Analytic formulation for J2 perturbed orbits

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Keywords: Orbit Propagation, Orbit Perturbations, J2 Effects, Formation Flying

Abstract. The paper deals with a technique developed along the years at the Scuola di Ingegneria Aerospaziale to provide an exact solution for J2 perturbed orbits, here applied to spacecraft formations. Analytic solutions are useful in the design phase and can help in operations to identify and to efficiently maintain a suitable configuration. The approach is based on the elaboration, conveniently performed by means of a symbolic software tool, of a set of equations analogous to the Lagrange planetary relations. Resulting parameters are expressed through Fourier series depending only on the initial conditions. Comparison with standard, longer to obtain and less accurate numerical propagation clarify the advantage of the technique, which is limited only by the number of terms taken into account in the expansion.

Introduction

Numerical propagation of orbits gained widespread acceptance due to the availability of large computation resources and to the possibility to include the effects of all perturbations. However, analytic formulations – when available - offer an exact and really fast solution and helps in the understanding of the problem, with obvious advantages in design. It is well known that Keplerian trajectories can be expressed as an expansion of terms, providing an analytical solution, even if practically limited by the number of terms taken into account. Taylor expansions in powers of the time or of the eccentricity and Fourier expansion in terms of the anomaly are possible, with a bound on eccentricity values in order to ensure convergence [1, 2]. It is interesting to similarly act for real orbits, where perturbations have to be considered. There is a large interval of orbital altitudes, between 600 and 900 km, where – for standard spacecraft, i.e. the ones missing extremely large appendages – the dominant perturbation is the one due to the aspherical gravitational potential of the Earth. Furthermore, the second harmonic of the Earth potential, the one representing the oblate or polar-flattened Earth and shortly indicated as J2, is definitely the most relevant term, so that the analysis can be conveniently limited to it. Interestingly, this interval of altitudes is highly important for Earth observation missions. In such a frame, an analytic solution – with obvious advantages with respect to numerical propagation in terms of time and accuracy – can be of significant interest. The present study is inspired to the original approach by Broglio [3], and has been step-by-step improved and applied to tracking and orbit determination and in previous works by present authors [4,5,6]. In this paper the focus is mostly on the computation of the distances among satellites, i.e. in the field of formation flying. The ultimate goal would be to obtain results similar to the ones provided by extensive numerical simulations aimed to identify J2 invariant formations [7] and to help in the relevant analysis [8], with the target to limit the effort required to control the configuration [9].

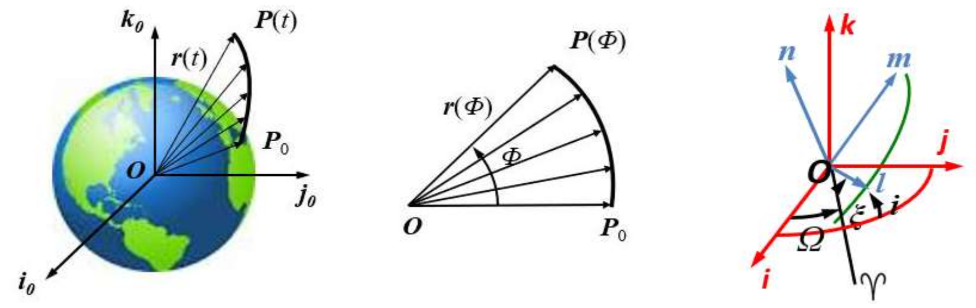

Fig. 1 – Sketch of a J2-perturbed orbit (not anymore laying on a plane) with the parameters adopted to describe the position of the satellite (adapted from [4]).

Approach

The position of the satellite along the orbit can be defined, according to the frame reported in Fig. 1, by the radius and the three angles Ω, i and ξ . The dynamics (Laplace planetary equations) can be written as

$$i' = a_n \frac{r^3}{K^2}\cos\xi \qquad \Omega' = a_n \frac{r^3}{K^2}\frac{\sin\xi}{\sin i} \qquad \xi' = 1 - a_n \frac{r^3}{K^2}\frac{\sin\xi}{\tan i} \tag{1}$$

$$\frac{K'}{K} = a_m \frac{r^3}{K^2} \qquad t' = \frac{r^2}{K} \qquad \left(\frac{1}{r}\right)'' + \frac{1}{r} = -\frac{r^2}{K^2}\left[a_l + r\left(\frac{1}{r}\right)' a_m\right]$$

where K is the angular momentum, t the time and derivatives, represented by the apex ($'$), refer to the angular variable and the a coefficients, if we limit to the case of the J2 effect, are simply given as

$$a_l = \frac{\mu}{r^2} - \frac{3}{2}\mu J_2 R_\oplus^2 \frac{1}{r^4}(1 - 3\,sin^2\, i\, sin^2\, \xi)$$

$$a_m = -\frac{3}{2}\mu J_2 R_\oplus^2 \frac{1}{r^4} sin^2\, i\, sin\, 2\,\xi \qquad a_n = -\frac{3}{2}\mu J_2 R_\oplus^2 \frac{1}{r^4} sin\, 2\, i\, sin\,\xi \tag{2}$$

