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Calculus of Variations

A continuous dependence estimate for viscous Hamilton–Jacobi equations on networks with applications

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Abstract

We study continuous dependence estimates for viscous Hamilton–Jacobi equations defined on a network Γ. Given two Hamilton–Jacobi equations, we prove an estimate of the C^2-norm of the difference between the corresponding solutions in terms of the distance among the Hamiltonians. We also provide two applications of the previous estimate: the first one is an existence and uniqueness result for a quasi-stationary Mean Field Games defined on the network Γ; the second one is an estimate of the rate of convergence for homogenization of Hamilton–Jacobi equations defined on a periodic network, when the size of the cells vanishes and the limit problem is defined in the whole Euclidean space.

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1 Introduction

In the recent years, there has been an increasing interest in the study of dynamical system on networks, in connection with problem such as vehicular traffic, data transmission, crowd motion, supply chains, etc. As consequence, many results for linear and nonlinear PDEs in the Euclidean case have been progressively extended to the network setting and also to more general geometric structures. Here, we are interested in continuous dependence estimates for viscous Hamilton–Jacobi (HJ for short) equations. Let us recall that such estimates play a crucial role in many contexts, for example for regularity results, error estimate for numerical schemes, rate of convergence in vanishing viscosity and homogenization [11, 17].

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Our analysis is inspired by the results in [20], where it is proved a continuous dependence estimate in the C^2-norm for solutions of a viscous HJ equations in the periodic setting with an explicit dependence on the distance of the coefficients and an explicit characterization of the constants. We prove an analogous result for viscous HJ equations defined on networks with Kirchhoff conditions at the vertices. To this end, we use some results concerning the study of these equations on networks [1, 2, 10] and suitably adapt the arguments in [20] to this specific setting.

Then, the previous continuous dependence estimate is applied to two problems:

(i) The well-posedness of a quasi-stationary Mean Field Games system defined on a network;
(ii) An estimate of the rate of convergence for homogenization of HJ equations defined on a periodic networks.

Mean Field Games (MFG for short), introduced in [18], modelize the interaction among a large number of agents. In this theory, the agents are assumed indistinguishable, infinitesimal and completely rational and their behaviour is influenced by the statistical distribution of the states of the other agents. In the classical formulation, MFG lead to the study of a coupled system of two evolutive PDEs, a backward HJ equation for the value function of the representative agent and a forward Fokker-Planck (FP for short) equation for the distribution of the agents. Recently, a different strategy mechanisms from classical MFG theory has been proposed in [22] (see also [12]): the agents are myopic and choose their strategy only according to the information available at present time, without forecasting the future evolution. In this case, the Nash equilibria for the distribution of the agents are characterized by a quasi-stationary MFG system, which is composed of a stationary HJ equation and an evolutive Fokker-Planck equation.

While classical MFG on networks have been studied in [1, 2, 9], here we consider a quasi-stationary MFG defined on a network and we prove existence and uniqueness of the corresponding solution. Existence is proved via a fixed point argument and the continuous dependence estimate is crucial since in this case it is not possible to exploit the regularizing effect of the parabolic HJ equation to show the continuity of the fixed point map. The continuous dependence estimate is also exploited to prove uniqueness of the solution, which, with respect to the classical case, requires no monotonicity assumption.

The second application of the continuous dependence estimate is to a homogenization problem. We show that the solution of a viscous HJ equation, defined on a periodic lattice of size ϵ, converges, as $\epsilon \to 0$, to the solution of an effective problem defined in all the Euclidean space and we also give an estimate of the rate of convergence. Moreover, we obtain a characterization of the corresponding effective operator in terms of the Hamiltonians defined on the edges of the lattice. We note that a similar problem was studied for first order HJ equations in [16] and for linear second order equations in [8].

The paper is organized as follows: in Sect. 2 we fix our setting and notations for the network. Section 3 is devoted to the main result, the continuous dependence estimate for the solution to an HJ equation on the network. In Sect. 4 we tackle quasi-stationary MFGs on the network: in particular, we obtain existence and uniqueness of a solution without requiring any monotonicity assumption. Section 5 concerns the homogenization of HJ equations on a lattice: the main result is a rate of convergence estimate.

2 The network Γ: notations and definitions

We consider a bounded network $\Gamma \subset \mathbb{R}^N$ composed by a finite collection of bounded straight edges $\mathcal{E} := \{\Gamma_\alpha, \alpha \in \mathcal{A}\}$, which connect a finite collection of vertices $\mathcal{V} := \{\nu_i, i \in I\}$. We assume that, for $\alpha, \beta \in \mathcal{A}$ with $\alpha \neq \beta$, $\Gamma_\alpha \cap \Gamma_\beta$ is either empty or made of a single vertex. For an edge $\Gamma_\alpha \in \mathcal{E}$ connecting two vertices ν_i and ν_j with $i < j$, we consider the parametrization $\pi_\alpha : [0, \ell_\alpha] \to \Gamma_\alpha$ given by

$$\pi_\alpha(y) = [y\nu_j + (\ell_\alpha - y)\nu_i]\ell_\alpha^{-1} \quad \text{for } y \in [0, \ell_\alpha],$$

where ℓ_α is the length of the edge. We denote with $\mathcal{A}_i = \{\alpha \in \mathcal{A} : \nu_i \in \Gamma_\alpha\}$ the set of indices of edges that are adjacent to the vertex ν_i.
For a function $v : \Gamma \to \mathbb{R}$, we denote with $v_\alpha : (0, \ell_\alpha) \to \mathbb{R}$ the restriction of v to $\Gamma_\alpha \setminus \mathcal{V}$, i.e.

$$v_\alpha(y) := v|_{\Gamma_\alpha} \circ \pi_\alpha(y), \quad \text{for all } y \in (0, \ell_\alpha).$$

Moreover we define for $x \in \Gamma_\alpha \setminus \mathcal{V}$ the derivative along the arc

$$\partial v(x) = \frac{dv_\alpha}{dy}(y) \quad \text{for } y = \pi_\alpha^{-1}(x).$$

Remark 2.1 The function v_α is defined only on $(0, \ell_\alpha)$; nevertheless, when it is possible, we denote v_α also its extension by continuity on 0 and on ℓ_α. Note that, in this way, v_α may not coincide with the original function v at the vertices when v is not continuous.

For $x = \nu_i \in \Gamma_\alpha$, we define the outward derivative at the vertex

$$\partial_\alpha v\left(\pi_\alpha^{-1}(\nu_i)\right) := \begin{cases} \lim\limits_{h\to 0^+} \dfrac{v_\alpha(0) - v_\alpha(h)}{h}, & \text{if } \nu_i = \pi_\alpha(0), \\ \lim\limits_{h\to 0^+} \dfrac{v_\alpha(\ell_\alpha) - v_\alpha(\ell_\alpha - h)}{h}, & \text{if } \nu_i = \pi_\alpha(\ell_\alpha). \end{cases}$$

Setting

$$n_{i\alpha} = \begin{cases} 1 \text{ if } \nu_i = \pi_\alpha(\ell_\alpha), \\ -1 \text{ if } \nu_i = \pi_\alpha(0), \end{cases}$$

we have

$$\partial_\alpha v(\nu_i) = n_{i\alpha}\, \partial v_\alpha(\pi_\alpha^{-1}(\nu_i)).$$

We introduce some functional spaces defined on the network Γ. The space $C(\Gamma)$ is composed of the continuous functions on Γ; the space

$$PC(\Gamma) := \{v : \Gamma \to \mathbb{R} : v_\alpha \in C([0, \ell_\alpha]), \quad \text{for all } \alpha \in \mathcal{A}\}$$

is composed of the piece-wise continuous functions on Γ, i.e. functions which are continuous inside the edges but not necessarily at the vertices. For $m \in \mathbb{N}$

$$C^m(\Gamma) := \left\{v \in C(\Gamma) : v_\alpha \in C^m([0, \ell_\alpha]) \text{ for all } \alpha \in \mathcal{A}\right\},$$

is the space of m-times continuously differentiable functions on Γ endowed with the norm

$$\|v\|_{C^m(\Gamma)} := \sum_{\alpha\in\mathcal{A}} \sum_{k\le m} \left\|\partial^k v_\alpha\right\|_{L^\infty(0,\ell_\alpha)}.$$

For $\sigma \in (0,1]$ and $w : A \to \mathbb{R}$, we write

$$[w]_{\sigma,A} = \sup_{\substack{y\neq z\\ y,z\in A}} \frac{|w(y)-w(z)|}{|y-z|^{\sigma}} \text{ and } \|w\|_{C^{0,\sigma}(A)} = \|w\|_{\infty} + [w]_{\sigma,A}.$$

For $\sigma \in (0,1]$, the space $C^{m,\sigma}(\Gamma)$ contains the functions $v \in C^m(\Gamma)$ such that $\partial^m v_\alpha \in C^{0,\sigma}([0,\ell_\alpha])$ for all $\alpha \in \mathcal{A}$ with the norm

$$\|v\|_{C^{m,\sigma}(\Gamma)} := \|v\|_{C^m(\Gamma)} + \sup_{\alpha\in\mathcal{A}} [\partial^m v_\alpha]_{\sigma,[0,\ell_\alpha]}.$$

The integral of a function v on Γ is defined by

$$\int_\Gamma v(x)dx = \sum_{\alpha\in\mathcal{A}} \int_0^{\ell_\alpha} v_\alpha(y)\,dy$$

and we set $\langle v\rangle = \int_\Gamma v(x)dx$. For $p \in [1,\infty]$, we define the Lebesgue space

$$L^p(\Gamma) = \left\{v : v_\alpha \in L^p((0,\ell_\alpha)) \text{ for all } \alpha \in \mathcal{A}\right\},$$

endowed with the standard norm. For any integer $m \in \mathbb{N}$, $m \geq 1$, and $p \in [1,\infty]$ we define the Sobolev space

$$W^{m,p}(\Gamma) := \left\{v \in C(\Gamma) : v_\alpha \in W^{m,p}((0,\ell_\alpha)) \text{ for all } \alpha \in \mathcal{A}\right\},$$

endowed with the norm

$$\|v\|_{W^{m,p}(\Gamma)} = \left(\sum_{k=1}^{m}\sum_{\alpha\in\mathcal{A}} \left\|\partial^k v_\alpha\right\|^p_{L^p(0,\ell_\alpha)} + \|v\|^p_{L^p(\Gamma)}\right)^{\frac{1}{p}}.$$

We also set $H^m(\Gamma) = W^{m,2}(\Gamma)$.

The geodesic distance $d_\Gamma(x,y)$ between two points $x, y \in \Gamma$ is the infimum of the length taken over all continuous, piecewise continuously differentiable curves $\xi : [a,b] \to \Gamma$ with $\xi(a) = x$ and $\xi(b) = y$. The couple (Γ, d_Γ) is a metric space.

