is a random variable with uniform law on [0, 1] that is drawn at time t, independently of the past of the chain. Interactions preserve the total mass,

$$W_N := \sum_{i=1}^{N} X^{i,N}, \tag{2.6}$$

and, therefore, the dynamics take place on Δ_{W_N}. We will consider two cases:

(i) Absolute wealth: $X^{i,N}$ represents the wealth of agent i and W_N represents the total wealth of the system;
(ii) Relative wealth: in this case $X^{i,N}$ represents the proportion of wealth of agent i and $W_N = 1$ for all N.

We are interested in studying the case when the number of agents grows large, i.e., $N \to \infty$. The first thing to observe is that agents are *exchangeable* by virtue of the non-preferential dynamics. Questions of interest also reflect that, in the sense that we want to know how much wealth the richest agent has, rather than who is the richest agent, since they all have the same probability of being the most rich. For this reason, we will focus our study on the empirical distribution

$$\mu_t^N(x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}(x). \tag{2.7}$$

The empirical distribution μ_t^N is a random probability measure on $\mathbb{R}_+$ that depends on the realisation of the Markov chain. For any interval $[a, b]$,

$$\begin{aligned}\mu_t^N([a,b]) &= \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}[a,b] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}\{a \le X_t^{i,N} \le b\}\\ &= \frac{\text{card}\{i : \text{agent } i\text{'s wealth } \in [a,b]\}}{N}.\end{aligned}$$

In general, for any measure μ on $\mathbb{R}_+$, and any μ-measurable function g, we define the brackets $\langle \cdot, \cdot \rangle$ by

$$\langle g, \mu \rangle := \int_{\mathbb{R}_+} g(x)\mu(dx). \tag{2.8}$$

When μ is a probability measure, the bracket notation is just another way to denote the expected value $\mathbb{E}_\mu(g)$. With this definition, when the measure is $N\mu_t^N(x_0)$ for some fixed x_0, the bracket $\langle 1, N\mu_t(x_0)\rangle$ gives the number of agents with wealth precisely x_0 at time t. Equivalently, keep the empirical measure as $\mu_t^N(x)$ and set $g(x) = N\sum_{i=1}^{N} \mathbb{1}\{X_t^{i,N} = x_0\}$ in order to obtain the same interpretation.

The total wealth in the system represented by W_N at time 0 as in Eq. (2.6), and we can write this fact in terms of the empirical distribution as

$$W_N = N\langle x, \mu_0^N\rangle. \tag{2.9}$$

The total wealth at time t is given by $N\langle x, \mu_t^N\rangle$ and it remains fixed for all $t \ge 0$ if we assume a conserved total wealth. Notice that if $W_N/N \to m$ as $N \to \infty$ then we also have that

$$\lim_{N\to\infty} \langle x, \mu_0^N\rangle = m. \tag{2.10}$$

If $\mu_0^N \Longrightarrow \mu_0$ weakly for some probability measure μ_0 and $m = 0$, Eq. (2.10) would imply that $\mu_0^N(x) \Longrightarrow \delta_0(x)$, as the measure has no support on the negative reals.

For a fixed t, the empirical measure μ_t^N is an element of the space of probability measures $\mathcal{M}_1$ on $\mathbb{R}_+$ and it only changes whenever an interaction event occurs. It is a function of the Markov chain $\mathbf{X}_t^N$ and it is also a Markov chain.

In order to describe its generator $\mathcal{G}$, we define the measure $\mu^{(x,y,r),N}$ after an interaction between an agent of wealth x (chosen first) and one of wealth y (chosen second) to be

$$\mu^{(x,y,r),N} = \mu^N - \frac{1}{N}\delta_x - \frac{1}{N}\delta_y + \frac{1}{N}\delta_{r(x+y)} + \frac{1}{N}\delta_{(1-r)(x+y)}.$$

Finally, we define the pair-measure $\mu^{(2,N)}$ on rectangles that generate the Borel σ-algebra $\mathcal{B}(\mathbb{R}\times\mathbb{R})$ to be

$$\mu^{(2,N)}(A\times B) = \mu^N(A)\mu^N(B) - \frac{1}{N}\mu^N(A\cap B), \qquad A, B \in \mathcal{B}(\mathbb{R}). \tag{2.11}$$

This is a natural choice of the pair measure, as it is a simplified version of the joint empirical measure for a pair of variables. Note that it is not a probability measure, but this does not matter, as we will only use it as $N\to\infty$. For more clarification and details see Remark 2.4 at the end of the section.

The generator for the evolution of μ_t^N, considering an interaction rate of $1/N$, is given by

$$\mathcal{G}F(\mu^N) = \int_0^1\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\{F(\mu^{(x,y,r),N}) - F(\mu^N)\}\mathbb{1}_{\{x+y\le W_N\}}N\mu^{(2,N)}(dx,dy)\,dr. \tag{2.12}$$

In the equation above, function F belongs to $C_b(\mathcal{M}_1)$, i.e., bounded measurable functions on the space of probability measures $\mathcal{M}_1$. Note that the evolution of the empirical measure under the law of the microscopic process is Markovian. We impose the term $\mathbb{1}_{\{x+y\le W_N\}}$ in the generator to ensure that the two masses created after the jump fulfil $r(x+y)\le W_N$ and $(1-r)(x+y)\le W_N$.

Remark 2.3. In this manner, we could consider that $\mu_t^N \in \mathcal{P}([0,W_N])$. However, to avoid having a functional space depending on the value of N, we will just consider that $\mu_t^N\in\mathcal{P}(\mathbb{R}_+)$. Notice that the generator can be also interpreted as representing a N-particle system with values in $\mathbb{R}_+$ where only pairs of values interact as long as their sum is below W_N.

Given the generator in (2.12), we have that the quantity M_t^F defined by

$$M_t^F = F(\mu_t^N) - F(\mu_0^N) - \int_0^t \mathcal{G}F(\mu_s^N)\,ds \tag{2.13}$$

is a martingale [29, Appendix], for any $F\in C_b(\mathcal{M}_1)$. In particular, for any function $g\in C_b(\mathbb{R}_+)$ (measurable bounded functions in $\mathbb{R}_+$), we define $F_g\in C_b(\mathcal{M}_1)$ by $F_g(\mu) = \langle g,\mu\rangle := \int g(x)\mu(dx)$. Expression (2.13) can now be re-written as

$$M_t^{g,N} = \langle g,\mu_t^N\rangle - \langle g,\mu_0^N\rangle - \int_0^t\langle g, Q^{(N)}(\mu_s^N)\rangle\,ds, \tag{2.14}$$

where we are denoting $\mathcal{G}(\langle g,\mu^N\rangle)$ by

$$\begin{aligned}\mathcal{G}(\langle g,\mu^N\rangle) &= \int_0^1\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\Big(g(r(x+y)) + g((1-r)(x+y)) - g(x) - g(y)\Big)\\ &\qquad\times\ \mathbb{1}_{\{x+y\le W_N\}}\mu^{(2,N)}(dx,dy)\,dr\\ &= \langle g, Q^{(N)}(\mu^N)\rangle.\end{aligned} \tag{2.15}$$

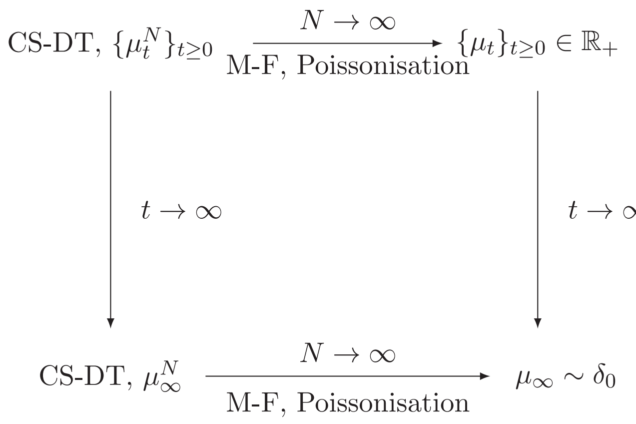

Fig. 2. Commutative diagram demonstrating the various limiting measures, depending on the order limits are taken, when the total wealth remains constant. There are two parameters that scale; the number of agents N and the time t. Time is discrete for the left down-arrow, but continuous in the right down-arrow. There is an intermediate step missing from the diagram in which discrete time events are changed with time events arising from a Poisson process of rate $1/N$ which simultaneously scales with N. That is called the Poissonisation step, and when the mean-field limits (M-F) are taken, the rate of the Poisson process also scales with N.

The last line in fact allows us to define $Q^{(N)}(\mu)$ implicitly via its brackets with bounded continuous functions g.

In the following sections we will see that μ_t^N converges in probability as $N \to \infty$ to a measure μ which is solution of the following kinetic equation in weak form:

$$\mu_t = \mu_0 + \int_0^t Q(\mu_s)\, ds, \tag{2.16}$$

or equivalently, for any $g \in C_b(\mathbb{R}_+)$

$$\langle g, \mu_t \rangle = \langle g, \mu_0 \rangle + \int_0^t \langle g, Q(\mu_s) \rangle\, ds. \tag{2.17}$$

The operator Q is defined as follows: for any $g \in C_b(\mathbb{R}_+)$

$$\begin{aligned} \langle g, Q(\mu) \rangle = \int_{[0,1]} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} & (g(r(x+y)) + g((1-r)(x+y)) - g(x) - g(y)) \\ & \times \mathbb{1}_{\{x+y \leq w_0\}} \mu(dx)\mu(dy)\, dr, \end{aligned} \tag{2.18}$$

with $w_0 = \lim_{N\to\infty} W_N$. We will also investigate the limit $t \to \infty$ and obtain different families of limiting invariant measures, in the process verifying the following commutative diagram of Fig. 2 in the simple case of fixed wealth $W_N = c$ for all N.

