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# Some Families of Random Fields Related to Multiparameter Lévy Processes

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**Abstract**
Let $\mathbb{R}_+^N = [0, \infty)^N$. We here make new contributions concerning a class of random fields $(X_t)_{t\in\mathbb{R}_+^N}$ which are known as multiparameter Lévy processes. Related multiparameter semigroups of operators and their generators are represented as pseudo-differential operators. We also provide a Phillips formula concerning the composition of $(X_t)_{t\in\mathbb{R}_+^N}$ by means of subordinator fields. We finally define the composition of $(X_t)_{t\in\mathbb{R}_+^N}$ by means of the so-called inverse random fields, which gives rise to interesting long-range dependence properties. As a byproduct of our analysis, we present a model of anomalous diffusion in an anisotropic medium which extends the one treated in Beghin et al. (Stoch Proc Appl 130:6364–6387, 2020), by improving some of its shortcomings.

**Keywords** Multiparameter Lévy processes · Subordination of random fields · Fractional operators · Semi-Markov processes · Anomalous diffusion

**Mathematics Subject Classification (2020)** 60G51 · 60G60 · 60K50

## 1 Introduction

In this paper, we consider multiparameter Lévy processes $(X_t)_{t\in\mathbb{R}_+^N}$ in the sense of [5, 38–40]. The reason they are called in this way is that they enjoy, in some sense, independence and stationarity of increments. Independence of increments is meant in the following way. First, a partial ordering on $\mathbb{R}_+^N$ is established, such that $a \preceq b$ in $\mathbb{R}_+^N$ if $a_i \le b_i$ for each $i = 1, \dots N$. Then, it is assumed that for any choice of ordered points $t^{(1)}, t^{(2)}, \dots, t^{(k)}$ in $\mathbb{R}_+^N$, we have that $X_{t^{(j+1)}} - X_{t^{(j)}}$, $j = 1, \dots, k-1$, is a set

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of independent random variables. On the other hand, stationarity of increments means that $X_{t+\tau} - X_t$ has the same distribution of $X_\tau$ for all $t, \tau \in \mathbb{R}_+^N$.

Such processes are not to be confused with other extensions of Lévy processes where the parameter is multidimensional. Among them, we recall a class of processes, including the Brownian sheet and the Poisson sheet, which have a different definition from ours, because in that case independence of increments is understood in another way (consult e.g., [1, 10, 18]).

Multiparameter Lévy processes are of interest in Analysis since they furnish a stochastic solution to some systems of differential equations, as will be recalled in Sect. 2. Roughly speaking, if the vector $G = (G_1, G_2, \dots, G_N)$ is the generator of a multiparameter Lévy process $(X_t)_{t\in\mathbb{R}_+^N}$, then provided that $u$ belongs to suitable function spaces, the function $\mathbb{E}u(x + X_t)$ ($\mathbb{E}$ denoting the expectation) solves the system

$$\frac{\partial}{\partial t_k} h(x,t) = G_k\, h(x,t) \qquad h(x,0) = u(x), \qquad k = 1, \dots, N \tag{1.1}$$

where $t = (t_1, \dots, t_N)$. Of course, for one-parameter Lévy processes, we have a single differential equation, as stated by the well-known Feller theory of one-parameter Markov processes and semigroups.

The idea of subordination for multiparameter Lévy processes is presented in [5, 38–40] (for the classical theory of subordination of one-parameter Lévy processes, see e.g., [43, chapter 6]). The construction is as follows. Let $(X_t)_{t\in\mathbb{R}_+^N}$ be a multiparameter Lévy process and let $(H_t)_{t\in\mathbb{R}_+^M}$ be a subordinator field, i.e., a multiparameter Lévy process with values in $\mathbb{R}_+^N$, such that it has non-decreasing paths in the sense of the partial ordering (i.e., $t_1 \preceq t_2$ in $\mathbb{R}_+^M$ implies $H_{t_1} \preceq H_{t_2}$ in $\mathbb{R}_+^N$). The subordinated field is defined by $(X_{H_t})_{t\in\mathbb{R}_+^M}$, and it is again a multiparameter Lévy process.

One of the main results of this paper is to provide a formula for the generator of the subordinated field. Indeed, we find an extension of the Phillips theorem to the multiparameter case, by involving the so-called multidimensional Bernstein functions. This gives rise to interesting systems of type (1.1). In those systems, the operator on the right side may possibly be pseudo-differential. For example, when the subordinator field is stable, such a system could be interesting for those studying fractional equations, since the operator on the right side involves the fractional Laplacian and the so-called fractional gradient; we recall that the fractional gradient is a generalization of the fractional Laplacian to the case where the jumps are not isotropically distributed (see e.g., [8, Example 2.2] and the references therein). Thus, while the existing theory provides a stochastic solution to single differential equations (where the operator on the right can be pseudo-differential or fractional), here we are able to provide a stochastic solution to systems of differential equations, which can be interesting in Analysis and in the study of fractional calculus (see e.g., the examples in Sect. 3.3).

The basic case of subordinator field is the one with $M = 1$. In this case, we have a one-parameter process $H_t = (H_1(t), \dots, H_N(t))$ which the authors in [5] call *multivariate subordinator*. This is nothing more than a one-parameter Lévy process with values in $\mathbb{R}_+^N$, where all the components $t \to H_j(t)$ are non-decreasing (namely, each $H_j$ is a subordinator). Using a multivariate subordinator, subordination of a multiparameter Lévy process gives a one-parameter Lévy process.

In the second part of the paper, by considering a multivariate subordinator $(H_1(t), \ldots H_N(t))$, we will construct a new random field

$$\mathcal{L}_t = \big(L_1(t_1), \ldots, L_N(t_N)\big), \qquad t = (t_1, \ldots, t_N) \tag{1.2}$$

where $L_j$ is the inverse, also said the hitting time, of the subordinator $H_j$, i.e.,

$$L_j(t_j) = \inf\{x > 0 : H_j(x) > t_j\}.$$

We will call (1.2) *inverse random field*. Now, let $(X_t)_{t\in\mathbb{R}_+^N}$ be a multiparameter Lévy process with values in $\mathbb{R}^d$, which is assumed to be independent of ((1.2)). We are interested in the subordinated random field $(Z_t)_{t\in\mathbb{R}_+^N}$ defined by

$$Z_t = X_{\mathcal{L}_t} \tag{1.3}$$

Of course, (1.2) and (1.3) are not multiparameter Lévy processes because they enjoy neither independence nor stationarity of increments with respect to the partial ordering on $\mathbb{R}_+^N$. However, they may be useful in applications in order to model spatial data exhibiting various correlation structures which cannot fall in the framework of multiparameter Lévy or Markov processes.

Our topic has been inspired by some existing literature. First of all, there are many papers (see e.g., [6, 22, 29–34, 48]) concerning semi-Markov processes of the form

$$Z(t) = X(L(t)), \qquad t \geq 0 \tag{1.4}$$

where $X$ is a (one-parameter) Lévy process in $\mathbb{R}^d$ and $L$ is the inverse of a subordinator $H$ independent of $X$, i.e.,

$$L(t) = \inf\{x > 0 : H(x) > t\}.$$

Processes of type (1.4) have an important role in statistical physics, since they model continuous time random walk scaling limits and anomalous diffusions. Moreover, it is known that (1.4) is not Markovian, and its density $p(x, t)$ is governed by an equation which is non-local in the time variable:

$$\mathcal{D}_t p(x, t) - \overline{\nu}(t) p(x, 0) = G^* p(x, t). \tag{1.5}$$

In the above equation, $G^*$ is the dual to the generator of $X$ and the operator $\mathcal{D}_t$ is the so-called generalized fractional derivative (in the sense of Marchaud), defined by

$$\mathcal{D}_t h(t) := \int_0^\infty \big(h(t) - h(t-\tau)\big)\nu(d\tau), \qquad t > 0, \tag{1.6}$$

where $\nu$ is the Lévy measure of $H$ and $\overline{\nu}(t) := \int_t^\infty \nu(dx)$ is the tail of the Lévy measure.

The main results regarding the random fields of type (1.3) are presented in Sect. 4; we will show that they have interesting correlation structures and that they are governed by particular integro-differential equations. Such equations are non-local in the $t_1, \dots, t_N$ variables and generalize equation (1.5) holding in the one-parameter case. Thus, while the existing theory of semi-Markov processes furnishes a stochastic solution to single equations that are non-local in time, we here furnish a stochastic solution to systems of equations which are non-local in the "time" variables $t_1, t_2, \dots, t_N$.

We also recall that the first idea of inverse random field appeared in [8, sect. 3], where the authors proposed a model of multivariate time change.

Another source of inspiration is the paper [24], even if it does not exactly fit into our context. Here, the authors considered a Poisson sheet $N(t_1, t_2)$, which is not a multiparameter Lévy process in the sense of this paper, and studied the composition

$$Z(t_1, t_2) = N(L_1(t_1), L_2(t_2)),$$

where $L_1$ and $L_2$ are two independent inverse stable subordinators, of index $\alpha_1$ and $\alpha_2$, respectively; the resulting random field showed interesting long-range dependence properties.

## 2 Basic Notions and Some Preliminary Results

We introduce the partial ordering on the set $\mathbb{R}_+^N = [0, \infty)^N$: the point $a = (a_1, \dots, a_N)$ precedes the point $b = (b_1, \dots, b_N)$, say $a \preceq b$, if and only if $a_j \leq b_j$ for each $j = 1, \dots, N$.

A sequence $\{x_i\}_{i=1}^\infty$ in $\mathbb{R}_+^N$ is said to be increasing if $x_i \preceq x_{i+1}$ for each $i$; it is said to be decreasing if $x_{i+1} \preceq x_i$ for each $i$.

Consider a function $f : \mathbb{R}_+^N \to \mathbb{R}^d$. We say that $f$ is right continuous at $x \in \mathbb{R}_+^N$ if, for any decreasing sequence $x_i \to x$ we have $f(x_i) \to f(x)$.

We say that $f : \mathbb{R}_+^N \to \mathbb{R}^d$ has left limits at $x \in \mathbb{R}_+^N/\{0\}$ if, for any increasing sequence $x_i \to x$, the limit of $f(x_i)$ exists; such a limit may depend on the choice of the sequence $x_i$.

Moreover, $f$ is said to be càdlàg if it is right continuous at each $x \in \mathbb{R}_+^N$ and has left limits at each $x \in \mathbb{R}_+^N/\{0\}$.

### 2.1 Multiparameter Lévy Processes

We here recall the notion of multiparameter Lévy process in the sense of [5, 38–40]. We also refer to [17] as a standard reference on Multiparameter Markov processes.

The parameter set is here assumed to be $\mathbb{R}_+^N$. An analogous (but more general) definition holds if the parameter set is any cone contained in $\mathbb{R}^N$, but this generalization is not essential for the aim of this paper.

**Definition 2.1** A random field $(X_t)_{t\in\mathbb{R}_+^N}$, with values in $\mathbb{R}^d$, is said to be a multiparameter Lévy process if

1. $X_0 = 0$ a.s.
2. It has independent increments with respect to the partial ordering on $\mathbb{R}_+^N$, i.e., for any choice of $0 = t^{(0)} \preceq t^{(1)} \preceq t^{(2)} \cdots \preceq t^{(k)}$, the random variables $X_{t^{(j)}} - X_{t^{(j-1)}}$, $j = 1, \ldots, k$, are independent.
3. It has stationary increments, i.e., $X_{t+\tau} - X_t \stackrel{d}{=} X_\tau$ for each $t, \tau \in \mathbb{R}_+^N$
4. It is càdlàg a.s.
5. It is continuous in probability, namely for any sequence $t^{(i)} \in \mathbb{R}_+^N$ such that $t^{(i)} \to t$, it holds that $X_{t^{(i)}}$ converges to $X_t$ in probability.

If (1), (2), (3), (5) hold, then $(X_t)_{t\in\mathbb{R}_+^N}$ is said to be a multiparameter Lévy process in law.

We present some examples of multiparameter Lévy processes, which are constructed from one-parameter ones. Such examples are taken from [5].

