OPTIMAL QUANTITATIVE STABILITY FOR A SERRIN-TYPE PROBLEM IN CONVEX CONES

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ABSTRACT. We consider a Serrin's type problem in convex cones in the Euclidean space and motivated by recent rigidity results we study the quantitative stability issue for this problem. In particular, we prove both sharp Lipschitz estimates for an L^2-pseudodistance and estimates in terms of the Hausdorff distance.

1. Introduction

The present paper deals with the quantitative stability of a rigidity result for a mixed boundary value Serrin-type problem in convex cones. Such a rigidity result was established in [35] (see also [14]) for convex cones that are smooth outside the origin. Our approach follows the spirit of [38], where quantitative stability results were obtained for (almost) constant mean curvature hypersurfaces and Heintze-Karcher-type inequalities. As in [38], our analysis allows non-smooth cones such as cones with singularities possibly different from the origin alone.

Given a cone Σ in $\mathbb{R}^N$, $N \geq 2$, with vertex at the origin, i.e.

$$\Sigma = \{tx \,:\, x \in \omega\,,\, t \in (0,\infty)\}\,,$$

where ω is an open connected set on the unit sphere $\mathbb{S}^{N-1}$; we consider a bounded domain (i.e., a bounded connected open set) $\Sigma \cap \Omega$ – where Ω is a smooth bounded domain in $\mathbb{R}^N$ – such that its boundary relative to the cone $\Gamma_0 := \Sigma \cap \partial\Omega$ is smooth, while $\partial\Gamma_0$ is a $N-2$-dimensional manifold and $\partial(\Sigma\cap\Omega)\setminus\Gamma_0$ is smooth enough outside a singular set $\mathcal{S} \subset \partial\Sigma$ of finite ℓ-dimensional upper Minkowski content for some $0 \leq \ell \leq N-2$. To simplify matters, we also assume that $\mathcal{H}^{N-2}(\partial^*\Sigma \cap \partial\Gamma_0) = \mathcal{H}^{N-2}(\partial\Gamma_0)$, where $\partial^*\Sigma$ denotes the smooth part of $\partial\Sigma$. For further details on the setting, we refer to [38, Setting A and Remark 2.2]. We also set $\Gamma_1 := \partial(\Sigma\cap\Omega)\setminus\left(\overline{\Gamma}_0 \cup \overline{\mathcal{S}}\right)$ and denote with ν the (exterior) unit normal vector field to $\Gamma_0 \cup \Gamma_1$. We consider the following mixed boundary value problem:

$$\begin{cases} \Delta u = N & \text{in } \Sigma\cap\Omega \\ u = 0 & \text{on } \Gamma_0 \\ u_\nu = 0 & \text{on } \Gamma_1. \end{cases} \tag{1.1}$$

As in [35], we assume that the solution u of (1.1) is of class

$$W^{1,\infty}(\Sigma\cap\Omega) \cap W^{2,2}(\Sigma\cap\Omega)\,; \tag{1.2}$$

such an assumption can be viewed as a gluing condition, and, as proved in [35, Section 6] for cones smooth outside of the vertex, it is surely satisfied if $\overline{\Gamma}_0$ and $\partial\Sigma$ intersect orthogonally.

As proved in [38] we have the following fundamental identity for Serrin's problem in Σ:

$$\int_{\Sigma\cap\Omega} (-u)\left\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\right\} dx + \int_{\Gamma_1} u\,\langle \nabla^2 u \nabla u, \nu\rangle\, dS_x = \frac{1}{2}\int_{\Gamma_0}\left(u_\nu^2 - R^2\right)\left(u_\nu - \langle x - z, \nu\rangle\right) dS_x, \tag{1.3}$$

for every $z \in \mathbb{R}^N$ such that

$$\langle x - z, \nu\rangle = 0 \quad \text{for any } x \in \Gamma_1, \tag{1.4}$$

where

$$R = \frac{N\,|\Sigma\cap\Omega|}{|\Gamma_0|}.$$

2020 *Mathematics Subject Classification.* Primary 35N25, 53A10, 35B35; Secondary 35A23.

Key words and phrases. Serrin's overdetermined problem, convex cones, symmetry, rigidity, integral identities, stability, quantitative estimates.

If the cone Σ is convex, such identity provides an alternative proof of [35, Theorem 1.1][1]. Indeed, if the following overdetermined condition is in force:

$$u_\nu = R \quad \text{on } \Gamma_0\,, \tag{1.5}$$

then (1.2) is satisfied (this can be deduced from [23]) and (1.3) reads as

$$\int_{\Sigma\cap\Omega} (-u) \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx + \int_{\Gamma_1} u \, \langle \nabla^2 u \nabla u, \nu \rangle \, dS_x = 0\,.$$

Moreover, being $u \leq 0$ in $\Sigma \cap \Omega$ (see e.g. [38, Lemma 4.1]) and using the convexity of the cone one has (see e.g. [35, Formula (3.9)])

$$\int_{\Gamma_1} u \, \langle \nabla^2 u \nabla u, \nu \rangle \, dS_x \geq 0\,, \tag{1.6}$$

and hence

$$\int_{\Sigma\cap\Omega} (-u) \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx \leq 0\,.$$

But, on the other hand, from Cauchy-Schwarz inequality we easily have

$$\int_{\Sigma\cap\Omega} (-u) \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx \geq 0\,.$$

Hence,

$$\int_{\Sigma\cap\Omega} (-u) \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx = 0\,,$$

and so, being as $u < 0$ in $\Sigma \cap \Omega$ (see e.g. [38, Lemma 4.2]), – similarly to [6, 35, 37, 43] – we deduce the following rigidity:

$$\Sigma \cap \Omega = \Sigma \cap B_R(z) \quad \text{and} \quad u(x) = \frac{\left(|x - z|^2 - R^2\right)^2}{2}\,. \tag{1.7}$$

Identity (1.3) provides the starting point of our quantitative analysis. We refer the reader to [1, 7, 11, 17, 28, 29, 31, 19, 34] and the surveys [15, 26] for results related to the quantitative stability of the classical Serrin's problem (i.e., the particular case where $\Sigma = \mathbb{R}^N$). Roughly speaking what we want to prove is the following: if (1.5) is "almost" satisfied then the domain Ω is "close" to the ball in a quantitative way.

We mention that, the point $z \in \mathbb{R}^N$ can be characterized in terms of the linear space generated by the normal vector field $\nu(x)$ for $x \in \Gamma_1$. In fact, being Σ a cone with vertex at the origin we have that $\langle x, \nu \rangle = 0$ on $\Gamma_1 \subset \partial\Sigma$, and hence (1.4) is equivalent to $\langle z, \nu \rangle = 0$ on Γ_1. That is, $z \in [\text{span}\,\{\nu(x) \,:\, x \in \Gamma_1\}]^\perp$, where $[\text{span}\,\{\nu(x) \,:\, x \in \Gamma_1\}]^\perp$ is the orthogonal complement in $\mathbb{R}^N$ of the vector subspace $\text{span}\,\{\nu(x) \,:\, x \in \Gamma_1\} \subseteq \mathbb{R}^N$. In particular,

$$\dim\left(\text{span}\,\{\nu(x) \,:\, x \in \Gamma_1\}\right) = N \tag{1.8}$$

is a sufficient condition that guarantees that z must be the origin. Moreover, condition (1.8) is surely verified if Γ_1 contains at least a transversally nondegenerate point, in the sense of the definition introduced in [36] (see also [38, Proposition 2.15]). In particular, this is always the case if Σ is a strictly convex cone (and $\Gamma_1 \neq \varnothing$). That a transversally nondegenerate point was sufficient to force z to be the origin was noticed in [36]. The condition in (1.8), used here following [38], is more general and successfully applies to the study of the stability issue. To avoid excessive technicalities, the stability results are presented under the additional assumption that Γ_0 and $\partial\Sigma$ intersect in a Lipschitz way so that $\Sigma \cap \Omega$ is a Lipschitz domain.

The tool that allows to fix the center z of the approximating ball in $[\text{span}\,\{\nu(x) \,:\, x \in \Gamma_1\}]^\perp$ (that is the origin whenever (1.8) is in force) is the following weighted Poincaré-type inequality established in [39]:

$$\|\mathbf{v}\|_{L^p(\Sigma\cap\Omega)} \leq \eta_{p,\alpha}(\Gamma_1, \Sigma\cap\Omega)^{-1} \, \|\delta_{\Gamma_0}^{\alpha} \nabla \mathbf{v}\|_{L^p(\Sigma\cap\Omega)}, \tag{1.9}$$

which holds true for any $0 \leq \alpha \leq 1$ and $\mathbf{v} : \Sigma \cap \Omega \to \text{span}\,\{\nu(x) \,:\, x \in \Gamma_1\} \subseteq \mathbb{R}^N$ such that $\mathbf{v} \in W^{1,p}_\alpha(\Sigma \cap \Omega)$ and $\langle \mathbf{v}, \nu \rangle = 0$ a.e. in Γ_1.

