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Hindman's theorem for sums along the full binary tree, Σ_2^0-induction and the Pigeonhole principle for trees

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Abstract

We formulate a restriction of Hindman's Finite Sums Theorem in which monochromaticity is required only for sums corresponding to rooted finite paths in the full binary tree. We show that the resulting principle is equivalent to Σ_2^0-induction over RCA_0. The proof uses the equivalence of this Hindman-type theorem with the Pigeonhole Principle for trees TT^1 with an extra condition on the solution tree.

Keywords Reverse mathematics · Hindman's theorem · Pigeonhole Principle · Induction

Mathematics Subject Classification 03D80 · 03B30 · 03F35 · 11B30 · 05D10

Introduction

Hindman's celebrated Finite Sums Theorem [12] states that however you color the positive integers in finitely many colours the coloring will be constant on an infinite set and on all finite sums of distinct elements from that set. Characterizing the logical and computational strength of Hindman's Finite Sums Theorem is one of the main open problems in Reverse Mathematics since the seminal work of Blass, Hirst and Simpson [1] who proved it to be weakly between ACA_0^+ (roughly the ω-th Turing Jump, in computability-theoretic terms) and ACA_0 (roughly, the Halting Problem).

Much recent research focused on restrictions of Hindman's Theorem in which only *some* finite sums are required to be monochromatic; see e.g. [3–6,10,11]. Starting

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from [2], much attention has been paid to restrictions based on the number of terms of monochromatic sums (see [5,10,11])—we call these *quantitative* restrictions. More general forms of restrictions—which we call *structural* restrictions—have been shown to possess interesting logical and computational properties [3–6]. For example, if one requires monochromaticity only for arbitrarily long finite sums of successive elements of an infinite set then one obtains a principle of strength roughly that of Ramsey's Theorem for pairs, see [4].

In this paper we are interested in a new structural restriction of Hindman's Theorem obtained by requiring monochromaticity only for sums selected by finite paths from the root of the full binary tree.

One motivation for investigating this restriction comes from the study of Ramsey's Theorem for trees, introduced by Chubb, Hirst and McNicholl in [8]. As observed by Hirst [16] the Pigeonhole Principle for trees (TT^1) follows from Hindman's Theorem by a simple proof. As Hirst [16] observes, the full strength of Hindman's Theorem is not required to prove TT^1. The latter is in fact provable from Σ^0_2-induction ([16], Theorem 1), while Hindman's Theorem is known to imply ACA_0 (see [1]). It is natural to ask whether there is a restriction of Hindman's Theorem that is optimal for proving TT^1. An inspection of Hirst's proof shows that monochromaticity is needed only for a very restricted subset of all possible finite sums. The subset in question essentially corresponds to finite paths from the root of the full binary tree.

We introduce the corresponding natural restriction of Hindman's Theorem and prove that it is slightly stronger than TT^1. In fact, our Hindman-type principle is equivalent to Σ^0_2-induction, while TT^1 was very recently shown to be strictly weaker by Chong et al. [7].

Our proof uses an auxiliary principle consisting of TT^1 with an extra condition on the solution tree. This condition is derived from a corresponding sparsity condition that plays a crucial role in Hindman-type theorems, called *apartness* in [3]. We first show that Hindman's Theorem restricted to sums along finite paths of the full binary tree with the apartness condition on the solution set is equivalent over RCA_0 to TT^1 with a corresponding extra condition on the solution tree. Then we show that the latter form of TT^1 is equivalent to Σ^0_2-induction, using a characterization of Σ^0_2-induction due to Hirst [16].

1 Hindman's theorem for binary tree paths

We start by recalling Hindman's Finite Sums Theorem from [12].

Definition 1 Let $k \geq 1$. HT_k is the following principle: For every $c : \mathbf{N} \to k$ there is an infinite set H such that for some $z < k$ all finite non-empty sums of elements of H have color z under c. We denote $(\forall k \geq 1)\mathsf{HT}_k$ by HT.

In recent literature [3–6,10,11] restrictions of Hindman's Theorem of the following general form have been investigated. Let $\mathcal{S}$ be a family of finite subsets of the positive integers. For $k \geq 1$ we denote by $\mathsf{HT}^{\mathcal{S}}_k$ the following principle: For all $c : \mathbf{N} \to k$ there exists $H \subseteq \mathbf{N}$ such that $H = \{h_1 < h_2 < h_3 < \dots\}$ is infinite and for some $i < k$

for all $J \in \mathcal{S}$, $c(\sum_{j \in J} h_j) = i$. The restriction of interest in the present paper is the one where $\mathcal{S}$ corresponds to finite paths from the root of the full binary tree.

