HOMOGENIZATION OF COMPOSITE MEDIA WITH NON-STANDARD TRANSMISSION CONDITIONS

M. AMAR[§] – A. AYUB[†] – R. GIANNI[‡]

§DIPARTIMENTO DI SCIENZE DI BASE E APPLICATE PER L'INGEGNERIA
SAPIENZA - UNIVERSITÀ DI ROMA
VIA A. SCARPA 16, 00161 ROMA, ITALY

‡GOVERNMENT DEGREE COLLEGE FOR WOMEN
AYUB RESEARCH, FAISALABAD, PUNJAB 38000, PAKISTAN

‡DIPARTIMENTO DI MATEMATICA ED INFORMATICA
UNIVERSITÀ DI FIRENZE
VIA SANTA MARTA 3, 50139 FIRENZE, ITALY

Abstract. In this paper we study the homogenization limits for the steady state of a diffusion problem in a composite medium made up by two different materials: a host material and the inclusion material which is disposed in a periodic array and has a typical length scale ε. On the interface separating the two phases two different sets of non-standard transmission conditions are assigned (thus originating two different systems of partial differential equations). As ε tends to zero a hierarchy of homogenization problem related to such interface conditions is studied and the physical properties of the various limits are discussed.

Keywords: Homogenization, asymptotic expansions, elliptic problems.
AMS-MSC: 35B27, 35C20, 35J47.

Acknowledgments: The first author is member of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). The second author wishes to thank *Dipartimento di Scienze di Base e Applicate per l'Ingegneria* for the warm hospitality and *Università "La Sapienza" of Rome* for the financial support, during her PhD program.

1. Introduction

The study of the transmission conditions through an interface separating two different media plays a fundamental role in the modelling of a large number of physical problems. As possible examples, we can refer to elasticity, electric conduction and heat diffusion, when two media with different material properties are bonded together and we have to investigate the evolution in time of the corresponding state variables (i.e. displacement, electric potential, temperature) in such a composite. From a mathematical point of view, we have to establish the interface conditions satisfied by the unknown on the surface separating the two phases. Clearly, these interface conditions will depend on how the junction is realized. For instance, in [28, 29, 31, 32, 33], several types of interface conditions are obtained, assuming that the two media are separated by a third one of vanishing thickness. Hence, the various transmission conditions are recovered in correspondence of the different physical properties of the vanishing phase. Adopting an alternative perspective, in [7, 8], new sets of interface conditions, related to the ones in [29, 31, 33], are derived via concentration, assuming that the interface is the limit of a thick membrane, which in turn is a composite: i.e. it is a wafer made up of two materials with different physical properties. These two new sets of transmission conditions are investigated, in the present paper, from the point of view of homogenization, following an approach perused, for instance, in [3, 4, 5, 11, 15, 16, 22, 23, 24, 25, 26, 27]. Namely, having in mind the study of heat diffusion (even though, for the sake of simplicity, in the time independent case), we will produce the rigorous homogenization limit for a composite periodic material made up of two different media on whose contact layer equations (3.1c)–(3.1d) or (3.2c)–(3.2d) are satisfied. We will study both the case in which the periodic inclusion of the embedded material, as well as the host material are connected (to which we refer as the connected/connected case) and the case in which the inclusion is made of disconnected periodic cells (to which we refer as the connected/disconnected case). As usual in this kind of problems, the interface conditions must be properly rescaled, multiplying them by a suitable power of ε (ε being the vanishing length scale of the inclusions). It is worthwhile noting that homogeneity reasons dictate that different power of ε multiply the various terms in the interface conditions, in a way such that the difference between the various powers matches the different orders of derivation of the unknown. Hence, if the terms accounting for the jump and the average of the state variable are multiplied by ε^m, the Beltrami Laplacian of the same quantities will be multiplied by ε^{m+2} (see (3.1d) and (3.2c)–(3.2d)). As already pointed out in the literature (see, for instance, [3, 4, 8, 9, 10, 12, 22, 23]), we recognize that only three different regimes are relevant in the homogenization, corresponding to the choice of $m = 0; \pm 1$. For this reason, in this paper, we will study a hierarchy of problems corresponding to these three choices of m and this will be done for both the two different sets of interface boundary conditions. It turns out that the case $m = -1$ is the only one that preserves in the limit all the physical properties of the interface. Consequently, in this situation, we also have that homogenizing the two different sets of transmission conditions leads to two different limit problems. On the contrary, in the case $m = 0, 1$, both sets of interface conditions lead to the same limit problem. More precisely, for $m = 0$ we obtain an elliptic equation in which no trace of the physical properties of the active interface is preserved; while, in the case $m = 1$, we get a bidomain model where the transversal diffusivity of the interface, represented by α (see equations (3.1d) and (3.2d) below), is the only physical property still appearing in the homogenization limit. We recall that having a bidomain problem in this homogenization procedure means that the unknown state variable in the inclusions tends to a limit (as ε tends to zero) which is different from the limit of the same state variable restricted to the hosting material. Both these limits solve different coupled partial differential equations. The previous results are in agreement with the literature cited above and the scalings used therein. We notice that an attractive part of this paper is due to the unexpected result that, for $m = 0$ (see Remark 6.5), as well as for $m = 1$ (see Remark 6.8), homogenizing the two different sets of boundary conditions (3.1c)–(3.1d) and (3.2c)–(3.2d) leads to the same limit problem. Only for $m = -1$, the resulting homogenized problems are different and maintain a strong affinity with the original microscopic models (see (5.8a)–(5.8e) and (6.2a)–(6.2e)). Moreover, only for such a scaling, the limits keep into account the transversal as well as the tangential diffusion properties of the interface.