In Fig. 2, the left down-arrow was obtained in [13]. The lower horizontal arrow is obtained in the present article in Proposition 2.8, and the remaining arrows in Sections 5 and 4.

Remark 2.4. Eq. (2.11) is a natural choice for the pair measure, as the following calculation demonstrates. We begin from the joint empirical measure

$$\nu_t^N(x, y) = \frac{1}{N(N-1)} \sum_{(i,j):i\neq j} \delta_{(X_t^{i,N}, X_t^{j,N})}(x, y).$$

On general product events $A \times B$ the measure can be computed as

$$\begin{aligned}
\nu_t^N(A \times B) &= \frac{1}{N(N-1)} \sum_{(i,j):i\neq j} \mathbb{1}\{X_t^{i,N} \in A, X_t^{j,N} \in B\} \\
&= \frac{1}{N(N-1)} \sum_{(i,j):i\neq j} \mathbb{1}\{X_t^{i,N} \in A\}\mathbb{1}\{X_t^{j,N} \in B\} \\
&= \frac{1}{N(N-1)} \sum_i \mathbb{1}\{X_t^{i,N} \in A\} \sum_j \mathbb{1}\{X_t^{j,N} \in B\} \\
&\quad - \frac{1}{N(N-1)} \sum_i \mathbb{1}\{X_t^{i,N} \in A \cap B\} \\
&= \frac{N}{N-1}\mu_t^N(A)\mu_t^N(B) - \frac{1}{N-1}\mu_t^N(A \cap B) = \frac{N}{N-1}\mu_t^{(2,N)}(A \times B).
\end{aligned}$$

As $N \to \infty$ the prefactor $N/(N-1) \to 1$ and the limiting measure has the same asymptotic properties. We choose to use the simplest form (2.11) without loss of generality.

We can now state our main theorem.

Theorem 2.5 (*Mean-Field Limit*). *Suppose that W_N is a non-decreasing sequence converging to $w_0 \in (0, \infty]$ as $N \to \infty$. Suppose that for a given measure μ_0 it holds that*

$$\langle x, \mu_0^N\rangle \leq \langle x, \mu_0\rangle < \infty, \tag{2.19}$$

and that as $N \to \infty$

$$\mu_0^N \Longrightarrow \mu_0 \qquad \text{weakly, as } N \to \infty. \tag{2.20}$$

Then the sequence of random measures $(\mu_t^N)_{t\geq 0}$ converges in probability in $D([0,\infty); \mathcal{M}_1(\mathbb{R}_+))$, as $N \to \infty$. The limit $(\mu_t)_{t\geq 0}$ is continuous in t and it satisfies the kinetic equation (2.16). In particular, for all $g \in C_b(\mathbb{R}_+)$ the following limits hold in probability, for any time t

(A) $\lim_{N\to\infty} \sup_{s\leq t} \langle g, \mu_s^N - \mu_s\rangle \stackrel{\mathbb{P}}{=} 0$,
(B) $\lim_{N\to\infty} \sup_{0\leq s\leq t} |M_s^{g,N}| \stackrel{\mathbb{P}}{=} 0$,
(C) $\lim_{N\to\infty} \int_0^t \langle g, Q^{(N)}(\mu_s^N)\rangle\, ds \stackrel{\mathbb{P}}{=} \int_0^t \langle g, Q(\mu_s)\rangle\, ds$.

As a consequence, Eq. (2.16) is obtained as the limit in probability of (2.14) as $N \to \infty$.

Some observations from Theorem 2.5 follow. From Eq. (2.9) we have that $N\langle x, \mu_0^N\rangle = W_N$. If we now assume that $\lim_{N\to\infty} N^{-1}W_N = m \in (0,\infty)$ then we see that W_N grows linearly in N and condition (2.19) implies

$$m \leq \langle x, \mu_0\rangle.$$

Now if W_N grows superlinearly, i.e. $\lim_{N\to\infty} N^{-1}W_N = \infty$, then condition (2.19) in Theorem 2.5 is violated and the theorem does not necessarily hold.

Finally, if either $\lim_{N\to\infty} W_N = w_0$ for some absolute constant w_0 or $W_N \to \infty$ as $N \to \infty$, but $\lim_{N\to\infty} N^{-1}W_N = 0$, we can actually study the asymptotic behaviour ($N \to \infty$) of the measures μ_t^N and show that the limiting measure is a δ mass as $N \to \infty$. This is discussed in Section 4.

Example 1. Assume $W_N \to \infty$, but so that

$$\lim_{N\to\infty} \frac{W_N}{N} = \infty.$$

As we mentioned above, when the wealth grows superlinearly, the theorem does not necessarily apply. However there is a way to scale using a random approximation to W_N so that the theorem works by renormalising the wealth, and so that we can verify its conditions.

Consider an i.i.d. sequence of geometric random variables $\{G_i(p)\}_{i\in\mathbb{N}}$ with mass function

$$\mathbb{P}\{G_i = k\} = p(1-p)^k, \quad k = 0, 1, 2, \ldots \tag{2.21}$$

In this case we fix

$$p_N = \frac{N}{W_N},$$

as the success probability of each independent geometric (so the sequence refreshes with every N). To denote this dependence we write $G_i^{(N)}$ for each geometric. Then define

$$(X_1^N, \ldots, X_N^N) = \begin{cases} (1, \ldots, 1), & \text{when } \sum_{i=1}^N G_i^{(N)} = 0, \\ N\Big(\frac{G_1^{(N)}}{\sum_{i=1}^N G_i^{(N)}}, \ldots, \frac{G_N^{(N)}}{\sum_{i=1}^N G_i^{(N)}}\Big), & \text{otherwise.} \end{cases} \tag{2.22}$$

This is a distribution on the simplex $N\Delta_{N-1}$. Conditional on the value of $\sum_{i=1}^N G_i^{(N)} = w_N$, this distribution is uniform on the discrete simplex with mesh N/w_N. As N grows, the geometric random variables are sharply concentrated around their mean $N^{-1}W_N$, so distribution (2.22) approximates the uniform distribution (as N grows) on the simplex since $\sum_{i=1}^N G_i^{(N)} \approx W_N$.

Let $(X_1^N, \ldots, X_N^N)$ be distributed as in (2.22), and define the empirical measure $\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}$. Then

$$\mu_0^N \Longrightarrow \mu_0 \sim \text{Exp}(1), \quad \mathbb{P} - a.s. \tag{2.23}$$

The details of showing this can be found in the first version of this article which is available in Arxiv, along with a longer discussion on integer partitions. As it turns out, the limiting measure in (2.23) is an invariant measure for the mean field limit. These results might be published separately in the near future. □

Remark 2.6. Note that our initial sequence of measures in the example is not uniform on the sequence of simplexes $N\Delta_N$, in contrast with Theorem 2 and Corollary 2 of [42]. However, as the geometric random variables are concentrated around their mean, the initial measures can be viewed as approximation of discrete uniform measures on mesh $1/W_N$ for Δ_N. Or, one can view them as measures on partitions of the random number $\sum_{i=1}^N G_i^{(N)}$.

2.3. Invariant measures for the mean field limit

In general, a measure $\tilde{\mu}$ is invariant (or stationary) for (2.16) if and only if when $\mu_0 = \tilde{\mu}$ then we have that $\mu_t = \tilde{\mu}$ for all $t > 0$.

One way to obtain invariant measures is to actually make some educated ansatz for μ_0 and show that it remains unchanged under the kinetic Eq. (2.16). It is immediate to check that for any value of w_0 (bounded or unbounded), the measure

$$\tilde{\mu}(x) = \delta_0(x) \tag{2.24}$$

is invariant for (2.16).

A more natural way to find invariant measures originates from the Markov chain perspective, where (limiting) *equilibrium* measures $\bar{\mu}$ are obtained by taking the limit (in the appropriate weak sense) of the measures μ_t as $t \to \infty$, i.e.

$$\bar{\mu} = \lim_{t\to\infty} \mu_t.$$

If such a limit exists then the measure $\bar{\mu}$ will be invariant. A sequence of measures however, may have many limit points; it is always an important and difficult task to decide whether those limits that are obtained include all possible equilibria for the system or are invariant probability measures. Moreover, the limiting measure(s) will depend on the initial measure μ_0 and other parameters of the evolution.

In this subsection, we discuss several invariant measures that can be obtained as equilibria. We begin with the case where W_N grows sublinearly and we show that under Theorem 2.5, δ_0 is the only possible candidate for invariant equilibrium measure. Proposition 2.7 indeed asserts that result, under the assumptions of Theorem 2.5, and Proposition 2.8 argues that the assumptions of Theorem 2.5 hold when the total wealth $w_0 = 1$ and we start from a uniform density on the simplex. Together, these propositions verify the commutativity of the diagram in Fig. 2.

Proposition 2.7 (*Sub-linear Growth for W_N*)**.** *Suppose the same assumptions on the initial data as in Theorem 2.5. If it holds that*

$$\langle x, \mu_0^N \rangle = \frac{W_N}{N} \to 0, \qquad as\ N \to \infty,$$

(which is in particular true if $w_0 < \infty$), then, we have that $\lim_{N\to\infty} \mu_t^N \overset{\mathbb{P}}{=} \delta_0$ in probability for all times t.

Proposition 2.8 (*Mean Field Limit of the Empirical Wealth Under Equilibrium Measures.*)**.** *Suppose $\mu_0^{\infty,N} \sim \mathrm{Unif}[\Delta_{N-1}]$ (therefore we assume the total wealth is fixed and equal to 1) for each $N \in \mathbb{N}$ and consider the empirical measure on $\mathbb{R}_+$*

$$\mu_0^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_0^{i,N}}, \quad (X_0^{1,N}, \dots, X_0^{N,N}) \sim \mu_0^{\infty,N}.$$

Then as $N \to \infty$,

$$\mu_0^N \Longrightarrow \delta_0, \quad a.s.$$

In particular the assumptions of Theorem 2.5 hold and, since $w_0 = 1$, Proposition 2.7 is in effect.