***Example 2.2*** If $(X_{t_1}^{(1)})_{t_1\in\mathbb{R}_+}, \ldots, (X_{t_N}^{(N)})_{t_N\in\mathbb{R}_+}$ are $N$ independent Lévy processes on $\mathbb{R}^d$, with laws $\nu_{t_1}^{(1)}, \ldots, \nu_{t_N}^{(N)}$, then

$$X_t := X_{t_1}^{(1)} + X_{t_2}^{(2)} + \cdots + X_{t_N}^{(N)}, \qquad t = (t_1, t_2, \ldots, t_N)$$

is an $N$-parameter Lévy process on $\mathbb{R}^d$, which is usually called additive Lévy process (see e.g., [19] and [17, p. 405]).

Here, $X_t$ has law

$$\mu_t = \nu_{t_1}^{(1)} * \cdots * \nu_{t_N}^{(N)}$$

where $*$ denotes the convolution. Examples of the sample paths are shown in Figs. 1 and 2.

***Example 2.3*** Let $(X_{t_1}^{(1)})_{t_1\in\mathbb{R}_+}, \ldots (X_{t_N}^{(N)})_{t_N\in\mathbb{R}_+}$ be independent $\mathbb{R}$-valued Lévy processes with laws $\nu_{t_1}^{(1)}, \ldots, \nu_{t_N}^{(N)}$. Then,

$$X_t = \left(X_{t_1}^{(1)}, X_{t_2}^{(2)}, \ldots, X_{t_N}^{(N)}\right), \qquad t = (t_1, t_2, \ldots, t_N)$$

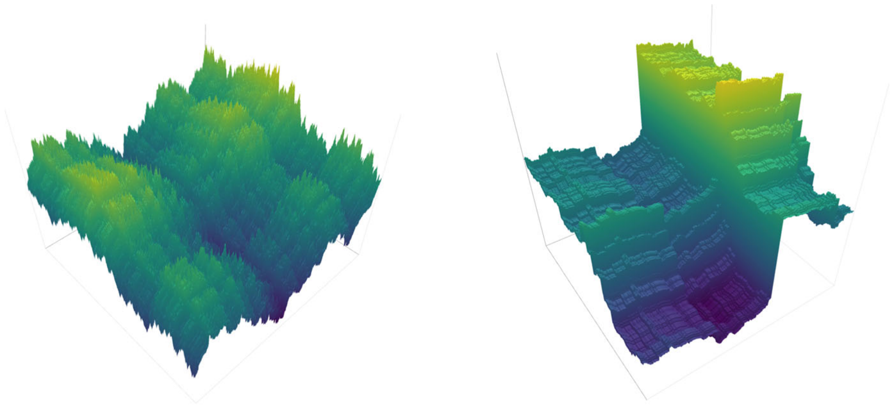
is a $\mathbb{R}^N$ valued Lévy process, which can be called product Lévy process (in the language of [17, p. 407]). Clearly, this is a particular case of Example 2.2 because

$$X_t = X_{t_1}^{(1)} e_1 + X_{t_2}^{(2)} e_2 + \cdots + X_{t_N}^{(N)} e_N,$$

where $\{e_1, \ldots, e_N\}$ denotes the canonical basis of $\mathbb{R}^N$.

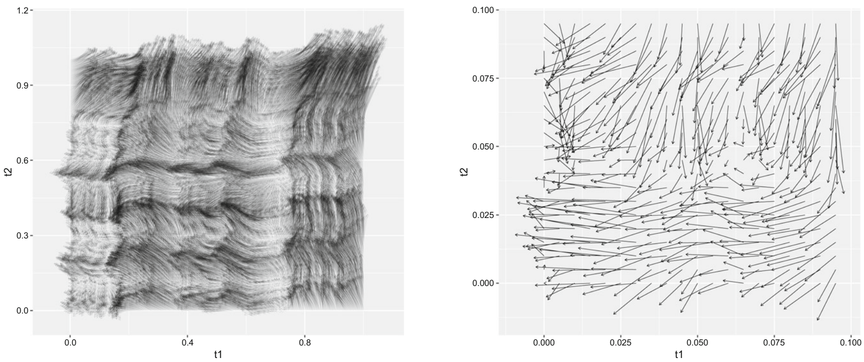
Here, $X_t$ has law

$$\mu_t = \nu_{t_1}^{(1)} \otimes \nu_{t_2}^{(2)} \cdots \otimes \nu_{t_N}^{(N)},$$



(A) Brownian additive field (B) Stable additive field

**Fig. 1** Sample paths of additive Lévy fields, as in Example 2.2




(A) Brownian vector field (B) Zoom of the previous figure on $[0, 1]^2$

**Fig. 2** Sample path of a $\mathbb{R}^2$-valued biparameter additive field (i.e., $d = N = 2$)

where $\otimes$ denotes the product of measures.

***Example 2.4*** Let $(V_t)_{t\in\mathbb{R}_+}$ be a Lévy process in $\mathbb{R}^d$. Then, $V_{c_1t_1+\cdots+c_Nt_N}$ is a multiparameter Lévy process for any choice of $(c_1, \ldots, c_N) \in \mathbb{R}_+^N$.

***Remark 2.5*** What we have presented is not the only way to extend the notion of independence of increments to the multiparameter case. A very common approach is to define independence of increments over disjoint rectangles (see [1] and [10]). This gives rise to a class of random fields, known as Lévy sheets (e.g., the Poisson sheet or the Brownian sheet).

In the following, $\delta_0$ will denote the probability measure concentrated at the origin. Moreover, $\{e_1, \ldots, e_N\}$ will denote the canonical basis of $\mathbb{R}^N$.

**Definition 2.6** A family $(\mu_t)_{t\in\mathbb{R}_+^N}$ of probability measures on $\mathbb{R}^d$ is said to be a $\mathbb{R}_+^N$-parameter convolution semigroup if
i) $\mu_{t+\tau} = \mu_t * \mu_\tau$, for all $t, \tau \in \mathbb{R}_+^N$
ii) $\mu_t \to \delta_0$ as $t \to 0$

By Definition 2.6, it follows that $\mu_t$ is infinitely divisible for each $t$.

The above notion of multiparameter convolution semigroup is related to multiparameter Lévy processes, as shown in Proposition 2.7, which is a special case of Theorem 4.5 of [5]. We underline that such Proposition will be crucial in the rest of our article.

We preliminarily observe that since $X_t$ is a multiparameter Lévy process, where $t = (t_1, \dots, t_N)$, it immediately follows that for each $j = 1, \dots, N$, the process $(X_{t_j e_j})_{t_j\in\mathbb{R}_+}$ is a classical one-parameter Lévy process. In other words, if $(\mu_t)_{t\in\mathbb{R}_+^N}$ is a multiparameter convolution semigroup, then $(\mu_{t_j e_j})_{t_j\in\mathbb{R}_+}$ is a one-parameter convolution semigroup which is the law of $X_{t_j e_j}$.

**Proposition 2.7** *Let $(X_t)_{t\in\mathbb{R}_+^N}$ be a multiparameter Lévy process on $\mathbb{R}^d$ and let $\mu_t$ be the law of the random variable $X_t$. Then,*

*i) the family $(\mu_t)_{t\in\mathbb{R}_+^N}$ is a $\mathbb{R}_+^N$-parameter convolution semigroup of probability measures.*

*ii) There exist independent random vectors $Y_{t_j}^{(j)}$, $j = 1, \dots, N$, with $Y_{t_j}^{(j)} \overset{d}{=} X_{t_j e_j}$, such that*

$$X_t \overset{d}{=} Y_{t_1}^{(1)} + \cdots + Y_{t_N}^{(N)}, \qquad t = (t_1, \dots, t_N).$$

***Proof*** By writing

$$X_{t+\tau} = (X_{t+\tau} - X_\tau) + X_\tau \qquad \text{for all } t, \tau \in \mathbb{R}_+^N$$

we observe that $X_{t+\tau} - X_\tau$ and $X_\tau$ are independent by the assumption of independence of increments along those sequences that are increasing with respect to the partial ordering. Moreover, $X_{t+\tau} - X_\tau$ has the same distribution of $X_t$ by stationarity. Hence, $\mu_{t+\tau} = \mu_t * \mu_\tau$. Moreover, stochastic continuity of $(X_t)_{t\in\mathbb{R}_+^N}$ gives $\mu_t \to \delta_0$ as $t \to 0$, and thus *i)* is proved. To prove *ii)*, it is sufficient to write $t = t_1 e_1 + \cdots + t_N e_N$ and apply the semigroup property just proved in point *i)*, to have

$$\mu_t = \mu_{t_1 e_1} * \cdots * \mu_{t_N e_N}$$

and the proof is complete since $\mu_{t_j e_j}$ is the law of $X_{t_j e_j}$. □

We stress that Proposition 2.7 is a statement about equality in law of random variables ($t$ is fixed) and not equality of processes.

We further observe that Proposition 2.7 says that to each multiparameter Lévy process in law there corresponds a unique convolution semigroup of probability measures. But, unlike what happens for classical Lévy processes (i.e., when $N = 1$), the converse is not true in general: a multiparameter convolution semigroup $(\mu_t)_{t\in\mathbb{R}_+^N}$ can be associated to different multiparameter Lévy processes in law, because $(\mu_t)_{t\in\mathbb{R}_+^N}$ does not completely determine all the finite-dimensional distributions. Indeed, only along $\mathbb{R}_+^N$-increasing sequences $0 \preceq \tau^{(1)} \preceq \cdots \preceq \tau^{(k)}$, the joint distribution of $(X_{\tau^{(1)}}, \ldots, X_{\tau^{(k)}})$ can be uniquely determined in terms of $\mu_t$ by using independence and stationarity of increments, but this is not possible if the points $\tau^{(1)}, \ldots, \tau^{(k)} \in \mathbb{R}_+^N$ are not ordered (in the sense of the partial ordering). As an example, consider two biparameter processes defined in the following way. The first one is $Z(t_1, t_2) = B_1(t_1) + B_2(t_2)$, where $B_1$ and $B_2$ are independent standard Brownian motions. The second one is $W(t_1, t_2) = B(t_1 + t_2)$ where $B(t)$ is a standard Brownian motion. Both $Z(t_1, t_2)$ and $W(t_1, t_2)$ have law $\mathcal{N}(0, t_1 + t_2)$. However, this law does not identify all the finite-dimensional distributions. Indeed, observe the processes at $(t_1, 0)$ and $(0, t_2)$, which are not ordered points (in the sense of the partial ordering). It is clear that $Z(t_1, 0)$ and $Z(0, t_2)$ are independent, unlike what happens for $W(t_1, 0)$ and $W(0, t_2)$.

### 2.1.1 Characteristic Function of Multiparameter Lévy Processes

Consider the $Y_{t_j}^{(j)}$ involved in Proposition 2.7. By the Lévy Khintchine formula, we have

$$\mathbb{E}e^{i\xi\cdot Y_{t_j}^{(j)}} = \int_{\mathbb{R}^d} e^{i\xi\cdot y}\mu_{t_j e_j}(\mathrm{d}y) = e^{t_j\psi_j(\xi)}, \qquad \xi \in \mathbb{R}^d, \tag{2.1}$$

the Lévy exponent $\psi_j$ having the form

$$\psi_j(\xi) = i\gamma_j \cdot \xi - \frac{1}{2}A_j\xi\cdot\xi + \int_{\mathbb{R}^d/\{0\}} (e^{i\xi\cdot z} - 1 - i\xi\cdot z I_{[-1,1]}(z))\nu_j(\mathrm{d}z), \tag{2.2}$$

where $\gamma_j \in \mathbb{R}^d$, $A_j$ is the Gaussian covariance matrix, $\nu_j$ denotes the Lévy measure and $\cdot$ denotes the scalar product. By the above considerations, we thus get the following statement.

**Proposition 2.8** *Let $(X_t)_{t\in\mathbb{R}_+^N}$ be a multiparameter Lévy process with values in $\mathbb{R}^d$. Then, $X_t$ has characteristic function*

$$\mathbb{E}e^{i\xi\cdot X_t} = e^{t_1\psi_1(\xi)+\cdots+t_N\psi_N(\xi)} = e^{t\cdot\Psi(\xi)}, \qquad \xi \in \mathbb{R}^d, \tag{2.3}$$

*where $t = (t_1, \ldots, t_N)$, the functions $\psi_j$ have been defined in (2.2), and*

$$\Psi(\xi) = (\psi_1(\xi), \ldots, \psi_N(\xi)). \tag{2.4}$$

We will call (2.4) the multidimensional Lévy exponent.