Hirst [16] presents a short proof of the so-called Pigeonhole Principle for Trees (TT^1) from Hindman's Theorem in RCA_0. To define the principle TT^1 we need to fix some notation and terminology for trees in RCA_0. We denote by $2^{<\mathbf{N}}$ the full binary tree of height ω, identified with the set of finite sequences of 0s and 1s ordered by initial segment ($\subseteq$). We call subsets of $2^{<\mathbf{N}}$ *subtrees*. A subtree S is *isomorphic* to $2^{<\mathbf{N}}$ if there exists a bijection $f : 2^{<\mathbf{N}} \rightarrow S$ such that for all $\sigma, \tau \in 2^{<\mathbf{N}}$, $\sigma \subseteq \tau$ if and only if $f(\sigma) \subseteq f(\tau)$. In other words, each node in S has exactly two children. We denote by $S \sim 2^{<\mathbf{N}}$ the fact that S is isomorphic to $2^{<\mathbf{N}}$.

Definition 2 Let $k \geq 1$. TT^1_k is the following principle: If $2^{<\mathbf{N}}$ is colored with k colors then there is a subtree S isomorphic to $2^{<\mathbf{N}}$ such that S is monochromatic . We denote $(\forall k \geq 1)\mathsf{TT}^1_k$ by TT^1.

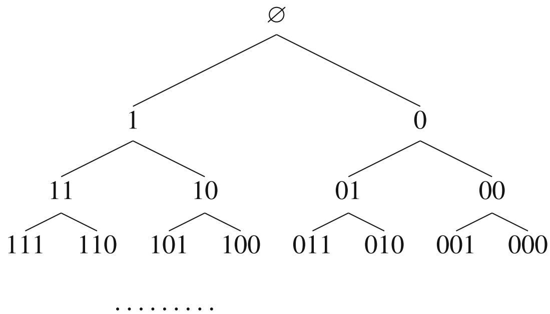
An inspection of the proof that $\mathsf{RCA}_0 + \mathsf{HT} \vdash \mathsf{TT}^1$ from [16] shows that only a few special sums need to be monochromatic. In general, a family of sums whose inclusion graph is order-isomorphic to the full binary tree is sufficient. To get a concrete Hindman-type principle of the form $\mathsf{HT}^{\mathcal{S}}$ we use as $\mathcal{S}$ a family of standard labels for finite paths in the full binary tree.

Definition 3 A finite non-empty set I of positive integers $i_1 < i_2 < \cdots < i_n$ is a *path* if and only if $i_1 = 1$ and, for all k such that $1 < k \leq n$, $i_k \in \{2i_{k-1}, 2i_{k-1} + 1\}$. We denote by bin the set of all paths.

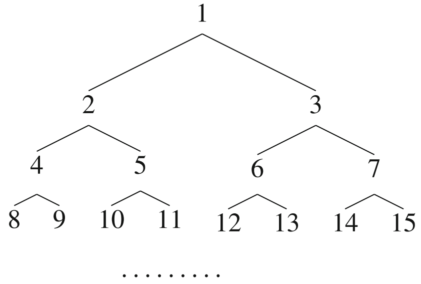
We can now formulate our Hindman-type principle for sums along finite paths in $2^{<\mathbf{N}}$.

Definition 4 Let $k \geq 1$. $\mathsf{HT}^{\mathsf{bin}}_k$ is the following principle: For every $c : \mathbf{N} \rightarrow k$ there is an infinite set $H = \{h_1 < h_2 < h_3 < \ldots\}$ such that for some $z < k$, for all $J \in \mathsf{bin}$, $c(\sum_{i \in J} h_i) = z$. We denote $(\forall k \geq 1)\mathsf{HT}^{\mathsf{bin}}_k$ by $\mathsf{HT}^{\mathsf{bin}}$.

To favor an intuitive understanding of the principle $\mathsf{HT}^{\mathsf{bin}}_k$ let us describe the set bin in a procedural way. Fix the following standard presentation of $2^{<\mathbf{N}}$, with extension-by-0 corresponding to right child and extension-by-1 corresponding to left child:



Consider the numbering of nodes determined by a level-by-level left-to-right visit of $2^{<\mathbf{N}}$:



The set bin collects the finite sets of integers naturally associated as labels to finite paths rooted at the root under the above labeling of nodes. We can enumerate bin by increasing last element as follows:

$$\mathsf{bin} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 6\}, \{1, 3, 7\}, \ldots\}.$$

We denote by $\mathsf{path}(n)$ the unique element S of bin such that $\max(S) = n$. Note that $\mathsf{path}(n)$ is also the nth element in the enumeration of bin by increasing last element. The children of $\mathsf{path}(n)$ in $(\mathsf{bin}, \subseteq)$ are $\mathsf{path}(2n)$ and $\mathsf{path}(2n+1)$. Note that in RCA_0 it is safe to identify $2^{<\mathbf{N}}$ with the set bin ordered by inclusion. These observations should convince the reader that the sums that are required to be monochromatic in $\mathsf{HT}_k^{\mathsf{bin}}$ correspond to finite paths from the root of the full binary tree. We introduce the following short-hand notation for the sums of interest in the $\mathsf{HT}_k^{\mathsf{bin}}$ principles:

$$h_n^+ := \sum_{i \in \mathsf{path}(n)} h_i.$$

The principle $\mathsf{HT}^{\mathsf{bin}}$ easily follows from RT^1 in RCA_0, where RT^1 denotes $(\forall k \geq 1)\mathsf{RT}_k^1$ and RT_k^1 denotes the Infinite Pigeonhole Principle for colorings of $\mathbf{N}$ in k colors.