The paper is organized as follows. Section 2 is devoted to introduce our notations and the geometrical, as well as the functional, settings. In Section 3, we introduce the two differential problems studied in this paper. Section 4 recalls the main properties of the unfolding operator. Finally, Sections 5 and 6 contain our main results, i.e. the homogenization of the two differential problems, in both the geometries and for all the scalings considered.

2. Notations and preliminaries

2.1. **Notations.** In the following, we will assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded open set with smooth boundary $\partial\Omega$ and $Y = (0,1)^N$ will denote the reference unit cell in $\mathbb{R}^N$.
The set $C_c^\infty(\Omega)$ will denote the subset of the functions belonging to the standard space $C^\infty(\Omega)$ with compact support in Ω, as well as $C_{per}^\infty(Y)$ will denote the set of the Y-periodic functions in $C^\infty(\mathbb{R}^N)$.
Moreover, $H^1(\Omega)$, $H_0^1(\Omega)$ and $H_{loc}^1(\Omega)$ will denote the usual Sobolev spaces; $L_{per}^p(Y)$ will denote the set of the Y-periodic functions in $L_{loc}^p(\mathbb{R}^N)$ and $H_{per}^1(Y)$ is the set of the Y-periodic functions in $H_{loc}^1(\mathbb{R}^N)$.
Finally, C will be a strictly positive constant, which may vary from line to line.

2.2. **Beltrami Differential Operators.** Given a function $\phi \in C^1(\Omega)$ and a smooth surface $S \subset \Omega \subseteq \mathbb{R}^N$, we denote by $\nabla^B\phi$ the tangential gradient of ϕ on S, i.e. the projection of $\nabla\phi$ on the tangent hyper-plane to S, defined by

$$\nabla^B\phi := \nabla\phi - (n \cdot \nabla\phi)\, n, \tag{2.1}$$

where n is the normal unit vector to S and ∇ is the classical gradient.
Taking into account the smoothness of S, the normal vector n can be naturally defined in a small neighbourhood of S as $\frac{\nabla d}{|\nabla d|}$, where d is the signed distance from S, so that, given a vector valued function $\Phi \in C^1(\Omega)$, we can define the tangential divergence of Φ on S as

$$\mathrm{div}^B\Phi := \mathrm{div}(\Phi - (n \cdot \Phi)n) = \mathrm{div}\,\Phi - (n \cdot \nabla\Phi_i)n_i - (\mathrm{div}\, n)(n \cdot \Phi). \tag{2.2}$$

Moreover, from (2.1) and (2.2), for a given scalar function $\phi \in C^2(\Omega)$, we can define the Laplace-Beltrami operator $\Delta^B\phi$ as

$$\begin{aligned}\Delta^B\phi &:= \mathrm{div}^B\left(\nabla^B\phi\right) = \Delta\phi - n^t\,\nabla^2\phi\, n - (n \cdot \nabla\phi)\,\mathrm{div}\, n \\ &= (\delta_{ij} - n_i n_j)\,\partial_{ij}^2\phi - (n_i\partial_i\phi)\,(\partial_j n_j),\end{aligned} \tag{2.3}$$

where $\nabla^2\phi$ denotes the Hessian matrix of ϕ, δ_{ij} is the Kronecker delta and, as usual, we sum with respect to repeated indexes.
Finally, we recall that, if S is a regular surface with no boundary, i.e. $\partial S = \emptyset$, we have

$$\int_S \mathrm{div}^B \Phi \, d\sigma = 0. \tag{2.4}$$

2.3. **Geometrical Settings.** In this paper, we will consider two different geometrical settings, according to the pictures displayed in Figure 1 and Figure 2. Let $E \subset \mathbb{R}^N$ be a periodic open set, i.e. $E + z = E$ for all $z \in \mathbb{Z}^N$. For the sake of simplicity, we will assume that the boundaries $\partial\Omega$ and ∂E are of class C^∞. Following the same notations as in [8], we set $E^{int} := E \cap Y$, $E^{out} := Y \backslash \bar{E}$ and $\Gamma := \partial E \cap Y$, so that $Y = E^{int} \cup E^{out} \cup \Gamma$; moreover, for the sake of simplicity, we will assume that E^{int} is connected. Let $\varepsilon \in (0, 1]$ be the small parameter accounting for the ratio between the micro and the macro length-scale, which will be let converge to zero. We denote the inner and the outer conductive phase by $\Omega_\varepsilon^{int} := \Omega \cap \varepsilon E$ and $\Omega_\varepsilon^{out} := \Omega \backslash \varepsilon \overline{E} = \Omega \backslash \overline{\Omega_\varepsilon^{int}}$, respectively, and the interface by $\Gamma_\varepsilon := \partial\Omega_\varepsilon^{int} \cap \Omega = \partial\Omega_\varepsilon^{out} \cap \Omega$, so that $\Omega = \Omega_\varepsilon^{int} \cup \Omega_\varepsilon^{out} \cup \Gamma_\varepsilon$.
We assume also that Ω_ε^{out} is connected at each step $\varepsilon > 0$, whereas Ω_ε^{int} will be connected or disconnected. Indeed, we will consider the following two different situations:

