

CR-TWISTOR SPACES OVER MANIFOLDS WITH G_2 -AND Spin(7)-STRUCTURES

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ABSTRACT. In 1984 LeBrun constructed a CR-twistor space over an arbitrary conformal Riemannian 3-manifold and proved that the CR-structure is formally integrable. This twistor construction has been generalized by Rossi in 1985 for m -dimensional Riemannian manifolds endowed with a $(m-1)$ -fold vector cross product (VCP). In 2011 Verbitsky generalized LeBrun's construction of twistor-spaces to 7-manifolds endowed with a G_2 -structure. In this paper we unify and generalize LeBrun's, Rossi's and Verbitsky's construction of a CR-twistor space to the case where a Riemannian manifold (M, g) has a VCP structure. We show that the formal integrability of the CR-structure is expressed in terms of a torsion tensor on the twistor space, which is a Grassmanian bundle over (M, g) . If the VCP structure on (M, g) is generated by a G_2 - or Spin(7)-structure, then the vertical component of the torsion tensor vanishes if and only if (M, g) has constant curvature, and the horizontal component vanishes if and only if (M, g) is a torsion-free G_2 or Spin(7)-manifold. Finally we discuss some open problems.

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1. INTRODUCTION

1.1. Motivations and prior works. In his papers [BG67], [Gray69], motivated by Calabi's work on almost complex structures on S^6 , Gray introduced the notion of a vector cross product (VCP for short) structure. By definition, a r -fold VCP structure χ on an Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ is a multilinear alternating map

$$\chi: \bigwedge^r V \rightarrow V$$

such that

$$\begin{aligned} \langle \chi(v_1, \dots, v_r), v_i \rangle &= 0 \text{ for } 1 \leq i \leq r, \\ \langle \chi(v_1, \dots, v_r), \chi(v_1, \dots, v_r) \rangle &= \|v_1 \wedge \dots \wedge v_r\|^2, \end{aligned}$$

where $\|\cdot\|$ is the induced metric on $\wedge^r V$. For a r -fold VCP χ on V , the associated VCP-form $\varphi_\chi: \wedge^{r+1} V \rightarrow \mathbb{R}$ is defined as

$$(1.1.1) \quad \varphi_\chi(v_1, \dots, v_{r+1}) = \langle \chi(v_1, \dots, v_r), v_{r+1} \rangle,$$

[Gray69, (4.1)]. As a matter of notation, once a r -fold VCP is fixed one often writes $v_1 \times v_2 \times \dots \times v_r$ for $\chi(v_1, \dots, v_r)$.

Remark 1.1.2. (1) The Brown-Gray classification [BG67] asserts that a r -fold VCP structure exists on \mathbb{R}^m if and only if one of the following possibilities holds (i) $r = 1$ and m is even; (ii) $r = m - 1$; (iii) $r = 2$ and $m = 7$; (iv) $r = 3$ and $m = 8$.

(2) A $(m - 1)$ -fold VCP structure χ on $(\mathbb{R}^m, \langle \cdot, \cdot \rangle)$ is defined uniquely by a given orientation on \mathbb{R}^m .

(3) For $m = 7$, $r = 2$ the VCP form φ_χ is called the *associative 3-form*. Its stabilizer in $\text{GL}(\mathbb{R}^7)$ is the exceptional group $G_2 \subset \text{SO}(7)$. For $m = 8$, $r = 3$ the VCP form φ_χ is called the *Cayley 4-form*. Its stabilizer in $\text{GL}(\mathbb{R}^8)$ is the subgroup $\text{Spin}(7) \subset \text{SO}(8)$. The VCP structures (χ, g) in these cases are in a 1-1 correspondence with their VCP-forms φ_χ . Given a VCP form φ on a 7-manifold an explicit formula for g_φ is given in [Hitchin00, §7.1]. Similarly, given a VCP form φ on a 8-manifold, a formula for g_φ can be

obtained using the relation $\varphi^2 = 8\text{vol}_{g_\varphi}$ and Hitchin's method, see similar results in [LPV08, §3].

One has an immediate notion of a r -fold VCP on a Riemannian manifold (M, g) as a smooth TM -valued r -form $\chi \in \Omega^r(M, TM)$ such that $\chi(x)$ is a r -fold VCP on $T_x M$ for all $x \in M$. The corresponding VCP-form will therefore be an element in $\Omega^{r+1}(M)$.

Remark 1.1.3. A VCP form φ_χ is parallel w.r.t. the Levi-Civita connection ∇_g^{LC} iff either (M^m, g) is an orientable Riemannian manifold and $r = m - 1$; or $m = 2n$, (M^{2n}, g) is a Kähler manifold and $r = 1$; or $m = 7$ and (M^7, g) is a torsion-free G_2 -manifold and $r = 2$; or $m = 8$ and (M^8, g) is a torsion-free Spin(7)-manifold and $r = 3$. This result singles out Kähler manifolds, torsion-free G_2 -and Spin(7)-manifolds as important classes of Riemannian manifolds with special holonomy [Joyce00]. Not unrelatedly, these classes play a prominent role in calibrated geometry, string theory and M-theory, and F-theory [Joyce07], [GS02], [BGP14].

Remark 1.1.4. The VCPs in dimension 3, 7, 8 can be expressed in terms of algebraic operations on normed algebras. Denote by $\text{Im } \mathbb{O}$ the imaginary part of the octonion algebra \mathbb{O} . Harvey and Lawson noticed that, identifying \mathbb{R}^7 with $\text{Im } \mathbb{O}$, the associative 3-form φ_χ on $\text{Im } \mathbb{O}$ has the following form [HL82, (1.1), p. 113]:

$$(1.1.5) \quad \varphi_\chi(x, y, z) = \langle x, yz \rangle.$$

Hence the 2-fold VCP χ on $\text{Im } \mathbb{O}$ is defined as follows [HL82, Definition B.1, p. 145]

$$(1.1.6) \quad y \times z = \text{Im}(yz).$$

The restriction of this 2-fold VCP to $\text{Im } \mathbb{H} \subset \text{Im } \mathbb{O}$ coincides with the 2-fold VCP on \mathbb{R}^3 [HL82, p. 145].

The 3-fold VCP on $\mathbb{R}^8 = \mathbb{O}$ can be expressed as follows [HL82, Definition B.3, p. 145]:

$$(1.1.7) \quad u \times v \times w = \frac{1}{2}((u\bar{v})w - (w\bar{v})u).$$

The relation between complex structures and VCP structures has been manifested also via CR-twistor spaces over manifolds endowed with a VCP structure. In 1984 LeBrun constructed a CR-twistor space over an arbitrary conformal Riemannian 3-manifold [LeBrun84]. LeBrun proved that the CR-twistor space of a conformal Riemannian 3-manifold is a CR-manifold, i.e. the CR-structure is integrable. This twistor construction has been generalized by Rossi in 1985 for m -dimensional Riemannian manifolds endowed with a $(m - 1)$ -fold VCP [Rossi85] and utilized further by LeBrun for his proof of the formal integrability of the almost complex structure J on the higher dimensional loop space over a Riemannian manifold (M^m, g) endowed with a $(m - 1)$ -fold VCP [LeBrun93], following a similar proof by Lempert

for the weak integrability of the almost complex structure on the loop space over a Riemannian 3-manifold [Lempert93]. In 2011 Verbitsky generalized LeBrun's construction of twistor-spaces to 7-manifolds endowed with the VCP 3-forms φ [Verbitsky11], which subsequently has been used by him for his proof of the formal integrability of the almost complex structure on the loop space over a holonomy G_2 -manifold [Verbitsky12].

1.2. Our main results. As a first result in this paper, we unify and generalize LeBrun's, Rossi's construction of a CR-twistor space over a conformal Riemannian manifold in dimension 3 and in arbitrary dimension respectively, as well as Verbitsky's construction of a CR-twistor space over a G_2 -manifold to the case when the underlying Riemannian manifold (M, g) has a VCP structure, see Definition 1.2.12 below. In order to state the result we need fixing notation.

Notation 1.2.1. Let (M, g) be an oriented Riemannian manifold.

- We denote by $\mathbb{G}r^+(r-1, M)$ the Grassmannian of oriented $(r-1)$ -planes in TM , which we shall identify with decomposable unit $(r-1)$ -vectors in $\bigwedge^{r-1} TM$. When no confusion is possible we will denote $\mathbb{G}r^+(r-1, M)$ simply by \mathbb{G} . We denote by $\pi : \bigwedge^{r-1} TM \rightarrow M$ the natural projection, which also induces the natural projection $\pi : \mathbb{G} \rightarrow M$. For any point $v \in \mathbb{G}$, the fiber of $\pi : \mathbb{G} \rightarrow M$ through v is naturally identified with the Grassmannian $\mathbb{G}r^+(r-1, T_{\pi(v)}M)$ of oriented $(r-1)$ -planes in $T_{\pi(v)}M$.
- For $v \in \mathbb{G}$ we denote by $E_v \subseteq T_{\pi(v)}M$ the oriented $(r-1)$ -plane associated to v and by E_v^\perp its orthogonal complement in $T_{\pi(v)}M$.

The Riemannian metric g induces a natural Riemannian metric on the vector bundle $\bigwedge^{r-1} TM \xrightarrow{\pi} M$ and so endows $\bigwedge^{r-1} TM$ with the corresponding Levi-Civita connection ∇^{LC} . This induces, for any $v \in \bigwedge^{r-1} TM$, a direct sum decomposition

$$(1.2.2) \quad T_v(\bigwedge^{r-1} TM) = \bigwedge^{r-1} T_{\pi(v)}M \oplus T_v^{\text{hor}}(\bigwedge^{r-1} TM),$$

where

$$T_v^{\text{hor}}(\bigwedge^{r-1} TM) \cong T_{\pi(v)}M$$

is the horizontal distribution in $T\bigwedge^{r-1} TM$ w.r.t. ∇^{LC} . Since $\mathbb{G}r^+(r-1, TM)$ is a fiber sub-bundle of the vector bundle $\bigwedge^{r-1} TM \xrightarrow{\pi} M$, for $v \in \mathbb{G}$ the orthogonal decomposition (1.2.2) induces the decomposition

$$(1.2.3) \quad T_v\mathbb{G} = T_v^{\text{vert}}\mathbb{G} \oplus^\perp T_v^{\text{hor}}\mathbb{G},$$

where

$$(1.2.4) \quad T_v^{\text{vert}}\mathbb{G} = T_v\mathbb{G}r^+(r-1, T_{\pi(v)}M)$$

and

$$(1.2.5) \quad T_v^{\text{hor}}\mathbb{G} = T_v^{\text{hor}}(\bigwedge^{r-1} TM) \cong T_{\pi(v)}M.$$

Notation 1.2.6. Let (M^m, g) be an m -dimensional Riemannian manifold. We denote by B the rank $m - r + 1$ distribution on \mathbb{G} defined at a point v of \mathbb{G} by

$$(1.2.7) \quad B_v := \{w \in T_v^{\text{hor}}\mathbb{G} \mid d\pi_v(w) \in E_v^\perp \subset T_{\pi(v)}M\} \subseteq T_v^{\text{hor}}\mathbb{G}.$$

An r -fold VCP structure χ on (M, g) endows the vector spaces E_v^\perp with a complex structure $J_{E_v^\perp}$ defined by

$$(1.2.8) \quad J_{E_v^\perp}(z) = \chi(v \wedge z),$$

for $z \in E_v^\perp$, see [FL21, Lemma 3.1], [LL07, p. 146], [Gray69, Theorem 2.6]. Since $d\pi_v : B_v \rightarrow E_v^\perp$ is an isometry, the complex structure $J_{E_v^\perp}$ induces a complex structure $J_{g,\chi}$ on B_v . It is defined by the equation

$$(1.2.9) \quad d\pi_v(J_{g,\chi}(w)) = J_{E_v^\perp}(d\pi_v(w)).$$

Definition 1.2.10. (cf. [DT06, Definitions 1.1, 1.2, p. 3]) An *almost CR-structure* on a manifold N is a pair (B, J_B) consisting of a distribution $B \subseteq TN$ and of an almost complex structure J_B on B . The triple (N, B, J_B) is called an *almost CR-manifold*. An almost CR-structure (B, J_B) on a manifold N is said to be *formally integrable* if the complex distribution $B^{1,0} \subset B \otimes \mathbb{C}$ is involutive, i.e., $[B^{1,0}, B^{1,0}] \subseteq B^{1,0}$. If (B, J_B) is integrable, then the almost CR-manifold (N, B, J_B) is called a *CR-manifold*.

Remark 1.2.11. The condition that the almost CR-structure (B, J_B) is formally integrable can be stated completely in terms of sections of the real vector bundle B , without going through its complexification, as follows. Denote by Π_B the orthogonal projection of $TGr^+(r-1, M)$ to B and by $\Gamma(B)$ the space of smooth sections of B .

