Letter

Semiparametric estimation of the Hong-Ou-Mandel profile

Valeria Cimini ,[1,2] Francesco Albarelli ,[3,4] Ilaria Gianani ,[1,*] and Marco Barbieri [1,5]

[1] *Dipartimento di Scienze, Universitá degli Studi Roma Tre, Via della Vasca Navale 84, 00146 Rome, Italy*
[2] *Dipartimento di Fisica, Sapienza Universitá di Roma, Piazzale Aldo Moro 5, I-00185 Roma, Italy*
[3] *Faculty of Physics, University of Warsaw, Pasteura 5, PL-02-093 Warszawa, Poland*
[4] *Department of Physics, University of Warwick, Coventry, United Kingdom*
[5] *Istituto Nazionale di Ottica - CNR, Largo Enrico Fermi 6, 50125 Florence, Italy*

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We apply the theory of semiparametric estimation to a Hong-Ou-Mandel interference experiment with a spectrally entangled two-photon state generated by spontaneous parametric down-conversion. Thanks to the semiparametric approach, we can evaluate the Cramér-Rao bound and find an optimal estimator for a particular parameter of interest without assuming perfect knowledge of the two-photon wave function, formally treated as an infinity of nuisance parameters. In particular, we focus on the estimation of the Hermite-Gauss components of the marginal symmetrized wave function, whose Fourier transform governs the shape of the temporal coincidence profile. We show that negativity of these components is an entanglement witness of the two-photon state.

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Introduction. Two-photon Hong-Ou-Mandel interference [1–3] is the key effect that enables many quantum technologies based on photons and their manipulation [4–11]. Its distinctive coincidence dip profile is a signature of the bosonic nature of the photons and, remarkably, its characteristic length is dictated by the two-photon wave packet, not by their wavelength, ensuring stable and reliable operation even with modest control of the path lengths, thus allowing for extensions to the multiphoton case [12–19].

This feature derives from a nontrivial dependence of the interferometric signal on the two-photon spectral wave function. However, since it ultimately relies on a symmetrization operation [2,20], it is an excellent test bed for verifying the degree of indistinguishability and spectral purity of two independent single-photon wave packets [21–34]. However complicated, this dependence can be inverted to obtain the wave function in the experiment, but this requires multiple Hong-Ou-Mandel (HOM) profile acquisitions [20]. With a single acquisition, some information can nevertheless be extracted, although in a limited amount [35–37]; this can still be an appropriate regime for estimating specific quantities relevant to the wave function.

Applying parameter estimation to such cases benefits from a generous pinch of salt when it comes to spelling out the statement of the problem. Even if we wish to isolate one particular parameter of the wave function, e.g., one of its moments, the estimation will unavoidably depend on the whole function, thus requiring, in principle, infinitely many other parameters for its full description. This apparently unsolvable problem has an elegant and efficient solution in semiparametric estimation [38,39].

In standard parameter estimation, one has to fix a statistical model, assuming a known dependence of the wave function from a finite number of parameters. On the contrary, the theory of semiparametric estimation deals with models with an infinity of degrees of freedom. The goal is to extract information about a finite number of parameters of interest, making as few assumptions as possible on the underlying model. A prototypical example is the estimation of the mean of an unknown probability distribution with finite variance [39]. In the context of quantum technologies, semiparametric methods have recently been applied to super-resolution imaging [40,41] and a fully quantum generalization of the theory has been derived [42]. The related task of estimating a small subset of a finite number of parameters, treating the others as a nuisance, has also been recently studied in the context of quantum estimation theory [43,44].

In this Letter, we apply semiparametric methods to estimating quantities pertinent to the frequency domain, based on the time profile of the coincidence dip in HOM interferometry. We show that certain quantities, e.g., certain raw moments, cannot be successfully estimated due to the Fourier transform needed to convert between the two domains. However, we find other interesting quantities, essentially regularized moments, that can be estimated and also provide useful information on entanglement of the two-photon state. In light of the possible applications of the HOM interference for time measurement [45–47], we demonstrate that semiparametric methods offer an intriguing solution for model-independent estimation.

