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No-Arbitrage Pricing with α-DS Mixtures in a Market with Bid-Ask Spreads

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Abstract

This paper introduces α-DS mixtures, which are normalized capacities that can be represented (generally not in a unique way) as the α-mixture of a belief function and its dual plausibility function. Assuming a finite state space, such capacities extend to a Choquet expectation functional that can be given a Hurwicz-like expression. In turn, α-DS mixtures and their Choquet expectations appear to be particularly suitable to model prices in a market with frictions, where bid-ask prices are usually averaged taking $\alpha = \frac{1}{2}$. For this, we formulate a no-arbitrage one-period pricing problem in the framework of α-DS mixtures and prove the analogues of the first and second fundamental theorems of asset pricing. Finally, we perform a calibration on market data to derive a market consistent no-arbitrage α-DS mixture pricing rule.

Keywords: α-DS mixture, no-arbitrage pricing, bid-ask spreads

1. Introduction

In the classical finite-state one-period no-arbitrage pricing theory [7, 27], uncertainty is quantified by probability measures. In detail, two probability measures appear in a no-arbitrage pricing problem:

(i) a "real-world" probability measures P, which encodes market agents' beliefs and is used to compute expected utilities;

(ii) a "risk-neutral" probability measure Q, which is an artificial measure singled out by the model to express market prices as a discounted expectation.

It turns out that the probability measure P does not play any role in pricing as no-arbitrage acts like a normative principle to derive a preference-free linear pricing rule. Indeed, P only fixes the set of plausible states of the world that form the entire Ω, as P is assumed to be strictly positive on the singletons. We have that the two probability measures are completely detached as they only share the property of being strictly positive on the singletons. In particular, the strict positivity assumption on Q has the purpose to assure that contracts with non-negative and non-null future payoff have strictly positive present price.

The "risk-neutral" probability Q is determined by market prices through the no-arbitrage principle, under some assumptions on the market that enforce linearity: the market is taken to be *competitive* and *frictionless*. Both assumptions are unrealistic but, in particular, as discussed in [1, 2], real markets show the presence of frictions, mainly in the form of bid-ask spreads, so, to model frictions one needs necessarily to give up on the linearity of the pricing rule.

To face the issue of frictions in the market, several papers tried to remove the linearity of the pricing rule, usually focusing either on bid prices or ask prices (see, e.g., [3, 8, 9, 10, 11]). Nevertheless, the standard approach in finance to deal with bid-ask prices and stick to linearity is to consider their $\frac{1}{2}$-mixture (see, e.g., [21]). Inspired by this last approach, we introduce α-DS mixtures which are normalized capacities that can be represented (generally not in a unique way) as the α-mixture of a belief function and its dual plausibility function. The parameter α acts like a *pessimism index* and permits to accommodate in a single class belief/plausibility functions, necessity/possibility measures, probability measures and their mixtures. Moreover, the Choquet expectation with respect to an α-DS mixture turns out to have a Hurwicz-like expression (see, e.g., [16, 18]).

The class of α-DS mixtures and their Choquet expectations reveals to be particularly suitable to model α-mixtures of bid-ask prices (without any request of linearity). For this, we consider no-arbitrage pricing by searching for a "risk-neutral" α-DS mixture to match market prices. All the theory we develop is based on a generalization of the classical no-arbitrage condition that rests upon the notion of *partially resolving uncertainty (PRU)* due to [17] and α-*pessimism* (namely α-*PRU* assumption). In this context we prove the analogues of the first and second fundamental theorems of asset pricing (FTAP) under α-PRU assumption. Finally, to show effectiveness of our proposal we calibrate a no-arbitrage α-DS mixture pricing model on market data.

The paper is structured as follows. In Section 2 we introduce and characterize the class of α-DS mixtures. Section 3 defines a finite-state one-period pricing problem in the context of α-DS mixtures and proves the analogues of the first and second FTAP. Next, Section 4 performs a calibration on market data. Finally, Section 5 collects our conclusions and future perspectives.

2. α-DS Mixtures

Let $\Omega = \{1, \ldots, n\}$ with $n \geq 1$ be a finite set of states of the world, and $\mathcal{P}(\Omega)$ its power set. In what follows, we denote by $\mathbb{R}^{\Omega}$ the set of all random variables and by $\mathbf{1}_A$ the *indicator* of each event $A \in \mathcal{P}(\Omega)$. To avoid cumbersome notation, we identify a with $a\mathbf{1}_{\Omega}$, for all $a \in \mathbb{R}$.

A *belief function* [12, 29] is a mapping $Bel : \mathcal{P}(\Omega) \to [0, 1]$ satisfying:

(i) $Bel(\emptyset) = 0$ and $Bel(\Omega) = 1$;

(ii) for every $k \geq 2$ and for every $A_1, \ldots, A_k \in \mathcal{P}(\Omega)$,

$$Bel\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right).$$

In particular, Bel is *additive* if *(ii)* holds as an equality, while is *minitive* if, for every $A, B \in \mathcal{P}(\Omega)$, it holds that

$$Bel(A \cap B) = \min\{Bel(A), Bel(B)\}. \tag{1}$$

As is well-known, every belief function is associated with a dual *plausibility function* $Pl : \mathcal{P}(\Omega) \to [0, 1]$, defined as $Pl(A) = 1 - Bel(A^c)$, for all $A \in \mathcal{P}(\Omega)$.

Moreover, a minitive/maxitive belief/plausibility function is called a *necessity/possibility measure* [14] (denoted by N and Π, respectively), while an additive belief/plausibility function is a *probability measure* (denoted by P). We also denote by

$$C_{Bel} = \{P : P \text{ is a probability measure}, P \geq Bel\}, \tag{2}$$

the *core* induced by Bel [15]. We recall that every belief function Bel is completely characterized by its *Möbius inverse* $\mu : \mathcal{P}(\Omega) \to [0, 1]$ that satisfies

$$\begin{array}{c} \mu(\emptyset) = 0, \quad \sum_{A \in \mathcal{P}(\Omega)} \mu(A) = 1, \\ Bel(A) = \sum_{B \subseteq A} \mu(B), \quad \text{for every } A \in \mathcal{P}(\Omega). \end{array} \tag{3}$$

Definition 1 *Let $\alpha \in [0, 1]$. A mapping $\varphi_\alpha : \mathcal{P}(\Omega) \to [0, 1]$ is called an* α**-DS mixture** *(where "DS" stands for Dempster-Shafer) if there exists a belief function $Bel : \mathcal{P}(\Omega) \to [0, 1]$ with dual plausibility function Pl such that, for all $A \in \mathcal{P}(\Omega)$,*

$$\begin{aligned} \varphi_\alpha(A) &= \alpha Bel(A) + (1 - \alpha) Pl(A) \\ &= \alpha Bel(A) + (1 - \alpha)(1 - Bel(A^c)). \end{aligned}$$

The belief function Bel is said to **represent** *the α-DS mixture φ_α.*

Every α-DS mixture is associated with a *dual* $(1 - \alpha)$-DS mixture $\varphi_{1-\alpha} : \mathcal{P}(\Omega) \to [0, 1]$ which can be represented by the same Bel of φ_α. Such function is defined, for all $A \in \mathcal{P}(\Omega)$, as

$$\begin{aligned} \varphi_{1-\alpha}(A) &= 1 - \varphi_\alpha(A^c) \\ &= 1 - [\alpha Bel(A^c) + (1 - \alpha) Pl(A^c)] \\ &= 1 - [\alpha(1 - Pl(A)) + (1 - \alpha)(1 - Bel(A))] \\ &= (1 - \alpha) Bel(A) + \alpha Pl(A). \end{aligned} \tag{4}$$

The notion of α-DS mixture can be further specialized as follows.

Definition 2 *Let $\alpha \in [0, 1]$. An α-DS mixture $\varphi_\alpha : \mathcal{P}(\Omega) \to [0, 1]$ is said to be:*

additive: *if it can be represented by a probability measure;*

consonant: *if it can be represented by a necessity measure.*

It is immediate to notice that belief functions are 1-DS mixtures, while plausibility functions are 0-DS mixtures. Analogously, necessity measures are consonant 1-DS mixtures, while possibility measures are consonant 0-DS mixtures. Moreover, probability measures turn out to be additive α-DS mixtures for every possible choice of the parameter α, and further φ_α itself is a representing belief function. We also notice that *credibility measures* according to [23] are consonant $\frac{1}{2}$-DS mixtures.