- (1) For any $X, Y \in \Gamma(B)$ one has $[J_B X, J_B Y] - [X, Y] \in \Gamma(B)$;
- (2) For any $X, Y \in \Gamma(B)$ one has

$$\Pi_B([J_B X, J_B Y] - [X, Y]) - J_B \circ \Pi_B([X, J_B Y] + [J_B X, Y]) = 0.$$

In literature [Bejancu86, p. 128], [DT06, p. 4] the condition (2) is replaced by the following condition

$$[J_B X, J_B Y] - [X, Y] - J_B([X, J_B Y] + [J_B X, Y]) = 0,$$

which has meaning only if the condition (1) holds. Clearly the conditions (1) and (2) are equivalent to the condition (1) and the classical condition stated above.

We shall call the condition (1) *the first CR-integrability condition*, and the condition (2) *the second CR-integrability condition*. In Cartan geometry, the condition (1) is also called the *partial integrability* of a CR-structure [CS09, p. 443].

Now we associate to each VCP-structure on a Riemannian manifold (M, g) an almost CR-manifold as follows.

Definition 1.2.12. Let (M, g, χ) be a Riemannian manifold endowed with a VCP structure χ . The almost CR-manifold $(\mathbb{G}, B, J_{g,\chi})$ consisting of the manifold \mathbb{G} together with the almost CR-structure given by the distribution B and the almost complex structure $J_{g,\chi}$ on B defined in Equations (1.2.7) and (1.2.9) will be called the *CR-twistor space over (M, g, χ)* .

Examples 1.2.13. (1) Let (χ, g) be a 1-fold VCP on a smooth manifold M^{2n} . This is equivalently an Hermitian almost complex structure on M . In this case one has $\mathbb{G} = M^{2n}$, the horizontal distribution B is identified with the tangent bundle to M^{2n} and the almost complex structure $J_{g,\chi}$ is identified with the almost complex structure on M .

(2) Let (χ, g) be a $(m - 1)$ -fold VCP on an oriented manifold M^m . Then the distribution B on \mathbb{G} is 2-dimensional and the CR-twistor structure $(\mathbb{G}, B, J_{g,\chi})$ coincides with the one constructed by LeBrun [LeBrun84] and extended by Rossi [Rossi85].

(3) Let χ be a 2-fold VCP on (M^7, g) . Then the CR-twistor structure on \mathbb{G} coincides with the one constructed by Verbitsky in [Verbitsky11].

Our main result in this paper concerns necessary and sufficient conditions for the first and the second CR-integrability of the CR-twistor space over a Riemannian manifold (M, g) endowed with a VCP structure. Let ∇_g^{LC} denote the Levi-Civita connection on (M, g) . We say that the VCP χ is parallel if $\nabla_g^{LC} \chi = 0$.

In this paper we prove the following

Theorem 1.2.14 (Main Theorem). *Let $\chi \in \Omega^{r+1}(M, TM)$ be a VCP structure on a Riemannian manifold (M, g) and $(\mathbb{G}, B, J_{g,\chi})$ the associated CR-twistor space. Then there exists a tensor $T \in \Gamma(\wedge^2 B^* \otimes T\mathbb{G})$ on the total space \mathbb{G} such that*

- (1) *The first CR-integrability (1) holds if and only if for any $v \in \mathbb{G}$ and $X, Y \in B(v)$ we have $T^{\text{vert}}(X, Y) = 0 \in T^{\text{vert}}\mathbb{G}$.*
- (2) *If $(r, m) = (1, 2n)$ or $(m - 1, m)$ then $T^{\text{vert}} = 0$ for any (M, g, χ) .*
- (3) *If $(r, m) = (2, 7)$ or $(3, 8)$ then $T^{\text{vert}} = 0$ if and only if (M, g) has constant curvature.*
- (4) *The second CR-integrability (2) holds, if and only if for any $v \in \mathbb{G}$ and $X, Y \in B(v)$ we have $T^{\text{hor}}(X, Y) = 0 \in T^{\text{hor}}\mathbb{G}$.*
- (5) *If $(r, m) = (1, 2n)$ then $T^{\text{hor}} = 0$ for (M, g, χ) if and only if χ is integrable.*
- (6) *If $(r, m) = (m - 1, m)$ then $T^{\text{hor}} = 0$ for any (M, g, χ) .*
- (7) *If $(r, m) = (2, 7)$ or $(r, m) = (3, 8)$ then $T^{\text{hor}} = 0$ if and only if χ is parallel.*

Remark 1.2.15. Parts (2&5) and (2&6) of the main theorem above combined, i.e., without decomposing the CR integrability condition into two independent conditions, are classical and we are including them only for

completeness. In particular, by combining Example 1.2.13 (1) with Example 2.2.10(1) we recover that a Riemannian manifold (M^{2n}, g) endowed with a 1-fold VCP χ is a CR-manifold if and only if the almost complex structure on M induced by χ is integrable. Part (6) is due to LeBrun, who proved that the CR-twistor space over a Riemannian manifold (M^m, g) with a $(m - 1)$ fold VCP χ is always a CR-manifold [LeBrun84], [LeBrun93]. Note that in this case χ is always parallel, see [Gray69, Proposition 4.5]. Part (7) for the case (2, 7) is due to Verbitsky [Verbitsky11]. Unfortunately his proof uses a wrong argument, see Remark 2.2.9. Finally, it was also known that the CR-twistor space over any flat Riemannian manifold (M, g) endowed with parallel VCP is a CR-manifold.

1.3. Organization of our paper. In the second section we study the first condition (1) for the formal integrability of the CR-twistor space over a Riemannian manifold (M, g) endowed with a VCP-structure χ . First, using a geometric characterization of the distribution B (Lemma 2.1.7), we express the condition (1) for the CR-twistor space over (M, g, χ) in terms of the vertical components of the Lie brackets $[J_B X, J_B Y]$ and $[X, Y]$, where $X, Y \in \Gamma(B)$, with respect to the decomposition (1.2.3) (Corollary 2.1.13). Using this, we prove that the first CR-integrability condition (1) for the CR-twistor space over (M, g, χ) holds if and only if the curvature $R(g)$ of the underlying Riemannian manifold (M, g) is a solution of an infinite system of linear equations (Proposition 2.2.8). Next, we study the first CR-integrability condition for the (2, 7) case using Proposition 2.2.8 and computer algebra. Using these results and ad-hoc methods in Section 3, we prove assertions (1) and (3) of Theorem 1.2.14. In Section 4 we study the second condition (2) for the formal integrability of the CR-twistor space over (M, g, χ) using the formalism of the Fröhlicher-Nijenhuis bracket (Proposition 4.1.1). Then we give the proof of Theorem 1.2.14 (4, 7). Finally we discuss our results and some open questions. We include an Appendix containing `sagemath` codes for solving the first and the second CR integrability condition in the (2, 7) case.

1.4. Notation and conventions.

- We keep notation in the introduction.
- For a vector bundle E over a manifold M and a smooth section $\alpha \in \Gamma(E)$, we also write α_x for the value $\alpha(x)$ to avoid possibly ugly notation like $\alpha(x)(v)$ occurring, e.g., when E is the endomorphism bundle of TM and v is a tangent vector at x .
- If ξ is an element in a vector space V with inner product $\langle \cdot, \cdot \rangle$, we denote by ξ^\sharp the element on V^* defined by $\xi^\sharp(v) = \langle \xi, v \rangle$ for all $v \in V$.
- Given a G -action on a space X , for $x \in X$, we denote by $\text{Stab}_G(x)$ the stabilizer of x in G .
- We consider in this paper the Killing metric on Lie algebra $\mathfrak{so}(\mathbb{R}^n)$ and any its Lie subalgebra defined as follows $\langle X, Y \rangle = -\frac{1}{2} \text{Tr}(XY)$.

- Let (V, \langle, \rangle) be an Euclidean vector space. We denote by $\mathcal{AC}(V)$ the vector subspace of $\bigwedge^2 V^* \otimes \mathfrak{so}(V)$ consisting of elements $R \in \bigwedge^2 V^* \otimes \mathfrak{so}(V)$ that satisfy the algebraic Bianchi identity, i.e.,

$$R(w_1, w_2)w_3 + R(w_2, w_3)w_1 + R(w_3, w_1)w_2 = 0,$$

The elements of $\mathcal{AC}(V)$ are called *algebraic curvature (operators)* on V . It is known that $\dim \mathcal{AC}(V) = \frac{1}{12}(\dim V)^2((\dim V)^2 - 1)$ [Gilkey01, Corollary 1.8.4, p. 45].

- It is known that the image R^{Id} of the operator $\text{Id} : \bigwedge^2 V^* \rightarrow \bigwedge^2 V^*$ in $\bigwedge^2 V^* \otimes \mathfrak{so}(V)$ via the identification $\bigwedge^2 V^*$ with $\mathfrak{so}(V)$ is an algebraic curvature of constant sectional curvature, see. e.g. [Gilkey01, Lemma 1.6.4, p. 31]. It is immediate to see that

$$(1.4.1) \quad R^{\text{Id}}(w_1, w_2)w_3 := \langle w_2, w_3 \rangle w_1 - \langle w_1, w_3 \rangle w_2,$$

for any w_1, w_2, w_3 , see [Gilkey01, p. 31]. By the Schur lemma if (M, g) is a connected Riemannian manifold of dimension at least 3, then the Riemannian curvature tensor of M is of the form $R = \lambda(x)R^{\text{Id}}$ at any point $x \in M$ if and only if (M, g) has constant curvature [KN63, Theorem 2.2, p. 202].

- Let $\text{Der}(\Omega^*(M))$ be the graded Lie algebra of graded derivations of $\Omega^*(M)$. For $K \in \Omega^*(M, TM)$ we denote by ι_K and by $\mathcal{L}_K = [d, \iota_K]$ the contraction with K and corresponding the Lie derivative, respectively. It is known that $\mathcal{L} : \Omega^*(M, TM) \rightarrow \text{Der}(\Omega^*(M))$ is injective, and moreover [FN56a, FN56b]

$$\mathcal{L}(\Omega^*(M, TM)) = \{D \in \text{Der}(\Omega^*(M)) \mid [D, d] = 0\}.$$

Hence $\mathcal{L}(\Omega^*(M, TM))$ is closed under the graded Lie bracket $[\cdot, \cdot]$ on $\text{Der}(\Omega^*(M))$ and one then defines the *Frölicher-Nijenhuis bracket* $[\cdot, \cdot]^{FN}$ on $\Omega^*(M, TM)$ as the pull-back of the graded Lie bracket on $\text{Der}(\Omega^*(M))$ via the linear embedding \mathcal{L} , i.e.,

$$\mathcal{L}_{[K, L]^{FN}} := [\mathcal{L}_K, \mathcal{L}_L].$$

2. A REFORMULATION OF THE FIRST CONDITION FOR THE FORMAL INTEGRABILITY OF CR-TWISTOR SPACES $(\mathbb{G}, B, J_{g, \chi})$

In this section we reformulate the first condition for the integrability of CR-twistor spaces $(\mathbb{G}, B, J_{g, \chi})$ in terms of a system of linear equations for the curvature tensor of (M, g) . This will in particular imply that the first integrability condition is automatically satisfied in the case $(r, m) = (1, 2n)$ or $(r, m) = (m - 1, m)$. The proof goes in two steps. First we express the first integrability condition as the condition $[J_{g, \chi}X, J_{g, \chi}Y]^{\text{vert}} = [X, Y]^{\text{vert}}$ for any $X, Y \in \Gamma(B)$ (Corollary 2.1.13). Then we translate this in a system of conditions on the curvature tensor of (M, g) (Proposition 2.2.8).

2.1. The equation $[J_{g,\chi}X, J_{g,\chi}Y]^{\text{vert}} = [X, Y]^{\text{vert}}$. Let (M, g, χ) be a Riemannian manifold endowed with a r -fold VCP structure and $(B, J_{g,\chi})$ the almost CR-structure on \mathbb{G} . Let E^* be the dual bundle of the tautological bundle E over \mathbb{G} . At every point v in \mathbb{G} , the Riemannian metric g induces a natural isomorphism $E_v = T_{\pi(v)}M/E_v^\perp$, where $\pi: \mathbb{G} \rightarrow M$ is the projection to the base. This gives a natural identification $E^* = \text{Ann}(E^\perp)$, where $\text{Ann}(E^\perp)$ is the vector bundle over \mathbb{G} whose fiber over v consists of all elements of $T_{\pi(v)}^*M$ that annihilate E_v^\perp .

Remark 2.1.1. Since E^* is a subbundle of π^*T^*M via the identification $E^* = \text{Ann}(E^\perp)$, any $\theta \in \Gamma(E^*)$ defines a map of fiber bundles over M ,

$$(2.1.2) \quad \hat{\theta}: \mathbb{G} \rightarrow T^*M,$$

mapping a point $v \in \mathbb{G}$ to the element θ_v seen as an element in $T_{\pi(v)}^*M$.

The Riemannian metric g induces a natural Riemannian metric on the vector bundle $T^*M \rightarrow M$ and so endows T^*M with the associated Levi-Civita connection ∇^{LC} and the corresponding splitting of the tangent bundle of the total space of T^*M into a vertical and a horizontal subbundle. The same applies to the bundle π^*T^*M and to its subbundle E^* .

