Convergence properties of symmetrization processes

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Abstract

Steiner symmetrization is well known for its rounding and general convergence properties. We identify a whole family of symmetrizations satisfying certain properties. In fact, we prove that all these symmetrizations share the same convergence behaviors.*

1 Introduction

Symmetrizations play a very important role in geometry and its applications, supplying many results with relatively easy and direct proofs. In particular geometric and analytic inequalities, like the isoperimetric, Blaschke-Santalò, Faber-Krahn inequalities, and many others have been proved using for example Steiner symmetrization. For some self-contained introductions on the subject see [9, Chapters 1 and 2], [10, Chapter 9], [12, Chapter 3], and [16, Chapter 10] and the references therein. All these results rely mainly on the fact that through Steiner symmetrization, for every compact convex set it is always possible to find a sequence of symmetrals converging to a ball while preserving the volume.

A *symmetrization process* is a sequence of symmetrizations applied to a subset of $\mathbb{R}^n$. These processes are the main focus of this work. In 1986 Mani-Levitska [14] showed that for Steiner symmetrization, a randomly chosen symmetrization process for a convex body (that is, a compact convex set with non-empty interior) converges almost surely to a ball. This result was later extended by Van Schaftingen [18] to general compact sets, then by Volčič [19] to measurable sets. Coupier and Davydov [7] later proved, thanks to the inclusion between Steiner and Minkowski symmetrals, that an analogous probabilistic property holds for Minkowski symmetrization. Other interesting results concerning convergence in probability can be found in the works of Bianchi, Burchard, Gronchi, and Volčič [1] and Burchard and Fortier [6].

Meanwhile, the study of deterministic convergence received a boost in 2012 with the work of Klain [13], which inspired a series of papers on the subject. In particular, Bianchi, Gardner, and Gronchi in [2] and [3] introduced a general framework for the study of

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symmetrizations, focusing on the relations between different symmetrizations and their properties. We mainly use their formalism, which now we introduce. Let $\mathcal{E}$ be a family of sets and let H be a subspace. An *H-symmetrization* is a map

$$\diamondsuit_H : \mathcal{E} \to \mathcal{E}_H,$$

where $\mathcal{E}_H$ is the subfamily of $\mathcal{E}$ of sets that are symmetric with respect to H (i.e. invariant under reflections with respect to H). We focus on subfamilies $\mathcal{E}$ of the family $\mathcal{C}^n$ of compact sets of $\mathbb{R}^n$ endowed with the Hausdorff metric (more details in the next section). Throughout this work, this is the metric we use whenever we refer to the convergence of a sequence of sets.

Symmetrizations may enjoy many different properties, some of which can completely characterize them, as is shown for example in [2, Section 9]. The properties we are interested in are monotonicity, invariance under reflections, and invariance under orthogonal translations (see the next section for the specific definitions). We denote by $\mathcal{F}$ the family of all the symmetrizations satisfying these three properties for every proper subspace of $\mathbb{R}^n$. The family $\mathcal{F}$ includes Minkowski and fiber symmetrizations (the latter corresponds for convex bodies to Steiner symmetrization when considered with respect to hyperplanes) and possesses a very strong structure, as proved in [2, Corollary 7.3]. This structure plays an essential role in the proof of our main results. Indeed it will allow us to characterize some convergence phenomena for $\mathcal{F}$ which are shared by the whole family. A convergence property that has been studied in the literature, starting from [2, Section 8], is the following. Here $\mathcal{E} = \mathcal{K}^n$, the family of compact convex sets of $\mathbb{R}^n$.

Definition 1.1. If $\diamondsuit$ is a symmetrization on $\mathcal{K}^n$, a sequence (H_m) of subspaces of $\mathbb{R}^n$ is said to be *weakly $\diamondsuit$-universal* if for every $k \in \mathbb{N}$, the sequence of sets

$$K_{m,k} = \diamondsuit_{H_m} \cdots \diamondsuit_{H_k} K$$

converges for every $K \in \mathcal{K}^n$ with non-empty interior to a ball of radius $r(K,k)$. This quantity can change with respect to k, but if $r(K,k)$ is independent of k, then (H_m) is said to be *universal* for $\diamondsuit$.

In contrast to the previous literature, we remark that the subspaces H_m do not necessarily have the same dimension except when explicitly stated. The definition of universal sequence was introduced in [7, Theorem 3.1], where the following result was obtained for the family $\mathcal{K}^n_n$ of convex bodies, i.e. compact convex sets of $\mathbb{R}^n$ with non-empty interior.

Theorem 1.2 (Coupier and Davydov)**.** *A sequence (H_m) of hyperplanes in $\mathbb{R}^n$ is universal for Steiner symmetrization in $\mathcal{K}^n_n$ if and only if it is universal for Minkowski symmetrization in $\mathcal{K}^n_n$.*

Later this theorem was extended by Bianchi, Gardner, and Gronchi together with the introduction of weakly universal sequences. In [2, Section 8] and [3, Section 6 and 7] many results were obtained in this direction. In particular, we observe that [3, Theorem 7.3 and 7.4] together with Theorem 1.2 generalize the latter result to the family $\mathcal{C}^n$.

In all these results the limit of the sequence is a ball. Here instead we study a wider class of sequences, without prescribing specific limit sets.

Definition 1.3. If $\diamondsuit$ is a symmetrization on $\mathcal{E}$, a sequence (H_m) of subspaces is said to be $\diamondsuit$-*stable* (or *stable for the symmetrization* $\diamondsuit$) if for every $k \in \mathbb{N}$, the sequence defined for $m \geq k$ by

$$K_{m,k} = \diamondsuit_{H_m} \cdots \diamondsuit_{H_{k+1}} \diamondsuit_{H_k} K$$

converges for every $K \in \mathcal{E}$.

Note that the limit might depend both on K and k, as we show in Examples 3.2 and 3.3. The generalization of Theorem 1.2 we prove is the following.

Theorem 1.4. *Let $\diamondsuit_0 \in \mathcal{F}$ be a symmetrization on $\mathcal{K}^n$. Then, if (H_m) is a $\diamondsuit_0$-stable sequence of subspaces of $\mathbb{R}^n$, it is $\diamondsuit$-stable for every symmetrization $\diamondsuit \in \mathcal{F}$.*
In particular, this holds for Steiner and Minkowski symmetrization when the H_m are hyperplanes.