- **connected-disconnected case**: in this case, we assume that $\Gamma \cap \partial Y = \emptyset$, that is the boundary of E does not touch the boundary of the unit cell Y (see Figure 1). Here, the domain Ω is the union of the connected domain Ω_ε^{out}, the disconnected domain Ω_ε^{int} and the common boundary Γ_ε. We also assume that the cells intersecting the boundary $\partial\Omega$ do not contain any inclusion, so that we have $dist(\Gamma_\varepsilon, \partial\Omega) \geq C_0\varepsilon$, for some suitable constant $C_0 > 0$ independent of ε.
- **connected-connected case**: in this case, we assume that $\partial E \cap Y \neq \emptyset$, but $|\partial E \cap Y|_{N-1} = 0$ (where $|\cdot|_{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure). In this situation, we stipulate that E^{int}, E^{out}, Ω_ε^{int}, and Ω_ε^{out} are connected and, without any loss of generality, that they have Lipschitz continuous boundary (at least for a suitable choice of a subsequence $\varepsilon_n \to 0$). In this case, at each level $\varepsilon > 0$, we have that both $\partial\Omega \cap \partial\Omega_\varepsilon^{int}$ and $\partial\Omega \cap \partial\Omega_\varepsilon^{out}$ are non empty.

Finally, let ν be the normal unit vector to Γ pointing into E^{out}, extended by periodicity to the whole of $\mathbb{R}^N$, so that, $\nu_\varepsilon(x) = \nu(\frac{x}{\varepsilon})$ denotes the normal unit vector to Γ_ε pointing into Ω_ε^{out}.

2.4. **The spaces $\mathcal{H}_{0,m}^\varepsilon(\Omega)$ and $\widehat{\mathcal{H}}_{0,m}^\varepsilon(\Omega)$.** In this subsection, we introduce the proper functional settings needed for the well-posedness of the ε-microscopic problems studied in this paper. Given a function $u : \Omega \to \mathbb{R}$, we denote by u^{int} and u^{out} its restriction to Ω_ε^{int} and Ω_ε^{out}, respectively, and, with abuse of notation, the same symbols will be used for the corresponding traces on Γ_ε. Furthermore, we denote by $[u]$ the jump of u across the interface Γ_ε, i.e.

$$[u] = u^{out} - u^{int}, \tag{2.5}$$

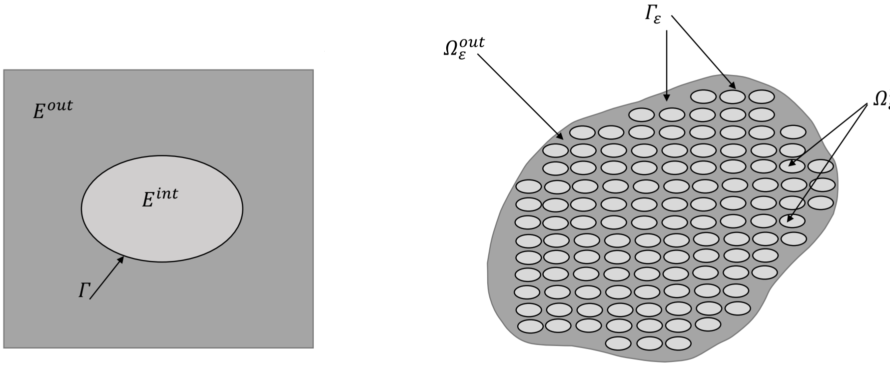

FIGURE 1. Macro and microscopic view of the periodic structure in the connected-disconnected geometrical settings. The region bounded by the polygonal will be denoted by $\widehat{\Omega}_\varepsilon$.

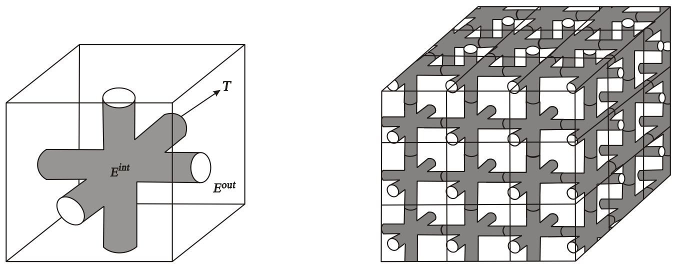

FIGURE 2. Macro and microscopic view of the periodic structure in the connected-connected geometrical settings.

and similarly, $\{u\}$ denotes the sum of u^{int} and u^{out} at the interface Γ_ε, i.e.

$$\{u\} = u^{out} + u^{int}. \tag{2.6}$$

The same notations will be used for other quantities. Notice that, from (2.5) and (2.6), u^{out} and u^{int} can be rewritten as

$$u^{out} = \frac{1}{2}\left(\{u\} + [u]\right), \qquad \text{and} \qquad u^{int} = \frac{1}{2}\left(\{u\} - [u]\right).$$