Notation 2.1.3. For $v \in \mathbb{G}$ we let

$$(2.1.4) \quad \Gamma_{\text{hor}(v)}(E^*) := \{\theta \in \Gamma(E^*) : d\hat{\theta}(T_v^{\text{hor}}\mathbb{G}) \subset T_{\hat{\theta}(v)}^{\text{hor}}T^*M\}.$$

In other words, $\Gamma_{\text{hor}(v)}(E^*)$ consists of elements in $\Gamma(E^*)$ that are ‘‘horizontal’’ at v . Using parallel transport in the Grassmann bundle $\mathbb{G} \rightarrow M$ and in the total space of the vector bundle E^* on \mathbb{G} seen as a fiber bundle over M one easily shows that every vector $\xi \in E_v^*$ can be extended to a section of E^* that is horizontal at v . For later reference, we state this fact as the following Lemma.

Lemma 2.1.5. *For any $\xi \in E_v^*$ there exists an element $\theta \in \Gamma_{\text{hor}(v)}(E^*)$ such that $\theta_v = \xi$.*

Notation 2.1.6. We write

$$\begin{aligned} \eta: \Gamma(E^*) &\rightarrow \Omega^1(\mathbb{G}), \\ \theta &\mapsto \eta[\theta] \end{aligned}$$

for the map sending a smooth section θ of E^* to the 1-form $\eta[\theta]$ given by

$$\eta[\theta]_v(w) = \theta_v(d\pi_v(w)),$$

for any $w \in T_v\mathbb{G}$. In other words, $\eta[\theta]$ is the section of $T^*\mathbb{G} \rightarrow \mathbb{G}$ given by the composition

$$\mathbb{G} \xrightarrow{\theta} E^* \hookrightarrow \pi^*T^*M \xrightarrow{(d\pi)^*} T^*\mathbb{G}.$$

Lemma 2.1.7. *Let us consider the subbundle $B \oplus^\perp T^{\text{vert}}\mathbb{G}$ of $T\mathbb{G}$. For any $v \in \mathbb{G}$ we have*

$$B_v \oplus^\perp T_v^{\text{vert}}\mathbb{G} = \bigcap_{\theta \in \Gamma_{\text{hor}(v)}(E^*)} \ker(\eta[\theta]_v).$$

Proof. Let $w = w_B + w_{\text{vert}} \in T_v\mathbb{G}$, where $w_B \in B_v$ and $w_{\text{vert}} \in T_v^{\text{vert}}\mathbb{G}$. Then for every $\theta \in \Gamma_{\text{hor}(v)}(E^*)$ we have

$$(2.1.8) \quad \eta[\theta]_v(w) = \theta_v(d\pi_v(w_B)) = 0.$$

Namely, by definition of B_v , equation (1.2.7), the vector $d\pi_v(w_B)$ is in E_v^\perp and so it is annihilated by $\theta_v \in E_v^* = \text{Ann}(E_v^\perp)$. Vice versa, let $w \in T_v\mathbb{G}$ be such that $\eta[\theta]_v(w) = 0$ for every $\theta \in \Gamma_{\text{hor}(v)}(E^*)$. Let us write $w = w_{\text{hor}} + w_{\text{vert}}$, with $w_{\text{hor/vert}} \in T^{\text{hor/vert}}\mathbb{G}$. Let $\xi \in E_v^*$. By Lemma 2.1.5, there exists $\theta \in \Gamma_{\text{hor}(v)}(E^*)$ such that $\theta_v = \xi$, and so

$$\xi(d\pi_v(w_{\text{hor}})) = \xi(d\pi_v(w)) = \theta_v(d\pi_v(w)) = \eta[\theta]_v(w) = 0.$$

Therefore,

$$d\pi_v(w_{\text{hor}}) \in \bigcap_{\xi \in \text{Ann}(E_v^\perp)} \ker(\xi) = E_v^\perp,$$

and so $w_{\text{hor}} \in B_v$. This completes the proof of Lemma 2.1.7. \square

Lemma 2.1.9. *Let $v \in \mathbb{G}$. For any $\theta \in \Gamma_{\text{hor}(v)}(E^*)$ one has*

$$(d\eta[\theta])_v|_{\wedge^2 T_v^{\text{hor}}\mathbb{G}} = 0$$

Proof. The bundle $E^* = \text{Ann}(E^\perp)$ over \mathbb{G} is a subbundle of the bundle π^*T^*M and therefore a section θ of $\Gamma(E^*)$ is a section of π^*T^*M . We have a commutative diagram

$$\begin{array}{ccccc} & & T^*\mathbb{G} & & \\ & & \uparrow (d\pi)^* & & \\ \eta[\theta] & \curvearrowright & \pi^*T^*M & \xrightarrow{\hat{\pi}} & T^*M \\ & & \downarrow \text{pr} & \nearrow \hat{\theta} & \downarrow \text{pr} \\ & & \mathbb{G} & \xrightarrow{\pi} & M, \end{array}$$

where $\hat{\theta}: \mathbb{G} \rightarrow T^*M$ is the map defined in Remark 2.1.1, and so

$$(2.1.10) \quad \eta[\theta] = \hat{\theta}^*(\Theta_{\text{Lio};M}),$$

where $\Theta_{\text{Lio};M}$ is the Liouville 1-form on T^*M . Let ω be the canonical symplectic form on T^*M . From (2.1.10) we obtain

$$(2.1.11) \quad d\eta[\theta] = \hat{\theta}^*(\omega) = \omega \circ (d\hat{\theta} \wedge d\hat{\theta}).$$

Since $\theta \in \Gamma_{\text{hor}(v)}(E^*)$, the differential $d\hat{\theta}$ maps the horizontal space $T_v^{\text{hor}}\mathbb{G}$ to $T_{\hat{\theta}(v)}^{\text{hor}}T^*M$. It is well-known that the restriction of the canonical 2-form

ω to $\bigwedge^2 T^{\text{hor}} T^* M$ identically vanishes. This concludes the proof of Lemma 2.1.9. \square

Lemma 2.1.12. *For any $X, Y \in \Gamma(B)$ we have*

$$[X, Y] \in \Gamma(B \oplus^\perp T^{\text{vert}} \mathbb{G}).$$

Proof. Let v be a point in \mathbb{G} . We have to show that $[X, Y]_v \in B_v \oplus^\perp T_v^{\text{vert}} \mathbb{G}$. By Lemma 2.1.7 this is equivalent to showing that for every $\theta \in \Gamma_{\text{hor}(v)}(E^*)$ we have $\eta[\theta]_v([X, Y]_v) = 0$. By the Cartan formula,

$$\eta[\theta]_v([X, Y]_v) = -(d\eta[\theta])_v(X_v, Y_v) + X_v(\eta[\theta](Y)) - Y_v(\eta[\theta](X)).$$

By definition of B , we have $B_v \subset T_v^{\text{hor}} \mathbb{G}$, and so $(d\eta[\theta])_v(X_v, Y_v) = 0$, by Lemma 2.1.9. By definition of $\Gamma_{\text{hor}(v)}(E^*)$, θ is in particular an element of $\Gamma(E^*) = \Gamma(\text{Ann}(E^\perp))$. Therefore, for any point v' in \mathbb{G} we have

$$\eta[\theta]_{v'}(X_{v'}) = \theta_{v'}(d\pi_{v'}(X_{v'})) = 0,$$

since $X \in \Gamma(B)$ and so $d\pi_{v'}(X_{v'}) \in E_{v'}^\perp$, by the defining equation (1.2.7). This means that $\eta[\theta](X)$ identically vanish on \mathbb{G} . By the same argument, also $\eta[\theta](Y) \equiv 0$, and so we have $\eta[\theta]_v([X, Y]_v) = 0$. \square

Corollary 2.1.13. *Let $J_{g,\chi}$ the complex structure on B induced by the VCP χ (equation (1.2.9)). For any $X, Y \in \Gamma(B)$ we have*

$$[J_{g,\chi} X, J_{g,\chi} Y] - [X, Y] \in \Gamma(B) \text{ if and only if } [J_{g,\chi} X, J_{g,\chi} Y]^{\text{vert}} = [X, Y]^{\text{vert}}.$$

Proof. By Lemma 2.1.12, we have $[X, Y] \in \Gamma(B \oplus T^{\text{vert}} \mathbb{G})$. Since $J_{g,\chi}$ is an vector bundle endomorphism of B , we also have $J_{g,\chi} X, J_{g,\chi} Y \in \Gamma(B)$ and so by Lemma 2.1.12 again, $[J_{g,\chi} X, J_{g,\chi} Y] \in \Gamma(B \oplus T^{\text{vert}} \mathbb{G})$. This gives $[J_{g,\chi} X, J_{g,\chi} Y] - [X, Y] \in \Gamma(B \oplus T^{\text{vert}} \mathbb{G})$ and so $[J_{g,\chi} X, J_{g,\chi} Y] - [X, Y] \in \Gamma(B)$ if and only if $([J_{g,\chi} X, J_{g,\chi} Y] - [X, Y])^{\text{vert}} = 0$. \square

2.2. A curvature reformulation of the first CR-integrability condition. By Corollary 2.1.13, the first CR-integrability condition (1) is equivalent to

$$[J_{g,\chi} X, J_{g,\chi} Y]^{\text{vert}} = [X, Y]^{\text{vert}}$$

for any $X, Y \in \Gamma(B)$. This latter condition can be conveniently expressed in terms of the curvature operator R of the Levi-Civita connection on (M, g) . If X, Y are horizontal vector fields on \mathbb{G} and $v \in \mathbb{G}$, we have $R_{\pi(v)}(d\pi_v(X_v), d\pi_v(Y_v)) \in \mathfrak{so}(T_{\pi(v)} M)$ and so a corresponding $\text{SO}(\dim M)$ -invariant vertical vector field $\mathfrak{r}_{\pi(v)}(d\pi_v(X_v), d\pi_v(Y_v))$ on the fiber of $\mathbb{G} \rightarrow M$ through v . Evaluating this vector field at the point v we obtain a vertical tangent vector $\mathfrak{r}_{\pi(v)}(d\pi_v(X_v), d\pi_v(Y_v))|_v \in T_v^{\text{vert}} \mathbb{G}$. It is a standard fact, that can be easily derived from see e.g. [Besse86, p. 290] or [KN63, p. 89] by noticing that $[-, -]^{\text{vert}} : T^{\text{hor}} \mathbb{G} \otimes T^{\text{hor}} \mathbb{G} \rightarrow T^{\text{vert}} \mathbb{G}$ is a tensor, that

$$(2.2.1) \quad [X, Y]_v^{\text{vert}} = -\mathfrak{r}_{\pi(v)}(d\pi_v(X_v), d\pi_v(Y_v))|_v.$$

We identify $T_v^{\text{vert}}\mathbb{G}$ with $\text{Hom}(E_v, E_v^\perp)$ and define a linear embedding $\epsilon : \text{Hom}(E_v, E_v^\perp) \rightarrow \mathfrak{so}(E_v \oplus E_v^\perp)$ by extending the following relations linearly for $\xi \in E_v$, $w \in E_v^\perp$, and $X \in E_v \oplus E_v^\perp$:

$$(2.2.2) \quad \epsilon(\xi^\# \otimes w)(X) := \xi(X) \cdot w - \langle w, X \rangle \cdot \xi,$$

where $\xi^\# \in E_v^*$ is dual to ξ w.r.t. $g|_{E_v}$. We shall use the shorthand notation $\xi^\# \hat{\otimes} w$ for $\epsilon(\xi^\# \otimes w)$. The decomposition

$$(2.2.3) \quad \mathfrak{so}(E_v \oplus E_v^\perp) = \mathfrak{so}(E_v) \oplus \mathfrak{so}(E_v^\perp) \oplus \epsilon(\text{Hom}(E_v, E_v^\perp))$$

is an orthogonal decomposition w.r.t. the Killing metric,¹ see e.g. [Helgason78, Theorem 1.1, p. 231]. Let $\Pi_{E_v^* \hat{\otimes} E_v^\perp} : \mathfrak{so}(E_v \oplus E_v^\perp) \rightarrow \epsilon(\text{Hom}(E_v, E_v^\perp))$ be the orthogonal projection. Then, under the identification $T_v^{\text{vert}}\mathbb{G} = \text{Hom}(E_v, E_v^\perp)$, we have $\mathfrak{r}_{\pi(v)}(w_1, w_2)|_v = \Pi_{E_v^* \hat{\otimes} E_v^\perp} R_{\pi(v)}(w_1, w_2)$ for any w_1, w_2 in $T_{\pi(v)}M$. Therefore, equation (2.2.1) can be rewritten as

$$(2.2.4) \quad [X, Y]_v^{\text{vert}} = -\Pi_{E_v^* \hat{\otimes} E_v^\perp} R_{\pi(v)}(d\pi_v(X_v), d\pi_v(Y_v)).$$

Lemma 2.2.5. *The following are equivalent*

(1) *For any $v \in \mathbb{G}$ and two vectors $w_1, w_2 \in E_v^\perp$ one has*

$$(2.2.6) \quad R_{\pi(v)}(w_1, w_2) - R_{\pi(v)}(J_{E_v^\perp} w_1, J_{E_v^\perp} w_2) \in \mathfrak{so}(E_v) \oplus \mathfrak{so}(E_v^\perp) \subset \mathfrak{so}(E_v \oplus E_v^\perp),$$

where $J_{E_v^\perp}$ is the complex structure on E_v^\perp defined by (1.2.8).