After a significant mathematical elaboration (see [5]), the set of Eq.(1) leads to an expression for r and for the three angles Ω, i and ξ as in following Eq. 3. Coefficients depend on the initial conditions only, and can be evaluated until the desired order. Notice that nowadays such an elaboration has been helped by symbolic software (e.g. MATLAB [10] in the present case).

$$r = R'_0 + \sum_h \left[R'_h \cos\left(h\frac{2\pi}{T}(t - t_{iniz})\right) + R''_h \sin\left(h\frac{2\pi}{T}(t - t_{iniz})\right)\right]$$

$$\Omega = \Omega'_0 - \frac{3}{2}J_2 \cos i_{iniz}\left(\frac{R_\oplus \mu_\oplus}{K_{iniz}^2}\right)^2 \frac{2\pi(t - t_{iniz})}{T} + \sum_h \left[\Omega'_h \cos\left(h\frac{2\pi}{T}(t - t_{iniz})\right) + \Omega''_h \sin\left(h\frac{2\pi}{T}(t - t_{iniz})\right)\right] \tag{3}$$

$$i = I'_0 + \sum_h \left[I'_h \cos\left(h\frac{2\pi}{T}(t - t_{iniz})\right) + I''_h \sin\left(h\frac{2\pi}{T}(t - t_{iniz})\right)\right]$$

$$\xi = \xi'_0 + \frac{2\pi}{T}(t - t_{iniz}) - \frac{3}{2}J_2 \cos i_{iniz}\left(\frac{R_\oplus \mu_\oplus}{K_{iniz}^2}\right)^2 \frac{2\pi(t - t_{iniz})}{T} + \sum_h \left[\xi'_h \cos\left(h\frac{2\pi}{T}(t - t_{iniz})\right) + \xi''_h \sin\left(h\frac{2\pi}{T}(t - t_{iniz})\right)\right]$$

The correctness of the solution can be easily estimated by the comparison with a standard numerical propagation (see Fig. 2 for examples relevant to two parameters of interest).

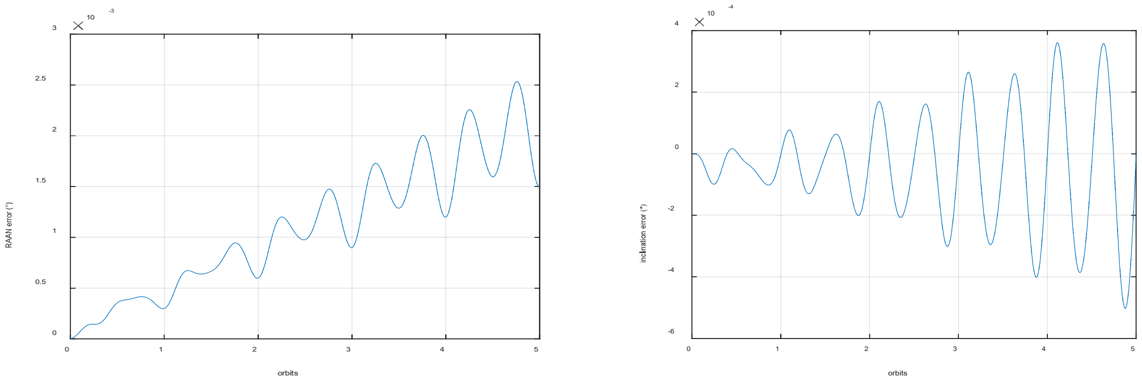

Fig. 2 – Differences between analytical approach and numerical integration.

Formations

The very same approach can be iterated for different spacecraft. However, it is extremely important to remark the relevance of the initial conditions to be imposed to the satellites belonging to the formation. A simple computation of the distance between generic, yet close initial conditions gives the results presented in Fig. 3 for two spacecraft. Notice that the distance is given by the difference between the two vectors representing the radii, with three backward rotations in the angles Ω_1, i_1, ξ_1 for the first spacecraft and Ω_2, i_2 and ξ_2 for the second one to obtain the components along the inertial frame's axes.

Within the concept of formation flying, it is desired that a configuration with limited inter-satellite distances should last in time. So, additional constraints can be applied among the parameters referred to the two - or more – spacecraft belonging to the formation. A first preliminary indication can be given by imposing the same energy to the two satellites (Fig. 4).

