AN EXPERIENCE OF EXPLORING THE BOUNDARY BETWEEN MATHEMATICS AND PHYSICS WITH PRESERVICE TEACHERS

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In this exploratory study, we investigate the potentiality of comparing mathematics and physics, from generic characterizations to interdisciplinary tasks and textbook analysis, as opportunities to foster secondary mathematics preservice teachers' awareness of the epistemic core of such disciplines. We designed and implemented a teaching sequence with Italian master students with a mathematical background, relying on a framework developed within a project about interdisciplinarity in preservice teacher education (IDENTITIES), and carried out three case studies analyzing data collected. We discuss the impact on students' conceptions and the development of learning processes at the boundary that our teaching sequence might have, as well as further reflections on how to make our activities more effective.

Keywords: Teaching and learning of mathematics in other fields, prospective teachers' education, rational behavior, learning potential, boundary crossing.

INTRODUCTION

The historical development of mathematics often reveals deep dialogues at the boundary with other scientific disciplines, particularly physics; stiffening the boundaries might lead to artificial and stereotyped views (see, for example, Boero et al., 2013; Branchetti et al., 2019), often accompanied by a characterization of the disciplines based, at least, on comparative definitions (Erduran & Dagher, 2014). Such an approach to the "disciplinarization" of knowledge not only hides the complexity of these dialogues but does not even mirror disciplinary authenticity, reachable by analyzing the scientific discourse, for example in articles or original texts (Branchetti et al., 2019). In previous works, it has been shown how a virtuous circle in preservice teacher education can be established: an interdisciplinary approach could help in understanding better the involved disciplines, while disciplinary knowledge could help in dealing with new problems not organized in a discipline yet (Satanassi et al., under review; Branchetti et al., 2019). Moreover, in Akkerman and Bakker (2011) it is stressed that while moving close to boundaries that separate/put in contact members of two different communities (in this case, disciplinary communities) there is a *learning potential* about the background of both communities, but whose fruitful activation depends on many contextual factors and is not trivial. Therefore, we address the following research problem: what processes might secondary mathematics preservice teachers enact when moving close to the boundary between mathematics and physics? Some activities to face and delve into this interdisciplinary exchange have been already explored (see Branchetti et al., 2019; Pollani et al., 2022). Among them, a teaching sequence about parabola, projectiles motion, and proof has been designed and implemented in different national and international contexts of secondary preservice teachers, to investigate whether and how learning potentials at the boundary might be exploited to make them develop an awareness of their view of mathematics and of the relationship between mathematics and physics. In this exploratory study, we qualitatively analyze the processes at the boundary between mathematics and physics that occurred in one of the national implementations, where the population consisted of preservice teachers with bachelor's in mathematics.

THEORETICAL FRAMEWORK

To explore our problem, we rely on a framework built on the notion of *boundary crossing mechanism* (Akkerman & Bakker, 2011), the *Family Resemblance Approach* (FRA) to the Nature of Science (Erduran & Dagher, 2014), and the *rational behavior* (Boero et al., 2013). Akkerman and Bakker highlight how in general the notion of *boundary* is ubiquitous and represents a dialogical phenomenon between communities, rather than a barrier. The authors then characterize four boundary crossing mechanisms: *identification*, occurring when a deep uncertainty of the line between disciplines leads first to question the core identity of intersections and then to renewed perspectives about disciplines; *coordination*, taking place if the cooperation between disciplines is required to keep the flow of work and the use of common tools; *reflection*, which happens through explaining and understanding the differences between disciplines, and thus enriching their identities; and *transformation*, which leads to a profound change, and even to new and in-between disciplines. We will use the terms *disciplinary* or *interdisciplinary learning potential* considering the increasing awareness respectively of disciplines, conveyed by identification and reflection, or of their interplay, conveyed by coordination and transformation. In designing our teaching sequence, to go beyond the stereotyped views of scientific disciplines, we referred to a characterization of disciplines developed within the FRA (Erduran & Dagher, 2014). According to the authors, the *epistemic core* of scientific disciplines is articulated in four networked categories, rather than disconnected fragments: *aims and values* (like objectivity, consistency, rationality, etc.), *practices* (like observation, argumentation, modeling, etc.), *methods* (like to generate reliable evidence and construct theories, laws, and models, etc.), and *knowledge* (like Euclidean geometry theory, atomic models, etc.). To identify the features of disciplinary and interdisciplinary discourses starting from the choices made in concrete examples, we also referred to the *rational behavior* (Boero et al., 2013; for textbooks and interdisciplinary contexts see Pollani et al., 2022), consisting of three interrelated dimensions: the *communicative* one for text presentation choices; the *epistemic* one for the choices related to identification and expounding of used facts; and the *teleological* one for pursued goals and strategies.