The issue of expressing a normalized capacity (i.e., a monotone and normalized set function) as an α-mixture of a pair of dual normalized capacities has been already faced in [20], in the context of α-maxmin expected utility. We refer to capacities with this property as *α-JP capacities* (where "JP" stands for Jaffray-Philippe). It turns out that α-DS mixtures are particular α-JP capacities in which we restrict to pairs of dual belief/plausibility functions.

The following proposition shows that, for a fixed $\alpha \in [0, 1]$, the belief function appearing in Definition 1 is unique when $\alpha \neq \frac{1}{2}$.

Proposition 3 *Let $\alpha \in [0, 1]$ with $\alpha \neq \frac{1}{2}$, and $\varphi_\alpha : \mathcal{P}(\Omega) \to [0, 1]$ be an α-DS mixture. Let Bel, Bel' be belief functions on $\mathcal{P}(\Omega)$. If both Bel and Bel' represent φ_α, then $Bel = Bel'$.*

Proof For all $A \in \mathcal{P}(\Omega)$, we have that

$$\begin{aligned} \varphi_\alpha(A) &= \alpha Bel(A) + (1 - \alpha)(1 - Bel(A^c)) \\ &= \alpha Bel'(A) + (1 - \alpha)(1 - Bel'(A^c)), \end{aligned}$$

and from this we get that

$$\alpha[Bel(A) - Bel'(A)] = (1 - \alpha)[Bel(A^c) - Bel'(A^c)].$$

Moreover, since the above equation must hold for all the events in $\mathcal{P}(\Omega)$, we also have that

$$\alpha[Bel(A^c) - Bel'(A^c)] = (1-\alpha)[Bel(A) - Bel'(A)].$$

Setting $x = Bel(A) - Bel'(A)$ and $y = Bel(A^c) - Bel'(A^c)$, the two equations above give rise to the homogeneous linear system

$$\begin{cases} \alpha x - (1-\alpha)y = 0, \\ (1-\alpha)x - \alpha y = 0. \end{cases}$$

The system has only the null solution for $\alpha \neq \frac{1}{2}$. So, by the arbitrariness of $A \in \mathcal{P}(\Omega)$, we get $Bel = Bel'$. ■

On the other hand, if $\alpha = \frac{1}{2}$, then the belief function representing $\varphi_{\frac{1}{2}}$ is generally not unique, as shown in the following example.

Example 1 *Let $\Omega = \{1,2\}$ and consider the additive $\frac{1}{2}$-DS mixture $\varphi_{\frac{1}{2}}$ on $\mathcal{P}(\Omega)$ such that $\varphi_{\frac{1}{2}}(\{1\}) = \varphi_{\frac{1}{2}}(\{2\}) = \frac{1}{2}$. In this case there are infinitely many belief functions representing $\varphi_{\frac{1}{2}}$. For $\beta \in \left[0, \frac{1}{2}\right]$, consider the belief function Bel_β with dual plausibility function Pl_β reported below (where i stands for $\{i\}$)*

$\mathcal{P}(\Omega)$	$\emptyset$	1	2	Ω
Bel_β	0	β	β	1
Pl_β	0	$1-\beta$	$1-\beta$	1
$\varphi_{\frac{1}{2}} = \frac{1}{2}Bel_\beta + \frac{1}{2}Pl_\beta$	0	$\frac{1}{2}$	$\frac{1}{2}$	1

♦

The non-unique representation of $\frac{1}{2}$-DS mixtures is not surprising since the non-unique representation of $\frac{1}{2}$-JP capacities has been already discussed in [20]. Nevertheless, Example 1 shows that the same continues to hold even if we restrict to pairs of dual belief/plausibility functions.

Incidentally, the previous example shows that an additive $\frac{1}{2}$-DS mixture can be represented also by a non-additive belief function. The following example shows that also consonant $\frac{1}{2}$-DS mixtures (i.e., credibility measures according to [23]) can be represented by non-minitive belief functions.

Example 2 *Let $\Omega = \{1,2,3\}$ and consider the consonant $\frac{1}{2}$-DS mixture $\varphi_{\frac{1}{2}}$ below (where i and ij stand for $\{i\}$ and $\{i,j\}$)*

$\mathcal{P}(\Omega)$	$\emptyset$	1	2	3	12	13	23	Ω
N	0	0	0	$\frac{1}{2}$	0	$\frac{3}{4}$	$\frac{1}{2}$	1
Π	0	$\frac{1}{2}$	$\frac{1}{4}$	1	$\frac{1}{2}$	1	1	1
$\varphi_{\frac{1}{2}}$	0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{3}{4}$	1

We have that $\varphi_{\frac{1}{2}}$ can be represented also by the non-minitive belief function below

$\mathcal{P}(\Omega)$	$\emptyset$	1	2	3	12	13	23	Ω
Bel	0	$\frac{1}{8}$	0	$\frac{5}{8}$	$\frac{1}{8}$	$\frac{6}{8}$	$\frac{5}{8}$	1
Pl	0	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{7}{8}$	$\frac{3}{8}$	1	$\frac{7}{8}$	1
$\varphi_{\frac{1}{2}}$	0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{3}{4}$	1

♦

The following proposition states that α-DS mixtures are particular normalized capacities.

Proposition 4 *Let $\alpha \in [0,1]$. An α-DS mixture $\varphi_\alpha : \mathcal{P}(\Omega) \to [0,1]$ satisfies the following properties:*

(i) $\varphi_\alpha(\emptyset) = 0$ and $\varphi_\alpha(\Omega) = 1$;

(ii) $\varphi_\alpha(A) \leq \varphi_\alpha(B)$, when $A \subseteq B$ and $A, B \in \mathcal{P}(\Omega)$;

(iii) φ_α is self-dual if and only if it is additive or $\alpha = \frac{1}{2}$;

(iv) φ_α is sub-additive if it is additive or $\alpha \in \left[0, \frac{1}{2}\right]$.

Proof Statements *(i)–(iii)* are immediate to prove. Statement *(iv)* is trivial when φ_α is additive. In general, for $A, B \in \mathcal{P}(\Omega)$ with $A \cap B = \emptyset$ and $\alpha \in \left[0, \frac{1}{2}\right]$, by Definition 1 and (3) we get that $\varphi_\alpha(A \cup B) - \varphi_\alpha(A) - \varphi_\alpha(B)$ is equal to

$$-(1-2\alpha) \sum_{\substack{C \subseteq A \cup B \\ C \not\subseteq A \\ C \not\subseteq B}} \mu(C) - (1-\alpha) \sum_{\substack{C \cap A \neq \emptyset \\ C \cap B \neq \emptyset}} \mu(C) \leq 0.$$

■

The following example shows that sub-additivity may fail for an α-DS mixture when $\alpha > \frac{1}{2}$.

Example 3 *Let $\Omega = \{1,2,3\}$, $\alpha = \frac{6}{10}$, and consider the consonant α-DS mixture φ_α below (where i and ij stand for $\{i\}$ and $\{i,j\}$)*

$\mathcal{P}(\Omega)$	$\emptyset$	1	2	3	12	13	23	Ω
N	0	0	0	0	1	0	0	1
Π	0	1	1	0	1	1	1	1
φ_α	0	$\frac{4}{10}$	$\frac{4}{10}$	0	1	$\frac{4}{10}$	$\frac{4}{10}$	1

The function φ_α is not sub-additive since

$$\varphi_\alpha(12) = 1 > \frac{4}{10} + \frac{4}{10} = \varphi_\alpha(1) + \varphi_\alpha(2).$$

♦

For $\alpha \in [0,1]$, denote by $\mathbf{M}_\alpha$ the set of all α-DS mixtures on $\mathcal{P}(\Omega)$ and by $\mathbf{P}$ the set of all probability measures on $\mathcal{P}(\Omega)$.