This turns out to be a simple consequence of a broader result we prove in Theorem 3.4, which involves the so-called *convergence in shape*, studied for the first time in [1]. This kind of convergence, properly defined in the next section, involves a sequence $(\mathbb{A}_m)$ of rotations, which corrects at every step the underlying symmetrization process. Indeed, this tool can allow the convergence of symmetrization processes which would not naturally have a limit, as the ones we study in Section 4. Using the notation of Theorem 1.4, we prove in Theorem 3.4 that, given two symmetrizations $\diamondsuit_1, \diamondsuit_2 \in \mathcal{F}$, $\diamondsuit_1$ is stable in shape if and only if $\diamondsuit_2$ is stable in shape.
This last result has many interesting consequences. One of them is a partial answer to the question: Does a converging sequence of hyperplanes induce a converging symmetrization process? We find a positive answer when one additional assumption is imposed.

Theorem 1.5. *Let (H_m) be a sequence of hyperplanes and consider the corresponding normals $u_m \in \mathbb{S}^{n-1}$. Let $\diamondsuit \in \mathcal{F}$. If the angles $\alpha_m \in [0, \pi/2]$ given by the relation $|u_m \cdot u_{m-1}| = \cos \alpha_m$ are such that*

$$\sum_{m \in \mathbb{N}} |\alpha_m| < \infty,$$

then (H_m) converges and is $\diamondsuit$-stable on $\mathcal{K}^n$.

We were not able to prove that the extra assumption is necessary, but the question is the subject of ongoing research.
The structure of this work is as follows. In Section 2 we introduce the basic notation and definitions, recalling some tools and instrumental results. In Section 3 we prove Theorem 3.4 and investigate some consequences. Finally in Section 4, after presenting a counterexample from [1], we exhibit new ones and show that the same procedures work for all the symmetrizations in the family $\mathcal{F}$.

2 Preliminaries

The families $\mathcal{E}$ with which we work are all subsets of the family $\mathcal{C}^n$. Given two subsets A, B of $\mathbb{R}^n$, the *Minkowski sum* of A and B is the set

$$A + B = \{x + y \mid x \in A, y \in B\}.$$

In this notation, the translate of a set A by a vector $x \in \mathbb{R}^n$ is written as $A + x$. The space $\mathcal{C}^n$ is a complete metric space with respect to the topology induced by the Hausdorff distance, which for two compact sets K, L is given by

$$d_{\mathcal{H}}(K, L) = \max\{\inf\{\varepsilon > 0,\ K \subset L + \varepsilon B^n\},\ \inf\{\varepsilon > 0,\ L \subset K + \varepsilon B^n\}\},$$

where B^n is the n-dimensional unit ball centered at the origin. We will always consider $\mathcal{C}^n$ and its subspaces as being endowed with this topology.

We denote by $\mathcal{K}^n$ the family of compact convex sets of $\mathbb{R}^n$, which is a closed subspace of $\mathcal{C}^n$. See [11], [12], and [16] for the classical theory of convex bodies. The space $\mathcal{K}^n_n$ denotes the family of convex bodies, i.e. elements of $\mathcal{K}^n$ with non-empty interior. Throughout the paper $B(x, r)$ denotes the n-dimensional ball centered at x of radius $r > 0$, and we write the scalar product of two vectors $x, y \in \mathbb{R}^n$ as $x \cdot y$. By $\mathbb{S}^{n-1}$ we denote the unit sphere in $\mathbb{R}^n$.

When dealing with symmetrizations a crucial role is played by some specific set functions on the family of sets we are considering. In particular, for $\mathcal{C}^n$ we are interested in the *volume*, given by the n-dimensional Lebesgue measure of a set which we write as $\lambda_n(\cdot)$. For convex sets, we can also consider the *mean width*, which for $K \in \mathcal{K}^n$ is

$$\mathcal{W}(K) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} [h_K(\nu) + h_K(-\nu)]\, d\mathcal{H}^{n-1}(\nu),$$

where ω_n is the $(n-1)$-dimensional measure of $\mathbb{S}^{n-1}$, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, and h_K is the *support function* of K, given by

$$h_K(x) = \sup\{x \cdot y \,|\, y \in K\}, \quad x \in \mathbb{S}^{n-1}.$$

An important property of the Minkowski sum is that

$$h_K + h_L = h_{K+L} \tag{2.1}$$

for every $K, L \in \mathcal{K}^n$.

The *symmetric difference* of two measurable sets A and B in $\mathbb{R}^n$ is

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

When restricted to $\mathcal{K}^n_n \times \mathcal{K}^n_n$, the map

$$(A, B) \mapsto \lambda_n(A \Delta B)$$

is called the *Nikodym distance* and it is equivalent to the Hausdorff distance in $\mathcal{K}^n_n$.

We denote by $\mathcal{G}(n, i)$ the sets of subspaces of $\mathbb{R}^n$ of dimension $1 \leq i \leq n-1$. We use the term *subspace* to refer to a linear subspace. For every subset A of $\mathbb{R}^n$ and subspace H, we write the projection of A onto H as $P_H A$. The *reflection* with respect to a subspace H, given by the map

$$x \mapsto x - 2P_{H^\perp}\{x\}, \tag{2.2}$$

is denoted by R_H, where $H^\perp$ is the subspace orthogonal to H. When a set A of $\mathbb{R}^n$ is such that $A = R_H A$, it is said to be symmetric with respect to H, or H-symmetric for short.

A fundamental result in the theory of convex bodies is the Brunn-Minkowski inequality. It states that for compact sets K and L the inequality

$$\lambda_n(K+L)^{1/n} \geq \lambda_n(K)^{1/n} + \lambda_n(L)^{1/n} \tag{2.3}$$

holds, where equality is achieved if and only if K and L are homothetic convex bodies or lower-dimensional convex sets lying in two parallel affine subspaces. For a complete survey regarding this inequality and its extensions see Gardner [8].

We now go back to the concept of H-symmetrization introduced in [2], [3]. Given a family $\mathcal{E}$ of sets and a subspace H, an H-symmetrization is a map

$$\diamondsuit_H : \mathcal{E} \to \mathcal{E}_H$$

with $\mathcal{E}_H = \{E \in \mathcal{E} | R_H E = E\}$, where R_H is the map given by (2.2).

Let $\diamondsuit$ be an H-symmetrization in $\mathcal{E}$. Of particular interest are the following properties.