Lemma 1 *$RCA_0 \vdash RT^1 \to HT^{bin}$.*

Proof We sketch the proof of $(\forall k \geq 1)(\mathsf{RT}_k^1 \to \mathsf{HT}_k^{\mathsf{bin}})$. Fix a coloring $c : \mathbf{N} \to k$. By RT_k^1 there exists an infinite homogeneous set of positive integers $\{h_1 < h_2 < \ldots\}$. First, it is easy to see that we can, if needed, thin out the sequence so as to ensure that the following set is strictly increasing

$$\{h_n - h_{\lfloor \frac{n}{2} \rfloor} : n \in \mathbf{N}\}.$$

Second, it is easy to verify that this set satisfies the homogeneity condition in the definition of $\mathsf{HT}_k^{\mathsf{bin}}$.

□

The above Lemma implies that $\mathsf{HT}^{\mathsf{bin}}$ cannot prove TT^1, since the latter is stronger than RT^1 by a result of Corduan, Groszek and Mileti [9].

Corollary 1 $RCA_0 + HT^{bin} \nvdash TT^1$.

Proof By Corollary 3.8 in [9], RT^1 does not imply TT^1 over RCA_0. □

While the set bin essentially contains the type of sums that are used in the proof of TT^1 from HT in [16] (see also the proof of Proposition 1, *infra*), an extra condition is needed for the proof to work. This condition, already implicit in [11,12], is called *apartness* in [5]. To define apartness we need the following notation. If $n = 2^{e_1} + \cdots + 2^{e_m}$, where $e_1 < e_2 < \cdots < e_m$, let $\lambda(n)$ denote e_1 and $\mu(n)$ denote e_m.

Definition 5 (Set Apartness) A set X of positive integers satisfies the *apartness condition*, or *is apart*, if for all $n, m \in X$ such that $n < m$, we have $\mu(n) < \lambda(m)$.

If P is a Hindman-type principle we denote by apP the same principle with the extra requirement that the solution set is apart and we call it "P with apartness". The apartness condition is built-in in the usual equivalent formulation of Hindman's Theorem in terms of finite unions. Most natural restrictions of the Finite Sums Theorem with apartness are computably interreducible with (and RCA_0-equivalent to) corresponding restrictions of the Finite Unions Theorem (see [6]). Yet it is nevertheless interesting to isolate the role of apartness and therefore we distinguish between $\mathsf{HT}_k^{\mathsf{bin}}$ and $\mathsf{apHT}_k^{\mathsf{bin}}$. To this extent, as observed in [3,4,6], restricted versions of Hindman's Theorem *with apartness* should be considered proper restrictions of Hindman's Theorem.

The apartness condition plays a crucial role in the investigation of restricted forms of Hindman's Theorem is illustrated in [3,4,6], suggesting that apartness increases the strength of Hindman-type theorems, or, at least, significantly simplifies proving lower bounds for such principles. The apartness condition also plays a key role in Hirst's proof of TT^1 from Hindman's Theorem. As we will see, a corresponding condition on TT^1 emerges when we look for a reversal. For σ a node in a tree T we denote by $\mathsf{parent}_T(\sigma)$ the immediate predecessor of σ in T and we omit the subscript when clear from context.

Definition 6 Let $S \sim 2^{<\mathbf{N}}$. We call an enumeration $\{\sigma_1, \sigma_2, \dots\}$ of S a *level-by-level enumeration* if for each $i \geq 1$ the children of σ_i in S are σ_{2i} and σ_{2i+1} or, equivalently, for each $i \geq 2$, $\mathsf{parent}_T(\sigma_i) = \sigma_{\lfloor \frac{i}{2} \rfloor}$.

We are now ready to define the analogue of the apartness condition for subtrees of $2^{<\mathbf{N}}$. We use $\frown$ to denote sequence concatenation.