[1]In [35] two proofs were provided, the first following the tracks of [6, Theorem 1] and the second following the tracks of [43, Theorem 1]. The proof in [38] instead, follows the tracks of [37, Theorems I.1, I.2] and their subsequent development in [28, Theorem 2.1].

Notice that, if we consider the function

$$h := q - u, \quad \text{where } q \text{ is the quadratic function defined as } q(x) = \frac{1}{2}|x - z|^2, \tag{1.10}$$

the choice $z = 0$ always guarantees that $\langle \nabla h, \nu \rangle = 0$ on Γ_1, by the homogeneous Neumann condition $u_\nu = 0$ on Γ_1 and $\langle x, \nu \rangle = 0$ on $\Gamma_1 \subset \partial\Sigma$; therefore, if $\operatorname{span}\{\nu(x) : x \in \Gamma_1\} = \mathbb{R}^N$ the new Poincaré-type inequality can be applied with $\mathbf{v} := \nabla h$. Such a weighted Poincaré-type inequality will allow us to deal with the weighted integral

$$\int_{\Sigma\cap\Omega} (-u) \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx\,,$$

which appears in (1.3). In the classical case $\Sigma = \mathbb{R}^N$, different approaches to deal with such a weighted integral can be found in [7, 17, 28, 29, 31].

The quantitative stability results provided in the present paper include, as particular case[2] and when (1.8) is in force, the following Lipschitz stability estimate for the L^2-pseudodistance of $\Sigma \cap \Omega$ to $\Sigma \cap B_R(0)$:

$$\| |x| - R \|_{L^2(\Gamma_0)} \le C \, \| u_\nu^2 - R^2 \|_{L^2(\Gamma_0)} \,. \tag{1.11}$$

The closeness in L^2-pseudodistance obtained here is stronger than the closeness in terms of the so called asymmetry in measure. In fact, clearly the L^2-pseudodistance is stronger than the L^1-pseudodistance, being as

$$\| |x| - R \|_{L^1(\Gamma_0)} \le |\Gamma_0|^{1/2} \| |x| - R \|_{L^2(\Gamma_0)},$$

by Hölder's inequality. In turn, [10, Proposition 6.1] informs us that the L^1-pseudodistance is stronger than the asymmetry in measure, that is

$$|(\Sigma \cap \Omega) \Delta (\Sigma \cap B_R(0))| \lesssim \| |x| - R \|_{L^1(\Gamma_0)}.$$

The constants in our quantitative estimates can be explicitly computed and estimated in terms of a few chosen geometrical parameters. At first, we obtain (1.11) for an explicit constant C only depending on $\eta_{2,1}(\Gamma_1, \Sigma \cap \Omega)$ and a lower bound $\underline{m}$ for u_ν on Γ_0.

In [38], new notions of uniform interior and exterior sphere conditions relative to the cone Σ were introduced. These return the classical known uniform sphere conditions in the case $\Sigma = \mathbb{R}^N$; when $\Sigma \subsetneq \mathbb{R}^N$ they are related to how $\overline{\Gamma}_0$ and $\partial\Sigma$ intersect (see [38, Sections 4.1 and 4.2]), and both of them are always satisfied if $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ intersect orthogonally. As in the classical case $\Sigma = \mathbb{R}^N$, these conditions revealed to be useful tools to perform barrier arguments in the mixed boundary value setting for $\Sigma \subset \mathbb{R}^N$ to obtain uniform lower and upper bound for the gradient. In particular, if Σ is convex, $\underline{r}_i$-uniform interior sphere condition relative to Σ guarantees the validity of Hopf-type estimates: in fact, [38, Lemma 4.4] gives that $u_\nu \ge \underline{r}_i$ on Γ_0 so that we can take $\underline{m} := \underline{r}_i$. Hence, whenever $\Sigma \cap \Omega$ satisfies the $\underline{r}_i$-uniform interior sphere condition relative to Σ, we obtain (1.11) with an explicit $C = C(\eta_{2,1}(\Gamma_1, \Sigma \cap \Omega), \underline{r}_i)$.

Similarly to [38], our method is robust enough to give a complete characterization of the stability issue even in absence of the assumption (1.8). In fact, in general we can set

$$k := \dim\left(\operatorname{span}\{\nu(x) : x \in \Gamma_1\}\right), \tag{1.12}$$

which may be any integer $0 \le k \le N$, and obtain closeness of $\Sigma \cap \Omega$ to $\Sigma \cap B_R(z)$ for some suitable point z whose components in the k directions spanned by $\operatorname{span}\{\nu(x) : x \in \Gamma_1\}$ are set to be 0. General statements containing Lipschitz stability estimates for the L^2-pseudodistance are presented in what follows.

Notice that the case $\Gamma_1 = \emptyset$ is included in our treatment (in that case, we have $k = 0$).

Up to changing orthogonal coordinates, we can assume that $\operatorname{span}\{\nu(x) : x \in \Gamma_1\}$ is the space generated by the first k axes $\mathbf{e}_1, \ldots, \mathbf{e}_k$. Notice that, in this way, if we set $z \in \mathbb{R}^N$ of the form

$$z = (0, \ldots, 0, z_{k+1}, \ldots, z_N) \in \mathbb{R}^N, \tag{1.13}$$

it surely satisfies (1.4). We also fix

$$z_i = \frac{1}{|\Sigma \cap \Omega|} \int_{\Sigma\cap\Omega} (x_i - u_i(x))\, dx \quad \text{for } i = k+1, \ldots, N, \tag{1.14}$$

[2] More precise general statements will be provided later on in this Introduction.

where u_i denotes the i-th partial derivative of u and x_i the i-th component of the vector $x \in \mathbb{R}^N$. With this choice of z, if we consider the harmonic function h defined in (1.10), we have that

$$(h_1, \dots, h_k, 0, \dots, 0) \in \operatorname{span}\{\nu(x) \,:\, x \in \Gamma_1\} \subseteq \mathbb{R}^N, \qquad \langle (h_1, \dots, h_k, 0, \dots, 0), \nu \rangle = \langle \nabla h, \nu \rangle = 0 \text{ on } \Gamma_1 \tag{1.15}$$

and

$$\int_{\Sigma\cap\Omega} h_i \, dx = 0 \quad \text{for } i = k+1, \dots, N. \tag{1.16}$$

The identity $\langle \nabla h, \nu \rangle = 0$ on Γ_1 easily follows by (1.4) and the Neumann condition $u_\nu = 0$ on Γ_1.

This will allow to use the Poincaré inequality (1.9) with $\mathbf{v} := (h_1, \dots, h_k, 0, \dots, 0)$ and the (classical) weighted Poincaré inequality for functions with zero mean

$$\|v\|_{L^p(\Sigma\cap\Omega)} \le \mu_{p,\alpha}(\Sigma\cap\Omega)^{-1} \|\delta^{\alpha}_{\partial(\Sigma\cap\Omega)} \nabla v\|_{L^p(\Sigma\cap\Omega)}, \ \text{ for } 0 \le \alpha \le 1, \quad v \in L^p(G) \cap W^{1,p}_{loc}(G) \text{ with } v_{\Sigma\cap\Omega} = 0, \tag{1.17}$$

with $v := h_i$ for $i = k+1, \dots, N$.

Setting

$$\Lambda_{p,\alpha}(k) := \begin{cases} \mu_{p,\alpha}(\Sigma\cap\Omega)^{-1} & \text{if } k = 0 \\ \eta_{p,\alpha}(\Gamma_1, \Sigma\cap\Omega)^{-1} & \text{if } k = N \\ \max\left[\mu_{p,\alpha}(\Sigma\cap\Omega)^{-1}, \, \eta_{p,\alpha}(\Gamma_1, \Sigma\cap\Omega)^{-1}\right] & \text{if } 1 \le k \le N-1, \end{cases} \tag{1.18}$$

where $\mu_{p,\alpha}(\Sigma\cap\Omega)$ and $\eta_{p,\alpha}(\Gamma_1, \Sigma\cap\Omega)$ are the constants in (1.17) and (1.9) (see also Lemma 2.1 and Theorem 2.5 below). We are now ready to present the sharp stability result for the L^2-pseudodistance.

Theorem 1.1 (Lipschitz stability for Serrin's problem in terms of an L^2-psudodistance)**.** *Let Σ be a convex cone and let $\Sigma\cap\Omega$ be as described above. Let u be a solution of* (1.1) *satisfying* (1.2) *and such that $u_\nu \ge \underline{m}$ on Γ_0, for some $\underline{m} > 0$. Given the point z defined in* (1.13) *and* (1.14)*, we have that*

$$\| |x - z| - R \|_{L^2(\Gamma_0)} \le C \, \|u_\nu^2 - R^2\|_{L^2(\Gamma_0)}, \tag{1.19}$$

where the positive constant C can be explicitly estimated as follows

$$C \le \frac{1}{2\,\underline{m}} \left(2N \, \Lambda_{2,1}(k)^2 + 3\right).$$

Whenever $\Sigma\cap\Omega$ satisfies the $\underline{r}_i$-uniform interior sphere condition relative to Σ, we can take $\underline{m} := \underline{r}_i$.