Definition 2.1. Given $\varepsilon \in (0,1]$ and recalling that the space $H^1(\Gamma_\varepsilon)$ is defined as the set of functions $v \in L^2(\Gamma_\varepsilon)$ such that $\nabla^B v \in L^2(\Gamma_\varepsilon)$, for $m = 0; \pm 1$, we set

$$\begin{aligned}\mathcal{H}^{\varepsilon}_{0,m}(\Omega) := \{u = (u^{int}, u^{out}) : u^{int} \in H^1(\Omega^{int}_\varepsilon),\ u^{out} \in H^1(\Omega^{out}_\varepsilon),\\ [u] \in H^1(\Gamma_\varepsilon),\ u = 0 \ on\ \partial\Omega\}\end{aligned} \tag{2.7}$$

and

$$\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega) := \{u = (u^{int}, u^{out}) \, : \, u^{int} \in H^1(\Omega^{int}_{\varepsilon}),\, u^{out} \in H^1(\Omega^{out}_{\varepsilon}), \\ [u], \{u\} \in H^1(\Gamma_{\varepsilon}), u = 0 \;\, on \; \partial\Omega\}, \tag{2.8}$$

endowed, respectively, with the norms

$$||u||^2_{\mathcal{H}^{\varepsilon}_{0,m}(\Omega)} := \, ||\nabla u^{int}||^2_{L^2(\Omega^{int}_{\varepsilon})} \, + \, ||\nabla u^{out}||^2_{L^2(\Omega^{out}_{\varepsilon})} \, + \, \varepsilon^m \, || \, [u] \, ||^2_{L^2(\Gamma_{\varepsilon})} \\ + \, \varepsilon^{m+2} \, || \, \nabla^B \, [u] \, ||^2_{L^2(\Gamma_{\varepsilon})} \tag{2.9}$$

and

$$||u||^2_{\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)} := \, ||\nabla u||^2_{L^2(\Omega^{int}_{\varepsilon})} \, + \, ||\nabla u||^2_{L^2(\Omega^{out}_{\varepsilon})} \, + \, \frac{\varepsilon^m}{2} \, || \, [u] \, ||^2_{L^2(\Gamma_{\varepsilon})} \\ + \, \frac{\varepsilon^{m+2}}{2} \, || \, \nabla^B [u] \, ||^2_{L^2(\Gamma_{\varepsilon})} + \, \frac{\varepsilon^{m+2}}{2} \, || \, \nabla^B \{u\} \, ||^2_{L^2(\Gamma_{\varepsilon})}. \tag{2.10}$$

□

Notice that the space $\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)$ coincides with the space of the piecewise H^1-functions in $\Omega^{int}_{\varepsilon}$ and $\Omega^{out}_{\varepsilon}$, with zero boundary value, whose traces on Γ_{ε} from $\Omega^{int}_{\varepsilon}$ and $\Omega^{out}_{\varepsilon}$ belong to the space $H^1(\Gamma_{\varepsilon})$. Moreover, $\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega) \subsetneqq \mathcal{H}^{\varepsilon}_{0,m}(\Omega)$. We recall also that, for $u \in \mathcal{H}^{\varepsilon}_{0,m}(\Omega)$ the following Poincaré inequality holds (see [30, Lemma 6 complemented with Lemma 4], for the connected/disconnected case, and [2, Lemma A.4], for the connected/connected case):

$$\|u\|^2_{L^2(\Omega)} \leq C \left(\|\nabla u\|^2_{L^2(\Omega^{out}_{\varepsilon})} + \|\nabla u\|^2_{L^2(\Omega^{int}_{\varepsilon})} + \varepsilon \|[u]\|^2_{L^2(\Gamma_{\varepsilon})} \right) , \tag{2.11}$$

where the constant C is independent of ε. Therefore, we have

$$\|u\|^2_{L^2(\Omega)} \leq C ||u||^2_{\mathcal{H}^{\varepsilon}_{0,m}(\Omega)} \quad \text{if } u \in \mathcal{H}^{\varepsilon}_{0,m}(\Omega); \tag{2.12}$$

$$\|u\|^2_{L^2(\Omega)} \leq C ||u||^2_{\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)} \quad \text{if } u \in \widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega). \tag{2.13}$$

Clearly, the spaces $\mathcal{H}^{\varepsilon}_{0,m}(\Omega)$ and $\widehat{\mathcal{H}}^{\varepsilon}_{0,m}(\Omega)$ are Banach spaces (see [13, Lemmas 3.2 and 4.1] and [8, Lemma 2.2.]).
For later use, we also define the periodic version of the previous spaces in the normalized unit cell by

$$\mathcal{H}_{per}(Y) := \{u = (u^{int}, u^{out}) \, : \, u^{int} \in H^1_{per}(E^{int}), \\ u^{out} \in H^1_{per}(E^{out}), \, [u] \in H^1_{per}(\Gamma)\}, \tag{2.14}$$

and

$$\widehat{\mathcal{H}}_{per}(Y) := \{u = (u^{int}, u^{out}) \, : \, u^{int} \in H^1_{per}(E^{int}), \\ u^{out} \in H^1_{per}(E^{out}), \, [u], \{u\} \in H^1_{per}(\Gamma)\}. \tag{2.15}$$

Here and in the following $H^1_{per}(E^{int})$ ($H^1_{per}(E^{out})$ and $H^1_{per}(\Gamma)$, respectively) denotes the space of the Y-periodic functions belonging to $H^1_{loc}(E)$ ($H^1_{loc}(\mathbb{R}^N \backslash \overline{E})$ and $H^1_{loc}(\partial E)$, respectively).