(2) *For any $v \in \mathbb{G}$, any $w_3 \in E_v^\perp$ and $w_4 \in E_v$ one has*

$$(2.2.7) \quad [\Pi_{\mathfrak{so}(E_v^\perp)} R_{\pi(v)}(w_3, w_4), J_{E_v^\perp}] = 0 \in \mathfrak{so}(E_v^\perp).$$

Proposition 2.2.8. *The first condition (1) for the CR-integrability of $(B, J_{g,\chi})$ is equivalent to (2.2.6) (and so to any of the conditions in Lemma 2.2.5).*

The proofs of Lemma 2.2.5 and Proposition 2.2.8 are straightforward and therefore omitted. Detailed proofs can be found in arXiv:2203.04233v2.

Remark 2.2.9. In [Verbitsky11] Verbitsky also expresses the integrability condition for the CR-twistor space over a Riemannian (M^7, g) endowed with an associative 3-form φ in terms of constraints on the curvature of the underlying Riemannian manifold (M^7, g) . The Condition (ii) in [Verbitsky11, Proposition 3.2] is equivalent to our condition (2.2.7). But his assertion in [Verbitsky11, Proposition 3.2] that this condition is equivalent to the condition that $R(w_i \wedge w_j)$ takes value in the Lie algebra \mathfrak{g}_2 is not correct. In fact, that assertion also contradicts a related statement in [SW17, Theorem 11.1].

Examples 2.2.10. (1) In the case $(r, m) = (1, 2n)$, the vector w_4 in equation (2.2.7) is necessarily 0, so (2.2.7) is trivially satisfied.

¹In what follows we shall often omit ‘‘Killing’’ when we talk about a metric on a compact Lie algebra.

(2) In the case $(r, m) = (m-1, m)$, the vector space E_v^\perp is of real dimension $m - (r - 1) = 2$. Therefore, $\mathfrak{so}(E_v^\perp)$ is an abelian Lie algebra and the second condition in Lemma 2.2.5 is trivially satisfied.

(3) Let E_w be the oriented 2-plane in $T_{\pi(v)}M$ spanned by the ordered basis (w_3, w_4) . If $R(w_3, w_4) \in \mathfrak{so}(E_w) \subset \mathfrak{so}(T_x M)$ for any $w_3, w_4 \in T_{\pi(v)}M$, then (2.2.7) is automatically satisfied. In the later part of this section, we will see that for 2-fold vector cross products on 7-dimensional manifolds and for 3-fold vector cross products on 8-dimensional manifolds the condition $R_{\pi(v)}(w_3, w_4) \in \mathfrak{so}(E_w)$ for the Riemannian curvature is also necessary.

It follows from Example 2.2.10 and Proposition 2.2.8 that the Condition (1) is non-trivial only for two cases $(r, m) = (2, 7)$ and $(r, m) = (3, 8)$.

2.3. An infinite system of linear conditions for R . Equation (2.2.7) can be interpreted as a system of linear conditions for a section of a certain vector bundle over M .

Let V be an Euclidean space endowed with an r -fold VCP. For any $w \in \mathbb{G}\mathbb{R}^+(2, V)$ let

$$(2.3.1) \quad \mathcal{R}_w := \{A_w \in \mathfrak{so}(V) \mid [\Pi_{\mathfrak{so}(E_v^\perp)} A_w, J_{E_v^\perp}] = 0\}$$

for any $v \in \mathbb{G}\mathbb{R}^+(r-1, V)$ with $\dim(E_v \cap E_w) = 1$ and $\dim(E_v^\perp \cap E_w) = 1$.

The following Lemma is immediate from the definition of the subspaces \mathcal{R}_w and Proposition 2.2.8.

Lemma 2.3.2. *The first condition (1) for the CR-integrability of the CR-twistor space $(\mathbb{G}, B, J_{g, \chi})$ over a manifold (M, g, χ) holds if and only if for any $w \in \mathbb{G}\mathbb{R}^+(2, TM)$ we have*

$$R(w) \in \mathcal{R}_w.$$

Remark 2.3.3. The condition $\dim(E_v \cap E_w) = 1$ and $\dim(E_v^\perp \cap E_w) = 1$ means that there exists an orthonormal frame (w_1, \dots, w_r) with (w_2, \dots, w_r) an orthonormal basis for E_v and (w_1, w_2) an orthonormal basis for E_w . Therefore, the first integrability condition (1) holds for (M, g, χ) if and only if for any $x \in M$, and any orthonormal frame (w_1, \dots, w_r) in $T_x M$ with (w_2, \dots, w_r) an orthonormal basis for E_v and (w_1, w_2) an orthonormal basis for E_w we have

$$(2.3.4) \quad [\Pi_{\mathfrak{so}(E_{w_2^\perp} \wedge \dots \wedge w_r)} R_g(x; w_1 \wedge w_2), J_{E_{w_2^\perp} \wedge \dots \wedge w_r}] = 0 \in \mathfrak{so}(E_{w_2^\perp \wedge \dots \wedge w_r}),$$

where $R_g(x; -)$ denotes the curvature tensor of (M, g) at the point x .

Definition 2.3.5. Let (V, \langle, \rangle) be an Euclidean vector space endowed with an r -fold VCP χ . We denote by $\mathcal{AC}_{CR1}(V, \chi) \subseteq \mathcal{AC}(V)$ the subspace of $\mathcal{AC}(V)$ consisting of those elements $R \in \mathcal{AC}(V)$ such that $R(w) \in \mathcal{R}_w$, for any $w \in \mathbb{G}\mathbb{R}^+(2, V)$, i.e., such that (2.3.4) holds for any orthonormal frame (w_1, \dots, w_r) in V .

Remark 2.3.6. The conditions defining the subspace $\mathcal{AC}_{CR1}(V, \chi)$ of $\mathcal{AC}(V)$ are an infinite system of linear equations. This is the infinite system the title of this Section alludes to.

Lemma 2.3.7. *We have $R^{\text{Id}} \in \mathcal{AC}_{CR1}(V, \chi)$.*

Proof. For any orthonormal r -frame w one has $\Pi_{\mathfrak{so}(E_{w_2}^\perp \wedge \dots \wedge w_r)} R_{w_1 \wedge w_2}^{\text{Id}} = 0$. Indeed, this identity is equivalent to the condition

$$\langle R^{\text{Id}}(w_1, w_2)z_1, z_2 \rangle = 0, \quad \forall z_1, z_2 \in E_{w_2 \wedge \dots \wedge w_r}^\perp$$

which in turn is immediate from the definition of R^{Id} (see Notation and Conventions). \square

Corollary 2.3.8. *If $\dim V \geq 2$ then $\dim \mathcal{AC}_{CR1}(V, \chi) \geq 1$.*

Remark 2.3.9. If I is a set of N orthonormal r -frames in V then we can consider the set

$$\mathcal{AC}_{CR1}^{[I]}(V, \chi) = \{R \in \mathcal{AC}(V) \mid [\Pi_{\mathfrak{so}(E_{w_2}^\perp \wedge \dots \wedge w_r)} R_{w_1 \wedge w_2}, J_{E_{w_2}^\perp \wedge \dots \wedge w_r}] = 0, \forall w \in I\}.$$

Clearly, for any I one has $\mathcal{AC}_{CR1}(V, \chi) \subseteq \mathcal{AC}_{CR1}^{[I]}(V, \chi)$ and so if $\dim V \geq 2$ then for any I one has $1 \leq \dim \mathcal{AC}_{CR1}(V, \chi) \leq \dim \mathcal{AC}_{CR1}^{[I]}(V, \chi)$. This paves the way to determining $\dim \mathcal{AC}_{CR1}(V, \chi)$ via Monte Carlo methods: one randomly picks a finite subset I and computes the corresponding dimension of $\mathcal{AC}_{CR1}^{[I]}(V, \chi)$. If this happens to be equal to 1, then one sees that necessarily $\dim \mathcal{AC}_{CR1}(V, \chi) = 1$.

Proposition 2.3.10. *Let V be a 7-dimensional Euclidean vector space endowed with a 2-fold VCP χ . Then $\dim \mathcal{AC}_{CR1}(V, \chi) = 1$. In particular, $\mathcal{AC}_{CR1}(V, \chi)$ is spanned by R^{Id} .*

Proof. The pair (V, χ) can be identified with the 7-dimensional space $\text{Im}\mathbb{O}$ of imaginary octonions endowed with their standard VCP. In this model one can easily implement a Monte Carlo computation of $\dim \mathcal{AC}_{CR1}(V, \chi) = 1$ as described in Remark 2.3.9. Implementation shows that already with 100 random points one generally obtains $\dim \mathcal{AC}_{CR1}(V, \chi) = 1$. A `sagemath` code implementing this computation is provided and commented in the Appendix. It runs in about 50 minutes on a 2.4 Ghz 8core. \square

Denote by \times the 2-fold VCP on $\text{Im}\mathbb{O}$, see (1.1.6). Proposition 2.3.10 implies the following corollary immediately

Corollary 2.3.11. *If $R \in \mathcal{AC}_{CR1}(\text{Im}\mathbb{O}, \times)$ then $R(w) \in \mathfrak{so}(E_w)$ for any $w \in \mathbb{G}r^+(2, \text{Im}\mathbb{O})$.*

3. PROOF OF THEOREM 1.2.14(1–3)

In this section we define the torsion tensor $T \in \Gamma(\wedge^2 B^* \otimes T\mathbb{G})$ on the total space \mathbb{G} over a Riemannian manifold (M, g) endowed with a VCP. Then, using the results in the previous section, we give a proof of Theorem

1.2.14 (1–2) and of Theorem 1.2.14 (3) for the (2,7) case. To prove Theorem 1.2.14 (3) for the (3,8) case we reduce the Equation (2.2.7) for the case $(r, m) = (3, 8)$ to the case $(r, m) = (2, 7)$ and utilize the symmetry of the equation (1) as well as ad-hoc techniques.

Let $(\mathbb{G}, B, J_{g,\chi})$ be the CR twistor space over a manifold (M, g) endowed with a VCP χ . Define a section

$$T : \mathbb{G} \rightarrow \bigwedge^2 B^* \otimes T\mathbb{G}$$

by

$$(3.0.1) \quad T_v(X, Y)^{\text{vert}} = ([J_{g,\chi}X, J_{g,\chi}Y] - [X, Y])_v^{\text{vert}},$$

$$(3.0.2) \quad T_v(X, Y)^{\text{hor}} = \Pi_B([J_{g,\chi}X, J_{g,\chi}Y] - [X, Y]) - J_{g,\chi} \circ \Pi_B([X, J_{g,\chi}Y] + [J_{g,\chi}X, Y])_v.$$

for any $v \in \mathbb{G}$ and any $X, Y \in \Gamma(B)$. By (2.2.1), the RHS of (3.0.1) depends only on $X(v), Y(v)$, thus T_v^{vert} is a well-defined tensor. We verify immediately, or utilize (4.1.3) below, to conclude that that the RHS of (3.0.2), like the Nijenhuis tensor, depends only on the value $X(v), Y(v)$. Thus T is a tensor on the manifold \mathbb{G} . Now,

- Theorem 1.2.14(1) follows immediately from Corollary 2.1.13, taking into account (3.0.1).
- Theorem 1.2.14(2) is Example 2.2.10 (1–2).
- Theorem 1.2.14(3) for the (2,7) case follows from Proposition 2.3.10, Lemma 2.3.7, (3.0.1), noting that if $\dim M \geq 3$ then a metric g on M which satisfies the condition $R_g(x) = \lambda(x)R_{\text{Id}}$ is a constant curvature metric by the Schur lemma.

To conclude the proof of Theorem 1.2.14 (3), i.e., to prove it for the (3,8) case, we need a preparatory result.