Proposition 5 *The following statements hold:*

(i) $\mathbf{P} \subseteq \mathbf{M}_\alpha$, for every $\alpha \in [0,1]$;

(ii) $\mathbf{M}_\alpha$ *is convex, for every* $\alpha \in [0,1]$.

Proof Statement *(i)* is trivial. Finally, to prove statement *(ii)* let $\varphi_\alpha, \varphi'_\alpha \in \mathbf{M}_\alpha$ be represented, respectively, by Bel, Bel', and take $\beta \in [0,1]$. Then, considering pointwise operations and equalities on $\mathcal{P}(\Omega)$, it holds that

$$\begin{aligned}
&\beta\varphi_\alpha + (1-\beta)\varphi'_\alpha \\
&\quad = \beta[\alpha Bel + (1-\alpha)Pl] + (1-\beta)[\alpha Bel' + (1-\alpha)Pl'] \\
&\quad = \alpha[\beta Bel + (1-\beta)Bel'] + (1-\alpha)[\beta Pl + (1-\beta)Pl'] \\
&\quad = \alpha Bel'' + (1-\alpha)Pl'',
\end{aligned}$$

where $Bel'' = \beta Bel + (1-\beta)Bel'$ and $Pl'' = \beta Pl + (1-\beta)Pl'$ are dual belief and plausibility functions. ∎

Every α-DS mixture φ_α uniquely extends to a functional $\mathbb{C}_{\varphi_\alpha} : \mathbb{R}^\Omega \to \mathbb{R}$ by setting, for every $X \in \mathbb{R}^\Omega$,

$$\mathbb{C}_{\varphi_\alpha}[X] = \oint X \,\mathrm{d}\varphi_\alpha, \tag{5}$$

where the integral on the right is of Choquet type [15].

Proposition 6 *Let* $\alpha \in [0,1]$ *and* Bel *be a belief function representing the* α*-DS mixture* φ_α*. For every* $X \in \mathbb{R}^\Omega$ *it holds that*

$$\mathbb{C}_{\varphi_\alpha}[X] = \alpha \min_{P \in C_{Bel}} \mathbb{E}_P[X] + (1-\alpha) \max_{P \in C_{Bel}} \mathbb{E}_P[X].$$

Proof Denote by $\mathbb{C}_{Bel}$ and $\mathbb{C}_{Pl}$ the Choquet integrals with respect to Bel and its dual plausibility function Pl. For every $X \in \mathbb{R}^\Omega$, the linearity of the Choquet integral with respect to the integrating capacity and the lower/upper expectation representation of $\mathbb{C}_{Bel}$ and $\mathbb{C}_{Pl}$ in terms of C_{Bel} (see [15, 28]) imply

$$\begin{aligned}
\mathbb{C}_{\varphi_\alpha}[X] &= \alpha\mathbb{C}_{Bel}[X] + (1-\alpha)\mathbb{C}_{Pl}[X] \\
&= \alpha \min_{P \in C_{Bel}} \mathbb{E}_P[X] + (1-\alpha) \max_{P \in C_{Bel}} \mathbb{E}_P[X].
\end{aligned}$$

∎

We notice that the above expression of $\mathbb{C}_{\varphi_\alpha}$ holds for every Bel representing it, so, all possible representations turn out to be equivalent.

Let $\mathcal{U} = \mathcal{P}(\Omega) \setminus \{\emptyset\}$. For a fixed $\alpha \in [0,1]$ and every $X \in \mathbb{R}^\Omega$, we call *α-DS mixture variable* the function $[\![X]\!]^\alpha : \mathcal{U} \to \mathbb{R}$ defined, for all $B \in \mathcal{U}$, as

$$[\![X]\!]^\alpha(B) = \alpha \min_{i \in B} X(i) + (1-\alpha) \max_{i \in B} X(i). \tag{6}$$

Proposition 7 *Let* $\alpha \in [0,1]$*. For every* α*-DS mixture* φ_α *on* $\mathcal{P}(\Omega)$ *there exists a (non-necessarily unique) function* $\mu : \mathcal{P}(\Omega) \to [0,1]$ *such that:*

(i) $\mu(\emptyset) = 0$ *and* $\sum_{B \in \mathcal{P}(\Omega)} \mu(B) = 1$*;*

(ii) for every $X \in \mathbb{R}^\Omega$*, it holds that*

$$\mathbb{C}_{\varphi_\alpha}[X] = \sum_{B \in \mathcal{U}} [\![X]\!]^\alpha(B)\mu(B).$$

Proof Let Bel be a belief function representing φ_α and μ its Möbius inverse, that is known to satisfy *(i)*. To prove *(ii)*, denote by $\mathbb{C}_{Bel}$ and $\mathbb{C}_{Pl}$ the Choquet integrals with respect to Bel and its dual plausibility function Pl. For every $X \in \mathbb{R}^\Omega$, the linearity of the Choquet integral with respect to the integrating capacity and the representation of $\mathbb{C}_{Bel}$ and $\mathbb{C}_{Pl}$ in terms of μ (see [15]) imply

$$\begin{aligned}
\mathbb{C}_{\varphi_\alpha}[X] &= \alpha\mathbb{C}_{Bel}[X] + (1-\alpha)\mathbb{C}_{Pl}[X] \\
&= \alpha \sum_{B \in \mathcal{U}} \left(\min_{i \in B} X(i)\right)\mu(B) \\
&\qquad + (1+\alpha) \sum_{B \in \mathcal{U}} \left(\max_{i \in B} X(i)\right)\mu(B) \\
&= \sum_{B \in \mathcal{U}} [\![X]\!]^\alpha(B)\mu(B).
\end{aligned}$$

∎

Propositions 6 and 7 show that $\mathbb{C}_{\varphi_\alpha}$ can be given a Hurwicz-like expression [16, 18, 19] where α acts like a *pessimism index* (see also [13]). The functional $\mathbb{C}_{\varphi_\alpha}$ turns out to be a particular instance of the objective ambiguity representation functional given in [26] for a finite setting, where the pessimism index is a function $\alpha : \mathcal{U} \to [0,1]$. In particular, the case $\alpha \equiv 1$ has been considered in [25].

3. No-Arbitrage with α-DS Mixtures

We consider a finite-state one-period financial market related to times $t = 0$ and $t = 1$, that presents frictions in the form of bid-ask spreads.

In such market there is a risk-free bond, which has a special role as it is used as numéraire. Such a bond is assumed to be frictionless and has price $S_0^0 = 1$ at time $t = 0$ and payoff $S_1^0 = 1 + r > 0$ at time $t = 1$.

In the market there are also m non-dividend paying stocks that have bid-ask prices $(\underline{S}_0^k, \overline{S}_0^k)$ at time $t = 0$, with $\underline{S}_0^k \leq \overline{S}_0^k$ and $\underline{S}_0^k, \overline{S}_0^k \in \mathbb{R}_{>0}$, and non-null payoff $S_1^k \in \mathbb{R}^\Omega_{\geq 0}$, for $k = 1, \dots, m$. Strict positivity of bid-ask prices is justified by the limited liability assumption on stocks. For a fixed $\alpha \in [0,1]$, we introduce the α-mixture prices $S_0^k = \alpha\underline{S}_0^k + (1-\alpha)\overline{S}_0^k$, for $k = 1, \dots, m$. Thus the market can be described by the vector of α-mixture prices $M_0 = (S_0^0, S_0^1, \dots, S_0^m)^T$ at time $t = 0$, and the vector of payoffs $M_1 = (S_1^0, S_1^1, \dots, S_1^m)^T$ at time $t = 1$.

A *portfolio* is a vector $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1}$ where λ_0 and λ_k are the numbers of units of bond and of k-th stock to buy/short sell.

According to Jaffray [17] we consider random payoffs at time $t = 1$ assuming *partially resolving uncertainty (PRU)*. Adopting such principle, we allow that an agent may only acquire the information that an event $B \neq \emptyset$ occurs, without knowing which is the true state of the world $i \in B$. In turn, this translates in considering payoffs of portfolios on every event $B \neq \emptyset$. This is in contrast to the usual *completely resolving uncertainty* assumption according to which the agent will always acquire which is the true state of the world in Ω. We further assume that the agent is *α-pessimistic*, that is he/she always considers the α-mixture between the minimum an the maximum of random payoffs on every $B \neq \emptyset$, where $\alpha \in [0, 1]$ is a constant pessimism index. The combination of partially resolving uncertainty and α-pessimism is referred to as *α-PRU* assumption in what follows.