### 2.2 Autocorrelation Function of Multiparameter Lévy Processes

Consider a multiparameter Lévy process $\{X_t\}_{t\in\mathbb{R}_+^N}$ with values in $\mathbb{R}$. In the following proposition, we will explicitly compute the autocorrelation function between two ordered points in the parameter space, i.e.,

$$\rho(X_s, X_t) := \frac{\mathrm{Cov}(X_s, X_t)}{\sqrt{\mathbb{V}ar X_s}\sqrt{\mathbb{V}ar X_t}}, \qquad s \preceq t. \tag{2.5}$$

Of course, (2.5) exists finite only in some cases, which will be specified in the following (e.g., the process must be non-deterministic). What we will find is the $N$-parameter extension of the well-known formula holding in the case $N = 1$, i.e., for classical Lévy processes (consult e.g., Remark 2.1 in [23]):

$$\rho(X_s, X_t) = \sqrt{\frac{s}{t}}, \qquad s \leq t.$$

**Proposition 2.9** *Let $\{X_t\}_{t\in\mathbb{R}_+^N}$ be an $N$-parameter Lévy process with values in $\mathbb{R}$, having multidimensional Lévy exponent $\Psi(\xi)$ defined in (2.3) and (2.4). For each $j = 1, \ldots, N$, let $\xi \to \psi_j(\xi)$ be twice differentiable in a neighborhood of $\xi = 0$, and such that $\psi_j''(0) \neq 0$. Then, the autocorrelation function defined in (2.5) reads*

$$\rho(X_s, X_t) = \sqrt{\frac{s\cdot\sigma^2}{t\cdot\sigma^2}}, \qquad s \preceq t, \tag{2.6}$$

*where $\cdot$ denotes the scalar product and $\sigma^2 := -\Psi''(0)$.*

***Proof*** Consider the decomposition of $X_t$ given in Proposition 2.7. Since $\psi_j''(0)$ exists, then $Y_{t_j}^{(j)}$ has finite mean and variance:

$$\begin{aligned}
\mathbb{E}Y_{t_j}^{(j)} &= -it_j\psi_j'(0) = t_j\,\mathbb{E}Y_1^{(j)} \\
\mathbb{E}\left(Y_{t_j}^{(j)}\right)^2 &= -t_j\psi_j''(0) - t_j^2\psi_j'(0)^2 \\
\mathbb{V}\mathrm{ar}Y_{t_j}^{(j)} &= -t_j\psi_j''(0) = t_j\mathbb{V}\mathrm{ar}Y_1^{(j)}
\end{aligned}$$

Letting $\mu := \left(\mathbb{E}Y_1^{(1)}, \ldots, \mathbb{E}Y_1^{(N)}\right)$ and $\sigma^2 := -\Psi''(0) = \left(\mathbb{V}\mathrm{ar}Y_1^{(1)}, \ldots, \mathbb{V}\mathrm{ar}Y_1^{(N)}\right)$, we get

$$\begin{aligned}
\mathbb{E}X_t &= -it\cdot\Psi'(0) = t\cdot\mu \\
\mathbb{V}\mathrm{ar}X_t &= -t\cdot\Psi''(0) = t\cdot\sigma^2
\end{aligned}$$

Moreover, for $s \preceq t$, we have

$$\begin{aligned}\mathbb{E}X_t X_s &= \mathbb{E}(X_t - X_s)X_s + \mathbb{E}(X_s)^2 \\ &= \mathbb{E}(X_t - X_s)\mathbb{E}X_s + \mathbb{E}(X_s)^2 \\ &= \mathbb{E}X_{t-s}\mathbb{E}X_s + \mathbb{E}(X_s)^2 \\ &= \big((t-s)\cdot\mu\big)\big(s\cdot\mu\big) + s\cdot\sigma^2 + (s\cdot\mu)^2,\end{aligned}$$

where we used independence and stationarity of the increments along $\mathbb{R}^N_+$ increasing sequences. We thus have

$$\mathrm{Cov}(X_t, X_s) := \mathbb{E}X_t X_s - \mathbb{E}X_t\mathbb{E}X_s = s\cdot\sigma^2$$

and the desired result immediately follows. □

***Remark 2.10*** Let $|v|$ denote the Euclidean norm of $v$. In the limit $|t| \to \infty$, we have that $\rho(X_s, X_t)$ behaves like $|t|^{-1/2}$. Indeed, consider the scalar product in the denominator of (2.6), i.e., $t\cdot\sigma^2 = |t|\,|\sigma^2|\cos\theta$, where $\theta$ is the angle between $t$ and $\sigma^2$. Now, observe that $\sigma^2$ is a fixed vector of $\mathbb{R}^N_+$, with strictly positive components by the assumption $\psi_j''(0) \neq 0$. Since $t$ is in $\mathbb{R}^N_+$ also, by simple geometric arguments, it follows that there exist two constants $c_1 > 0$ and $c_2 > 0$, which do not depend on $t$, such that $c_1 \leq \cos\theta \leq c_2$. Then, $k_1|t|^{-1/2} \leq \rho(X_s, X_t) \leq k_2|t|^{-1/2}$ for two suitable constants $k_1 > 0$ and $k_2 > 0$ both independent of $t$.

### 2.3 Multiparameter Semigroups of Operators and their Generators

Let $\mathbb{B}$ be a Banach space equipped with the norm $||\cdot||_{\mathbb{B}}$. An $N$-parameter family $(T_t)_{t\in\mathbb{R}^N_+}$ of bounded linear operators on $\mathbb{B}$ is said to be an $N$-parameter semigroup of operators if $T_0$ is the identity operator and the following property holds:

$$T_{s+t} = T_s \circ T_t \qquad \forall s, t \in \mathbb{R}^N_+. \tag{2.7}$$

We say that $(T_t)_{t\in\mathbb{R}^N_+}$ is strongly continuous if

$$\lim_{t\to 0} ||T_t u - u||_{\mathbb{B}} = 0 \qquad \forall u \in \mathbb{B}.$$

Moreover, we say that $(T_t)_{t\in\mathbb{R}^N_+}$ is a contraction semigroup if, for any $t \in \mathbb{R}^N_+$, we have $||T_t u||_{\mathbb{B}} \leq ||u||_{\mathbb{B}}$.

***Example 2.11*** Let $G_1, G_2, \ldots, G_N$ be bounded operators on $\mathbb{B}$, such that $[G_i, G_k] := G_iG_k - G_kG_i = 0$ for all $i \neq k$. Consider the vector

$$G = (G_1, \ldots, G_N).$$

Then, for all $t = (t_1, \dots, t_N)$, the family

$$T_t = e^{t_1 G_1} \circ \dots \circ e^{t_N G_N} = e^{G \cdot t}$$

defines a strongly continuous semigroup on $\mathbb{B}$. In light of the following Definition 2.13, we will call the vector $G$ the generator of the multiparameter semigroup.

***Example 2.12*** Let $(\mu_t)_{t\in\mathbb{R}^N_+}$ be a multiparameter convolution semigroup of probability measures on $\mathbb{R}^d$ (in the sense of Definition 2.6) and let $\mathcal{C}_0(\mathbb{R}^d)$ be the space of continuous functions vanishing at infinity, equipped with the sup-norm. Then,

$$T_t q(x) = \int_{\mathbb{R}^d} q(x-y)\mu_t(\mathrm{d}y) = \mu_t * q(x), \qquad q \in \mathcal{C}_0(\mathbb{R}^d), \qquad t \in \mathbb{R}^N_+$$

defines a strongly continuous contraction multiparameter semigroup.

Let $t = (t_1, \dots, t_N) \in \mathbb{R}^N_+$ and let $\{e_1, \dots, e_N\}$ be the canonical basis of $\mathbb{R}^N$. For each $j = 1, \dots, N$, we refer to the one-parameter semigroups $T_{t_j e_j}$ as the marginal semigroups. By the property (2.7), it follows that the marginal semigroups commute, i.e., $[T_{t_i e_i}, T_{t_j e_j}] = 0$ for $i \neq j$ and the following relation holds:

$$T_t = T_{t_1 e_1} \circ T_{t_2 e_2} \circ \dots \circ T_{t_N e_N}$$

Now, let $G_i$ be the generator of $T_{t_i e_i}$, defined on $\mathrm{Dom}(G_i)$. It is well-known that if $u \in \mathrm{Dom}(G_i)$, then $T_{t_i e_i} u \in \mathrm{Dom}(G_i)$ and the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t_i} w(t_i) = G_i w(t_i), \qquad w(0) = u$$

is solved by $w(t_i) = T_{t_i e_i} u$. We here recall the notion of generator of a multiparameter semigroup (see [9, chapter 1]).

**Definition 2.13** Let $(T_t)_{t\in\mathbb{R}^N_+}$ be a strongly continuous $N$-parameter semigroup on $\mathbb{B}$ and let $G_i$, $i = 1, \dots, N$, be the generators of the marginal semigroups, each defined on $\mathrm{Dom}(G_i)$. We say that the vector

$$G = (G_1, \dots, G_N)$$

is the generator of $(T_t)_{t\in\mathbb{R}^N_+}$, defined on $\mathrm{Dom}(G) = \bigcap_{j=1}^N \mathrm{Dom}(G_j)$.

The above definition is intuitively motivated by the following result.

**Proposition 2.14** *Let $(T_t)_{t\in\mathbb{R}^N_+}$ be a strongly continuous $N$-parameter semigroup with generator $G$ according to Definition 2.13. Then, for $u \in \bigcap_{j=1}^N \mathrm{Dom}(G_j)$, the function $w(t) = T_t u$ solves the following system of differential equations*

$$\nabla_t w(t) = G w(t), \qquad w(0) = u, \tag{2.8}$$

*where* $\nabla_t$ *denotes the gradient with respect to* $t = (t_1, \dots, t_N)$*. Namely, we have*

$$\frac{\partial}{\partial t_i} w(t) = G_i w(t), \qquad i = 1 \dots N \tag{2.9}$$

*subject to* $w(0) = u$*.*

***Proof*** Let us introduce a compact notation to denote the composition of operators, namely

$$\bigcirc_{k=1}^{N} A_k := A_1 \circ A_2 \circ \cdots A_N$$

Let us fix $i = 1, \dots, N$. For $q \in \mathrm{Dom}(G_i)$, it is true that $T_{t_i e_i} q \in \mathrm{Dom}(G_i)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t_i} T_{t_i e_i} q = G_i T_{t_i e_i} q \tag{2.10}$$

By using Propositions 1.1.8 and 1.1.9 in [9], we know that if $u \in \mathrm{Dom}(G_i)$, then $T_t u \in \mathrm{Dom}(G_i)$ for any $t \in \mathbb{R}_+^N$. In particular, we have $\bigcirc_{k=1, k\neq i}^{N} T_{t_k e_k} u \in \mathrm{Dom}(G_i)$. Hence, Eq. (2.10) holds for $q = \bigcirc_{k=1, k\neq i}^{N} T_{t_k e_k} u$:

$$\frac{\mathrm{d}}{\mathrm{d}t_i} T_{t_i e_i} \bigcirc_{k=1, k\neq i}^{N} T_{t_k e_k} u = G_i T_{t_i e_i} \bigcirc_{k=1, k\neq i}^{N} T_{t_k e_k} u \tag{2.11}$$

and Eq. (2.9) for a fixed $i$ is found by using property (2.7). By choosing $u \in \bigcap_{j=1}^{N} \mathrm{Dom}(G_j)$, it is possible to repeat the same argument for all $i = 1, \dots, N$, and the system of differential equations is obtained. □

By putting $t = 0$ in Eq. (2.8), it follows that the generator $G$ can also be found by

$$Gu = \nabla_t T_t u\big|_{t=0}, \qquad u \in \bigcap_{j=1}^{N} \mathrm{Dom}(G_j). \tag{2.12}$$

For other results concerning multiparameter semigroups and generators consult [9]. Moreover, for a general discussion on operator semigroups related to multiparameter Markov processes, we refer to [17].