Denote with $\mathcal{M}$ the space of Borel probability measures on Γ. For $1 \leq p < \infty$, the L^p-Wasserstein distance $\mathbf{d}_p$ between $\sigma, \tau \in \mathcal{M}$ is defined by the Monge-Kantorovich transport problem

$$\mathbf{d}_p(\sigma,\tau) = \min_{\Sigma\in\Pi(\sigma,\tau)} \left\{\int_{\Gamma\times\Gamma} (d_\Gamma(x,y))^p\, d\Sigma(x,y)\right\}$$

where $\Pi(\sigma,\tau)$ denotes the set of transport plans, i.e. Borel probability measures on $\Gamma\times\Gamma$ with marginals σ and τ (see [7]). Since Γ is compact, the Wasserstein distance $\mathbf{d}_p$ metrises the topology of weak convergence of probability measures on Γ. In particular, for $p = 1$, we have

$$\mathbf{d}_1(\sigma,\tau) = \sup\left\{\int_\Gamma f(x)d(\sigma-\tau) : \ f : \Gamma \to \mathbb{R},\ |f(x)-f(y)| \leq d_\Gamma(x,y)\right\}. \quad (2.1)$$

We shortly recall the definition of diffusion process on the network Γ (see [14, 15] for details). Consider the linear differential operator $\mathcal{L}$ defined on the edges by

$$\mathcal{L}_\alpha u(x) = \mu_\alpha \partial^2 u(x) + B_\alpha(x)\partial u(x), \quad x \in \Gamma_\alpha,\ \alpha \in \mathcal{A}$$

with domain

$$D(\mathcal{L}) = \left\{ u \in C^2(\Gamma) : \sum_{\alpha \in \mathcal{A}_i} p_{i,\alpha} \partial_\alpha u(\nu_i) = 0, \ i \in I \right\}$$

where $p_{i,\alpha} \in (0,1)$, $\sum_{\alpha \in \mathcal{A}_i} p_{i,\alpha} = 1$. Then, the operator $\mathcal{L}$ is the infinitesimal generator of a Feller-Markov process (X_t, α_t), with $X_t \in \Gamma_{\alpha_t}$, such that, for $x_t = \pi_{\alpha_t}^{-1}(X_t)$, we have

$$dx_t = B_{\alpha_t}(x_t)dt + \mu_{\alpha_t} dW_t + d\ell_{i,t} + dh_{i,t}. \tag{2.2}$$

In (2.2), W_t is a one dimensional Wiener process; $\ell_{i,t}$ and $h_{i,t}$, $i \in I$, are continuous non-decreasing and, respectively, non-increasing processes, measurable with respect to the σ-field generated by (X_t, α_t) and satisfying

$$\begin{aligned} &\ell_{i,t} \ \text{ increases only when } X_t = \nu_i \ \text{ and } x_t = 0, \\ &h_{i,t} \ \text{decreases only when } X_t = \nu_i \text{ and } x_t = \ell_\alpha. \end{aligned}$$

Moreover, the following Itô formula holds true: for every $u \in C^2(\Gamma)$,

$$\begin{aligned} u(x_t) = u(x_0) &+ \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} p_{i,\alpha} \partial_\alpha u(\nu_i) \left(\ell_{i,t} + h_{i,t}\right) \\ &+ \sum_{\alpha \in \mathcal{A}} \int_0^t 1_{\{x_s \in \Gamma_\alpha \setminus \mathcal{V}\}} \left[\left(\mu_\alpha \partial^2 u(x_s) + B_\alpha(x_s) \partial u(x_s)\right) ds + \sqrt{2\mu_\alpha} \partial u(x_s) dW_s \right]. \end{aligned}$$

3 The continuous dependence estimate

We consider the following ergodic HJ equation on Γ

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x, \partial v) + \rho = 0, & x \in (\Gamma_\alpha \setminus \mathcal{V}), \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) = 0, & \nu_i \in \mathcal{V}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}, \\ \langle v \rangle = 0. \end{cases} \tag{3.1}$$

Let us notice that, when H is given by

$$H_\alpha(x, p) = \sup_{a \in A_\alpha} \{-b_\alpha(x, a)p - f_\alpha(x, a)\} \qquad \text{for } x \in \Gamma_\alpha \setminus \mathcal{V}, \tag{3.2}$$

problem (3.1) represents the dynamic programming equation for the optimal control problem with long-run average cost functional

$$\rho = \inf_a \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T f(X_t, a_t) dt \right]$$

where a_t is a feedback control law of form $a_t = a(X_t)$ and X_t is a diffusion process on Γ such that $x_t = \pi_{\alpha_t}(X_t)$ satisfies (2.2) with $B_\alpha(x) = B_\alpha(x, a(x))$ (see [1, Section 1.3] for more details). Connected with the optimal control interpretation of (3.1), the second equation is a Kirchhoff transmission condition, where the quantity $p_{i\alpha} = \gamma_{i\alpha}\mu_\alpha (\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha)^{-1}$ represents the probability that the trajectories of the diffusion process enter in edge Γ_α, $\alpha \in \mathcal{A}_i$, from the vertex ν_i; it can be also interpreted as a Neumann boundary condition if

$\sharp(\mathcal{A}_i) = 1$. The third equation implies continuity of the solution at the vertices and the last one is a normalization condition.
We make the following assumptions

(H1) μ_α and $\gamma_{i,\alpha}$ are positive constants and

$$\sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha = 1, \qquad \forall i \in I.$$

(H2) $H : \Gamma \times \mathbb{R} \to \mathbb{R}$, with $H(x, p) = H_\alpha(x, p)$ if $x \in \Gamma_\alpha \setminus \mathcal{V}$ and there exist positive constants $K, L, \theta \in (0, 1]$ and $\widetilde{H}_\alpha \in C(\Gamma_\alpha \times \mathbb{R})$, with $\widetilde{H}_\alpha(x, 0) = 0$ such that

$$\begin{aligned} &\sup_{x\in\Gamma_\alpha} |H_\alpha(x, 0)| \le K \\ &H_\alpha(\cdot, p) \in C^{0,1}(\Gamma_\alpha) \text{ with } \|H_\alpha(\cdot, p)\|_{C^{0,1}(\Gamma_\alpha)} \le L(1 + |p|) \\ &H_\alpha(x, \cdot) \in C^{1,\theta}(\mathbb{R}) \text{ with } \|\partial_p H_\alpha\|_{C^{0,\theta}(\Gamma_\alpha\times\mathbb{R})} \le L \\ &\lim_{\xi\to\infty} \frac{H_\alpha(x, \xi p)}{\xi} = \widetilde{H}_\alpha(x, p) \end{aligned} \tag{3.3}$$

for all $x \in \Gamma_\alpha$ and $\alpha \in \mathcal{A}$.

For the study of the ergodic problem (3.1), it is expedient to introduce for $\lambda \in (0, 1)$ the discount approximation

$$\begin{cases} -\mu_\alpha\partial^2 v^\lambda + H\left(x, \partial v^\lambda\right) + \lambda v^\lambda = 0, & x \in (\Gamma_\alpha\backslash\mathcal{V}), \alpha \in \mathcal{A}, \\ \sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha\partial_\alpha v^\lambda(\nu_i) = 0, & \nu_i \in \mathcal{V}, \\ v^\lambda|_{\Gamma_\alpha}(\nu_i) = v^\lambda|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}. \end{cases} \tag{3.4}$$

The following statement concerns existence, uniqueness and regularity of classical solutions to the HJ Eqs. (3.1) and (3.4). The proof is an adaptation of previous papers ([5, Theorem II.2], [10] and [1, Proposition 3.2 and Theorem 3.7]); hence, we shall only sketch it in the Appendix.

Proposition 3.1 *There exists a unique classical solution v^λ to the Eq. (3.4). Moreover,*

(i) there exists a positive constant C_1, independent of λ, such that

$$\|\lambda v^\lambda\|_{L^\infty(\Gamma)} \le K, \tag{3.5}$$

$$\|v^\lambda - \langle v^\lambda\rangle\|_{C^{2,\theta}(\Gamma)} \le C_1(1 + K + L) =: \bar{K}, \tag{3.6}$$

where K, L and θ as in (3.3);

(ii) for $\lambda \to 0^+$, $\lambda v^\lambda \to \rho$, $v^\lambda - \langle v^\lambda\rangle \to v$ and the couple (v, ρ) is the unique classical solution to (3.1). Moreover

$$\|v\|_{C^{2,\theta}(\Gamma)} \le \bar{K}. \tag{3.7}$$

Remark 3.2 The statement of Proposition 3.1 holds also when H is replaced by $H + F$ and $F_\alpha \in C^{0,\theta}(\Gamma_\alpha)$. Moreover, for $i = 1, 2$, let v_i^λ be the unique bounded solution to (3.4) where H_α is replaced by $H_\alpha + F_\alpha^i$, with $F_\alpha^i \in C^{0,\theta}(\Gamma)$, and set: $\rho_i := \lim_{\lambda\to0^+} \lambda v_i^\lambda$; then,

$$\|\lambda(v_1^\lambda - v_2^\lambda)\|_\infty \le \sup_{\alpha\in\mathcal{A}} \|F_\alpha^1 - F_\alpha^2\|_\infty \quad \text{ and } \quad |\rho - \rho_1| \le \sup_{\alpha\in\mathcal{A}} \|F_\alpha^1 - F_\alpha^2\|_\infty.$$

We give some preliminary results for Eqs. (3.1) and (3.4). The first result is a strong maximum principle for the linear HJ equation (see [1, Lemma 2.8] or [10, Theorem 3.1]).

Lemma 3.3 *For $g \in PC(\Gamma)$, the solutions of*

$$\begin{cases} -\mu_\alpha \partial^2 v + g\partial v = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, i \in I \end{cases}$$

are the constant functions on Γ.

The second result is a comparison principle for (3.4) (see [1, Lemma 3.6] and [10, Corollary 3.1]).

Lemma 3.4 *If $u, v \in C^2(\Gamma)$ satisfy*

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x, \partial v) + \lambda v \geq -\mu_\alpha \partial^2 u + H(x, \partial u) + \lambda u, & \text{if } x \in \Gamma_\alpha \backslash \mathcal{V}, \alpha \in A, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) \geq \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha u(\nu_i), & \text{if } \nu_i \in \mathcal{V}, \end{cases}$$

then $v \geq u$.

We now give a continuous dependence estimate for the solution of (3.1) and (3.4) with respect to the data of the problem.

