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N-Soliton Matrix mKdV Solutions: Some Special Solutions Revisited

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ABSTRACT

In this article, a general solution formula is derived for the d × d-matrix modified Korteweg–de Vries equation. Then, a solution class corresponding to special parameter choices is examined in detail. Roughly, this class can be described as *N*-solitons (in the sense of Goncharenko) with common phase matrix. It turns out that such a solution even takes values in a *commutative* subalgebra of the d × d-matrices. We arrive at a rich picture of possibilities for generalized 1-solitons and at visual patterns of *N*-solitons which combine nonlinear with linear features. The impact of the phase matrix is visualized in computer plots.

1 | Introduction

Solitons are termed fascinating solutions to certain nonlinear partial differential equations, sometimes called *soliton equations*. These special solutions enjoy the characteristic property to maintain their shape as they propagate over time without changing amplitude. They are commonly encountered in applied mathematics, particularly in areas like fluid dynamics, nonlinear optics, acoustics [43], and even in fields like theoretical physics and biophysics, for an up-to-date introduction to the subject, see [22, 42]; recent results on soliton equations are collected in the special issue of the journal Studies in Applied Mathematics [3]. Solitons offer a rich area of study in applied mathematics due to their unique properties and the insight they can provide into complex systems. A variety of solutions, starting from the pioneering work by Gardner et al. [31], is present in the literature which go from special solutions admitted by particular problems, see, e.g., [34] where a shallow water problem is addressed to. Soliton solutions are of as interest also in nonlinear optics, according to the review on the subject in [70], special cases are studied in [4, 23], as well as in theoretical physics [52, 57] and biophysics [7, 39, 45, 73] and, lately also in the area of *metamaterials* [71]. The wide variety of applications motivates the theoretical study which allows to construct solutions of soliton equations. Accordingly, soliton equations and their solutions via inverse scattering transform (IST) are studied in the two well-known books by Ablowitz and Segur [2], and by Calogero and Degasperis [9] who were pioneering the subject. Further approaches are represented by symmetry properties [53, 54, 56], by the Hirota method [40, 41] and by Bäcklund transformations. The latter represent a key tool in our investigation: their importance is testified by many books, such as [1, 2, 9, 35, 60, 62, 63], to mention those ones which are more closely connected to this study. Indeed, via Bäcklund transformations, it is possible to both reveal structural properties,

such as symmetry properties, and/or the Hamiltonian structure [24–26, 32, 49] enjoyed by a nonlinear evolution equation, on one side. On the other side, via Bäcklund transformations, it is possible to construct new solutions a nonlinear differential equation admits: many applicative examples are comprised in the books by Rogers and Shadwick [63], and by Rogers and Ames [60]. Our research project concerns applications of Bäcklund transformations aiming not only to investigate structural properties of nonlinear evolution equations, in the commutative setting as well as in the noncommutative, but also to construct solutions admitted by nonlinear evolution equations both in the commutative setting [30] as well as in the noncommutative one [15, 17, 19, 59, 66, 68]. The results in this article follow the lines of long-term research program which aims not only to investigate structural properties, but also to construct solutions. Results concerning structural properties of nonlinear evolution equations, the commutative case, are given in [11, 29, 61], while results in the noncommutative one are obtained in [12–14, 16, 18, 69]. Notably, in [18] the recursion operator admitted by the noncommutative Burgers equation is obtained via the Cole–Hopf transformation, i.e., a special Bäcklund transformation, recovering the same recursion operator obtained by Kuperschmidt [44] following a different approach. Noncommutative solutions are comprised in [47], where matrix solutions are studied. Also in [36–38] noncommutative Burgers equations are investigated. In addition, via Bäcklund transformations, new solutions can be constructed, in the commutative case [30], and, more recently, in the noncommutative setting, see, e.g., [15, 17, 19]. The research project aims to provide a classification of solutions: the results presented in this article represent a step in this direction. Classification studies of integrable nonlinear evolution equations are provided in [5, 8, 50, 51, 55, 72], the latter classifies linearizable nonlinear evolution equations. In this framework, the presented results are concerned about a special class of solutions. The aim of this study, which continues previous investigations, in particular in [15–17, 19] where, to start with, the theoretical setting in the space of bounded linear operators on some Banach space is provided [16]. Then in [17] the special case of a finite-dimensional Banach space is considered and, hence, hierarchies of matrix equations are studied. Later, in [15, 19] some explicit examples are given. Now, some special subclasses of solutions admitted by the $\mathsf{d}\times\mathsf{d}$-matrix modified Korteweg–deVries (matrix mKdV) equation are studied in detail. More precisely, our article aims at contributing to the classification of N-solitons in the sense of Goncharenko who in [33] gives a convincing derivation of N-solitons from the viewpoint of the IST (Theorem 1); Goncharenko, in addition, provided an asymptotic analysis only for a very special subclass of these solutions.

In this article, we study generalized N-soliton solutions to the $\mathsf{d}\times\mathsf{d}$-matrix mKdV

$$V_t = V_{xxx} + 3\{V^2, V_x\}, \tag{1}$$

where $\{\cdot,\cdot\}$ denotes the anti-commutator.

In the first part (Section 2), we present an explicit solution formula to (1) in Theorem 2.1. The theorem was announced by the authors in [19]. It was used in [15, 19] for generating computer graphics, and in [20] to obtain the asymptotics of 2-soliton solutions with invertible phase matrices.[1] The proof of the solution formula builds on work in [16, 17], formulated in the framework of quasi-Banach operator ideals. The link to these more general results is established in the Appendix. Note that, for the matrix KdV, a corresponding solution formula was derived in [17].

The parameters the solution formula in Theorem 2.1 depends on are (a priori) complex numbers $k_1,\dots,k_N$ and $\mathsf{d}\times\mathsf{d}$ matrices $C_1,\dots,C_N$. By analogy with formulas from ISM (in the reflectionless case), we call these the *spectral data*, with *eigenvalues* k_n and *phase matrices* C_n, and the resulting solutions *generalized N-solitons*. Actually, these solutions comprise non-localized and even singular solutions. We conclude the first part by settling the regularity question for generalized 1-solitons with real eigenvalue k.

The second part (Sections 3 and 4) is dedicated to the particular class of generalized N-solitons in Proposition 3.1, where, roughly speaking, the phase matrices $C_1,\dots,C_N$ are nontrivial multiples of a common matrix C, compare Remark 3.2. The study of this solution class was initiated in [15]. Observe that for $\mathsf{d}=1$ it contains all classical N-solitons.

In Section 3, a detailed description of this class is given. We observe that after a change of basis C can be assumed to be in Jordan form. Under the assumption that this can be done with real matrices, we will proceed to a detailed description of the resulting solutions in three steps of increasing complexity:

1. For generalized 1-solitons, the solution has a block-wise upper band structure with scalar 1-solitons emerging on the diagonal, and we give explicit formulas for its entries.
2. In the case that C is diagonal, so is the solution. Its diagonal entries are N-solitons, all characterized by the $k_1,\dots,k_N$. However, the jth eigenvalue of C contributes an *overall shift* to the N-soliton at the jth diagonal entry. Note that this overall shift effects the paths of the solitons individually (cf. Remark 3.6).
3. The general case is treated by extending the arguments from 1 and 2 above.

Section 4 starts with the observation that the solution class in Proposition 3.1 is globally regular. However, the main purpose of the section is to visualize our results, providing computer plots for prototypical examples. In passing we illustrate that, in the matrix case, the eigenvalues $k_1,\dots,k_N$ cannot be chosen positive without loss of generality.

2 | Generalized N-Soliton Solutions of the Matrix mKdV

The aim of this section is to give a proof of Theorem 2.1, a solution formula for the matrix mKdV equation (1) announced in [19]. After the proof we will briefly discuss regularity and reality of the obtained solutions.

Theorem 2.1. *Let $k_1,\dots,k_N$ be complex numbers with $k_m+k_n\neq 0$ for all m,n, and let $C_1,\dots,C_N$ be arbitrary complex $\mathsf{d}\times\mathsf{d}$-matrices.*

Define the $N\mathsf{d}\times N\mathsf{d}$-matrix function $L=L(x,t)$ as block matrix $L=(L_{mn})_{m,n=1}^{N}$ with the $\mathsf{d}\times\mathsf{d}$-blocks

$$L_{mn}=\frac{\ell_m}{k_m+k_n}C_n,$$

where $\ell_m=\ell_m(x,t)=\exp(k_mx+k_m^3t)$.