***Remark 2.15*** A different definition of generator for multiparameter semigroups is given in [14] and [47]. Here, the authors defined the generator as the composition of the marginal generators, i.e.,

$$G = G_1 \circ G_2 \circ \cdots \circ G_N.$$

The motivation for such definition is that for $u \in \mathrm{Dom}(G_1 \circ \cdots \circ G_N)$, the authors prove that $w(t) = T_t u$ solves the partial differential equation

$$\frac{\partial^N}{\partial t_1 \dots \partial t_N} w(t) = G w(t), \qquad w(0) = u, \tag{2.13}$$

where $t = (t_1, \dots t_N)$. Also this approach seems to be very interesting, especially in the field of partial differential equations as it allows to find probabilistic solutions to equations of type (2.13), containing a mixed derivative.

### 2.4 Semigroups Associated with Multiparameter Lévy Processes

Let $(X_t)_{t\in\mathbb{R}_+^N}$ be a multiparameter Lévy process on $\mathbb{R}^d$ and let $(\mu_t)_{t\in\mathbb{R}_+^N}$ be the associated convolution semigroup of probability measures, i.e., $\mu_t$ is the law of $X_t$ for each $t$. Consider the operator

$$T_t h(x) := \mathbb{E}\, h(x + X_t) = \int_{\mathbb{R}^d} h(x+y)\mu_t(\mathrm{d}y), \qquad h \in \mathcal{C}_0(\mathbb{R}^d), \qquad t \in \mathbb{R}_+^N, \tag{2.14}$$

where $\mathcal{C}_0(\mathbb{R}^d)$ denotes the space of continuous functions vanishing at infinity. By using the properties of $\{\mu_t\}_{t\in\mathbb{R}_+^N}$, it immediately follows that the family $(T_t)_{t\in\mathbb{R}_+^N}$ is a strongly continuous contraction semigroup on $\mathcal{C}_0(\mathbb{R}^d)$; it is also positivity preserving, hence it is a Feller semigroup. We now give a representation of this semigroup and its generator by means of pseudo-differential operators. We restrict to the Schwartz space of functions $\mathcal{S}(\mathbb{R}^d)$.

We define the Fourier transform by

$$\hat{h}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} h(x)\mathrm{d}x, \qquad \xi \in \mathbb{R}^d.$$

Since $h \in \mathcal{S}(\mathbb{R}^d)$, the following Fourier inversion formula holds:

$$h(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} \hat{h}(\xi)\mathrm{d}\xi, \qquad x \in \mathbb{R}^d.$$

**Theorem 2.16** *Let $(X_t)_{t\in\mathbb{R}_+^N}$ be a multiparameter Lévy process with Lévy exponent $\Psi$ defined in (2.3) and (2.4). Let $(T_t)_{t\in\mathbb{R}_+^N}$ be the associated semigroup defined in (2.14) and let $G = (G_1, \dots, G_N)$ be its generator. Then,*

1. *For any $t \in \mathbb{R}_+^N$, $T_t$ is a pseudo-differential operator with symbol $e^{t\cdot\Psi}$, i.e.,*

$$T_t h(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} e^{t\cdot\Psi(\xi)} \hat{h}(\xi)\mathrm{d}\xi, \qquad h \in \mathcal{S}(\mathbb{R}^d). \tag{2.15}$$

2. *$G$ is a pseudo-differential operator with symbol $\Psi$, i.e., for each $i = 1, \dots, N$ we have*

$$G_i h(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} \psi_i(\xi) \hat{h}(\xi)\mathrm{d}\xi, \qquad h \in \mathcal{S}(\mathbb{R}^d).$$

***Proof*** 1. Since (2.14) is a convolution integral, its Fourier transform can be computed as

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} T_t h(x) \mathrm{d}x = \hat{h}(\xi) \mathbb{E} e^{i\xi\cdot X_t},$$

where $\mathbb{E}e^{i\xi\cdot X_t} = e^{t\cdot\Psi(\xi)}$ by using (2.3). Then, Fourier inversion gives the result.

2. By applying formula (2.12), we have that

$$\begin{aligned} G_i h(x) &= \frac{\partial}{\partial t_i} T_t u(x)\bigg|_{t=0} \\ &= \left[\lim_{t_i\to 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} \frac{e^{t_i\psi_i(\xi)} - 1}{t_i} \prod_{k=1,k\neq i}^{N} e^{t_k\psi_k(\xi)} \hat{h}(\xi) \mathrm{d}\xi\right]_{t=0}. \end{aligned}$$

The limit can be taken inside the integral due to dominated convergence theorem. Indeed, $|e^{t_k\psi_k(\xi)}| \leq 1$ for each $k$ because $e^{t_k\psi_k(\xi)}$ is the characteristic function of $\mu_{t_k e_k}$ (see (2.1)); moreover,

$$\left|\frac{e^{t_i\psi_i(\xi)} - 1}{t_i}\right| \leq |\psi_i(\xi)| \leq C_i(1 + |\xi|^2),$$

where for the last inequality we used [4, p. 31]. Thus, the absolute value of the integrand is dominated by $(1 + |\xi|^2)\hat{h}(\xi)$. But the last function is independent of $t_i$ and is integrable on $\mathbb{R}^d$ because $\hat{h}$ is a Schwartz function. Then, by exchanging the limit and the integral, the result immediately follows.

□

## 3 Composition of Random Fields

### 3.1 Subordinator Fields

In order to treat the composition of random fields, the main object is provided by the following definition.

**Definition 3.1** A multiparameter Lévy process $(H_t)_{t\in\mathbb{R}_+^M}$ is said to be a subordinator field if, for some positive integer $N$, it takes values in $\mathbb{R}_+^N$ almost surely.

The above definition means that almost surely, $t \to H_t$ is a non-decreasing function with respect to the partial ordering, i.e., $t_1 \preceq t_2$ on $\mathbb{R}_+^M$ implies $H_{t_1} \preceq H_{t_2}$ on $\mathbb{R}_+^N$.

***Example 3.2*** **(Classical subordinators)** If $N = M = 1$, then $(H_t)_{t\in\mathbb{R}_+}$ is a classical subordinator, i.e., a non-decreasing Lévy process with values in $\mathbb{R}_+$. Hence, it is such that

$$\mathbb{E}e^{-\lambda H_t} = e^{-tf(\lambda)}, \qquad \lambda \geq 0,$$

where the Laplace exponent $f$ is a so-called Bernstein function. Thus, it is defined by

$$f(\lambda) = b\lambda + \int_{\mathbb{R}_+} (1 - e^{-\lambda x})\phi(\mathrm{d}x),$$

where $b \geq 0$ is the drift coefficient and $\phi$ is the Lévy measure, which is supported on $\mathbb{R}_+$ and satisfies $\int_{\mathbb{R}_+} \min(x, 1)\phi(\mathrm{d}x) < \infty$. For more details on this subject, consult [45].

***Example 3.3*** (**Multivariate subordinators**) If $M = 1$ and $N \geq 1$, then $(H_t)_{t\in\mathbb{R}_+}$ is a multivariate subordinator in the sense of [5]. Thus, it is a one-parameter Lévy process with values in $\mathbb{R}_+^N$, i.e., it is non-decreasing in each marginal component. Here, $H_t$ has Laplace transform

$$\mathbb{E}e^{-\lambda\cdot H_t} = e^{-tS(\lambda)}, \qquad \lambda \in \mathbb{R}_+^N,$$

where the Laplace exponent $S$ is a multivariate Bernstein function. Hence, it is defined by

$$S(\lambda) = b \cdot \lambda + \int_{\mathbb{R}_+^N} (1 - e^{-\lambda\cdot x})\phi(\mathrm{d}x), \qquad \lambda \in \mathbb{R}_+^N$$

where $b \in \mathbb{R}_+^N$, and the Lévy measure $\phi$ is supported on $\mathbb{R}_+^N$ and satisfies

$$\int_{\mathbb{R}_+^N} \min(|x|, 1)\phi(\mathrm{d}x) < \infty.$$

It is known (see e.g., Sect. 2 in [8]) that if $H_t$ has a density $p(x, t)$, then it solves

$$\partial_t p(x, t) = b \cdot \nabla_x p(x, t) - \mathcal{D}_x p(x, t), \qquad x \in \mathbb{R}_+^N, \qquad t > 0,$$

where $\mathcal{D}_x$ denotes the $N$-dimensional version of the generalized fractional derivative defined in (1.6), i.e.,

$$\mathcal{D}_x h(x) = \int_{\mathbb{R}_+^N} \big(h(x) - h(x - y)\big)\phi(\mathrm{d}y), \qquad x \in \mathbb{R}_+^N. \tag{3.1}$$

***Example 3.4*** (**Multivariate stable subordinators**) We here consider a special subcase of Example 3.3, in which the multivariate subordinator is stable. In order to define this process by means of its Lévy measure, we need to use the spherical coordinates $r$ and $\hat{\theta}$, which respectively denote the length and the direction of jumps. Clearly, $\hat{\theta}$ takes values in the set $\mathcal{C}^{N-1} = \{\hat{\theta} \in \mathbb{R}_+^N : |\hat{\theta}| = 1\}$ because, by definition, all the marginal components make positive jumps. So, a multivariate subordinator $(H_t)_{t\in\mathbb{R}_+}$ is said to be $\alpha$-stable if its Lévy measure can be written in spherical coordinates as

$$\phi(\mathrm{d}r, \mathrm{d}\hat{\theta}) = \frac{\mathrm{d}r}{r^{\alpha+1}}\sigma(\mathrm{d}\hat{\theta}), \qquad r > 0, \qquad \hat{\theta} \in \mathcal{C}^{N-1},$$

where $\alpha \in (0, 1)$ denotes the stability index and $\sigma$ is the so-called spectral measure, which is proportional to the probability distribution of the jump direction $\hat{\theta}$. By simple calculations, it is easy to see that in this case, the Laplace exponent takes the form

$$S^{\alpha,\sigma}(\lambda) = k \int_{\mathcal{C}^{N-1}} (\lambda \cdot \hat{\theta})^{\alpha} \sigma(\mathrm{d}\hat{\theta}), \qquad \lambda \in \mathbb{R}_+^N \tag{3.2}$$

for a suitable $k > 0$. It is known that $H_t$ has a density $p(x, t)$ solving the following equation:

$$\partial_t p(x, t) = -\mathcal{D}_x^{\alpha,\sigma} p(x, t), \tag{3.3}$$

where $\mathcal{D}_x^{\alpha,\sigma}$ is the so-called *fractional gradient*, i.e., a pseudo-differential operator defined by

$$\mathcal{D}_x^{\alpha,\sigma} h(x) = k \int_{\mathcal{C}^{N-1}} (\nabla \cdot \hat{\theta})^{\alpha} h(x) \sigma(\mathrm{d}\hat{\theta}). \tag{3.4}$$

Note that (3.4) represents the average under $\sigma(\mathrm{d}\hat{\theta})$ of the fractional power of the directional derivative along the direction $\hat{\theta}$. For some theory and applications about this operator, consult Example 2.2 in [8], chapter 6 in [33] and also [13, 28].