Basics of HOM interference. The HOM effect consists of a two-photon interference occurring when these arrive at the same time on a beam splitter (BS) with reflectivity R and transmittivity $T = 1 - R$ from separate ports [1]. This results in a suppression of the observed coincidence rate C, as measured by photon detectors at the two BS outputs. The effect is generally studied by scanning the relative delay τ in the arrival

*ilaria.gianani@uniroma3.it

times, producing an interference figure $C(\tau)$, modulated from $C(\tau) = C_0$ for long delays to a minimum achieved for $\tau = 0$:

$$C(\tau) = C_0[1 - v\tilde{f}(\tau)], \tag{1}$$

where $v = 2RT/(R^2 + T^2)$. We can write $\tilde{f}(\tau) = \int e^{i\omega\tau} f(\omega)d\omega$, in which the function $f(\omega)$ is intimately related to the spectral properties of the emission (see Supplemental Material [48]).

Semiparametric estimation. The estimation of any parameter θ connected to $f(\omega)$ without assuming a specific form for it implicitly relies on the knowledge of many other parameters $\boldsymbol{\eta} = [\eta_1, \eta_2, \dots, \eta_M]$ needed for describing the spectral function. An example could be the estimation of one given moment of $f(\omega)$, with all the others acting as nuisance parameters. Here the problem lies in the fact that M could be too large for practical purposes, with the genuine semiparametric setting being achieved when $M \to \infty$. In the standard parametric approach, it would be natural to try to write the Fisher information matrix $\mathcal{F}$ of the vector of parameters $[\theta, \boldsymbol{\eta}]$ and obtain the Cramér-Rao bound (CRB) for an unbiased estimator of θ by means of its inversion,

$$\Delta^2\check{\theta} \geqslant \frac{1}{N}(\mathcal{F}^{-1})_{\theta\theta}, \tag{2}$$

with N being the number of repetitions of the experiment. When the dimension of the Fisher matrix M is large, inversion could be difficult and prone to numerical instabilities, or even unfeasible in the semiparametric limit due to its large size. This bound is written in terms of the classical Fisher information and thus pertains to a specific choice of the measurement, and should not be confused with the quantum version, which is independent of the setting of the experiment.

The theory of semiparametric estimation assists us in obtaining an expression for the bound (2) without manipulating large, formally infinite-dimensional matrices. The evaluation of the CRB is based on geometrical considerations: instead of evaluating and inverting the Fisher information matrix of multiple parameters, the optimal bound is obtained by Hilbert space methods [38,39]. While the complete details of the theory are quite technical, we rely on the treatment in Sec. II-III of Ref. [40], which deals with the semiparametric estimation of the moments of an incoherent light source with an arbitrary spatial distribution. Here we adopt the same approach, but we have the time-delay variable τ rather than a spatial distribution.

Consider a generic parameter defined as $\theta = \int f(\omega)\vartheta(\omega)d\omega$, by means of a known function $\vartheta(\omega)$. Our aim is to estimate θ relying only on its definition, but without assuming a particular functional form for $f(\omega)$. Since in practice we have access to the function $\tilde{f}(\tau)$ in the conjugate domain of times, we can write

$$\begin{aligned} \theta &= \int d\omega\, \vartheta(\omega)\frac{1}{2\pi}\int d\tau\, e^{-i\omega\tau}\tilde{f}(\tau) \\ &= \int d\tau\, \tilde{f}(\tau)\tilde{\vartheta}(\tau), \end{aligned} \tag{3}$$

meaning that the parameter θ can equivalently be determined by the known function $\tilde{\vartheta}(\tau)$ in the time domain. This parameter is, strictly speaking, defined by integrating in τ over the whole real line, but in practice we will approximate this with an integral in a finite range $[-T, T]$, symmetric around 0. In the following, for convenience, instead of θ we consider the parameter

$$\begin{aligned} \theta' &= \int d\tau\, \tilde{\vartheta}(\tau)\frac{C(\tau)}{C_0} \\ &= -v\,\theta + \int d\tau\, \tilde{\vartheta}(\tau), \end{aligned} \tag{4}$$

which is more closely related to the experimental data, i.e., the coincide profile $C(\tau)$.

Formally we can introduce a "detector space" $\mathcal{T} \subset \mathbb{R}$ [40], which describes the possible settings of the detection: in our case, the time delay $\tau \in \mathcal{T}$. We then introduce two measures on this space, i.e., two ways of weighting the settings: $d\mu(\tau)$ which considers the actual experimental choices, and a random measure $dn(\tau)$, which accounts for the registered intensities, and presents Poisson statistics with mean $d\bar{n}(\tau)$ [50]. The semiparametric estimation theory ensures that given the distribution $p(\tau) = d\bar{n}(\tau)/d\mu(\tau)$, the CRB is readily found as

$$\Delta^2\theta' \geqslant \int d\tau\, p(\tau)[\tilde{\vartheta}'(\tau)]^2, \tag{5}$$

without resorting to matrix inversion. In our case, the natural description of the detector space is a continuous set $d\mu(\tau) = d\tau$, but the data are collected at discrete delays. An interpolation $\check{C}(\tau)$ can then employed, but care must be taken as this is not a genuine distribution, and simply using an integral would lead to miscalculating the number of resources N. The correct expression is found by the analogy with the discrete case (see Supplemental Material [48]).