Regarding the last sentence of the theorem we refer to Remark 3.2.

Our second quantitative stability result is presented in the following theorem. Before stating it, we need to introduce the following notations. Given the point $z \in \mathbb{R}^N$ chosen in (1.13) and in (1.14) we define

$$\rho_e = \max_{x \in \overline{\Gamma}_0} |x - z| \quad \text{ and } \quad \rho_i = \min_{x \in \overline{\Gamma}_0} |x - z|, \tag{1.20}$$

so that we have

$$\Gamma_0 \subseteq \left(\overline{B}_{\rho_e}(z) \setminus B_{\rho_i}(z)\right) \cap \Sigma.$$

Given $\theta \in (0, \pi/2]$ and $\tilde{a} > 0$, we say that a set G satisfies the $(\theta, \tilde{a})$*-uniform interior cone condition*, if for every $x \in \partial G$ there is a unit vector $\omega = \omega_x$ such that the cone with vertex at the origin, axis ω, opening width θ, and height $\tilde{a}$ defined by

$$\mathcal{C}_\omega = \{y \,:\, \langle y, \omega \rangle > |y| \cos(\theta), \, |y| < \tilde{a}\}$$

is such that

$$w + \mathcal{C}_\omega \subset G \ \text{ for every } \ w \in B_{\tilde{a}}(x) \cap \overline{G}.$$

Such a condition is equivalent to Lipschitz-regularity of the domain; more precisely, it is equivalent to the strong local Lipschitz property of Adams [2, Pag 66].

The landmark result of the next theorem is to estimate the difference $\rho_e - \rho_i$ in terms of the L^2-norm of the function $u_\nu - R$. Explicitly we have the following

Theorem 1.2 (Stability in terms of $\rho_e - \rho_i$ for Serrin's problem in cones)**.** *Let Σ be a convex cone and let $\Sigma \cap \Omega$ be as described above. Let u be a solution of* (1.1) *satisfying* (1.2) *and assume that $\Sigma \cap \Omega$ satisfies the $(\theta, \tilde{a})$-uniform interior cone condition.*

Let $z \in \mathbb{R}^N$ be the point chosen in (1.13)-(1.14)*. Then, we have that*

$$\rho_e - \rho_i \leq C \begin{cases} \|u_\nu - R\|_{L^2(\Gamma_0)} \max\left[\log\left(\frac{1}{\|u_\nu - R\|_{L^2(\Gamma_0)}}\right), 1\right], & \text{if } N = 2, \\ \|u_\nu - R\|_{L^2(\Gamma_0)}^{\frac{2}{N}}, & \text{if } N \geq 3. \end{cases} \tag{1.21}$$

The constant C can be explicitly estimated only in terms of $N, \tilde{a}, \theta$, the constant $\eta_{2,1}(\Gamma_1, \Sigma \cap \Omega)$ from Theorem 2.5, the diameter $d_{\Sigma\cap\Omega}$, $\underline{m}$ defined in (3.3), and $\|\nabla u\|_{L^\infty(\Sigma\cap\Omega)}$.

Whenever $\Sigma \cap \Omega$ satisfies the $\underline{r}_i$-uniform interior sphere condition relative to Σ, we can take $\underline{m} := \underline{r}_i$. If $\Sigma \cap \Omega$ satisfies the $\underline{r}_e$-uniform exterior sphere condition relative to Σ, $\|\nabla u\|_{L^\infty(\Sigma\cap\Omega)}$ can be explicitly estimated in terms of N, $d_{\Sigma\cap\Omega}$ and $\underline{r}_e$.

Regarding the last two sentences of the theorem, we refer to Remarks 3.2 and 4.7. In addition to the previous theorem, in Theorem 4.6 we also show that, under certain geometrical assumptions (i.e., Definition 2.8 and (2.8)), the stability profile in (1.21) can be improved. As noticed in Remark 2.11 such additional assumptions are automatically satisfied when $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ intersect orthogonally, and they are also trivially satisfied when $\partial\Gamma_0 = \varnothing$.

We point out that different choices of the point z can lead to alternative stability results. For instance, we may avoid using (1.9) and hence completely remove the dependence on $\eta_{p,\alpha}(\Gamma_1, \Sigma\cap\Omega)^{-1}$ for any $0 \leq k \leq N$, at the cost of leaving the point z free to have non-zero components also in the directions spanned by ν on Γ_1: we refer to Subsection 5.1 for details.

When $\Sigma = \mathbb{R}^N$, Theorem 1.1 returns a sharp stability result for the classical Serrin's problem in the spirit of [17], whereas Theorem 4.6 returns variants of the stability estimates established in [29] (for $N \neq 3$) and [31] (for $N = 3$). We refer to Subsection 5.2 and Theorem 5.2 for details.

We conclude the introduction by mentioning that the hypothesis that the cone is convex is motivated by the rigidity result of [35] which uses the inequality (1.6) which, in turn, relies on the convexity of the cone. The fact that the convexity of the cone plays a role to get the rigidity theorems is clear by analogous results for constant mean curvature surfaces and for the isoperimetric problem (see e.g. [8, 18, 24, 40]); the convexity has also been very important to prove Liouville-type results for the critical $p-$Laplace equation (see [12, 25]) and rigidity results such as radial symmetry à la Gidas-Ni-Nirenberg (see [16] and also the recent paper [13] where it is shown that the result does not hold in general nonconvex cones).

Let us observe that in [4] the isoperimetric inequality is also obtained for almost convex cones; we believe that, similarly, the rigidity result of [35] should hold for almost convex cones and, consequently, our quantitative estimates should be extended to this case. However, it is important to remark that rigidity results, both from overdetermined torsion problem and for the soap bubble one, cannot be obtained in general non-convex cones as shown in [22].

Organization of the paper. The paper is organized as follows. In Section 2 we collect some preliminary estimates, in particular the Poincaré-type inequalities that are useful to obtain our stability results and Lipschitz growth estimates for u from Γ_0. Section 3 contains the stability analysis in terms of the L^2-pseudodistance, including the proof of Theorem 1.1. Section 4 provides the stability results in terms of $\rho_e - \rho_i$ and contains the proof of Theorem 1.2 and its improved version given in Theorem 4.6. Finally, in Section 5 we discuss the corresponding stability results for alternative choices of the point z and the classical case $\Sigma = \mathbb{R}^N$.

2. Preliminary estimates

In this section we collect some preliminary estimates that we are going to use in the sequel. In particular, we start recalling some weighted Poincaré-type inequalities and then we prove some Lipschitz growth estimates for the function u from the boundary Γ_0.

In what follows, for a set $G \subset \mathbb{R}^N$ and a function $v : G \to \mathbb{R}$, v_G denotes the *mean value of v in G*, that is

$$v_G = \frac{1}{|G|} \int_G v \, dx.$$

Also, denoting with $\delta_{\partial G}(x)$ the distance of a point x in G to the boundary ∂G, for a function $v : G \to \mathbb{R}$ we define

$$\|\delta_{\partial G}^{\alpha}\, \nabla v\|_{L^p(G)} = \left(\sum_{i=1}^{N} \|\delta_{\partial G}^{\alpha}\, v_i\|_{L^p(G)}^p \right)^{\frac{1}{p}} \quad \text{and} \quad \|\delta_{\partial G}^{\alpha}\, \nabla^2 v\|_{L^p(G)} = \left(\sum_{i,j=1}^{N} \|\delta_{\partial G}^{\alpha}\, v_{ij}\|_{L^p(G)}^p \right)^{\frac{1}{p}},$$

for $0 \le \alpha \le 1$ and $p \in [1, \infty)$.

We first recall the following *weighted Poincaré-type inequality* which can be found in [5].

Lemma 2.1. *Let $G \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with boundary ∂G of class $C^{0,\alpha}$, $0 \le \alpha \le 1$ and consider $p \in [1, \infty)$. Then, there exists a positive constant, $\mu_{p,\alpha}(G)$ such that*

$$\|v - v_G\|_{L^p(G)} \le \mu_{p,\alpha}(G)^{-1} \|\delta_{\partial G}^{\alpha}\, \nabla v\|_{L^p(G)}, \tag{2.1}$$

for every function $v \in L^p(G) \cap W^{1,p}_{loc}(G)$.

In particular, if G has a Lipschitz boundary, the number α can be replaced by any exponent in $[0, 1]$.

Remark 2.2. When $\alpha = 0$ we understand the boundary of G to be locally the graph of a continuous function.

Secondly, we recall that in [21] inequality (2.1) has been strengthened, provided $p(1-\alpha) < N$. We report here a reformulation of the result in [21] and we refer to [29, Lemma 2.1] for a proof.