Theorem 3.5 *For $i = 1, 2$, consider $H^i : \Gamma \times \mathbb{R} \to \mathbb{R}$ and $F^i : \Gamma \to \mathbb{R}$. Assume that*

(i) H^1 and H^2 satisfy (H2) with the same constants K, L and θ;
(ii) the functions $F^i : \Gamma \to \mathbb{R}$, $i = 1, 2$, fulfill: for some $K_F > 0$,

$$\|F^i_\alpha\|_{C^{0,\theta}(\Gamma_\alpha)} \leq K_F \qquad \forall \alpha \in \mathcal{A}.$$

For $i = 1, 2$, let v^λ_i be the solution of (3.4) with Hamiltonian $H(x, p) = H^i(x, p) + F^i(x)$ and set $w^\lambda_i := v^\lambda_i - \langle v^\lambda_i \rangle$. Then, there exists a positive constant C_0, independent of λ, such that

$$\begin{aligned} \|w^\lambda_1 - w^\lambda_2\|_{C^2(\Gamma)} \leq C_0 \max_{\alpha \in \mathcal{A}} \left(\max_{(x,p) \in \Gamma_\alpha \times [-\bar{K}, \bar{K}]} |H^1_\alpha - H^2_\alpha| + \max_{x \in \Gamma_\alpha} |F^1_\alpha - F^2_\alpha| \right) \\ + \max_{\alpha \in \mathcal{A}} [H^1_\alpha - H^2_\alpha]_{1, \Gamma_\alpha \times [-\bar{K}, \bar{K}]} + \max_{\alpha \in \mathcal{A}} [F^1_\alpha - F^2_\alpha]_{\theta, \Gamma_\alpha}, \end{aligned} \tag{3.8}$$

where $\bar{K}$ is defined in (3.6). Estimate (3.8) also holds for v_i, $i = 1, 2$, solution to (3.1) corresponding to $H(x, p) = H^i(x, p) + F^i(x)$.

Proof We shall proceed by contradiction. We assume that, for $k \to \infty$, there exist sequences $\lambda_k \to 0$, $H^{i,k}$, $F^{i,k}$, $i = 1, 2$, satisfying (i) and (ii) with the same constants K, L, θ, K_F and $v^{\lambda_k}_i$, solution to (3.4) with discount λ_k, Hamiltonian $H^{i,k}$ and term $F^{i,k}$ such that

$$\begin{aligned} c_k :=& \|w^{\lambda_k}_1 - w^{\lambda_k}_2\|_{C^2(\Gamma)} \\ \geq& k \max_{\alpha \in \mathcal{A}} \left(\max_{(x,p) \in \Gamma_\alpha \times [-\bar{K}, \bar{K}]} |H^{1,k}_\alpha - H^{2,k}_\alpha| + \max_{x \in \Gamma_\alpha} |F^{1,k}_\alpha - F^{2,k}_\alpha| \right) \end{aligned}$$

$$+\max_{\alpha\in\mathcal{A}}[H_\alpha^{1,k}-H_\alpha^{2,k}]_{1,\Gamma_\alpha\times[-\bar K,\bar K]}+\max_{\alpha\in\mathcal{A}}[F_\alpha^{1,k}-F_\alpha^{2,k}]_{\theta,\Gamma_\alpha}$$

where $w_i^{\lambda_k}=v_i^{\lambda_k}-\langle v_i^{\lambda_k}\rangle$. The function $w_i^{\lambda_k}$, $i=1,2$, solves the equation

$$\begin{cases}-\mu_\alpha\partial^2 w_i^{\lambda_k}+H^{i,k}\big(x,\partial w_i^{\lambda_k}\big)+F^{i,k}(x)+\lambda_k w_i^{\lambda_k}\\ \qquad +\lambda_k\langle v_i^{\lambda_k}\rangle=0, & x\in(\Gamma_\alpha\backslash\mathcal{V}),\alpha\in\mathcal{A},\\ \sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha w_i^{\lambda_k}(\nu_i)=0, & \nu_i\in\mathcal{V},\\ w_i^{\lambda_k}|_{\Gamma_\alpha}(\nu_i)=w_i^{\lambda_k}|_{\Gamma_\beta}(\nu_i), & \alpha,\beta\in\mathcal{A}_i,\nu_i\in\mathcal{V}.\end{cases}$$

Hence the function $W^k=c_k^{-1}(w_1^{\lambda_k}-w_2^{\lambda_k})$ fulfills $\|W^k\|_{C^2(\Gamma)}\|\le 1$ and solves the equation

$$\begin{cases}-\mu_\alpha\partial^2 W^k+c_k^{-1}\big(H^{1,k}(x,\partial w_1^{\lambda_k})\\ \qquad -H^{1,k}(x,\partial w_2^{\lambda_k})\big)+R^k=0, & x\in(\Gamma_\alpha\backslash\mathcal{V}),\alpha\in\mathcal{A},\\ \sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha W^k(\nu_i)=0, & \nu_i\in\mathcal{V},\\ W^k|_{\Gamma_\alpha}(\nu_i)=W^k|_{\Gamma_\beta}(\nu_i), & \alpha,\beta\in\mathcal{A}_i,\nu_i\in\mathcal{V},\end{cases}\tag{3.9}$$

where

$$R^k=\lambda_k c_k^{-1}(\langle v_1^{\lambda_k}\rangle-\langle v_2^{\lambda_k}\rangle)+c_k^{-1}\big(H^{1,k}(x,\partial w_2^{\lambda_k})-H^{2,k}(x,\partial w_2^{\lambda_k})\big)$$
$$+\lambda_k W^k+c_k^{-1}\Big(F^{1,k}(x)-F^{2,k}(x)\Big).$$

By the third line in (3.3), we can rewrite (3.9) as

$$\begin{cases}-\mu_\alpha\partial^2 W^k+g^k\partial W^k+R^k=0, & x\in(\Gamma_\alpha\backslash\mathcal{V}),\alpha\in\mathcal{A},\\ \sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha W^k(\nu_i)=0, & \nu_i\in\mathcal{V},\\ W^k|_{\Gamma_\alpha}(\nu_i)=W^k|_{\Gamma_\beta}(\nu_i), & \alpha,\beta\in\mathcal{A}_i,\nu_i\in\mathcal{V},\end{cases}\tag{3.10}$$

where

$$g_\alpha^k(x)=\int_0^1\partial_p H_\alpha^{1,k}\Big(x,t\partial w_{1,\alpha}^{\lambda_k}(x)+(1-t)\partial w_{2,\alpha}^{\lambda_k}(x)\Big)\,dt$$

and we aim to pass to the limit in (3.10) for $k\to\infty$.

We first recall from the bound (3.6) and the third line in (3.3) that $w_i^{\lambda_k}\in C^{2,\theta}(\Gamma)$ with $\|w_i^{\lambda_k}\|_{C^{2,\theta}(\Gamma)}\le\bar K$ and respectively $\partial_p H_\alpha^{i,k}\in C^{0,\theta}(\Gamma_\alpha\times\mathbb{R})$ with $\|\partial_p H_\alpha^{i,k}\|_{C^{0,\theta}(\Gamma_\alpha\times\mathbb{R})}\le L$.

Then, the functions g_α^k, $\alpha\in\mathcal{A}$, are uniformly bounded and uniformly Hölder continuous of exponent θ. Hence, there exists $g:\Gamma\to\mathbb{R}$ such that for any $\alpha\in\mathcal{A}$

$$g_\alpha^k\to g_\alpha\qquad\text{for } k\to\infty,\quad\text{uniformly in }\ \Gamma_\alpha.\tag{3.11}$$

On the other hand, we claim that the functions R_α^k are uniformly θ-Hölder continuous, i.e. $[R_\alpha^k]_{\theta,\Gamma_\alpha}$ are uniformly bounded. In order to prove this claim, we first observe that, by assumptions (3.3), for every $x,y\in\Gamma_\alpha$, the function $h(x)=H_\alpha^{1,k}(x,\partial w_2^{\lambda_k}(x))-H_\alpha^{2,k}(x,\partial w_2^{\lambda_k}(x))$ fulfills

$$|h(x)-h(y)|/d_\Gamma(x,y)\le[H_\alpha^{1,k}(\cdot,\partial w_2^{\lambda_k}(x))-H_\alpha^{2,k}(\cdot,\partial w_2^{\lambda_k}(x))]_{1,\Gamma_\alpha}$$
$$+\|w_2^{\lambda_k}\|_{C^2(\Gamma)}[H_\alpha^{1,k}(y,\cdot)-H_\alpha^{2,k}(y,\cdot)]_{1,[-\bar K,\bar K]}$$
$$\le(1+\|w_2^{\lambda_k}\|_{C^2(\Gamma)})[H_\alpha^{1,k}(\cdot,\cdot)-H_\alpha^{2,k}(\cdot,\cdot)]_{1,\Gamma_\alpha\times[-\bar K,\bar K]}$$

where the first inequality is due to estimate (3.6). Hence, by (3.6) and the definition of c_k, we have

$$c_k^{-1}\left[H_\alpha^{1,k}(\cdot,\partial w_2^{\lambda_k}(\cdot))-H_\alpha^{2,k}(\cdot,\partial w_2^{\lambda_k}(\cdot))\right]_{1,\Gamma_\alpha}=c_k^{-1}[h]_{1,\Gamma_\alpha}$$
$$\leq\left(1+\|w_2^{\lambda_k}\|_{C^2(\Gamma)}\right)c_k^{-1}[H^1-H^2]_{1,\Gamma_\alpha\times[-\bar K,\bar K]}\leq 1+\|w_2^{\lambda_k}\|_{C^{2,\theta}(\Gamma)}\leq 1+\bar K.$$

On the other hand, by our choice of c_k, we have

$$\left[c_k^{-1}\left(F_\alpha^{1,k}-F_\alpha^{2,k}\right)\right]_{\theta,\Gamma_\alpha}=c_k^{-1}\left[F_\alpha^{1,k}-F_\alpha^{2,k}\right]_{\theta,\Gamma_\alpha}\leq 1;$$

hence, our claim is proved, i.e. $[R_\alpha^k]_{\theta,\Gamma_\alpha}$ are uniformly bounded.
We now claim that

$$\|R^k\|_{L^\infty(\Gamma)}=o_k(1)\qquad\text{as } k\to\infty \tag{3.12}$$

where $\lim_{k\to\infty}o_k(1)=0$, uniformly in x and may change from line to line. Indeed, we have

$$\lambda_k W^k=\lambda_k\frac{w_1^{\lambda_k}-w_2^{\lambda_k}}{\|w_1^{\lambda_k}-w_2^{\lambda_k}\|_{C^2(\Gamma)}}=o_k(1). \tag{3.13}$$

Moreover, we claim that, for $\bar K$ as in (3.6), there holds

$$\lambda_k\|v_1^{\lambda_k}-v_2^{\lambda_k}\|_{L^\infty}\leq\max_{\alpha\in\mathcal A}\left(\max_{(x,p)\in\Gamma_\alpha\times[-\bar K,\bar K]}|H_\alpha^{1,k}-H_\alpha^{2,k}|+\max_{x\in\Gamma_\alpha}|F_\alpha^{1,k}-F_\alpha^{2,k}|\right). \tag{3.14}$$