Then,

$$V=(C_1\ \dots\ C_N)\,(I_{N\mathsf{d}}+L^2)^{-1}\begin{pmatrix}\ell_1 I_{\mathsf{d}}\\ \vdots\\ \ell_N I_{\mathsf{d}}\end{pmatrix}$$

is a solution of the matrix mKdV (1) with values in the $\mathsf{d}\times\mathsf{d}$-matrices on every open set Ω on which $\det(I_{N\mathsf{d}}+L^2)\neq 0$.

Remark 2.2. For the matrix KdV equation $U_t=U_{xxx}+3\{U,U_x\}$, it is shown in [17, Section VIII] that the solution class corresponding to Theorem 2.1 contains the multisoliton solutions as constructed by Goncharenko [33] using the inverse scattering method.

Proof. The main ingredient of the proof is Theorem A.1, a solution formula for the matrix mKdV (1) established within the framework of operator theory. In order to make the link to Theorem A.1 we introduce the following notations.

Let A be the block diagonal matrix

$$A=\begin{pmatrix}k_1I_{\mathsf{d}} & & 0\\ & \ddots & \\ 0 & & k_NI_{\mathsf{d}}\end{pmatrix}$$

of size $N\mathsf{d}\times N\mathsf{d}$. Observe that $0\notin\operatorname{spec}(A)+\operatorname{spec}(A)$ by assumption.[2]Furthermore, let J, C be the block matrices

$$J=\begin{pmatrix}I_{\mathsf{d}} & \dots & I_{\mathsf{d}}\end{pmatrix},\qquad D=\begin{pmatrix}C_1 & \dots & C_N\end{pmatrix}$$

of size $\mathsf{d}\times N\mathsf{d}$.

Next, we denote by $e^{(j)}$ the jth row of J, which we identify with the functional given by multiplication with the row vector $e^{(j)}$ from the left, and by $d^{(j)}$ the jth column vector of D^T, where the superscript T stands for transposition. With this notation we find that the $N\mathsf{d}\times N\mathsf{d}$-matrix D^TJ can be written as

$$D^TJ=\begin{pmatrix}C_1^T & \dots & C_1^T\\ \vdots & & \vdots\\ C_N^T & \dots & C_N^T\end{pmatrix}=\sum_{j=1}^{\mathsf{d}}d^{(j)}e^{(j)}=\sum_{j=1}^{\mathsf{d}}e^{(j)}\otimes d^{(j)}.$$

In particular, D^TJ is d-dimensional. Note also that the functionals $e^{(j)}$, $j=1,\dots,\mathsf{d}$, are linearly independent. Finally, it is straightforward to verify that

$$B=\left(\frac{1}{k_m+k_n}C_m^T\right)_{m,n=1}^{N}$$

is the (unique) solution of the d-dimensionality condition $AB+BA=D^TJ$. Note that existence and uniqueness of B follow from classical results on the Sylvester equation. Concretely, we use that the spectrum of the operator $X\mapsto AX+XA$ equals $\operatorname{spec}(A)+\operatorname{spec}(A)$, see, e.g., [6].

Now, we are in the position to apply Theorem A.1, which implies that

$$W=\left(\left\langle(I_{N\mathsf{d}}+F^2)^{-1}\widehat{F}\,d^{(j)},e^{(i)}\right\rangle\right)_{i,j=1}^{\mathsf{d}},$$

with $F=\widehat{F}B$, $\widehat{F}=\exp(Ax+A^3t)$, is a solution of the matrix mKdV (1). Since $(I_{N\mathsf{d}}+F^2)\widehat{F}=\left(I_{N\mathsf{d}}+(\widehat{F}B)^2\right)\widehat{F}=\widehat{F}\left(I_{N\mathsf{d}}+(B\widehat{F})^2\right)$, and hence

$$\left(I_{N\mathsf{d}}+F^2\right)^{-1}\widehat{F}\ =\ \widehat{F}\left(I_{N\mathsf{d}}+(B\widehat{F})^2\right)^{-1},$$

we get

$$W=\left(\left\langle\widehat{F}\left(I_{N\mathsf{d}}+(B\widehat{F})^2\right)^{-1}d^{(j)},e^{(i)}\right\rangle\right)_{i,j=1}^{\mathsf{d}}.$$

As we are in the finite-dimensional case, we can rewrite application of the functional $e^{(j)}$ in terms of multiplication with the corresponding row vector from the left, which yields

$$\begin{aligned}W&=\left(e^{(i)}\widehat{F}\left(I_{N\mathsf{d}}+(B\widehat{F})^2\right)^{-1}d^{(j)}\right)_{i,j=1}^{\mathsf{d}}\\&=J\left(\widehat{F}\left(I_{N\mathsf{d}}+(B\widehat{F})^2\right)^{-1}\right)D^T.\end{aligned}$$

In order to recover the solution V stated in the theorem, we finally observe that the matrix mKdV (1) is invariant under transposition, which means that if W is a solution of (1) then also $V:=W^T$. Moreover,

$$\begin{aligned}V&=D\left(\widehat{F}\left(I_{N\mathsf{d}}+(B\widehat{F})^2\right)^{-1}\right)^TJ^T\\&=D\left(I_{N\mathsf{d}}+(\widehat{F}^TB^T)^2\right)^{-1}\widehat{F}^TJ^T\\&=D\left(I_{N\mathsf{d}}+(\widehat{F}B^T)^2\right)^{-1}\widehat{F}J^T,\end{aligned}$$

since $\widehat{F}^T=\widehat{F}$, with

$$\widehat{F}B^T=\left(\frac{\ell_m}{k_m+k_n}C_n\right)_{m,n=1}^{N}\quad\text{and}\quad\widehat{F}J^T=\begin{pmatrix}\ell_1I_{\mathsf{d}}\\ \vdots\\ \ell_NI_{\mathsf{d}}\end{pmatrix}$$

as claimed. □

Remark 2.3. In [68], solution formulas in the spirit of Theorem 2.1 are given for the matrix mKdV equation $V_t=V_{xxx}+3\,(V\,V^{\mathrm{T}}V_x+V_x\,V^{\mathrm{T}}V)$, based on a unified treatment of the AKNS system. Note that here V is no longer assumed to be a square matrix. As for applications, [68] proceeds to an asymptotic description of scalar multiple-pole solutions[3] and a complete asymptotic description of vector 2-solitons.

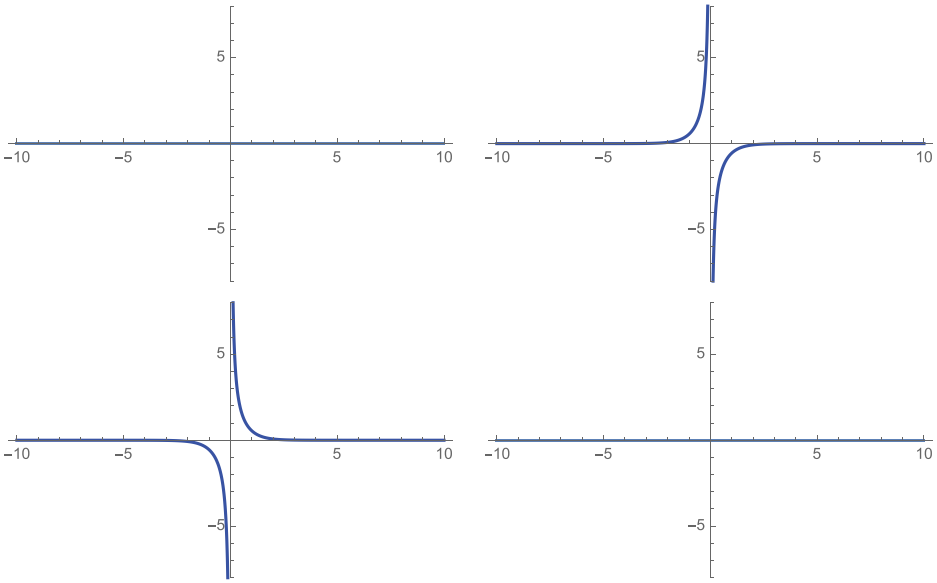

FIGURE 1 | A snapshot (at the time $t = 0$) of the generalized 1-soliton in Example 2.4 for $k = 2$, $c = 1$.