When $N = 2$, the Lévy measure has the form

$$\phi(\mathrm{d}r, \mathrm{d}\theta) = \frac{\mathrm{d}r}{r^{\alpha+1}} \sigma(\mathrm{d}\theta), \qquad r > 0, \qquad 0 \le \theta \le \frac{\pi}{2}$$

and, by denoting $\lambda = (\lambda_1, \lambda_2)$, the Laplace exponent can be written as

$$S^{\alpha,\sigma}(\lambda_1, \lambda_2) = k \int_0^{\pi/2} (\lambda_1 \cos\theta + \lambda_2 \sin\theta)^{\alpha} \sigma(\mathrm{d}\theta),$$

whence the fractional gradient, acting of a function $(x, y) \to h(x, y)$, has the form

$$\mathcal{D}_{x,y}^{\alpha,\sigma} h(x, y) = k \int_0^{\pi/2} \left( \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \right)^{\alpha} h(x, y) \sigma(\mathrm{d}\theta), \tag{3.5}$$

#### 3.1.1 The General Case

In the general case where $N$ and $M$ are any positive integers, the Laplace transform of $H_t$ can be computed as follows. Let $t = (t_1, \ldots, t_M) \in \mathbb{R}_+^M$ and let $\{e_1, \ldots, e_M\}$ be the canonical basis of $\mathbb{R}^M$. We can use Proposition 2.7 to say that there exist independent random vectors $Z_{t_k}^{(k)}$, $k = 1, \ldots, M$, with $Z_{t_k}^{(k)} \overset{d}{=} H_{t_k e_k}$, such that

$$H_t \overset{d}{=} Z_{t_1}^{(1)} + \ldots Z_{t_M}^{(M)}.$$

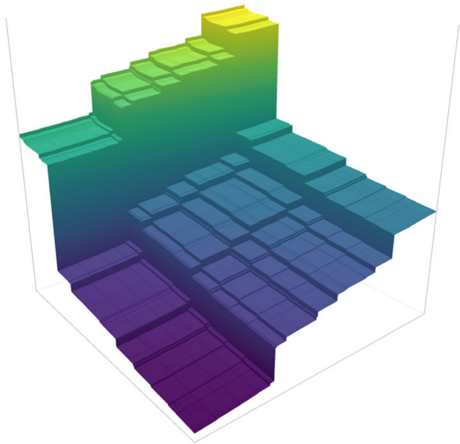


**Fig. 3** Sample path of a stable subordinator field

But, by the construction of $(H_t)_{t\in\mathbb{R}_+^M}$, it follows that for each $k = 1, \ldots, M$, the process $(H_{t_k e_k})_{t_k\in\mathbb{R}_+}$ is a multivariate subordinator in the sense explained in the previous Example 3.3. Hence, there exist $b_k \in \mathbb{R}_+^N$ and a Lévy measure $\phi_k$ on $\mathbb{R}_+^N$ (satisfying $\int_{\mathbb{R}_+^N} \min(|x|, 1)\phi_k(\mathrm{d}x) < \infty$) such that $H_{t_k e_k}$ has Laplace transform

$$\mathbb{E}e^{-\lambda\cdot H_{t_k e_k}} = e^{-t_k S_k(\lambda)}, \qquad \lambda \in \mathbb{R}_+^N,$$

where $S_k$ is a multivariate Bernstein functions, defined by

$$S_k(\lambda) = b_k \cdot \lambda + \int_{\mathbb{R}_+^N} (1 - e^{-\lambda\cdot x})\phi_k(\mathrm{d}x). \tag{3.6}$$

Hence, the Laplace transform of $H_t$ can be compactly written as

$$\mathbb{E}e^{-\lambda\cdot H_t} = e^{-t_1 S_1(\lambda)\cdots - t_M S_M(\lambda)} = e^{-t\cdot S(\lambda)}, \tag{3.7}$$

where $t = (t_1, \ldots, t_M)$ and

$$S(\lambda) = \big(S_1(\lambda), \ldots, S_M(\lambda)\big). \tag{3.8}$$

We call (3.8) the multidimensional Laplace exponent of the subordinator field. The above decomposition of a subordinator field into the sum (in distribution) of independent multivariate subordinators will play a decisive role in the following.

A sample path of a stable subordinator field is shown in Fig. 3.

### 3.2 Subordinated Fields

Let $(X_s)_{s\in\mathbb{R}_+^N}$ be an $N$-parameter Lévy process with values in $\mathbb{R}^d$ and let $(H_t)_{t\in\mathbb{R}_+^M}$ be a subordinator field (in the sense of Sect. 3.1) with values in $\mathbb{R}_+^N$. In the following,

$(X_s)_{s\in\mathbb{R}^N_+}$ and $(H_t)_{t\in\mathbb{R}^M_+}$ are assumed to be independent. We consider the subordinated random field

$$Z_t := X_{H_t}, \qquad t \in \mathbb{R}^M_+. \tag{3.9}$$

It is known that (3.9) is also a multiparameter Lévy process (see [39, Thm. 3.12]). Let $\mu_s$, $\rho_t$ and $\nu_t$, respectively, denote the probability laws of $X_s$, $H_t$ and $Z_t$. Then, by conditioning, for any Borel set $B \subset \mathbb{R}^d$, we have

$$\nu_t(B) = \int_{\mathbb{R}^N_+} \mu_s(B)\, \rho_t(\mathrm{d}s). \tag{3.10}$$

Processes of type (3.9) have also been studied in the literature.

In [5], the authors study the case $M = 1$ and prove that $(Z_t)_{t\in\mathbb{R}^+}$ is again a Lévy process and find the characteristic triplet.

In [38–40], the authors consider the general case $M \geq 1$; actually their study is more general, since they consider cone-parameter Lévy processes subordinated by cone-valued Lévy processes.

Now, let $(T_t)_{t\in\mathbb{R}^N_+}$ be the Feller semigroup associated with $X_t$, defined in (2.14), with generator $G = (G_1, \ldots, G_N)$. Moreover, let $(T^Z_t)_{t\in\mathbb{R}^M_+}$ be the Feller semigroup associated with $Z_t$, i.e.,

$$T^Z_t h(x) := \mathbb{E}\, h(x + Z_t) = \int_{\mathbb{R}^d} h(x + y)\nu_t(\mathrm{d}y), \qquad h \in \mathcal{C}_0(\mathbb{R}^d), \qquad t \in \mathbb{R}^M_+, \tag{3.11}$$

where $\nu_t$ is the law of $Z_t$ defined in (3.10), whence we can rewrite (3.11) as a subordinated semigroup:

$$T^Z_t h(x) = \int_{\mathbb{R}^N_+} T_s h(x)\rho_t(\mathrm{d}s), \qquad t \in \mathbb{R}^M_+. \tag{3.12}$$

In the following theorem, we determine the form of the generator $G^Z = (G^Z_1, \ldots, G^Z_M)$ for the subordinated semigroup, by restricting to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. We obtain a multiparameter generalization of the well-known Phillips formula (see e.g., [43, p. 212]) holding for one-parameter subordinated semigroups.

**Theorem 3.5** *For each $k = 1, \ldots, M$, we have*

$$G^Z_k h(x) = b_k \cdot G\, h(x) + \int_{\mathbb{R}^N_+} \big(T_z h(x) - h(x)\big)\phi_k(\mathrm{d}z), \qquad h \in \mathcal{S}(\mathbb{R}^d). \tag{3.13}$$

*where $b_k$ and $\phi_k$ have been defined in (3.6).*

***Proof*** We first compute the characteristic function of $Z_t = X_{H_t}$, following the lines of [5]. By conditioning, and using (2.3) and (3.7), we have

$$\begin{aligned}\mathbb{E}e^{i\xi\cdot X_{H_t}} &= \int_{\mathbb{R}_+^N} \mathbb{E}e^{i\xi\cdot X_u} P(H_t \in \mathrm{d}u) \\ &= \int_{\mathbb{R}_+^N} e^{u\cdot\Psi(\xi)} P(H_t \in \mathrm{d}u) \\ &= \mathbb{E}e^{-(-\Psi(\xi))\cdot H_t} \\ &= e^{-t\cdot S(-\Psi(\xi))}, \qquad \xi \in \mathbb{R}^d,\end{aligned}$$

where $t = (t_1, \ldots, t_M)$ and

$$-S(-\Psi(\xi)) := \begin{pmatrix} -S_1\big(-\psi_1(\xi), \ldots, -\psi_N(\xi)\big) \\ \vdots \\ -S_M\big(-\psi_1(\xi), \ldots, -\psi_N(\xi)\big) \end{pmatrix}.$$

Thus, by using Theorem 2.16, it follows that $T_t^Z$ is a pseudo-differential operator with symbol $e^{-t\cdot S(-\Psi)}$, i.e.,

$$T_t^Z h(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} e^{-t\cdot S(-\Psi(\xi))} \hat{h}(\xi)\mathrm{d}\xi, \qquad h \in \mathcal{S}(\mathbb{R}^d) \tag{3.14}$$

while for each $k = 1, \ldots, M$, $G_k^Z$ is a pseudo-differential operator with symbol

$$-S_k(-\Psi(\xi)) = -S_k(-\psi_1(\xi), \ldots, -\psi_N(\xi)).$$

This means that

$$G_k^Z h(x) = -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} S_k(-\psi_1(\xi), \ldots, -\psi_N(\xi))\, \hat{h}(\xi)\mathrm{d}\xi, \qquad h \in \mathcal{S}(\mathbb{R}^d). \tag{3.15}$$

But, using (3.6), we have that

$$-S_k(-\Psi(\xi)) = b_k \cdot \Psi(\xi) + \int_{\mathbb{R}_+^N} (e^{z\cdot\Psi(\xi)} - 1)\phi_k(\mathrm{d}z). \tag{3.16}$$

Then, after substituting (3.16) in (3.15), we can solve the inverse Fourier transform, and taking into account the representation of $T_t$ given in (2.15), we obtain the result. □

***Remark 3.6*** In the spirit of operational functional calculus, the well-known Phillips Theorem (see e.g., [43, p. 212]) can be informally stated as follows. Let a Markov process $(X_t)_{t\in\mathbb{R}_+}$ have generator $G$ and let a subordinator $(H_t)_{t\in\mathbb{R}_+}$ have Bernstein function $f$. Then, the subordinated process $(X_{H_t})_{t\in\mathbb{R}_+}$ has generator $-f(-G)$.

In a similar way, our Theorem 3.5 can be stated as follows.

Let $(X_t)_{t\in\mathbb{R}^N_+}$ be a multiparameter Lévy process with generator $G=(G_1,\ldots,G_N)$ and let $(H_t)_{t\in\mathbb{R}^M_+}$ be a subordinator field associated with the multivariate Bernstein functions $S_1, S_2,\ldots,S_M$, namely its Laplace exponent is $S=(S_1,S_2,\ldots,S_M)$. Then, the subordinated field $(X_{H_t})_{t\in\mathbb{R}^M_+}$ has generator

$$-S(-G):=\begin{pmatrix} -S_1(-G_1,-G_2,\ldots,-G_N)\\ -S_2(-G_1,-G_2,\ldots,-G_N)\\ \cdot\\ \cdot\\ -S_M(-G_1,-G_2,\ldots,-G_N)\end{pmatrix}.$$

### 3.3 Stochastic Solution to Systems of Integro-Differential Equations

Our extension of the Phillips theorem, given in Theorem 3.5, provides a stochastic solution to some systems of differential equations.

Indeed, let $(X_t)_{t\in\mathbb{R}^N_+}$ be a Multiparameter Lévy process with values in $\mathbb{R}^d$. Moreover, let $(H_t)_{t\in\mathbb{R}^M_+}$ be a subordinator field with values in $\mathbb{R}^N_+$ and let $(Z_t)_{t\in\mathbb{R}^M_+}=(X_{H_t})_{t\in\mathbb{R}^M_+}$ be the subordinated field. Then, by virtue of Proposition 2.14, and using the symbolic notation of Remark 3.6, we have that, for any $u\in\mathcal{S}(\mathbb{R}^d)$, the function $\mathbb{E}u(x+Z_t)$ solves the system

$$\begin{cases} \frac{\partial}{\partial t_1}h(x,t)=-S_1(-G_1,-G_2,\ldots,-G_N)h(x,t)\\ \frac{\partial}{\partial t_2}h(x,t)=-S_2(-G_1,-G_2,\ldots,-G_N)h(x,t)\\ \cdot\\ \cdot\\ \frac{\partial}{\partial t_M}h(x,t)=-S_M(-G_1,-G_2,\ldots,-G_N)h(x,t)\\ \\ h(x,0)=u(x)\end{cases} \qquad x\in\mathbb{R}^d,\quad t\in\mathbb{R}^M_+, \tag{3.17}$$

where $t=(t_1,\ldots,t_M)$, $G=(G_1,\ldots,G_N)$ denotes the generator of $(X_t)_{t\in\mathbb{R}^N_+}$ and $S_1,\ldots,S_M$ are the multivariate Bernstein functions, i.e., the components of the Laplace exponent of $(H_t)_{t\in\mathbb{R}^M_+}$ defined in (3.8).