Indeed, to prove (3.14), it is sufficient to observe that

$$v_\pm(x):=v_2^{\lambda_k}(x)\pm\lambda_k^{-1}\max_{\alpha\in\mathcal A}\left(\max_{(x,p)\in\Gamma_\alpha\times[-\bar K,\bar K]}|H_\alpha^{1,k}-H_\alpha^{2,k}|+\max_{x\in\Gamma_\alpha}|F_\alpha^{1,k}-F_\alpha^{2,k}|\right)$$

are a subsolution and a supersolution of the equation satisfied by $v_1^{\lambda_k}$ and to apply Lemma 3.4. By (3.14) and $(H2)$, we have

$$\begin{aligned}|\lambda_k c_k^{-1}(\langle v_1^{\lambda_k}\rangle-\langle v_2^{\lambda_k}\rangle)|\leq c_k^{-1}\max_{\alpha\in\mathcal A}\Big(&\max_{(x,p)\in\Gamma_\alpha\times[-\bar K,\bar K]}|H_\alpha^{1,k}-H_\alpha^{2,k}|\\&+\max_{x\in\Gamma_\alpha}|F_\alpha^{1,k}-F_\alpha^{2,k}|\Big)\int_\Gamma dx=o_k(1).\end{aligned} \tag{3.15}$$

Furthermore, taking into account (3.6) and $(H2)$, we have

$$\begin{aligned}\frac{\left|H^{1,k}(x,\partial w_2^{\lambda_k})-H^{2,k}(x,\partial w_2^{\lambda_k})\right|}{c_k}&\leq\max_{\alpha\in\mathcal A}\max_{(x,p)\in\Gamma_\alpha\times[-\bar K,\bar K]}\frac{|H_\alpha^{1,k}-H_\alpha^{2,k}|}{c_k}\\&=o_k(1);\end{aligned}$$

by our choice of c_k, we also have

$$\left|c_k^{-1}\left(F_\alpha^{1,k}(x)-F_\alpha^{2,k}(x)\right)\right|\leq c_k^{-1}\max_{x\in\Gamma_\alpha}\left|F_\alpha^{1,k}(x)-F_\alpha^{2,k}(x)\right|\leq 1/k.$$

By these estimates, (3.13) and (3.15), we obtain the claim (3.12) and we conclude that for any $\alpha\in\mathcal A$

$$R_\alpha^k\to 0\qquad\text{for } k\to\infty,\ \text{ uniformly in }\Gamma_\alpha. \tag{3.16}$$

By definition, the functions W^k satisfy $\|W^k\|_{C^2(\Gamma)} = 1$. On the other hand, by (3.10), they are also uniformly bounded in $C^{2,\theta}(\Gamma)$. Hence, possibly passing to a subsequence that we still denote W^k, we have that, as $k \to \infty$, W^k uniformly converges to a function $W \in C^2(\Gamma)$ along with all its derivatives up to order 2. Moreover, taking into account (3.11) and (3.16), W is a solution to

$$\begin{cases} -\mu_\alpha \partial^2 W + g\partial W = 0, & x \in (\Gamma_\alpha \backslash \mathcal{V}), \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha W(\nu_i) = 0, & \nu_i \in \mathcal{V}, \\ W|_{\Gamma_\alpha}(\nu_i) = W|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}. \end{cases}$$

By Lemma 3.3, it follows that W is constant and, since $\langle W^k \rangle = 0$ for all k, then also $\langle W \rangle = 0$. It follows that $W \equiv 0$ which gives a contradiction to $\|W^k\|_{C^2(\Gamma)} = 1$ for all $k \in \mathbb{N}$.

The estimate for the solutions of the ergodic problem (3.1) follows immediately from Proposition 3.1-(ii) and since (3.8) is independent of λ. □

Remark 3.6 It is also possible to prove a L^∞-continuous dependence estimate. More precisely, there exists a constant C_0 such that

$$\|w_1^\lambda - w_2^\lambda\|_{L^\infty(\Gamma)} \le C_0 \max_{\alpha \in \mathcal{A}} \left(\max_{(x,p) \in \Gamma_\alpha \times [-\bar{K}, \bar{K}]} |H_\alpha^1 - H_\alpha^2| + \max_{x \in \Gamma_\alpha} |F_\alpha^1 - F_\alpha^2| \right). \quad (3.17)$$

Estimate (3.17) also holds for v_i, $i = 1, 2$, solution to (3.1) corresponding to $H(x, p) = H^i(x, p) + F^i(x)$.

The proof is similar (and simpler) as the one of Theorem 3.5 so we shall omit it and we refer the reader to [20, Theorem 2.1].

4 Quasi-stationary Mean Field Games on networks

Quasi-stationary Mean Field Games, introduced in [22] (see also [12]), modelize the case when the agent cannot predict the evolution of the population in the future, as in the classical MFG theory, but, at each instant, it decides its behaviour only on the basis of the information available at the current time. This feature leads to a system given by an evolutive Fokker-Planck equation and a stationary HJ equation (which in fact depends on time through the cost):

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x, \partial v) + \rho = F[m(t)](x), & (x,t) \in (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \alpha \in \mathcal{A}, \\ \partial_t m - \mu_\alpha \partial^2 m - \partial \left(m \partial_p H(x, \partial v) \right) = 0, & (x,t) \in (\Gamma_\alpha \backslash \mathcal{V}) \times (0,T), \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i, t) = 0, & (\nu_i, t) \in \mathcal{V} \times (0,T), \\ \sum_{\alpha \in \mathcal{A}_i} \mu_\alpha \partial_\alpha m(\nu_i, t) + n_{i\alpha} \partial_p H_\alpha(\nu_i, \partial v_\alpha(\nu_i, t)) m|_{\Gamma_\alpha}(\nu_i, t) = 0, & (\nu_i, t) \in \mathcal{V} \times (0,T), \\ v|_{\Gamma_\alpha}(\nu_i, t) = v|_{\Gamma_\beta}(\nu_i, t), \ \dfrac{m|_{\Gamma_\alpha}(\nu_i, t)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i, t)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i, (\nu_i, t) \in \mathcal{V} \times (0,T), \\ \langle v \rangle = 0, \ m(x, 0) = m_0(x), & x \in \Gamma. \end{cases} \quad (4.1)$$

Here, H is as in (3.2), $m_0 \in \mathcal{M}$ describes the initial distribution of the players and F is the nonlocal coupling cost (see below for the precise assumptions and see the Appendix for examples); moreover, at each time $t \in [0, T]$, given the distribution of the population $m(t)$, the representative agent assumes that it will not change in the future and solves an optimal control problem with long-run average cost functional

$$\rho(t)=\inf_a \liminf_{T\to\infty}\frac{1}{T}\mathbb{E}_{y,t}\left[\int_t^T (f(Y_s,a_s)+F[m(t)](Y_s))\,ds\right]$$

where Y_s is a "fictitious" dynamics on the network such $Y_t = x$ and F is an additional cost term which depends on the distribution of the agents. Note that Y_s is "fictitious" because it is *not* the trajectory really followed by the agent: is only the trajectory that they would follow if the distribution m of the population does not change after time t. If the corresponding HJ equation admits a smooth solution $v(t)$, then the optimal feedback law $a_t^\star(x) = -\partial_p H(x, \partial v(t))$ gives the vector field governing the evolution of the distribution of the population at time t and X_s obeys to the stochastic Eq. (2.2) with $B_{\alpha_s} = b_{\alpha_s}(X_s, \alpha_s^*(X_s))$.

In system (4.1), the first two lines are the standard differential equations for quasi-stationary MFG systems, the second two relations are the vertex transition conditions and the last two lines prescribe the behaviour of v and m at the vertices, the standard normalization condition for v and the initial datum of m. Note that the well-posedness of differential equations on networks relies on suitably chosen transition conditions; here these conditions for the FP equation are obtained by duality with respect to the corresponding ones for the HJ equation and express conservation of the flux and, respectively, a rule for the distribution of the density. Clearly, dealing with such transition conditions is the main novelty in the study of these equations on networks.

These quasi-stationary systems loss the standard forward-backward structure of MFG. In order to establish the existence of a solution, it is crucial to have some regularity in time for the value function v. In the classical approach for MFG, such a regularity follows from the parabolicity of HJ equation; here, it will be retrieved using the continuous dependence estimate of Sect. 3.

We first recall some basic results concerning the Fokker-Planck equation

$$\begin{cases}\partial_t m-\mu_\alpha\partial^2 m-\partial\,(bm)=0, & \text{in } (\Gamma_\alpha\backslash\mathcal{V})\times(0,T),\ \alpha\in\mathcal{A},\\ \sum_{\alpha\in\mathcal{A}_i}\mu_\alpha\partial_\alpha m(\nu_i,t)+n_{i\alpha}b_\alpha(\nu_i,t)m|_{\Gamma_\alpha}(\nu_i,t)=0, & t\in(0,T),\ \nu_i\in\mathcal{V},\\ \dfrac{m|_{\Gamma_\alpha}(\nu_i,t)}{\gamma_{i\alpha}}=\dfrac{m|_{\Gamma_\beta}(\nu_i,t)}{\gamma_{i\beta}}, & t\in(0,T),\ \alpha,\beta\in\mathcal{A}_i,\ \nu_i\in\mathcal{V}\backslash\partial\Gamma,\\ m\,(x,0)=m_0(x), & x\in\Gamma.\end{cases}\tag{4.2}$$

We introduce suitable parabolic spaces for weak solution of the FP equation. We set $V = H^1(\Gamma)$ and

$$H_b^1(\Gamma):=\left\{v:\Gamma\to\mathbb{R}\text{ s.t. } v_\alpha\in H^1(0,\ell_\alpha)\text{ for all }\alpha\in\mathcal{A}\right\},$$

(unlike V, continuity at the vertices is not required), endowed with the norm

$$\|v\|_{H_b^1(\Gamma)}=\left(\sum_{\alpha\in\mathcal{A}}\|\partial v_\alpha\|^2_{L^2(0,\ell_\alpha)}+\|v\|^2_{L^2(\Gamma)}\right)^{\frac{1}{2}}.$$

By Remark 2.1, for $v\in H_b^1(\Gamma)$, we still denote v_α the extension by continuity of v_α on the whole interval $[0,\ell_\alpha]$. We also define

$$W:=\left\{w:\Gamma\to\mathbb{R}:\ w\in H_b^1\,(\Gamma)\ \text{and}\ \frac{w|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}}=\frac{w|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}\quad\forall i\in I,\alpha,\beta\in\mathcal{A}_i\right\},$$

$$PC(\Gamma\times[0,T]):=\{v:\Gamma\times[0,T]\to\mathbb{R}:\ v(\cdot,t)\in PC(\Gamma)\text{ for all } t\in[0,T]\ \text{ and }\ v|_{\Gamma_\alpha\times[0,T]}\in C(\Gamma_\alpha\times[0,T])\ \text{ for all }\alpha\in\mathcal{A}\}.$$

Definition 4.1 For $m_0 \in L^2(\Gamma)$, a weak solution of (4.2) is a function $m \in L^2(0,T;W) \cap C([0,T];L^2(\Gamma))$ such that $\partial_t m \in L^2\left(0,T;V'\right)$ and

$$\begin{cases} \langle \partial_t m, v\rangle_{V',V} + \int_\Gamma \mu\, \partial m \partial v dx + \int_\Gamma bm\partial v dx = 0 \quad \text{for all } v \in H^1(\Gamma),\ \text{a.e. } t \in (0,T), \\ m(\cdot,0) = m_0. \end{cases}$$

The following result concerns existence, uniqueness and stability for the solution of (4.2) (see [2, Theorem 3.1 and Lemma 3.1]).