Let χ be the 3-fold VCP on $\mathbb{R}^8 = \mathbb{O}$, see (1.1.7) and \times the 2-fold VCP on $\mathbb{R}^7 = \text{Im } \mathbb{O}$, and let

$$t : \bigwedge^2 \mathbb{O}^* \otimes \mathfrak{so}(\mathbb{O}) \rightarrow \bigwedge^2 (\text{Im } \mathbb{O})^* \otimes \mathfrak{so}(\text{Im } \mathbb{O})$$

be the restriction/projection operator defined by

$$t(\alpha \otimes A) = \alpha|_{\text{Im } \mathbb{O}} \otimes \Pi_{\mathfrak{so}(\text{Im } \mathbb{O})} A.$$

It induces by restriction a map

$$t : \mathcal{AC}_{CR1}(\mathbb{O}, \chi) \rightarrow \mathcal{AC}_{CR1}(\text{Im } \mathbb{O}, \times).$$

Proposition 3.0.3. *Assume that $R \in \mathcal{AC}_{CR1}(\mathbb{O}, \chi)$. Then for any $w_1 \wedge w_2 \in \mathbb{G}\mathbb{r}^+(2, \mathbb{O})$ we have*

$$R(w_1 \wedge w_2) \in \mathfrak{so}(E_{w_1 \wedge w_2}).$$

Proof. Since Spin(7) acts transitively on $\mathbb{G}\mathbb{r}^+(2, \mathbb{O})$ and the space $\mathcal{AC}_{CR1}(\mathbb{O}, \chi)$ is invariant under the Spin(7)-action, it suffices to prove Proposition 3.0.3

for $w_1 = i$ and $w_2 = j$. Since $t(R) \in \mathcal{AC}_{CR1}(\mathbb{O}, \times)$, taking into account Corollary 2.3.11, we have $R(i \wedge j) \in \Pi_{\mathfrak{so}(\text{Im } \mathbb{O})}^{-1}(\mathfrak{so}(E_{i \wedge j}))$ and so

$$(3.0.4) \quad R(i \wedge j) = R_1(i \wedge j) + R_2(i \wedge j),$$

with $R_1(i \wedge j)$ in $\mathfrak{so}(E_{i \wedge j})$ and $R_2(i \wedge j)$ in $\epsilon(\text{Hom}(E_1, \text{Im } \mathbb{O}))$. We shall show that $R_2(i \wedge j) = 0$. Let \mathcal{U} be the $\text{Stab}_{\text{Spin}(7)}(i \wedge j)$ -invariant subspace of $\mathfrak{so}(\mathbb{O})$ defined by

$$(3.0.5) \quad \mathcal{U} = \{A \in \mathfrak{so}(\mathbb{O}) \mid [\Pi_{\mathfrak{so}(E_{j \wedge w}^\perp)} A, J_{E_{j \wedge w}^\perp}] = 0, \forall w \in E_{i \wedge j}^\perp\}.$$

Since $R \in \mathcal{AC}_{CR1}(\mathbb{O}, \chi)$, we have $R(i \wedge j) \in \mathcal{U}$ and by Example 2.2.10 (3), we have $R_1(i \wedge j) \in \mathcal{U}$. Therefore $R_2(i \wedge j) \in \mathcal{U}$. Now we write

$$R_2(i \wedge j) = a_i 1^\# \hat{\otimes} i + a_j 1^\# \hat{\otimes} j + a_k 1^\# \hat{\otimes} k + a_l 1^\# \hat{\otimes} \ell + b_i 1^\# \hat{\otimes} li + b_j 1^\# \hat{\otimes} lj + b_k 1^\# \hat{\otimes} lk,$$

where recall that $1^\# \in E_1^*$ is dual to 1 in E_1 , and $a_i, a_j, a_k, a_l, b_i, b_j, b_k, b_l \in \mathbb{R}$. From [Bryant82, Proposition 2.1, p. 196] and other assertions therein one obtains that the stabilizer $\text{Stab}_{\text{Spin}(7)}(i \wedge j)$ acts transitively on the product of unit spheres $S^1(E_{i \wedge j}) \times S^5(E_{i \wedge j}^\perp)$ so we can assume without loss of generality that $a_j = a_l = b_i = b_j = b_k = 0$. Picking $w = \ell$ in (3.0.5) we obtain

$$\Pi_{\mathfrak{so}(E_{j \wedge \ell}^\perp)} R_2(i \wedge j) = \Pi_{\mathfrak{so}(E_{j \wedge \ell}^\perp)} (a_i 1^\# \hat{\otimes} i + a_k 1^\# \hat{\otimes} k) = a_i 1^\# \hat{\otimes} i + a_k 1^\# \hat{\otimes} k.$$

Using the explicit form of the Cayley 4-form φ_χ as given in [KLS18, (4.1)]

$$\begin{aligned} \varphi_\chi = & e^{0123} + e^{0145} + e^{0167} + e^{0246} - e^{0257} - e^{0347} - e^{0356} \\ & + e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}, \end{aligned}$$

where e^{abcd} is a shorthand notation for $e^a \wedge e^b \wedge e^c \wedge e^d$, and (e^i) is the dual basis of the standard orthonormal basis of \mathbb{O} , one computes

$$[J_{E_{j \wedge \ell}^\perp}, (a_i 1^\# \hat{\otimes} i + a_k 1^\# \hat{\otimes} k)] = a_i (\ell j)^\# \hat{\otimes} i + a_k (\ell j)^\# \hat{\otimes} k - a_i 1^\# \hat{\otimes} \ell k - a_k 1^\# \hat{\otimes} \ell i.$$

This vanishes if and only if $a_i = a_k = 0$. \square

Theorem 1.2.14(3) in the (3,8) case now follows immediately by noticing that the Riemannian curvature tensor R of a connected Riemannian manifold (M, g) with $\dim M \geq 3$ satisfies $R(v, w) \in \mathfrak{so}(E_{v \wedge w})$ for any two linearly independent tangent vectors if and only if (M, g) has constant sectional curvature.²

Remark 3.0.6. Given Riemannian manifolds (M^7, g) , or (M^8, g) of constant curvature, the existence of a VCP product on (M^7, g) and (M^8, g) is equivalent to the existence of a section of the associated $SO(7)/G_2$ -bundle over M^7 and of the associated $SO(8)/\text{Spin}(7)$ -bundle over M^8 , respectively. A section of the associated $SO(7)/G_2$ -bundle over M^7 exists if and only if the manifold M^7 is orientable and spinnable, i.e., equivalently, if and only

²At least the ‘‘if’’ assertion seems well known, see e.g. [Gilkey01, p. 31] for an equivalent formulation, which we also utilize below. The ‘‘only if’’ part is an easy consequence of Schur’s lemma for the Ricci tensor. A detailed proof can be found in arXiv:2203.04233v2.

if the first and the second Stiefel-Whitney classes $w_1(M^7)$ and $w_2(M^7)$ of M^7 vanish, see [LM89, Theorem 10.6, Chapter IV] or [FKMS97, Proposition 3.2]. A section of the associated $SO(8)/\text{Spin}(7)$ -bundle over M^8 exists if and only if $w_1(M^8) = w_2(M^8) = 0$ and for any choice of orientation of M^8 one has $p_1(M^8)^2 - 4p_2(M^8) \pm 8\chi(M^8) = 0$, where p_1 and p_2 are the first two Pontryagin classes and χ is the Euler class [GG1970, Theorem 3.4, Corollary 3.5]. A family of $\text{Sp}(2)$ -invariant G_2 -structure on homogeneous 7-sphere $\text{Sp}(2)/\text{Sp}(1)$ of constant curvature is given in Remark at the end of Section 2 in [LMES21]. It follows from [ACFR20] that there is no homogeneous Spin(7)-structure on the sphere S^8 .

4. THE SECOND CR-INTEGRABILITY CONDITION

In this section (M, g, χ) is a Riemannian manifold with a VCP structure χ and $(\mathbb{G}, B, J_{g,\chi})$ is its CR-twistor space. In Subsection 4.1 we express the second integrability condition (2) in terms of the Frölicher-Nijenhuis tensor and compute this tensor in terms of (M, g, χ) in later subsections. Using this we complete the proof of the Main Theorem 1.2.14. For this purpose, we consider the natural metric \tilde{g} on the total space $\wedge^{r-1}TM$ such that

- (i) for any $v \in \wedge^{r-1}TM$, $T_v^{\text{vert}} \wedge^{r-1}TM$ is orthogonal to $T_v^{\text{hor}} \wedge^{r-1}TM$,
- (ii) for any $v \in \wedge^{r-1}TM$ the restriction of \tilde{g} to $T_v^{\text{vert}} \wedge^{r-1}T_{\pi(v)}M$ coincides with the metric on $\wedge^{r-1}T_{\pi(v)}M$ defined by $g(\pi(v))$,
- (iii) The projection $\pi : (\wedge^{r-1}TM, \tilde{g}) \rightarrow (M, g)$ is a Riemannian submersion.

If $r = 2$ then \tilde{g} is the Sasaki metric on TM [Sasaki58]. Abusing notation, we also denote by \tilde{g} the restriction of \tilde{g} to \mathbb{G} . Let us extend the operator $J_{g,\chi} : B \rightarrow B$ to an operator $\tilde{J}_B : T\mathbb{G} \rightarrow T\mathbb{G}$ on the whole space $T\mathbb{G}$ by setting

$$(\tilde{J}_B)|_B = J_{g,\chi}, \quad (\tilde{J}_B)|_{B^\perp} = 0,$$

where B^\perp is the orthogonal complement to B in $T\mathbb{G}$.

4.1. The second CR-integrability condition and the Frölicher-Nijenhuis tensor.

Proposition 4.1.1. *The second CR-integrability condition is equivalent to the following condition*

$$(4.1.2) \quad \Pi_B([\tilde{J}_B, \tilde{J}_B]_B^{FN}) = 0.$$

Proof. By [KMS93, Corollary 8.12, p. 73], for any two vector fields X, Y on $T\mathbb{G}$ we have

$$(4.1.3) \quad \Pi_B\left(\frac{1}{2}[\tilde{J}_B, \tilde{J}_B]^{FN}(X, Y)\right) = \Pi_B([\tilde{J}_B X, \tilde{J}_B Y] - [X, Y] - \tilde{J}_B([X, \tilde{J}_B Y] + [\tilde{J}_B X, Y])).$$

Taking into account $\Pi_B \circ \tilde{J}_B = \tilde{J}_B \circ \Pi_B$, this proves Proposition 4.1.1. \square

Now we are going to express Condition (4.1.2) in terms of the Levi-Civita covariant derivative $\tilde{\nabla}$ on the Riemannian manifold (\mathbb{G}, \tilde{g}) . Let $\tilde{\omega}$ be the 2-form on \mathbb{G} defined by $\tilde{\omega}(X, Y) = \tilde{g}(\tilde{J}_B X, Y)$. Equivalently, $\tilde{\omega}_v(X, Y) = \varphi_\chi(v \wedge d\pi_v X \wedge d\pi_v Y)$. In particular, we have

$$(4.1.4) \quad \tilde{\omega}_v(X, Y) = (\pi^* \varphi_\chi)(Z, X, Y)$$

for any Z with $d\pi_v Z = v$. Notice that, by construction, $\tilde{\omega}_v$ only depends on the horizontal parts of the tangent vectors X and Y in $T_v \mathbb{G}$.

Denote by (e_i) an orthonormal basis of $T_v \mathbb{G}$. By [KLS18, Proposition 2.2], we have

$$[\tilde{J}_B, \tilde{J}_B]_v^{FN} = 2 \sum_{i,j} ((\iota_{e_i} \tilde{\omega}) \wedge (\iota_{e_j} \tilde{\nabla}_{e_i} \tilde{\omega}) + \sum_k (\iota_{e_j} \iota_{e_i} \tilde{\omega}) \wedge e^k \wedge (\iota_{e_i} \tilde{\nabla}_{e_k} \tilde{\omega})) \otimes e_j.$$

Let $m = \dim M$ and $N = \dim \mathbb{G}$. We can choose (e_i) in such a way that e_1, \dots, e_{m-r+1} is a basis of $B(v)$. With such a choice one has that $\Pi_B([\tilde{J}_B, \tilde{J}_B]_{B(v)}^{FN}) = 0$ if and only if for all $j, p, q \in [1, m+r-1]$ one has

$$\iota_{e_p} \iota_{e_q} \left(\sum_{i \in [1, m-r+1]} (\iota_{e_i} \tilde{\omega}) \wedge (\iota_{e_j} \tilde{\nabla}_{e_i} \tilde{\omega}) + \sum_{i, k \in [1, m-r+1]} (\iota_{e_j} \iota_{e_i} \tilde{\omega}) \wedge e^k \wedge (\iota_{e_i} \tilde{\nabla}_{e_k} \tilde{\omega}) \right) = 0.$$

We can choose (e_1, \dots, e_{m-r+1}) to be a unitary frame with respect to the pair $(\tilde{g}|_{B(v)}, J_{g, \chi})$, i.e., in such a way that $e_{\frac{m+r-1}{2}+k} = J_{g, \chi} e_k$, for $k \in [1, \frac{m+r-1}{2}]$. The vectors $(e_1, \dots, e_{\frac{m+r-1}{2}})$ will be called a *Hermitian basis*. With this choice, for $a, b \in [1, m+r-1]$ with $a < b$ one has

$$\tilde{\omega}(e_a, e_b) = \begin{cases} 1 & \text{if } e_b = J_{g, \chi} e_a \\ 0 & \text{elsewhere} \end{cases}$$

The second CR-intergrability condition is therefore equivalent to the system

$$(4.1.5) \quad \begin{aligned} & \iota_{e_p} \iota_{e_q} \left(\iota_{J_{g, \chi} e_p} \tilde{\omega} \wedge (\iota_{e_j} \tilde{\nabla}_{J_{g, \chi} e_p} \tilde{\omega}) + \iota_{J_{g, \chi} e_q} \tilde{\omega} \wedge (\iota_{e_j} \tilde{\nabla}_{J_{g, \chi} e_q} \tilde{\omega}) \right. \\ & \left. + \sum_{k \in [1, m-r+1]} e^k \wedge (\iota_{J_{g, \chi} e_j} \tilde{\nabla}_{e_k} \tilde{\omega}) \right) = 0 \end{aligned}$$

for any $j, p, q \in [1, m-r+1]$. The term involving e^k in the last sum in LHS of (4.1.5) vanishes unless $k \in \{p, q\}$. So we can rewrite (4.1.5) as follows

$$(4.1.6) \quad \begin{aligned} & -\iota_{e_q} (\iota_{e_j} \tilde{\nabla}_{J_{g, \chi} e_p} \tilde{\omega}) + \iota_{e_p} (\iota_{e_j} \tilde{\nabla}_{J_{g, \chi} e_q} \tilde{\omega}) + \iota_{e_q} (\iota_{J_{g, \chi} e_j} \tilde{\nabla}_{e_p} \tilde{\omega}) - \iota_{e_p} \iota_{J_{g, \chi} e_j} \tilde{\nabla}_{e_q} \tilde{\omega} = 0 \\ & \text{for any } j, p, q \in [1, m-r+1]. \end{aligned}$$

We can now complete the proof of Theorem 1.2.14.