In the case of the scalar mKdV, one is mainly interested in real solutions. We therefore add two comments on the matrix case:

1. Obviously, for real spectral data $k_1, \dots, k_N$, $C_1, \dots, C_N$ the solution V in Theorem 2.1 is real.
2. In the scalar case, it is well-known that also complex conjugate pairs in the spectral data lead to real solutions (the corresponding building blocks being breathers). This is still true in the matrix case. For a formal proof in the case $N = 2$ with $k_2 = \overline{k}_1, C_2 = \overline{C}_1$, we refer to [19].

The following example of (real) generalized 1-solitons with values in the 2×2-matrices shows that the solutions in Theorem 2.1 may become singular.

Example 2.4. Let $N = 1$, $\mathsf{d} = 2$, and $k \in \mathbb{R}$ with $k \neq 0$. In this example, we consider a solution class for which the phase matrix C has a pair of purely imaginary eigenvalues. Specifically, let

$$C = 2kc\,D \text{ with } D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } c \in \mathbb{R}.$$

Observe that $D^2 = -I_2$. Hence, with the abbreviation $f(x,t) = c\exp(kx + k^3 t)$, from Theorem 2.1 we get the solution

$$V = 2k\, fD\Big(I_2 + (fD)^2\Big)^{-1} = 2k\,\frac{f}{1-f^2}D,$$

meaning that the soliton is singular along the lines $kx + k^3 t + \log|c| = 0$, see Figure 1 for an illustration.

We conclude the section with the following proposition clarifying regularity for generalized 1-solitons in the case that k is real.

Proposition 2.5. *For $N = 1$ and $k \in \mathbb{R}$, the solution in Theorem 2.1 is regular if and only if* $\operatorname{spec}(C) \cap \mathrm{i}\mathbb{R} = \emptyset$.

Proof. For $N = 1$, and with the abbreviation $f = \ell/(2k)$, the solution obtained from Theorem 2.1 reads

$$V = 2k\, fC\big(I + (fC)^2\big)^{-1}.$$

Since bringing C to Jordan form does not change regularity of V, we may assume that C is in Jordan form. This shows that V is regular if and only if

$$0 \neq \det\big(I + (fC)^2\big) = \prod_{c \in C} \big(1 + (cf)^2\big)^{\mu_c},$$

where C is the set of eigenvalues of C, and where, for $c \in C$, μ_c denotes the algebraic multiplicity of c. From this it is easy to see that factors $1 + (cf)^2$ with $\operatorname{Re}(c) \neq 0$ do not vanish. In contrast, for $c = \mathrm{i}\gamma, \gamma \in \mathbb{R}$, the factor $1 + (cf)^2$ vanishes along the line $kx + k^3 t + \log|\gamma/(2k)| = 0$. □

3 | A Particular Solution Class

This section presents a systematic treatment of a solution class that includes some examples (with $N = 2$) examined in [15] as special cases.

Proposition 3.1 (Generalized N-solitons with common phase matrix). *Let $k_1, \dots, k_N$ be real numbers such that $k_m + k_n \neq 0$ for all m, n, and let C be a real $\mathsf{d} \times \mathsf{d}$-matrix. Furthermore, let $\varphi_m \in \mathbb{R}$ and $\epsilon_m = \pm 1$. Then*

$$V = (C \ \dots \ C)\,(I_{N\mathsf{d}} + F^2)^{-1} \begin{pmatrix} f_1 I_{\mathsf{d}} \\ \vdots \\ f_N I_{\mathsf{d}} \end{pmatrix}, \quad F = \left(\frac{f_m}{k_m + k_n} C\right)_{m,n=1}^{N},$$

and $f_m(x,t) = \epsilon_m \exp\big(k_m(x + k_m^2 t + \varphi_m)\big)$, solves the matrix mKdV (1) with values in the $\mathsf{d} \times \mathsf{d}$-matrices on open sets in $\{(x,t) \mid \det(I_{N\mathsf{d}} + F^2) \neq 0\}$.

Here, as usual, φ_m assigns an initial position shift to the mth soliton, and the sign ϵ_m can be used to switch between soliton and antisoliton.

Proof. The solution class in the proposition is a subclass of the solution class in Theorem 2.1, obtained from choosing $C_m = \gamma_m C$, where $\gamma_m := \epsilon_m \exp(k_m \varphi_m)$. In fact, this choice leads to the solution

$$V = (\gamma_1 C \ \dots \ \gamma_N C)\,(I_{Nd} + L^2)^{-1} \begin{pmatrix} \ell_1 I_d \\ \vdots \\ \ell_N I_d \end{pmatrix}, \quad L = \left(\frac{\gamma_n \ell_m}{k_m + k_n} C \right)_{m,n=1}^{N}. \tag{2}$$

Introducing the block diagonal matrix $\Gamma = \mathrm{diag}\{\gamma_1 I_d, \dots, \gamma_N I_d\}$, we observe that $L = \Gamma^{-1} F \Gamma$ and hence

$$\begin{aligned} (I_{Nd} + L^2)^{-1} &= \left(I_{Nd} + (\Gamma^{-1} F \Gamma)^2\right)^{-1} = (I_{Nd} + \Gamma^{-1} F^2 \Gamma)^{-1} \\ &= \left(\Gamma^{-1}(I_{Nd} + F^2)\Gamma\right)^{-1} = \Gamma^{-1}(I_{Nd} + F^2)^{-1}\Gamma. \end{aligned}$$

Furthermore,

$$(\gamma_1 C \ \dots \ \gamma_N C) = (C \ \dots \ C)\Gamma, \qquad \begin{pmatrix} \ell_1 I_d \\ \vdots \\ \ell_N I_d \end{pmatrix} = \Gamma^{-1} \begin{pmatrix} f_1 I_d \\ \vdots \\ f_N I_d \end{pmatrix}.$$

Using the above identities in (2), we have rewritten V in the form claimed in the proposition. □

Remark 3.2. Note that the beginning of the proof also shows that replacing the occurrencies of C in the statement of the proposition by non-zero multiples $C_1, \dots, C_N$ of some common matrix C would not enlarge the solution class.

Lemma 3.3. *Up to a change of basis[4] the phase matrix C in Proposition 3.1 can be assumed to be in Jordan form.*

Proof. Since C is real, there is a real, invertible matrix T such that $T^{-1} C T = J$, where J is in Jordan form. Let $\mathcal{T}$ be the block diagonal matrix with the N identical blocks T on the diagonal. Then

$$F = \mathcal{T} \left(\frac{f_m}{k_m + k_n} J \right)_{m,n=1}^{N} \mathcal{T}^{-1} =: \mathcal{T} F_J \mathcal{T}^{-1}$$

and hence, by a similar manipulation as in the proof of Proposition 3.1, we obtain $(I_{Nd} + F^2)^{-1} = \mathcal{T}\left(I_{Nd} + F_J^2\right)^{-1} \mathcal{T}^{-1}$. Furthermore,

$$(C \ \dots \ C) = T\,(J \ \dots \ J)\,\mathcal{T}^{-1}, \qquad \begin{pmatrix} f_1 I_d \\ \vdots \\ f_N I_d \end{pmatrix} = \mathcal{T} \begin{pmatrix} f_1 I_d \\ \vdots \\ f_N I_d \end{pmatrix} T^{-1}.$$

Putting the above identities together, we find that $V = T V_J T^{-1}$ with V_J the solution one gets by replacing C with J in Proposition 3.1. □

A similar argument can be applied to phase matrices $C_1, \dots, C_N$ which can be brought to Jordan form simultaneously.

Lemma 3.3 allows us to make the following

Assumption 3.4. C is in Jordan form.

Starting from a real matrix C, its Jordan form still may contain pairs of complex conjugated eigenvalues. For simplicity, we assume in the following that this is not the case.

Assumption 3.5. The eigenvalues of C are real.

By now our assumptions for the solution class under consideration are complete. Our analysis in the rest of the section proceeds in three steps. In Section 3.1, we first look at the case $N = 1$, in which we derive explicit formulas for the entries of the solution. Next, in Section 3.2, we consider the case that C is diagonal. Then, this is also true for the solution, the diagonal entries being N-solitons which are identical up to an *overall shift*. Observe that such an overall shift effects the individual solitons differently (see Remark 3.6 and Example 3.7). Finally, we turn to the general case. The result is an explicit entry-wise description of the solution.