Proposition 4.2 *We have*

(i) For $b \in L^\infty(\Gamma \times (0,T))$, $m_0 \in L^2(\Gamma)$, there exists a unique weak solution to (4.2). Moreover, there exists $C = C(\|b\|_\infty)$ such that

$$\|m\|_{L^2(0,T;W)} + \|m\|_{L^\infty(0,T;L^2(\Gamma))} + \|\partial_t m\|_{L^2(0,T;V')} \le C\, \|m_0\|_{L^2(\Gamma)}. \tag{4.3}$$

Moreover, if $m_0 \in \mathcal{M}$, then $m(t) \in \mathcal{M}$ for all $t \in [0,T]$.

(ii) Let b^n be such that

$$b^n \longrightarrow b \text{ in } L^2(\Gamma \times (0,T)), \quad \|b\|_{L^\infty(\Gamma\times(0,T))}, \|b^n\|_{L^\infty(\Gamma\times(0,T))} \le K$$

with K independent of n. Let m^n (resp. m) be the solution of (4.2) corresponding to the coefficient b^n (resp. b). Then, the sequence (m^n) converges to m in $L^2(0,T;W) \cap L^\infty\left(0,T;L^2(\Gamma)\right)$, and the sequence $(\partial_t m^n)$ converges to $\partial_t m$ in $L^2\left(0,T;V'\right)$.

Proposition 4.3 *For $m_0 \in L^2(\Gamma) \cap \mathcal{M}$, let m be the solution of (4.2) found in Proposition 4.2. Then there exists a constant C_W, depending only on $\|b\|_{L^\infty}$ and $\|m_0\|_{L^2}$, such that*

$$\mathbf{d}_1(m(t), m(s)) \le C_W |t-s|^{\frac{1}{2}}. \tag{4.4}$$

Proof Let $\phi : \Gamma \to \mathbb{R}$ with $|\phi(x) - \phi(y)| \le d_\Gamma(x,y)$, hence $\phi \in H^1(\Gamma)$. For $s, t \in [0,T]$ with $s < t$, by Definition 4.1 and regularity of m, we have

$$\begin{aligned} &\int_\Gamma \phi(x)(m(t) - m(s))dx \le \int_s^t \int_\Gamma (\mu|\partial m|\,|\partial\phi| + m|b|\,|\partial\phi|)dxdr \\ &\le \|\mu\|_\infty \int_s^t \int_\Gamma |\partial m| dxdr + \|b\|_{L^\infty} \int_s^t \int_\Gamma m\, dxdr \\ &\le \|\mu\|_\infty \left[\int_s^t \int_\Gamma |\partial m|^2 dxdr\right]^{\frac{1}{2}} \left[\int_s^t \int_\Gamma 1 dxdr\right]^{\frac{1}{2}} + \|b\|_{L^\infty} \int_s^t \int_\Gamma m dxdr. \end{aligned}$$

Exploiting $\int_\Gamma m(r)dx = 1$ for any $r \in [0,T]$, (4.3) and (2.1), by the previous inequality we get (4.4). □

We now prove the well posedness of system (4.1).

Theorem 4.4 *Assume (H1), (H2) with $\theta = 1$ in the third line of (3.3), $m_0 \in L^2(\Gamma) \cap \mathcal{M}$, and $F : \mathcal{M} \to L^2(\Gamma)$ satisfying*

(F) $F_\alpha : \mathcal{M} \to C^{0,\theta}(0,\ell_\alpha)$, $\alpha \in \mathcal{A}$, and there exist $C_F > 0$ and $\theta \in (0,1]$ s.t.

$$\begin{aligned} &\max_\alpha \|F_\alpha[m]\|_{C^{0,\theta}(0,\ell_\alpha)} \le C_F, \\ &\max_\alpha \|F_\alpha[m_1] - F_\alpha[m_2]\|_{C^{0,\theta}(0,\ell_\alpha)} \le C_F\, \mathbf{d}_1(m_1, m_2) \end{aligned}$$

for all $m, m_1, m_2, \alpha \in \mathcal{A}$ (see Sect. 2 for the definition of $\|\cdot\|_{C^{0,\theta}}$).

Then, the system (4.1) *admits a unique solution* (u, ρ, m), *where* $(u, \rho) \in C([0, T], C^2(\Gamma)) \times C([0, T])$ *is a classical solution to the HJ equation for any* $t \in [0, T]$ *and* $m \in L^2(0, T; W) \cap C([0, T]; L^2(\Gamma) \cap \mathcal{M})$ *with* $\partial_t m \in L^2\left(0, T; V'\right)$ *is a weak solution to the FP equation.*

Proof

Existence: We consider the convex, compact set

$$\mathcal{X} = \left\{ m \in C([0, T]; \mathcal{M}) : \mathbf{d}_1(m(t), m(s)) \leq C_W |t - s|^{\frac{1}{2}}, s, t \in [0, T] \right\},$$

where C_W as in (4.4), and we define a map $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ in the following way: given $m \in \mathcal{X}$, let $(u(t), \rho(t))$, $t \in [0, T]$, be the solution of the HJ equation

$$\begin{cases} -\mu_\alpha \partial^2 u + H(x, \partial u) + \rho = F[m(t)](x), & x \in (\Gamma_\alpha \backslash \mathcal{V}), \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha u(\nu_i) = 0, & \nu_i \in \mathcal{V}, \\ u|_{\Gamma_\alpha}(\nu_i) = u|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}, \\ \langle u \rangle = 0 & x \in \Gamma. \end{cases} \tag{4.5}$$

Then, $\bar{m} = \mathcal{T}(m)$ solves the FP Eq. (4.2) with $b = \partial_p H(x, \partial u)$.

Note first that the map $\mathcal{T}$ is well defined. Indeed Proposition 3.1 ensures that, for each $t \in [0, T]$, there exists a unique couple $(u(t), \rho(t)) \in C^2(\Gamma) \times \mathbb{R}$, solution to the HJ in (4.5). Let us now prove: $(u, \rho) \in C([0, T], C^2(\Gamma)) \times C([0, T])$. To this end, we note that, by definition of $\mathcal{X}$ and assumption (F), there holds

$$\max_\alpha \|F_\alpha[m(t)] - F_\alpha[m(s)]\|_{C^{0,\theta}(0,\ell_\alpha)} \leq C_F C_W |t - s|^{\frac{1}{2}} \qquad \forall t, s \in [0, T], m \in \mathcal{X}.$$

Remark 3.2 and Theorem 3.5 entail respectively the continuity of ρ and of u w.r.t. time. Moreover, by Proposition 4.2, there exists a unique solution $\bar{m}$, in the sense of Definition 4.1, to problem (4.2) with $b = \partial_p H(\partial u)$. Since $\bar{m} \in C([0, T]; L^2(\Gamma))$ can be identified with the corresponding Borel measure with density $\bar{m}(t)$ on Γ at time t, by Proposition 4.3, we also have that $\mathcal{T}$ maps $\mathcal{X}$ into itself.

We prove that $\mathcal{T}$ is continuous. Given $m_n, m \in \mathcal{X}$, let $(u^n(t), \rho^n(t))$, $(u(t), \rho(t))$ be the solutions, for any $t \in [0, T]$, of the HJ Eq. (4.5) with right hand side $F[m^n(t)]$ and, respectively, $F[m(t)]$ and let $\bar{m}^n = \mathcal{T}(m^n)$, $\bar{m} = \mathcal{T}(m)$. If $m^n \to m$ in $\mathcal{X}$, then $\mathbf{d}_1(m^n(t), m(t)) \to 0$ uniformly for $t \in [(0, T]$. Invoking again Theorem 3.5, by (3.8) and (F), for any $t \in [0, T]$ there holds

$$\begin{aligned} \|u^n(t) - u(t)\|_{C^2(\Gamma)} &\leq C_0 \max_{\alpha \in \mathcal{A}} \|F_\alpha[m^n(t)] - F_\alpha[m(t)]\|_{C^{0,\theta}} \\ &\leq C \max_{\alpha \in \mathcal{A}} \mathbf{d}_1(m^n_\alpha(t), m_\alpha(t)), \end{aligned}$$

with C independent of m^n, m. The previous estimate and Proposition 4.2.(ii) with $b^n = \partial_p H(x, \partial u^n)$, $b = \partial_p H(x, \partial u)$ imply that $\bar{m}^n$ converges to $\bar{m}$ in $\mathcal{X}$ and therefore the map $\mathcal{T}$ is continuous.

By Schauder fixed point theorem, we conclude that there exists a fixed point of $\mathcal{T}$ and therefore a solution of (4.1).

Uniqueness: Suppose that there are two solutions (u_1, ρ_1, m_1), (u_2, ρ_2, m_2) of (4.1).

As in [2], we introduce the function $\varphi : \Gamma \to \mathbb{R}$ as

$$\varphi_\alpha \text{ is affine on } [0, \ell_\alpha], \qquad \varphi_\alpha(\nu_i) = \gamma_{i\alpha} \quad \text{if } \alpha \in \mathcal{A}_i.$$

Note that $\varphi \in H^1_b(\Gamma)$ is strictly positive and bounded. Hence the reciprocal φ^{-1} is well defined, positive and bounded; this property will play a crucial role in our argument.