Hermite-Gauss parameter estimation. For a straightforward characterization of $f(\omega)$, it would be convenient to extract its moments, defined as the integrals of $\omega^n f(\omega)$ over

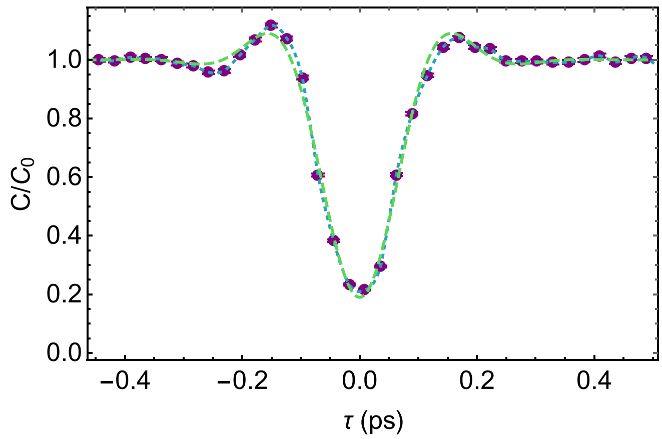

FIG. 1. Coincidence dip profile, normalized to C_0 coincidences, obtained scanning the relative delay between the two photons when arriving at the BS. The photon pairs are detected through avalanche photodiodes after passing two interference filters (fourth-order super-Gaussian profile, 7.3 nm width). The coincidence counts are collected in 5 s, with $C_0 = 4653$ coinc. The dotted line is the interpolation; the dashed line is a fit with the function reported in (see Supplemental Material and Refs. [48,49]).

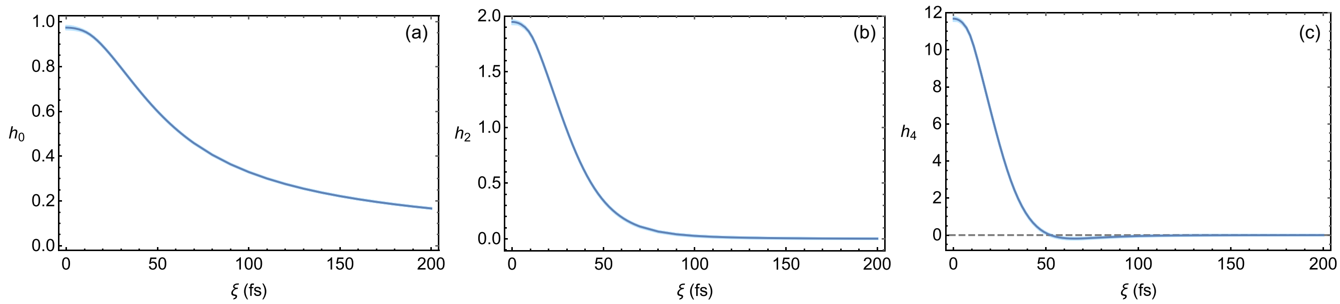

FIG. 2. Estimation of HG parameters obtained with the semiparametric method from the measured data as a function of the width of the HG functions determined by ξ. The plots show the estimate for (a) the zero-order one, (b) the second-order term, and (c) the fourth-order one, which shows a negative region witnessing the presence of entanglement. The shading under the curve indicates the uncertainty on the estimated parameters.

the frequency space. In this case, the associated semiparametric estimators are the nth derivative of the Dirac δ function, according to (3). Since the corresponding experimental estimate would not be a well-defined quantity, the semiparametric approach cannot be applied in similar instances. Instead, we can focus on the decomposition of $f(\omega)$ in Hermite-Gauss (HG) functions,

$$HG_n(\omega) = e^{-\xi^2\omega^2} H_n(\xi\omega), \tag{6}$$

where $H_n(x)$ is the nth Hermite polynomial and we have $\tilde{HG}_n(\tau) = (-i)^n e^{-\frac{\tau^2}{4\xi^2}} \frac{\tau^n}{\xi^{n+1}}$ [51], and ξ is a positive scale parameter with units of time. These quantities measure the contribution of modulating terms in the spectral function $f(\omega)$, with an exponential providing the necessary regularity. Due to the Hermitianity of $f(\omega)$, we expect all odd-n terms to vanish. The extraction of any particular HG component, independently of the others by making no assumptions on the density profile $C(\tau)$, is a semiparametric problem: the abstract parameter θ will correspond to h_{2n} for different n.