3.1 | Generalized 1-Solitons

In the case $N = 1$, the solution can be easily be rewritten in the form

$$V = 2k\,(gC)\left(I_d + (gC)^2\right)^{-1},$$

where $g(x,t) = f(x,t)/(2k)$.

First, we consider the case that C is a single Jordan block with eigenvalue c. Observe

$$\left(I_d + (gC)^2\right)^{-1} = \frac{1}{2}\left((I_d + \mathrm{i}gC)^{-1} + (I_d - \mathrm{i}gC)^{-1}\right).$$

Let us denote by J the Jordan block with eigenvalue 0 such that we can write $C = cI_d + J$. Then,

$$I_d \pm \mathrm{i}gC = (1 \pm \mathrm{i}cg)\left(I_d - \frac{\mp \mathrm{i}g}{1 \pm \mathrm{i}cg} J\right)$$

$$\Longrightarrow \ (I_d \pm \mathrm{i}gC)^{-1} = \frac{1}{1 \pm \mathrm{i}cg} \sum_{j \geq 0} \left(\frac{\mp \mathrm{i}g}{1 \pm \mathrm{i}cg} \right)^j J^j$$

from the Neumann series. Note that the series terminates since $J^d = 0$. This shows that $\left(I_d + (gC)^2\right)^{-1} = \sum_{j \geq 0} \gamma_j J^j$, where

$$\gamma_j = \mathrm{Re}\left(\frac{(-\mathrm{i}g)^j}{(1 + \mathrm{i}cg)^{j+1}} \right).$$

Hence,

$$V = 2k\,g(cI_d + J) \sum_{j \geq 0} \gamma_j J^j = 2k\,g\left(\gamma_0 c\, I_d + \sum_{j \geq 1} (\gamma_j c + \gamma_{j-1}) J^j \right),$$

and we compute

$$\gamma_j c + \gamma_{j-1} = \mathrm{Re}\left(\frac{(-\mathrm{i}g)^{j-1}}{(1 + \mathrm{i}cg)^{j+1}} \right) = \frac{(-1)^j g^{j-1}}{(1 + (cg)^2)^{j+1}} \mathrm{Re}\left((cg + \mathrm{i})^{j+1}\right).$$

Summing up, the generalized 1-soliton V is an upper band matrix with entries v_j on the diagonal ($j = 0$) respective jth superdiagonal ($j \geq 1$), where

$$v_0 = 2k\frac{cg}{1+(cg)^2}, \tag{3}$$

$$v_j = 2k\frac{(-g)^j}{(1+(cg)^2)^{j+1}}\,\mathrm{Re}\big((cg+\mathrm{i})^{j+1}\big),\ j \geq 1. \tag{4}$$

It should be mentioned that the real part appearing in the above formula can be evaluated using the fact that

$$\mathrm{Re}\big((cg+\mathrm{i})^{j+1}\big) = \sum_{j'=0}^{[j/2]}(-1)^{j'}\binom{j}{2j'}(cg)^{j-2j'}. \tag{5}$$

If C consists of several Jordan blocks, $C = \mathrm{diag}\{C_1, \dots, C_M\}$, it is easy to see that $V = \mathrm{diag}\{V_1, \dots, V_M\}$, the block V_m corresponding to the Jordan block C_m, and of the form described above.

3.2 | Diagonal Phase Matrix

Next, we consider the case that C is diagonal, with N arbitrary. In the following, we use a permutation argument to see that the solution V is diagonal, and to identify its diagonal entries.

Let $C = \mathrm{diag}\{c_1, \dots, c_\mathsf{d}\}$. Denoting by $e_1, \dots, e_\mathsf{d}$ the standard unit vectors in $\mathbb{R}^\mathsf{d}$ (considered as row vectors), we introduce the $N\mathsf{d} \times N\mathsf{d}$ permutation matrix

$$\Pi = \begin{pmatrix}\Pi_1\\ \vdots \\ \Pi_\mathsf{d}\end{pmatrix} \text{with the blocks } \Pi_j = \begin{pmatrix} e_j & 0_{1\times\mathsf{d}} & \dots & 0_{1\times\mathsf{d}}\\ 0_{1\times\mathsf{d}} & e_j & \dots & 0_{1\times\mathsf{d}}\\ \vdots & & \ddots & \vdots\\ 0_{1\times\mathsf{d}} & \dots & \dots & e_j\end{pmatrix}$$

of size $N \times N\mathsf{d}$.

Then, $\Pi F \Pi^{-1}$ is block diagonal, more precisely $\Pi F \Pi^{-1} = G = \mathrm{diag}\{G_1, \dots, G_\mathsf{d}\}$, where

$$G_j = c_j\left(\frac{f_m}{k_m + k_n}\right)_{m,n=1}^{N},$$

yielding $(I_{N\mathsf{d}} + F^2)^{-1} = (I_{N\mathsf{d}} + (\Pi^{-1}G\Pi)^2)^{-1} = (I_{N\mathsf{d}} + \Pi^{-1}G^2\Pi)^{-1}$ $= (\Pi^{-1}(I_{N\mathsf{d}} + G^2)\Pi)^{-1} = \Pi^{-1}(I_{N\mathsf{d}} + G^2)^{-1}\Pi = \Pi^{-1}\,\mathrm{diag}\{(I_N + G_j^2)^{-1} \mid j = 1, \dots, \mathsf{d}\}\,\Pi$. Furthermore,

$$(C \dots C)\,\Pi^{-1} = \mathrm{diag}\{c_j e \mid j = 1, \dots, \mathsf{d}\},$$

$$\Pi\begin{pmatrix} f_1 I_\mathsf{d}\\ \vdots \\ f_N I_\mathsf{d}\end{pmatrix} = \mathrm{diag}\{f \mid j = 1, \dots, \mathsf{d}\},$$

where $e \in \mathbb{R}^N$ is the row vector with all entries equal to 1 and $f = (f_1, \dots, f_N)^T$.

As a result, the solution

$$V = \mathrm{diag}\left\{\, v_j \mid j = 1, \dots, \mathsf{d}\right\} \text{ with } v_j = c_j\, e\Big(I_N + G_j^2\Big)^{-1} f$$

is diagonal. For $c_j \neq 0$, by comparison with the scalar solution formula, v_j is the N-soliton with $k_1, \dots, k_N$ determining the asymptotic speeds, initial position shifts $\varphi_1 + (\log|c_j|)/k_1, \dots, \varphi_N + (\log|c_j|)/k_N$, and signs $\epsilon_1\,\mathrm{sgn}(c_j), \dots, \epsilon_N\,\mathrm{sgn}(c_j)$. In contrast, for $c_j = 0$, the entry v_j vanishes identically.

Remark 3.6. Note that the sign $\mathrm{sgn}(c_j)$ allows to switch *all* involved solitons to antisolitons and vice versa. In addition, $|c_j|$ induces an *overall shift* affecting the N-soliton appearing at the jth entry of the solutions diagonal, which amounts to *individual* shifts of the paths of the involved solitons.

Example 3.7. For an illustration, we look at the 3-soliton solution $v = v(x,t)$, the involved solitary waves no. 1, no. 2, no. 3 determined by $k_1 = 2$, $k_2 = 1.5$, $k_3 = 1$, respectively. Moreover, we choose

a. $\epsilon_1 = 1, \epsilon_2 = -1, \epsilon_3 = 1$,

b. $\varphi_1 = (\log 84)/2, \varphi_2 = (\log 105)/1.5, \varphi_3 = \log 30$.

Note that *a.* means that the solitary waves no. 1, no. 3 are solitons, and wave no. 2 an antisoliton. As for *b.*, an asymptotic analysis shows that it implies that the phase-shifts the solitary waves experience during collision are equally distributed in $t \approx \pm\infty$. In fact, it can be verified that $v(x,t) = v(-x,-t)$.

Let us now look at the solution V obtained from Proposition 3.1 with the same parameters as above and phase matrix

$$C = \begin{pmatrix} c_1 & 0\\ 0 & c_2\end{pmatrix}, \text{ where } c_1 = 1, c_2 = -e^{10}.$$

Then, $V = \mathrm{diag}\{v_1, v_2\}$ with $v_1 = v$ as described above. Note that, since $c_1 = 1$, there is no additional effect on v_1. However, for v_2 we get an overall shift caused by $|c_2|$, which results in different position shifts for the solitary waves, and, since $\mathrm{sgn}(c_2) = -1$, solitons switch to antisolitons and vice versa. For an illustration, see Figure 2.