Definition 7 (Tree Apartness) A subtree $S \subseteq 2^{<\mathbf{N}}$ is *apart* if the following conditions hold:

1. $S \sim 2^{<\mathbf{N}}$.
2. For each $n \in \mathbf{N}$ there is at most one sequence of length n in S.
3. The length-increasing enumeration of $S = \{\sigma_1, \sigma_2, \dots\}$ is a level-by-level enumeration of S.
4. For all $i \geq 1$ there exists $\sigma \in 2^{<\mathbf{N}} \setminus \{0\}^{<\mathbf{N}}$ such that

$$\sigma_{i+1} = \mathsf{parent}(\sigma_{i+1}) \frown 0^{|\sigma_i| - |\mathsf{parent}(\sigma_{i+1})|} \frown \sigma.$$

In brief, a full binary tree is apart if each sequence extends its parent with zeros until it joins the length of its predecessor, then it can be extended arbitrarily but with at least a 1.

Remark 1 Requiring at least a 1 in σ in the above definition is a need dictated by a later proof: we will associate increasing numbers to increasing sequences according to the binary representation of numbers (if τ extends σ by zeros, then σ and τ represent the same number). Note that, except for the root and one of its children, the predecessor of a node in an apart tree is never also its parent.

Definition 8 Let $k \geq 1$. We denote by apTT^1_k the principle TT^1_k with the extra constraint that the monochromatic subtree is apart as in Definition 7. We denote $(\forall k \geq 1)\mathsf{apTT}^1_k$ by apTT^1.

The next two propositions establish the equivalence of apTT^1 and $\mathsf{apHT}^{\mathsf{bin}}$ over RCA_0.

Proposition 1 *$RCA_0 \vdash apHT^{bin} \rightarrow apTT^1$.*

Proof We show in RCA_0 that $(\forall k \geq 1)(\mathsf{apHT}^{\mathsf{bin}}_k \rightarrow \mathsf{apTT}^1_k)$. The proof is similar to Hirst's proof of TT^1 from Hindman's Finite Unions Theorem [16], yet a different labeling of nodes of $2^{<\mathbf{N}}$ with sums is used here.

Fix $k \in \mathbf{N}$ and $c : 2^{<\mathbf{N}} \rightarrow k$. We define a coloring $c' : \mathbf{N} \rightarrow k$ of the natural numbers in a very intuitive fashion. Let $seq : \mathbf{N} \rightarrow 2^{<\mathbf{N}}$ be the function mapping 0 to the empty sequence and, for each $n \geq 1$, if $n = 2^{e_1} + \cdots + 2^{e_m}$ with $e_1 < e_2 < \cdots < e_m$, then $seq(n) = \sigma_{\{e_1,\ldots,e_m\}}$, where if X is a finite subset of $\mathbf{N}$, $\sigma_X \in 2^{<\mathbf{N}}$ is the shortest binary string representing the characteristic function of X. We set

$$c'(n) := c(seq(n)).$$

By $\mathsf{apHT}^{\mathsf{bin}}$ there exists an infinite apart set $H = \{h_1 < h_2 < \ldots\}$ of positive integers such that all sums h^+_n, for $n \geq 1$, have the same c'-color. Let $z < k$ be this color. We define $T_H \subseteq 2^{<\mathbf{N}}$ as the set of τ_n defined as follows, for $n \geq 1$:

$$\tau_n := seq\left(\sum_{j \in \mathsf{path}(n)} h_j\right) = seq(h^+_n).$$

It is easy to prove that T_H is monochromatic for c. By definition of c' and τ_n we have

$$c(\tau_n) = c\left(seq\left(\sum_{j \in \mathsf{path}(n)} h_j\right)\right) = c(seq(h^+_n)) = c'(h^+_n) = z.$$

We then need to show that T_H is an apart subtree (see Definition 7). This is where the apartness condition on H is crucially used. It is easy to verify that, since H is apart, $|seq(h_n^+)| = |seq(h_n)|$. Thus, for any $0 < i < j$, $|\tau_i| < |\tau_j|$ as required by tree apartness.

We next show that T_H isomorphic to $2^{<\mathbf{N}}$ (or, equivalently, to $(\mathsf{bin}, \subseteq)$). Since H is apart we have that for any $n < m$, $\mathsf{path}(n) \subset \mathsf{path}(m)$ if and only if $\tau_n \subset \tau_m$. In fact, $\tau_n = seq(h_n^+)$ and $\tau_m = seq(h_m^+)$, where

$$h_n^+ = \sum_{j \in \mathsf{path}(n)} h_j$$

. If $\mathsf{path}(n) \subset \mathsf{path}(m)$, then

$$h_m^+ = \sum_{j \in \mathsf{path}(m)} h_j = \sum_{j \in \mathsf{path}(n)} h_j + \sum_{j \in \mathsf{path}(m) \setminus \mathsf{path}(n)} h_j.$$

Since H is apart, $h_m^+ = h_n^+ + b$, for some b with $\mu(h_n^+) < \lambda(b)$. From this we easily conclude $\tau_n \subset \tau_m$. For the other direction, let $\oplus$ be the XOR binary operation on binary strings where the shortest binary string is extended by zeros to reach the length of the longest binary string. By apartness, for any $h < h'$ in H the position of the last 1 in $seq(h)$ is strictly smaller than the position of the first 1 in $seq(h')$. Thus, $seq(h_n^+) = \bigoplus_{j \in \mathsf{path}(n)} seq(h_j)$ and $seq(h_m^+) = \bigoplus_{j \in \mathsf{path}(m)} seq(h_j)$. If $seq(h_n^+) \subset seq(h_m^+)$ it must necessarily be the case that $\mathsf{path}(n) \subset \mathsf{path}(m)$.