Set $M = \varphi^{-1}(m_1 - m_2)$. The transition condition of m_i and the definition of φ ensure that $M(t) \in H^1(\Gamma)$ for a.e. $t \in (0, T)$. Hence, we can use Definition 4.1 for m_1 and m_2 with $M(t)$ as a test function obtaining

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(m_1 - m_2)(t)(\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)} + \|\partial(m_1 - m_2)(t)(\mu\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)} \\
&= -\int_\Gamma \mu\partial(m_1 - m_2)(t)(m_1 - m_2)(t)\partial(\varphi^{-1})dx \\
&- \int_\Gamma b_1(m_1 - m_2)(t)\partial M(t)dx - \int_\Gamma (b_1 - b_2)m_2(t)\partial M(t)dx
\end{aligned} \tag{4.6}$$

where $b_i = \partial_p H(\cdot, \partial u_i(\cdot, t))$ for $i = 1, 2$. We now estimate the three integrals in the right hand side of equality (4.6). Since now on, C will denote a constant that may change from line to line but is always independent of M. By Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&-\int_\Gamma \mu\partial(m_1 - m_2)(t)(m_1 - m_2)(t)\partial(\varphi^{-1})dx \\
&\le \int_\Gamma \left|\mu\partial(m_1 - m_2)(t)(m_1 - m_2)(t)\partial(\varphi^{-1})\varphi^{1/2}\varphi^{-1/2}\right| dx \\
&\le \frac{1}{2}\|\partial(m_1 - m_2)(t)(\mu\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_\Gamma \mu|m_1 - m_2|^2(t)|\partial(\varphi^{-1})|^2\varphi dx \\
&\le \frac{1}{2}\|\partial(m_1 - m_2)(t)(\mu\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)} + C\|(m_1 - m_2)(t)(\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)}
\end{aligned} \tag{4.7}$$

where the last inequality is due to the boundedness of μ and to the properties of φ. Moreover, by the boundedness of b_1 and of μ, again using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&-\int_\Gamma b_1(m_1 - m_2)(t)\partial M(t)dx \\
&= -\int_\Gamma b_1(m_1 - m_2)(t)[\partial(m_1 - m_2)(t)\varphi^{-1} + (m_1 - m_2)(t)\partial(\varphi^{-1})]dx \\
&\le \int_\Gamma \left|b_1(m_1 - m_2)(t)\partial(m_1 - m_2)(t)\varphi^{-1}\right| dx + \int_\Gamma \left|b_1(m_1 - m_2)^2(t)\partial(\varphi^{-1})\right| dx \\
&\le \frac{1}{4}\|\partial(m_1 - m_2)(t)(\mu\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)} + C\|(m_1 - m_2)(t)(\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)}.
\end{aligned} \tag{4.8}$$

Let us also assume for the moment the following estimate

$$\begin{aligned}
-\int_\Gamma (b_1 - b_2)m_2(t)\partial M(t)dx &\le \frac{1}{4}\|\partial(m_1 - m_2)(t)(\mu\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)} \\
&+C\|(m_1 - m_2)(t)(\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)}
\end{aligned} \tag{4.9}$$

whose proof is postponed at the end.

Replacing relations (4.7), (4.8), (4.9) in (4.6), we get

$$\frac{d}{dt}\|(m_1 - m_2)(t)(\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)} \le C\|(m_1 - m_2)(t)(\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)}.$$

Since $m_1(0) = m_2(0)$, by the previous inequality we get $m_1(t) = m_2(t)$ for all $t \in [0, T]$, hence $u_1 = u_2$ in $\Gamma \times [0, T]$ and $\rho_1 = \rho_2$.

It remains only to prove inequality (4.9). To this end, we first estimate

$$\|b_1 - b_2\|_{L^\infty(\Gamma)} = \|\partial_p H(\cdot, \partial u_1(\cdot, t)) - \partial_p H(\cdot, \partial u_2(\cdot, t))\|_{L^\infty(\Gamma)}$$

$$\leq C\|\partial u_1(\cdot,t)-\partial u_2(\cdot,t)\|_{L^\infty(\Gamma)}.$$

Moreover, applying Theorem 3.5 on $H^i(x,p)=H(x,p)-F[m_i(t)]$,we get

$$\|\partial u_1(\cdot,t)-\partial u_2(\cdot,t)\|_{L^\infty(\Gamma)}\leq C\,\mathbf{d}_1(m_1(t),m_2(t)).$$

By the last two inequalities, for $\delta=\mathbf{d}_1(m_1(t),m_2(t))$, we get $\|b_1-b_2\|_{L^\infty(\Gamma)}\leq C\delta$ and we deduce

$$\begin{aligned}\int_\Gamma |(b_1-b_2)m_2(t)\partial M(t)|\,dx &\leq C\int_\Gamma \delta\,|m_2(t)\partial M(t)|\,dx\\ &\leq \frac{1}{8}\int_\Gamma \mu|\partial M(t)|^2\varphi dx+C\int_\Gamma \frac{\delta^2|m_2(t)|^2}{\mu\varphi}dx.\end{aligned}\tag{4.10}$$

We denote I_1 and I_2 respectively the two integrals in the right hand side of the last inequality. We have

$$\begin{aligned}I_1 &\leq 2\int_\Gamma \left[\mu|\partial(m_1-m_2)|^2\varphi^{-1}+\mu|m_1-m_2|^2|\partial(\varphi^{-1})|^2\varphi\right]dx\\ &\leq 2\|\partial(m_1-m_2)(\mu\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)}+C\|(m_1-m_2)(\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)}.\end{aligned}$$

Moreover, since $m_2\in C([0,T],L^2(\Gamma))$, we have

$$I_2=C\delta^2\int_\Gamma |m_2|^2dx\leq C\delta^2\leq C\|m_1-m_2\|^2_{L^2(\Gamma)}\leq C\|(m_1-m_2)(\varphi^{-1})^{1/2}\|^2_{L^2(\Gamma)}$$

where we used the definition of δ and the properties of φ. Replacing these estimates for I_1 and I_2 in (4.10), we accomplish the proof of inequality (4.9). □

5 Homogenization of HJ equations defined on a lattice structure

In this section, we describe an application of the continuous dependence estimate in Sect. 3 to the study of a homogenization problem for a HJ equation defined on a periodic network.

For $\varepsilon\in(0,1]$, let Γ^ε be the periodic network generated by the lattice $\varepsilon\mathbb{Z}^N$. Hence $\mathcal{V}^\varepsilon=\varepsilon\mathbb{Z}^N$ and $\mathcal{E}^\varepsilon=\left\{\Gamma^\varepsilon_\alpha,\alpha\in\mathcal{A}^\varepsilon\right\}$, where

$$\Gamma^\varepsilon_\alpha=\{ym+(\varepsilon-y)n:\ y\in[0,\varepsilon]\}$$

for some $m,n\in\mathbb{Z}^N$ with $|m-n|=1$. Since Γ^ε is a lattice, there are $2N$ edges coming out of each vertex $\nu_i\in\mathcal{V}^\varepsilon$, in the directions of the vectors e_k of the canonical basis of $\mathbb{R}^N$ and in the opposite directions e_{-k}.

For $k\in\mathbb{Z}^N$, we define $\Gamma^\varepsilon_\alpha+k=\{y(m+k)+(\varepsilon-y)(n+k):\ y\in[0,\varepsilon]\}$ and we say that a function $\phi:\Gamma^1\to\mathbb{R}$ is Γ^1-periodic if

$$\phi_\beta=\phi_\alpha\quad\text{if}\ \ \Gamma^1_\beta=\Gamma^1_\alpha+k,k\in\mathbb{Z}^N.$$

On the network Γ^ε, we consider the problem

$$\begin{cases}-\mu_\alpha\partial^2u^\varepsilon+H_\alpha\left(x,\frac{x}{\varepsilon},\partial u^\varepsilon\right)+u^\varepsilon=0, & x\in\left(\Gamma^\varepsilon_\alpha\backslash\mathcal{V}^\varepsilon\right),\alpha\in\mathcal{A}^\varepsilon,\\ \sum\limits_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha u^\varepsilon(\nu_i)=0, & \nu_i\in\mathcal{V}^\varepsilon,\\ u^\varepsilon|_{\Gamma_\alpha}(\nu_i)=u^\varepsilon|_{\Gamma_\beta}(\nu_i), & \alpha,\beta\in\mathcal{A}^\varepsilon_i,\nu_i\in\mathcal{V}^\varepsilon.\end{cases}\tag{5.1}$$

We assume $H_\alpha:\mathbb{R}^N\times\Gamma^1_\alpha\times\mathbb{R}\to\mathbb{R}$ and

(i) there exist positive constants K, L, $\theta \in (0,1]$ and $\widetilde{H}_\alpha \in C(\mathbb{R}^N \times \Gamma^1_\alpha \times \mathbb{R})$, with $\widetilde{H}_\alpha(x,y,0) = 0$ such that

$$\sup_{x\in\mathbb{R}^N}\sup_{y\in\Gamma_\alpha}|H_\alpha(x,y,0)| \le K$$
$$H_\alpha(\cdot,\cdot,p) \in C^{0,1}(\mathbb{R}^N\times\Gamma_\alpha) \text{ with } [H_\alpha(\cdot,\cdot,p)]_{1,\mathbb{R}^N\times\Gamma_\alpha} \le L(1+|p|)$$
$$H_\alpha(x,y,\cdot) \in C^{1,\theta}(\mathbb{R}) \text{ with } \|\partial_p H_\alpha\|_{C^{0,\theta}} \le L$$
$$\lim_{\xi\to\infty}\frac{H_\alpha(x,y,\xi p)}{\xi} = \widetilde{H}_\alpha(x,y,p);$$

(ii) for every $(x,p) \in \mathbb{R}^N \times \mathbb{R}$, $H(x,\cdot,p)$ is Γ^1-periodic;
(iii) μ_α and $\gamma_{i,\alpha}$ only depend on the direction e_k parallel to Γ_α and $\gamma_{i,\alpha} = \gamma_\alpha$, for $\alpha \in \mathcal{A}$ and $i \in I$ and fulfill $(H1)$.

We denote with $\mathbb{S}^N$ the space of the symmetric $N \times N$ matrices.
We consider the effective problem

$$u + \bar{H}(x, Du, D^2u) = 0 \qquad x \in \mathbb{R}^N, \tag{5.2}$$

where the effective Hamiltonian $\bar{H}$ is defined as follows: for every $(x,P,X) \in \mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^N$ fixed, the value $\bar{H}(x,P,X)$ is equal to $-\rho$, where ρ is the unique constant (see Lemma 5.1 below) for which there exists a couple (v,ρ), with v Γ^1-periodic and $\rho \in \mathbb{R}$, solution to

$$\begin{cases} -\mu_\alpha\partial^2(v + Xy\cdot y/2) + H(x,y,\partial(P\cdot y)) + \rho = 0, & y \in \left(\Gamma^1_\alpha\backslash\mathcal{V}^1\right), \alpha \in \mathcal{A}^1, \\ \sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha v(\nu_i) = 0, & \nu_i \in \mathcal{V}^1, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha,\beta \in \mathcal{A}^1_i, \nu_i \in \mathcal{V}^1 \\ \int_{\Gamma^1} v(x)dx = 1. \end{cases} \tag{5.3}$$

To display the dependence of v with respect to (x,P,X), we will denote with $v(\cdot;x,P,X)$ the solution to (5.3).

We need some preliminary results whose proof are postponed to the Appendix; note that the bound (5.6) relies on the continuous dependence estimates of Sect. 3.

Lemma 5.1 *For any $(x,P,X) \in \mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^N$, there is a unique $\rho\in\mathbb{R}$ for which there exists a Γ^1-periodic solution to (5.3). Moreover*

$$\rho = -\frac{\sum_{k=1}^N \gamma_k\Big[-Xe_k\cdot e_k + \int_{e_k} H(x,y,P\cdot e_k)dy\Big]}{\sum_{k=1}^N\gamma_k} \tag{5.4}$$

and there exists a constant $\bar{C}_1$ such that

$$\|v(\cdot;x_1,P_1,X_1)\|_{C^{2,\theta}(\Gamma)} \le \bar{C}_1(1+|P_1|)$$
$$\|v(\cdot;x_1,P_1,X_1) - v(\cdot;x_2,P_2,X_2)\|_{L^\infty(\Gamma)} \le \bar{C}_1|P_1-P_2| \tag{5.5}$$
$$+\bar{C}_1|x_1-x_2|(1+|P_1|\wedge|P_2|) \tag{5.6}$$

for every $(x_1,P_1,X_1), (x_2,P_2,X_2) \in \mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^N$, where $\theta\in(0,1]$ as in (3.7).