3. Position of the problems $\mathcal{A}_\varepsilon^m$ and $\mathcal{B}_\varepsilon^m$

The ε-microscopic models, which we are interested in, are given by

$$\mathcal{A}_\varepsilon^m : \begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon^{int}) = f & in \quad \Omega_\varepsilon^{int}, & (3.1a)\\ -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon^{out}) = f & in \quad \Omega_\varepsilon^{out}, & (3.1b)\\ [A_\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] = 0 & on \quad \Gamma_\varepsilon, & (3.1c)\\ \alpha\, \varepsilon^m\, [u_\varepsilon] - \beta\, \varepsilon^{m+2} \triangle^B [u_\varepsilon] = A_\varepsilon \nabla u_\varepsilon^{out} \cdot \nu_\varepsilon & on \quad \Gamma_\varepsilon, & (3.1d)\\ u_\varepsilon = 0 & on \quad \partial\Omega, & (3.1e) \end{cases}$$

and

$$\mathcal{B}_\varepsilon^m : \begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon^{int}) = f & in \quad \Omega_\varepsilon^{int}, & (3.2a)\\ -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon^{out}) = f & in \quad \Omega_\varepsilon^{out}, & (3.2b)\\ -\gamma\, \varepsilon^{m+2} \triangle^B \{u_\varepsilon\} = [A_\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] & on \quad \Gamma_\varepsilon, & (3.2c)\\ \alpha\, \varepsilon^m\, [u_\varepsilon] - \beta\, \varepsilon^{m+2} \triangle^B [u_\varepsilon] = \{A \nabla u_\varepsilon \cdot \nu_\varepsilon\} & on \quad \Gamma_\varepsilon, & (3.2d)\\ u_\varepsilon = 0 & on \quad \partial\Omega, & (3.2e) \end{cases}$$

where $m = 0; \pm 1$ and α, β, γ are strictly positive constants, whose physical meaning is referable to the tangential and the transversal diffusivities in the concentrated layer, when we consider heat diffusion in multilayered media on which a concentration limit has been performed (see [8, Section 4]).
The source term $f \in L^2(\Omega)$ and the diffusivity matrix A_ε is given by $A_\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$ where A is a measurable, Y-periodic symmetric matrix satisfying

$$\lambda\, |\zeta|^2 \leq\, (A(y)\zeta, \zeta)\, \leq\, \Lambda\, |\zeta|^2 \quad \textit{for a.e. } y \in Y \textit{ and any } \zeta \in \mathbb{R}^N, \tag{3.3}$$

for two suitable constants $0 < \lambda < \Lambda < +\infty$.
More precisely, the mathematical description of our problems is given by an elliptic equation in each phase Ω_ε^{int} and Ω_ε^{out} complemented with homogenous Dirichlet boundary conditions on $\partial\Omega$. The thermal potentials u_ε^{int} and u_ε^{out} of the two phases are coupled by means of two interface conditions. In the first model the flux of the solution u_ε is continuous across the interface and the jump $[u_\varepsilon]$ is governed by an equation involving the Laplace-Beltrami operator with the flux of the solution as a source term. In the second model the flux has a jump proportional to the Laplace-Beltrami of $\{u_\varepsilon\}$ and the jump $[u_\varepsilon]$ is governed by an equation involving the Laplace-Beltrami operator with a source term given by $\{A \nabla u_\varepsilon \cdot \nu_\varepsilon\}$.
In the following, we will consider the problems $\mathcal{A}_\varepsilon^m$ and $\mathcal{B}_\varepsilon^m$ for different scalings of the parameter ε, by taking into account the exponent $m = 0; \pm 1$. This is consistent with what has been done in the literature (see, for instance, [3, 4, 9, 10, 12, 22, 23]). As a consequence, the problems $\mathcal{A}_\varepsilon^m$ and $\mathcal{B}_\varepsilon^m$ will be split in three independent subproblems.

Definition 3.1. Given $\varepsilon \in (0,1]$ and $m = 0, \pm 1$, we say that $u_\varepsilon \in \mathcal{H}^\varepsilon_{0,m}(\Omega)$ is a weak solution of the problem $\mathcal{A}^m_\varepsilon$ given in (3.1), if

$$\begin{aligned} &\int_{\Omega^{int}_\varepsilon} A_\varepsilon \nabla u^{int}_\varepsilon \cdot \nabla \varphi \, dx + \int_{\Omega^{out}_\varepsilon} A_\varepsilon \nabla u^{out}_\varepsilon \cdot \nabla \varphi \, dx \\ &+\alpha\, \varepsilon^m \int_{\Gamma_\varepsilon} [u_\varepsilon]\,[\varphi]\, d\sigma + \beta\, \varepsilon^{m+2} \int_{\Gamma_\varepsilon} \nabla^B [u_\varepsilon] \nabla^B [\varphi]\, d\sigma = \int_\Omega f \varphi \, dx, \end{aligned} \tag{3.4}$$

for every test function $\varphi \in \mathcal{H}^\varepsilon_{0,m}(\Omega)$.