As for the last property required for tree apartness, note that $\{\tau_n : n \in \mathbf{N}\}$ is a level-by-level enumeration of T_H because $\{\mathsf{path}(n) : n \in \mathbf{N}\}$ is a level-by-level enumeration of bin, and the isomorphism maps $\mathsf{path}(n)$ to τ_n. Using the definition $\tau_n := seq(h_n^+)$ it is easy to check that, for each $i \geq 1$:

$$\tau_{i+1} = \mathsf{parent}(\tau_{i+1}) \frown 0^{|\tau_i| - |\mathsf{parent}(\tau_{i+1})|} \frown \tau,$$

for some τ containing at least a 1.

□

Proposition 2 *$RCA_0 \vdash apTT^1 \rightarrow apHT^{bin}$.*

Proof We show in RCA_0 that $(\forall k \geq 1)(\mathsf{apTT}^1_k \rightarrow \mathsf{apHT}^{\mathsf{bin}}_k)$. Let $num : 2^{<\mathbf{N}} \rightarrow \mathbf{N}$ be the surjective mapping $\sigma \mapsto 2^{e_1} + \cdots + 2^{e_t}$ where $\{e_1, \ldots, e_t\}$ are the positions on which σ has value 1, and all $\sigma \in \{0\}^{<\mathbf{N}}$ are mapped to 0. The empty sequence is mapped to 0. Fix a coloring $c : \mathbf{N} \rightarrow k$. Consider the coloring $c' : 2^{<\mathbf{N}} \rightarrow k$ defined as follows:

$$c'(\sigma) := c(num(\sigma)).$$

By apTT^1 there is an apart tree T homogeneous for c'. Let $T = \{\tau_1, \tau_2, \ldots\}$ be T listed in length-increasing order. Since T is apart this ordering is a level-by-level ordering. We can furthermore assume w.l.o.g. that $\tau_1 \notin \{0\}^{<\mathbf{N}}$. Let $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ where $\sigma_1 := \tau_1$ and, for $i > 1$, $\sigma_i := 0^{|\tau_{i-1}|} \frown \tau$ where τ is such that $\tau_i = \mathsf{parent}(\tau_i) \frown 0^{|\tau_{i-1}| - |\mathsf{parent}(\tau_i)|} \frown \tau$. We observe that:

- τ_i and σ_i have the same length.
- $\tau_i = \mathsf{parent}(\tau_i) \oplus \sigma_i$.
- $\tau_i = \bigoplus_{j \in \mathsf{path}(i)} \sigma_j$.

Let $h_i := num(\sigma_i)$ and consider the set $H_T := \{h_1, h_2, \dots\}$. By the tree apartness of T and definition of Σ we have that $h_1 < h_2 < \dots$, since $|\tau_1| < |\tau_2| < \dots$ and $|\tau_i| = |\sigma_i|$ for all $i \geq 1$.

We can prove by (Π^0_1-) induction (on the formula $(\forall m < n)(\mu(h_m) < \lambda(h_n))$) that H_T is apart. For each $i > 0$, since $\sigma_i = 0^{|\tau_{i-1}|} \frown \tau$, $|\sigma_{i-1}| = |\tau_{i-1}|$ and $h_i = num(\sigma_i)$, we have that $\mu(h_{i-1}) < \lambda(h_i)$ by definition of num. Since the apartness condition on integers is transitive, by induction we are done.

Next we prove that H_T satisfies the required monochromaticity condition for c. Let $z < k$ be the color witnessing that T is homogeneous for c'. We show that for all $n \geq 1$, $h^+_n = \sum_{i \in \mathsf{path}(n)} h_i$ has color z. In fact

$$num(\tau_n) = num(\bigoplus_{i \in \mathsf{path}(n)} \sigma_i) = \sum_{i \in \mathsf{path}(n)} num(\sigma_i) = \sum_{i \in \mathsf{path}(n)} h_i = h^+_n.$$

Thus, $c(h^+_n) = c(num(\tau_n)) = c'(\tau_n) = z$. □

From Proposition 1 and Proposition 2 we get the following corollary.