1. (*Monotonicity*): For every $K, L \in \mathcal{E}$ if $K \subseteq L$, then $\diamondsuit_H K \subseteq \diamondsuit_H L$.

2. (*Idempotence*): For every $K \in \mathcal{E}$, we have $\diamondsuit_H K = \diamondsuit_H \diamondsuit_H K$.

3. (*$H^\perp$-translation invariance for H-symmetric sets*): If $K \in \mathcal{E}$ and $R_H K = K$, then for every $x \in H^\perp$ we have $\diamondsuit_H(K + x) = K$.

4. (*Invariance for H-symmetric sets*): If $K \in \mathcal{E}$ and $R_H K = K$, then $\diamondsuit_H K = K$.

5. (*F-invariance*): There exist a function $F : \mathcal{E} \to \mathbb{R}$ such that $F(K) = F(\diamondsuit_H K)$ for every $K \in \mathcal{E}$.

When we refer to $\diamondsuit$ as a symmetrization it is understood that $\diamondsuit_H$ is an H-symmetrization for every subspace H of $\mathbb{R}^n$. Of particular interest is the family of symmetrizations

$$\mathcal{F} = \{ \text{ symmetrizations } \diamondsuit \mid \text{ Properties 1, 3, and 4 hold } \}.$$

The results obtained in this paper concern mainly Schwarz, Minkowski, and fiber symmetrizations, which we proceed now to present.

Schwarz Symmetrization Let $K \in \mathcal{C}^n$, for a fixed $H \in \mathcal{G}(n,i), 1 \leq i \leq n-1$. The Schwarz symmetral of K is the set

$$S_H K = \bigcup_{x \in H} B(x, r_x),$$

where r_x is such that $\lambda_{n-i}(K \cap (H^\perp + x)) = \lambda_{n-i}(B(x, r_x))$ if $\lambda_{n-i}(K \cap (H^\perp + x)) > 0$. If the measure of the section at $x \in H$ is zero, but the section is non-empty, we replace it with x. Otherwise, we replace this section with the empty set. From Fubini's Theorem, it follows that this symmetrization preserves the volume, thus satisfying Property 5 for $F(\cdot) = \lambda_n(\cdot)$. When $i = n-1$ this is better known as *Steiner symmetrization*, and in general it decreases intrinsic volumes (see [11, Satz XI, p. 260] or [16, Theorem 10.4.1]). Both in $\mathcal{C}^n$ and $\mathcal{K}^n$ Schwarz symmetrization satisfies Properties 1, 2, and 5, while 3 and 4 hold only for convex sets in the case $i = n-1$.

Minkowski Symmetrization Let $K \in \mathcal{C}^n$ and let $H \in \mathcal{G}(n,i), 1 \leq i \leq n$. The Minkowski symmetral of K is the set

$$M_H K = \frac{K + R_H K}{2}.$$

Clearly from the definition of $\mathcal{W}(K)$ and (2.1) it preserves the mean width when K is convex, thus Property 5 holds for $F = \mathcal{W}$. From the Brunn-Minkowski inequality (2.3) it follows that

$$\lambda_n(M_H K) \geq \lambda_n(K).$$

It may be useful to consider *central Minkowski symmetrization*, i.e.

$$\triangle K = \frac{K - K}{2},$$

which is origin symmetric. If K lies in an affine subspace $H + x, x \in H^\perp$, then we write $\triangle_x K$ for central Minkowski symmetrization in $H + x$, i.e.

$$\triangle_x K = \frac{K + R_{H^\perp} K}{2}.$$

In $\mathcal{K}^n$ Minkowski symmetrization satisfies all the listed properties, but in $\mathcal{C}^n$ only Property 1 holds.

Fiber Symmetrization Let $K \in \mathcal{C}^n$ and let $H \in \mathcal{G}(n,i)$, $1 \leq i \leq n$. Then the fiber symmetral of K is the set

$$F_H K = \bigcup_{x \in H} \triangle_x (K \cap (H^\perp + x)).$$

This symmetrization can be seen as a hybrid between Schwarz and Minkowski symmetrization, and the underlying operation was introduced by McMullen [15]. Like Minkowski symmetrization, it increases the volume, and in $\mathcal{K}^n$ it satisfies Properties 1, 2, 3, and 4, while 2, 3, and 4 fail to hold in $\mathcal{C}^n$. See [17] for some specific examples.

Fiber and Minkowski symmetrization can be considered the *extremals* of the family $\mathcal{F}$, in a sense that Theorem 2.1 will make clear. Moreover, other symmetrizations, different from fiber or Minkowski, can be found in $\mathcal{F}$; see for example the remarks after Corollary 7.3 in [2], where the following fact is proved.

Theorem 2.1 (Bianchi, Gardner, and Gronchi)**.** *Let $H \in \mathcal{G}(n,i), 1 \leq i \leq n-1$ and let $\mathcal{E} = \mathcal{K}^n$ or $\mathcal{K}^n_n$. If $\diamondsuit \in \mathcal{F}$, then*

$$F_H K \subseteq \diamondsuit_H K \subseteq M_H K \tag{2.4}$$

for every $K \in \mathcal{E}$.

In the last section of this paper, we study some counterexamples to the convergence of symmetrization processes. We will present one in Example 4.1, which was given previously in [5] and [6]. This example shows that a dense sequence of directions, if accurately chosen, leads to a non-converging symmetrization process. We show in particular that the same construction works for the whole family of symmetrizations considered in Theorem 2.1.

For this family of non-converging sequences, when dealing with Steiner symmetrization of compact sets, convergence is still possible in a weaker sense, called *convergence in shape*. The next section will be devoted to the study and the generalization of this kind of convergence (see Definition 3.2). The first result in this direction was achieved in [1, Theorem 2.2] and reads as follows.

Theorem 2.2 (Bianchi, Burchard, Gronchi, and Volčič)**.** *Let (u_m) be a sequence in $\mathbb{S}^{n-1}$ such that $u_m \cdot u_{m-1} = \cos\alpha_m$, where $\alpha_m \in (0, 2\pi)$ and $\sum_{m\in\mathbb{N}} \alpha_m^2 < \infty$. Let (H_m) be the corresponding sequence of hyperplanes given by $H_m = u_m^\perp$ for every $m \in \mathbb{N}$.*

Then there exists a sequence $(\mathbb{A}_m)$ of rotations such that for every non-empty compact set $K \subset \mathbb{R}^n$ the sets

$$K_m = \mathbb{A}_m S_{H_m} \cdots S_{H_1} K$$

converge in Hausdorff metric to a compact convex set L.