***Example 3.7*** Let $\{e_1,\ldots,e_M\}$ be the canonical basis of $\mathbb{R}^M$. Assume that the subordinator field $(H_t)_{t\in\mathbb{R}^M_+}$ is such that, for each $i=1,\ldots,M$, the component $H_{t_ie_i}$ is a multivariate stable subordinator in the sense of Example 3.4, with index $\alpha_i\in(0,1)$, whose multivariate Bernstein function reads

$$S_i^{\alpha_i,\sigma_i}(\lambda)=k_i\int_{\mathcal{C}^{N-1}}(\lambda\cdot\hat{\theta})^{\alpha_i}\sigma_i(\mathrm{d}\hat{\theta}). \tag{3.18}$$

Then, the system (3.17) takes the form

$$\begin{cases} \frac{\partial}{\partial t_1} h(x,t) = -k_1 \int_{\mathcal{C}^{N-1}} (-G \cdot \hat{\theta})^{\alpha_1} h(x,t) \sigma_1(\mathrm{d}\hat{\theta}) \\ \frac{\partial}{\partial t_2} h(x,t) = -k_2 \int_{\mathcal{C}^{N-1}} (-G \cdot \hat{\theta})^{\alpha_2} h(x,t) \sigma_2(\mathrm{d}\hat{\theta}) \\ \cdots \\ \frac{\partial}{\partial t_M} h(x,t) = -k_M \int_{\mathcal{C}^{N-1}} (-G \cdot \hat{\theta})^{\alpha_M} h(x,t) \sigma_M(\mathrm{d}\hat{\theta}) \end{cases} \qquad x \in \mathbb{R}^d, \quad t \in \mathbb{R}^M_+, \tag{3.19}$$

where, on the right side, the fractional powers $(-G \cdot \hat{\theta})^{\alpha_i}$ are well-defined because $-G \cdot \hat{\theta}$ is the generator of a contraction semigroup.

***Example 3.8*** Let $N = M = 2$. Consider the biparameter, additive Lévy process

$$X(t_1, t_2) = X_1(t_1) + X_2(t_2), \tag{3.20}$$

where $X_1$ and $X_2$ are independent isotropic stable processes with indices $\alpha_1 \in (0, 2]$ and $\alpha_2 \in (0, 2]$, respectively. Let

$$H(t_1, t_2) = (H_1(t_1, t_2), H_2(t_1, t_2))$$

be a subordinator field, such that $H(t_1, 0)$ and $H(0, t_2)$ are two bivariate stable subordinators in the sense of Example 3.4, respectively, having indices $\beta_1 \in (0, 1)$ and $\beta_2 \in (0, 1)$ and spectral measures $\sigma_1$ and $\sigma_2$. Let

$$Z(t_1, t_2) = X_1(H_1(t_1, t_2)) + X_2(H_2(t_1, t_2))$$

be the subordinated field. Then, for any $u \in \mathcal{S}(\mathbb{R}^d)$, the function $\mathbb{E}u(x + Z(t_1, t_2))$ solves the system

$$\begin{cases} \frac{\partial}{\partial t_1} h(x,t) = -k_1 \int_0^{\pi/2} \left((-\Delta)^{\alpha_1/2} \cos\theta + (-\Delta)^{\alpha_2/2} \sin\theta\right)^{\beta_1} h(x,t) \sigma_1(\mathrm{d}\theta) \\ \\ \frac{\partial}{\partial t_2} h(x,t) = -k_2 \int_0^{\pi/2} \left((-\Delta)^{\alpha_1/2} \cos\theta + (-\Delta)^{\alpha_2/2} \sin\theta\right)^{\beta_2} h(x,t) \sigma_2(\mathrm{d}\theta) \\ \\ h(x,0) = u(x) \end{cases} \tag{3.21}$$

where $-(-\Delta)^{\alpha_i/2}$ denotes the fractional Laplacian. To write the system (3.21), we used that for $i = 1, 2$, the generator of the isotropic stable process $X_i$ is $G_i = -(-\Delta)^{\alpha_i/2}$ (see e.g., [4, p. 166]).

***Example 3.9*** Consider again Example 3.8. In the special case where $\alpha_1 = \alpha_2 = 2$, the process (3.20) is a so-called additive Brownian motion (see e.g., [17, p. 394]) and the above system simplifies to

$$\begin{cases} \frac{\partial}{\partial t_1} h(x,t) = -C_1(-\Delta)^{\beta_1} h(x,t) \\ \frac{\partial}{\partial t_2} h(x,t) = -C_2(-\Delta)^{\beta_2} h(x,t) \\ h(x,0) = u(x) \end{cases} \tag{3.22}$$

for suitable constants $C_1, C_2 > 0$.

***Example 3.10*** Let $N = 1$ and $M > 1$ (so that subordination increases the number of parameters). So let $(X_t)_{t\in\mathbb{R}_+}$ be a one-parameter Lévy process and let $(H_t)_{t\in\mathbb{R}_+^M}$ be a subordinator field with values in $\mathbb{R}_+$. For example, assume that $(X_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion in $\mathbb{R}^d$ and, for each $k = 1, \ldots, M$, $H_{t_k e_k}$ is a stable subordinator of index $\beta_k \in (0,1)$ ($e_k$ denoting the $k$-th vector of the canonical basis). Let $(Z_t)_{t\in\mathbb{R}_+^M} = (X_{H_t})_{t\in\mathbb{R}_+^M}$ be the subordinated field. Then, $\mathbb{E}u(x+Z_t)$ solves

$$\begin{cases} \frac{\partial}{\partial t_1} h(x,t) = -(-\Delta)^{\beta_1} h(x,t) \\ \frac{\partial}{\partial t_2} h(x,t) = -(-\Delta)^{\beta_2} h(x,t) \\ . \\ . \\ \frac{\partial}{\partial t_M} h(x,t) = -(-\Delta)^{\beta_M} h(x,t) \\ \\ h(x,0) = u(x) \end{cases}$$

## 4 Subordination by the Inverse Random Field

Let $(H_t)_{t\in\mathbb{R}_+}$ be a multivariate subordinator in the sense of Example 3.3, which takes values in $\mathbb{R}_+^N$. Hence, it is defined by $H_t = (H_1(t), \ldots, H_N(t))$, where each marginal component $H_j(t)$ is a classical subordinator. Consider a new random field $(\mathcal{L}_t)_{t\in\mathbb{R}_+^N}$ defined by

$$\mathcal{L}_t = \big(L_1(t_1), \ldots, L_N(t_N)\big), \qquad t = (t_1, \ldots, t_N), \tag{4.1}$$

where $L_j$ is the inverse hitting time of the subordinator $H_j$, i.e.,

$$L_j(t_j) = \inf\{x > 0 : H_j(x) > t_j\}.$$

As stated in the introduction, we will call (4.1) *inverse random field*.

Now, let $(X_t)_{t\in\mathbb{R}_+^N}$ be an $N$-parameter Lévy process with values in $\mathbb{R}^d$. We are interested in the subordinated random field $(Z_t)_{t\in\mathbb{R}_+^N}$ defined by

$$Z_t = X_{\mathcal{L}_t}, \qquad t \in \mathbb{R}_+^N. \tag{4.2}$$

This topic has many sources of inspiration. Above all, there is a well-established theory (consult e.g., [6, 7, 22, 27, 29–34, 48]) concerning semi-Markov processes of the form

$$Z(t) = X(L(t)), \qquad t \geq 0, \tag{4.3}$$

where $X$ is a Lévy process in $\mathbb{R}^d$ and $L$ is the inverse hitting time of a subordinator $H$, i.e.,

$$L(t) = \inf\{x > 0 : H(x) > t\}.$$

Such processes have a great interest in statistical physics, as they arise as scaling limits of suitable continuous time random walks.

***Example 4.1*** A special case (see e.g., [2, 3, 21, 25, 26]) is the process

$$Z(t) = B(L^\alpha(t)), \tag{4.4}$$

where $B$ is a $d$-dimensional standard Brownian motion and $L^\alpha$ is the inverse of a $\alpha$-stable subordinator independent of $B$, where $\alpha \in (0, 1)$. The process (4.4) is a so-called subdiffusion: the mean square displacement behaves as $C t^\alpha$, i.e., the motion is delayed with respect to the Brownian behavior. This models the case where the moving particle is trapped by inhomogeneities or perturbations in the medium; thus, the particle runs on Brownian paths, but, for arbitrary time intervals, it is forced to be at rest, which gives rise to a sub-diffusive dynamics. Diffusions in porous media and penetration of a pollutant in the ground have this type of motion (see [35] for other applications of anomalous diffusions). The random variable $B(L^\alpha(t))$ has a density solving the following anomalous diffusion equation

$$\mathcal{D}_t^\alpha q(x,t) - \frac{t^\alpha}{\Gamma(1-\alpha)}\delta(x) = \frac{1}{2}\Delta q(x,t), \tag{4.5}$$

where $\Delta$ denotes the Laplacian operator and $\mathcal{D}_t^\alpha$ is the Marchaud fractional derivative, defined by

$$\mathcal{D}_t^\alpha h(t) := \int_0^\infty \big(h(t) - h(t-\tau)\big)\frac{\alpha\tau^{-\alpha-1}}{\Gamma(1-\alpha)}\mathrm{d}\tau. \tag{4.6}$$

See also [11, 12] for a tempered version of such operator and its connections to drifted Brownian motions. We finally recall that recent models of anomalous diffusion in heterogeneous media, where the fractional order $\alpha$ is space-dependent, have been developed in [20, 41, 44] (see also [16] for a related model).

Equation (4.5) is a special case of a more general theory. Indeed, as anticipated in the Introduction, if $X$ and $L$ are independent, the connection of the process (4.3) with integro-differential equations is given by the following facts. Let $X$ have a density $p(x,t)$ solving

$$\partial_t p(x,t) = G^* p(x,t),$$

where $G^*$ is the dual to the Markov generator. Moreover, let $L$ be the inverse of a subordinator with Lévy measure $\nu$. If $L$ has a density $l(x,t)$, then, by conditioning, $X(L(t))$ has a density

$$p^*(x,t) = \int_0^\infty p(x,u)l(u,t)\mathrm{d}u.$$

Such a density solves

$$\mathcal{D}_t p^*(x,t) - \overline{\nu}(t)p^*(x,0) = G^* p^*(x,t), \tag{4.7}$$

where $\overline{\nu}(t) = \int_t^\infty \nu(\mathrm{d}x)$ and the operator $\mathcal{D}_t$, usually called generalized Marchaud fractional derivative, is defined by

$$\mathcal{D}_t h(t) := \int_0^\infty \big(h(t) - h(t-\tau)\big)\nu(\mathrm{d}\tau). \tag{4.8}$$

Concerning the link between semi-Markov processes and non-local in time equations, consult also [36, 37] for a discrete-time model and [42] for the theory of abstract equations related to semi-Markov Random evolutions.

The rest of this section will be structured as follows. A special case of biparameter Lévy processes will be treated in Sect. 4.1 and a related model of anisotropic subdiffusion will be presented in Sect. 4.2. Finally, the special case where the $L_j$, $j = 1, \ldots, N$, are independent will be presented in Sect. 4.3 and some long-range dependence properties will be analyzed.

### 4.1 Subordination of Some Two-Parameter Lévy Processes

Consider the following biparameter Lévy process with values in $\mathbb{R}^d$:

$$X(t_1,t_2) = (X_1(t_1), X_2(t_2)), \tag{4.9}$$

where $X_1$ and $X_2$ are (possibly dependent) Lévy processes with values in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively, with $d_1 + d_2 = d$.

Consider now a bivariate subordinator $(H_1(t), H_2(t))$ and the related bivariate inverse random field $(L_1(t_1), L_2(t_2))$ as defined in (4.1).

We will consider the following assumptions:

*A1)* $X_1(t_1)$ and $X_2(t_2)$ have marginal densities $p_1(x_1,t)$ and $p_2(x_2,t)$ satisfying the following forward equations:

$$\frac{\partial}{\partial t} p_i(x_i,t) = G_i^* p_i(x_i,t), \qquad i = 1, 2,$$

where $G_1^*$ and $G_2^*$ are the duals to the generators of $X_1$ and $X_2$.

*A2)* $X(t_1, t_2)$ has density $p(x_1, x_2, t_1, t_2)$ satisfying the system

$$\frac{\partial}{\partial t_i} p(x_1, x_2, t_1, t_2) = G_i^* p(x_1, x_2, t_1, t_2), \qquad i = 1, 2.$$

*A3)* For all $t_1, t_2 > 0$, the random vector $(H_1(t_1), H_2(t_2))$ has a density $q(x_1, x_2, t_1, t_2)$.[1]

We now consider the subordinated random field

$$Z(t_1, t_2) = X(L_1(t_1), L_2(t_2)) \tag{4.10}$$

The following proposition gives a generalization of equation (4.7) adapted to the random field (4.10).