Corollary 2 $RCA_0 \vdash apTT^1 \leftrightarrow apHT^{bin}$.

2 Equivalence with Σ^0_2-induction

We prove that $\mathsf{apHT}^{\mathsf{bin}}$ is equivalent to Σ^0_2-induction. To this aim we use the intermediate principle apTT^1 and an equivalent of Σ^0_2-induction from [16], the Eventually Constant Tails principle.

2.1 Upper bound

We adapt the upper bound proof from Lemma 1.1 in [8], taking some extra care to ensure tree-apartness.

Proposition 3 $RCA_0 \vdash apTT^1_2$.

Proof Fix $c : 2^{<\mathbf{N}} \to \{\mathsf{red}, \mathsf{blue}\}$. First we observe that there is a recursive procedure `FIND_RED` that given any $\tau \in 2^{<\mathbf{N}}$ returns the shortest $\sigma \in 2^{<\mathbf{N}}$ such that $\tau \subseteq \sigma$ and $c(\sigma) = \mathsf{red}$ if any, and otherwise loops. Indeed it is sufficient to iterate level-by-level over the subtree of $2^{<\mathbf{N}}$ rooted at σ and return the first string that is colored red.

For any $\tau \in 2^{<\mathbf{N}}$ we denote by $sub(\tau)$ the subtree of $2^{<\mathbf{N}}$ rooted at τ. We define a recursive procedure that builds a c-monochromatic apart red tree $T = \{\tau_1, \tau_2, \dots\}$ in length-increasing order.

– Let τ_1 be the least red node of $2^{<\mathbf{N}}$.

- Let τ_2 be the least red node of $sub(\tau_1 \frown 1)$.
- For any $i \geq 3$ let τ_i be the least red node of the following subtree:

$$sub(\tau_{\lfloor \frac{i}{2} \rfloor} \frown 0^{|\tau_{i-1}|-|\tau_{\lfloor \frac{i}{2} \rfloor}|} \frown 1).$$

Note that $\lfloor \frac{i}{2} \rfloor$ is the index of $\mathsf{parent}_T(\tau_i)$. Our procedure is recursive since it only uses FIND_RED as a sub-procedure. If our procedure defines τ_i for all $i \geq 1$ then $T = \{\tau_1, \tau_2, \dots\}$ is a level-by-level enumeration of a monochromatic red tree in length-increasing order and T is apart by construction. The procedure ensures that T is computably enumerable. Since the enumeration is in length-increasing order T is also computable. Now suppose that for some step i the procedure FIND_RED loops before it defines τ_{i+1}. To find the least such i we need the Π^0_1-least element principle. This principle can be proved in RCA_0 as it follows from $I\Sigma^0_1$ induction (see Theorem A in [18]). If $i = 1$ then $sub(\tau_1 \frown 1)$ is a subtree all coloured blue. Similarly if $i > 1$ then $sub(\tau_{\lfloor \frac{i}{2} \rfloor} \frown 0^{|\tau_{i-1}|-|\tau_{\lfloor \frac{i}{2} \rfloor}|} \frown 1)$ is a subtree all coloured blue. In both cases we can apply our procedure (using the blue colour instead of red) on the blue subtree, and this time the procedure won't fail to build a blue apart tree. □

Remark 2 In the above proof, the construction can be carried out starting from any string of $2^{<\mathbf{N}}$. Indeed the proof actually shows that, given any 2-coloring of $2^{<\mathbf{N}}$, for all $\sigma \in 2^{<\mathbf{N}}$ either there exists an infinite red apart tree whose strings extend σ or there exists a full binary subtree colored blue whose strings extend σ. This is the real analogue of Lemma 1.1 in [8].

Proposition 4 *$RCA_0 + \Sigma^0_2$-$IND \vdash apTT^1$.*

Proof The proof is modeled after the proof of Theorem 1.2 in [8], using Proposition 3 instead of Lemma 1.1 in [8]. We repeat it here for completeness. Moreover, the idea is just the same as in the proof of Proposition 3 with the only difference that for arbitrary colours we need to operate on the set C defined below whose existence is guaranteed by Σ^0_2-IND, while for a fixed number of colours, RCA_0 is sufficient.

Fix $c : 2^{<\mathbf{N}} \to k$. Consider the set

$$C = \{j < k : (\exists \sigma)(\forall \tau \supseteq \sigma)(j \leq c(\tau))\}.$$

By bounded Σ^0_2-comprehension (provable from Σ^0_2-IND, see [20], p. 72) C exists. C is a non-empty finite set since $0 \in C$ and every $j \in C$ is less than k. Let j be the maximum of C. Let σ witness j. Note that every node extending σ is colored with a color greater than or equal to j. Define a 2-coloring c' of the subtree T rooted at σ as follows: $c'(\tau) :=$ red if $c(\tau) = j$, $c'(\tau) :=$ blue otherwise. By Remark 2 there either exists a red apart subtree—and in that case we are done—or there exists a full binary subtree T' colored blue. In that case the minimum color used by c to color T' is greater than j and belongs to C, a contradiction. □

From Proposition 3, Proposition 4 and Corollary 2 we get the following corollary.