Lemma 2.3. *Let $G \subset \mathbb{R}^N$ be a bounded b_0-John domain, and consider three numbers r, p, α such that*

$$1 \le p \le r \le \frac{Np}{N - p(1-\alpha)}, \quad p(1-\alpha) < N, \quad 0 \le \alpha \le 1. \tag{2.2}$$

Then, there exists a positive constant $\mu_{r,p,\alpha}(G)$ such that

$$\|v - v_G\|_{L^r(G)} \le \mu_{r,p,\alpha}(G)^{-1} \|\delta_{\partial G}^{\alpha}\, \nabla v\|_{L^p(G)}, \tag{2.3}$$

for every function $v \in L^1_{loc}(G)$ such that $\delta_{\partial G}^{\alpha}\, \nabla v \in L^p(G)$.

The class of *John domain* is huge: it contains Lipschitz domains, but also very irregular domains with fractal boundaries as, e.g., the Koch snowflake. Roughly speaking, a domain is a b_0-John domain if it is possible to travel from one point of the domain to another without going too close to the boundary. The formal definition is the following: a domain G in $\mathbb{R}^N$ is a b_0*-John domain*, $b_0 \ge 1$, if each pair of distinct points a and b in G can be joined by a curve $\gamma : [0, 1] \to G$ such that

$$\delta_{\partial G}(\gamma(t)) \ge b_0^{-1} \min \{|\gamma(t) - a|, |\gamma(t) - b|\}.$$

The notion could be also defined through the so-called b_0*-cigar* property (see [42]).

Remark 2.4 (Explicit estimates of the constants and geometric dependence)**.** The best constant is characterized by the (solvable) variational problem

$$\mu_{r,p,\alpha}(G) = \min \left\{ \|\delta_{\partial G}^{\alpha}\, \nabla v\|_{L^p(G)} : \|v\|_{L^r(G)} = 1 \text{ in } G, v_G = 0 \right\}.$$

Explicit estimates are provided by [29, Remark 2.4], exploiting the fact that the proofs in [20, 21] have the benefit of giving an explicit upper bound for the Poincaré constants.

(i) For $\mu_{r,p,\alpha}(G)^{-1}$, we have that

$$\mu_{r,p,\alpha}(G)^{-1} \le k_{N,\, r,\, p,\, \alpha}\, b_0^N |G|^{\frac{1-\alpha}{N} + \frac{1}{r} + \frac{1}{p}}.$$

(ii) In the sequel we will also need an explicit estimate for the constant $\mu_{p,0}(G)$ appearing in (2.1) in the case $\alpha = 0$. By putting together [20, Theorem 8.5] and [33, Theorem 8.5], [29, item (ii) of Remark 2.4] informs that

$$\mu_{p,0}(G)^{-1} \le k_{N,\, p}\, b_0^{3N(1+\frac{N}{p})}\, d_G.$$

(iii) If G satisfies the $(\theta, \tilde{a})$-uniform interior cone condition (defined in the Introduction), then it is a b_0-John domain and b_0 can be explicitly estimated in terms of θ, $\tilde{a}$, and d_G: see [32, Lemma A.2].

We now turn our attention to *weighted Poincaré-type inequalities for vector fields,* in particular the next two theorems can be found in [39].

Theorem 2.5. *Let Σ be a cone and let Ω a smooth bounded domain in $\mathbb{R}^N$. Given $1 \le p < +\infty$ and $0 \le \alpha \le 1$, let $\Sigma \cap \Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then, there exists a positive constant $\eta_{p,\alpha}(\Gamma_1, \Sigma \cap \Omega)$ (depending on N, p, α, Γ_1 and $\Sigma \cap \Omega$) such that*

$$\|\mathbf{v}\|_{L^p(\Sigma\cap\Omega)} \le \eta_{p,\alpha}(\Gamma_1, \Sigma \cap \Omega)^{-1} \|\delta_{\Gamma_0}^{\alpha} D\mathbf{v}\|_{L^p(\Sigma\cap\Omega)}, \tag{2.4}$$

for every $\mathbf{v} : \Sigma \cap \Omega \to \operatorname{span}\{\nu(x) : x \in \Gamma_1\} \subseteq \mathbb{R}^N$ belonging to $W^{1,p}_{\alpha}(\Sigma \cap \Omega)$ and such that $\langle \mathbf{v}, \nu\rangle = 0$ a.e. on Γ_1. Here and in the following, $W^{1,p}_{\alpha}(\Sigma \cap \Omega)$ denotes the weighted Sobolev space with norm given by $\|\mathbf{v}\|_{L^p(\Sigma\cap\Omega)} + \|\delta_{\Gamma_0}^{\alpha} D\mathbf{v}\|_{L^p(\Sigma\cap\Omega)}$.

Remark 2.6. If (1.8) is in force, then (2.4) holds true for any vector field $\mathbf{v} : \Sigma \cap \Omega \to \mathbb{R}^N$ belonging to $W^{1,p}_{\alpha}(\Sigma \cap \Omega)$ such that $\langle \mathbf{v}, \nu\rangle = 0$ a.e. on Γ_1. The existence of a transversally nondegenerate point on Γ_1 (in the sense of the definition introduced in [36]) is sufficient for the validity of (1.8). In particular, this is always the case if Σ is a strictly convex cone (and $\Gamma_1 \neq \varnothing$).

As before, in the case $p(1 - \alpha) < N$, we have the following strengthened version of (2.4).

Theorem 2.7. *Let $\Sigma \cap \Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain as before. Let r, p, α be three numbers satisfying (2.2). If $\langle \mathbf{v}, \nu\rangle = 0$ a.e. in Γ_1, then there exists a positive constant $\eta_{r,p,\alpha}(\Gamma_1, \Sigma \cap \Omega)$ (depending on N, r, p, α, Γ_1 and $\Sigma \cap \Omega$) such that*

$$\|\mathbf{v}\|_{L^r(\Sigma\cap\Omega)} \le \eta_{r,p,\alpha}(\Gamma_1, \Sigma \cap \Omega)^{-1} \|\delta_{\Gamma_0}^{\alpha} D\mathbf{v}\|_{L^p(\Sigma\cap\Omega)}, \tag{2.5}$$

for for every $\mathbf{v} : \Sigma \cap \Omega \to \operatorname{span}\{\nu(x) : x \in \Gamma_1\} \subseteq \mathbb{R}^N$ belonging to $W^{1,p}_{\alpha}(\Sigma \cap \Omega)$. Moreover, we have that

$$\eta_{r,p,\alpha}(\Gamma_1, \Sigma \cap \Omega)^{-1} \le \max\left\{ |\Sigma \cap \Omega|^{\frac{1}{r} - \frac{1}{p}}\, \eta_{p,\alpha}(\Gamma_1, \Sigma \cap \Omega)^{-1},\ \mu_{r,p,\alpha}(\Sigma \cap \Omega)^{-1} \right\},$$

where $\mu_{r,p,\alpha}(\Sigma \cap \Omega)^{-1}$ and $\eta_{p,\alpha}(\Gamma_1, \Sigma \cap \Omega)^{-1}$ are those appearing in Lemma 2.3 and Theorem 2.5.

We now prove some Lipschitz growth estimates for the function u from the boundary Γ_0. We firstly recall the following definition, which was introduced in [38].

Definition 2.8. *We say that $\Sigma \cap \Omega$ satisfies the $\underline{r}_i$-uniform interior sphere condition relative to the cone Σ, if for each $x \in \overline{\Gamma}_0$ there exists a touching ball of radius $\underline{r}_i$ such that*
(i) its center x_0 is contained in $\overline{\Sigma \cap \Omega}$
and
(ii) its closure intersects $\overline{\Gamma}_0$ only at x.

We secondly recall the following Lemma, proved in [38, Lemma 4.2]

Lemma 2.9. *Let Σ be a cone and let u be the solution of (1.1). We have that*

$$-u(x) \ge \frac{1}{2}\, \delta_{\partial(\Sigma\cap\Omega)}(x)^2 \quad \text{for every} \quad x \in \overline{\Sigma \cap \Omega}, \tag{2.6}$$

where $\delta_{\partial(\Sigma\cap\Omega)}(x)$ denotes the distance of x to $\partial(\Sigma \cap \Omega)$.

If Σ is a convex cone, then we have that

$$-u(x) \ge \frac{1}{2}\, \delta_{\Gamma_0}(x)^2 \quad \text{for every} \quad x \in \overline{\Sigma \cap \Omega}, \tag{2.7}$$

where $\delta_{\Gamma_0}(x)$ denotes the distance of x to Γ_0.

We are now ready to prove the following finer version of the previous Lemma, under suitable additional assumptions.