Given a portfolio $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1}$ we define:

Price at time $t = 0$:

$$V_0^{\boldsymbol{\lambda}} = \lambda_0 + \sum_{k=1}^{m} \lambda_k S_0^k.$$

Payoff under α-PRU at time $t = 1$:

$$V_1^{\boldsymbol{\lambda}} = \lambda_0(1 + r) + \sum_{k=1}^{m} \lambda_k [\![S_1^k]\!]^{\alpha}.$$

Let us notice that $V_0^{\boldsymbol{\lambda}}$ is a number while $V_1^{\boldsymbol{\lambda}}$ is a real-valued function defined on $\mathcal{U}$.

Introducing the discounted payoffs $\tilde{S}_1^k = (1 + r)^{-1} S_1^k$, for $k = 0, \ldots, m$, the discounted payoff of the portfolio under α-PRU is

$$\tilde{V}_1^{\boldsymbol{\lambda}} = \lambda_0 + \sum_{k=1}^{m} \lambda_k [\![\tilde{S}_1^k]\!]^{\alpha}.$$

Theorem 8 (No-Dutch book under α-PRU) *Let $\alpha \in [0, 1]$. The following conditions are equivalent:*

(i) there exists an α-DS mixture $\widehat{\varphi_\alpha}$ such that $\mathbb{C}_{\widehat{\varphi_\alpha}}[\tilde{S}_1^k] = S_0^k$, for $k = 1, \ldots, m$;

(ii) for every $\boldsymbol{\lambda} \in \mathbb{R}^{m+1}$ it holds that

$$\min_{B \in \mathcal{U}} (\tilde{V}_1^{\boldsymbol{\lambda}}(B) - V_0^{\boldsymbol{\lambda}}) \leq 0 \leq \max_{B \in \mathcal{U}} (\tilde{V}_1^{\boldsymbol{\lambda}}(B) - V_0^{\boldsymbol{\lambda}}).$$

Proof Fix an enumeration of $\mathcal{U} = \{B_1, \ldots, B_{2^n-1}\}$. Condition *(i)* is equivalent to the solvability of the following system

$$\begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}, \end{cases}$$

where $\mathbf{x} = (\widehat{\mu}(B_1), \ldots, \widehat{\mu}(B_{2^n-1}))^T \in \mathbb{R}^{2^n-1}$ is an unknown column vector, $\mathbf{A} \in \mathbb{R}^{(m+1)\times(2^n-1)}$ is the coefficient matrix with

$$\mathbf{A} = \begin{pmatrix} [\![\mathbf{1}_\Omega]\!]^{\alpha}(B_1) & \cdots & [\![\mathbf{1}_\Omega]\!]^{\alpha}(B_{2^n-1}) \\ [\![\tilde{S}_1^1]\!]^{\alpha}(B_1) & \cdots & [\![\tilde{S}_1^1]\!]^{\alpha}(B_{2^n-1}) \\ \vdots & & \vdots \\ [\![\tilde{S}_1^m]\!]^{\alpha}(B_1) & \cdots & [\![\tilde{S}_1^m]\!]^{\alpha}(B_{2^n-1}) \end{pmatrix},$$

and $\mathbf{b} = (S_0^0, S_0^1, \ldots, S_0^m)^T \in \mathbb{R}^{m+1}$.

By Farkas' lemma [24], the system above is compatible if and only if the following system is not compatible

$$\begin{cases} \mathbf{A}^T \mathbf{y} \leq \mathbf{0}, \\ \mathbf{b}^T \mathbf{y} > 0, \end{cases}$$

where $\mathbf{y} = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1}$ is an unknown column vector. It holds that $\mathbf{A}^T \mathbf{y} \in \mathbb{R}^{2^n-1}$ and, for $i = 1, \ldots, 2^n - 1$, the i-th component of constraint $\mathbf{A}^T \mathbf{y} \leq \mathbf{0}$ is

$$\lambda_0 + \sum_{k=1}^{m} \lambda_k [\![\tilde{S}_1^k]\!]^{\alpha}(B_i) \leq 0,$$

moreover, subtracting the strictly positive quantity $\mathbf{b}^T \mathbf{y} = V_0^{\boldsymbol{\lambda}}$ we get

$$\sum_{k=1}^{m} \lambda_k \left([\![\tilde{S}_1^k]\!]^{\alpha}(B_i) - S_0^k \right) < 0.$$

Thus, condition *(i)* is equivalent to the existence of $i \in \{1, \ldots, 2^n - 1\}$ such that the above inequality does not hold, which, in turn, is equivalent to *(ii)*. ■

The previous theorem provides a necessary and sufficient condition to the existence of an α-DS mixture whose discounted Choquet expectation agrees with the α-mixtures of market bid-ask prices. Nevertheless, no particular property is asked to $\widehat{\varphi_\alpha}$. Nevertheless, in agreement with the limited liability assumption for stocks, a desideratum in finance is that a contract with a non-negative and non-null payoff should have positive bid-ask prices. In turn, if α-mixture market prices have to be consistent with the discounted Choquet expectation with respect to an α-DS mixture $\widehat{\varphi_\alpha}$, then $\widehat{\varphi_\alpha}$ must be represented by a belief function, which is strictly positive on $\mathcal{U}$.

The next theorem is a version of the *first fundamental theorem of asset pricing (FTAP)* in the framework of α-DS framework.

Theorem 9 (First FTAP under α-PRU) *Let $\alpha \in [0, 1]$. The following conditions are equivalent:*

(i) there exists an α-DS mixture $\widehat{\varphi_\alpha}$ represented by a belief function strictly positive on $\mathcal{U}$ and such that $\mathbb{C}_{\widehat{\varphi_\alpha}}[\tilde{S}_1^k] = S_0^k$, for $k = 1, \ldots, m$;

(ii) *for every* $\boldsymbol{\lambda} \in \mathbb{R}^{m+1}$ *none of the following conditions holds:*

(a) $V_1^{\lambda}(\{i\}) = 0$, *for* $i = 1, \ldots, n$, $V_1^{\lambda}(B) \geq 0$, *for all* $B \in \mathcal{U} \setminus \{\{i\} : i \in \Omega\}$ *and* $V_0^{\lambda} < 0$;

(b) $V_1^{\lambda}(\{i\}) \geq 0$, *for* $i = 1, \ldots, n$, *with at least a strict inequality,* $V_1^{\lambda}(B) \geq 0$, *for all* $B \in \mathcal{U} \setminus \{\{i\} : i \in \Omega\}$, *and* $V_0^{\lambda} \leq 0$.

Proof Statement *(i)* is equivalent to the existence of a function $\widehat{\mu} : \mathcal{P}(\Omega) \to [0,1]$ that satisfies (3), $\widehat{\mu}(\{i\}) > 0$, for all $i \in \Omega$, and

$$\mathbb{C}_{\widehat{\varphi_\alpha}}[\tilde{S}_1^k] = \sum_{B \in \mathcal{U}} [\![\tilde{S}_1^k]\!]^{\alpha}(B)\widehat{\mu}(B) = S_0^k, \quad \text{for } k = 1, \ldots, m.$$