Remark 2.9. Propositions 2.7 and 2.8 interpreted in the context of wealth, imply that if the wealth in the system does not grow proportional to the number of agents, there will be a condensation phenomenon. In that case, all the wealth of the system is accumulated to a single (or small number) of agents.

Corollary 2.10. *Let $\lim_{N\to\infty} W_N = w_0 \in (0, \infty]$. Assume that μ_t is a solution of (2.16) which has a density f_t for all t. Then*

(1) *If* $w_0 = \infty$*, the exponential distributions*

$$\tilde{f}(x) = \frac{e^{-x/m}}{m}, \tag{2.25}$$

are equilibria for the operator Q *and remain invariant under* (2.16)*. In particular, if* f_0 *is of the form* (2.25) *with* $\langle x, f_0 \rangle = m_0 > 0$*, then the distribution* (2.25) *with* $m = m_0$ *is a stationary solution of* (2.16)*.*

(2) *If* $0 < w_0 < \infty$*, then the following distributions are compactly supported on* $[0, w_0]$ *and are equilibria for the operator* Q

$$\tilde{f}(x) = \frac{e^{-x/m}}{m(1 - e^{-w_0/m})} \mathbb{1}_{\{x \le w_0\}}. \tag{2.26}$$

(3) *(Uniqueness of the invariant family at* $w_0 = \infty$*) Moreover, under the extra assumption that the density* f_t *is differentiable on* $\mathbb{R}_+$*, then measures with density* (2.25) *are the unique equilibria.*

Remark 2.11. Equilibria with exponential tails like (2.25) have been observed in several wealth models. For example, in the model of [8] where the mean wealth is conserved, slim tail equilibria can be observed when the market risk is low. Sometimes this is called "socialistic behaviour" because this is what actual wealth distributions in socialistic countries looked like. Tuning the risk parameter can also lead to delta equilibria or even equilibria with Pareto tails. Moreover, in the case of heavy tailed equilibria, all agents have a portion of the wealth, but it is just much more unevenly distributed and there is no small "upper class" which holds proportionally large fraction of the wealth. All these behaviours can be observed in real data and it is comforting that they can also be captured by this simple exchange model.

Uniqueness of equilibrium is also not surprising. In [16] one can find quantitative estimates about rates of convergence to the unique equilibrium of several wealth models.

The next proposition tells us that invariant distributions (2.26) cannot be obtained as limits of the discrete measures when the wealth remains bounded, therefore they are extraneous, while invariant distributions of the form (2.25) are possible.

Proposition 2.12. *Let* $\lim_{N\to\infty} W_N = w_0 \in (0, \infty]$.

(1) *(*$w_0 = \infty$*) Consider an infinite i.i.d. sequence* $\{X_i\}_{i\ge 1}$ *of* $\mathrm{Exp}(1/m_0)$ *variables. For every* $N \in \mathbb{N}$*, define*

$$W_N = \sum_{i=1}^{N} X_i, \quad \text{and} \quad \mu_0^N(x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i}(x),$$

i.e. the initial wealth of each agent is an independent exponential random variable as we increase the number of agents, but always fixed across the N *index. Then as* $N \to \infty$*,* $\mu_0^N \Longrightarrow \mu_0$ *where* $\mu_0(x) = \frac{1}{m_0} e^{-x/m_0}\,dx$ *and therefore Theorem 2.5 holds. Then by Corollary 2.10,* μ_0 *remains invariant in time.*

(2) *(*$w_0 < \infty$*) There does not exist a sequence of measures* $\{\mu_0^N\}_{N\in\mathbb{N}}$ *so that* $\mu_0^N \Longrightarrow \mu_0$ *with* μ_0 *having a density* (2.26)*.*

Remark 2.13. To see that (2) above is enough to guarantee that equilibria with density (2.26) cannot be obtained, apply Proposition 2.7. It is possible that equilibria (2.26) correspond to a metastability of condensates; conceivably they could occur when $W_N \to \infty$ (at some speed) but most of the wealth is concentrated by a single (or finitely many) agent(s). Whether this heuristic can be made rigorous merits further investigation and it is left as an open question for the moment.

3. Process level convergence to a discrete-time continuous space model

This section is dedicated to proving the process level convergence of the DS-DT model to the CS-DT model, thus completing Proposition 7.3 in [13] where convergence of the one dimensional marginals was shown. We need an equivalent, alternative description of the DS-DT model, so we begin this section with it. The number of agents N remains fixed throughout this section, so we will omit it from the notation, and we will write $\mathbf{Y}^{(n)}$, $\mathbf{X}^{(n)}$ and $\mathbf{X}^{(\infty)}$ instead of $\mathbf{Y}^{n,N}$, $\mathbf{X}^{n,N}$ and $\mathbf{X}^{\infty,N}$ respectively.

Proof of Theorem 2.1. Since we may embed the sequence of processes $\{\mathbf{Y}^{(n)}\}_{n\in\mathbb{N}}$ in $(\Delta_{N-1})^{\mathbb{N}}$, which is compact, the collection of their induced measures $\{\mathcal{P}^n\}_{n\in\mathbb{N}}$ is tight. Therefore, for process-level convergence, it suffices to show that finite dimensional marginals converge weakly. See the Appendix for references on convergence in the Skorokhod space and [3] for a general reference. We show this for vectors of the form $(\mathbf{Y}_0^{(n)}, \mathbf{Y}_1^{(n)}, \dots, \mathbf{Y}_k^{(n)})$ with law denoted by (2.1). In the calculation below, we denote by ν_U the law of the random variable U. For any generic measure μ, we denote by $\mathbb{E}_\mu$ the expectation operator with respect to that measure.

For any bounded continuous function f

$$
\begin{aligned}
\mathbb{E}_{\mathcal{P}_k^n}(f(\mathbf{Y}_0^{(n)}, \mathbf{Y}_1^{(n)}, \dots, \mathbf{Y}_k^{(n)})) &= \sum_{\mathbf{y}_0}\cdots\sum_{\mathbf{y}_k} f(\mathbf{y}_0,\dots,\mathbf{y}_k)\mathcal{P}_k^n\{\mathbf{Y}_0^{(n)}=\mathbf{y}_0,\dots,\mathbf{Y}_k^{(n)}=\mathbf{y}_k\}\\
&= \sum_{\mathbf{y}_0}\cdots\sum_{\mathbf{y}_k} f(\mathbf{y}_0,\dots,\mathbf{y}_k)\mathcal{P}_{k-1}^n\{\mathbf{Y}_0^{(n)}=\mathbf{y}_0,\dots,\mathbf{Y}_{k-1}^{(n)}=\mathbf{y}_{k-1}\}\\
&\qquad \times \mathbb{P}\{\mathbf{Y}_k^{(n)}=\mathbf{y}_k|\mathbf{Y}_{k-1}^{(n)}=\mathbf{y}_{k-1}\}\\
&= \sum_{\mathbf{y}_0}\cdots\sum_{\mathbf{y}_{k-1}} \mathcal{P}_{k-1}^n\{\mathbf{Y}_0^{(n)}=\mathbf{y}_0,\dots,\mathbf{Y}_{k-1}^{(n)}=\mathbf{y}_{k-1}\}\\
&\qquad \times \sum_{\mathbf{y}_k} f(\mathbf{y}_0,\dots,\mathbf{y}_k)\mathbb{P}\{\mathbf{Y}_k^{(n)}=\mathbf{y}_k|\mathbf{Y}_{k-1}^{(n)}=\mathbf{y}_{k-1}\}\\
&= \sum_{\mathbf{y}_0}\cdots\sum_{\mathbf{y}_{k-1}} \mathcal{P}_{k-1}^n\{\mathbf{Y}_0^{(n)}=\mathbf{y}_0,\dots,\mathbf{Y}_{k-1}^{(n)}=\mathbf{y}_{k-1}\}\\
&\qquad \times \frac{1}{N(N-1)}\sum_{(i,j):i\neq j}\int_0^1 f(\mathbf{y}_0,\dots,\mathbf{y}_{k-1}, g_{i,j}(\mathbf{y}_{k-1},u))\,du\\
&= \frac{1}{N(N-1)}\sum_{\mathbf{y}_0}\cdots\sum_{\mathbf{y}_{k-1}} \mathcal{P}_{k-1}^n\{\mathbf{Y}_0^{(n)}=\mathbf{y}_0,\dots,\mathbf{Y}_{k-1}^{(n)}=\mathbf{y}_{k-1}\}\\
&\qquad \times \sum_{(i,j):i\neq j}\mathbb{E}_{\nu_{U_{i,j}^{(n)}(k-1)}}\Big(f(\mathbf{y}_0,\dots,\mathbf{y}_{k-1}, g_{i,j}(\mathbf{y}_{k-1}, U_{i,j}^{(n)}(k-1)))\Big)\\
&= \frac{1}{N(N-1)}\mathbb{E}_{\mathcal{P}_{k-1}^n}\Big(\sum_{(i,j):i\neq j}\mathbb{E}_{\nu_{U_{i,j}^{(n)}(k-1)}}\Big(f(\mathbf{Y}_0^{(n)},\dots,\mathbf{Y}_{k-1}^{(n)}, g_{i,j}(\mathbf{Y}_{k-1}^{(n)}, U_{i,j}^{(n)}(k-1)))\Big)\Big)\\
&= \frac{1}{N(N-1)}\sum_{(i,j):i\neq j}\mathbb{E}_{\mathcal{P}_{k-1}^n\otimes\nu_{U_{i,j}^{(n)}(k-1)}}\Big(f(\mathbf{Y}_0,\dots,\mathbf{Y}_{k-1}, g_{i,j}(\mathbf{Y}_{k-1}, U_{i,j}^{(n)}(k-1)))\Big).
\end{aligned}
$$