In Corollary 3.6 we obtain a generalization of Theorem 2.2 in $\mathcal{K}^n$ for all the symmetrizations in $\mathcal{F}$ with respect to sequences of hyperplanes.

3 Shape-Stable Symmetrization Processes

We start by noting some monotonicity properties for volume and mean width with respect to symmetrizations in $\mathcal{F}$.

Lemma 3.1. *Consider $H \in \mathcal{G}(n, i), 1 \leq i \leq n-1$, and a symmetrization $\diamondsuit_H : \mathcal{K}^n \to (\mathcal{K}^n)_H$ such that $\diamondsuit \in \mathcal{F}$. Then for every $K \in \mathcal{K}^n$ we have*

$$\lambda_n(\diamondsuit_H K) \geq \lambda_n(K), \quad \mathcal{W}(K) \geq \mathcal{W}(\diamondsuit_H K).$$

Proof. The first inequality is a consequence of the Brunn-Minkowski inequality (2.3). Indeed by definition if $K \in \mathcal{K}^n$, then every H-orthogonal section of $F_H K$ is a Minkowski symmetral of an H-orthogonal section of K, thus $\lambda_n(F_H K) \geq \lambda_n(K)$. Now, by (2.4) we have

$$\lambda_n(\diamondsuit_H K) \geq \lambda_n(F_H K) \geq \lambda_n(K).$$

For the second inequality, again in view of (2.4), we have $\diamondsuit_H K \subseteq M_H K$. Thus clearly $h_{\diamondsuit_H K}(u) \leq h_{M_H K}(u)$ for every $u \in \mathbb{S}^{n-1}$ and consequently

$$\mathcal{W}(\diamondsuit_H K) \leq \mathcal{W}(M_H K) = \mathcal{W}(K),$$

completing the proof. □

We now need a weaker concept of convergence, which generalizes the phenomenon studied in Theorem 2.2.

Definition 3.2 (Convergence in shape)**.** Consider a family $\mathcal{E}$ of sets. Given $K \in \mathcal{E}$, a symmetrization $\diamondsuit$ on $\mathcal{E}$ and a sequence (H_m) of subspaces, the sequence of symmetrals

$$\diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K$$

is said to *converge in shape* if there exist a sequence $(\mathbb{A}_m)$ of rotations such that

$$\mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K \tag{3.1}$$

converges.

Definition 3.3 (Shape-stable sequences)**.** If in Definition 3.2 for every $k \in \mathbb{N}, m \geq k$ the sequence

$$\mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_k} K$$

converges, where $(\mathbb{A}_m)$ is independent of K and k, the sequence (H_m) of subspaces is *shape-stable* in $\mathcal{E}$ for $\diamondsuit$.

Examples of shape-stable sequences were presented in Theorem 2.2. Note that if $\mathbb{A}_m$ is the identity for every $m \in \mathbb{N}$, then a shape-stable sequence (H_m) is stable. To better understand these new concepts, we exhibit some examples.

Example 3.1 (Klain's Theorem)**.** In [13] Klain proved that for a finite family $\mathcal{D}$ of hyperplanes, a sequence (H_m) such that $H_m \in \mathcal{D}$ for every $m \in \mathbb{N}$ is stable for Steiner symmetrization on $\mathcal{K}^n$. This result is extended in [3] to Minkowski, fiber, Schwarz, and more generic symmetrizations, again in the family $\mathcal{K}^n$. Moreover, it was proved in [1] for Steiner symmetrization in $\mathcal{C}^n$ and in [3] for the Schwarz symmetrization in $\mathcal{C}^n$. In [17], we proved an analogous result for Minkowski symmetrization in $\mathcal{C}^n$.

Example 3.2 (Stable sequences)**.** Consider in $\mathbb{R}^2$ the square Q with vertices $(1,0)$, $(0,-1)$, $(-1,0)$, $(0,1)$ as in Figure 1. Consider the sequence (H_m) of lines where $H_m = span\{(1,0)\}$ when m is even, $H_m = span\{(0,1)\}$ when m is odd, for $m \geq 2$, while $H_1 = span\{(\sqrt{2}+1,1)\}$. Then (H_m) is stable in $\mathcal{K}^2$ for Minkowski symmetrization thanks to [3, Theorem 5.7] (see Example 3.1). Now, observe that $M_{H_1}Q$ is the red octagon in the figure, and all the other symmetrizations leave this body unchanged so that the limit is exactly $M_{H_1}Q$. If we start from $m \geq 2$ instead, the limit is always Q.

Example 3.3 (Shape-stable sequences)**.** Consider in $\mathbb{R}^2$ an ellipse E centered at the origin and a sequence of lines as in Theorem 2.2. It is known that Steiner symmetrization preserves ellipses, and we can choose a direction v such that the symmetrization with respect to the line H_1 parallel to v gives a ball.

If we consider a sequence (H_m) of lines starting from H_1 and then continuing as the sequence of Theorem 2.2, we infer that (H_m) is shape-stable in $\mathcal{K}^2$. Moreover, since $S_{H_1}E$ is a ball centered at the origin the limit is of course $S_{H_1}E$. If we skip the first symmetrizations, as was proved in [1, Example 2.1] (which we recall here in Example 4.1) we can choose the remaining directions such that the limit of the convergence in shape is not a ball.

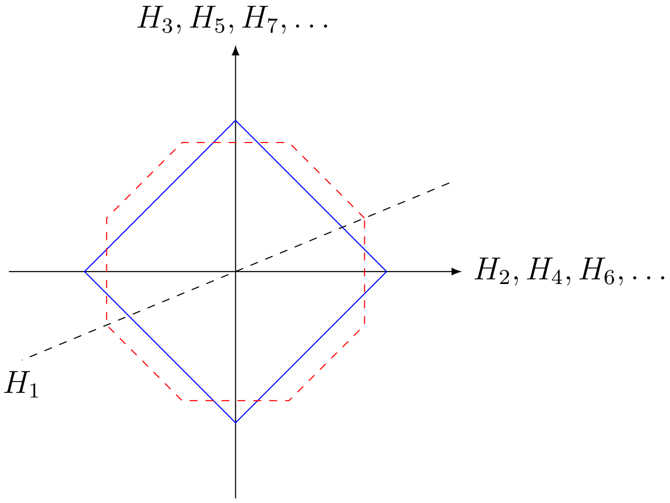

Figure 1: Different limits may arise from stable sequences.