Lemma 2.10. *Let u be the solution of (1.1). If Σ is convex and $\Sigma \cap \Omega$ satisfy the $\underline{r}_i$-uniform interior sphere condition with radius $\underline{r}_i$, and*

$$\begin{aligned} &\text{for any } x \in \overline{\Sigma \cap \Omega} \text{ such that its closest point } \underline{x} \text{ to } \overline{\Gamma}_0 \text{ belongs to } \partial\Gamma_0, \\ &\text{the ball } B_{\underline{r}_i}\left(\underline{x} + \underline{r}_i \frac{x - \underline{x}}{|x - \underline{x}|}\right) \text{ is a touching ball at } \underline{x} \text{ relative to } \Sigma \text{ (as in Definition 2.8),} \end{aligned} \tag{2.8}$$

then we have that

$$-u(x) \ge \frac{\underline{r}_i}{2}\, \delta_{\Gamma_0}(x) \quad \text{for every} \quad x \in \overline{\Sigma \cap \Omega}. \tag{2.9}$$

Proof. By (2.7), (2.9) certainly holds if $\delta_{\Gamma_0}(x) \geq \underline{r}_i$. If $\delta_{\Gamma_0}(x) < \underline{r}_i$, instead, let $\underline{x}$ be the closest point in $\overline{\Gamma}_0$ to x and call $B := B_{\underline{r}_i}\left(\underline{x} + \underline{r}_i \frac{x-\underline{x}}{|x-\underline{x}|}\right)$ the touching ball at $\underline{x} \in \overline{\Gamma}_0$ which contains x. The existence of such a ball is guaranteed by Definition 2.8 and either (2.8) (if $\underline{x} \in \partial\Gamma_0$) or the fact that Ω is C^1 (if $\underline{x} \in \Gamma_0$). By Definition 2.8, the center $\underline{x}_0 := \underline{x} + \underline{r}_i \frac{x-\underline{x}}{|x-\underline{x}|}$ of the touching ball belongs to $\overline{\Sigma} \cap \Omega$. Setting $w(y) = \left(|y - \underline{x}_0|^2 - \underline{r}_i^2\right)/2$, we get that

$$\begin{cases} \Delta(w-u) = 0 & \text{in } \Sigma \cap B \\ w - u \geq 0 & \text{on } \Sigma \cap \partial B \\ w_\nu - u_\nu \geq 0 & \text{on } \partial\Sigma \cap B. \end{cases} \tag{2.10}$$

The last boundary condition holds being as $\Sigma \cap B$ star-shaped with respect to $\left(x_0 + \underline{r}_i \frac{x-x_0}{|x-x_0|}\right) \in \overline{\Sigma} \cap B$. By comparison ([38, Lemma 4.1] with $f := w - u$) we have that $w \geq u$ in $\Sigma \cap B$, and hence, being as $x \in \Sigma \cap B$,

$$-u(x) \geq \frac{1}{2}\left(|x - \underline{x}_0|^2 - \underline{r}_i^2\right) = \frac{1}{2}\left(\underline{r}_i + |x - \underline{x}_0|\right)\left(\underline{r}_i - |x - \underline{x}_0|\right) \geq \frac{1}{2}\,\underline{r}_i\left(\underline{r}_i - |x - \underline{x}_0|\right).$$

This implies (2.9), since $\underline{r}_i - |x - \underline{x}_0| = \delta_{\Gamma_0}(x)$. □

Remark 2.11. As noticed in [38, Section 4.1], Definition 2.8, returns the classical uniform interior sphere condition[3] in the case $\Sigma = \mathbb{R}^N$, whereas when $\Sigma \subsetneq \mathbb{R}^N$ it is related to how $\overline{\Gamma}_0$ and $\partial\Sigma$ intersect; in fact, it is surely satisfied if $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ intersect orthogonally. We point out that also the additional assumption in (2.8) is automatically satisfied whenever $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ intersect orthogonally, and it is trivially satisfied whenever $\partial\Gamma_0 = \varnothing$: in the last case, Lemma 2.10 reduces to [28, (3.4)].

3. Sharp quantitative stability in terms of an L^2-pseudodistance: proof of Theorem 1.1

From now on, we consider Σ and Ω as in the setting described at the beginning of the Introduction, and in addition we assume the cone Σ to be convex and that Σ and Ω intersect in a Lipschitz way so that $\Sigma \cap \Omega$ is a Lipschitz domain.

We set k as in (1.12) and $z \in \mathbb{R}^N$ of the form (1.13) such that (1.14) holds. As already observed in the Introduction, with this choice of z, if we consider the harmonic function h defined in (1.10) we have that (1.15) and (1.16) hold true. Moreover, by direct computation, it is easy to check that $|\nabla^2 h|^2$ equals the Cauchy-Schwarz deficit for $\nabla^2 u$, that is,

$$|\nabla^2 h|^2 = |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \quad \text{in } \Sigma \cap \Omega. \tag{3.1}$$

With the previous notations, we can now establish the following.

Lemma 3.1. *For $0 \leq \alpha \leq 1$ and $1 \leq p < \infty$, we have that*

$$\|\nabla h\|_{L^p(\Sigma\cap\Omega)} \leq C\, \|\delta_{\Gamma_0}^\alpha\, \nabla^2 h\|_{L^p(\Sigma\cap\Omega)},$$

for some positive constant C satisfying $C \leq \Lambda_{p,\alpha}(k)$, where the constant $\Lambda_{p,\alpha}(k)$ is defined in (1.18).

Proof. In light of (1.15), we can apply (2.4) in Theorem 2.5 with $\mathbf{v} := (h_1, \dots, h_k, 0, \dots, 0)$ to get that

$$\left(\sum_{i=1}^k \|h_i\|_{L^p(\Sigma\cap\Omega)}^p\right)^{1/p} \leq \eta_{p,\alpha}(\Gamma_1, \Sigma \cap \Omega)^{-1} \left(\sum_{i,j=1}^k \|\delta_{\Gamma_0}^\alpha\, h_{ij}\|_{L^p(\Sigma\cap\Omega)}^p\right)^{1/p}. \tag{3.2}$$

In light of (1.16), we can apply (2.1) to each first partial derivative h_i of h, $i = k+1, \dots, N$; notice that in those applications of (2.1) we can replace $\delta_{\partial(\Sigma\cap\Omega)}$ with δ_{Γ_0}, being as $\delta_{\partial(\Sigma\cap\Omega)}(x) \leq \delta_{\Gamma_0}(x)$.

Raising to the power of p those inequalities and (3.2), and then summing up, the conclusion easily follows. □

In what follows we show that we can obtain explicit ad hoc trace-type inequalities for h and ∇h whenever we have at our disposal a positive lower bound $\underline{m}$ for $|\nabla u|$ on Γ_0, i.e.,

$$u_\nu \geq \underline{m} > 0 \quad \text{on } \Gamma_0. \tag{3.3}$$

[3]Since, in general, we assume Ω to be smooth (say, at least, C^2), it surely satisfies the classical uniform sphere conditions: see also Remark 5.3.

Remark 3.2. We mention that a geometric condition that guarantees the validity of (3.3) is the uniform interior sphere condition relative to Σ of Definition 2.8; indeed if $\Sigma \cap \Omega$ satisfies the $\underline{r_i}$-uniform interior sphere condition relative to Σ, then [38, Lemma 4.4] ensures that (3.3) holds true with $\underline{m} := \underline{r_i}$.

Lemma 3.3 (Weighted trace inequality for $h - h_{\Sigma\cap\Omega}$ and ∇h)**.** *For any $z \in \mathbb{R}^N$ satisfying* (1.4)*, consider $h = q - u$ defined as in* (1.10)*. Let $\underline{m}$ be the lower bound defined in* (3.3)*. We have that*

$$\|h - h_{\Sigma\cap\Omega}\|^2_{L^2(\Gamma_0)} \le \frac{2}{\underline{m}} \left(\frac{N}{\mu_{2,1}(\Sigma\cap\Omega)^2} + 1 \right) \|(-u)^{\frac{1}{2}} \nabla h\|^2_{L^2(\Sigma\cap\Omega)}, \tag{3.4}$$

where $\mu_{2,1}(\Sigma\cap\Omega)$ is the best constant in the Poincaré inequality (2.1) *(with $p = 2$, $\alpha = 1$).*

Moreover, we have that

$$\|\nabla h\|^2_{L^2(\Gamma_0)} \le C \left(\|(-u)^{\frac{1}{2}} \nabla^2 h\|^2_{L^2(\Sigma\cap\Omega)} + \int_{\Gamma_1} u \langle \nabla^2 u \nabla u, \nu \rangle dS_x \right), \tag{3.5}$$

where the positive constant C in (3.5) *satisfies*

$$C \le \frac{2}{\underline{m}} \left(N\, \Lambda_{2,1}(k)^2 + 1 \right),$$

where $\Lambda_{2,1}(k)$ is the constant defined in (1.18) *(with $p = 2$, $\alpha = 1$).*

Remark 3.4. Note that from the convexity of the cone, the error term

$$\int_{\Gamma_1} u \langle \nabla^2 u \nabla u, \nu \rangle dS_x$$

is non-negative (see e.g. [35, Formula (3.9)] for a proof). Such an error term will be re-absorbed later, as it appears in the left-hand side of the integral identity for Serrin's problem (1.3).

Proof of Lemm 3.3. Combining (2.1) (used here with $G := \Sigma\cap\Omega$, $p := 2$, $\alpha := 1$) and (2.6) we find that

$$\int_{\Sigma\cap\Omega} (h - h_{\Sigma\cap\Omega})^2 dx \le 2\, \mu_{2,1}(\Sigma\cap\Omega)^{-2} \int_{\Sigma\cap\Omega} (-u) |\nabla h|^2 dx.$$

Putting together the last inequality, [38, (5.12)], and (3.3), (3.4) easily follows.