At this point, we have to deal with a small technical issue. The function $g_{i,j}$ is not immediately continuous on its arguments, since it can create jumps of order $1/n \geq U^{(n)}_{i,j}(k)(y_i(k)+y_j(k)) - [U^{(n)}_{i,j}(k)(y_i(k)+y_j(k))]_n$ for all k. However f is a bounded continuous function on a compact space Δ^{k+1}_{N-1}, and it is uniformly continuous in the last coordinate. Then define on $\Delta_{N-1}\times[0,1]$ the bounded continuous function

$$g^{\mathrm{cont}}_{i,j}(\mathbf{y},u) = (y_1,\dots,u(y_i+y_j),\dots,(1-u)(y_i+y_j),\dots,y_N).$$

Fix a $\delta > 0$ and let $n = n(\delta)$ be large enough so that

$$\sup_{\mathbf{y}\in\Delta_{N-1}(n)}\ \sup_{u\in[0,1]}\ \sup_{(i,j)} \|g^{\mathrm{cont}}_{i,j}(\mathbf{y},u) - g_{i,j}(\mathbf{y},u)\|_\infty < \delta.$$

Fix an $\varepsilon > 0$ and choose δ so that for any $\|(\mathbf{z}_1,\dots,\mathbf{z}_k) - (\mathbf{y}_1,\dots,\mathbf{y}_k)\|_\infty < \delta$

$$\|f(\mathbf{z}_1,\dots,\mathbf{z}_k) - f(\mathbf{y}_1,\dots,\mathbf{y}_k)\|_\infty < \varepsilon(kN)^{-2}.$$

Then we proceed with the computation for n large enough:

$$\begin{aligned}
&\mathbb{E}_{\mathcal{P}^n_k}(f(\mathbf{Y}^{(n)}_0,\mathbf{Y}^{(n)}_1,\dots,\mathbf{Y}^{(n)}_k))\\
&\quad= \frac{1}{N(N-1)}\sum_{(i,j):i\neq j} \mathbb{E}_{\mathcal{P}^n_{k-1}\otimes \nu_{U^{(n)}_{i,j}(k-1)}}\Big(f(\mathbf{Y}_0,\dots,\mathbf{Y}_{k-1},g_{i,j}(\mathbf{Y}_{k-1},U^{(n)}_{i,j}(k-1)))\Big)\\
&\quad= \frac{1}{N(N-1)}\sum_{(i,j):i\neq j} \mathbb{E}_{\mathcal{P}^n_{k-1}\otimes \nu_{U^{(n)}_{i,j}(k-1)}}\Big(f(\mathbf{Y}_0,\dots,\mathbf{Y}_{k-1},g^{\mathrm{cont}}_{i,j}(\mathbf{Y}_{k-1},U^{(n)}_{i,j}(k-1)))\Big)\\
&\qquad+ O(\varepsilon)\\
&\quad= \frac{1}{N(N-1)}\sum_{(i,j):i\neq j} \mathbb{E}_{\mathcal{P}^n_{k-1}\otimes \nu_{U^{(n)}_{i,j}(k-1)}}\Big(\tilde{f}_{i,j}(\mathbf{Y}_0,\dots,\mathbf{Y}_{k-1},U^{(n)}_{i,j}(k-1))\Big) + O(\varepsilon).
\end{aligned}$$

Above, $\tilde{f}_{i,j}$ is a bounded continuous function. By iterating the same argument using the Markov property iteratively, we conclude, for n large enough that

$$\begin{aligned}
\mathbb{E}_{\mathcal{P}^n_k}(f(\mathbf{Y}^{(n)}_0,\mathbf{Y}^{(n)}_1,\dots,\mathbf{Y}^{(n)}_k)) &= \left(\frac{1}{N(N-1)}\right)^k\\
&\times\sum_{\substack{(i_0,j_0)\\ i_0\neq j_0}}\cdots\sum_{\substack{(i_{k-1},j_{k-1})\\ i_{k-1}\neq j_{k-1}}} \mathbb{E}_{\mu^{(n)}_0\otimes_{\ell=0}^{k-1}\nu_{U^{(n)}_{i_\ell,j_\ell}(\ell)}}\\
&\quad\times\Big(\tilde{f}_{(i_0,j_0),\dots,(i_{k-1},j_{k-1})}(\mathbf{Y}_0,U^{(n)}_{i_0,j_0}(0),\dots,U^{(n)}_{i_{k-1},j_{k-1}}(k-1))\Big)\\
&\quad+ O(\varepsilon).
\end{aligned}$$

The sums above are finitely many, so the accumulated error is bounded by $C\varepsilon$. Each multiindexed $\tilde{f}$ is a bounded continuous function on all its arguments. Finally, the assumptions of the theorem imply the joined weak convergence

$$(\mathbf{Y}_0,U^{(n)}_{i_0,j_0}(0),\dots,U^{(n)}_{i_{k-1},j_{k-1}}(k-1)) \Longrightarrow (\mathbf{X}^{(\infty)}_0,U^{(\infty)}_{i_0,j_0}(0),\dots,U^{(\infty)}_{i_{k-1},j_{k-1}}(k-1)).$$

The limiting vector can be used to uniquely construct the CS-DT process using the indices of the associated function $\tilde{f}$. By reversing the decomposition above, therefore

$$\Big|\lim_{n\to\infty}\mathbb{E}_{\mathcal{P}^n}(f(\mathbf{Y}^{(n)}_0,\mathbf{Y}^{(n)}_1,\dots,\mathbf{Y}^{(n)}_k)) - \mathbb{E}_{\mathcal{P}^\infty}(f(\mathbf{X}^{(\infty)}_0,\mathbf{X}^{(\infty)}_1,\dots,\mathbf{X}^{(\infty)}_k))\Big| = O(\varepsilon). \qquad (3.1)$$

Let $\varepsilon \to 0$ to finish the proof. □

Remark 3.1 (*Almost Sure Convergence for Finite Sample Paths*)**.** Assume that the initial distributions satisfy $\mathbf{Y}_0^{(n)} \to \mathbf{X}_0^{(\infty)}$ a.e. as $n \to \infty$ and that we use common uniforms for each time step k, i.e.

$$U_{i,j}(k) \equiv U_{i,j}^{(n)}(k) = U_{i,j}^{(m)}(k) = U_{i,j}^{(\infty)}(k), \qquad \text{for all } n, m \in \mathbb{N},$$

while maintaining the independence across the time index. Then for any fixed $k \in \mathbb{N}$

$$(\mathbf{Y}_0^{(n)}, \dots, \mathbf{Y}_k^{(n)}) \xrightarrow{a.s.} (\mathbf{X}_0^{(\infty)}, \dots, \mathbf{X}_k^{(\infty)}),$$

provided the same indices (i, j) are selected at each step. This is because of the compact state space for these processes. For any fixed n, the construction using now the common (in n) uniform random variables $U_{i,j}^{(n)}(\ell)$ creates an error of at most $2/n$ per step in the supremum norm of the state space, so the total error is $2k/n$, which vanishes as $n \to \infty$.

4. Kinetic equations as thermodynamic limit of the Markov chain with continuous state space

In this section we prove that Eq. (2.16) is obtained as the limit in probability of (2.14) as $N \to \infty$, (see Theorem 2.5). Before stating the result rigorously, we need to mention some terminology and basic facts.

Definition 4.1 (*Solutions*)**.** We say that a measure $(\mu_t)_{t<T}$ is local solution if it satisfies (2.17) for all functions g which are bounded and measurable. If T can be taken to be $+\infty$, then we say we have a (global) solution of (2.17).

It is important to ascertain that solutions do exist, and this is the content of the next proposition. The proof of it follows the same arguments as in the proof for Smoluchowski's equation in [32, Proposition 2.2], and it is omitted from this manuscript.

Proposition 4.2 (*Existence and Uniqueness of Solutions*)**.** *Suppose that $\mu_0 \in \mathcal{M}_1(\mathbb{R}_+)$. The kinetic Eq.* (2.16) *has a unique solution $(\mu_t)_{t\geq 0}$ with initial data μ_0.*

Above we introduced $\mathcal{M}_1(\mathbb{R}_+)$ as the space of probability measures with support on the non-negative reals. In general, $\mathcal{M}_1(K)$ denotes the set of probability measures on the set K. We have already discussed how the empirical measure μ_t^N is an element of $\mathcal{M}_1(\mathbb{R}_+)$. In particular, for any $t \geq 0$, μ_t^N is a *random element* of $\mathcal{M}_1(\mathbb{R}_+)$, and its distribution is solely dictated by the distribution of the Markov chain at time t.

The next proposition states the two main conservation properties that we are using throughout the manuscript. First we show that the support of the initial measure dictates the support of all μ_t without exiting the class of probability measures, and the second property is the conservation of total wealth. Recall the notation introduced in Section 2.2.

Proposition 4.3. *Suppose that $w_0 < \infty$. Assume that $\mu_0^N \in \mathcal{M}_1([0, w_0])$, then $\mu_t^N \in \mathcal{M}_1([0, w_0])$ for all times. Moreover, if $\langle x, \mu_0^N \rangle = m_0 \in \mathbb{R}_+$, then $\langle x, \mu_t^N \rangle = m_0$ for all times.*

Proof. To check the proposition one just needs to notice that

$$\langle \mathbb{1}\{x \leq w_0\}, Q(\mu) \rangle = 0,$$

for any measure μ. Therefore, by (2.16), we have that

$$\langle \mathbb{1}\{x \leq w_0\}, \mu_t^N \rangle = \langle \mathbb{1}\{x \leq w_0\}, \mu_0^N \rangle = 1,$$

and so $\mu_t^N(\{x \; ; \; x \leq w_0\}) = 1$. This implies we can write $\mu_t^N \in \mathcal{M}_1([0, w_0])$ for any t.