In this paper, the research question is: can any *learning potential* at the boundary of mathematics and physics as disciplines be actualized by our teaching sequence? In particular: what *boundary crossing mechanisms* might preservice teachers activate while characterizing mathematics and physics in different tasks? Whether and how do they become aware of their personal view about the *epistemic core* of the disciplines and able to question it? To address it, we analyze reports of 3 preservice teachers collected during the implementation of the teaching sequence carried with 25 master students in mathematics attending a mathematics education course, held at the University of Milan by LB.

In the following, we resume the teaching sequence of five two-hour lessons. In the first one, we brainstormed *What characterizes mathematics as a discipline? What are common aspects and the main differences with physics?* Then LB held the lecture "the FRA and the epistemic core of disciplines", followed by a questionnaire about the topic of the lesson, and the task *Deliver a written personal report on the characterization of mathematics as a discipline, also in comparison with physics, considering what emerged in the classroom, but also stressing your point of view*. We asked as homework: "Read this excerpt of a discussion between students about the task *Which curve is represented in the following images?* (see Fig. 1) *Gianni argues that the trajectory of the first image certainly represents a parabola, while we cannot say anything for sure about the others, but Francesca is not convinced: she says that we do not have enough information to establish that the first is a parabola, while on the others it is certain. Amina intervenes by saying that unknowing what context the images are placed in, we can never conclude.* Do you agree with one of the three? Which aspects of each position can be interesting, and which are questionable? How would you enter the debate and make it evolve to take a position?"

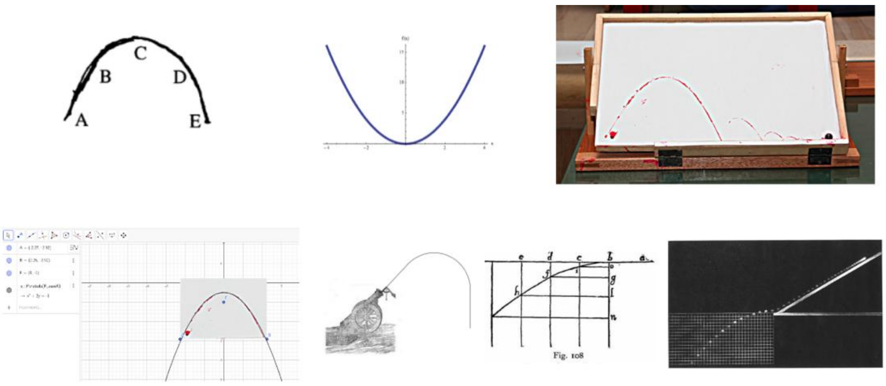

Figure 1: Images proposed to students to discuss curves and trajectories.

In the following 3 lessons, after an initial discussion about the tasks and the homework using FRA tools, three lectures were held: "The parabolic motion and the birth of physics as a discipline" (by Olivia Levrini, physicist), "Parabola in the history of mathematics and physics" (by LB), and "Habermas' dimensions of rationality" (by LP). In the fifth lesson, LP presented the analysis of an Italian physics textbook excerpt on the motion of projectiles using rationality, and then the task *Analyze in small groups with the lens of rationality the first part of the paragraph about horizontal initial speed. Deliver a text explaining in detail your* *group analysis*. Collective discussions were audio recorded, and the homework and two reports were collected. The first report led us to understand, at least partially, what were the students' points of view, so we could triangulate their answers in the second report, about textbook analysis. To help preservice teachers to catalyze and organize their conceptions, they were provided with the lenses of FRA and rationality, which we posit act as scaffolding factors. We hypothesized that the FRA could encourage them to characterize disciplines in terms of resemblances or differences, rather than definitions. Moreover, we conjecture that asking them to analyze a simulated debate and to detect in the first person the rationality in textbooks could bind and switch on their aims and values of disciplines, going beyond generic and stereotyped sentences, like "mathematics is the science of numbers" and "physics is the science of phenomena". In our analysis, we rely on the explicit information to search for boundary mechanisms, referring: to *identification* with differentiating phrases like "mathematics/physics is/is not"; to *reflection* with those like "from a physical point of view it is relevant", showing more awareness of a different relative disciplinary point of view; to *coordination* when opportunities for using mathematics to solve a physical problem are pointed out. The *transformation* is not considered relevant in this case. We identify how they are matched with aspects of the epistemic core and rational choices stressed as personally relevant. Referring to the FRA, we deepen our coding by searching if preservice teachers undertake a definitory or more blurred characterization of disciplines, if they show awareness of and in their processes, and if they refer to stereotyped views, meaning that they refer to generic and external terms or praxes.