3.3 | The General Case

Finally, we show how (generalizations of) the arguments used in Sections 3.1 and 3.2, together with an additional permutation argument, can be combined to cover the general case, $N \geq 1$ and C in Jordan form.

To this end, let $C = \mathrm{diag}\{C_1, \dots, C_D\}$ with $\mathsf{d}_j \times \mathsf{d}_j$-Jordan blocks C_j with eigenvalue c_j. Note that $\sum_{j=1}^{D}\mathsf{d}_j = \mathsf{d}$.

STEP 1 Our first aim is to extend the permutation argument in Section 3.2 to the case that the phase matrix C is block diagonal. The result will be that then also the solution V is block diagonal.

In fact, let us denote by E_j, for $j = 1, \dots D$, the $\mathsf{d}_j \times \mathsf{d}$ matrices

$$E_j = \Big(\underbrace{0}_{\mathsf{d}_j\times\sum_{i=1}^{j-1}\mathsf{d}_i}\ I_{\mathsf{d}_j}\ \underbrace{0}_{\mathsf{d}_j\times\sum_{i=j+1}^{D}\mathsf{d}_i}\Big),$$

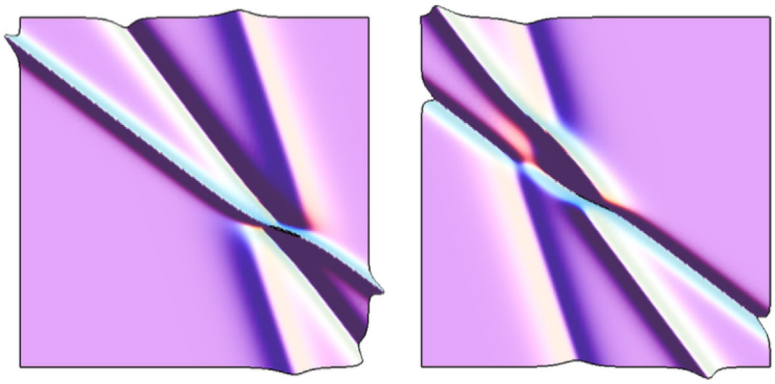

FIGURE 2 | The diagonal entries of the 3-soliton solution from Example 3.7 at $t = 0$. The plot shows the region $-22 \le x \le 8$, $-4 \le t \le 6$, and a range between $-5/2$ and $5/2$.

and introduce the $N\mathsf{d} \times N\mathsf{d}$ permutation matrix

$$\Pi = \begin{pmatrix} \Pi_1 \\ \vdots \\ \Pi_D \end{pmatrix}, \quad \text{with the blocks } \Pi_j = \begin{pmatrix} E_j & 0_{\mathsf{d}_j \times \mathsf{d}} & \dots & 0_{\mathsf{d}_j \times \mathsf{d}} \\ 0_{\mathsf{d}_j \times \mathsf{d}} & E_j & \dots & 0_{\mathsf{d}_j \times \mathsf{d}} \\ \vdots & & \ddots & \vdots \\ 0_{\mathsf{d}_j \times \mathsf{d}} & \dots & \dots & E_j \end{pmatrix}$$

of size $N\mathsf{d}_j \times N\mathsf{d}$. It is straightforward to verify that $\Pi F \Pi^{-1}$ is block diagonal. More precisely, $\Pi F \Pi^{-1} = G = \operatorname{diag}\{G_1, \dots, G_D\}$, where

$$G_j = \left(\frac{f_m}{k_m + k_n} C_j \right)_{m,n=1}^{N}.$$

Furthermore,

$$(C \dots C)\,\Pi^{-1} = \operatorname{diag}\{ \underbrace{(C_j \dots C_j)}_{N} \mid j = 1, \dots, D\},$$

$$\Pi \begin{pmatrix} f_1 I_{\mathsf{d}} \\ \vdots \\ f_N I_{\mathsf{d}} \end{pmatrix} = \operatorname{diag}\left\{ \begin{pmatrix} f_1 I_{\mathsf{d}_j} \\ \vdots \\ f_N I_{\mathsf{d}_j} \end{pmatrix} \mid j = 1, \dots, D \right\}.$$

This shows that

$$V = \operatorname{diag}\left\{ (C_j \dots C_j) \left(I_N + G_j^2 \right)^{-1} \begin{pmatrix} f_1 I_{\mathsf{d}_j} \\ \vdots \\ f_N I_{\mathsf{d}_j} \end{pmatrix} \mid j = 1, \dots, D \right\}.$$

Hence, we can restrict to studying the case that the phase matrix C consists of a single Jordan block.

STEP 2 Let now C be a single Jordan block with eigenvalue c, of size $\mathsf{d} \times \mathsf{d}$. Roughly speaking, the next aim is to rewrite G, which is an $N\mathsf{d} \times N\mathsf{d}$-matrix built from $\mathsf{d} \times \mathsf{d}$-blocks, in terms of $N \times N$-blocks.

To this end, we apply again a permutation argument. Considering the permutation matrix Π from Section 3.2, it is not hard to see that $\mathcal{G} = \Pi G \Pi^{-1}$ is built from the $N \times N$-blocks $c\mathcal{F}$ on the diagonal, the blocks $\mathcal{F}$ on the first superdiagonal, and zero-blocks else, where

$$\mathcal{F} = \left(\frac{f_m}{k_m + k_n} \right)_{m,n=1}^{N}.$$

In other words,

$$\mathcal{G} = \mathcal{F}\mathcal{C},$$

where $\mathcal{C}$ is the Jordan-type block matrix of size $N\mathsf{d} \times N\mathsf{d}$ with the blocks cI_N on the diagonal, the blocks I_N on the first superdiagonal and zero blocks else.

Furthermore, the $\mathsf{d} \times N\mathsf{d}$-matrix $(C \dots C)\Pi^{-1}$ is built from the $1 \times N$-blocks $c\,e$ on the diagonal, the blocks e on the first superdiagonal, and zero-blocks else, where $e \in \mathbb{R}^N$ is the row vector with all entries equal to 1, or

$$(C \dots C)\Pi^{-1} = \operatorname{diag}\{e \mid j = 1, \dots, \mathsf{d}\}\, C.$$

Finally, the $N\mathsf{d} \times \mathsf{d}$-matrix $\Pi(f_1 I_{\mathsf{d}} \dots f_N I_{\mathsf{d}})^T$ is block diagonal with the $N \times 1$-blocks f on the diagonal, where $f = (f_1, \dots, f_N)^T$, that is,

$$\Pi \begin{pmatrix} f_1 I_{\mathsf{d}} \\ \vdots \\ f_N I_{\mathsf{d}} \end{pmatrix} = \operatorname{diag}\{f \mid j = 1, \dots, \mathsf{d}\}.$$

As a result,

$$V = \operatorname{diag}\{e \mid j = 1, \dots, \mathsf{d}\}\, C \left(I_{N\mathsf{d}} + \mathcal{F}^2 \mathcal{C}^2 \right)^{-1} \operatorname{diag}\{f \mid j = 1, \dots, \mathsf{d}\}.$$

STEP 3 After this preparation, we are now in the position to extend the arguments in Section 3.1 to the situation at hand. Below we sketch the main steps.