Remark 5.2 The formula (5.4) entails that the effective operator $\bar{H}$ is convex and uniformly elliptic in X and there exists a constant $\bar{C}_1$ such that

$$\left|\bar{H}(x_1,P_1,X_1) - \bar{H}(x_2,P_2,X_2)\right| \le \bar{C}_1(|P_1-P_2| + |X_1-X_2|)$$

$$+\bar{C}_1|x_1 - x_2|(1 + |P_1| \wedge |P_2|)$$

for every $(x_1, P_1, X_1), (x_2, P_2, X_2) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^N$.

Lemma 5.3 *The problems* (5.1) *and* (5.2) *admit a unique bounded solution u^ε and, respectively, u. Moreover, there exists a constant C_0 such that*

$$\|u^\varepsilon\|_{C^{2,\theta}(\Gamma^\varepsilon)} \le C_0, \qquad \|u\|_{C^{2,\delta}(\mathbb{R}^N)} \le C_0 \tag{5.7}$$

for some θ, $\delta \in (0, 1]$.

Lemma 5.4 *If $g : \mathbb{R}^N \to \mathbb{R}$ is a smooth function, then it satisfies the Kirchhoff condition at $v_i \in \mathcal{V}^\varepsilon$ in* (5.1)*, i.e.*

$$\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha g(v_i) = 0.$$

Theorem 5.5 *Let u^ε and u be respectively the solution of* (5.1) *and* (5.2)*. Then, there exists a constant M such that for ε sufficiently small*

$$\|u^\varepsilon - u\|_{L^\infty(\Gamma^\varepsilon)} \le M\varepsilon^\delta,$$

where δ as in (5.7).

Proof Given $\varepsilon \in (0, 1)$, for $\eta \in (0, \infty)$ define the function

$$\phi(x) := u^\varepsilon(x) - u(x) - \varepsilon^2 v\left(\frac{x}{\varepsilon}; [u](x)\right) - \frac{\eta}{2}|x|^2 \qquad \forall x \in \Gamma^\varepsilon,$$

where $v(y; [u](x)) := v(y; x, Du(x), D^2u(x))$ is the solution of (5.3) with $(x, P, X) = (x, Du(x), D^2u(x))$. Since u, u^ε and v are bounded, there exists $\hat{x} \in \Gamma^\varepsilon$ where the function ϕ attains its maximum.

Set $c := 4\bar{C}_1(1 + 2C_0)\varepsilon^\delta$ (where $\bar{C}_1$, C_0 and δ are the constants introduced respectively in Lemma 5.1 and Lemma 5.3) and introduce the function

$$\tilde{\phi}(x) := u^\varepsilon(x) - u(x) - \varepsilon^2 v\left(\frac{x}{\varepsilon}; [u](\hat{x})\right) - \frac{\eta}{2}|x|^2 - c|x - \hat{x}|^2$$

where $|x - \hat{x}|$ is the standard Euclidean distance between x and $\hat{x}$. We have $\tilde{\phi}(\hat{x}) = \phi(\hat{x})$ and also, by the definition of $\hat{x}$,

$$\begin{aligned}\tilde{\phi}(\hat{x}) - \tilde{\phi}(x) &= [\tilde{\phi}(\hat{x}) - \phi(x)] + [\phi(x) - \tilde{\phi}(x)] \ge \phi(x) - \tilde{\phi}(x) \\ &\ge -\varepsilon^2 \left[v(\frac{x}{\varepsilon}; [u](x)) - v(\frac{x}{\varepsilon}; [u](\hat{x}))\right] + c\varepsilon^2\end{aligned}$$

for every $x \in \partial B(\hat{x}, \varepsilon) \cap \Gamma_\varepsilon$, where $B(\hat{x}, \varepsilon) = \{x \in \mathbb{R}^N : |x - \hat{x}| < \varepsilon\}$. Using the estimates in (5.6), Lemma 5.3 and recalling the definition of c, we get

$$\begin{aligned}\tilde{\phi}(\hat{x}) - \tilde{\phi}(x) \ge &- \bar{C}_1[2C_0\varepsilon^\delta + (1 + \|Du\|_\infty + \|D^2u\|_\infty)\varepsilon]\varepsilon^2 \\ &+ 4\bar{C}_1(1 + 2C_0)\varepsilon^{2+\delta} > 0\end{aligned}$$

for every $x \in \partial B(\hat{x}, \varepsilon) \cap \Gamma^\varepsilon$. Therefore, $\tilde{\phi}$ attains a maximum at some point $\tilde{x} \in B(\hat{x}, \varepsilon) \cap \Gamma^\varepsilon$. Let us prove that there exists a constant $M_1 > 0$ (independent of ε and η) such that

$$\eta^{\frac{1}{2}}|\tilde{x}| \le M_1. \tag{5.8}$$

By Lemma 5.1, Lemma 5.3 and the inequality $\phi(\hat{x}) \geq \phi(0)$, we obtain

$$\frac{\eta}{2}|\hat{x}|^2 \leq 4C_0 + 2\bar{C}_1(1+2C_0)\varepsilon^2.$$

We deduce that, for M_1 sufficiently large, we have $\eta^{1/2}|\hat{x}| \leq M_1/2$ and therefore

$$\eta^{\frac{1}{2}}|\tilde{x}| \leq \eta^{\frac{1}{2}}|\hat{x}| + \eta^{\frac{1}{2}}|\tilde{x}-\hat{x}| \leq \eta^{\frac{1}{2}}\frac{M_1}{2} + \eta^{\frac{1}{2}}\varepsilon \leq \eta^{\frac{1}{2}}M_1,$$

hence (5.8).
We claim that there exists a constant M (independent of ε and η) such that

$$u^\varepsilon(\tilde{x}) - u(\hat{x}) \leq M\left[\varepsilon^\delta + \eta^{1/2}\right]. \tag{5.9}$$

We first show that $\tilde{x} \notin \mathcal{V}^\varepsilon$. Indeed, assume by contradiction that $\tilde{x} = \nu_i \in \mathcal{V}^\varepsilon$. By adding the term $-d_\Gamma(x,\tilde{x})^2$, where d_Γ is the geodesic distance on the network, it is not restrictive to assume that $\tilde{x}$ is a strict maximum point for $\tilde{\phi}$ and therefore $\partial_\alpha\tilde{\phi}(\nu_i) > 0$ for all $\alpha \in \mathcal{A}_i^\varepsilon$ (recall the definition of ∂_α as the outward derivative at the vertex). Since u^ε and v solve respectively (5.1) and (5.2), by Lemma 5.4 we have

$$0 < \sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha\phi(\nu_i) = \sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha\left(u + \frac{\eta}{2}|x|^2 + c|x-\hat{x}|^2\right)_{x=\nu_i} = 0,$$

a contradiction and therefore $\tilde{x} \in (B(\hat{x},\varepsilon)\cap\Gamma^\varepsilon)\backslash\mathcal{V}^\varepsilon$. Let $\alpha \in \mathcal{A}$ be such that $\tilde{x} \in \Gamma_\alpha^\varepsilon$ and e_α a unit vector parallel to $\Gamma_\alpha^\varepsilon$. Since u^ε satisfies (5.1) and $\tilde{x}$ is a maximum point for $u^\varepsilon(x) - [u(x) + \varepsilon^2 v(x/\varepsilon;[u](\hat{x})) + \eta|x|^2/2 + c|x-\hat{x}|^2]$, we have

$$\begin{aligned} 0 \geq u^\varepsilon(\tilde{x}) &- \mu_\alpha\partial^2\left[u(x) + \varepsilon^2 v(\frac{x}{\varepsilon};[u](\hat{x})) + \frac{\eta}{2}|x|^2 + c|x-\hat{x}|^2\right]_{x=\tilde{x}} + \\ &H\Big(\tilde{x}, \frac{\tilde{x}}{\varepsilon}, \partial\left[u(x) + \varepsilon^2 v(\frac{x}{\varepsilon};[u](\hat{x})) + \frac{\eta}{2}|x|^2 + c|x-\hat{x}|^2\right]_{x=\tilde{x}}\Big). \end{aligned} \tag{5.10}$$

We compute

$$\begin{aligned} &\partial\big[u(x) + \varepsilon^2 v(\frac{x}{\varepsilon};[u](\hat{x})) + \frac{\eta}{2}|x|^2 + c|x-\hat{x}|^2\big]_{x=\tilde{x}} \\ =&Du(\tilde{x})\cdot e_\alpha + \varepsilon\partial_y v(\frac{\tilde{x}}{\varepsilon};[u](\hat{x})) + \eta\tilde{x}\cdot e_\alpha + 2c(\tilde{x}-\hat{x})\cdot e_\alpha, \end{aligned}$$

and

$$\begin{aligned} &\partial^2\big[u(x) + \varepsilon^2 v(\frac{x}{\varepsilon};[u](\hat{x})) + \frac{\eta}{2}|x|^2 + c|x-\hat{x}|^2\big]_{x=\tilde{x}} = \\ =&D^2u(\tilde{x})e_\alpha\cdot e_\alpha + \partial_y^2 v(\frac{\tilde{x}}{\varepsilon};[u](\hat{x})) + \eta + 2c. \end{aligned}$$