Let us now prove (3.5). Combining Lemma 3.1 (used here with $p := 2$, $\alpha := 1$) and (2.6) we find that

$$\int_{\Sigma\cap\Omega} |\nabla h|^2 dx \le 2\, \Lambda_{2,1}(k)^2 \int_{\Sigma\cap\Omega} (-u) |\nabla^2 h|^2 dx,$$

where $\Lambda_{2,1}(k)$ is the constant defined in (1.18) (with $p := 2$ and $\alpha := 1$). The conclusion easily follows putting together the last inequality, [38, (5.13)], and (3.3). □

The last Lemma that we need in order to prove Theorem 1.1 is the following

Lemma 3.5. *Let $\underline{m}$ be the lower bound defined in* (3.3)*. We have that*

$$\|\nabla h\|_{L^2(\Gamma_0)} \le \frac{C}{2} \|u_\nu^2 - R^2\|_{L^2(\Gamma_0)}, \tag{3.6}$$

where C is the same constant appearing in (3.5)*.*

Proof. By putting together (3.5), (3.1), and (1.3), we find that

$$\|\nabla h\|^2_{L^2(\Gamma_0)} \le \frac{C}{2} \int_{\Gamma_0} \left(u_\nu^2 - R^2 \right) h_\nu \, dS_x,$$

and, since by using Hölder's inequality we have that

$$\int_{\Gamma_0} \left(u_\nu^2 - R^2 \right) h_\nu \, dS_x \le \|u_\nu^2 - R^2\|_{L^2(\Gamma_0)} \|h_\nu\|_{L^2(\Gamma_0)} \le \|u_\nu^2 - R^2\|_{L^2(\Gamma_0)} \|\nabla h\|_{L^2(\Gamma_0)}, \tag{3.7}$$

the conclusion easily follows. □

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. By using the triangle inequality, we compute:

$$
\begin{aligned}
\| |x-z| - R \|_{L^2(\Gamma_0)} &\le \| |x-z| - |\nabla u| \|_{L^2(\Gamma_0)} + \| |\nabla u| - R \|_{L^2(\Gamma_0)} \\
&\le \| (x-z) - \nabla u \|_{L^2(\Gamma_0)} + \| |\nabla u| - R \|_{L^2(\Gamma_0)} \\
&= \| \nabla h \|_{L^2(\Gamma_0)} + \| u_\nu - R \|_{L^2(\Gamma_0)} .
\end{aligned}
\tag{3.8}
$$

Estimating the first summand by using (3.6), and the second summand by using that

$$
|u_\nu - R| \le \frac{1}{\underline{m} + R} \, |u_\nu^2 - R^2| \le \frac{1}{2\,\underline{m}} \, |u_\nu^2 - R^2| ,
$$

the conclusion easily follows. In the last inequality we used that $R = (u_\nu)_{\Gamma_0} \ge \underline{m}$. □

4. Stability estimates in terms of $\rho_e - \rho_i$: proof of Theorem 1.2

In this section we use the same notations as in Section 3.

Theorem 4.1. *Let $\Sigma \cap \Omega$ be a bounded domain satisfying the $(\theta, \tilde{a})$-uniform interior cone condition. Let $z \in \mathbb{R}^N$ be the point chosen as in* (1.13)-(1.14).

Then, there exists an explicit positive constant C such that

$$
\rho_e - \rho_i \le C \begin{cases} \| \delta_{\Gamma_0} \nabla^2 h \|_{L^2(\Sigma\cap\Omega)} \max \left[\log \left(\dfrac{e \, \| \nabla h \|_{L^\infty(\Sigma\cap\Omega)}}{\| \delta_{\Gamma_0} \nabla^2 h \|_{L^2(\Sigma\cap\Omega)}} \right), 1 \right], & \text{for } N = 2; \\ \| \nabla h \|_{L^\infty(\Sigma\cap\Omega)}^{\frac{N-2}{N}} \, \| \delta_{\Gamma_0} \nabla^2 h \|_{L^2(\Sigma\cap\Omega)}^{\frac{2}{N}}, & \text{for } N \ge 3. \end{cases}
$$

The constant C can be explicitly estimated only in terms of $N, \tilde{a}, \theta$, the constant $\eta_{2,1}(\Gamma_1, \Sigma \cap \Omega)$ from Theorem 2.5, and the diameter $d_{\Sigma\cap\Omega}$.

Proof. In what follows, we use the letter C to denote a constant whose value may change line by line. All the constants C can be explicitly computed (by following the steps of the proof) and estimated in terms of the parameters declared in the statement only (by recalling Remark 2.4).

(i) Let $N = 2$. We use [38, Lemma 6.4] with $p := N = 2$ and get:

$$
\rho_e - \rho_i \le C \, \max \left\{ \| \nabla h \|_{L^2(\Sigma\cap\Omega)} \log \left(\frac{e \, \| \nabla h \|_{L^\infty(\Sigma\cap\Omega)}}{\| \nabla h \|_{L^2(\Sigma\cap\Omega)}} \right), \| \nabla h \|_{L^2(\Sigma\cap\Omega)} \right\} .
$$

Next, Lemma 3.1 with $p := 2$ and $\alpha := 1$ gives:

$$
\| \nabla h \|_{L^2(\Sigma\cap\Omega)} \le C \, \| \delta_{\Gamma_0} \nabla^2 h \|_{L^2(\Sigma\cap\Omega)} .
$$

Thus, the desired conclusion ensues by invoking the monotonicity of the function $t \mapsto t \max\{\log(A/t), 1\}$ for every $A > 0$.

(ii) When $N \ge 3$, we can use [38, Lemma 6.4] with $p := 2$ and put it together with Lemma 3.1 with $p := 2$ and $\alpha := 1$. □

By coupling the previous theorem with a suitable upper bound for $\| \nabla h \|_{L^\infty(\Sigma\cap\Omega)}$, we easily obtain the following.

Corollary 4.2. *Let $\Sigma \cap \Omega$ be a bounded domain satisfying the $(\theta, \tilde{a})$-uniform interior cone condition. Let $z \in \mathbb{R}^N$ be the point chosen as in* (1.13)-(1.14).

Then, there exists an explicit positive constant C such that

$$
\rho_e - \rho_i \le C \begin{cases} \| \delta_{\Gamma_0} \nabla^2 h \|_{L^2(\Sigma\cap\Omega)} \max \left[\log \left(\dfrac{e}{\| \delta_{\Gamma_0} \nabla^2 h \|_{L^2(\Sigma\cap\Omega)}} \right), 1 \right], & \text{for } N = 2; \\ \| \delta_{\Gamma_0} \nabla^2 h \|_{L^2(\Sigma\cap\Omega)}^{\frac{2}{N}}, & \text{for } N \ge 3. \end{cases}
$$

The constant C can be explicitly estimated only in terms of $N, \tilde{a}, \theta$, the constant $\eta_{2,1}(\Gamma_1, \Sigma \cap \Omega)$ from Theorem 2.5, the diameter $d_{\Sigma\cap\Omega}$, and $\| \nabla u \|_{L^\infty(\Sigma\cap\Omega)}$.

Proof. The proof is analogous to that of [38, Corollary 6.7] with the only difference that to obtain the upper bound for $\| \nabla h \|_{L^\infty(\Sigma\cap\Omega)}$ we now use (3.2) with $\alpha := 1$ and $p := 2$ (instead of $\alpha := 0$ and $p := 2$), hence obtaining [38, (6.8)] with $\eta_{2,0}(\Gamma_1, \Sigma \cap \Omega)$ and $\| \nabla^2 h \|_{L^2(\Sigma\cap\Omega)}$ replaced by $\eta_{2,1}(\Gamma_1, \Sigma \cap \Omega)$ and $\| \delta_{\Gamma_0} \nabla^2 h \|_{L^2(\Sigma\cap\Omega)}$. □

We are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. By putting together (3.1) and (1.3) we find that

$$\int_{\Sigma\cap\Omega}(-u)|\nabla^2 h|^2\,dx+\int_{\Gamma_1}u\,\langle\nabla^2 u\nabla u,\nu\rangle\,dS_x=\frac{1}{2}\int_{\Gamma_0}\left(u_\nu^2-R^2\right)h_\nu\,dS_x,$$

Discarding the second summand in the left-hand side (which is non-negative) and using (3.7) and (3.6) to estimate the right-hand side, we obtain that

$$\int_{\Sigma\cap\Omega}(-u)|\nabla^2 h|^2\,dx\le C\,\|u_\nu^2-R^2\|^2_{L^2(\Gamma_0)}. \tag{4.1}$$

The conclusion follows by putting together the last inequality, (2.6) and Corollary 4.2. □

The stability profile obtained in Theorem 1.2 can be improved whenever (2.6) can be replaced with the finer estimate (2.9) relating u and δ_{Γ_0}. To this aim, we will use the following strengthened version of Lemma 3.1 in the case where $p(1-\alpha)<N$.