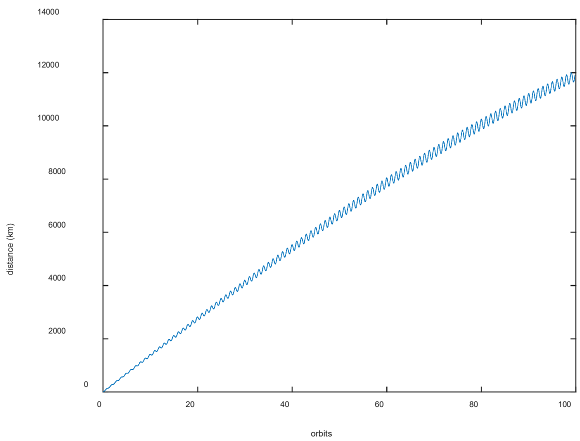

Fig. 3 – Distance between two close satellites.

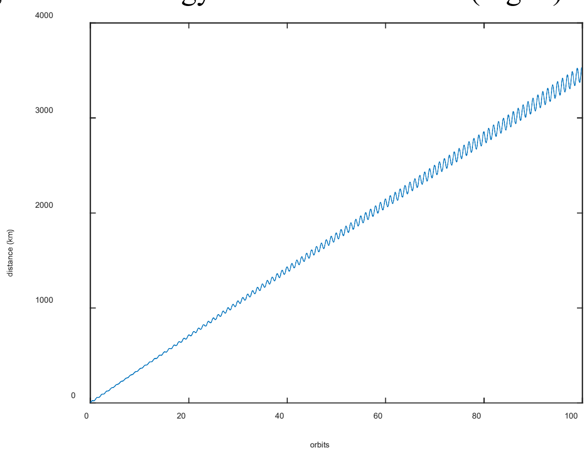

Fig. 4 – Distance imposing equal energy.

A first constraint is given by equal period, that is a requirement to avoid divergence. A second constraint is related to the inclination, that has to be assumed as quite close for the satellites to stay in formation: in fact, even 1 degree if difference would create a distance in the order of 120 km for orbits of 600 km altitude. Furthermore, larger differences end up in a different environment in terms of other perturbations, so an almost equal inclination can be reasonably assessed. A third constraint is given by the equal precession of the ascending nodes. Note that the analytical solution gives a secular term: this term vanishes for some critical inclinations. Once all of these constraints are imposed, the results plotted in Fig. 5-6 can be obtained.

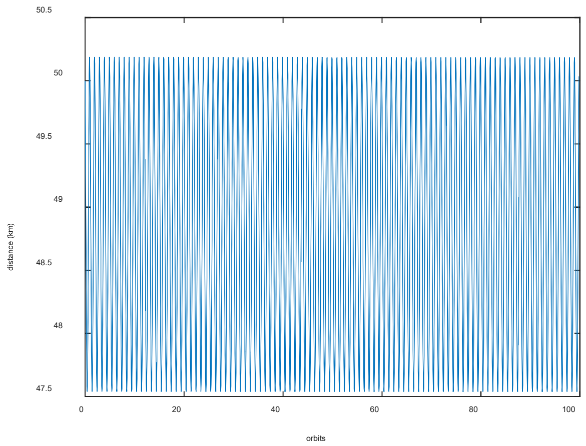

Fig. 5 – Distance imposing the condition of an equal location after a short time interval (8s).

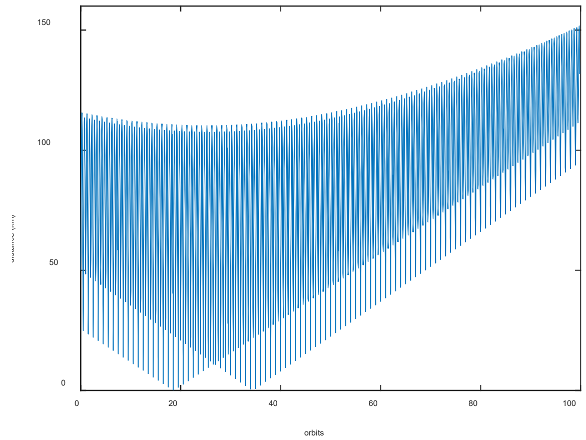

Fig. 6 Initial conditions as per Fig.4 adding constraints of equal period and secular drift.

Final Remarks

Design and operations phases of formation flying missions can be helped by the availability of analytic solutions taking into account the oblateness effects. The work, following the path pursued along the years by the authors, proposes analytic formulas for the distances between satellites considering the dominant effect of the J2 term, and prove their correctness and their appeal even if only a limited number of terms in the expansion should be used.

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