DATA ANALYSIS

S1

FRA report	
The first words I would use straight away to characterize mathematics are: rigor, logic, abstraction, truth, and utility.	*Identification* of rather stereotyped *aims and values* of mathematics by listing single nouns.
Trying to describe a 'mathematical' process: we start from a real or invented problem, we try to identify the variables that influence the problem and isolate them, […] then we try to make the problem itself the most general and abstract as possible, […] finally we look for the limits revealed to us about the problem itself.	Non-definitory *identification* of stereotyped epistemic aspects of mathematics, again as a list; the repeated verb "to try" and the shift from "process" to "we" might show awareness.
Mathematics is the universal language to explain nature (quoting Galileo), it aims to respond to various practical problems, simplify the life of man, look for an order or a rule where there seems to be none, and explain what seems to be outside our control.	Definitory and stereotyped *identification* of mathematics, referring to a list of *aims* and *methods*.

Looking at the first words with which I characterized mathematics it is obvious that they also belong to other disciplines, such as physics.	The action of "looking at the first words" and recognizing resemblances with other disciplines shows an aware process of comparison of the epistemic cores.
Throughout history, mathematics and physics have interacted for a long time, […] geniuses have contributed to both, precisely because so close.	Historical generic *coordination* is pointed out, based on the "closeness" of the disciplines, without declining it epistemically.
Some of the main differences between mathematics and physics are that the former chose to be independent of reality, while the second has a continuous comparison and denial or verification with reality as appropriate.	*Reflection*, explaining some of the main differences in their willing choices, in one case with stereotyped *aims*, in the other with stereotyped *practices*.
However much the sciences need language mathematical and depend on it, mathematics has the same dependence on the sciences, since if it were not such a useful and versatile language, if it were not able to speak with and for the sciences, the mathematics would see its *raison d'être* disappear.	Instrumental stereotyped *coordination* and mutual need based on *aims* of utility and versatility of mathematical as a language, emphasized with "such a" utterance, and on its ontological status.
Homework	
The first image is a parabola (or perhaps a branch of a hyperbola?), and the second (probably) represents a parabola along the initial line ABC and then becomes a vertical line. The third has various parabolas, of different sizes and openings. The fourth again begins as a parabola and then assumes the course of what appears to be a horizontal line. The fifth consists of a branch of a parabola and then of an oblique line.	In her first statement a doubt is insinuated, that later becomes the possibility to state for a curve to be a parabola with a grade of uncertainty/probability, which seems to start to blur the initial "to be". All the statements are rather definitory and absolute, without much explanation.
I can't completely agree with any of the three, however how much Gianni and Francesca have positions that I partly share. Amina's position is the one with which the more I disagree, as context doesn't matter, it could certainly help us have more elements, but don't depend on it. Gianni is too rigid in excluding the parabola for the other images, it also takes a certain degree of adaptation/approximation. Francesca, on the other hand, approximates a little too much.	The first utterance reveals an aware attempt of *identification* of mathematics' *aims and values*. She justifies why she disagrees with definitory claims of the absolute truth of mathematics, and she considers truth independent from the context. She consciously identifies a new *value* ("a certain degree of adaptation"), but she does not state explicitly a standard for its acceptability.

Table 1: On the left are the original excerpts by S1, and on the right is our analysis.

In the questionnaire, she declared she had never reflected on this topic before. She was impressed by the choice to characterize and not define, and the composition of the *epistemic core*. What most impressed her is the inclusion of *aims and values*, which are terms used to describe a "real and alive person". She stressed that she appreciated that this framework does not only refer to concrete everyday actions but to less visible values, the essence of actions, which counts most. She complained that *practices* and *methods* are not easy to distinguish, and *knowledge* is too static, and does not consider that disciplines are knowledge in evolution (time-dynamicity), not an object that can be divided into fixed pieces (space dynamicity). In her textbook analysis, she referred to what the text does visually "put in evidence" (e.g., "it puts in evidence the paths", "[it could have shown] visually what it tells only by words"), as traces of *communicative rationality* and a *value* to be pursued. However, she focused more on "lack of rationality", up to questioning the text directly: for example, about the *teleological* dimension she criticized the statement "let us isolate *t* [the time variable]" with "why must *t* be isolated? […] it looks like a magic trick […] Why not write that you want to prove what is the motion path […] instead of leaving the reader unaware of reasonings behind the undertaken calculations?", where we can point out an *identification* mechanism and almost a defense of mathematics epistemology from being "a magic trick"; or, for example, about *epistemic* and *teleological* dimensions she pointed out how the "reasonable hypothesis" and the proof are not mentioned as such, implicitly referring to the hypothetical-deductive system, an aspect of epistemic core of mathematics.