Replacing the previous identities in (5.10) and using Lemma 5.1, Lemma 5.3, (5.8) and $\tilde{x} \in B(\hat{x},\varepsilon)\cap\Gamma^\varepsilon$, we get

$$\begin{aligned} 0 \geq\, & u^\varepsilon(\tilde{x}) - \mu_\alpha\big(D^2u(\tilde{x})e_\alpha\cdot e_\alpha + \partial_y^2 v(\tilde{x}/\varepsilon;[u](\hat{x}))\big) + H\Big(\tilde{x}, \frac{\tilde{x}}{\varepsilon}, Du(\tilde{x})\cdot e_\alpha\Big) \\ &- M_2\big(\varepsilon\bar{C}_2(1+2C_0) + \eta^{1/2}M_1 + 2c\varepsilon + \eta + 2c\big) \geq u^\varepsilon(\tilde{x}) \\ &- \mu_\alpha\big(D^2u(\hat{x})e_\alpha\cdot e_\alpha + \partial_y^2 v(\tilde{x}/\varepsilon;[u](\hat{x}))\big) + H\Big(\hat{x}, \frac{\tilde{x}}{\varepsilon}, Du(\hat{x})\cdot e_\alpha\Big) \\ &- M_2(\varepsilon^\delta + \eta^{1/2}) = u^\varepsilon(\tilde{x}) + \bar{H}(\hat{x}, Du(\hat{x}), D^2u(\hat{x})) - M_2(\varepsilon^\delta + \eta^{1/2}) \\ &= u^\varepsilon(\tilde{x}) - u(\hat{x}) - M_2(\varepsilon^\delta + \eta^{1/2}) \end{aligned}$$

for some M_2, which may change from line to line but is always independent of ε and γ; hence (5.9). For every $x \in \Gamma^\varepsilon$, by $\tilde{\phi}(\tilde{x}) \geq \tilde{\phi}(\hat{x}) = \phi(\hat{x}) \geq \phi(x)$, we get by (5.9), Lemma 5.3 and Lemma 5.1

$$\begin{aligned} u^\varepsilon(x) - u(x) &\leq [u^\varepsilon(\tilde{x}) - u(\hat{x})] + [u(\hat{x}) - u(\tilde{x})] + \varepsilon^2 \left[v(x/\varepsilon; [u](x)) - v(\tilde{x}/\varepsilon; [u](\hat{x})) \right] \\ &\quad + \frac{\eta}{2}|x|^2 \\ &\leq M_2 \left[\varepsilon^\delta + \eta^{1/2} \right] + C_0\varepsilon + 2\bar{C}_2(1 + 2C_0)\varepsilon^2 + \frac{\eta}{2}|x|^2. \end{aligned}$$

Letting $\eta \to 0^+$, we deduce

$$u^\varepsilon(x) - u(x) \leq M_2\varepsilon^\delta \qquad \forall x \in \Gamma^\varepsilon.$$

Reversing the role of u and u^ε, we get the statement. □

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A Appendix

Proof of Proposition 3.1 We just sketch this proof because it is an adaptation of the previous results [5, Theorem II.2], [10] and [1, Proposition 3.2 and Theorem 3.7].

The result in [1, Proposition 3.2] ensures that problem (3.4) admits a solution $v^\lambda \in C^{2,\theta}(\Gamma)$.

(i). The comparison principle yields: $\|\lambda v^\lambda\|_\infty \leq K$. The function $w^\lambda := v^\lambda - \langle v^\lambda \rangle$ solves

$$\begin{cases} -\mu_\alpha \partial^2 w^\lambda + H\left(x, \partial w^\lambda\right) + \lambda w^\lambda + \lambda \langle v^\lambda \rangle = 0, & x \in (\Gamma_\alpha \backslash \mathcal{V}), \alpha \in \mathcal{A}, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha w^\lambda(\nu_i) = 0, & \nu_i \in \mathcal{V}, \\ w^\lambda|_{\Gamma_\alpha}(\nu_i) = w^\lambda|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}. \end{cases} \tag{A.1}$$

We claim that there exists $K_* \in \mathbb{R}$ such that $\|w^\lambda\|_\infty \leq K_*(1 + K)$. Indeed, in order to prove this claim, we proceed by contradiction assuming that there exist λ_k, with $\lambda_k \to 0$ as $k \to \infty$, such that $\|w^{\lambda_k}\|_\infty \geq k(1 + K)$. Then, the function $W_k = w^{\lambda_k}/\|w^{\lambda_k}\|_\infty$ is zero somewhere, fulfills $\|W_k\|_\infty = 1$ and, for $x \in (\Gamma_\alpha \backslash \mathcal{V})$ and $\alpha \in \mathcal{A}$,

$$-\mu_\alpha \partial^2 W_k + \partial W_k \int_0^1 \partial_p H(x, t\partial w^{\lambda_k})dt + \frac{H(x, 0) + \lambda_k \langle v^{\lambda_k} \rangle}{\|w^{\lambda_k}\|_\infty} + \lambda_k W_k = 0$$

with the same transition conditions as in (A.1). We observe that

$$\lim_{k\to\infty} \frac{H(x, 0) + \lambda_k \langle v^{\lambda_k} \rangle}{\|w^{\lambda_k}\|_\infty} + \lambda_k W_k = 0$$

uniformly in x and that, by standard arguments, the W_k are uniformly bounded in C^2. Hence, eventually passing to a subsequence, W_k converges to some function W such that it is zero somewhere and $\|W\|_\infty = 1$. On the other hand, dividing the differential equation in (A.1) and letting $k \to \infty$, by stability, we get

$$-\mu_\alpha \partial^2 W + \widetilde{H}_\alpha(x, \partial W) = 0$$

with the same transition conditions as in (A.1). The maximum principle entails that W is constant; the desired contradiction is achieved. In conclusion, arguing as before, we have that $-\mu_\alpha \partial^2 w_\lambda + \partial w_\lambda \int_0^1 \partial_p H(x, t\partial w^\lambda)dt$ are uniformly bounded in $C^{0,\theta}$ uniformly in λ. By standard arguments we achieve the proof.
(ii). It is an easy consequence of point (i). □

Proof of Lemma 5.1 The proofs of existence and uniqueness of such a ρ and of relation (5.5) rely on an easy adaptation of standard techniques; we refer the reader to [4, 5, 13]. Relation (5.6) is due to (3.17) and (5.5).

We prove an explicit formula for ρ. Recall that, by the assumptions, there are N different Hamiltonians H_k and N viscosity coefficients $\mu_k, k = 1, \dots, N$, corresponding to the vectors e_k. Integrating the HJ equation in (5.3) along the N arcs Γ_k parallel to e_k exiting from ν_i and denoting with μ_k, γ_k the corresponding coefficients in the Kirchhoff condition, we have

$$0 = \sum_{k=1}^{N} \gamma_k \int_{e_k} [-\mu_k \partial^2_{e_k}(v + Xy \cdot y/2) + H(x, y, \partial_{e_k}(P \cdot y)) + \rho]dy. \tag{A.2}$$

By periodicity of v, we have

$$\int_{e_k} \partial^2_{e_k} v(y)dy = \partial_{e_k} v(1) - \partial_{e_k} v(0) = -\partial_{-e_k} v(0) - \partial_{e_k} v(0).$$

Replacing the previous identity in (A.2), we have

$$\begin{aligned} 0 = &\sum_{k=1}^{N} \gamma_k \mu_k [\partial_{-e_k} v(0) + \partial_{e_k} v(0)] + \sum_{k=1}^{N} \gamma_k [X e_k \cdot e_k \\ &+ \int_{e_k} H(x, y, P \cdot e_k)dy] + \rho \sum_{k=1}^{N} \gamma_k. \end{aligned}$$

Therefore, taking into account the Kirchhoff condition in (5.3) at $\nu_i = 0$ and observing that $\gamma_k = \gamma_{-k}$, where γ_{-k} is the coefficient γ_α for the arc $-e_k$, we get (5.4).

□

Proof of Lemma 5.3 The statement is obtained by standard arguments. For problem (5.1), we refer the reader to [6, Theorem 14.5.1], [19] and [21] (note that these papers tackle infinite networks). For problem (5.2) we refer to [4, 5]. □

Proof of Lemma 5.4 Let $g : \mathbb{R}^N \to \mathbb{R}$ be a smooth function. Since $\gamma_{i\alpha} = \gamma_\alpha$ and $\gamma_\alpha, \mu_\alpha$ only depend on the direction e_k, parallel to Γ_α, we have

$$\begin{aligned} \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha g(\nu_i) &= Dg(\nu_i) \cdot \sum_{k=1}^{N} (\gamma_k \mu_k e_k + \gamma_k \mu_k e_{-k}) \\ &= Dg(\nu_i) \cdot \sum_{k=1}^{N} (\gamma_k \mu_k e_k - \gamma_k \mu_k e_k) = 0. \end{aligned}$$

□

Example A.1 [Example of coupling F] We provide an example of coupling for the MFG system (4.1) which fulfills the assumption (F) of Theorem 4.4. To this end, we shall borrow some ideas of [3, Example 3.4] and of [18, pag.238].

Consider the coupling F defined by: for any $m \in \mathcal{M}$,

$$F[m](x) = \int_{\Gamma} K(x, y)m(dy) \qquad \forall x \in \Gamma$$

where the function $K : \Gamma \times \Gamma \to \mathbb{R}$ fulfills, for some positive constants k_0, k_1, k_2,

$$\begin{aligned}
&K(x, y) = K(y, x), \quad 0 \le K(x, y) \le k_0 \\
&K(x, \cdot) \in C^1(\Gamma) \text{ with } \|K(x, \cdot)\|_{C^1(\Gamma)} \le k_1 \\
&\frac{K(x, \cdot) - K(x', \cdot)}{d_\Gamma(x, x')} \in C^{0,1}(\Gamma) \text{ with } \|K(x, \cdot) - K(x', \cdot)\|_{C^{0,1}(\Gamma)} \le k_1 d_\Gamma(x, x')
\end{aligned}$$

for every $x, x', y \in \Gamma$.

Then, the coupling F satisfies assumption (F) with $\theta = 1$ and $C_F = k_0 + k_1 + k_2$.

Example A.2 [Example of Hamiltonian H] We provide an example of Hamiltonian H which fulfills the assumptions of Theorem 4.4 (in particular, assumption $(H2)$). Consider H_α as in (3.2) with

$$\begin{aligned}
&A_\alpha = \{a \in \mathbb{R}^d : |a| \le R_\alpha\} \\
&b_\alpha(x, a) = \widetilde{b}_\alpha(x) - a \qquad \text{and} \qquad f_\alpha(x, a) = \frac{|a|^2}{2\widetilde{f}_\alpha(x)}.
\end{aligned}$$

where $R_\alpha > 0$ and $\widetilde{b}_\alpha, \widetilde{f}_\alpha \in C^{1,\theta}(\Gamma_\alpha)$, $\widetilde{f}_\alpha > 0$. By standard calculations, the Hamiltonian H_α reads

$$H_\alpha(x, p) = \begin{cases} \widetilde{f}_\alpha(x)\dfrac{|p|^2}{2} - \widetilde{b}_\alpha(x)p & \text{for } \; |p| \le \dfrac{R_\alpha}{\widetilde{f}_\alpha(x)} \\ R_\alpha|p| - \dfrac{R^2}{2\widetilde{f}_\alpha(x)} - \widetilde{b}_\alpha(x)p & \text{for } \; |p| > \dfrac{R_\alpha}{\widetilde{f}_\alpha(x)}. \end{cases}$$

and fulfills the assumptions of Theorem 4.4. Moreover, in this case, we also have an explicit formula for the optimal feedback operator: for $x \in \Gamma_\alpha$

$$a^\star(x, p) = \begin{cases} \widetilde{f}_\alpha(x)p - \widetilde{b}_\alpha(x) & \text{for } \; |p| \le \dfrac{R_\alpha}{\widetilde{f}_\alpha(x)} \\ R_\alpha\dfrac{p}{|p|} - \widetilde{b}_\alpha(x) & \text{for } \; |p| > \dfrac{R_\alpha}{\widetilde{f}_\alpha(x)}. \end{cases}$$

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