Lemma 4.3. *Let $z\in\mathbb{R}^N$ be the point chosen as in* (1.13)-(1.14)*.*
If r,p,α are as in (2.2)*, then we have that*

$$\|\nabla h\|_{L^r(\Sigma\cap\Omega)}\le C\,\|\delta_{\Gamma_0}^{\alpha}\,\nabla^2 h\|_{L^p(\Sigma\cap\Omega)},$$

for some positive constant C satisfying $C\le\Lambda_{r,p,\alpha}(k)$, where we have set

$$\Lambda_{r,p,\alpha}(k):=\begin{cases}\mu_{r,p,\alpha}(\Sigma\cap\Omega)^{-1} & \text{if } k=0\\ \eta_{r,p,\alpha}(\Gamma_1,\Sigma\cap\Omega)^{-1} & \text{if } k=N\\ \max\left[\mu_{r,p,\alpha}(\Sigma\cap\Omega)^{-1},\,\eta_{r,p,\alpha}(\Gamma_1,\Sigma\cap\Omega)^{-1}\right] & \text{if } 1\le k\le N-1,\end{cases} \tag{4.2}$$

where $\mu_{r,p,\alpha}(\Sigma\cap\Omega)$ and $\eta_{r,p,\alpha}(\Gamma_1,\Sigma\cap\Omega)$ are those in (2.3) *and Theorem 2.7.*

Proof. In light of (1.15), we can apply (2.5) in Theorem 2.7 with $G:=\Sigma\cap\Omega$, $A:=\Gamma_1$, and $\mathbf{v}:=(h_1,\dots,h_k,0,\dots,0)$ to get that

$$\left(\sum_{i=1}^k\|h_i\|_{L^r(\Sigma\cap\Omega)}\right)^{1/r}\le\eta_{r,p,\alpha}(\Gamma_1,\Sigma\cap\Omega)^{-1}\left(\sum_{i=1}^k\sum_{j=1}^N\|\delta_{\Gamma_0}^{\alpha}\,h_{ij}\|^p_{L^p(\Sigma\cap\Omega)}\right)^{1/p}. \tag{4.3}$$

In light of (1.16), we can apply (2.3) (with $G:=\Sigma\cap\Omega$) to each first partial derivative h_i of h, $i=k+1,\dots,N$. Raising to the power of r those inequalities and (4.3), and then summing up, the conclusion easily follows by using the inequality

$$\sum_{i=1}^N x_i^{\frac{r}{p}}\le\left(\sum_{i=1}^N x_i\right)^{\frac{r}{p}}, \tag{4.4}$$

which holds for every $(x_1,\dots,x_N)\in\mathbb{R}^N$ with $x_i\ge0$ for $i=1,\dots,N$, since $r/p\ge1$. □

Theorem 4.4. *Let $\Sigma\cap\Omega$ be a bounded domain satisfying the $(\theta,\tilde a)$-uniform interior cone condition. Let $z\in\mathbb{R}^N$ be the point chosen as in* (1.13)-(1.14)*. Then, there exists an explicit positive constant C such that*

$$\rho_e-\rho_i\le C\begin{cases}\|\delta_{\Gamma_0}^{1/2}\nabla^2 h\|_{L^2(\Sigma\cap\Omega)} & \text{if } N=2;\\ \|\delta_{\Gamma_0}^{1/2}\nabla^2 h\|_{L^2(\Sigma\cap\Omega)}\max\left[\log\left(\dfrac{e\,\|\nabla h\|_{L^\infty(\sigma\cap\Omega)}}{\|\delta_{\Gamma_0}^{1/2}\nabla^2 h\|_{L^2(\Sigma\cap\Omega)}}\right),1\right] & \text{if } N=3;\\ \|\nabla h\|_{L^\infty(\Sigma\cap\Omega)}^{(N-3)/(N-1)}\|\delta_{\Gamma_0}^{1/2}\nabla^2 h\|^{2/(N-1)}_{L^2(\Sigma\cap\Omega)} & \text{if } N\ge4.\end{cases}$$

The constant C can be explicitly estimated only in terms of $N,\tilde a,\theta$, the constant $\eta_{2,1/2}(\Gamma_1,\Sigma\cap\Omega)$ from Theorem 2.5, and the diameter $d_{\Sigma\cap\Omega}$.

Proof. As usual, we use the letter C to denote a constant whose value may change line by line. All the constants C can be explicitly computed (by following the steps of the proof) and estimated in terms of the parameters declared in the statement only.

In particular, in the following proof we are going to apply Lemma 4.3, which introduces (for some choices of r, p, α) the constant $\Lambda_{r,p,\alpha}(k)$ defined in (4.2). Notice that $\Lambda_{r,p,\alpha}(k)$ can be estimated in terms of $N,r,p,\tilde a,\theta,d_{\Sigma\cap\Omega}$, and, if $1\le k\le N$, $\eta_{p,\alpha}(\Gamma_1,\Sigma\cap\Omega)$. In fact, $\mu_{r,p,\alpha}(\Sigma\cap\Omega)$ (which appears in (4.2) if $0\le k\le N-1$) can be estimated in terms of $N,p,\tilde a,\theta,d_{\Sigma\cap\Omega}$ by recalling Remark 2.4. Moreover, from the statement of Theorem 2.7 (and recalling Remark 2.4) we have that $\eta_{r,p,\alpha}(\Gamma_1, \Sigma \cap \Omega)$ (which appears in (4.2) if $1 \le k \le N$) can be estimated in terms of $N, p, \tilde{a}, \theta, d_{\Sigma\cap\Omega}$ and $\eta_{p,\alpha}(\Gamma_1, \Sigma \cap \Omega)$.

(i) Let $N = 2$. By using [38, Lemma 6.4] with $p := 4$ we have that

$$\rho_e - \rho_i \le C \, \|\nabla h\|_{L^4(\Sigma\cap\Omega)}.$$

By applying Lemma 4.3 (with $r := 4$, $p := 2$, and $\alpha := 1/2$) we obtain that

$$\|\nabla h\|_{L^4(\Sigma\cap\Omega)} \le C \, \|\delta_{\Gamma_0}^{1/2} \nabla^2 h\|_{L^2(\Sigma\cap\Omega)},$$

and the conclusion follows.

(ii) Let $N = 3$. By using Lemma 4.3 with $r := 3$, $p := 2$, $\alpha := 1/2$, we get

$$\|\nabla h\|_{L^3(\Sigma\cap\Omega)} \le C \, \|\delta_{\Gamma_0}^{1/2} \nabla^2 h\|_{L^2(\Sigma\cap\Omega)}.$$

The conclusion follows by using [38, Lemma 6.4] with $p := N = 3$.

(iii) When $N \ge 4$, we use [38, Lemma 6.4] with $p := 2N/(N-1)$ and put it together with Lemma 4.3 with $r := \frac{2N}{N-1}$, $p := 2$, $\alpha := 1/2$. □

By coupling the previous theorem with a suitable upper bound for $\|\nabla h\|_{L^\infty(\Sigma\cap\Omega)}$, we easily obtain the following.

Corollary 4.5. *Let $\Sigma \cap \Omega$ be a bounded domain satisfying the $(\theta, \tilde{a})$-uniform interior cone condition. Let $z \in \mathbb{R}^N$ be the point chosen as in* (1.13)-(1.14). *Then, there exists an explicit positive constant C such that*

$$\rho_e - \rho_i \le C \begin{cases} \|\delta_{\Gamma_0}^{1/2} \nabla^2 h\|_{L^2(\Sigma\cap\Omega)} & \text{if } N = 2; \\ \|\delta_{\Gamma_0}^{1/2} \nabla^2 h\|_{L^2(\Sigma\cap\Omega)} \max\left[\log\left(\dfrac{e}{\|\delta_{\Gamma_0}^{1/2} \nabla^2 h\|_{L^2(\Sigma\cap\Omega)}} \right), 1 \right] & \text{if } N = 3; \\ \|\delta_{\Gamma_0}^{1/2} \nabla^2 h\|_{L^2(\Sigma\cap\Omega)}^{2/(N-1)} & \text{if } N \ge 4. \end{cases}$$

The constant C can be explicitly estimated only in terms of $N, \tilde{a}, \theta$, the constant $\eta_{2,1/2}(\Gamma_1, \Sigma \cap \Omega)$ from Theorem 2.5, the diameter $d_{\Sigma\cap\Omega}$, and, if $N \ge 3$, $\|\nabla u\|_{L^\infty(\Sigma\cap\Omega)}$.

Proof. The proof is analogous to that of [38, Corollary 6.7] with the only difference that to obtain the upper bound for $\|\nabla h\|_{L^\infty(\Sigma\cap\Omega)}$ we now use (3.2) with $\alpha := 1/2$ and $p := 2$ (instead of $\alpha := 0$ and $p := 2$), hence obtaining [38, (6.8)] with $\eta_{2,0}(\Gamma_1, \Sigma \cap \Omega)$ replaced by $\eta_{2,1/2}(\Gamma_1, \Sigma \cap \Omega)$. □

We are now ready to prove the following improved version of Theorem 1.2, under the additional geometrical assumption (2.8).