- Theorem 1.2.14(4) follows from Lemma 2.1.12, Proposition 4.1.1 and (3.0.2).
- Theorem 1.2.14(5–6) is classical, see Remark 1.2.15.
- A proof of Theorem 1.2.14(7) is the content of the following two Sections.

4.2. The second CR-integrability condition for a 7-manifold (M, g) with a VCP structure. Since the complex structure $J_{g, \chi}$ on $B(v)$ is given by the vector cross product with the unit vector v , and the 2-form $\tilde{\omega}$ is defined in terms of the 3-form φ_χ , one should expect that the expressions appearing in (4.1.6) can be written in terms of φ_χ and of $v \times -$. This is precisely the content of the following Lemma.

Lemma 4.2.1. *Let x be a point in M and let (v, w_1, w_2, w_3) be a orthonormal quadruple in $T_x M$ such that (w_1, w_2, w_3) is a Hermitian basis of $(E_v^\perp, J_{E_v^\perp})$. Let $w_{3+k} = v \times w_k$ for $k \in [1, 3]$. Let (e_1, \dots, e_6) be an orthonormal basis in $B(v)$ with $w_a = d\pi_v(e_a)$ for $a \in [1, 6]$. Then the following identities hold for $j, p, q \in [1, 6]$*

$$(4.2.2) \quad \iota_{e_q}(\iota_{e_j} \tilde{\nabla}_{J_{g, \chi} e_p} \tilde{\omega}) = (\nabla_{v \times w_p} \varphi_\chi)(v, w_j, w_q)$$

$$(4.2.3) \quad \iota_{e_q}(\iota_{J_{g, \chi} e_j} \tilde{\nabla}_{e_p} \tilde{\omega}) = (\nabla_{w_p} \varphi_\chi)(v, v \times w_j, w_q)$$

In order to prove Lemma 4.2.1 we will need some preparation. Let $v \in \mathbb{G} = \mathbb{G}\mathbb{r}_1^+(1, M) \subset TM$. For a tangent vector X in $T_{\pi(v)}M$, we denote by $X^{\text{h.l.}}|_v$ and by $X^{\text{v.l.}}|_v$ the horizontal and vertical lifts of X to horizontal and tangent vectors in $T_v TM$. Since $T_v^{\text{hor}} TM = T_v^{\text{hor}} \mathbb{G}$, for any horizontal tangent vector $Y \in T_v^{\text{hor}} \mathbb{G}$ we have $Y = (d\pi_v(Y))^{\text{h.l.}}|_v$.

Lemma 4.2.4. *Let $v \in \mathbb{G}$ and let (e_i) be a orthonormal basis of $T_v^{\text{hor}} \mathbb{G}$ with $d\pi_v(e_7) = v$. Let $w_1, \dots, w_6, v \in T_{\pi(v)}M$ be images of e_1, \dots, e_7 via $d\pi_v$, and let (W_i) be the vector fields on a neighborhood U of $\pi(v)$ in M corresponding to the normal coordinates defined by the exponential map $\exp_{\pi(v)}: T_{\pi(v)}M \rightarrow M$ such that $W_i|_{\pi(v)} = w_i$ for $i = 1, \dots, 6$ and $W_7|_{\pi(v)} = v$. Finally let (\hat{e}_i) be the vector fields in the neighborhood $\pi^{-1}(U)$ of v in \mathbb{G} defined by horizontal lifting of (W_i) , i.e., $\hat{e}_i|_y = W_{i; \pi(y)}^{\text{h.l.}}|_y$ for any $y \in \pi^{-1}(U)$. Then $\hat{e}_i|_v = e_i$ and*

$$(\tilde{\nabla}_{\hat{e}_i} \hat{e}_j)^{\text{hor}}|_v = 0$$

for any $i, j \in [1, 6]$.

Proof. The first statement is immediate by the definition of \hat{e}_i : we have

$$\hat{e}_i|_v = W_{i; \pi(v)}^{\text{h.l.}}|_v = w_i^{\text{h.l.}}|_v = (d\pi_v(e_i))^{\text{h.l.}}|_v = e_i.$$

For the second statement, let ∇ denote the Levi-Civita connection of (M, g) . Since $\nabla_{W_i} W_j|_{\pi(v)} = 0$, by [GK02, Proposition 7.2 (i)] we have

$$\tilde{\nabla}_{\hat{e}_i} \hat{e}_j|_v = -\frac{1}{2}(R_g(W_i, W_j)|_{\pi(v)} v)^{\text{v.l.}}|_v,$$

where R_g denotes the curvature tensor of (M, g) . The conclusion immediately follows. \square

Lemma 4.2.5. *In the same notation as in Lemma 4.2.4, let Z be the vector field on the neighborhood U of $\pi(v)$ in M defined by $Z_x = \text{tra}_{\pi(v), x}(v)$, where $\text{tra}_{\pi(v), x}: T_{\pi(v)}M \rightarrow T_x M$ is the parallel transport along the unique geodesics*

in U from $\pi(v)$ to x , and let \mathcal{H} be the horizontal lift of Z to the neighborhood $\pi^{-1}(U)$ of v in \mathbb{G} , i.e., $\mathcal{H}_y = Z_{\pi(y)}^{\text{h.l.}}|_y$ for any $y \in \mathbb{G}$ with $\pi(y) \in U$. Then $d\pi_v(\mathcal{H}_v) = v$ and

$$(\tilde{\nabla}_{\hat{e}_i} \mathcal{H})^{\text{hor}}|_v = 0$$

for any $i, j \in [1, 6]$.

Proof. The first statement is immediate from $\mathcal{H}_v = Z_{\pi(v)}^{\text{h.l.}}|_v = v^{\text{h.l.}}|_v$. For the second statement, since Z is defined by parallel transport along geodesics stemming from $\pi(v)$ and the vector fields W_i correspond to normal coordinates at $\pi(v)$, we have $\nabla_{W_i} Z|_{\pi(v)} = 0$, where ∇ denotes the Levi-Civita connection on M . The conclusion then follows from [GK02, Proposition 7.2 (i)], by the same reasoning as in the proof of Lemma 4.2.4. \square

Proof of Lemma 4.2.1. By assumption, the triple (e_1, e_2, e_3) is a Hermitian basis of $(B(v), J_{g,\chi})$, so that setting $e_4 = J_{g,\chi}e_1$, $e_5 = J_{g,\chi}e_2$ and $e_6 = J_{g,\chi}e_3$ we obtain a orthonormal basis for $B(v)$. We complete it to a orthonormal basis for $T_v^{\text{hor}}\mathbb{G}$ by adding a horizontal vector e_7 with $d\pi_v(e_7) = v$. Let us also write

$$u_p = \begin{cases} w_{p+3} & \text{if } p \in [1, 3] \\ -w_{p-3} & \text{if } p \in [4, 6] \end{cases}; \quad f_p = \begin{cases} e_{p+3} & \text{if } p \in [1, 3] \\ -e_{p-3} & \text{if } p \in [4, 6] \end{cases}$$

and, accordingly, $\hat{f}_p = \hat{e}_{p+3}$ if $p \in [1, 3]$ and $\hat{f}_p = -\hat{e}_{p-3}$ if $p \in [4, 6]$. We are in the assumptions of Lemma 4.2.4 and 4.2.5 and so, in the same notation as there, we have $(\tilde{\nabla}_{\hat{e}_i} \hat{e}_j)^{\text{hor}}|_v = (\tilde{\nabla}_{\hat{e}_i} \mathcal{H})^{\text{hor}}|_v = 0$. Therefore, recalling that $\tilde{\omega}$ only depends on the horizontal components of its arguments,

$$\begin{aligned} (\tilde{\nabla}_{\hat{f}_p} (\tilde{\omega}(\hat{e}_j, \hat{e}_q)))|_v &= ((\tilde{\nabla}_{\hat{f}_p} \tilde{\omega})(\hat{e}_j, \hat{e}_q))|_v = (\iota_{\hat{e}_q} \iota_{\hat{e}_j} \tilde{\nabla}_{\hat{f}_p} \tilde{\omega})|_v = \iota_{e_q} \iota_{e_j} \left((\tilde{\nabla}_{\hat{f}_p} \tilde{\omega})|_v \right) \\ &= \iota_{e_q} \iota_{e_j} (\tilde{\nabla}_{f_p} \tilde{\omega}). \end{aligned}$$

On the other hand,

$$(\tilde{\nabla}_{\hat{f}_p} (\tilde{\omega}(\hat{e}_j, \hat{e}_q)))|_v = \tilde{\nabla}_{f_p} (\tilde{\omega}(\hat{e}_j, \hat{e}_q)) = \frac{d}{dt} \Big|_{t=0} \tilde{\omega}_{\tilde{\gamma}(t)}(\hat{e}_j|_{\tilde{\gamma}(t)}, \hat{e}_q|_{\tilde{\gamma}(t)}),$$

where $\tilde{\gamma}$ is any path in $Gr^{+1}(1, M)$ with $\tilde{\gamma}(0) = v$ and $\frac{d}{dt} \Big|_{t=0} \tilde{\gamma} = f_p$. In particular, we can choose as $\tilde{\gamma}$ the horizontal lift of the geodesics $\gamma_{\pi(v); u_p}$ stemming from the point $\pi(v)$ of M with tangent vector $u_p = d\pi_v f_p$. By definition of parallel transport, this lift is $Z_{\gamma_{\pi(v); u_p}}$ so that

$$(\tilde{\nabla}_{\hat{f}_p} (\tilde{\omega}(\hat{e}_j, \hat{e}_q)))|_v = \frac{d}{dt} \Big|_{t=0} \tilde{\omega}_{Z_{\gamma_{\pi(v); u_p}(t)}}(\hat{e}_j|_{Z_{\gamma_{\pi(v); u_p}(t)}}, \hat{e}_q|_{Z_{\gamma_{\pi(v); u_p}(t)}}).$$

By definition of \mathcal{H} , we have $d\pi_y \mathcal{H}_y = Z_{\pi(y)}$ for any $y \in \pi^{-1}(U)$, so that, in particular, $d\pi_{Z_{\gamma_{\pi(v); u_p}(t)}} \mathcal{H}_{Z_{\gamma_{\pi(v); u_p}(t)}} = Z_{\gamma_{\pi(v); u_p}(t)}$. By (4.1.4) we therefore

have

$$\begin{aligned}\tilde{\omega}_{Z_{\gamma_{\pi(v);u_p}(t)}}(\hat{e}_j|_{Z_{\gamma_{\pi(v);u_p}(t)}}, \hat{e}_q|_{Z_{\gamma_{\pi(v);u_p}(t)}}) &= (\pi^* \varphi_\chi)(\mathcal{H}, \hat{e}_j, \hat{e}_q)|_{Z_{\gamma_{\pi(v);u_p}(t)}} \\ &= \varphi_\chi(Z, W_j, W_q)|_{\gamma_{\pi(v);u_p}(t)},\end{aligned}$$

and so

$$\begin{aligned}(\tilde{\nabla}_{\hat{f}_p}(\tilde{\omega}(\hat{e}_j, \hat{e}_q)))|_v &= \frac{d}{dt}\Big|_{t=0} \varphi_\chi(Z, W_j, W_q)|_{\gamma_{\pi(v);u_p}(t)} = \nabla_{u_p}(\varphi_\chi(Z, W_j, W_q)) \\ &= (\nabla_{u_p} \varphi_\chi)(v, w_j, w_q),\end{aligned}$$

where in the last identity we used the fact that, by construction, $\nabla_{u_p} Z = \nabla_{u_p} W_j = \nabla_{u_p} W_q = 0$. Since $f_p = J_{g,\chi} e_p$ we have

$$u_p = d\pi_v f_p = d\pi_v(J_{g,\chi} e_p) = J_{E_v^\perp}(d\pi_v e_p) = J_{E_v^\perp}(w_p) = v \times w_p,$$

so we have finally found

$${}_{\iota} e_q \iota e_j(\tilde{\nabla}_{J_{g,\chi} e_p} \tilde{\omega}) = (\nabla_{v \times w_p} \varphi_\chi)(v, w_j, w_q).$$

This proves (4.2.2). The proof of (4.2.3) is analogue. \square

From (4.1.6) and Lemma 4.2.1 we get the following.