We can exploit the fact that for a separable wave function, the following condition must hold (see Supplemental Material [48]):

$$h_{2n} = (-1)^n \int HG_{2n}(\omega) f(\omega) \geqslant 0. \tag{7}$$

The semiparametric estimation of one parameter h_{2n} violating the positivity condition acts as a witness of spectral entanglement. We notice that this is equivalent to observed values $\tilde{f}(\tau) < 0$, i.e., $C(\tau) > C_0$, which is a well-known fact [35,52]. However, our approach recasts this entanglement criterion in different quantitative terms, remarkably, by considering the whole shape of the interferogram, rather than individual points.

Results. Our setup is the standard HOM interferometer in which two photons from a down-conversion crystal (β barium borate, 3 mm length, degenerate type-I phase matching at $\lambda = 810$ nm) arrive on a beam splitter; this was chosen with reflectivity $R \sim 2/3$, thus setting the visibility in (1) to $v = 0.81$. The use of a CW pump makes the wave function almost monochromatic along Ω, as enforced by energy conservation, while two interference filters define the wave function in the ω direction, since the intrinsic bandwidth of the down-conversion emission, as dictated by the crystal length, is much wider. The HOM dip profile $C(\tau)/C_0$ has been reconstructed at different points, as shown in Fig. 1. The delay τ was controlled by means of a translation stage. The interference figure is collected at a sampling rate of $\delta\tau = 13.4$ fs, as reported in Fig. 1, and then interpolated by means of third-order polynomials. This constitutes the data set we use for estimation of generalized momenta of the order of $n = 0, 2, 4$, seconding the expected symmetry.

In Fig. 2, we plot the semiparametric estimates of h_n, as a function of the parameter ξ, as obtained by the integral estimator based on the interpolated function: the semiparametric method offers reliable estimates, and h_4 is the first to witness the presence of entanglement in the state, taking negative values for a wide range of ξ.

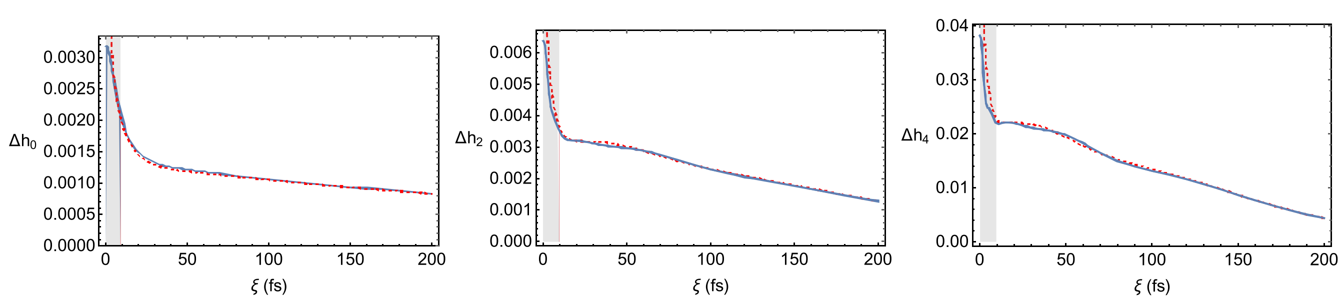

FIG. 3. Uncertainty on the estimation of HG parameters obtained with a Monte Carlo simulation over 1000 repetitions considering Poissonian noise to the measured data. The red dotted line shows the relative CRB. The shaded gray area identifies the one in which a bias can be expected, and may show as a violation of the CRB.

The corresponding uncertainties are analyzed in Fig. 3, which show the standard deviation Δh_n on the estimated parameters. This is assessed by means of a bootstrap method, consisting of Monte Carlo repetitions of the experiment, based on the registered experimental counts; this smoothly accounts for the contribution to the uncertainty due to the interpolation step. This reveals that although the curves of the average values appear regular, a bias occurs, manifesting as a violation of the semiparametric CRB (5). We hence conclude that our estimator is not an unbiased estimator of the parameter since the *discrete* nature of the original data still affects it in the interpolation needed to obtain the estimator $\check{C}(\tau)$ for the continuous density. This means that if we keep spending resources to increase the precision on the punctual estimates of the rates $C(\tau_i)$ without reducing $\delta\tau$, the approximated parameter will eventually reveal its difference from the true one. Our statistical model assumes that uncertainties on the data are purely statistical, whereas the interpolation is affected by errors of a different kind. This idea that one needs to balance between interpolation error and punctual statistical errors is quite general, and was recently reported in the context of function estimation with multiple phase measurements [53].