Fix an enumeration of $\mathcal{U} = \{B_1, \ldots, B_{2^n-1}\}$ such that $B_i = \{i\}$, for $i = 1, \ldots, n$, and consider the matrices $\mathbf{A} \in \mathbb{R}^{(2(m+1)+2^n-(n+1)) \times (2^n-1)}$ and $\mathbf{B} \in \mathbb{R}^{n \times (2^n-1)}$ defined as

$$\mathbf{A} = \left(\frac{\mathbf{C}}{\mathbf{O}_1 | -\mathbf{I}_{(2^n-(n+1))}} \right) \quad \text{and} \quad \mathbf{B} = (-\mathbf{I}_n | \mathbf{O}_2),$$

where $\mathbf{C} \in \mathbb{R}^{2(m+1) \times (2^n-1)}$ is defined as

$$\mathbf{C} = \begin{pmatrix} [\![\mathbf{1}_\Omega]\!]^{\alpha}(B_1) & \cdots & [\![\mathbf{1}_\Omega]\!]^{\alpha}(B_{2^n-1}) \\ -[\![\mathbf{1}_\Omega]\!]^{\alpha}(B_1) & \cdots & -[\![\mathbf{1}_\Omega]\!]^{\alpha}(B_{2^n-1}) \\ [\![\tilde{S}_1^1]\!]^{\alpha}(B_1) & \cdots & [\![\tilde{S}_1^1]\!]^{\alpha}(B_{2^n-1}) \\ -[\![\tilde{S}_1^1]\!]^{\alpha}(B_1) & \cdots & -[\![\tilde{S}_1^1]\!]^{\alpha}(B_{2^n-1}) \\ \vdots & & \vdots \\ [\![\tilde{S}_1^m]\!]^{\alpha}(B_1) & \cdots & [\![\tilde{S}_1^m]\!]^{\alpha}(B_{2^n-1}) \\ -[\![\tilde{S}_1^m]\!]^{\alpha}(B_1) & \cdots & -[\![\tilde{S}_1^m]\!]^{\alpha}(B_{2^n-1}) \end{pmatrix},$$

in which $\mathbf{I}_{(2^n-(n+1))} \in \mathbb{R}^{(2^n-(n+1)) \times (2^n-(n+1))}$ and $\mathbf{I}_n \in \mathbb{R}^{n \times n}$ are identity matrices, and $\mathbf{O}_1 \in \mathbb{R}^{(2^n-(n+1)) \times n}$ and $\mathbf{O}_2 \in \mathbb{R}^{n \times (2^n-(n+1))}$ are null matrices. Take the vector

$$\mathbf{b} = (1, -1, S_0^1, -S_0^1, \ldots, S_0^m, -S_0^m, 0, \ldots, 0)^T$$

with $\mathbf{b} \in \mathbb{R}^{(2(m+1)+2^n-(n+1))}$ and consider the unknown vector

$$\mathbf{x} = (\widehat{\mu}(B_1), \ldots, \widehat{\mu}(B_{2^n-1}))^T$$

with $\mathbf{x} \in \mathbb{R}^{2^n-1}$. Condition *(i)* turns out to be equivalent to the solvability of the following system

$$\begin{cases} \mathbf{Ax} \leq \mathbf{b}, \\ \mathbf{Bx} < \mathbf{0}. \end{cases}$$

By a well-known version of Motzkin's theorem of the alternative (see, e.g., Theorem 1 in [5]) the above system is solvable if and only if, for every $\mathbf{y} = (y_0, y_0', y_1, y_1', \ldots, y_m, y_m', \alpha_{n+1}, \ldots, \alpha_{2^n-1})^T \in \mathbb{R}^{(2(m+1)+2^n-(n+1))}$ and $\mathbf{z} = (z_1, \ldots, z_n)^T \in \mathbb{R}^n$ with $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{z} \geq \mathbf{0}$, none of the following conditions holds:

- $\mathbf{A}^T\mathbf{y} + \mathbf{B}^T\mathbf{z} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} < 0$;
- $\mathbf{A}^T\mathbf{y} + \mathbf{B}^T\mathbf{z} = \mathbf{0}$, $\mathbf{z} \neq \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} \leq 0$.

In turn, setting $\lambda_k = y_k - y_k'$, for $k = 0, \ldots, m$, and considering $\bar{\mathbf{y}} \in \mathbb{R}^{((m+1)+2^n-(n+1))}$, $\bar{\mathbf{A}} \in \mathbb{R}^{((m+1)+2^n-(n+1)) \times (2^n-1)}$ and $\bar{\mathbf{b}} \in \mathbb{R}^{((m+1)+2^n-(n+1))}$, with

$$\bar{\mathbf{y}} = (\lambda_0, \ldots, \lambda_m, \alpha_{n+1}, \ldots, \alpha_{2^n-1})^T \quad \text{such that } \alpha_j \geq 0,$$

$$\bar{\mathbf{A}} = \left(\frac{\bar{\mathbf{C}}}{\mathbf{O}_1 | -\mathbf{I}_{(2^n-(n+1))}} \right) \quad \text{and}$$

$$\bar{\mathbf{b}} = (1, S_0^1, \ldots, S_0^m, 0, \ldots, 0)^T,$$

where $\bar{\mathbf{C}} \in \mathbb{R}^{(m+1) \times (2^n-1)}$ is defined as

$$\bar{\mathbf{C}} = \begin{pmatrix} [\![\mathbf{1}_\Omega]\!]^{\alpha}(B_1) & \cdots & [\![\mathbf{1}_\Omega]\!]^{\alpha}(B_{2^n-1}) \\ [\![\tilde{S}_1^1]\!]^{\alpha}(B_1) & \cdots & [\![\tilde{S}_1^1]\!]^{\alpha}(B_{2^n-1}) \\ \vdots & & \vdots \\ [\![\tilde{S}_1^m]\!]^{\alpha}(B_1) & \cdots & [\![\tilde{S}_1^m]\!]^{\alpha}(B_{2^n-1}) \end{pmatrix},$$

the above conditions can be rewritten as:

- $\bar{\mathbf{A}}^T\bar{\mathbf{y}} + \mathbf{B}^T\mathbf{z} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$ and $\bar{\mathbf{b}}^T\bar{\mathbf{y}} < 0$;
- $\bar{\mathbf{A}}^T\bar{\mathbf{y}} + \mathbf{B}^T\mathbf{z} = \mathbf{0}$, $\mathbf{z} \neq \mathbf{0}$ and $\bar{\mathbf{b}}^T\bar{\mathbf{y}} \leq 0$.

Denoting $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1}$, we have that

$$(\bar{\mathbf{A}}^T\bar{\mathbf{y}} + \mathbf{B}^T\mathbf{z})_i = \begin{cases} \tilde{V}_1^{\lambda}(B_i) - z_i, & \text{for } i = 1, \ldots, n, \\ \tilde{V}_1^{\lambda}(B_i) - \alpha_i, & \text{for } i = n+1, \ldots, 2^n - 1, \end{cases}$$

and further $\bar{\mathbf{b}}^T\bar{\mathbf{y}} = V_0^{\lambda}$.

Hence, for every $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^{m+1}$, since $\tilde{V}_1^{\lambda}(B) \geq 0 \iff V_1^{\lambda}(B) \geq 0$, for all $B \in \mathcal{U}$, the above conditions can be rewritten as *(a)* and *(b)* of statement *(ii)*. ■

We interpret V_0^{λ} as the hypothetical price of the portfolio $\boldsymbol{\lambda}$ as if we were in a situation of completely resolving uncertainty. In this light, conditions *(ii.a)* and *(ii.b)* of previous theorem can be interpreted as two forms of arbitrage under α-PRU. Avoiding condition *(ii.a)* assures that we cannot find a portfolio $\boldsymbol{\lambda}$ whose hypothetical price V_0^{λ} is negative (that is we are paid for it), resulting in a uniformly non-negative payoff V_1^{λ} in all the possible events in $\mathcal{U}$, with null value on the singletons (i.e., on those events where we have completely resolving uncertainty). Avoiding condition *(ii.b)* assures that we cannot find a portfolio $\boldsymbol{\lambda}$ whose hypothetical price V_0^{λ} is negative or null (that is we are paid or we do not pay anything for it), resulting in a uniformly non-negative payoff V_1^{λ} in all the possible events in $\mathcal{U}$, with at least a strictly positive value on the singletons (i.e., on those events where we have completely resolving uncertainty).

It is easy to see that the no-arbitrage principle under α-PRU expressed by statement *(ii)* of Theorem 9 implies the no-Dutch book condition under α-PRU in statement *(ii)* of Theorem 8. In particular, if a portfolio $\boldsymbol{\lambda}$ satisfies condition *(ii.a)* of Theorem 9 then it violates the no-Dutch book condition under α-PRU.