Lemma 4.2.6. *The second CR-integrability holds for $(\mathbb{G}r^+(1, M^7), B, J_{g,\chi})$ if and only if for any $x \in M, v \in \mathbb{G}$ and some (and hence any) Hermitian basis w_1, w_2, w_3 of $(E_v^\perp, J_{E_v^\perp})$ the following conditions hold*

$$(4.2.7) \quad \begin{aligned} &(\nabla_{v \times w_q} \varphi_\chi)(v, w_j, w_p) - (\nabla_{v \times w_p} \varphi_\chi)(v, w_j, w_q) \\ &+ (\nabla_{w_q} \varphi_\chi)(v, v \times w_j, w_p) - (\nabla_{w_p} \varphi_\chi)(v, v \times w_j, w_q) = 0 \end{aligned}$$

for any $j, p, q \in [1, 6]$, where $w_{3+k} = v \times w_k$ for $k \in [1, 3]$.

Definition 4.2.8. Let V be a 7-dimensional Euclidean space endowed with a 2-fold vector cross product \times . The space of *algebraic intrinsic torsions* for (V, \times) is the subspace $\mathcal{T}(V)$ of $V^* \otimes \bigwedge^3 V^*$ consisting of those elements A such that $A(\eta_1; \eta_2 \wedge \eta_3 \wedge (\eta_2 \times \eta_3)) = 0$ for any $\eta_1, \eta_2, \eta_3 \in V$. The G_2 -invariant subspace $\mathcal{T}_{CR2}(V)$ of $\mathcal{T}(V)$ is defined as the subspace of $\mathcal{T}(V)$ consisting of those elements A such that for any $v \in \mathbb{G}r^+(1, V)$ and some (and hence any) Hermitian basis w_1, w_2, w_3 of $(E_v^\perp, J_{E_v^\perp})$ the condition

$$(4.2.9) \quad \begin{aligned} &A(v \times w_q; v \wedge w_j \wedge w_p) - A(v \times w_p; v \wedge w_j \wedge w_q) \\ &+ A(w_q; v \wedge (v \times w_j) \wedge w_p) - A(w_p; v \wedge (v \times w_j) \wedge w_q) = 0 \end{aligned}$$

holds for any $j, p, q \in [1, 6]$, where $w_{3+k} = v \times w_k$ for $k \in [1, 3]$.

Lemma 4.2.10. *We have $\dim(\mathcal{T}(V)) = 49$ and $\dim(\mathcal{T}_{CR2}(V)) = 0$.*

Proof. It is well known that $\dim(\mathcal{T}(V)) = 49$ [FG82, Lemma 4.1]. Then (4.2.9) imposes an infinite system of linear equations on $\mathcal{T}(V)$, parametrized by quadruples (v, w_1, w_2, w_3) as in the statement of the Lemma. These are

obviously not linearly independent. Yet it turns out that it is generally sufficient to sample ten random quadruples to impose 49 linearly independent equations. A `sagemath` code doing this is provided in the Appendix. It runs in about 1 hour on a 2.4 Ghz 8core.

□

Remark 4.2.11. Fernandez and Gray provide in [FG82] an explicit decomposition of $\mathcal{T}(V)$ into a orthogonal direct sum of four irreducible G_2 representations, of dimensions, 1, 7, 14, 27, respectively. It is likely that working with these and with standard basis quadruples in $(\text{Im}(\mathbb{O}), \times)$ one can obtain $\dim(\mathcal{T}_{CR2}(V)) = 0$ by imposing by hand a suitable subset of the equations (4.2.9) without relying on computer algebra. This would however presumably take much more than an hour to be done.

Proposition 4.2.12. *The second CR-integrability condition holds for a 2-fold VCP structure (g, χ) on a 7-manifold M^7 if and only if the VCP structure (g, χ) is parallel.*

Proof. Immediate from Lemmas 4.2.6 and 4.2.10. □

4.3. The second CR-integrability for a 8-manifold (M, g) with a VCP structure. The analysis of the second CR-integrability for a 8-manifold (M, g) with a VCP structure goes precisely along the same lines as for the 7-dimensional case. The only change is that now v is an orthonormal frame representing an element in $\mathbb{G} = \mathbb{G}r^+(2, M)$ instead of a unit vector representing an element in $\mathbb{G}r^+(1, M)$. With the same proof, we have the following 8-dimensional analogue of Lemma 4.2.6.

Proposition 4.3.1. *The second CR-integrability holds for $(\mathbb{G}r^+(2, M^8), B, J_{g, \chi})$ if and only if for any $x \in M, v \in \mathbb{G}$ and some (and hence any) Hermitian basis w_1, w_2, w_3 of $(E_v^\perp, J_{E_v^\perp})$ the following conditions hold*

$$(4.3.2) \quad \begin{aligned} & (\nabla_{v \times w_q} \varphi_\chi)(v \wedge w_j \wedge w_p) - (\nabla_{v \times w_p} \varphi_\chi)(v \wedge w_j \wedge w_q) \\ & + (\nabla_{w_q} \varphi_\chi)(v \wedge (v \times w_j) \wedge w_p) - (\nabla_{w_p} \varphi_\chi)(v \wedge (v \times w_j) \wedge w_q) = 0 \end{aligned}$$

for any $j, p, q \in [1, 6]$, where $w_{3+k} = v \times w_k$ for $k \in [1, 3]$.

Theorem 1.2.14(7) for the (3,8) case can then be proved by the following reduction argument. First we recall that for any $x \in M^8$, one has $(\nabla \varphi_\chi)_x \in \mathcal{W}(T_x M) \subset T_x M^* \otimes \bigwedge^4 T_x M^*$, where

$$\mathcal{W}(V) = \left\{ A \in V^* \otimes \bigwedge^4 V^* \mid A(\eta_1; \eta_2 \wedge \eta_3 \wedge \eta_4 \wedge \chi(\eta_2, \eta_3, \eta_4)) = 0, \right. \\ \left. \forall \eta_1, \eta_2, \eta_3, \eta_4 \in V \right\},$$

see [Fernandez1986, §4].³ For any $\xi \in V$, we have a restriction operator

$$|_{E_\xi^\perp} : \mathcal{W}(V) \rightarrow \mathcal{T}(E_\xi^\perp)$$

given by

$$A|_{E_\xi^\perp}(\eta_1; \eta_2 \wedge \eta_3 \wedge \eta_4) := A(\eta_1; \xi \wedge \eta_2 \wedge \eta_3 \wedge \eta_4).$$

Let us denote by $\mathcal{W}_{CR2}(V)$ the subspace of $\mathcal{W}(V)$ consisting of those elements A such that for any $v \in \mathbb{G}r^+(2, V)$ and some (and hence any) Hermitian basis w_1, w_2, w_3 of $(E_v^\perp, J_{E_v^\perp})$ the condition

$$(4.3.3) \quad \begin{aligned} & A(v \times w_q; v \wedge w_j \wedge w_p) - A(v \times w_p; v \wedge w_j \wedge w_q) \\ & + A(w_q; v \wedge (v \times w_j) \wedge w_p) - A(w_p; v \wedge (v \times w_j) \wedge w_q) = 0 \end{aligned}$$

holds for any $j, p, q \in [1, 6]$, where $w_{3+k} = v \times w_k$ for $k \in [1, 3]$. It is straightforward to check that the restriction operator induces a restriction operator

$$|_{E_\xi^\perp} : \mathcal{W}_{CR2}(V) \rightarrow \mathcal{T}_{CR2}(E_\xi^\perp).$$

By Lemma 4.2.10 we therefore have that if $A \in \mathcal{W}_{CR2}(V)$, then $A|_{E_\xi^\perp} = 0$ for any ξ , and this means $A = 0$.

5. CONCLUSIONS AND FINAL REMARKS

(1) In this paper we unified and extended the construction of a CR-twistor space by LeBrun, Rossi and Verbitsky respectively, to the case when the underlying Riemannian manifold admits a VCP structure. We solved the question of the formal integrability of the CR-structure on the twistor space, recovering the results by LeBrun and Rossi respectively, and correcting the result by Verbitsky.

(2) We expressed the formal integrability of a CR-structure in terms of a torsion tensor on the underlying space. Our method can be applied for expressing the formal integrability of a CR-structure (B, J_B) on a smooth manifold M as follows. First we pick up any complement B^\perp of B in TM . Then we choose a metric g on M such that (i) B^\perp is orthogonal to B , (ii) $g|_B$ is a Hermitian metric with respect to J_B . Denote by Π_{B^\perp} and Π_B the orthogonal projections to B^\perp and B respectively. Define the tensor $T \in \bigwedge^2 B^* \otimes TM$ as

$$\begin{aligned} T(X, Y) = & \underbrace{\Pi_{B^\perp}([J_B X, J_B Y] - [X, Y])}_{T^{\text{vert}}(X, Y)} \\ & + \underbrace{\Pi_B([J_B X, J_B Y] - [X, Y]) - J_B \circ \Pi_B([X, J_B Y] + [J_B X, Y])}_{T^{\text{hor}}(X, Y)}. \end{aligned}$$

Then the CR-structure is formally integrable if and only if T vanishes. More precisely, the first CR-integrability condition for (B, J_B) holds if and only

³To emphasize the fact that $(\nabla\varphi_\chi)_x \in \mathcal{W}(T_x M)$, Fernandez calls \mathcal{W} “the space of covariant derivatives of the fundamental 4-form” in [Fernandez1986].

the tensor T^{vert} vanishes and the second CR-integrability condition holds if and only if the tensor T^{hor} that vanishes. Note that the integrability of a CR-structure has been investigated from the point of views of integrability and formal integrability of G -structures, see [DT06, §1.6.1, Theorem 1.14] for a detailed discussion. They also showed that the integrability of a CR-structure, viewed as a G -structure, implies that the associated Levi form vanishes [DT06, p. 71], see also [BHLN20, p.76]. We would like to mention that our expression of the integrability of a CR-structure in terms of the vertical and horizontal component of a torsion tensor is reminiscent to O'Brian- Rawnsley's expression of the Nijenhuis tensor of an almost complex structure on certain twistor spaces in terms of the curvature and torsion tensor of the associated connection [OR85].

(3) The characterization of metric of constant curvature on a Riemannian manifold of dimension at least 3 used at the end of the proof of Theorem 1.2.14(3) can be reformulated in terms of representation theory by saying that, given $w_0 \in \mathbb{G}\mathbb{r}^+(2, \mathbb{R}^n)$ and $R \in \mathcal{A}\mathcal{C}(\mathbb{R}^n)$, if for any $\gamma \in \text{SO}(n)$, $n \geq 3$

$$(5.0.1) \quad \gamma^* R(w_0) \in \mathfrak{so}(E_{w_0})$$

then $R = \lambda R^{\text{Id}}$ for some $\lambda \in \mathbb{R}$. The whole statement of Theorem 1.2.14(3) can be reformulated in a similar way: given $w_0 \in \mathbb{G}\mathbb{r}^+(2, \mathbb{R}^n)$ and $R \in \mathcal{A}\mathcal{C}(\mathbb{R}^n)$, $n = 7$ or $n = 8$, if for any $\gamma \in \text{G}_2$ or $\gamma \in \text{Spin}(7)$ respectively, we have

$$(5.0.2) \quad \gamma^* R(w_0) \in \mathcal{R}_{w_0} \subset \mathfrak{so}(\mathbb{R}^n),$$

where \mathcal{R}_{w_0} is the linear subspace of $\mathfrak{so}(\mathbb{R}^n)$ defined by (2.3.1), then $R = \lambda R^{\text{Id}}$. Using representation theory, it is not hard to see that the space of G_2 -invariant algebraic curvatures on \mathbb{R}^7 has dimension 1 and the space of $\text{Spin}(7)$ -invariant algebraic curvature on \mathbb{R}^8 also has dimension 1. Thus the equations (5.0.2) has only invariant solutions. It would be interesting to find a proof using only representation theory for Theorem 1.2.14(3) in the (2,7) case. For this case, one can compute that the dimension of \mathcal{R}_w in the RHS of (5.0.2) is 10.

(4) Like in the (2,7) case, also in the (3,8) case Theorem 1.2.14 (3) can be proved by directly using computer algebra, but this takes much longer machine time with respect to the 7-dimensional case (approximately 11 hours on a 2.4 Ghz 8core). One could shorten the machine time by investing human time on writing code for decomposition of the space of algebraic curvatures as G_2 -and $\text{Spin}(7)$ -modules.

(5) We gave a new characterization of torsion-free G_2 and $\text{Spin}(7)$ -structures. As we noted in Remark 4.2.11, it would be interesting to find a proof of Lemma 4.2.10 which would be only based on representation theory.