The discrepancy with the CRB depends strongly on the value of ξ. We can derive an argument illustrating what the region is in which we can neglect our bias. In fact, the width of $HG_{2n}(\omega)$ increases as ξ is reduced, and the spacing $\delta\tau$ eventually becomes too large to capture variations—this effectively imposes a low-pass filter. Since these functions have no compact support, we cannot use a Shannon-Nyquist criterion rigorously to evaluate the quality of the sampling; however, we can take as a guiding principle the fact that the sampling frequency $1/\delta\tau$ must exceed the width of the Gaussian in (6), leading to $\xi > \delta\tau/\sqrt{2}$. Below this value, we cannot rely on our estimate, even if an interpolation is used. Conversely, for a target parameter, the bandwidth of the corresponding function determines what sampling step can be judged satisfactory. The interval in which h_4 witness entanglement, on the other hand, is safely outside this unreliability region, revealing the validity of our witness.

The estimates of the parameters (6) can serve the purpose of using the HOM profile for the measurement of small time delays, without resorting to fitting the coincidence curve to a specific model [45]. Considering an extra delay τ_0 shifting the coincidence curve as $\tilde{f}(\tau - \tau_0)$, we can evaluate what uncertainty $\Delta\tau_0$ can be obtained by measuring the three Hermite-Gauss parameters h_0, h_2, or h_4. For small shifts, these can be evaluated as $\Delta\tau_0 = |\Delta h_i/\partial_{\tau_0} h_i|_{\tau_0=0}|$, which are shown in Fig. 4 as a function of ξ. As a general rule, higher modes provide lower uncertainties, when *xi* is properly set; notice that this optimization for h_0 is akin to a standard fit enforcing a Gaussian shape. It should be noted that higher HG terms would be less reliable: since they consider modulations with shorter periods in τ, they would be more affected by fluctuations of the level of the signal. Further, since the bandwidth of the HG functions grows with the order, finer sampling could be required. Our analysis shows that with the collected number of events, one can reduce the statistical uncertainty to the point where instrumental effects—notably, the reproducibility of the translation stage movements—become the main source of error. On the other hand, this technique requires a complete scan of the coincidence profile, as well as a calibration step at $\tau_0 = 0$, differently from model-dependent techniques [9].

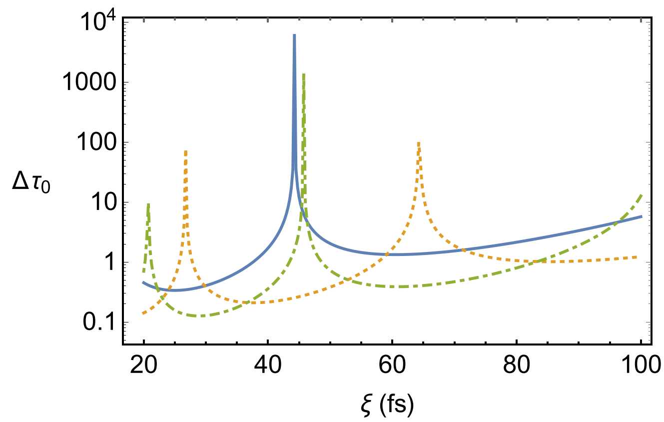

FIG. 4. Uncertainty on the estimation of an extra delay τ_0 obtained through h_0 (blue solid line), h_2 (orange dotted line), and h_4 (green dash-dotted line) as a function of ξ.

Conclusions. We have presented a semiparametric analysis of the Hong-Ou-Mandel interference profile. The use of Fourier transforms curtails the adoption of semiparametric methods; nevertheless, an analysis in terms of Hermite-Gauss functions can be effectively carried out. This brings about a reinterpretation of a standard entanglement witness in terms of spectral properties. The possible use of this analysis for delay measurements has been illustrated.

In this respect, tools from statistical classical and quantum estimation theory have already proven to be extremely useful for analyzing and engineering metrological schemes based on HOM interference [9,46,47,54–56]. Accordingly, we expect that delving into more advanced statistical methods will also bring further insights to such applications.

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