Let us stress that the no-arbitrage principle under α-PRU of Theorem 9 is actually weaker than the classical no-arbitrage principle. This is due to the fact that if a portfolio $\boldsymbol{\lambda}$ gives rise to an arbitrage under α-PRU of the form *(ii.a)* or *(ii.b)* then it also gives rise to a classical arbitrage, while a portfolio $\boldsymbol{\lambda}$ giving rise to a classical arbitrage does not generally give rise to an arbitrage under α-PRU.

We first notice that the generalized no-arbitrage condition presented in [11], corresponds to no-arbitrage under 1-PRU, that is to no-arbitrage under partially resolving uncertainty and complete pessimism. We also notice that the classical no-arbitrage principle (see, e.g., [7, 27]) is obtained by replacing the set $\mathcal{U}$ in Theorem 9, used to model partially resolving uncertainty, with the set $O = \{\{i\} : i \in \Omega\}$ that identifies a situation of completely resolving uncertainty. Under completely resolving uncertainty, α does not play any role as index of pessimism. In turn, switching to completely resolving uncertainty amounts to restrict to the class of additive α-DS mixtures (i.e., probability measures).

Example 4 *Let $\Omega = \{1,2,3\}$, $1+r = 1$, and consider the market with payoffs $M_1 = (S_1^0, S_1^1, S_1^2, S_1^3)^T$ and $\frac{1}{2}$-mixture prices $M_0 = (S_0^0, S_0^1, S_0^2, S_0^3)^T$, where*

Ω	1	2	3
S_1^0	1	1	1
S_1^1	2	1	1
S_1^2	1	2	1
S_1^3	3	3	2

and $S_0^0 = 1$, $S_0^1 = S_0^2 = \frac{11}{8}$ and $S_0^3 = \frac{21}{8}$. We first notice that, it cannot exist any additive $\frac{1}{2}$-DS mixture $\widehat{\varphi_{\frac{1}{2}}}$ consistent with the given $\frac{1}{2}$-mixture prices. Indeed, this would imply the linearity of $\mathbb{C}_{\widehat{\varphi_{\frac{1}{2}}}}$, which cannot hold since $S_1^3 = S_1^1 + S_1^2$ but $S_0^3 \neq S_0^1 + S_0^2$.

The portfolio $\boldsymbol{\lambda} = (0,-1,-1,1)^T$ gives rise to a classical arbitrage since we have

O	$\{1\}$	$\{2\}$	$\{3\}$
$V_1^{\boldsymbol{\lambda}}$	0	0	0

and $V_0^{\boldsymbol{\lambda}} = -\frac{1}{8} < 0$. That is, assuming completely resolving uncertainty, according to $\boldsymbol{\lambda}$, we are paid at time $t = 0$ to lose or gain nothing at time $t = 1$.

On the other hand, the market $\frac{1}{2}$-mixture prices in M_0 satisfy the no-arbitrage condition under $\frac{1}{2}$-PRU since taking the $\frac{1}{2}$-DS mixture such that $\widehat{\varphi_{\frac{1}{2}}}(i) = \frac{3}{8}$ and $\widehat{\varphi_{\frac{1}{2}}}(ij) = \frac{5}{8}$, we have that

$$\mathbb{C}_{\widehat{\varphi_{\frac{1}{2}}}}[S_1^k] = S_0^k, \quad \textit{for } k = 1,2,3.$$

In other terms, by Theorem 9 we cannot find a portfolio $\boldsymbol{\lambda} \in \mathbb{R}^4$ that gives rise to an arbitrage of type (ii.a) or (ii.b) under $\frac{1}{2}$-PRU. ♦

Similarly to the classical no-arbitrage theory [7, 27], an α-DS mixture $\widehat{\varphi_\alpha}$ meeting condition *(i)* of Theorem 9 can be dubbed a *"risk-neutral" α-DS mixture* since

$$\mathbb{C}_{\widehat{\varphi_\alpha}}\left[\frac{S_1^k}{S_0^k}\right] = 1 + r, \quad \text{for } k = 1,\ldots,m,$$

that is all risky assets have Choquet expected return that coincides with the risk-free return. We also notice that by Theorem 9, $\widehat{\varphi_\alpha}$ can be also called an *equivalent Choquet martingale α-DS mixture*.

Nevertheless, Theorem 9 does not assure the uniqueness of $\widehat{\varphi_\alpha}$. To reach uniqueness a suitable notion of completeness of the market must be introduced (see [7, 27] for the classical definition). Fix the enumeration $\mathcal{U} = \{B_1,\ldots,B_{2^n-1}\}$. The market is said to be *α-PRU complete* if $m \geq 2^n - 1$ and $S_1^k = \mathbf{1}_{B_k}$, for $k = 1,\ldots,2^n-1$, where payoffs $\mathbf{1}_{B_k}$'s are the *generalized Arrow-Debreu's securities*. The following theorem is a version of the *second fundamental theorem of asset pricing (FTAP)* in the framework of α-DS mixtures.

Theorem 10 (Second FTAP under α-PRU) *Let $\alpha \in [0,1]$. If the market satisfies condition (ii) of Theorem 9 and is α-PRU complete, then the α-DS mixture $\widehat{\varphi_\alpha}$ in condition (i) of Theorem 9 is unique.*

Proof By Theorem 9, there exists a belief function $\widehat{Bel}$ strictly positive on $\mathcal{U}$ that represents an α-DS mixture $\widehat{\varphi_\alpha}$ and such that $\mathbb{C}_{\widehat{\varphi_\alpha}}[\tilde{S}_1^k] = S_0^k$, for $k = 1,\ldots,m$. The α-PRU completeness of the market implies that

$$\widehat{\varphi_\alpha}(B_k) = \mathbb{C}_{\widehat{\varphi_\alpha}}[\mathbf{1}_{B_k}] = (1+r)S_0^k, \quad \text{for } k = 1,\ldots,m,$$

and since the α-mixture prices S_0^k's are fixed, we get the uniqueness of $\widehat{\varphi_\alpha}$. ■

Given $\widehat{\varphi_\alpha}$ as in condition *(i)* of Theorem 9, represented by $\widehat{Bel}$ strictly positive on $\mathcal{U}$, for every payoff $X_1 \in \mathbb{R}^\Omega$ we can define its no-arbitrage bid-ask prices as

$$\underline{X}_0 = (1+r)^{-1} \min_{Q \in \mathcal{C}_{\widehat{Bel}}} \mathbb{E}_Q[X_1], \tag{7}$$

$$\overline{X}_0 = (1+r)^{-1} \max_{Q \in \mathcal{C}_{\widehat{Bel}}} \mathbb{E}_Q[X_1]. \tag{8}$$

Therefore, the *α-DS mixture no-arbitrage price* is

$$X_0 = (1+r)^{-1}\mathbb{C}_{\widehat{\varphi_\alpha}}[X_1] = \alpha\underline{X}_0 + (1-\alpha)\overline{X}_0. \tag{9}$$

As a particular case, let us consider the European call and put options with strike price $K > 0$ on the k-th stock, whose payoff at time $t = 1$ is

$$C_1 = \max\{S_1^k - K, 0\}, \tag{10}$$

$$P_1 \quad = \quad \max\{K - S_1^k, 0\}. \tag{11}$$

It is well known that

$$C_1 - P_1 = S_1^k - K. \tag{12}$$

Since C_1 and $-P_1$ are comonotonic, by the comonotonic additivity and the translation invariance of the Choquet integral [15] we get

$$C_0 + (1+r)^{-1}\mathbb{C}_{\widehat{\varphi_\alpha}}[-P_1] = S_0^k - (1+r)^{-1}K,$$

and if $\widehat{\varphi_\alpha}$ is additive or $\alpha = \frac{1}{2}$ we get the *put-call parity relation*

$$C_0 - P_0 = S_0^k - (1+r)^{-1}K.$$

We recall that, working with bid-ask prices, different forms of the put-call parity relation may arise [8, 10]. In particular, taking $\alpha = 0$ we get the form of [8].

4. Calibration on Market Data

The aim of this section is to calibrate a "risk-neutral" α-DS mixture on market data. Data are taken from `Yahoo! Finance` and accessed in Python through the `yfinance` library [30]. Optimization tasks are performed through the `Bonmin` solver [6].