We can now prove the following equivalence result. Note that, for this result, the dimension of the subspaces in the sequence is not relevant.

Theorem 3.4. *Let $\diamondsuit_0 \in \mathcal{F}$. If a sequence (H_m) of subspaces is shape-stable for $\diamondsuit_0$ in $\mathcal{K}^n$ with rotations $(\mathbb{A}_m)$, then for any $\diamondsuit \in \mathcal{F}$, (H_m) is shape-stable for $\diamondsuit$ in $\mathcal{K}^n$ with rotations $(\mathbb{A}_m)$.*

In particular, a sequence (H_m) of subspaces is shape-stable for $\diamondsuit \in \mathcal{F}$ if and only if the same property holds for fiber or Minkowski symmetrization. If each H_m is a hyperplane, the same conclusion holds for Steiner symmetrization.

Proof of Theorem 3.4. The outline of the proof is the following. We proceed by applying (2.4) multiple times. First, proving that if (H_m) is shape stable for $\diamondsuit_0$ in $\mathcal{K}^n$, then it is shape-stable for M in $\mathcal{K}^n$. After that, we show that if the same sequence is shape-stable for M in $\mathcal{K}^n$, then the same holds for any $\diamondsuit \in \mathcal{F}$. For a subspace H we denote by $\diamondsuit_{0,H}$ the symmetrization $\diamondsuit_0$ with respect to H.

Let (H_m) be a shape-stable sequence of subspaces for $\diamondsuit_0$ in $\mathcal{K}^n$ with rotations $(\mathbb{A}_m)$. We want to prove that for every $K \in \mathcal{K}^n$ the sequence of sets

$$K_m = \mathbb{A}_m M_{H_m} \cdots M_{H_1} K \tag{3.2}$$

converges. Suppose on the contrary that there exists $K \in \mathcal{K}^n$ such that for two subsequences (K_{m_j}) and (K_{m_l}) obtained by (3.2) one has $K_{m_j} \to L_1, K_{m_l} \to L_2$ where $L_1 \neq L_2$, $L_1, L_2 \in \mathcal{K}^n$.

Consider the sequence of bodies obtained by the same process starting from $K_r = K + B(0,r)$ instead of K, $r > 0$ fixed. This is done to cover both the full- and lower-dimensional cases at the same time.

Note that if H is a subspace and $\mathbb{A}$ is a rotation, for every $K \in \mathcal{K}^n$ we have

$$M_H(K + B(0,r)) = M_H K + B(0,r), \quad \mathbb{A}(K + B(0,r)) = \mathbb{A}K + B(0,r)$$

and thus

$$K_{r,m} = \mathbb{A}_m M_{H_m} \cdots M_{H_1}(K + B(0,r)) = \mathbb{A}_m M_{H_m} \cdots M_{H_1} K + B(0,r),$$

that is $K_{r,m} = K_m + B(0,r) = (K_m)_r$ and instead of L_1 and L_2 we have the limits $L_1 + B(0,r)$ and $L_2 + B(0,r)$. Note that $L_1 \neq L_2$ if and only if $L_1 + B(0,r) \neq L_2 + B(0,r)$ (see for example [16, Theorem 1.7.5 (a)]).

Since $\lambda_n(K_r) > 0$, thanks to Lemma 3.1 the sequence of volumes $\lambda_n(K_m + B(0,r))$ is increasing and strictly positive. Moreover, it is bounded; indeed, from the compactness of K, there exists a ball $B(0,R)$ with $R > 0$ such that $K_r \subseteq B(0,R)$. From the monotonicity and symmetry invariance of Minkowski symmetrization $K_m + B(0,r) \subseteq B(0,R)$ for every $m \in \mathbb{N}$. Therefore $\lambda_n((K_m)_r)$ converges to a certain value $c_r > 0$.

Since $L_1 \neq L_2$, $\lambda_n((L_1)_r \Delta (L_2)_r) = \delta > 0$. Fix $0 < \varepsilon < \delta/2$. There exists an index ν such that $c_r - \lambda_n((K_m)_r) < \varepsilon$ for every $m \geq \nu$. If for $m > \nu$ we define

$$J_m = \mathbb{A}_m \diamondsuit_{0,H_m} \cdots \diamondsuit_{0,H_{\nu+1}} \mathbb{A}_\nu^{-1} (K_\nu)_r$$

$$= \mathbb{A}_m \diamondsuit_{0,H_m} \cdots \diamondsuit_{0,H_{\nu+1}} M_{H_\nu} \cdots M_{H_1} K_r,$$

then thanks to Theorem 2.1, $J_m \subseteq (K_m)_r$ and in particular we have $J_{m_j} \subseteq (K_{m_j})_r$ and $J_{m_l} \subseteq (K_{m_l})_r$. From the hypothesis the sequence (H_m) is shape-stable in $\mathcal{K}^n$ for $\diamondsuit_0$, so there exists $J \in \mathcal{K}^n$ such that $J_m \to J$. Clearly the same holds for (J_{m_j}) and (J_{m_l}). In particular $J \subseteq (L_1)_r$ and $J \subseteq (L_2)_r$ and for Lemma 3.1 $\lambda_n(J) \geq \lambda_n((K_\nu)_r)$. We infer

$$\lambda_n((L_1)_r \Delta (L_2)_r) = \lambda_n((L_1)_r \setminus (L_2)_r) + \lambda_n((L_2)_r \setminus (L_1)_r) \leq$$
$$\lambda_n((L_1)_r \setminus J) + \lambda_n((L_2)_r \setminus J) = 2c_r - 2\lambda_n(J) \leq 2c_r - 2\lambda_n((K_\nu)_r) < 2\varepsilon < \delta,$$

which is a contradiction, so $L_1 = L_2$. The same argument can be repeated for every truncated sequence

$$\mathbb{A}_m M_{H_m} \cdots M_{H_k} K$$

and consequently (H_m) is shape-stable for Minkowski symmetrization.