S2

FRA report	
One of the first things that in my opinion characterizes mathematics, and which distinguishes it clearly from all other sciences, is the fact that it is not necessarily a pragmatic knowledge […] mathematics also makes sense to exist by itself, free from all its innumerable applications.	Conscious *identification* of a type of knowledge reached by mathematics, justified by its ontological status. The use of "not necessarily" avoids an overall definitory approach, unlike "clearly" could have made think.
Other characteristic that I would associate with mathematics is consistency, which is not exactness, but the fact that everything is consistent within a well-defined and defined axiomatic system.	Non-definitory and first-person conscious *identification* of and reflection on a *value* and its related knowledge structure.
Another characteristic that unites it to knowledge traditionally considered 'humanistic' is the fact that it is 'argumentative'.	Recognizing the resemblance of *practice* between mathematics and "humanistic knowledge".
The figure of the mathematician, for me, is that of a person who studies the mathematical world, which does not always coincide with the real one, although it may be its model;	Aware first-person *reflection* using rhetoric negation on the generic example of "the figure of the

interest in this mathematical world, however, would exist even without the link with reality, feeding on the sole desire to investigate the nature of abstract objects. […] mathematics is more akin to philosophy than to other purely scientific knowledge such as chemistry, biology, and others	mathematician", re-affirming the ontological status of mathematics and recognizing resemblances with philosophy.
Both mathematics and physics provide some of what Jürgen Habermas calls the dimensions of rational behavior.	Resemblance using the tool of rationality between mathematics and physics.
Homework	
The group agreed with Amina's statement, without having information on the context, on the reference system, it is not possible to have certain information. The statements of Gianni and Francesca are not motivated, to make the debate evolve it could be observed that in the first image there is no additional information concerning the graph represented, while in the other images there is information that accompanies the graph. Amina's position seemed to us the most reasonable, it underlines and highlights the importance of contextualizing each representation.	Conscious *identification* that the context and the information one can gain from it carries a certain degree of (un)certainty, and that contextualizing and motivating statements is necessary and help to increase their reasonability. The statements are formulated in general, without referring to a specific discipline.

Table 2: On the left are the original excerpts by S2, and on the right is our analysis.

In the questionnaire, she declared she had already reflected on the epistemology of mathematics, through personal readings and in a university course. She was impressed by the external rings concerning the institutional and social dimension of science: she considered important not only to be immersed in a discipline but to understand also how society sees scientific disciplines from the outside. In her physics textbook analysis, she started evaluating the coherence of choices with the authors' explicit aims (she says, "as authors promised"), and some generic visual criteria like "shortness, lightness, slenderness, clarity of images". All along the paragraph she stressed mainly the lack of rationality and explanations/motivations, with long and detailed arguments based on previous knowledge and considering possible student's point of view. She seems aware of the undertaken evaluating process (she often said, "[here] I point out/do not point out"), but not of her disciplinary point of view: indeed, she said, for example "It would have been more rational [first to have a general case and then particular cases because] they would have been reunited under a general theory", where she opposed a deductive approach as "more rational" to an inductive one. Furthermore, even if she also identified strengths, she did not motivate them in terms of the epistemic core of either physics or mathematics; for example, she said, "from this choice, it is easy for students to mistake of thinking that some physical laws are valid only in a specific case and not in others" and "without this concept to have been ever defined".