We take 2023-01-23 as our valuation date, identified with time $t = 0$, and consider $t = 1$ as 2023-02-24, so the length of the period is 32 days equal to $\frac{32}{365}$ years under ACT/ACT time convention.

Let us consider a market formed by the `META` stock and a frictionless risk-free bond that we identify with a US T-Bill maturing in 1 month, such that $1 + r = (1.0469)^{\frac{32}{365}}$. We use the daily time series of `META` closing prices for the period from 2022-01-24 to 2023-01-23 (whose plot is reported in Figure 1) to determine the range of S_1^1. The continuous range $[89, 323]$ is divided into 5 sub-intervals, each mapped to its midpoint.

Therefore, we get that S_1^1 ranges in the set $\mathcal{S}_1^1 = \{112.4, 159.2, 206.0, 252.8, 299.6\}$ and $\Omega = \{1, \ldots, 5\}$ is determined by $\mathcal{S}_1^1$.

We consider a set of call and put options on `META` stock with different strike prices and common maturity on 2023-02-24, of which we know the bid-ask prices on 2023-01-23, reported in Figure 2.

Denote by $\mathcal{K}_{call}, \mathcal{K}_{put}$, the sets of available strike prices, and by $(\underline{C}_0^K, \overline{C}_0^K)$, $(\underline{P}_0^K, \overline{P}_0^K)$ the available bid-ask prices of options. We also denote by $C_1^K = \max\{S_1^1 - K\}$ and $P_1^K = \max\{K - S_1^1\}$ the payoffs at time $t = 1$.

For a fixed $\alpha \in [0, 1]$, we associate to bid-ask prices the α-mixture prices $C_0^{K,\alpha} = \alpha\underline{C}_0^K + (1-\alpha)\overline{C}_0^K$ and $P_0^{K,\alpha} = \alpha\underline{P}_0^K + (1-\alpha)\overline{P}_0^K$.

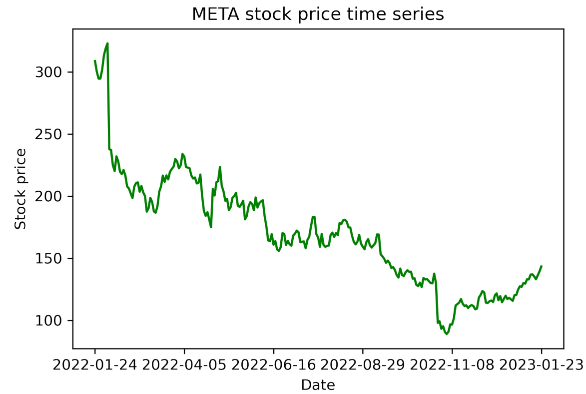

Figure 1: `META` closing prices from 2022-01-24 to 2023-01-23.

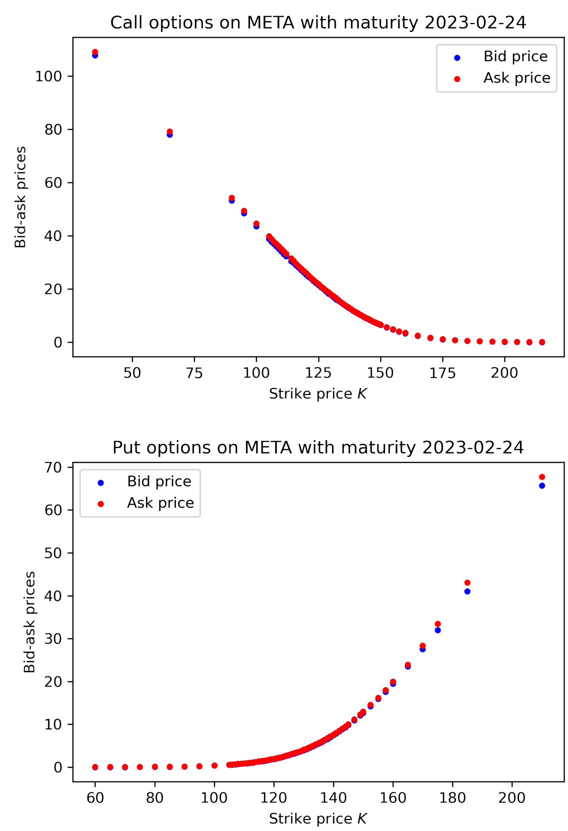

Figure 2: `META` call and put bid-ask prices

For an α-DS mixture $\widehat{\varphi_\alpha}$ let the *squared error* be defined as

$$E(\widehat{\varphi_\alpha}) = \sum_{K \in \mathcal{K}_{call}} \left(C_0^{K,\alpha} - \frac{\mathbb{C}_{\widehat{\varphi_\alpha}}[C_1^K]}{1+r} \right)^2 + \sum_{K \in \mathcal{K}_{put}} \left(P_0^{K,\alpha} - \frac{\mathbb{C}_{\widehat{\varphi_\alpha}}[P_1^K]}{1+r} \right)^2. \quad (13)$$

Our aim is to solve the optimization problem

$$\text{minimize } E(\widehat{\varphi_\alpha})$$
$$\text{subject to:}$$
$$\begin{cases} \widehat{\varphi_\alpha} \in \mathbf{M}_\alpha, \\ \widehat{\varphi_\alpha} \text{ is represented by } \widehat{Bel}, \\ \widehat{Bel}(\{i\}) \geq \epsilon, \text{ for all } i \in \Omega, \end{cases} \quad (14)$$

where $\epsilon = 0.0001$.

We initially assume that $\alpha = 0.7$, that singles out a quite pessimistic market agent, for which we have that the optimal $\widehat{\varphi_\alpha}$ is represented by the belief function $\widehat{Bel}$ with Möbius inverse such that $\widehat{\mu}(\{1\}) = 0.3168$, $\widehat{\mu}(\{1,2\}) = 0.0394$, $\widehat{\mu}(\{2\}) = 0.6425$, $\widehat{\mu}(\{3\}) = \widehat{\mu}(\{4\}) = 0.0001$, $\widehat{\mu}(\{5\}) = 0.0011$, and 0 otherwise.

Let us consider the contract with payoff $X_1 = \sqrt{S_1^1}$ at time $t = 1$. The belief $\widehat{Bel}$ gives rise to the bid-ask prices

$$\underline{X}_0 = 11.8547,$$
$$\overline{X}_0 = 11.9338,$$

so, the bid-ask spread is $\overline{X}_0 - \underline{X}_0 = 0.0791$ and the no-arbitrage α-DS mixture price is $X_0 = 11.8785$.

We point out that in this calibration scheme we did not constrain $\widehat{\varphi_\alpha}$ on the bid-ask prices of the stock $(\underline{S}_0^1, \overline{S}_0^1)$ observed on the market, but we used a large set of bid-ask prices of options on `META` instead. Indeed, we chose to use the bid-ask prices of `META` on the market as a benchmark. It turns out that, since $\underline{S}_0^1 = 143.17$ and $\overline{S}_0^1 = 143.25$, the α-mixture of bid-ask prices on the market is

$$S_0^1 = 143.1940,$$

while the no-arbitrage α-DS mixture price obtained by the calibration is

$$\frac{\mathbb{C}_{\widehat{\varphi_\alpha}}[S_1^1]}{1+r} = 142.6448,$$

which is reasonably close to S_0^1, taking into account the simplifying assumptions we made.

In the above calibration, we supposed, somehow arbitrarily, that α is fixed at 0.7. Nevertheless, also the parameter α can be tuned according to market data, so as to achieve the minimum value of $E(\widehat{\varphi_\alpha})$. To show this, we repeated the above calibration by varying α in $[0, 1]$ with a 0.1 step. Moreover, to have a clearer picture of the possible behaviors of the optimal value of $E(\widehat{\varphi_\alpha})$ seen as a function of α, we carried out three analogous single-stock calibration procedures for other two assets, namely `AMZN` and `TSLA`, besides `META`.

Figure 3 shows the optimal squared error as a function of α for `AMZN`, `META`, and `TSLA` stocks. Values are normalized in $[0, 1]$ to favor a comparison.