We say that $u_\varepsilon \in \widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$ is a weak solution of the problem $\mathcal{B}^m_\varepsilon$ given in (3.2), if

$$\begin{aligned} &\int_{\Omega^{int}_\varepsilon} A_\varepsilon \nabla u^{int}_\varepsilon \cdot \nabla \varphi \, dx + \int_{\Omega^{out}_\varepsilon} A_\varepsilon \nabla u^{out}_\varepsilon \cdot \nabla \varphi \, dx + \alpha \frac{\varepsilon^m}{2} \int_{\Gamma_\varepsilon} [u_\varepsilon]\,[\varphi]\, d\sigma \\ &+ \beta \frac{\varepsilon^{m+2}}{2} \int_{\Gamma_\varepsilon} \nabla^B [u_\varepsilon] \cdot \nabla^B [\varphi]\, d\sigma + \gamma \frac{\varepsilon^{m+2}}{2} \int_{\Gamma_\varepsilon} \nabla^B \{u_\varepsilon\} \cdot \nabla^B \{\varphi\}\, d\sigma \\ &= \int_\Omega f \varphi dx, \end{aligned} \tag{3.5}$$

for every test function $\varphi \in \widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$. □

The well-posedness of the ε-problems is an easy consequence of the Lax-Milgram Lemma (see [13, Theorems 3.2 and 4.2] and [8, Theorem 3.4]).
By choosing $\varphi = u_\varepsilon$ in the weak formulation (3.4) and (3.5) and using (2.12) and (2.13), respectively, we get the two energy inequalities

$$\begin{aligned} ||\, u_\varepsilon \,||^2_{\mathcal{H}^\varepsilon_{0,m}(\Omega)} &= ||\nabla u_\varepsilon||^2_{L^2(\Omega^{int}_\varepsilon \cup \Omega^{out}_\varepsilon)} + \varepsilon^m \,||\,[u_\varepsilon]\,||^2_{L^2(\Gamma_\varepsilon)} \\ &+ \varepsilon^{m+2} \,||\,\nabla^B [u_\varepsilon]\,||^2_{L^2(\Gamma_\varepsilon)} \le C\,||f||^2_{L^2(\Omega)}, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} ||\, u_\varepsilon \,||^2_{\widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)} &= ||\nabla u_\varepsilon||^2_{L^2(\Omega^{int}_\varepsilon \cup \Omega^{out}_\varepsilon)} + \frac{\varepsilon^m}{2} \,||\,[u_\varepsilon]\,||^2_{L^2(\Gamma_\varepsilon)} \\ + \frac{\varepsilon^{m+2}}{2} \,||\,\nabla^B [u_\varepsilon]\,||^2_{L^2(\Gamma_\varepsilon)} &+ \frac{\varepsilon^{m+2}}{2} \,||\,\nabla^B \{u_\varepsilon\}\,||^2_{L^2(\Gamma_\varepsilon)} \le C\,||\,f\,||^2_{L^2(\Omega)}, \end{aligned} \tag{3.7}$$

where C is independent of ε.

Remark 3.2. If $u_\varepsilon \in \mathcal{H}^\varepsilon_{0,m}(\Omega)$ is the solution of (3.4), by taking into account (3.6) and the Poincaré inequalities (2.12), it follows that there exists a function $u_0 \in L^2(\Omega)$, such that, up to a subsequence, $u_\varepsilon \rightharpoonup u_0$ weakly in $L^2(\Omega)$.

Moreover, there exists a constant $C \geq 0$, independent of ε, such that

$$
\begin{aligned}
\| \, u_\varepsilon^{int} \, \|_{H^1(\Omega_\varepsilon^{int})} &\leq C, \\
\| \, u_\varepsilon^{out} \, \|_{H^1(\Omega_\varepsilon^{out})} &\leq C, \\
\| \, [u_\varepsilon] \, \|_{L^2(\Gamma_\varepsilon)} &\leq C \, \varepsilon^{-\frac{m}{2}}, \\
\| \, \nabla^B [u_\varepsilon] \, \|_{L^2(\Gamma_\varepsilon)} &\leq C \, \varepsilon^{-\frac{m+2}{2}}.
\end{aligned}
\tag{3.8}
$$

On the other hand, if $u \in \widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$ is the solution of (3.5), due to (3.7) and (2.13), we have that also in this case there exists a function $u_0 \in L^2(\Omega)$, such that, up to a subsequence, $u_\varepsilon \rightharpoonup u_0$ weakly in $L^2(\Omega)$ and, in addition to (3.8), we also have that

$$
\| \, \nabla^B \{u_\varepsilon\} \, \|_{L^2(\Gamma_\varepsilon)} \leq C \, \varepsilon^{-\frac{m+2}{2}}. \tag{3.9}
$$

The main purpose of this paper is to characterize such a limit u_0 as the solution of a suitable differential problem. □

4. The unfolding operator

In this section we recall the definitions and the main properties of the unfolding operator $\mathcal{T}_\varepsilon$ (see, for instance, [11, 17, 18, 19, 20, 21, 22, 23]).