Observe first that, since $\mathcal{F}\mathcal{C}$ is real,

$$\left(I_{N\mathsf{d}} + \mathcal{F}^2 \mathcal{C}^2 \right)^{-1} = \operatorname{Re}\left((I_{N\mathsf{d}} + \mathrm{i}\mathcal{F}\mathcal{C})^{-1} \right),$$

(the real part taken entrywise). Next, we introduce the $N\mathsf{d} \times N\mathsf{d}$-matrix $\mathcal{J}$ with the blocks I_N on the first superdiagonal and zero blocks else, such that we can write $\mathcal{C} = cI_{N\mathsf{d}} + \mathcal{J}$. Then

$$I_{N\mathsf{d}} + \mathrm{i}\mathcal{F}\mathcal{C} = I_{N\mathsf{d}} + \mathrm{i}\mathcal{F}(cI_{N\mathsf{d}} + \mathcal{J}) = (I_N + \mathrm{i}c\mathcal{F})I_{N\mathsf{d}} + \mathrm{i}\mathcal{F}\mathcal{J}$$

$$= (I_N + \mathrm{i}c\mathcal{F})\left(I_{N\mathsf{d}} + \mathrm{i}\mathcal{F}(I_N + \mathrm{i}c\mathcal{F})^{-1}\mathcal{J} \right).$$

Using the Neumann series, we find that $(I_{Nd} + \mathcal{F}^2\mathcal{C}^2)^{-1} = \sum_{j\geq 0} \Gamma_j \mathcal{J}^j$ with $\Gamma_j = \mathrm{Re}\left((-\mathrm{i}\mathcal{F})^j (I_N + \mathrm{i}c\mathcal{F})^{-(j+1)}\right)$, and hence

$$C(I_{Nd} + \mathcal{F}^2\mathcal{C}^2)^{-1} = c\Gamma_0 I_{Nd} + \sum_{j\geq 1} \left(c\Gamma_j + \Gamma_{j-1}\right)\mathcal{J}^j.$$

From this it is straightforward to see that the solution V is an upper band matrix with the entries with entries v_j on the diagonal ($j = 0$) respective jth superdiagonal ($j \geq 1$), where

$$v_0 = e\left(I_N + (c\mathcal{F})^2\right)^{-1}(cf), \tag{6}$$

$$v_j = (-1)^j e\left((I_N + (c\mathcal{F})^2\right)^{-(j+1)} \mathcal{F}^{j-1}\,\mathrm{Re}\left((\mathrm{i}I_N + c\mathcal{F})^{j+1}\right))f, \quad j \geq 1. \tag{7}$$

Note that, for the real part, a similar expression as in (5) holds since the binomial theorem can be applied to commuting matrices.

For $c \neq 0$, by comparison with the scalar formula, the diagonal entry is the N-soliton with $k_1, \ldots, k_N$ determining the asymptotic speeds, initial position shifts $\varphi_1 + (\log|c|)/k_1, \ldots, \varphi_N + (\log|c|)/k_N$, and signs $\epsilon_1 \mathrm{sgn}(c), \ldots, \epsilon_N \mathrm{sgn}(c)$.

For $c = 0$, the diagonal entry vanishes identically. However, this is not the case for the first superdiagonal entry, showing that also in this case the solution is non-trivial.

The following remark is due to one of the referees.

Remark 3.8.

a. The previous analysis uncovers additional structural properties, which are hard to predict from Proposition 3.1 and Assumptions 3.4 and 3.5. A condensed expression of this, pointed out to us by one of the referees, is that the investigated solutions all satisfy $VV_x = V_xV$ and are therefore even solutions of the equation $V_t = V_{xxx} + 6V^2V_x$ (without anticommutator). This property follows from the rewriting of our solutions in terms of blocks with Toeplitz matrices in Section 3.3.

b. For each block, one may view the functions on the upper diagonals as unknown functions of a differential system. In fact, systems of this type were already considered in [28, 46].

4 | Further Comments and Visualization

In this section, we elaborate on the results from Section 3. The first observation concerns regularity of the solution class in Proposition 3.1. We continue with its properties, illustrated by examples and computer plots, in particular addressing the question whether, without loss of generality, the k_m always can be chosen positive (as one usually does in the scalar case).

4.1 | Regularity

First, we would like to stress that the results in Section 3 imply that the solution class studied in this article is regular.

Proposition 4.1. *The members of the solution class described in Proposition 3.1, subject to Assumption 3.5, are globally regular.*

Proof. Note first that formulas (3), (4) for the generalized 1-soliton solution in Section 3.1 show that all of its entries are globally regular, independently of the eigenvalues of the phase matrix C.

In the case that C is diagonal, by Section 3.2 also the solution is diagonal, its diagonal entries being (scalar) N-solitons, and hence globally regular.

In the general case, regularity follows from formulas (6) and (7) by the following observation: in both formulas the same inverse $(I_N + c\mathcal{F})^{-1}$ appears. However, since (6) is a (scalar) N-soliton, this inverse necessarily exists for all (x, t). □

4.2 | Examples and Computer Plots

We start with a discussion of generalized 1-solitons, focusing on the case that C is a single Jordan block with eigenvalue c.

4.2.1 | The Case $c \neq 0$

Writing out the explicit formulas for the entries v_j of the diagonal ($j = 0$) and of the jth superdiagonal for $j = 1, 2, 3$, we get

$$\begin{aligned}
j = 0 &: \quad v_0 = \sigma\,|2k|\,\frac{\ell}{1+\ell^2},\\
j = 1 &: \quad v_1 = -\sigma\,\mathrm{sgn}(c)\left|\frac{2k}{c}\right|\ell\,\frac{\ell^2-1}{(1+\ell^2)^2},\\
j = 2 &: \quad v_2 = \sigma\left|\frac{2k}{c^2}\right|\ell^2\frac{\ell^3-3\ell}{(1+\ell^2)^3},\\
j = 3 &: \quad v_3 = -\sigma\,\mathrm{sgn}(c)\left|\frac{2k}{c^3}\right|\ell^3\,\frac{\ell^4-6\ell^2+1}{(1+\ell^2)^4},
\end{aligned}$$

where $\ell(x,t) = \exp\left(k(x + k^2t + \psi)\right)$ with $\psi \in \mathbb{R}$, and $\sigma = \pm 1$. Here, σ, ψ are defined by the parameters k, φ, ϵ, and c by the relations

$$\sigma = \epsilon\,\mathrm{sgn}(c), \qquad \psi = \varphi + (\log|c| - \log|2k|)/k.$$

Hence, on the diagonal appears the scalar 1-soliton of the mKdV with initial position shift ψ, the sign σ determining whether its shape is of the form +sech (*soliton*) or −sech (*antisoliton*).

Note that the modulus of c can be used to rescale the jth superdiagonal entries by the factor $|c|^{-j}$, and that changing the sign of c has the effect of reflecting all entries of jth superdiagonals with j odd in the x-axis.

In contrast, changing the sign σ reflects *all* entries in the x-axis.

For an illustration, see Figure 3.

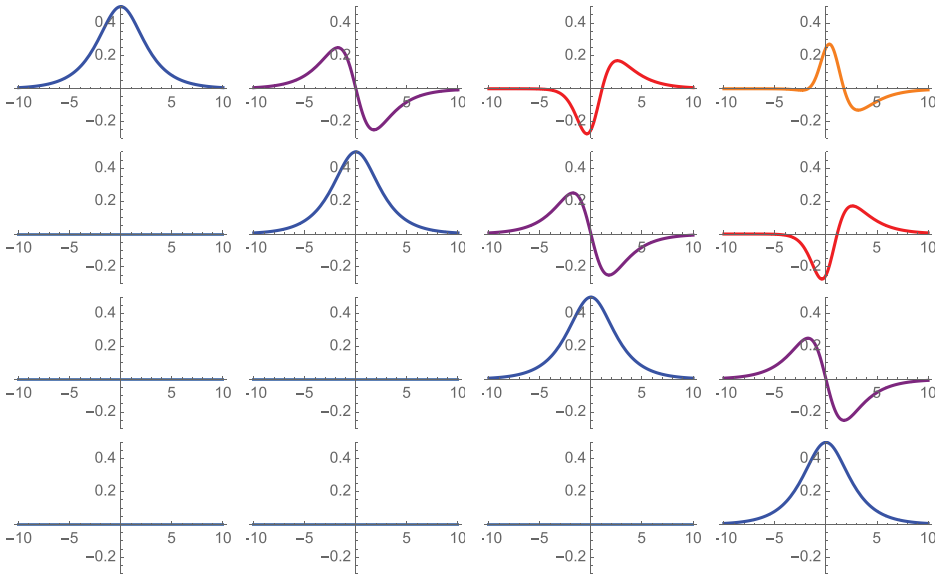

FIGURE 3 | A snapshot ($t = 0$) of a generalized 1-soliton solution with values in the 4×4-matrices. The phase matrix C is a 4×4-Jordan block with eigenvalue $c = 1$, and the parameters are $k = 1/2$, $\psi = 0$, $\sigma = 1$.