**Proposition 4.2** *Under the assumptions A1), A2), A3), the random vector $X(L_1(t_1), L_2(t_2))$ has a density $h(x_1, x_2, t_1, t_2)$ satisfying*

$$\mathcal{D}_{t_1,t_2} h(x_1, x_2, t_1, t_2) = (G_1^* + G_2^*) h(x_1, x_2, t_1, t_2), \qquad x_1 \neq 0, x_2 \neq 0, \tag{4.11}$$

*where $\mathcal{D}_{t_1,t_2}$ is the bidimensional version of the generalized fractional derivative, defined in (3.1), i.e.,*

$$\mathcal{D}_{t_1,t_2} h(t_1, t_2) = \int_{\mathbb{R}_+^2} \big(h(t_1, t_2) - h(t_1 - \tau_1, t_2 - \tau_2)\big) \phi(\mathrm{d}\tau_1, \mathrm{d}\tau_2).$$

***Proof*** Under assumption *A3)*, the distribution of $(L_1(t_1), L_2(t_2))$ is the sum of two components (see [8, sect. 3.1]): the first one is absolutely continuous with respect to the bidimensional Lebesgue measure, with density $l$, namely

$$P(L_1(t_1) \in \mathrm{d}x_1, L_2(t_2) \in \mathrm{d}x_2) = l(x_1, x_2, t_1, t_2)\mathrm{d}x_1\mathrm{d}x_2, \qquad x_1 \neq x_2,$$

while the second one has support on the bisector line $x_1 = x_2$, with one-dimensional Lebesgue density $l_*(x, t_1, t_2)$ (i.e., $P(L_1(t_1) = L_2(t_2)) = \int_0^\infty l_*(x, t_1, t_2)\mathrm{d}x$).

Then, by using a simple conditioning argument, the random vector $X(L_1(t_1), L_2(t_2))$ has density

$$\begin{aligned} h(x_1, x_2, t_1, t_2) = &\int_0^\infty \int_0^\infty p(x_1, x_2, u, v) l(u, v, t_1, t_2) \mathrm{d}u\, \mathrm{d}v \\ &+ \int_0^\infty p(x_1, x_2, u, u) l_*(u, t_1, t_2) \mathrm{d}u. \end{aligned}$$

[1] Observe that the random field $(t_1, t_2) \to (H_1(t_1), H_2(t_2))$ is not a biparameter Lévy process even if $t \to (H_1(t), H_2(t))$ is a multivariate subordinator, unless the two marginal components are independent.

By applying $\mathcal{D}_{t_1,t_2}$ to both sides and by using [8, Thm 3.6], we have

$$\begin{aligned}
&\mathcal{D}_{t_1,t_2} h(x_1, x_2, t_1, t_2) \\
&= -\int_0^\infty \int_0^\infty p(x_1, x_2, u, v) \frac{\partial}{\partial u} l(u, v, t_1, t_2) \mathrm{d}u \, \mathrm{d}v \\
&\quad - \int_0^\infty \int_0^\infty p(x_1, x_2, u, v) \frac{\partial}{\partial v} l(u, v, t_1, t_2) \mathrm{d}u \, \mathrm{d}v \\
&\quad - \int_0^\infty p(x_1, x_2, u, u) \frac{\partial}{\partial u} l_*(u, t_1, t_2) \mathrm{d}u. \qquad (4.12)
\end{aligned}$$

Now, we integrate by parts by using assumptions *A1* and *A2*. We also use that $X_1(0) = 0$ and $X_2(0) = 0$ almost surely, which implies that $P(X_1(0) \in A, X_2(t_2) \in B) = \mathcal{I}_{(0\in A)} P(X_2(t_2) \in B)$ and $P(X_1(t_1) \in A, X_2(0) \in B) = P(X_1(t_1) \in A)\mathcal{I}_{(0\in B)}$; thus we get

$$\begin{aligned}
&\mathcal{D}_{t_1,t_2} h(x_1, x_2, t_1, t_2) \\
&= G_1^* \int_0^\infty \int_0^\infty p(x_1, x_2, u, v) l(u, v, t_1, t_2) \mathrm{d}u \, \mathrm{d}v + \delta(x_1) \int_0^\infty p_2(x_2, v) l(0, v, t_1 t_2) \mathrm{d}v \\
&\quad + G_2^* \int_0^\infty \int_0^\infty p(x_1, x_2, u, v) l(u, v, t_1, t_2) \mathrm{d}u \, \mathrm{d}v + \delta(x_2) \int_0^\infty p_1(x_1, u) l(u, 0, t_1 t_2) \mathrm{d}u \\
&\quad + (G_1^* + G_2^*) \int_0^\infty p(x_1, x_2, u, u) l_*(u, t_1, t_2) \mathrm{d}u + \delta(x_1)\delta(x_2)\overline{\phi}(t_1, t_2),
\end{aligned}$$

where

$$\overline{\phi}(t_1, t_2) = \int_{t_1}^\infty \int_{t_2}^\infty \phi(\mathrm{d}x_1, \mathrm{d}x_2).$$

In the above calculations, we have taken into account that

$$\frac{\partial p(x_1, x_2, u, u)}{\partial u} = (G_1^* + G_2^*) p(x_1, x_2, u, u)$$

since the total derivative of $p(x_1, x_2, t_1, t_2)$, with $t_1 = u$ and $t_2 = u$, is given by

$$\frac{\partial p}{\partial t_1} \frac{\partial t_1}{\partial u} + \frac{\partial p}{\partial t_2} \frac{\partial t_2}{\partial u} = G_1^* p + G_2^* p.$$

In the region $x_1 \neq 0, x_2 \neq 0$, we have

$$\begin{aligned}
&\mathcal{D}_{t_1,t_2} h(x_1, x_2, t_1, t_2) \\
&= (G_1^* + G_2^*) \int_0^\infty \int_0^\infty p(x_1, x_2, u, v) l(u, v, t_1, t_2) \mathrm{d}u \, \mathrm{d}v \\
&\quad + (G_1^* + G_2^*) \int_0^\infty p(x_1, x_2, u, u) l_*(u, t_1, t_2) \mathrm{d}u
\end{aligned}$$

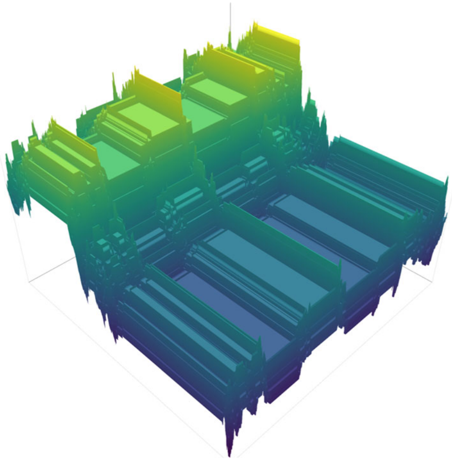


**Fig. 4** Sample path of a time-changed additive Brownian field with an inverse stable field

$$= (G_1^* + G_2^*)h(x_1, x_2, t_1, t_2),$$

which concludes the proof. □

A sample path of a time-changed field is shown in Fig. 4.

### 4.2 Anomalous Diffusion in Anisotropic Media

As a byproduct of the results of Sect. 4.1, we here propose another model of subdiffusion which extends the one treated in Example (4.1), by including it as a special case.

As explained, the process (4.4) models a subdiffusion through an isotropic medium, i.e., the trapping effect is the same in all coordinate directions (e.g., all components of the Brownian motion are delayed by the same random time process). Hence, the subordinated process (4.4) is isotropic as well as the Brownian motion.

Thus, it is natural to search for a model of subdiffusion in the case where the external medium is not isotropic.

Actually, a first model of anisotropic subdiffusion has been proposed in [8, sect. 5]. Here, the authors defined a process

$$Z(t) = \big(B_1(L_1(t)), B_2(L_2(t))\big), \qquad t \geq 0,$$

where $(B_1, B_2)$ is a bidimensional standard Brownian motion with independent components and $(L_1, L_2)$ is the inverse random field of $(H_1, H_2)$, the last one being a bivariate stable subordinator of index $\alpha$ (in the sense of Example 3.4). The authors in [8] prove that the law of $Z(t)$ is not rotationally invariant; hence, the process is anisotropic. However, the mean square displacement along the direction $\theta$ grows as $C_\theta\, t^\alpha$; hence, the spreading rate $\alpha$ is the same for all coordinate directions (that is, anisotropy is given by the constant $C_\theta$ alone). Thus, from a physical point of view, the model presented in [8] differs little from the isotropic model, which was previously existing in the literature (where $C_\theta$ is independent of $\theta$). In a sense, this is due to the fact that the subordinator $H(t) = (H_1(t), H_2(t))$ used in [8] is characterized by the isotropic scaling $H(ct) \stackrel{d}{=} c^{1/\alpha} H(t)$. In the following, we will improve these shortcomings, by considering a bivariate operator stable subordinator, which is characterized by the anisotropic scaling $H(ct) \stackrel{d}{=} c^A H(t)$, where $A$ is a linear operator (whose eigenvalues determine the spreading rates in different directions). On this points, see the discussion before formula (4.16), and also the Examples from 4.3 onward.

We need to recall some notions on operator stability (consult [15] and [46]). A random vector $X$ with values in $\mathbb{R}^d$ is said to be *operator stable* if, for any positive integer $n$, there exist a vector $c_n \in \mathbb{R}^d$ and a $d \times d$ matrix $A$ such that $n$ indipendent copies $X_1, \ldots, X_n$ of $X$ satisfy

$$X_1 + \cdots + X_n \stackrel{d}{=} n^A X + c_n, \tag{4.13}$$

where the matrix power $n^A$ is defined by

$$n^A = e^{A \ln n} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k (\ln n)^k.$$

In the special case $A = \frac{1}{\alpha} I$, with $\alpha \in (0, 2]$ and $I$ denoting the identity matrix, we have that $X$ is $\alpha$-stable. In the general case, $A$ has eigenvalues whose real parts have the form $1/\alpha_i$, with $\alpha_i \in (0, 2]$, $i = 1, \ldots, d$. We stress that the matrix $A$ is not unique, i.e., there may be different $n \times n$ matrices satisfying (4.13) (unlike what happens in the stable case, where the index $\alpha$ is uniquely defined).

Operator stable laws are infinite divisible, hence they correspond to some Lévy processes. A Lévy process $X(t), t \geq 0$ is said to be an operator stable Lévy motion if $X(1)$ is an operator stable random vector. Note that such a process is characterized by the anisotropic scaling $X(ct) \stackrel{d}{=} c^A X(t)$. This property is a generalization of self-similarity of $\alpha$-stable processes where the scaling is the same for all coordinates, i.e., $X(ct) \stackrel{d}{=} c^{1/\alpha} X(t)$.

We are now ready to present the model of anisotropic subdiffusion. So, let us consider a bivariate subordinator $(H_1(t), H_2(t))$ which is constructed as an operator stable Lévy motion with values in $\mathbb{R}^2_+$. In this case, $A$ has eigenvalues whose real parts have the form $1/\alpha_i$, with $\alpha_i \in (0, 1)$, $i = 1, 2$. Now, let $r > 0$ and $\theta \in [0, \frac{\pi}{2}]$ be the so-called Jurek coordinates (see e.g., [15] and [33, p.185]) which are defined by the mapping $\mathbb{R}^2_+ \ni x = r^A \hat{\theta}$, where $\hat{\theta} = (\cos\theta, \sin\theta)$. In this new coordinates, the bidimensional Lévy measure can be expressed as

$$\phi^{A,M}(\mathrm{d}r, \mathrm{d}\theta) = C \frac{\mathrm{d}r}{r^2} M(\mathrm{d}\theta), \qquad r > 0, \qquad \theta \in \left[0, \frac{\pi}{2}\right],$$

where $M$ is a probability measure on the angular component. Then, the operator $\mathcal{D}_x$, $x \in \mathbb{R}^2_+$, defined in formula (3.1) of Example 3.3, takes the form

$$\mathcal{D}_x^{A,M} h(x) = C \int_0^{\pi/2} \int_0^{\infty} \big(h(x) - h(x - r^A \hat{\theta})\big) \frac{\mathrm{d}r}{r^2} M(\mathrm{d}\theta). \tag{4.14}$$

If $(H_1(t), H_2(t))$ is a bivariate stable subordinator (see Example 3.4), i.e., $A = \frac{1}{\alpha} I$, by a simple change of variables, one re-obtains the fractional gradient defined in formula (3.4).