For each $x \in \mathbb{R}^N$ and any $\varepsilon \in (0,1]$, we set $x = \varepsilon\Big(\big[\dfrac{x}{\varepsilon}\big]_Y + \big\{\dfrac{x}{\varepsilon}\big\}_Y\Big)$, where $[\cdot]_Y$ denotes the unique integer combination of the periods such that $\big\{\dfrac{x}{\varepsilon}\big\}_Y = \dfrac{x}{\varepsilon} - \big[\dfrac{x}{\varepsilon}\big]_Y$ belongs to the unit cell Y. For later use, we set

$$
\Xi^\varepsilon := \big\{\xi \in \mathbb{Z}^N, \quad \varepsilon(\xi + Y) \subset \Omega\big\} \quad \text{and} \quad \widehat{\Omega}_\varepsilon = \text{interior}\left\{ \bigcup_{\xi \in \Xi^\varepsilon} \varepsilon(\xi + \overline{Y}) \right\}.
$$

Definition 4.1 (Unfolding Operator $\mathcal{T}_\varepsilon$)**.**
For a Lebesgue measurable function w defined in Ω, the unfolding operator $\mathcal{T}_\varepsilon$ is defined as

$$
\mathcal{T}_\varepsilon(w)(x,y) := \begin{cases} w\left(\varepsilon\big[\dfrac{x}{\varepsilon}\big]_Y + \varepsilon y\right) & \mathit{if}\ (x,y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \mathit{otherwise}. \end{cases}
$$

□

We recall that $\mathcal{T}_\varepsilon$ is a linear and continuous operator and that, for any two Lebesgue measurable functions w_1, w_2, we have $\mathcal{T}_\varepsilon(w_1 w_2) = \mathcal{T}_\varepsilon(w_1)\mathcal{T}_\varepsilon(w_2)$.

In the following, for a general set O, $\mathcal{M}_O(\cdot)$ will denote the integral average on O.

Proposition 4.2. *Let $w \in L^2(\Omega)$, then $\mathcal{T}_\varepsilon(w) \to w$ strongly in $L^2(\Omega \times Y)$. Moreover, if $w_\varepsilon \to w$ strongly in $L^2(\Omega)$, then $\mathcal{T}_\varepsilon(w_\varepsilon) \to w$ strongly in $L^2(\Omega \times Y)$. Finally, if $w_\varepsilon \rightharpoonup w$ weakly in $L^2(\Omega)$, then there exists $\widehat{w} \in L^2(\Omega \times Y)$ such that $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w}$ weakly in $L^2(\Omega \times Y)$ and $w = \mathcal{M}_Y(\widehat{w})$.*

We recall that, for every $w \in H^1(\Omega)$, it follows

$$
\nabla_y(\mathcal{T}_\varepsilon(w))(x,\, y) \;=\; \varepsilon\, \mathcal{T}_\varepsilon(\nabla w)(x,\, y) \quad \mathit{for\ a.e} \quad (x,y) \,\in\, \Omega \times Y, \tag{4.1}
$$

and $\mathcal{T}_\varepsilon$ maps $H^1(\Omega)$ into $L^2(\Omega;\, H^1(Y))$. Moreover, the only classes for which the strong convergence of the unfolding $\mathcal{T}_\varepsilon(w_\varepsilon)$ is known to hold in $L^2(\Omega \times Y)$, without assuming the strong convergence of the sequence w_ε, are sums of the following cases: $w_\varepsilon(x) = w(x, \varepsilon^{-1}x)$ Y-periodic in the second variable, such that $w(x, \varepsilon^{-1}x) = f_1(x)f_2(\varepsilon^{-1}x)$ with f_1, f_2 suitable Lebesgue-measurable functions or $w \in L^2(Y; C^0(\overline{\Omega}))$ or $w \in L^2(\Omega; C^0(\overline{Y}))$ (see [1, 18, 19]).

Definition 4.3 (Boundary Unfolding Operator)**.**
For a Lebesgue-measurable function w defined on Γ_ε, the boundary unfolding operator is defined as

$$\mathcal{T}^b_\varepsilon(w)(x,y) := \begin{cases} w(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y) & if\ (x,y) \in \widehat{\Omega}_\varepsilon \times \Gamma, \\ 0 & otherwise. \end{cases}$$

□

Clearly, also $\mathcal{T}^b_\varepsilon$ is linear and continuous and $\mathcal{T}^b_\varepsilon(w_1 w_2) = \mathcal{T}^b_\varepsilon(w_1)\mathcal{T}^b_\varepsilon(w_2)$. Notice that $\mathcal{T}^b_\varepsilon(w)$ is the trace of the unfolding operator $\mathcal{T}_\varepsilon(w)$ on $\widehat{\Omega}_\varepsilon \times \Gamma$, when both the operators are defined.

Proposition 4.4. *For $w \in L^2(\Gamma_\varepsilon)$, we have*

$$||\mathcal{T}^b_\varepsilon(w)||_{L^2(\Omega\times\Gamma)} \leq \sqrt{\varepsilon}\, ||w||_{L^2(\Gamma_\varepsilon)}, \tag{4.2}$$

and

$$\Big| \int_{\Gamma_\varepsilon} w\, d\sigma \; - \; \frac{1}{\varepsilon} \int_\Omega \int_\Gamma \mathcal{T}^b_\varepsilon(w) d\sigma\, dx \Big| \leq \int_{\Gamma_\varepsilon \setminus \widehat{\Omega}_\varepsilon} |w|\, d\sigma. \tag{4.3}$$

We remark that in the connected-disconnected case, $\Gamma_\varepsilon \setminus \widehat{\Omega}_\varepsilon = \emptyset$.