The second statement can be proven analogously substituting $g(x) = x\mathbb{1}\{x \le w_0\}$ in (2.15). □

The symbol $D(K, S)$ denotes the space of càdlàg (right continuous with left limit) functions from K to S, called the Skorokhod space. We wish to study the process of the empirical measures $\{\mu_t^N\}_{t\ge 0}$ as a sequence in N. For any fixed N, the sequence $\{\mu_t^N\}_{t\ge 0}$ is an element of $D([0,\infty); \mathcal{M}_1(\mathbb{R}_+))$. All necessary background information for Skorokhod spaces that will be used in the section can be found in Appendix.

With the notation set, we can now proceed and prove Theorem 2.5. Technical proofs are left to the end of the section to not mar the exposition. Again, recall the notation from Section 2.2.

4.1. Proof of Theorem 2.5

The main idea for the proof is to take the limit as $N \to \infty$ in the martingale formulation (2.14) by following the methodology presented in [32].

The theorem can be proven directly from the following three propositions. We do that right after these propositions are proven.

Proposition 4.4 (*Martingale Convergence*). *For any $g \in C_b(\mathbb{R}_+)$, $t \ge 0$, it holds that*

$$\lim_{N\to\infty} \sup_{0\le s\le t} |M_s^{g,N}| = 0 \qquad \text{in } \mathcal{L}^2(\mathbb{R}),$$

where $M_t^{g,N}$ is defined in (2.14). In particular, the limit also holds in probability.

Proposition 4.5 (*Weak Convergence for the Measures*). *The sequence of laws $\mathcal{P}_N$ of the elements $\{\mu_t^N\}_{t\in\mathbb{R}_+}$ is tight. Therefore there exists a weakly convergent subsequence $(\mu^{N_k})_{k\in\mathbb{N}}$ in $D([0,\infty); \mathcal{M}_1(\mathbb{R}_+))$ as $k \to \infty$.*

Proposition 4.6 (*Convergence for the Trilinear Term*). *For any converging subsequence $\{\mu^{N_k}\}_{k\in\mathbb{N}}$ (and particularly for those established in Proposition 4.5), it holds that*

$$\int_0^t \langle f, Q^{(N_k)}(\mu_s^{N_k})\rangle\, ds \to \int_0^t \langle f, Q(\mu_s)\rangle\, ds \quad \text{weakly,}$$

as $k \to \infty$.

4.1.1. Proof of Proposition 4.4

Keep in mind that $M_t^{g,N}$ is a martingale. From Proposition 8.7 in [10] (a consequence of Doob's $\mathcal{L}^2$ inequality) we have that for any finite T,

$$\mathbb{E}\left[\sup_{s\le T} |M_s^{g,N}|^2\right] \le 4\mathbb{E}\int_0^T \alpha^{g,N}(\mu_s^{(2,N)})ds, \tag{4.1}$$

where in this case

$$\begin{aligned}\alpha^{g,N}(\mu_s^{(2,N)}) &= \int_{[0,1]}\int_{\mathbb{R}_+^2} \left(\frac{1}{N}\Big(g(r(x+y)) + g((1-r)(x+y)) - g(x) - g(y)\Big)\right)^2 \\ &\qquad\qquad \times\ \mathbb{1}_{\{x+y\le W_N\}}\, N\mu_s^{(2,N)}(dx,dy)\,dr \\ &\le \frac{N}{N^2}\frac{N-1}{N} 16\|g\|_\infty^2 \le \frac{16}{N}\|g\|_\infty^2. \end{aligned} \tag{4.2}$$

Use this estimate in (4.1) to obtain

$$\mathbb{E}\left[\sup_{s\le T}|M_s^{g,N}|^2\right]\le \frac{1}{N}64\|g\|_\infty^2 T. \tag{4.3}$$

This gives the convergence of the supremum towards 0 in $\mathcal{L}^2$ as $N\to\infty$, which implies also the convergence in probability. □

4.1.2. Proof of Proposition 4.5

The results stated in the proposition will be proven at the end of this subsection, and they follow from two lemmas.

Lemma 4.7. *Fix an $f\in C_b(\mathbb{R}_+)$. Then the sequence of laws of $(\langle f,\mu^N\rangle)_{N\in\mathbb{N}}$ on $D([0,\infty);\mathbb{R}_+)$ is tight.*

Proof of Lemma 4.7. We use Theorem A.8 in Appendix. Thus, we need to verify the two conditions of the Theorem.

To prove condition (i) of the Theorem we use that for any fixed $f\in C_b(\mathbb{R}_+)$

$$|\langle f,\mu_t^N\rangle|=\left|\frac{1}{N}\sum_{i=1}^N f(X_t^{i,N})\right|\le\frac{1}{N}\sum_{i=1}^N|f(X_t^{i,N})|\le\|f\|_\infty$$

so for all $t\ge 0$, $\langle f,\mu_t^N\rangle\in[-\|f\|_\infty,\|f\|_\infty]$. To directly see the connection with Theorem A.8, set $\Lambda_{\eta,t}=[-\|f\|_\infty,\|f\|_\infty]$ (fixed for any η) and $X_N(t)=\langle f,\mu_t^N\rangle$.

The verify the second condition (ii) of Theorem A.8 we make use of the following inequalities:

$$\mathbb{E}\left[\sup_{r\in[s,t)}|M_r^{f,N}-M_s^{f,N}|^2\right]\le\frac{1}{N}64\|f\|_\infty^2(t-s) \tag{4.4}$$

and

$$\mathbb{E}\left[\sup_{r\in[s,t)}\left(\int_s^r\langle f,Q^{(N)}(\mu_u^N)\rangle\,du\right)^2\right]\le 16\|f\|_\infty^2(t-s)^2. \tag{4.5}$$

To see inequality (4.4) recall that since $M_t^{f,N}$ is an $\mathcal{F}_t$-martingale, then $\tilde{M}_t^{f,N}=M_{t+s}^{f,N}-M_s^{f,N}$ is an $\tilde{\mathcal{F}}_t=\mathcal{F}_{t+s}$ martingale. Therefore

$$\mathbb{E}\left[\sup_{r\in[s,t)}|M_r^{f,N}-M_s^{f,N}|^2\right]=\mathbb{E}\left[\sup_{r\in[0,t-s)}|\tilde{M}_r^{f,N}|^2\right]\le\frac{1}{N}64\|f\|_\infty^2(t-s),$$

just like in Eq. (4.3). Inequality (4.5) follows from (2.15) and a bound similar to the one used in (4.2).

Eqs. (4.4) and (4.5) together give the bound

$$\mathbb{E}\left[\sup_{r\in[s,t)}|\langle f,\mu_r^N-\mu_s^N\rangle|^2\right]\le A\left((t-s)^2+\frac{(t-s)}{N}\right) \tag{4.6}$$

for some $A>0$ depending only on $\|f\|_\infty$. With these estimates the proof follows as in [31] where further details can be found. □

Lemma 4.8. *The sequence of laws* $\{\mathcal{P}_N\}_{N\in\mathbb{N}}$ *of the elements* $(\mu_t^N)_{t\in\mathbb{R}_+} \in D([0,\infty); \mathcal{M}_1(\mathbb{R}_+))$ *is tight.*

Proof of Lemma 4.8. We will use Theorem A.6 in the Appendix to prove this result. To check condition (i), we find a suitable compact set $W \in \mathcal{M}_{\le 1}(\mathbb{R}_+)$, where $\mathcal{M}_{\le 1}(\mathbb{R}_+)$ is the set of all sub-probability measures on $\mathbb{R}_+$, which is a separable, compact metric space, and therefore a completely regular topological space. Any closed subset W of $\mathcal{M}_{\le 1}(\mathbb{R}_+)$ will be compact with respect to the topology induced by the weak convergence of measures, and it will be metrisable as a subset of a metric space.

We define for some positive constant C the set

$$W_C := \left\{ \tau \in \mathcal{M}_1(\mathbb{R}_+) \,:\, \int_{\mathbb{R}_+} x\, \tau(dx) \le C \right\},$$

which is closed (and therefore compact). Assume that $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of measures in W_C that converge weakly to τ. Then for any $M \in \mathbb{R}_+$

$$\begin{aligned}\int_{\mathbb{R}_+} x\tau(dx) &= \lim_{M\to\infty} \int_{\mathbb{R}_+} (x\wedge M)\tau(dx) = \lim_{M\to\infty} \lim_{n\to\infty} \int_{\mathbb{R}_+} (x\wedge M)\tau_n(dx)\\ &\le \lim_{n\to\infty} \int_{\mathbb{R}_+} x\tau_n(dx) \le C,\end{aligned}$$

and therefore the limit point τ is also in W_C.

In our case, from the conservation of the total mass (wealth) and the fact that we can find a c_1 so that $W_N < c_1 N$, for all $N \in \mathbb{N}$, we have for any $t \in \mathbb{R}_+$

$$\int_{\mathbb{R}_+} x\mu_t^N(dx) = \frac{1}{N}\sum_{i=0}^{N} X_t^{i,N} = \frac{1}{N}\sum_{i=0}^{N} X_0^{i,N} = \int_{\mathbb{R}_+} x\mu_0^N(dx) \le c_1 \quad \text{a.s.}$$

Consider $(\mathcal{P}_N)_{N\in\mathbb{N}}$ the family of probability measures in $\mathcal{M}_1(D([0,\infty); W_{c_1}))$ which are the laws of $(\mu_t^N)_{t\in\mathbb{R}_+}$. For any $T > 0$, the above discussion and Remark 4.5 in [28] give

$$\mathcal{P}_N(D([0,\infty); W_{c_1})) = \mathcal{P}_N(D([0,T]; W_{c_1})) = 1, \qquad \text{for all } N \in \mathbb{N}.$$

This verifies condition (i) of Theorem A.6.