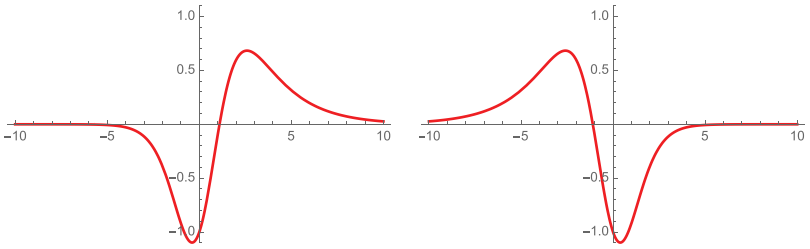

FIGURE 4 | The plots show the second superdiagonal entries of the generalized 1-soliton (at $t = 0$) in the case that C is a single Jordan block, and with parameters $\sigma = 1$, $\psi = 0$. To the left $k = 1/2$, $c = 1/2$, and to the right $k = -1/2$, $c = -1/2$. Note that the entries on the diagonal coincide, and that switching the sign of c also makes the first superdiagonal entries coincide.

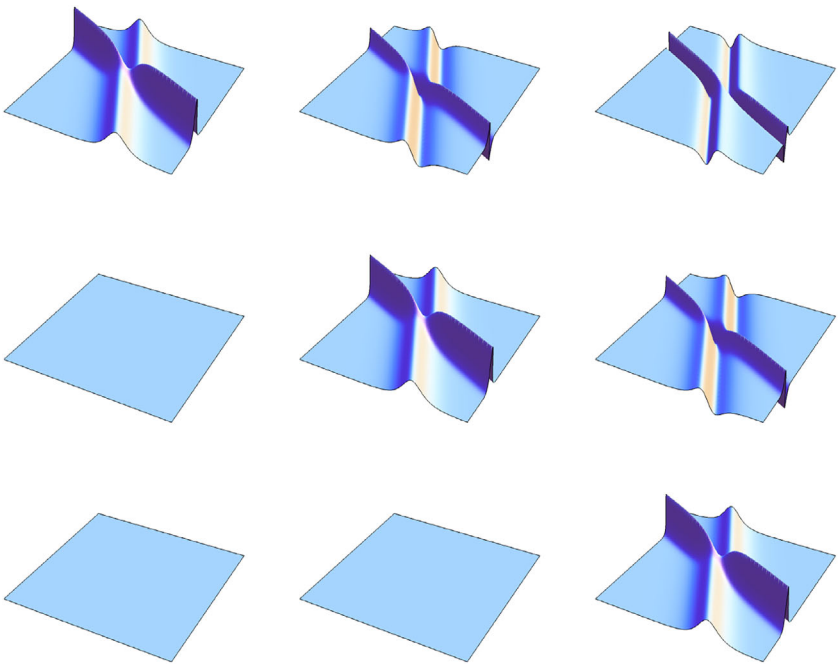

FIGURE 5 | The 2-soliton solution with values in the 3×3-matrices from Example 4.3 at $t = 0$. The plot shows the region $-10 \leq x \leq 10$, $-5 \leq t \leq 5$, and a range between -1 and 2.

4.2.2 | The Case $c = 0$

Observe first that the entries v_j on the diagonal and on all jth superdiagonals with j even vanish. However, on the jth superdiagonal with j odd, the entries are still present and become purely exponential,

$$v_j = (-1)^{(j-1)/2}\epsilon\,|2k|\,\ell^j, \tag{8}$$

where $\ell(x,t) = \exp\left(k(x + k^2t + \psi)\right)$ with $\psi = \varphi - (\log|2k|)/k$. It is remarkable that 1-solitons in the sense of [33] thus include non-localized solutions (see also [17] for the case of the matrix KdV equation).

Remark 4.2. In the scalar case $\mathrm{d} = 1$, one can restrict to $k > 0$ without loss of generality[5]: in fact, replacing k by $-k$ amounts to replacing ℓ by $1/\ell$ (up to a possible change of the initial position shift ψ), a transformation which leaves the solution invariant. In contrast, in the matrix-valued case $\mathrm{d} > 1$, the situation stepwise becomes more complicated:

For $\mathrm{d} = 2$, this is still true provided that $c \neq 0$. In fact, replacing ℓ by $1/\ell$ transforms the first superdiagonal entry v_1 to $-v_1$, an effect which also can be achieved by changing the sign of c. However, for $c = 0$, this is no longer the case. Indeed, looking at (8), replacing k by $-k$ amounts to a reflection along the y-axis which cannot be counteracted by changing the sign of ϵ.

For $\mathrm{d} > 2$ we also have to take into account the second superdiagonal entry v_2. As can be seen from Figure 4, in this case the effect of replacing ℓ by $1/\ell$ cannot be achieved by changing parameters.

We conclude with an example on 2-solitons in the case that the phase matrix is a 3×3-Jordan block.

Example 4.3. For the 2-soliton solution with values in the 3×3-matrices and parameters $k_1 = 2$, $k_2 = 1$, and C a single Jordan block with eigenvalue $c = 1$, the resulting solution is depicted in Figure 5.

By a similar asymptotic reasoning as in Example 3.7, we have chosen $\varphi_1 = (\log 12)/2$, $\varphi_2 = \log 6$, and $\epsilon_1 = \epsilon_2 = 1$ for the remaining parameters.

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Conflicts of Interest

The authors have no conflicts of interest.

Data Availability Statement

The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

Endnotes

[1] Note that the terminology *spectral matrix* is used in [33], and in earlier work of the authors [15, 20].

[2] For two sets L and K, $L + K$ denotes their Minkowski sum $\{\ell + k \mid \ell \in L,\, k \in K\}$.

[3] It should be mentioned that this result is complete in the sense that multiple-pole solutions built from both solitons, antisolitons and breathers are included. The essential assumption needed is that different wave packages move with different velocities, excluding degeneracies as in [48].

[4] That is, there is an invertible matrix T, real (and not depending on x, t), such that $T^{-1}VT$ is in the form as claimed in the lemma. It should be mentioned that conjugation of V with T preserves the solution property of V.

[5] This is well-known for N-soliton solutions, and also holds for multipole solutions, see [67] for a rigorous proof in the case of the Nonlinear Schrödinger equation.

[6] With the obvious replacement of L by iL in [10, Theorem 7.2(a)].

[7] The determinant formulas in Proposition A.2 and [17, Theorem 11 (b)] differ by a minus sign coming from a corresponding difference in a minus sign in the underlying operator solutions.

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Appendix A: Proof of Theorem A.1 and Background Material

In this Appendix, we provide the proof of Theorem A.1, the main ingredient for the solution formula in Theorem 2.1, and establish the link to results for the matrix modified KdV *hierarchy* in [16, 17]. The framework of generalized determinants on quasi-Banach ideals in the sense of [58], which is not strictly necessary for this work, is mainly kept for consistency.

Let us first recall some notation. For a Banach space E, we denote by E' the dual Banach space, by $\mathcal{L}(E)$ the Banach algebra of bounded linear operators on E, and by I_E the identity operator on E. Furthermore, for $d \in E'$, $c \in E$, the one-dimensional operator $d \otimes c$ is given by $(d \otimes c)x = \langle x, d\rangle c$ where the dual pairing $\langle x, d\rangle$ stands for the evaluation of d on x.

Theorem A.1. *Let E be a Banach space and $A \in \mathcal{L}(E)$ with $0 \notin \operatorname{spec}(A) + \operatorname{spec}(A)$. Moreover, let $d^{(j)} \in E'$ be linearly independent, $c^{(j)} \in E$ $(j = 1, \dots, \mathsf{d})$, and $B \in \mathcal{L}(E)$ a solution of the* d*-dimensionality condition*

$$AB + BA = \sum_{j=1}^{\mathsf{d}} d^{(j)} \otimes c^{(j)}.$$

Denote by $\widehat{L} = \widehat{L}(x,t)$, $L = L(x,t)$ the operator functions $\widehat{L}(x,t) = \exp(Ax + A^3 t)$ and $L = \widehat{L}B$. Then

$$V = \left(\left\langle (I_E + L^2)^{-1} \widehat{L}\, c^{(j)}, d^{(i)} \right\rangle \right)_{i,j=1}^{\mathsf{d}}$$

is a solution of the matrix mKdV (1) with values in the $\mathsf{d} \times \mathsf{d}$*-matrices on every open subset Ω on which $I_E + L^2$ is invertible.*

Proof. The first step in the proof consists of the verification that the operator function $\mathcal{V} = (I_E + L^2)^{-1}(AL + LA)$ solves the non-commutative modified KdV equation (1) with values in the bounded linear operators on E. This is done in [10], Theorem 7.2 (a).[6]