Theorem 4.6 (Improved stability in terms of $\rho_e - \rho_i$ for Serrin's problem in cones)**.** *Let $\Sigma \cap \Omega$ be a bounded domain and assume that Σ is a convex cone and $\Sigma \cap \Omega$ satisfies the $(\theta, \tilde{a})$-uniform interior cone condition. Assume that $\Sigma \cap \Omega$ satisfies the $\underline{r}_i$-uniform interior sphere condition relative to the cone Σ (as in Definition 2.8) together with* (2.8)*. Let $z \in \mathbb{R}^N$ be the point chosen in* (1.13)-(1.14)*. Then, we have that*

$$\rho_e - \rho_i \le C \begin{cases} \|u_\nu - R\|_{L^2(\Gamma_0)}, & \text{if } N = 2 \\ \|u_\nu - R\|_{L^2(\Gamma_0)} \max\left[\log\left(\frac{1}{\|u_\nu - R\|_{L^2(\Gamma_0)}} \right), 1 \right], & \text{if } N = 3, \\ \|u_\nu - R\|_{L^2(\Gamma_0)}^{\frac{2}{N-1}}, & \text{if } N \ge 4. \end{cases} \tag{4.5}$$

The constant C can be explicitly estimated only in terms of $N, \tilde{a}, \theta$, the constant $\eta_{2,1/2}(\Gamma_1, \Sigma \cap \Omega)$ from Theorem 2.5, the diameter $d_{\Sigma\cap\Omega}$, $\underline{r}_i$, and, if $N \ge 3$, $\|\nabla u\|_{L^\infty(\Sigma\cap\Omega)}$.

Proof. The conclusion follows by putting together (4.1), (2.9) and Theorem 4.4. □

Remark 4.7. Whenever $\Sigma \cap \Omega$ satisfies the $\underline{r}_e$-uniform exterior sphere condition relative to Σ in the sense of the definition introduced in [38, Definition 4.6], $\|\nabla u\|_{L^\infty(\Sigma\cap\Omega)}$ can be explicitly estimated in terms of N, $d_{\Sigma\cap\Omega}$ and $\underline{r}_e$ (see [38, Lemma 4.7 and Lemma 4.8]).

5. Additional remarks

5.1. **Alternative choices for the point z.** As already mentioned in the Introduction, different choices of the point z lead to alternative stability results. For instance, we can avoid using (2.4) (and (2.5)) and hence completely remove the dependence on $\eta_{p,\alpha}(\Gamma_1, \Sigma \cap \Omega)^{-1}$ (and $\eta_{r,p,\alpha}(\Gamma_1, \Sigma \cap \Omega)^{-1}$) for any $0 \le k \le N$, at the cost of leaving the point z free to have non-zero components also in the directions spanned by ν on Γ_1. A suitable choice to do this may be the following

$$z = \frac{1}{|\Sigma \cap \Omega|} \int_{\Sigma \cap \Omega} (x - \nabla u)\, dx. \tag{5.1}$$

Thanks to this choice of z we have the following result which is a modification of the results contained in Theorem 1.1 and in Theorems 1.2, 4.6.

Theorem 5.1. *Setting $z \in \mathbb{R}^N$ as in* (5.1) *we have that:*

(i) *Theorem 1.1 remains true with $\Lambda_{2,1}(k)$ replaced simply by $\mu_{2,1}(\Sigma \cap \Omega)^{-1}$.*
(ii) *Theorem 1.2 holds true with an explicit constant C only depending on N, $\tilde{a}$, θ, $d_{\Sigma \cap \Omega}$, $\underline{m}$, and $\|\nabla u\|_{L^\infty(\Sigma \cap \Omega)}$. Moreover, Theorem 4.6 holds true with an explicit constant C only depending on N, $\tilde{a}$, θ, $d_{\Sigma \cap \Omega}$, $\underline{r}_i$, and, if $N \ge 3$, $\|\nabla u\|_{L^\infty(\Sigma \cap \Omega)}$.*

5.2. **The classical case $\Sigma = \mathbb{R}^N$.** The following theorem analyzes Theorems 1.1 and 4.6 in the particular case where $\Sigma = \mathbb{R}^N$, returning variants of the results established in [17, 29, 31].

Notice that, when $\Sigma = \mathbb{R}^N$, we have that $\Sigma \cap \Omega = \Omega$ is a smooth, say C^2, bounded domain in $\mathbb{R}^N$. Such a domain always satisfies the classical uniform interior and exterior sphere conditions in $\mathbb{R}^N$. Moreover, as already mentioned, when $\Sigma = \mathbb{R}^N$ the uniform interior and exterior sphere conditions relative to Σ reduce to the classical uniform interior and exterior sphere conditions in $\mathbb{R}^N$.

When $\Sigma = \mathbb{R}^N$ the choice of z in (1.13) and (1.14) agrees with that in (5.1), and reduces to the center of mass of Ω, being as

$$z = \frac{1}{|\Omega|} \int_\Omega (x - \nabla u)\, dx = \frac{1}{|\Omega|} \left[\int_\Omega x\, dx - \int_{\Gamma_0} u\nu\, dx \right] = \frac{1}{|\Omega|} \int_\Omega x\, dx.$$

We point out that, in the particular case $\Sigma = \mathbb{R}^N$, many other choices for the point z are admissible: we refer the interested reader to [17, 27, 28, 29].

We are now in position to prove the following

Theorem 5.2 (Sharp stability for the classical Serrin's problem in $\mathbb{R}^N$)**.** *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain of class C^2. Then, we have that:*

(i) (1.19) *holds true for an explicit constant C only depending on N, $\underline{r}_i$, and d_Ω.*

(ii) (4.5) *holds true for an explicit constant C only depending on N, $\underline{r}_i$, $\underline{r}_e$, and d_Ω. If $\Gamma_0 = \partial\Omega$ is mean convex, then the dependence on $\underline{r}_e$ can be dropped.*

Proof of Theorem 5.2. As already noticed, (i) and (ii) immediately follow from Theorem 1.1 and Theorem 4.6 (recalling Remark 2.11).

Being as $\Sigma = \mathbb{R}^N$ and hence $\Gamma_1 = \varnothing$, we have that $k = 0$ and hence $\Lambda_{p,\alpha}(0) = \mu_{p,\alpha}(\Sigma \cap \Omega)^{-1}$ and $\Lambda_{r,p,\alpha}(0) = \mu_{r,p,\alpha}(\Sigma \cap \Omega)^{-1}$. In turn, being as Ω a C^2 domain, $\mu_{p,\alpha}(\Sigma \cap \Omega)^{-1}$ and $\mu_{r,p,\alpha}(\Sigma \cap \Omega)^{-1}$ can be explicitly estimated in terms of $\underline{r}_i$ and $d_{\Sigma \cap \Omega}$ only (see [29, (iii) of Remark 2.4]).

Being as $\Sigma = \mathbb{R}^N$, $\|\nabla u\|_{L^\infty(\Sigma \cap \Omega)} = \|\nabla u\|_{L^\infty(\Omega)}$ can be estimated by using [27, Theorem 3.10], which informs us that

$$\|\nabla u\|_{L^\infty(\Omega)} \le \frac{\max\{3, N\}}{2} \frac{d_\Omega (d_\Omega + \underline{r}_e)}{\underline{r}_e}.$$

In the particular case where $\Gamma_0 = \partial\Omega$ is mean convex, a better estimate is available, that is, $\|\nabla u\|_{L^\infty(\Omega)}$ can be estimated in terms of N and $\max_{\overline{\Omega}}(-u)$ only (see, e.g, [30, Lemma 2.2]). In turn, $\max_{\overline{\Omega}}(-u)$ can be easily estimated (e.g., applying [38, (ii) of Lemma 4.9] in the special case $\Gamma_1 = \varnothing$) by means of

$$\max_{\overline{\Omega}}(-u) \le \frac{d_\Omega^2}{2}.$$

We mention that a finer bound for $\max_{\overline{\Omega}}(-u)$ in terms of the volume $|\Omega|$ holds true thanks to a classical result on radially decreasing rearrangements due to Talenti ([41]). □

Remark 5.3. We recall that the uniform interior and exterior touching ball condition is equivalent to the $C^{1,1}$ regularity of $\partial\Omega$ (see, for instance, [3, Corollary 3.14]). Nevertheless, (ii) of Theorem 5.2 remains true by replacing $\underline{r}_i$, $\underline{r}_e$ (and hence relaxing the $C^{1,1}$ uniform regularity) with the $C^{1,\gamma}$

regularity, for $0<\gamma<1$: we refer to [9] for details. For $\gamma \geq 1$ instead, the stability exponent in (4.5) can be improved for $N \geq 4$, as proved in [31, Theorem 4.4].

Acknowledgements

The authors are members of the Gruppo Nazionale Analisi Matematica Probabilità e Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). Giorgio Poggesi is supported by the Australian Research Council (ARC) Discovery Early Career Researcher Award (DECRA) DE230100954 "Partial Differential Equations: geometric aspects and applications" and is member of the Australian Mathematical Society (AustMS).

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