In order to check condition (ii) we will use the family of continuous functions on $\mathcal{M}_{\le 1}(\mathbb{R}_+)$ defined as

$$\mathbb{F} = \{F \,:\, \mathcal{M}_{\le 1}(\mathbb{R}_+) \to \mathbb{R} \,:\, F(\tau) = \langle f, \tau\rangle \text{ for some } f \in C_b(\mathbb{R}_+)\}.$$

This family is closed under addition since $C_b(\mathbb{R}_+)$ is, it is continuous in $\mathcal{M}_{\le 1}(\mathbb{R}_+)$, and separates points in $\mathcal{M}_{\le 1}(\mathbb{R}_+)$: if $F(\tau) = F(\bar\tau)$ for all $F \in \mathbb{F}$ then

$$\int_{\mathbb{R}_+} f(x)d(\tau - \bar\tau)(x) = 0 \quad \forall f \in C_b(\mathbb{R}_+)$$

hence $\tau \equiv \bar\tau$, since we can approximate indicator functions for any Borel set A using functions from $C_b(\mathbb{R}_+)$. So we are left with proving that for every $f \in C_b(\mathbb{R}_+)$ the sequence $\{\langle f, \mu^N\rangle\}_{N\in\mathbb{N}}$ is tight. This was proven in Lemma 4.7. □

Now Proposition 4.5 follows immediately.

Proof of Proposition 4.5. The result follows from Lemma 4.8 and Prokhorov's theorem. □

To prove Proposition 4.6 we need the following three lemmas. Throughout we are assuming that $\{\mu^{N_k}\}$ is a converging sequence in the space $D([0,\infty); \mathcal{M}_1(\mathbb{R}_+))$.

Lemma 4.9 (*Continuity of the Limit*)**.** *The weak limit of* $(\mu_t^{N_k})_{t\geq 0}$ *as* $k\to\infty$ *is continuous in time a.e.*

Proof of Lemma 4.9. We have that for any $f\in C_b(\mathbb{R}_+)$

$$|\langle f,\mu_t^{N_k}\rangle - \langle f,\mu_{t-}^{N_k}\rangle| \leq \frac{4}{N_k}\|f\|_\infty,$$

when a jump happens in the process only the wealth of two individuals is altered. Then we may apply Theorem A.9 of the Appendix to obtain that $\langle f,\mu_t\rangle$ is continuous for any $f\in C_b(\mathbb{R}_+)$ and this implies the continuity of $(\mu_t)_{t\geq 0}$. □

Lemma 4.10 (*Uniform Convergence*)**.** *For all* $f\in C_b(\mathbb{R}_+)$*, and finite* $t\geq 0$ *we have*

$$\sup_{s\leq t}|\langle f,\mu_s^{N_k}-\mu_s\rangle| \to 0 \quad \textit{weakly}$$

as $k\to\infty$.

Proof of Lemma 4.10. By Lemma 4.9, the limit of $(\mu_t^{N_k})_{N\in\mathbb{N}}$ is continuous in time. The statement is consequence of the Continuous Mapping Theorem in the Skorokhod space and the fact that $g(X)(t)=\sup_{s\leq t}|X|$ is a continuous function in this space. □

Lemma 4.11. *For all* $f\in C_b(\mathbb{R}_+)$*, and finite* $t\geq 0$ *we have*

$$\sup_{s\leq t}|\langle f,Q^{(N_k)}(\mu_s^{N_k})-Q(\mu_s)\rangle| \to 0 \quad \textit{weakly}$$

as $k\to\infty$.

Proof of Lemma 4.11. We abuse notation and denote by $(\mu_t^N)_{N\in\mathbb{N}}$ the convergent subsequence. The result will manifest itself when we show that for all $f\in C_b(\mathbb{R}_+)$:

(i) $\sup_{s\leq t}|\langle f,\left(Q-Q^{(N)}\right)(\mu_s^N)\rangle| \to 0$ as $N\to\infty$,
(ii) $\sup_{s\leq t}\left|\langle f,Q\left(\mu_s^N\right)-Q\left(\mu_s\right)\rangle\right| \to 0$ as $N\to\infty$.

We will use the fact that the product measures also converge weakly, i.e. $\mu_t^N\otimes\mu_t^N \Longrightarrow \mu_t\otimes\mu_t$.

Item (i) is then a consequence of

$$\begin{aligned}
&|\langle f,\left(Q-Q^{(N)}\right)(\mu_s^N)\rangle| \\
&\leq \left|\int_{[0,1]}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}[f(r(x+y))+f((1-r)(x+y))-f(x)-f(y)]\right. \\
&\qquad \left.\times\ d(\mu_s^N\otimes\mu_s^N-\mu_s^{(2,N)})dr\right| \\
&\leq 4\|f\|_\infty\left|\int_{\mathbb{R}_+^2}d(\mu_s^N\otimes\mu_s^N-\mu_s^{(2,N)})\right| = 4\|f\|_\infty\left|\frac{1}{N}\int_{\mathbb{R}_+}d\mu_s^N\right| \leq \frac{4}{N}\|f\|_\infty.
\end{aligned} \tag{4.7}$$

The bound is true for any s and therefore for the supremum up to a finite time as well. Now for (ii), we compute

$$\begin{aligned}
&\sup_{s\leq t}\left|\langle f,Q(\mu_s^N)-Q(\mu_s)\rangle\right| \\
&\qquad\leq \sup_{s\leq t}\int_{[0,1]}\int_{\mathbb{R}_+^2}|f(r(x+y))+f((1-r)(x+y))-f(x)-f(y)|
\end{aligned}$$

$$\left|\mathbb{1}_{\{x+y\le W_N\}}\mu_s^N(dx)\mu_s^N(dy) - \mathbb{1}_{\{x+y\le w_0\}}\mu_s(dx)\mu_s(dy)\right| dr$$
$$\le 4\|f\|_\infty \sup_{s\le t}\int_{\mathbb{R}_+^2}\left|\mathbb{1}_{\{x+y\le W_N\}}\mu_s^N(dx)\mu_s^N(dy) - \mathbb{1}_{\{x+y\le w_0\}}\mu_s(dx)\mu_s(dy)\right|$$
$$\le 4\|f\|_\infty \sup_{s\le t}\int_{\mathbb{R}_+^2}\left|\mathbb{1}_{\{x+y\le w_0\}}\mu_s^N(dx)\mu_s^N(dy) - \mathbb{1}_{\{x+y\le w_0\}}\mu_s(dx)\mu_s(dy)\right|$$
$$\le 4\|f\|_\infty \sup_{s\le t}\int_{\mathbb{R}_+^2}\left|\mu_s^N(dx)\mu_s^N(dy) - \mu_s(dx)\mu_s(dy)\right|. \tag{4.8}$$

We conclude (ii) with an argument analogous to Lemma 4.10 applied to the function $f = 1$. □

Proof of Proposition 4.6. By Lemma 4.11 we can pass the limit inside the time integral. □

We are now in position to prove Theorem 2.5:

Proof of Theorem 2.5. The weak form of item (A) is proven in Lemma 4.10, item (B) is proven in Proposition 4.4, and item (C) is the content of Proposition 4.6. Since all those weak convergences in the previous propositions were to 0, they can be upgraded to convergence in probability.

Then (and also by using the assumptions of the theorem) we have that for any $f \in C_b(\mathbb{R}_+)$ and any converging subsequence of measures,

$$0 \stackrel{\mathcal{D}}{=} \lim_{N\to\infty} M_t^{g,N}$$
$$= \lim_{N\to\infty}\langle g, \mu_t^N\rangle - \langle g, \mu_0^N\rangle - \int_0^t \langle g, Q^{(N)}(\mu_s^N)\rangle ds \stackrel{\mathcal{D}}{=} \langle g, \mu_t\rangle - \langle g, \mu_0\rangle - \int_0^t \langle g, Q(\mu_s)\rangle,$$

and therefore the limit of the subsequence of measures must satisfy Eq. (2.16). Using the uniqueness of the kinetic Eq. (2.16), we have that all the convergent subsequences from Proposition 4.5 converge to the same limit. Hence the whole sequence converges (if a tight sequence has every weakly convergent subsequence converging to the same limit, then the whole sequence converges weakly to that limit [3]).

Now, we have that the weak limit of $(\mu_t^N)_{N\in\mathbb{N}}$ satisfies the kinetic Eq. (2.16) (thanks to Prop. 4.4, 4.6), so it is deterministic. Therefore, we actually have convergence in probability. □

5. Invariant measures for the mean field limit

In this section we discuss the invariant measures.

Proof of Proposition 2.7. If $\langle x, \mu_0^N\rangle \to 0$ as $N \to \infty$, by positivity of the support of the measures and conditions (2.19)–(2.20), it follows that

$$\langle x, \mu_0\rangle = 0.$$

On the other hand, μ_0 is a probability measure, so the above implies that $\mu_0(x) = \delta_0(x)$. Then it follows that $\mu_t(x) = \delta_0(x)$ since we already argued that the delta distribution is an invariant solution of Eq. (2.16). □

Proof of Proposition 2.8. This proof does not need the technicalities associated with martingales, as the initial distributions of the process are invariant, and every time an interaction event occurs their distribution remains unchanged. The theorem can be proven in a direct way, without even the Poissonisation trick.

Consider a continuous function g on [0, 1] and assume that $\|g\|_\infty \le B$. Let $\varepsilon > 0$ and select a $\delta > 0$ so that $\delta < \varepsilon/2 \wedge B$. Furthermore assume that N is large enough so that for a fixed β, $0 < \beta < 1$ we have that

$$\sup_{x\in[0,N^{-\beta}]} |g(0) - g(x)| < \delta.$$

In order to prove the result we just need to show that $\langle g, \mu_0^N\rangle \to g(0)$ as $N \to \infty$. We will show that this happens $\mathbb{P}$- a.s., when $\mathbb{P} = \otimes_{N=2}^{\infty}\mu_0^{\infty,N}$ the product measure on the space $\otimes_{N=2}^{\infty}\Delta_{N-1}$.