APPENDIX: A CODE FOR THE CR CONDITIONS ON A 7-DIMENSIONAL
MANIFOLD WITH VCP

To begin with, we define the command `randomorthogonal2`. It produces a random element in $O(7)$ with rational entries as follows: two random 6-dimensional vectors with entries in $\{0, 1\}$ are generated and stereographically projected on S^7 so to produce two unit vectors in \mathbb{R}^7 . Then we produce the orthogonal reflections with respect to the orthogonal hyperplanes to these two vectors and multiply them. Elements in $SO(7)$ obtained as the multiplication of two reflections are general enough so that their first two columns can be an arbitrary orthonormal pair in \mathbb{R}^7 . The pairs we produce are not uniformly distributed on the Stiefel manifold of pairs of orthonormal vectors in \mathbb{R}^7 , but for our aims this is not important. For later use we also define the orthogonal projection on the hyperplane orthogonal to a unit vector in \mathbb{R}^7 .

```

def stereographic7(myvector):
    v=vector(QQ,[0,0,0,0,0,0,0])
    somma=sum(myvector[j]^2 for j in [0..5])
    denom=1+somma
    v[6]=(-1+somma)/denom
    for i in [0..5]:
        v[i]=2*myvector[i]/denom
    return v

def reflection7(myvector):
    M=matrix(QQ,7)
    v=matrix(QQ,1,7)
    for h in [0..6]:
        v[0,h]=myvector[h]
    for h in [0..6]:
        M[h]=matrix.identity(7)[h]-2*(matrix.identity(7)[h]*
            transpose(v))*v
    N=transpose(M)
    return N

def randomorthogonal2():
    A=matrix(QQ,7)
    s1=[randint(0, 1) for i in [0..5]]
    t1=stereographic7(s1)
    s2=[randint(0, 1) for i in [0..5]]
    t2=stereographic7(s2)
    A=reflection7(t1)*reflection7(t2)
    return A

def projection7(myvector):
    M=matrix(QQ,7)

```

```

v=matrix(QQ,1,7)
for h in [0..6]:
    v[0,h]=myvector[h]
for h in [0..6]:
    M[h]=matrix.identity(7)[h]-(matrix.identity(7)[h]*
        transpose(v))*v
    N=transpose(M)
return N

```

We define the 2-fold VCP on \mathbb{R}^7 by means of the 3-form

$$e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

and define the operator $J_v = v \times - : \mathbb{R}^7 \rightarrow \mathbb{R}^7$. When v is a norm one vector, the operator J_v induces a complex multiplication on the hyperplane v^\perp .

```

def unpermutede(myvector):
    a=0
    if myvector==[1,2,3]:
        a=1
    elif myvector==[1,4,5]:
        a=1
    elif myvector==[1,6,7]:
        a=1
    elif myvector==[2,4,6]:
        a=1
    elif myvector==[2,5,7]:
        a=-1
    elif myvector==[3,4,7]:
        a=-1
    elif myvector==[3,5,6]:
        a=-1
    return a

def e(myvector):
    a=0
    Pp=[]
    P=Permutations([1,2,3])
    segni=[]
    for p in P:
        segni.append(Permutation(p).sign())
    P=Permutations(myvector)
    for p in P:
        v=[]
        for j in [0..2]:
            v.append(p[j])
        Pp.append(v)
    a=sum(segni[i]*unpermutede(Pp[i]) for i in [0..5])
    return a

```

```

def Mult(n):
    M=matrix(ZZ,7,7)
    for i in [1..7]:
        for j in [1..7]:
            if Permutations([n,i,j]).cardinality()==6:
                M[i-1,j-1]=e([n,i,j])
    return M

def J(v):
    M=sum(v[i]*Mult(i+1) for i in [0..6])
    return M

```

Next we impose the algebraic curvature equations on the set of variables R_{ijkl} giving the coefficients of a 4-index tensor in \mathbb{R}^7 . We collect all these equations in a list of equations named `listone` and we extract the matrix of coefficients of the linear system given by the algebraic curvature equations in the variables R_{ijkl} and call this matrix `MAC`. In order to extract this matrix we use the fact that the coefficient a_k of the variable x^k in the linear equation $a_i x^i$ is $a_k = \partial_{x^k}(a_i x^i)$. To check we have correctly implemented the algebraic curvature equations we compare the dimension of the space of solution of the algebraic curvature equations obtained from the rank of the matrix `MAC` with the dimension given by [Gilkey01, Corollary 1.8.4, p. 45], finding they match.

```

dim=7
listone=[]
for i in [0..dim-1]:
    for j in [0..dim-1]:
        for k in [0..dim-1]:
            for l in [0..dim-1]:
                listone.append(var('R_%d%d%d%d' % (i,j,k,l))+
                    var('R_%d%d%d%d' % (i,j,l,k)))
                listone.append(var('R_%d%d%d%d' % (i,j,k,l))+
                    var('R_%d%d%d%d' % (j,i,k,l)))
                listone.append(var('R_%d%d%d%d' % (i,j,k,l))+
                    var('R_%d%d%d%d' % (i,k,l,j))+var('R_%d%d
                    %d%d' % (i,l,j,k)))
                listone.append(var('R_%d%d%d%d' % (i,j,k,l))-
                    var('R_%d%d%d%d' % (k,l,i,j)))

n=0
MAC = matrix(ZZ,4*7^4,7^4)
for c in listone:
    for i in [0..6]:
        for j in [0..6]:
            for k in [0..6]:
                for l in [0..6]:

```



```

|| sort=100
|| listone=[]
|| for p in [1..sort]:
||     A=randomorthogonal2()
||     w1=A[0]
||     w2=A[1]
||     d1=lcm(denominators(w1)[0])
||     d2=lcm(denominators(w2)[0])
||     d=d2^4*d1
||     Pro=projection7(w2)
||     Rr=ERRE(w1,w2)
||     Jay=J(w2)
||     M=d*(Pro*Rr*Pro*Jay-Jay*Pro*Rr*Pro)
||     for l in [0..dim-1]:
||         for h in [0..dim-1]:
||             listone.append(M[l,h])
||
|| n=0
|| MCR = matrix(ZZ,sort*7^2,7^4)
|| for c in listone:
||     for i in [0..6]:
||         for j in [0..6]:
||             for k in [0..6]:
||                 for l in [0..6]:
||                     MCR[n,l+7*k+7^2*j+7^3*i]=derivative(c,var
||                         ('R_%d%d%d%d' % (i,j,k,l)))
||
||     n=n+1
||
|| M=MAC.stack(MCR)
||
|| print(7^4-M.rank());

```

One can write a code solving the first CR condition on \mathbb{R}^8 in essentially the same way: one uses the 4-form

$$\begin{aligned}
 & e^{0123} + e^{0145} + e^{0167} + e^{0246} - e^{0257} - e^{0347} - e^{0356} \\
 & + e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}
 \end{aligned}$$

to define the 3-fold VCP on \mathbb{R}^8 and uses the multiplication of three reflections to produce random orthonormal triples in \mathbb{R}^8 . We omit the details.

Now, we provide a code implementing the second CR condition on \mathbb{R}^7 . It uses a few of the functions defined in the code implementing the second CR condition that are not repeated here. The strategy is very similar to what we did for the first CR condition. First we impose the antisymmetry conditions on the structure constants A_{ijkl} of an element in $\otimes^4 V^*$, where $V = \mathbb{R}^7$, in order to have it be an element in $V^* \otimes \wedge^3 V^*$. As a check, at

the end we compare the dimension of the space of solutions found this way with the expected dimension of $7 \cdot \binom{7}{3} = 245$.

```

dim=7
listone2=[]
for i in [0..dim-1]:
    for j in [0..dim-1]:
        for k in [0..dim-1]:
            for l in [0..dim-1]:
                listone2.append(var('A_%d%d%d%d' % (i,j,k,l))
                    +var('A_%d%d%d%d' % (i,k,j,l)))
                listone2.append(var('A_%d%d%d%d' % (i,j,k,l))
                    +var('A_%d%d%d%d' % (i,j,l,k)))
n=0
MACA = matrix(ZZ,4*7^4,7^4)
for c in listone2:
    for i in [0..6]:
        for j in [0..6]:
            for k in [0..6]:
                for l in [0..6]:
                    MACA[n,l+7*k+7^2*j+7^3*i]=derivative(c,
                        var('A_%d%d%d%d' % (i,j,k,l)))
    n=n+1
print(7^4-MACA.rank(),7*7*5);

```

We implement the multilinear function $w_1 \otimes w_2 \otimes w_3 \otimes w_4 \mapsto w_1^i w_2^j w_3^k w_4^l A_{ijkl}$, randomly produce 300 orthonormal triples η_1, η_2, η_3 in \mathbb{R}^7 and for each of these triples we impose the equation $A(\eta_1; \eta_2 \wedge \eta_3 \wedge (\eta_2 \times \eta_3)) = 0$. As a check, at the end we compare the dimension of the resulting space of solutions with the expected value of 49, the dimension of the space of algebraic intrinsic torsions on V .

```

def AAA(w1,w2,w3,w4):
    M=sum(sum(sum(w1[i]*w2[j]*w3[k]*w4[l]*var('A_%d%d%d%d'
        '% (i,j,k,l)) for l in [0..6]) for k in [0..6]) for
        j in $
    return M

dim=7
sort=300
listaccia=[]
for p in [1..sort]:
    A=randomorthogonal3()
    x=A[0]
    y=A[1]
    z=A[2]
    d1=lcm(denominators(x)[0])

```

```

d2=lcm(denominators(y)[0])
d3=lcm(denominators(z)[0])
d=d1*d2^2*d3^2
Jay=J(y)
B=d*AAA(x,y,z,Jay*z)
listaccia.append(B)

n=0
MCRA = matrix(ZZ,sort,7^4)
for c in listaccia:
    for i in [0..6]:
        for j in [0..6]:
            for k in [0..6]:
                for l in [0..6]:
                    MCRA[n,l+7*k+7^2*j+7^3*i]=derivative(c,
                        var('A_%d%d%d%d' % (i,j,k,l)))
                n=n+1

MA=MACA.stack(MCRA)

print(7^4-MA.rank(),49);

```

Finally we produce 10 random orthonormal bases $(w_1, w_2, w_3, w_4, w_5, w_6, v)$ with (w_1, w_2, w_3) a Hermitian basis with respect to $J-v = v \times -$. To do this, we first produce a random orthonormal basis $(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$; then using a reflection (if needed) we change the third vector into $u_1 \times u_2$, while keeping u_1 and u_2 fixed. This way we obtain a orthonormal basis of the form $(u_1, u_2, u_1 \times u_2, \tilde{u}_4, \tilde{u}_5, \tilde{u}_6, \tilde{u}_7)$. We set $w_1 = u_1, w_2 = u_2, w_3 = u_1 \times u_2, v = \tilde{u}_7$ and $w_4 = v \times w_1, w_5 = v \times w_2, w_6 = v \times w_3$. Next we implement the equations defining $\mathcal{T}_{CR1}(V)$ for each of these 7-ples and print the dimension of the space of solutions. Since in the first part of the code the reflection matrix was defined only when the input was a norm one vector, we implement at the beginning of the code here its version for arbitrary nonzero vectors.

```

def reflectionunnorm7(myvector):
    somma=sum(myvector[j]^2 for j in [0..6])
    M=matrix(QQ,7)
    v=matrix(QQ,1,7)
    for h in [0..6]:
        v[0,h]=myvector[h]
    for h in [0..6]:
        M[h]=matrix.identity(7)[h]-(2/somma)*(matrix.
            identity(7)[h]*transpose(v))*v
        N=transpose(M)
    return N

def randomg2():
    A=matrix(QQ,7)
    s1=[randint(0, 1) for i in [0..5]]

```

```

t1=stereographic7(s1)
s2=[randint(0, 1) for i in [0..5]]
t2=stereographic7(s2)
s3=[randint(0, 1) for i in [0..5]]
t3=stereographic7(s3)
A=reflection7(t1)*reflection7(t2)*reflection7(t3)
At=transpose(A)
u1=At[0]
u2=At[1]
u3=At[2]
J1=J(u1)
u3good=J1*u2
if u3good==u3:
    B=matrix.identity(7)
else:
    z=u3good-u3
    B=reflectionunnorm7(z)
C=transpose(B*A)
v=C[6]
D=C
D[3]=J(v)*C[0]
D[4]=J(v)*C[1]
D[5]=J(v)*C[2]
return D

def randomsimplex2():
    A=matrix(QQ,7)
    A=matrix.identity(7)
    At=transpose(A)
    u1=At[0]
    u2=At[1]
    u3=At[2]
    J1=J(u1)
    u3good=J1*u2
    if u3good==u3:
        B=matrix.identity(7)
    else:
        z=u3good-u3
        B=reflectionunnorm7(z)
    C=transpose(B*A)
    v=C[6]
    D=C
    D[3]=J(v)*C[0]
    D[4]=J(v)*C[1]
    D[5]=J(v)*C[2]
    return D

dim=7

```



```

sort=10
listuccia=[]
for p in [1..sort]:
    A=randomg2()
    v=A[0]
    d0=lcm(denominators(v)[0])
    peco=[A[1],A[2],A[3],A[4],A[5],A[6]]
    Jv=J(v)
    for wq in peco:
        for wj in peco:
            for wp in peco:
                d1=lcm(denominators(wq)[0])
                d2=lcm(denominators(wj)[0])
                d3=lcm(denominators(wp)[0])
                d=d0^2*d1*d2*d3
                B=d*(AAA(Jv*wq,v,wj,wp)-AAA(Jv*wp,v,wj,wq)+
                    AAA(wq,v,Jv*wj,wp)-AAA(wp,v,Jv*wj,wq))
                listuccia.append(B)

n=0
MCRAg2 = matrix(ZZ,sort*6^3,7^4)
for c in listuccia:
    for i in [0..6]:
        for j in [0..6]:
            for k in [0..6]:
                for l in [0..6]:
                    MCRAg2[n,l+7*k+7^2*j+7^3*i]=derivative(c,
                        var('A_%d%d%d%d' % (i,j,k,l)))
    n=n+1

MAg2=MA.stack(MCRAg2)

print(7^4-MAg2.rank());

```

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