Corollary 3 *$RCA_0 \vdash apHT^{bin}_2$ and $RCA_0 + \Sigma^0_2$-$IND \vdash apHT^{bin}$.*

2.2 Lower bound

We next show that apTT^1 implies Σ^0_2-induction over RCA_0. In [16] the following principle—called Eventually Constant Tails—is proved equivalent to Σ^0_2-IND over RCA_0.

Definition 9 (Hirst [16]) $\mathsf{ECT}(\mathbf{N})$ is the following principle: For any $c : \mathbf{N} \to k$ the following holds:

$$(\exists b)(\forall n \geq b)(\exists m > n)(c(n) = c(m)).$$

Our proof that apTT^1 implies $\mathsf{ECT}(\mathbf{N})$ uses a non-trivial adaptation of the parity argument inaugurated in the study of the strength of Hindman's Theorem in [1] and simplified in [5]. Note that both these proofs in their original form show an implication to ACA_0 and are designed for 2-colorings. We need a preliminary definition.

Definition 10 Let $\sigma \in 2^{<\mathbf{N}}$. We call σ a *good sequence* if it cointains at least two 1s with some 0s in-between. For a good sequence σ we define $I(\sigma) \subseteq (\mathbf{N} \setminus \{0\})^{<\mathbf{N}}$, called the *interval sequence of* σ, to be the set of consecutive intervals of 0-entries of σ, i.e. an interval $[i+1, j-1]$ is in $I(\sigma)$ if and only if the following three points are satisfied.

1. $i + 1 < j$,
2. $\sigma(i) = \sigma(j) = 1$,
3. $\sigma(k) = 0$ for all $k \in [i+1, j-1]$.

If σ is not a good sequence we set $I(\sigma) := \emptyset$. The elements of $I(\sigma)$ are called the *intervals of* σ.

We can naturally order $I(\sigma)$ as follows: for $I, J \in I(\sigma)$ we let $I < J$ if $\max(I) < \min(J)$. So we can write $I(\sigma) = \{I_1 < I_2 < \cdots < I_\ell\}$ for some $\ell \geq 1$. If $c : \mathbf{N} \to k$ and $I \subseteq \mathbf{N}$ we denote by $c(I)$ the set $\{c(i) : i \in I\}$.

Definition 11 Let $k \geq 1$ and $c : \mathbf{N} \to k$, $z < k$, and $\sigma \in 2^{<\mathbf{N}}$ a good sequence. Suppose $I(\sigma) = \{I_1, I_2, \ldots, I_\ell\}$. We define the predicate "j is z-important in σ", denoted $\mathrm{imp}(j, z, \sigma)$, as follows: $2 \leq j \leq \ell$ and $z \in c(I_{j-1})$ and $z \notin c(I_j)$. If σ is not a good sequence, the predicate is always false.

Theorem 1 *$RCA_0 \vdash apTT^1 \to ECT(\mathbf{N})$.*

Proof Fix $c : \mathbf{N} \to k$. Define $c' : 2^{<\mathbf{N}} \to 2^{k \times \{0,1\}}$ as follows:

$$c'(\sigma) := \{\langle z, \mathrm{card}\{j : \mathrm{imp}(j, z, \sigma)\} \bmod 2\rangle : z < k\}.$$

Intuitively the color assigned by c' to a sequence σ is the set of ordered pairs (z, p_z) where $z < k$ is a c-color and p_z is the parity of the set of indices j such that z appears as a c-color of some element of interval I_{j-1} but not as a c-color of some element of the successive interval I_j.

By apTT^1 there exists a color $w \in 2^{k\times\{0,1\}}$ and a w-monochromatic apart subtree T. Let $T = \{\sigma_1, \sigma_2, \ldots\}$ be enumerated in length-increasing order. Since T is apart, this is also a level-by-level enumeration. Without loss of generality, since T is apart, we can assume that all σ_is are good sequences.