S3

FRA report	
Contrary to what is seen in advanced university courses in mathematics, in high school I always saw the latter as a subject completely different from any other, being then almost essentially practical. […] Mathematics is, on the other hand, exercises on exercises, a small theoretical introduction, and then again exercises on exercises. Personally, I think I did 20% of theory and 80% of exercises in the whole period of high school.	First a personal generic differentiation between mathematics and physics without traces of resemblances ("completely different"), based on a type of *knowledge*. Then a definitory *identification* based on stereotyped scholastic *practices* and his scholastic experience. Disciplines looked only as school subjects.
The similarity with a subject such as physics is almost evident: formulas and problems. However, physics on its part sees a more historical approach than mathematics. […] Unlike mathematics, however, the theoretical part of the subject appears much more present in physics (always in secondary school). The concept of proof was often associated, by any high school student who had not privately explored mathematics, with the physical realm. The structure of the verifications was perhaps the accomplice of this: many times, requests for proofs in physics appeared, seldom in mathematics.	Resemblance with physics rather stereotyped, based on scholastic *knowledge* and *practices*. Blurred attempts of reflection on differentiation based on epistemic aspects like *knowledge* and *practice*, but with disciplines looked only as school subjects.
I can summarize one last huge difference between mathematics and other subjects in the following sentence: the exception proves the rule, except in mathematics.	Hyperbolical and stereotyped differentiating *identification* of different *methods* between mathematics and other subjects.
Homework	
Reading the proposed discussion, we are more likely to agree with Amina, noting how important the context in which these images are placed is for us too, to feel confident in affirming whether or not they are parabolas.	Aware *identification* of the need of the context to reassure and "feel confident" in definitory classifying objects, referred to personal reasoning and not a specific discipline.
In class, we had thought of 'excluding' those that in our opinion were not parabolas (or not necessarily at least) by intervening on the drawings both manually and with GeoGebra […] For the fourth slide [I thought] to trace function profiles with GeoGebra that resemble that of the parabola.	Classification of curves is blurred ("not necessarily", "resemble"). Contextualizing can convince students about the "truth" behind the drawings: implicit *identification* of *values* and features of mathematical *knowledge*.

Table 3: On the left are the original excerpts by S3, and on the right is our analysis.

In the questionnaire, he declared he had never reflected on the topic before. He was impressed by the epistemic core of the disciplines, which considered something that students should learn from the very beginning because what appears simple and basic could be not so solid. He did not understand well the difference between characterizing and defining sciences, referring explicitly to the example of the mathematical definition of continuous functions, where defining is characterizing and vice versa. In his textbook analysis, he deemed each of the rational dimensions separately, starting with quantitative utterances about how much each rational dimension is in the excerpt. Through qualifying adjectives, he referred to generic and stereotyped *values* or *practices*, such as "explanations are simple and immediate", also with negative and comparative forms, like "choices do not seem unusual" and "to be more precise, it would have been more correct to write […] (domain, limitedness and compactness change)", even if the comparison (mathematical) term remains implicit. He also pointed out that "the setting is well-represented" and "even if the thesis is missed", which refers, again implicitly, to the mathematical practice of building a hypothetical deductive system.

DISCUSSION AND CONCLUSIONS

We observed three different cases during the same teaching sequence in the same context: S1 seemed to develop more personal reflections about mathematics and interdisciplinarity, S2 seemed to reinforce some views of mathematics but did not elaborate on her previous knowledge, while S3 seemed to refer only to school practices and not to deepen into epistemic issues. During our sequence, the FRA report fostered mainly an *identification* mechanism, as could have been expected from the questions, leading to characterize mathematics and physics as disciplines, but also as school subjects, as in the case of S3. This process did not always involve all the aspects of the epistemic core, was not always non-definitory and aware, and often led to explaining more differences, rather than resemblances. The homework, asking them to take a stand and being formulated in an interdisciplinary way, led to more aware observations, and made new epistemic aspects being mentioned, sometimes not so consistently with their declared view of the disciplines. Only in the case of S1 did our sequence seem to trigger a learning process at the boundary, blurring general and external sentences into more personal and contingent with the materials, supported both by the FRA and rationality. What we find particularly interesting is the fact that during the physics textbook analysis they seemed to use mathematical *values* to look at the presentation, up to questioning it openly, as in the case of S1. All of them seemed unconscious that they were referring implicitly to personal mathematical standards, which they assumed and defended as absolute and correct also analyzing a physics textbook. In conclusion, we provide preliminary answers to our question: from our analysis, it seems to emerge that some students grasped important epistemic aspects that we identified as *learning potentials*, but this did not work for all the students (see S3). The task of characterizing might foster itself mainly *identification*, and more rarely *reflection* and *coordination*. However, only in the case of S1, it seemed to occur a *learning process* at the boundary, triggered using both the FRA and rationality: indeed, S1 (critically) analyzed the textbook by questioning it openly, showing a more personal approach than the initial one, even if partially implicit and unaware. The overall weak awareness of the disciplinary point of view could be due to the homogeneity of the population's bachelors. To discuss this hypothesis, data from two other contexts, one national (Bologna, physics education course) and one international (summer school of the European project IDENTITIES, https://identitiesproject.eu), where the population was composed of secondary prospective teachers with a bachelor in mathematics *or physics*, are being analyzed, but we conjecture that this heterogeneous context could have led them to develop more "disciplinary awareness" and deeper reflections. These data might suggest a need for further reflections concerning a characterization of rationality in terms of disciplines. We will carry out further studies to check how these results are significant and generalizable, and how we can improve our teaching sequence to make it effective for most preservice teachers.

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