Now we prove that if a sequence is shape-stable in $\mathcal{K}^n$ for Minkowski symmetrization, then it is shape-stable for $\diamondsuit \in \mathcal{F}$ as well. Consider for $Z \in \mathcal{K}^n$ the sequence of sets

$$Z_m = \mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_1} Z.$$

If Z_m does not converge, we can find two different subsequences (Z_{m_j}) and (Z_{m_l}) converging respectively to W_1 and $W_2 \in \mathcal{K}^n$ with $W_1 \neq W_2$.

Thanks to Lemma 3.1 the sequence $\mathcal{W}(Z_m)$ is non-negative and non-increasing, thus $\mathcal{W}(Z_m) \to b$ for some $b \geq 0$. Then, since $W_1 \neq W_2$, we have $\mathcal{W}(\mathrm{conv}(W_1 \cup W_2)) > b$. Note that the cases $W_1 \subset W_2$ and vice versa are automatically excluded since $b = \mathcal{W}(W_1) = \mathcal{W}(W_2)$ and the mean width is strictly monotone. Now, for every $\varepsilon > 0$ we can find $\nu \in \mathbb{N}$ such that $\mathcal{W}(Z_m) - b < \varepsilon$ for every $m \geq \nu$. For every $m \geq \nu$, define the sequence of sets

$$V_m = \mathbb{A}_m M_{H_m} \cdots M_{H_{\nu+1}} \mathbb{A}_\nu^{-1} Z_\nu$$

$$= \mathbb{A}_m M_{H_m} \cdots M_{H_{\nu+1}} \diamondsuit_{H_\nu} \cdots \diamondsuit_{H_1} Z.$$

Then the sets V_m converge to some $V \in \mathcal{K}^n$ because (H_m) is shape-stable for Minkowski symmetrization. Minkowski symmetrization preserves the mean width, thus $\mathcal{W}(V) = \mathcal{W}(Z_\nu)$. Moreover, thanks to Theorem 2.1 we have $W_1, W_2 \subseteq V$ and since V is convex, $\mathrm{conv}(W_1 \cup W_2) \subseteq V$ and hence $\mathcal{W}(V) \geq \mathcal{W}(\mathrm{conv}(W_1 \cup W_2))$. Therefore

$$\mathcal{W}(\mathrm{conv}(W_1 \cup W_2)) - b \leq \mathcal{W}(V) - b = \mathcal{W}(Z_\nu) - b < \varepsilon.$$

Since ε is arbitrary this inequality contradicts $W(\mathrm{conv}(W_1 \cup W_2)) > b$ and thus $W_1 = W_2$. Again the same process can be applied to the truncated sequences, concluding the proof. □

Now Theorem 1.4 is just an easy corollary.

Proof of Theorem 1.4. Observe that if (H_m) is stable, then it is shape-stable with $\mathbb{A}_m$ equal to the identity for every m. The proof is then a straightforward application of Theorem 3.4. □

A second consequence is the following extension of Theorem 1.2. Note that the extension is twofold: The result is valid for the whole family $\mathcal{F}$ and the respective family of objects is $\mathcal{K}^n$ instead of $\mathcal{K}^n_n$.

Theorem 3.5. *Let $\diamondsuit_0 \in \mathcal{F}$. A sequence (H_m) of subspaces is weakly $\diamondsuit_0$-universal in $\mathcal{K}^n$ if and only if it is weakly $\diamondsuit$-universal for every $\diamondsuit \in \mathcal{F}$.*

Proof. The strategy is the same as Theorem 3.4, with the advantage of using Theorem 1.4.
If (H_m) is weakly $\diamondsuit_0$-universal in $\mathcal{K}^n$, then it is stable in $\mathcal{K}^n$. Using Theorem 1.4, this implies that (H_m) is stable for Minkowski symmetrization. Thus we only need to prove that for every $K \in \mathcal{K}^n$ and $k \in \mathbb{N}$, the limit L of the corresponding sequence of sets

$$K_m = M_{H_m} \cdots M_{H_k} K$$

is a ball, where $m \geq k$. Again the sequence $\lambda_n(K_m)$ is bounded and increasing, and therefore it converges to a certain $c \geq 0$. By the same argument employed in Theorem 3.4, we can suppose that $c > 0$, i.e. considering $K + B(0, r)$ for arbitrarily small $r > 0$ instead of K.
Since (H_m) is weakly $\diamondsuit_0$-universal, for every $\nu \geq k$ the sequence of sets

$$\diamondsuit_{0,H_m} \cdots \diamondsuit_{0,H_{\nu+1}} M_{H_\nu} \cdots M_{H_k} K$$

converges to a ball B_ν. Moreover $\lambda_n(B_\nu) \geq \lambda_n(K_\nu)$ by Lemma 3.1 and $B_\nu \subseteq L$ for every ν thanks to Theorem 2.1. Since $\lambda_n(K_m)$ increases to c, for every $\varepsilon > 0$ exists $\nu \in \mathbb{N}$ such that $\lambda_n(L \Delta B_\nu) < \varepsilon$. Therefore L is a ball.
Suppose now that (H_m) is weakly universal for Minkowski symmetrization in $\mathcal{K}^n$. This implies that (H_m) is stable for Minkowski symmetrization and hence also for $\diamondsuit$, by Theorem 1.4. Consider for $Z \in \mathcal{K}^n$ and $k \in \mathbb{N}$ the limit W of the sequence of sets

$$Z_m = \diamondsuit_{H_m} \cdots \diamondsuit_{H_k} Z$$

for $m \geq k$. Again $(\mathcal{W}(Z_m))$ is a non-negative and non-increasing sequence; thus it converges to a value $b \geq 0$.

The sequence (H_m) is weakly universal for Minkowski symmetrization, thus for every $\nu \geq k$ we have a ball B_ν as the limit of the sequence of sets

$$M_{H_m} \cdots M_{H_{\nu+1}} \diamondsuit_{H_\nu} \cdots \diamondsuit_{H_k} Z.$$

Then $\mathcal{W}(B_\nu) = \mathcal{W}(Z_\nu)$ thanks to the properties of Minkowski symmetrization and for every ν Theorem 2.1 gives $W \subseteq B_\nu$. Since $\mathcal{W}(B_\nu)$ decreases to $\mathcal{W}(W)$, W must be a ball. □

Theorem 3.4 allows us to extend many known results for Steiner symmetrization to all $\diamondsuit \in \mathcal{F}$, in particular, Minkowski symmetrization with respect to hyperplanes. For example, we immediately have the following generalization of Theorem 2.2.