Suppose by way of contradiction that $\mathsf{ECT}(\mathbf{N})$ fails for c, i.e.,

$$(\forall b)(\exists n \geq b)(\forall m > n)(c(n) \neq c(m)).$$

For $b = |\sigma_1| + 1$ this gives:

$$(\exists n > |\sigma_1|)(\forall m > n)(c(n) \neq c(m)).$$

Let $j = \min_i |\sigma_i| > n$. Such a j exists since T is infinite. Clearly $j \neq 1$. Consider σ_{j+1} and $\mathsf{parent}(\sigma_{j+1})$. Since $j \neq 1$ we have that

$$|\mathsf{parent}(\sigma_{j+1})| \;<\; |\sigma_j|.$$

By minimality of j it must be $|\mathsf{parent}(\sigma_{j+1})| \leq n$. In particular $\mathsf{parent}(\sigma_{j+1})(n)$ is undefined. By tree apartness it must be that $\sigma_{j+1}(n) = 0$. In fact, σ_{j+1} has the form

$$\mathsf{parent}(\sigma_{j+1}) \frown 0^{|\sigma_j| - |\mathsf{parent}(\sigma_{j+1})|} \frown \sigma$$

for some σ, and, by choice of j, $|\sigma_j| > n$. Furthermore if

$$I(\mathsf{parent}(\sigma_{j+1})) = \{I_1 < I_2 < \cdots < I_\ell\}$$

then

$$I(\sigma_{j+1}) = \{I_1 < I_2 < \cdots < I_\ell < I_{\ell+1} < \cdots < I_{\ell+r}\}.$$

We can assume w.l.o.g. $r \geq 2$ by taking any extension of σ_{j+1} in T instead of σ_{j+1} if needed. Note that, for any c-color $z < k$, if $i \in [2, \ell]$ is z-important in $\mathsf{parent}(\sigma_{j+1})$ then i is also z-important in σ_{j+1}. Clearly $n \in I_{\ell+1}$ and thus by our hypothesis on n, $\ell + 2$ is $c(n)$-important in σ_{j+1}. On the other hand $\ell + 1$ is not $c(n)$-important in σ_{j+1} since $n \in I_{\ell+1}$. By homogeneity of T we have $c'(\mathsf{parent}(\sigma_{j+1})) = c'(\sigma_{j+1})$. Then, by definition of c' and a parity argument, there must be at least one $w > 2$ such that $\ell + w$ is $c(n)$-important in σ_{j+1}. This implies $c(n) \in c(I_{\ell+w-1})$, contradicting our choice of n. □

From Theorem 1 and Corollary 2 we get the following corollary.

Corollary 4 *$RCA_0 \vdash apHT^{bin} \leftrightarrow apTT^1 \leftrightarrow \Sigma^0_2$-IND.*

3 Conclusion

Inspired by an elegant and simple proof of the Pigeonhole Principle for Trees (TT^1) from Hindman's Theorem in [16] we formulated a natural restriction of Hindman's Theorem, $\mathsf{HT}^{\mathsf{bin}}$, according to which only sums along finite paths from the root of the full binary tree are required to be monochromatic. The proof of TT^1 from HT in [16] crucially uses a sparsity condition on the solution to Hindman's Theorem. In our setting, this gives a proof in RCA_0 of TT^1 from $\mathsf{apHT}^{\mathsf{bin}}$, that is $\mathsf{HT}^{\mathsf{bin}}$ with the so-called apartness condition on the solution set. We proved that $\mathsf{apHT}^{\mathsf{bin}}$ is equivalent to Σ^0_2-IND over RCA_0. To obtain this result we formulated a principle resulting from TT^1 with an extra structural condition on the solution monochromatic subtree. This condition is modeled on the apartness condition in Hindman's Theorem and we called it (tree) apartness. The corresponding principle is apTT^1 and we proved it equivalent to $\mathsf{apHT}^{\mathsf{bin}}$. As is the case for TT^1, the principle apTT^1 is provable from Σ^0_2-IND but, perhaps surprisingly, the reverse implication also holds. This should be contrasted with the very recent result by Chong et al. [7] showing that TT_1 doesn't prove Σ^0_2-IND. Thus the principles TT^1 and apTT^1 are in different classes of strength. This means that the apartness condition, that is so crucial in Hindman's Theorem, plays a role also in Ramsey-type principles for trees, boosting the strength of TT^1 to the level of Σ^0_2-induction. By the equivalence of apTT^1 with $\mathsf{apHT}^{\mathsf{bin}}$, this also shows that $\mathsf{HT}^{\mathsf{bin}}$ and $\mathsf{apHT}^{\mathsf{bin}}$ are in different classes of strength: while $\mathsf{HT}^{\mathsf{bin}}$ follows from RT^1, $\mathsf{apHT}^{\mathsf{bin}}$ is equivalent to Σ^0_2-IND. This is the first example showing that the apartness condition can strictly increase the strength of restrictions of Hindman-type Finite Sums Theorems with respect to provability in RCA_0. Furthermore $\mathsf{apHT}^{\mathsf{bin}}$ is a "weak but yet strong" natural restriction of Hindman's Theorem in the sense of [3] which entirely exploits the particular structure of its sums to prove Σ^0_2-IND. A natural direction for future research is to investigate the relations between Hindman-type theorems and Ramsey-type theorems for trees for colorings of n-tuples with $n > 1$, starting from TT^2.

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