We have that $\langle g, \mu_0^N\rangle = N^{-1}\sum_{i=1}^N g(X_i^{N_0})$, so for the $\mathbb{P}-$ a.s. convergence we estimate

$$\begin{aligned}
\mathbb{P}\Big\{\Big|\frac{1}{N}\sum_{i=1}^N g(X_i) - g(0)\Big| > \varepsilon\Big\} &= \mathbb{P}\Big\{\Big|\sum_{i=1}^N (g(X_i) - g(0))\Big| > N\varepsilon\Big\}\\
&\le \mathbb{P}\Big\{\sum_{i=1}^N \big|g(X_i) - g(0)\big| > N\varepsilon\Big\}\\
&\le e^{-\varepsilon N}\mathbb{E}\Big(\exp\Big\{\sum_{i=1}^N \big|g(X_i) - g(0)\big|\Big\}\Big)\\
&= e^{-\varepsilon N}\mathbb{E}\ \Big(\ \exp\Big\{\sum_{i=1}^N \big|g(X_i) - g(0)\big|\Big\}\\
&\quad \times \sum_{I\subseteq[N]} \mathbb{1}\{X_i \ge N^{-\beta}, i \in I\}\mathbb{1}\{X_i < N^{-\beta}, i \notin I\}\ \Big)\\
&= e^{-\varepsilon N}\mathbb{E}\ \Big(\sum_{I\subseteq[N]} e^{\sum_{i\in I}|g(X_i)-g(0)|}\mathbb{1}\{X_i \ge N^{-\beta}, i \in I\}\\
&\quad \times\ e^{\sum_{i\notin I}|g(X_i)-g(0)|}\mathbb{1}\{X_i < N^{-\beta}, i \notin I\}\ \Big)\\
&\le e^{-\varepsilon N}\mathbb{E}\Big(\sum_{I\subseteq[N]} e^{2B|I|}\mathbb{1}\{X_i \ge N^{-\beta}, i \in I\}e^{(N-|I|)\delta}\mathbb{1}\{X_i < N^{-\beta}, i \notin I\}\Big)\\
&\le e^{-\varepsilon N}\sum_{k=0}^N \binom{N}{k} e^{2Bk+(N-k)\delta}\mathbb{E}\big(\mathbb{1}\{X_i \ge N^{-\beta} \text{ for } k \text{ indices}\}\big).
\end{aligned}$$

The last line has the simplified sum index because of exchangeability of the coordinates, and it is an upper bound, because we dropped the second indicator function. Before proceeding with the calculation, we just bound the last expectation when k is not zero. Note that if $k > [N^{1-\beta}]$, the indicator inside is identically zero, otherwise the total wealth cannot be one. We also restrict the index of summation to $[N^{1-\beta}]$ as the indicator vanishes otherwise.

$$\begin{aligned}
\mathbb{P}\Big\{\Big|\frac{1}{N}\sum_{i=1}^N g(X_i) - g(0)\Big| > \varepsilon\Big\} &\le e^{(\delta-\varepsilon)N}\sum_{k=0}^{[N^{1-\beta}]}\binom{N}{k}e^{(2B-\delta)k}\\
&\le e^{-\varepsilon N/2}N^{1-\beta}\binom{N}{[N^{1-\beta}]}e^{(2B-\delta)N^{1-\beta}}. \qquad (5.1)
\end{aligned}$$

The last line follows because eventually δ will vanish and the exponent $(2B-\delta)$ will be eventually positive. therefore the maximum term in the sum is the last one, when $k=[N^{1-\beta}]$ as combinations are also increasing until around $N/2$. Finally, one can use Stirling's formula to see that asymptotically there exists a constant c so that

$$\binom{N}{[N^{1-\beta}]}\sim e^{cN^{1-\beta}}.$$

Therefore the upper bound in Eq. (5.1) is summable over N. A final application of the Borel–Cantelli lemma completes the proof. □

In the remaining part of this subsection, we discuss invariant measures that are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. The blanket assumption is that for each $t\ge 0$, there exists a probability density function f_t so that

$$\mu_t(x)=f_t(x)\,dx,$$

and we can find invariant measures with this property. We will show that one family of such measures can be obtained as limits of the empirical measures and (it is therefore a true invariant measure) that the other family cannot and therefore the kinetic Eq. (2.16) does give extraneous solutions.

The first step is to find the restriction of the operator Q to the class of absolutely continuous measures, which we will call $\bar{Q}$. Using $\bar{Q}$, we can formally write an equation for the evolution of the assumed densities f_t. Assume that $\lim_{N\to\infty}W_N=w_0\in(0,\infty]$. For any value of w_0 we will denote the restricted operator by $\bar{Q}_{w_0}$, and $\bar{Q}_{w_0}$ acts on probability densities f on $\mathbb{R}_+$. In other words, for any $g\in C_b(\mathbb{R}_+)$

$$\langle g,\ Q(\mu)\rangle=\langle g,\ \bar{Q}_{w_0}(f)\rangle,\qquad \text{whenever } \mu(x)=f(x)\,dx.$$

First notice that when the measure μ has a density f we can write

$$\int_0^1\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}g((1-r)(x+y))\mathbb{1}_{\{x+y\le w_0\}}f(x)f(y)dx\,dy\,dr = \int_0^1\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}g(r(x+y))\mathbb{1}_{\{x+y\le w_0\}}f(x)f(y)dx\,dy\,dr$$

with a change of variables $r\mapsto 1-r$. Therefore, expression (2.18) can be rewritten as

$$\langle g,\ Q(\mu)\rangle=\int_{[0,1]}\int_{\mathbb{R}_+^2}[2g(r(x+y))-g(x)-g(y)]\mathbb{1}_{\{x+y\le w_0\}}f(x)f(y)dx\,dy\,dr. \tag{5.2}$$

Now it follows that

$$\begin{aligned}\int_{[0,1]}\int_{\mathbb{R}_+^2}&g(r(x+y))\mathbb{1}_{\{x+y\le w_0\}}f(x)f(y)dx\,dy\,dr\\ &=\int_{[0,1]}\int_{\mathbb{R}_+^2}g(u+p)\mathbb{1}_{\{u+p\le rw_0\}}f(u/r)f(p/r)du\,dp\,\frac{dr}{r^2}\\ &=\int_{[0,1]}\int_{\mathbb{R}_+^2}g(z)\mathbb{1}_{\{z\ge p\}}\mathbb{1}_{\{z\le rw_0\}}f((z-p)/r)f(p/r)dz\,dp\frac{dr}{r^2}\\ &=\int_{[0,1]}\int_{\mathbb{R}_+^2}g(x)\mathbb{1}_{\{x\ge y\}}\mathbb{1}_{\{x\le rw_0\}}f((x-y)/r)f(y/r)dx\,dy\frac{dr}{r^2}\end{aligned}$$

where in the first equality we made the change of variables $rx = u$, $ry = p$; in the second equality we made the change of variables $z = u + p$; in the last equality we just changed the name of the labels $z = x$, $p = y$. With similar computations, we obtain that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} g(x)\mathbb{1}_{\{x+y\le w_0\}} f(x)f(y)dx\,dy = \int_{\mathbb{R}_+} g(x)f(x)\left(\int_0^{(w_0-x)_+} f(y)dy\right)dx,$$

where $(w_0 - x)_+ = (w_0 - x)\mathbb{1}_{(w_0-x)\ge 0}$.

Combine these calculations into (5.2) to obtain that $\bar{Q}_{w_0}$ is given by

$$\bar{Q}_{w_0}(f) := 2\int_{x/w_0}^1\int_0^x f\left(\frac{y}{r}\right) f\left(\frac{x-y}{r}\right) dy\frac{dr}{r^2} - 2f(x)\int_0^{(w_0-x)_+} f(y)dy, \tag{5.3}$$

Similarly, the evolution of the density functions can be obtained (in a weak sense) from

$$f_t = f_0 + \int_0^t \bar{Q}_{w_0}(f_s)\,ds. \tag{5.4}$$

Note that when $w_0 < \infty$ and $x > w_0$, then $\bar{Q}_{w_0}(f)(x) = 0$. When $w_0 = \infty$ the operator $\bar{Q}_\infty$ reads

$$\bar{Q}_\infty(f) = 2\int_0^1\int_0^x f\left(\frac{y}{r}\right) f\left(\frac{x-y}{r}\right) dy\frac{dr}{r^2} - 2f(x), \tag{5.5}$$

since $f(x)$ is a probability density. In order to prove Corollary 2.10, it suffices to show that the proposed equilibria annihilate $\bar{Q}$. We re-state the corollary, using this observation.

Corollary 5.1. *Let* $\lim_{N\to\infty} W_N = w_0 \in (0, \infty]$. *Assume that* μ_t *is a solution of* (2.16) *which has a density* f_t *for all* t, *satisfying Eq.* (5.4) *Then*

(1) *If* $w_0 = \infty$, *the exponential distributions*

$$\tilde{f}(x) = \frac{e^{-x/m}}{m}, \tag{5.6}$$

are equilibria for the operator $\bar{Q}_\infty$, *(i.e.* $\bar{Q}_\infty(\tilde{f}) = 0$*) and remain invariant under* (5.4). *In particular, if* f_0 *is of the form* (2.25) *with* $\langle x, f_0\rangle = m_0 > 0$, *then the distribution* (2.25) *with* $m = m_0$ *is a stationary solution of* (5.4).

(2) *If* $0 < w_0 < \infty$, *then the following distributions are compactly supported in* $[0, w_0]$ *and are equilibria for the operator* $\bar{Q}_{w_0}$

$$\tilde{f}(x) = \frac{e^{-x/m}}{m(1-e^{-w_0/m})}\mathbb{1}_{\{x\le w_0\}}. \tag{5.7}$$

(3) *(Uniqueness of the invariant family at* $w_0 = \infty$*) Moreover, under the extra assumption that the density* f_t *is differentiable on* $\mathbb{R}_+$, *then measures with density* (2.25) *or, equivalently,* (5.6) *are the unique equilibria of* $\bar{Q}_\infty$.