Corollary 3.6. *Let (H_m) be a sequence of hyperplanes with corresponding normals $u_m \in \mathbb{S}^{n-1}$. Consider $\diamondsuit \in \mathcal{F}$ and angles $\alpha_m \in (0, 2\pi)$ such that $u_m \cdot u_{m-1} = \cos \alpha_m$. If $\sum_{m \in \mathbb{N}} \alpha_m^2 < \infty$, then there exist rotations $\mathbb{A}_m$ such that for every non-empty compact convex set $K \subset \mathbb{R}^n$ the sets*

$$K_m = \mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K$$

converge in Hausdorff metric to a set $L \in \mathcal{K}^n$.

For Minkowski symmetrization with respect to hyperplanes, we have a stronger result. First, we need the following convergence criterion from [17, Theorem 1.2].

Theorem 3.7. *Consider $K \in \mathcal{K}^n$ and a sequence $(\mathbb{A}_m)$ of isometries. If the sets*

$$K_m = \frac{1}{m} \sum_{j=1}^{m} \mathbb{A}_j K$$

converge, then the same happens for every compact set $C \in \mathcal{C}^n$ such that conv $C = K$. Moreover, the two sequences converge to the same limit.

Together with Theorem 1.4, this implies the following.

Corollary 3.8. *If (H_m) is a shape-stable sequence of hyperplanes for Steiner symmetrization on $\mathcal{C}^n$, then it is shape-stable on $\mathcal{C}^n$ for Minkowski symmetrization. In particular, Theorem 2.2 holds for Minkowski symmetrization as well.*

Proof. First, observe that since $\mathcal{K}^n$ is closed in $\mathcal{C}^n$, the sequence (H_m) is shape-stable for Steiner symmetrization on $\mathcal{K}^n$ and by Theorem 3.4 it is shape-stable for Minkowski symmetrization on $\mathcal{K}^n$.

Now, to conclude the proof, we only have to express the shape-stable sequence as a sequence of means of isometries so that we can apply Theorem 3.7. To see this, note that for every $C \in \mathcal{C}^n$

$$C_1 = \mathbb{A}_1 M_{H_1} C = \mathbb{A}_1 \left(\frac{C + R_{H_1} C_1}{2} \right) = \frac{\mathbb{A}_1 C + \mathbb{A}_1 R_{H_1} C}{2}.$$

Iterating this process, we see that every C_m is a Minkowski mean of 2^m isometries of C.

Theorem 2.2 provides shape-stable sequences for Steiner symmetrization in $\mathcal{C}^n$, thus the same sequences are shape-stable in $\mathcal{C}^n$ for Minkowski symmetrization. □

We conclude this section with the proof of Theorem 1.5.

Proof of Theorem 1.5. We shall apply Corollary 3.6. The rotations $\mathbb{A}_m$ there are those constructed in Theorem 2.2; see the proof of [1, Theorem 2.2] which shows that $\mathbb{A}_m = \phi_m \cdots \phi_1$, where ϕ_m is a planar rotation by α_m degrees that fixes $u_m^\perp \cap e_1^\perp$ and is such that $\phi_m \mathbb{A}_{m-1} u_m = e_1$, for a fixed basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$.

First, we show that $(\mathbb{A}_m)$ is a Cauchy sequence in $GL(n)$ with the usual norm

$$\|\phi\| = \sup_{z \in \mathbb{S}^{n-1}} \|\phi z\|.$$

From the properties of the norm and the triangle inequality we obtain

$$\|\mathbb{A}_{m+k} - \mathbb{A}_m\| = \|\phi_{m+k} \cdots \phi_{m+1} \mathbb{A}_m - \mathbb{A}_m\| \leq \|\mathbb{A}_m\| \|\phi_{m+k} \cdots \phi_{m+1} - Id\| =$$
$$\|\phi_{m+k} \cdots \phi_{m+1} - Id\| \leq \|\phi_{m+k} \cdots \phi_{m+1} - \phi_{m+1}\| + \|\phi_{m+1} - Id\| \leq$$
$$\|\phi_{m+1}\| \|\phi_{m+k} \cdots \phi_{m+2} - Id\| + 2\sin(|\alpha_{m+1}|/2) \leq \cdots \leq 2 \sum_{j=m+1}^{m+k} \sin(|\alpha_j|/2).$$

From the hypothesis, the series $\sum |\alpha_m|$ converges, proving the claim. Since $\mathbb{A}_m$ is a composition of rotations, $\mathbb{A}_m \in SO(n)$ for every m. As a subspace of $GL(n)$, $SO(n)$ is compact, and therefore it is complete. Thus the sequence $(\mathbb{A}_m)$ converges to some $\mathbb{A} \in SO(n)$.

By Corollary 3.6 the sequence of sets

$$\mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K$$

converges for every $K \in \mathcal{K}^n$ to a certain set L. We then have the estimate

$$d_{\mathcal{H}}(\diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K, \mathbb{A}^{-1} L) \leq d_{\mathcal{H}}(\diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K, \mathbb{A}_m^{-1} L) + d_{\mathcal{H}}(\mathbb{A}_m^{-1} L, \mathbb{A}^{-1} L).$$

By the isometry invariance of the Hausdorff distance

$$d_{\mathcal{H}}(\mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K, L) = d_{\mathcal{H}}(\diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K, \mathbb{A}_m^{-1} L)$$

and thus from the convergence of $\mathbb{A}_m \diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K$ and $\mathbb{A}_m$ we infer that $\diamondsuit_{H_m} \cdots \diamondsuit_{H_1} K$ converges to $\mathbb{A}^{-1} L$, concluding the proof.

□

4 Counterexamples to convergence

As we have seen, Theorem 3.4 lets us extend Theorem 2.1 to all the symmetrizations $\diamondsuit \in \mathcal{F}$. The latter result arises from the study of a peculiar counterexample which we now briefly show. It can be found in different versions in [1], [5], and [6]. In the following examples the vectors $\{e_1, e_2\}$ form an orthonormal basis of $\mathbb{R}^2$.