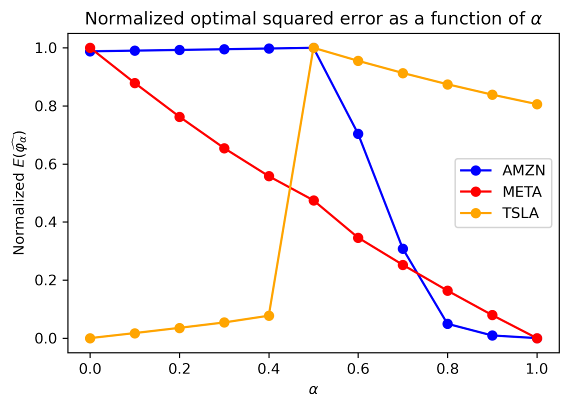

Figure 3: Normalized $E(\widehat{\varphi_\alpha})$ as a function of α.

The three stocks show quite different behaviors and in the three cases the minimum value of $E(\widehat{\varphi_\alpha})$ is achieved for extreme values of α. More in detail, `AMZN` and `META` achieve the minimum for $\alpha = 1$, that expresses a completely pessimistic attitude, while `TSLA` achieves the minimum for $\alpha = 0$, that expresses a completely optimistic attitude. The quoted difference can be attributed to market's feelings hidden in market prices. We also notice that the difference in between `AMZN`, `META`, and `TSLA` can be motivated by the different market sectors they refer to. In passing, we notice that for all the three stocks the usual choice $\alpha = 0.5$ is not optimal.

5. Conclusion

This paper introduces α-DS mixtures, which are normalized capacities that can be represented (generally not in a unique way) as the α-mixture of a belief function and its dual plausibility function. The parameter α acts like a *pessimism index* that permits to accommodate in a single class belief/plausibility functions, necessity/possibility measures, probability measures and their mixtures. In turn, the Choquet expectation functional generated by an α-DS mixture has a Hurwicz-like expression and is suitable to face pricing in a market with frictions.

As in classical no-arbitrage pricing theory, we are interested in deriving a “risk-neutral” α-DS mixture so as to obtain a preference-free pricing rule under a suitable notion of no-arbitrage. It turns out that the searched no-arbitrage principle in the framework of α-DS mixtures rests upon partially resolving uncertainty and α-pessimism (namely, α-PRU assumption). This allows us to prove the analogues of the first and second fundamental theorems of asset pricing under α-PRU. We finally perform a calibration procedure on market data showing that the parameter α can be tuned so as to reveal market pessimism hidden in market bid-ask prices.

Due to space limitations, we focused our empirical analysis on a single time period, thus an interesting development is to perform our calibration procedure referring to different time periods, where the market is bullish/bearish. Indeed, the target is to verify if the calibration of the pessimism index α can capture the different attitudes of the market. Moreover, yet in the one-period case, we aim at designing a joint calibration scheme on more risky assets.

As an aim of future research we plan to extend the α-DS mixture framework to the multi-period case. In this concern, the link to dynamic α-maxmin expected utility and dynamic consistency is worth exploring [4]. Still connected to α-maxmin expected utility, an interesting line of future research is the investigation of the collapse to the mean phenomenon, as studied in [22] for α-JP capacities.

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References

[1] Y. Amihud and H. Mendelson. Asset pricing and the bid-ask spread. *Journal of Financial Economics*, 17 (2):223–249, 1986.

[2] Y. Amihud and H. Mendelson. Liquidity, Maturity, and the Yields on U.S. Treasury Securities. *The Journal of Finance*, 46(4):1411–1425, 1991.

[3] L. Bastianello, A. Chateauneuf, and B. Cornet. Put-Call Parities, absence of arbitrage opportunities and non-linear pricing rules, 2022. URL https://arxiv.org/abs/2203.16292.

[4] P. Beissner, Q. Lin, and F. Riedel. Dynamically consistent alpha-maxmin expected utility. *Mathematical Finance*, 30(3):1073–1102, 2020.

[5] A. Ben-Israel. Motzkin's transposition theorem, and the related theorems of Farkas, Gordan and Stiemke. *Encyclopedia of Mathematics, Supplement III*, 2001.

[6] Bonmin. URL https://www.coin-or.org/Bonmin/.

[7] A. Černý. *Mathematical Techniques in Finance: Tools for Incomplete Markets*. Princeton University Press, 2nd edition, 2009.

[8] S. Cerreia-Vioglio, F. Maccheroni, and M. Marinacci. Put–Call Parity and market frictions. *Journal of Economic Theory*, 157:730–762, 2015.

[9] A. Chateauneuf and B. Cornet. Submodular financial markets with frictions. *Economic Theory*, 73(2):721–744, 2022.

[10] A. Chateauneuf, R. Kast, and A. Lapied. Choquet pricing for financial markets with frictions. *Mathematical Finance*, 6(3):323–330, 1996.

[11] A. Cinfrignini, D. Petturiti, and B. Vantaggi. Envelopes of equivalent martingale measures and a generalized no-arbitrage principle in a finite setting. *Annals of Operations Research*, 321(1):103–137, 2023.

[12] A.P. Dempster. Upper and Lower Probabilities Induced by a Multivalued Mapping. *Annals of Mathematical Statistics*, 38(2):325–339, 1967.

[13] T. Denœux. Decision-making with belief functions: A review. *International Journal of Approximate Reasoning*, 109:87–110, 2019.

[14] D. Dubois and H. Prade. *Possibility Theory: An Approach to Computerized Processing of Uncertainty*. Plenum Press, New York and London, 1988.

[15] M. Grabisch. *Set Functions, Games and Capacities in Decision Making*. Springer, Cham, 2016.

[16] L. Hurwicz. The Generalized Bayes Minimax Principle: A Criterion for Decision Making Under Uncertainty. *Cowles Commission Discussion Paper*, 355, 1951.

[17] J.-Y. Jaffray. Coherent bets under partially resolving uncertainty and belief functions. *Theory and Decision*, 26(2):99–105, 1989.

[18] J.-Y. Jaffray and M. Jeleva. Information Processing under Imprecise Risk with the Hurwicz criterion. In *ISIPTA '07: Proceeedings of the 5th International Symposium on Imprecise Probability: Theories and Applications, Prague, Czech Republic, 2007*, pages 233–244, 2007.

[19] J.-Y. Jaffray and M. Jeleva. Information processing under imprecise risk with an insurance demand illustration. *International Journal of Approximate Reasoning*, 49(1):117–129, 2008.

[20] J.-Y. Jaffray and F. Philippe. On the Existence of Subjective Upper and Lower Probabilities. *Mathematics of Operations Research*, 22(1):165–185, 1997.

[21] H.E. Leland. Option Pricing and Replication with Transactions Costs. *The Journal of Finance*, 40(5): 1283–1301, 1985.

[22] F.-B. Liebrich and C. Munari. Law-Invariant Functionals that Collapse to the Mean: Beyond Convexity. *Mathematics and Financial Economics*, 16(3):447–480, 2022.

[23] B. Liu and Y.K. Liu. Expected value of fuzzy variable and fuzzy expected value models. *IEEE Transactions on Fuzzy Systems*, 10(4):445–450, 2002.

[24] O.L. Mangasarian. *Nonlinear Programming*, volume 10 of *Classics in Applied Mathematics*. SIAM, 1994.

[25] D. Petturiti and B. Vantaggi. Modeling agent's conditional preferences under objective ambiguity in Dempster-Shafer theory. *International Journal of Approximate Reasoning*, 119:151–176, 2020.

[26] D. Petturiti and B. Vantaggi. Conditional decisions under objective and subjective ambiguity in Dempster-Shafer theory. *Fuzzy Sets and Systems*, 447:155–181, 2022.

[27] S.R. Pliska. *Introduction to Mathematical Finance: Discrete Time Models*. Blackwell Publishing, Oxford, 1997.

[28] D. Schmeidler. Integral representation without additivity. *Proceedings of the American Mathematical Society*, 97(2):255–261, 1986.

[29] G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, NJ, 1976.

[30] yfinance. URL https://pypi.org/project/yfinance/.