In the second step, the projection technique from [66, Section 3.1], see also [17, Section IV], is applied. We briefly sketch the argument. Let

$$S_{d^{(1)},\dots,d^{(\mathsf{d})}}(E) = \left\{ \sum_{j=1}^{\mathsf{d}} d^{(j)} \otimes x^{(j)} \;\middle|\; x^{(1)}, \dots, x^{(\mathsf{d})} \in E \right\}.$$

Observe that $S_{d^{(1)},\dots,d^{(\mathsf{d})}}(E)$ is a Banach algebra, and a left ideal in $\mathcal{L}(E)$. Moreover, the map σ defined by

$$\sigma\left(\sum_{j=1}^{\mathsf{d}} d^{(j)} \otimes x^{(j)} \right) = \left(\langle x^{(j)}, d^{(i)} \rangle \right)_{i,j=1}^{\mathsf{d}},$$

which takes its values in the $\mathsf{d} \times \mathsf{d}$-matrices, is an algebra homomorphism on $S_{d^{(1)},\dots,d^{(\mathsf{d})}}(E)$. Now, by the d-dimensionality condition, $\mathcal{V} \in S_{d^{(1)},\dots,d^{(\mathsf{d})}}(E)$, and since application of σ preserves the solution property, $V = \sigma(\mathcal{V})$ solves (1) in the $\mathsf{d} \times \mathsf{d}$-matrices. □

The following proposition gives a reformulation of the solution in Theorem A.1 in terms of generalized determinants on quasi-Banach operator ideals (in the sense of [58]).

Proposition A.2. *Let $\mathcal{A}$ be a quasi-Banach operator ideal admitting a continuous determinant δ. Then, for the solution V of the matrix mKdV (1) in Theorem A.1, it holds*

$$V = \frac{\mathrm{i}}{2} \left(\frac{\delta\big(I_E - \mathrm{i}(L + L^{(i,j)})\big)}{\delta(I_E - \mathrm{i}L)} - \frac{\delta\big(I_E + \mathrm{i}(L + L^{(i,j)})\big)}{\delta(I_E + \mathrm{i}L)} \right)_{i,j=1}^{\mathsf{d}},$$

where

$$L^{(i,j)} = \exp(Ax + A^3 t)\big(d^{(i)} \otimes c^{(j)}\big).$$

Remark A.3. In [17], Theorem 11 (b), a natural counterpart[7] of Proposition A.2 is given for the whole matrix mKdV hierarchy

$$V_{t_{2k-1}} = \Psi(V)^{k-1} V_x, \quad k \geq 1,$$

where

$$\Psi(V) = (D + C_V D^{-1} C_V)(D + A_V D^{-1} A_V),$$

and C_V, A_V denote the commutator, anti-commutator with V, respectively. Note that the use of the recursion operator $\Psi(V)$, in particular of D^{-1}, requires that V decreases sufficiently fast for $x \to -\infty$. This is guaranteed by posing the stronger spectral assumption $\mathrm{spec}(A) \subseteq \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > 0\}$ on A, and restricting the domain of definition of V.

Proof. We start with the observation that all determinants in Proposition A.2 are defined. To this end, it suffices to show $B \in \mathcal{A}(E)$.

Indeed, since $0 \notin \mathrm{spec}(A) + \mathrm{spec}(A)$, the d-dimensionality condition, considered on the quasi-Banach ideal $\mathcal{A}$, has a (unique) solution $B_0 \in \mathcal{A}(E)$, see [10, Section 4] for more details. Obviously, as $\mathcal{A}(E) \subseteq \mathcal{L}(E)$, the operator B_0 also solves the d-dimensionality condition on $\mathcal{L}$. But so does the bounded operator B, and hence, by uniqueness $B = B_0 \in \mathcal{A}(E)$.

In the subsequent computations, we use that

1. on operators $I_E + T$, $T \in \mathcal{A}(E)$, the determinant δ is multiplicative,
2. for one-dimensional operators T it holds that $\delta(I_E + T) = 1 + \tau(T)$, where τ is the trace corresponding to the determinant δ according to the trace-determinant theorem [58],
3. the trace τ is linear,
4. the identity $(I_E - \mathrm{i}L)^{-1} + (I_E + \mathrm{i}L)^{-1} = 2(I_E + L^2)^{-1}$ holds.

Hence,

$$\begin{aligned}\frac{\delta\big(I_E \pm \mathrm{i}(L + L^{(i,j)})\big)}{\delta(I_E \pm \mathrm{i}L)} &= \delta\big((I_E \pm \mathrm{i}L)^{-1}\big((I_E \pm \mathrm{i}L) \pm \mathrm{i}L^{(i,j)}\big)\big) \\ = \delta\big(I_E \pm \mathrm{i}(I_E \pm \mathrm{i}L)^{-1} L^{(i,j)}\big) &= 1 \pm \mathrm{i}\,\tau\big((I_E \pm \mathrm{i}L)^{-1} L^{(i,j)}\big),\end{aligned}$$

showing that

$$\begin{aligned}&\frac{\delta\big(I_E - \mathrm{i}(L + L^{(i,j)})\big)}{\delta(I_E - \mathrm{i}L)} - \frac{\delta\big(I_E + \mathrm{i}(L + L^{(i,j)})\big)}{\delta(I_E + \mathrm{i}L)} \\ &= \big(1 - \mathrm{i}\,\tau\big((I_E - \mathrm{i}L)^{-1} L^{(i,j)}\big)\big) - \big(1 + \mathrm{i}\,\tau\big((I_E + \mathrm{i}L)^{-1} L^{(i,j)}\big)\big) \\ &= -\mathrm{i}\,\tau\left(\big((I_E - \mathrm{i}L)^{-1} + (I_E + \mathrm{i}L)^{-1}\big) L^{(i,j)}\right) \\ &= -2\mathrm{i}\,\tau\left((I_E + L^2)^{-1} L^{(i,j)}\right) \\ &= -2\mathrm{i}\,\tau\Big((I_E + L^2)^{-1} \widehat{L}\,\big(d^{(i)} \otimes c^{(j)}\big)\Big) \\ &= -2\mathrm{i}\,\tau\left(d^{(i)} \otimes \left((I_E + L^2)^{-1} \widehat{L}\,\big(c^{(j)}\big)\right)\right).\end{aligned}$$

To conclude the proof, it remains to verify

$$\tau\Big(d^{(i)} \otimes \big((I_E + L^2)^{-1} \widehat{L}\big(c^{(j)}\big)\big)\Big) = \big\langle (I_E + L^2)^{-1} \widehat{L}\big(c^{(j)}, d^{(i)}\big\rangle.$$

To this end, observe that any trace τ on a quasi-Banach ideal $\mathcal{A}$ by definition coincides with the unique trace tr on the operator ideal of finite-dimensional operators. Hence in particular $\tau(d \otimes c) = \mathrm{tr}(d \otimes c) = \langle c, d\rangle$ holds for one-dimensional operators $d \otimes c$. □

As an alternative formulation of the solution, in which the complex unit does not appear explicitly (however, at the cost of involving the square of the operator function L), we state the following.

Proposition A.4. *Let $\mathcal{A}$ be a quasi-Banach operator ideal admitting a continuous determinant δ. Then, for the solution V of the matrix mKdV (1) in Theorem A.1, it holds*

$$V = \left(\frac{\delta\big(I_E + L^2 + L^{(i,j)}\big)}{\delta(I_E + L^2)} - 1\right)^{\mathrm{d}}_{i,j=1},$$

where

$$L^{(i,j)} = \exp(Ax + A^3 t)\big(d^{(i)} \otimes c^{(j)}\big).$$

Proof. As the proof follows analogous arguments as the proof of Proposition A.2, we refer to the details of the arguments there. As for the computations, we observe that

$$\begin{aligned}\frac{\delta(I_E + L^2 + L^{(i,j)})}{\delta(I_E + L^2)} &= \delta\Big((I_E + L^2)^{-1}\big((I_E + L^2) + L^{(i,j)}\big)\Big) \\ = \delta\Big(I_E + (I_E + L^2)^{-1} L^{(i,j)}\Big) &= 1 + \tau\Big((I_E + L^2)^{-1} L^{(i,j)}\Big) \\ = 1 + \Big\langle (I_E + L^2)^{-1} \widehat{L}\, c^{(j)}, d^{(i)}\Big\rangle,&\end{aligned}$$

yielding the desired reformulation. □