Example 4.1. Consider a sequence of angles $\alpha_m \in (0, \pi/2)$ such that

$$\sum_{m\in\mathbb{N}} \alpha_m = \infty, \qquad \sum_{m\in\mathbb{N}} \alpha_m^2 < \infty. \tag{4.1}$$

We consider the sequence of directions in $\mathbb{R}^2$ given by $u_m = (\cos\beta_m, \sin\beta_m)$ where

$$\beta_m = \sum_{j=1}^{m} \alpha_j$$

with corresponding orthogonal lines $H_m = u_m^\perp$.

Let $0 < \gamma = \prod_{m\in\mathbb{N}} \cos\alpha_m$ (which converges because of the second condition in (4.1)). We consider a compact set $K \subset \mathbb{R}^2$ with area $0 < \lambda_2(K) < \pi(\gamma/2)^2$ and containing a vertical unit segment ℓ centered at the origin. We claim that the sets

$$K_m = S_{H_m} \cdots S_{H_1} K$$

do not converge. Indeed, consider the segments

$$\ell_m = P_{H_m} K_{m-1},$$

where the length of $\ell_m \subseteq H_m$ converges to $\gamma > 0$. The sequence (u_m) of directions is dense in $\mathbb{S}^1$, thanks to (4.1), and it does not converge. The same holds for the sequence (H_m) of lines. Thus for every $\nu \in \mathbb{S}^1$, we can find a subsequence $H_{m_k}^\perp$ such that the normal directions converge to ν. Then ℓ_{m_k} converges to a segment of length $\gamma > 0$ parallel to ν.

Now, if K_m converges, by the monotonicity of Steiner symmetrization the limit set must contain all these subsequences of diameters, and consequently a ball B of diameter γ centered at the origin. But we supposed $\lambda_2(K) < \pi(\gamma/2)^2$, thus K_m cannot converge.

The peculiarity of the sequence involved in this example is that the corresponding directions are dense in $\mathbb{S}^1$, which could seem a reasonable sufficient condition for convergence to a ball. As was shown, this is not the case, even though in [5] it was proved for compact convex sets that a dense sequence of hyperplanes can be reordered to obtain a universal sequence. This was generalized in [20] to generic compact sets.

In [4], a characterization is proved of the symmetry that a convex body needs to be a ball. The form we present includes the statements from [4, Theorem 3.2] for one-dimensional subspaces.

Theorem 4.1 (Bianchi, Gardner, and Gronchi)**.** *Let $H_j \in \mathcal{G}(n,1), j = 1, \ldots, n,$ be such that*
(i) at least two of them form an angle that is an irrational multiple of π,
(ii) $H_1 + \cdots + H_n = \mathbb{R}^n$ and
(iii) $H_1, \ldots, H_n$ cannot be partitioned into two mutually orthogonal non-empty subsets.
If $E \subseteq \mathbb{S}^{n-1}$ is non-empty, closed and such that $R_{H_j}E = E, j = 1, \ldots, n,$ then $E = \mathbb{S}^{n-1}$.
Hence, if $K \in \mathcal{K}_n^n$ satisfies $R_{H_j}K = K$ for $j = 1, \cdots, n,$ then K is a ball centered at the origin.

We can use this theorem to find sequences of lines such that the corresponding symmetrization process, if it converges, tends to a ball. Indeed, consider a sequence (v_m) of directions with n accumulation points generating a family of lines $H_1, \dots, H_n$ as in the statement of Theorem 4.1. Consider a sequence (K_m) of convex bodies such that every K_m is symmetric with respect to $v_m^\perp$. Then, if the sequence converges, the limit must necessarily be a ball. We can use this fact to provide a new kind of counterexample.

Example 4.2. Consider in $\mathbb{R}^2$ the two directions $w_1 = (1,0), w_2 = (\cos\alpha, \sin\alpha)$ such that $\alpha > 0$ is an irrational multiple of π. We consider a $\gamma_m \in [0,\alpha]$, $m \in \mathbb{N}$, such that $\alpha_m = |\gamma_{m+1} - \gamma_m|$ is as in (4.1). Moreover we want α and 0 to be accumulation points of (γ_m).
Consider the sequence (H_m) of lines given by $H_m = \text{span}\{(\cos\gamma_m, \sin\gamma_m)\}$. Then the corresponding sequence of directions has w_1 and w_2 as accumulation points.
Let K be a compact body centered at the origin with a diameter of unit length parallel to w_1 and consider the sequence of symmetrals

$$K_m = S_{H_m} \cdots S_{H_1} K.$$

As in Example 4.1 we can consider a sequence of segments

$$\ell_m = K_m \cap H_m$$

such that $\lambda_1(\ell_{m+1}) \geq \lambda_1(\ell_m) \cos\alpha_{m+1}$, thus $\lambda_1(\ell_m)$ converges to a value $\gamma > 0$ and in particular the two limits of the converging subsequences of (ℓ_m) respectively parallel to w_1 and w_2 have length greater than γ.
Using Theorem 4.1, if K_m converges, the limit must be a ball. If we choose $\lambda_2(K) < \pi(\gamma/2)^2$, the limit ball should contain a diameter of length γ, which is not possible. Therefore (K_m) cannot converge.

We conclude proving that Example 4.1 can be generalized for other symmetrizations, again thanks to Theorem 2.1.

Example 4.3. Consider a set $K \in \mathcal{K}_2^2$ that contains a horizontal unit segment and has mean width $1/2\pi < \mathcal{W}(K) < \gamma$, where γ is as in Example 4.1. In the hypothesis of Theorem 2.1, for $\diamondsuit \in \mathcal{F}$

$$S_{U_m} \cdots S_{U_1} K \subseteq \diamondsuit_{U_m} \cdots \diamondsuit_{U_1} K \subseteq M_{U_m} \cdots M_{U_1} K,$$

where $U_j = \text{span}\{u_j\}$, and we used Steiner symmetrization because it is equivalent to fiber symmetrization relative to a hyperplane, which is our case working in $\mathbb{R}^2$.
In this way, we can exploit the first counterexample and the inclusion chain of Theorem 2.1 to guarantee that, if a limit exists for $\diamondsuit_{U_m} \cdots \diamondsuit_{U_1} K$ and $M_{U_m} \cdots M_{U_1} K$, proceeding as before it must contain a ball of diameter γ, and therefore this limit must have mean width greater than γ. In particular, this holds for the sequence of Minkowski symmetrals. But Minkowski symmetrization preserves mean width, which we supposed to be less than γ. This is a contradiction, and therefore there cannot be a limit.

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