Proof of Corollary 2.10. It is straightforward to check that $\bar{Q}_{w_0}(\tilde{f}) = 0$ for both $w_0 = \infty$ and $w_0 < \infty$. Also, if $f_0 = \tilde{f}$ with $\langle x, f_0\rangle = m_0$ this implies that f_0 is stationary solution of (5.4) with $m = m_0$. It remains to show item (3). Select any invariant f and for that, recall that $\bar{Q}_\infty(f) = 0$. Let X, Y be independently distributed with density f, and U a uniform r.v. on $[0, 1]$. Start from Eq. (5.2), and observe that a different way to write it is

$$0 = \langle g, \bar{Q}_\infty(f)\rangle = \langle g, Q(\mu)\rangle = 2\mathbb{E}_{(U,X,Y)}[g(U(X+Y))] - 2\mathbb{E}_X[g(X)],$$

and therefore, the distribution of $U(X+Y)$ is the same as the distribution of X. If we now condition on the value of $X+Y := S = s$, we have that the conditional distribution of X given $S = s$ is that of a uniform r.v. on $[0, s]$. Let f_S denote the density of the sum $X+Y$ and $f_{X|S}$ the conditional density of X given $S = s$. We can write

$$f(x) = \int_x^\infty f_{X|S}(x|s) f_S(s)\,ds = \int_x^\infty \frac{1}{s} f_S(s)\,ds.$$

Now use the fundamental theorem of calculus to differentiate both sides with respect to x in order to obtain

$$f'(x) = -\frac{1}{x} f_S(x) \Longleftrightarrow x f'(x) = -f_S(x).$$

Take the Laplace transform of the equation above; denote by $\bar{g}(t)$ the Laplace transform of $g(x)$ and use basic properties on the equation in the last display, to argue that

$$LHS = \overline{x f'(x)} = -\frac{d}{dt}\overline{f'(x)}(t) = -\frac{d}{dt}(t\bar{f}(t)) = -\bar{f} - t\frac{d\bar{f}}{dt},$$

while the Laplace transform of the convolution that gives the density of S is

$$RHS = -(\bar{f})^2.$$

These give rise to the differential equation

$$\frac{d\bar{f}}{\bar{f}(\bar{f}-1)} = \frac{dt}{t}.$$

The solution to the differential equation, for some constant m, is

$$\log\left|\frac{\bar{f}-1}{\bar{f}}\right| = \log mt.$$

Keep in mind that since $t > 0$ and $\bar{f}(t) < 1$, we can solve

$$\bar{f}(t) = \frac{m^{-1}}{m^{-1}+t},$$

where we identify the Laplace transform of an exponential distribution with mean m. □

Proof of Proposition 2.12. Here is the proof of the two points.

(1) We only need to show the convergence of the initial measures. Consider a function $g \in C_b(\mathbb{R}_+)$ and compute

$$\langle g, \mu_0^N \rangle = \frac{1}{N}\sum_{i=1}^N g(X_i) \longrightarrow \mathbb{E}_{\mathrm{Exp}(1/m_0)} g(X_1) = \int_0^\infty g(x)\mu_0(dx),$$

by the law of large numbers. This verifies the definition of weak convergence $\mu_0^N \Longrightarrow \mu_0$.

(2) Assume the contrary, and consider a sequence of converging initial measures. Since $w_0 < \infty$, Proposition 2.7 gives that μ_0^N should converge to δ_0, which does not have a density (2.26). This gives the desired contradiction. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We would like to thank Christina Goldschmidt and Stefan Grosskinsky for valuable and interesting discussions and for suggesting related references. ES further wishes to thank Fei Cao for a useful discussion.

Appendix. Some properties of the Skorokhod space

Theorem A.1 (*Prohorov's Theorem [22], Chapter 3*)**.** *Let (S, d) be complete and separable, and let $\mathcal{M} \in \mathcal{M}_1(S)$. Then the following are equivalent:*

(1) *$\mathcal{M}$ is tight.*
(2) *For each $\varepsilon > 0$, there exists a compact $K \in S$ such that*

$$\inf_{P \in \mathcal{M}} P(K^{\varepsilon}) \geq 1 - \varepsilon$$

where $K^{\varepsilon} := \{x \in S : \inf_{y \in K} d(x, y) < \varepsilon\}$.
(3) *$\mathcal{M}$ is relatively compact.*

Let (E, r) be a metric space. The space $D([0, \infty); E)$ of càdlàg functions taking values in E is widely used in stochastic processes. In general we would like to study the convergence of measures on this space, however, most of the tools known for convergence of measures are for measures in $\mathcal{M}_1(S)$ for S a complete separable metric space. Therefore, it would be very useful to find a topology in $D([0, \infty); E)$ such that it is a complete and separable metric space. This can be done when E is also complete and separable; and the metric considered is the Skorokhod one. This is why in this case the space of càdlàg functions is called Skorokhod space.

Some important properties of this space are the following:

Proposition A.2 (*[22], Chapter 3*)**.** *If $x \in D([0, \infty); E)$, then x has at most countably many points of discontinuity.*

Theorem A.3 (*[22], Chapter 3*)**.** *If E is separable, then $D([0, \infty); E)$ is separable. If (E, r) is complete, then $(D([0, \infty); E), d)$ is complete, where d is the Skorokhod metric.*

Theorem A.4. *The Skorokhod space is a complete separable metric space.*

Theorem A.5 (*The a.s. Skorokhod representation theorem, [22], Theorem 1.8, Chapter 3*)**.** *Let (S, d) be a separable metric space. Suppose P_n, $n = 1, 2, \ldots$ and P in $\mathcal{M}_1(S)$ satisfy $\lim_{n \to \infty} \rho(P_n, P) = 0$ where ρ is the metric in $\mathcal{M}_1(S)$. Then there exists a probability space $(\Omega, \mathcal{F}, \nu)$ on which are defined S- valued random variable X_n, $n = 1, 2, \ldots$ and X with distributions P_n, $n = 1, 2, \ldots$ and P, respectively such that $\lim_{n \to \infty} X_n = X$ almost surely.*

Theorem A.6 (*Tightness Criteria for Measures on the Skorokhod Space*)**.** *See [28] Remark 4.5, and Theorem 4.6 Let $(S, \mathcal{T})$ be a completely regular topological space with metrisable compact sets. Let $\mathbb{G}$ be a family of continuous functions on S taking values in $\mathbb{R}$. Suppose that $\mathbb{G}$ separates points in S and that it is closed under addition. Then a family $\{\mathcal{L}^n\}_{n \in \mathbb{N}}$ of probability measures in $\mathcal{M}_1(D([0, \infty); S))$ is tight iff the two following conditions hold:*

(i) For each $T > 0$ and $\varepsilon > 0$ there exists a compact set $K_{T,\varepsilon} \subset S$ such that

$$\mathcal{L}^n(D([0, T]; K_{T,\varepsilon})) > 1 - \varepsilon, \quad n \in \mathbb{N}.$$

(ii) The family $\{\mathcal{L}^n\}_{n\in\mathbb{N}}$ is $\mathbb{G}$-weakly tight, i.e., for any $g \in \mathbb{G}$ the family $\{\mathcal{L}^n \circ (\tilde{g})^{-1}\}_{n\in\mathbb{N}}$ of probability measures on $D([0, \infty); \mathbb{R})$ is tight; where $\tilde{g}$ is defined as follows:

$$\tilde{g} : D([0, \infty); S) \to D([0, \infty); \mathbb{R})$$

with $[\tilde{g}(\nu)](t) = g(\nu(t))$ for $\nu \in D([0, \infty); S)$ (so that $\nu(t) \in S$).

Remark A.7. [28] only states the results when the time index is in [0, 1] (i.e. in a compact set) and the space is $D([0, 1]; S)$. However, when the sequence of measures is tight, the result of [3] allows the result to generalise when the space is $D([0, \infty); S)$.

Theorem A.8 (*Criteria for Tightness in Skorokhod Spaces ([22], Corollary 7.4, Chapter 3)*)**.** *Let (E, r) be a complete and separable metric space, and let $\{X_n\}$ be a family of processes with sample paths in $D([0, \infty); E)$. Then $\{X_n\}$ is relatively compact iff the two following conditions hold:*

(i) For every $\eta > 0$ and rational $t \geq 0$, there exists a compact set $\Lambda_{\eta,t} \subset E$ such that

$$\liminf_{n\to\infty} \mathbb{P}\{X_n(t) \in \Lambda_{\eta,t}\} \geq 1 - \eta.$$

(ii) For every $\eta > 0$ and $T > 0$, there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\{w'(X_n, \delta, T) \geq \eta\} \leq \eta.$$

where we have used the ***modulus of continuity*** *w' defined as follows: for $x \in D([0, \infty) \times E)$, $\delta > 0$, and $T > 0$:*

$$w'(x, \delta, T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} r(x(s), x(t)),$$

where $\{t_i\}$ ranges over all partitions of the form $0 = t_0 < t_1 < \ldots < t_{n-1} < T \leq t_n$ with $\min_{1\leq i\leq n}(t_i - t_{i-1}) > \delta$ and $n \geq 1$

Theorem A.9 (*Continuity Criteria for the Limit in Skorokhod Spaces ([22], Theorem 10.2, Chapter 3)*)**.** *Let (E, r) be a metric space. Let X_n, $n = 1, 2, \ldots$, and X be processes with sample paths in $D([0, \infty); E)$ and suppose that X_n converges in distribution to X. Then X is a.s. continuous if and only if $J(X_n)$ converges to zero in distribution, where*

$$J(x) = \int_0^\infty e^{-u}[J(x, u) \wedge 1]\, du$$

for

$$J(x, u) = \sup_{0\leq t\leq u} r(x(t), x(t-)).$$

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