Now, let $(L_1(t_1), L_2(t_2))$ be the inverse random field of $(H_1(t), H_2(t))$ and let $(B_1(t), B_2(t))$ be a bidimensional standard Brownian motion with independent components. Consider the time-changed process

$$Z(t) = \big(B_1(L_1(t)), B_2(L_2(t))\big), \qquad t \geq 0. \tag{4.15}$$

The process (4.15) is a model of anisotropic subdiffusion. Indeed consider the random variable

$$Z_\theta(t) = Z(t) \cdot \hat{\theta}$$

representing the displacement along the direction $\hat{\theta} = (\cos\theta, \sin\theta)$. By conditioning, the mean square displacement can be written as

$$\mathbb{E} Z_\theta^2(t) = \mathbb{E} L_1(t) \cos^2\theta + \mathbb{E} L_2(t) \sin^2\theta \tag{4.16}$$

which, in general, depends on $\theta$ because of anisotropy.

In the spirit of [8, Sect. 4], a governing equation for the process (4.15) can be obtained by considering the related random field $(B_1(L_1(t_1)), B_2(L_2(t_2)))$. Indeed, by applying Proposition 4.2 of the previous Section, it has a density $h(x_1, x_2, t_1, t_2)$ satisfying the anomalous diffusion equation

$$\mathcal{D}_t^{A,M} h(x_1, x_2, t_1, t_2) = \frac{1}{2} \Delta\, h(x_1, x_2, t_1, t_2), \qquad x_1 \neq 0, x_2 \neq 0,$$

where the operator $\mathcal{D}_t^{A,M}$, defined in (4.14), now acts on $t = (t_1, t_2)$.

***Example 4.3*** If $L_1(t) = L_2(t) = L(t)$, where $L(t)$ is the inverse of a $\alpha$-stable subordinator, the process (4.15) reduces to the isotropic subdiffusion (4.4). In this case, we have $\mathbb{E} L(t) = C t^\alpha$. Thus, $\mathbb{E} Z_\theta^2(t) = C t^\alpha$, which is independent of $\theta$ because of isotropy.

***Example 4.4*** If $A$ has the form $\frac{1}{\alpha} I$, where $I$ is the identity matrix, then $Z(t)$ reduces to the anisotropic subdiffusion considered in [8, sect. 5]. In this case, $\mathbb{E} L_1(t) = C_1 t^\alpha$ and $\mathbb{E} L_2(t) = C_2 t^\alpha$. Hence, following formula (4.16), the mean square displacement along the direction $\theta$ has the form $\mathbb{E} Z_\theta^2(t) = C_\theta t^\alpha$, where $C_\theta = C_1 \cos^2\theta + C_2 \sin^2\theta$.

***Example 4.5*** If $H_1(t)$ and $H_2(t)$ are independent stable subordinators, then the matrix $A$ is diagonal with elements $1/\alpha_1$ and $1/\alpha_2$. If $\alpha_1 \neq \alpha_2$, the process (4.15) is anisotropic, in such a way that $\alpha_1$ and $\alpha_2$ represent the spreading rates along the two coordinate directions. Indeed, since $\mathbb{E}L_i(t) = C_i t^{\alpha_i}$ for $i = 1, 2$, then the mean square displacement along a direction $\hat{\theta}$ has the form $\mathbb{E}Z_\theta^2(t) = C_1 t^{\alpha_1} \cos^2\theta + C_2 t^{\alpha_2} \sin^2\theta$ which depends on $\hat{\theta}$ (and, asymptotically, it behaves like $t^{\max(\alpha_1,\alpha_2)}$).

***Example 4.6*** If $A$ is a symmetric matrix with eigenvalues $1/\alpha_1$ and $1/\alpha_2$, where $\alpha_1$ and $\alpha_2$ are in $(0, 1)$, then a rigid rotation of the coordinate system allows to find the two eigenvectors, along which the spreading rates are $\alpha_1$ and $\alpha_2$, respectively, which corresponds to the situation explained in Example 4.5.

### 4.3 Subordination by Independent Inverses

In the following, let $X(t_1, \ldots, t_N)$ be an $N$-parameter Lévy process with density $p(x, t)$ satisfying the system

$$\partial_{t_j} p(x, t) = G_j^* p(x, t), \qquad j = 1, \ldots, N$$

with the usual notation $t = (t_1, \ldots, t_N)$. Assume that the marginal components $L_j(t_j)$ of the inverse random field (4.1) are mutually independent, each having density $l_j(x, t_j)$ and Lévy measure $\nu_j$. Consider the subordinated random field

$$Z(t) := X(L_1(t_1), \ldots, L_N(t_N)). \tag{4.17}$$

Before stating the next result, we introduce the following notation: For a given vector $v = (v_1, \ldots, v_N)$, we introduce the vector $v^{(j)}$ defined by $v^{(j)} = (v_1, \ldots, v_{j-1}, 0, v_{j+1}, \ldots, v_N)$.

**Proposition 4.7** *Under the above assumptions, the subordinated field (4.17) has a density $p^*(x, t)$ satisfying the system*

$$\mathcal{D}_{t_j}^{(\nu_j)} p^*(x, t) - \overline{\nu}_j(t_j)\, p^*(x, t^{(j)}) = G_j^*\, p^*(x, t), \qquad j = 1, \ldots, N,$$

*where $\mathcal{D}_{t_j}^{(\nu_j)}$ denotes the generalized fractional derivative defined in (4.8) with Lévy measure $\nu_j$, and $\overline{\nu}_j(t_j) = \int_{t_j}^{\infty} \nu_j(\mathrm{d}\tau)$.*

***Proof*** By conditioning, (4.17) has a density

$$p^*(x, t) = \int_{R_+^N} p(x, u_1, \ldots, u_N) \prod_{i=1}^{N} l_i(u_i, t_i) \mathrm{du}_1 \cdots \mathrm{du}_\mathrm{N}.$$

By applying $\mathcal{D}_{t_j}^{(\nu_j)}$ to both members and taking into account that such operator commutes with the integral, we have

$$\mathcal{D}_{t_j}^{(\nu_j)} p^*(x,t) = -\int_{R_+^N} p(x,u_1,\dots,u_N) \frac{\partial}{\partial u_j} l_j(u_j,t_j) \prod_{i=1, i\neq j}^{N} l_i(u_i,t_i)\, \mathrm{d}u_1 \cdots \mathrm{d}u_N,$$

where we used that the density $l_j(x,t_j)$ of an inverse subordinator satisfies the equation $\mathcal{D}_{t_j}^{(\nu_j)} l_j(x,t_j) = -\partial_x l_j(x,t_j)$ under the condition $l_j(0,t_j) = \overline{\nu}_j(t_j)$ (see e.g., [22]).

Integrating by parts, we have

$$\mathcal{D}_{t_j}^{(\nu_j)} p^*(x,t) = G_j^* p^*(x,t) + \overline{\nu}_j(t_j) \int_{\mathbb{R}_+^{N-1}} p(x,u^{(j)}) \prod_{i=1, i\neq j}^{N} l_i(u_i,t_i) \mathrm{d}u_i,$$

where the last integral can be written as

$$\int_{\mathbb{R}_+^{N-1}} p(x,u^{(j)}) \prod_{i=1, i\neq j}^{N} l_i(u_i,t_i) du_i = p^*(x,t^{(j)})$$

because $l_j(u_j,0) = \delta(u_j)$. This completes the proof. □

#### 4.3.1 Long-Range Dependence

Consider a process of type (4.17). For each $k = 1,\dots,N$, let $L_k(t_k)$ be the inverse of a $\alpha$-stable subordinator. The subordinated field exhibits a power law decay of the autocorrelation function which is slower with respect to the $|t|^{-\frac{1}{2}}$ decay holding for multiparameter Lévy processes (which was discussed in Remark 2.10 ). This can be useful in applied fields, where spatial data exhibit long-range dependence properties.

So, let $s \preceq t$. By using the results of Sect. 2.2, we have

$$\begin{aligned}
&\mathrm{Cov}(X_{L_s}, X_{L_t}) \\
&= \mathbb{E}\big[\mathrm{Cov}(X_{L_s}, X_{L_t}) \big| L_s, L_t\big] + \mathrm{Cov}\big(\mathbb{E}[X_{L_s} | L_s, L_t], \mathbb{E}[X_{L_t} | L_s, L_t]\big) \\
&= \mathbb{E}[L_s \cdot \sigma^2] + \mathrm{Cov}(L_t \cdot \mu, L_s \cdot \mu) \\
&= \mathbb{E}\left[\sum_{k=1}^{N} \sigma_k^2 L_k(s_k)\right] + \mathrm{Cov}\left(\sum_{k=1}^{N} \mu_k L_k(t_k), \sum_{i=1}^{N} \mu_i L_i(s_i)\right) \\
&= \sum_{k=1}^{N} \sigma_k^2 \mathbb{E} L_k(s_k) + \sum_{k=1}^{N}\sum_{i=1}^{N} \mu_k \mu_i \,\mathrm{Cov}\big(L_k(t_k), L_i(s_i)\big) \\
&= \sum_{k=1}^{N} \sigma_k^2 \mathbb{E} L_k(s_k) + \sum_{k=1}^{N} \mu_k^2 \,\mathrm{Cov}\big(L_k(t_k), L_k(s_k)\big),
\end{aligned}$$

where in the last step we used independence between $L_i$ and $L_k$ when $i \neq k$. Putting $s = t$, we have

$$\mathbb{V}ar X_{L_t} = \sum_{k=1}^{N} \sigma_k^2 \mathbb{E} L_k(t_k) + \sum_{k=1}^{N} \mu_k^2 \, \mathbb{V}ar L_k(t_k).$$

By self-similarity of the inverse stable subordinator (consult e.g., Proposition 3.1 in [31]), we have $L_k(t_k) \overset{d}{=} t_k^\alpha L_k(1)$. Hence,

$$\mathbb{E} L_k(t_k) = t_k^\alpha \, \mathbb{E} L_k(1) \qquad \mathbb{V}\mathrm{ar} L_k(t_k) = t_k^{2\alpha} \, \mathbb{V}\mathrm{ar} L_k(1).$$

Thus, by using the notation $t^\beta := \left(t_1^\beta, \dots, t_N^\beta\right)$, we can write

$$\mathbb{V}\mathrm{ar} X_{L_t} = w \cdot t^\alpha + v \cdot t^{2\alpha},$$

where we defined $w_k = \sigma_k^2 \, \mathbb{E} L_k(1)$ and $v_k = \mu_k^2 \, \mathbb{V}\mathrm{ar} L_k(1)$.

Moreover, by using Formula 10 in [23], we have

$$\mathrm{Cov}\big(L_k(t_k), L_k(s_k)\big) \sim \frac{s_k^{2\alpha}}{\Gamma(2\alpha + 1)} \qquad \text{as } t_k \to \infty.$$

In summary, for $|t| \to \infty$, we have

$$\rho(X_{L_s}, X_{L_t}) \sim \begin{cases} \frac{1}{|t^\alpha|^{1/2}} & \text{if } \mu = 0 \\ \frac{1}{|t^{2\alpha}|^{1/2}} & \text{if } \mu \neq 0. \end{cases} \tag{4.18}$$

***Remark 4.8*** What we found in (4.18) is the multiparameter extension of the known formula holding in the $N = 1$ case, see e.g., Example 3.2 in [23]. Here, the authors considered the subordinated process $(X_{L(t)})_{t \in \mathbb{R}_+}$, where $(X_t)_{t \in \mathbb{R}_+}$ is a Lévy process and $(L(t))_{t \in \mathbb{R}_+}$ is the inverse of a $\alpha$-stable subordinator, with $\alpha \in (0, 1)$. By considering two times $s$ and $t$, such that $s < t$, and letting $t \to \infty$, they show that the autocorrelation $\rho(X_{L(t)}, X_{L(s)})$ behaves like $t^{-\alpha}$ if $\mathbb{E} X_1 \neq 0$ and $t^{-\frac{\alpha}{2}}$ if $\mathbb{E} X_1 = 0$. It is interesting to note that the same power law behavior is observed in the corresponding discrete-time models (see Proposition 4 in [37]).

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## Declarations

**Conflict of interest** The authors have no conflict of interest or other interests that might be perceived to influence the results and discussion presented in this paper.



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