Proposition 4.5. (1) *If $w \in C^0(\overline{\Omega})$, then*

$$\mathcal{T}^b_\varepsilon(w) \to w \quad strongly\ in \quad L^2(\Omega \times \Gamma).$$

(2) *If $w \in H^1(\Omega)$, then*

$$\mathcal{T}^b_\varepsilon(w) \to w \quad strongly\ in \quad L^2(\Omega \times \Gamma).$$

(3) *If $w_\varepsilon \rightharpoonup w$ weakly in $H^1_0(\Omega)$, then*

$$\mathcal{T}^b_\varepsilon(w_\varepsilon) \rightharpoonup w \quad weakly\ in \quad L^2(\Omega \times \Gamma).$$

Proposition 4.6. *Let $\phi : Y \to \mathbb{R}$ be a function extended by Y-periodicity to the whole $\mathbb{R}^N$ and define the sequence*

$$\phi_\varepsilon(x) = \phi(\frac{x}{\varepsilon}), \qquad x \in \mathbb{R}^N. \tag{4.4}$$

Clearly,

$$\mathcal{T}_\varepsilon(\phi_\varepsilon)(x,y) = \begin{cases} \phi(y) & for\, a.e. \quad (x,y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & otherwise, \end{cases} \tag{4.5}$$

and

$$\mathcal{T}^b_\varepsilon(\phi_\varepsilon)(x,y) = \begin{cases} \phi(y) & for\, a.e. \quad (x,y) \in \widehat{\Omega}_\varepsilon \times \Gamma, \\ 0 & otherwise. \end{cases} \tag{4.6}$$

Then, as $\varepsilon \to 0$, *if* $\phi \in L^2(Y)$, *we have*

$$\mathcal{T}_\varepsilon(\phi_\varepsilon) \to \phi \quad \textit{strongly in} \quad L^2(\Omega \times Y), \tag{4.7}$$

if $\phi \in H^1(Y)$, *we have*

$$\nabla_y \mathcal{T}_\varepsilon(\phi_\varepsilon) \to \nabla_y \phi \quad \textit{strongly in} \quad L^2(\Omega \times Y), \tag{4.8}$$

and, if $\phi \in L^2(\Gamma)$, *we have*

$$\mathcal{T}^b_\varepsilon(\phi_\varepsilon) \to \phi \quad \textit{strongly in} \quad L^2(\Omega \times \Gamma). \tag{4.9}$$

Notice that, if $w \in \mathcal{H}^\varepsilon_{0,m}(\Omega)$ (or $w \in \widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$), then

$$\nabla^B_y \mathcal{T}^b_\varepsilon([w]) = \varepsilon\, \mathcal{T}^b_\varepsilon(\nabla^B[w]) \qquad \text{(and} \qquad \nabla^B_y \mathcal{T}^b_\varepsilon(\{w\}) = \varepsilon\, \mathcal{T}^b_\varepsilon(\nabla^B\{w\})). \tag{4.10}$$

4.1. **Compactness results for the connected-connected geometry.** In this subsection we focus on some well-known results that we will use in the following for the connected-connected geometrical structure of a periodic media.

Theorem 4.7. *Let* $m = 0, \pm 1$ *and* $w_\varepsilon = (w^{int}_\varepsilon, w^{out}_\varepsilon)$ *be a sequence in* $\mathcal{H}^\varepsilon_{0,m}(\Omega)$. *Assume that there exists* $C > 0$ *(independent of* ε*) such that*

$$\int_\Omega |w_\varepsilon|^2\, dx + \int_\Omega |\nabla w_\varepsilon|^2\, dx \le C\,, \qquad \forall \varepsilon > 0. \tag{4.11}$$

Then, there exist $w^{int}, w^{out} \in H^1_0(\Omega)$, $\hat{w}^{int} \in L^2(\Omega; H^1_{per}(E^{int}))$ *and* $\hat{w}^{out} \in L^2(\Omega; H^1_{per}(E^{out}))$ *such that, up to subsequence, as* $\varepsilon \to 0$, *we have*

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{int}_\varepsilon} w_\varepsilon) \to \chi_{E^{int}} w^{int} = w^{int}\,, \qquad \textit{strongly in } L^2(\Omega; H^1_{per}(E^{int}))\,; \tag{4.12}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{out}_\varepsilon} w_\varepsilon) \rightharpoonup \chi_{E^{out}} w^{out} = w^{out}\,, \qquad \textit{strongly in } L^2(\Omega; H^1_{per}(E^{out}))\,; \tag{4.13}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{int}_\varepsilon} \nabla w_\varepsilon) \rightharpoonup \chi_{E^{int}} \left(\nabla w^{int} + \nabla_y \hat{w}^{int}\right)\,, \qquad \textit{weakly in } L^2(\Omega \times Y)\,; \tag{4.14}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{out}_\varepsilon} \nabla w_\varepsilon) \rightharpoonup \chi_{E^{out}} \left(\nabla w^{out} + \nabla_y \hat{w}^{out}\right)\,, \qquad \textit{weakly in } L^2(\Omega \times Y)\,. \tag{4.15}$$

Moreover, we also have

$$\varepsilon \int_{\Gamma_\varepsilon} [w_\varepsilon]^2\, d\sigma \le 2\varepsilon \int_{\Gamma_\varepsilon} \left(|w^{int}_\varepsilon|^2 + |w^{out}_\varepsilon|^2\right)\, d\sigma \le C\,, \qquad \forall \varepsilon > 0\,, \tag{4.16}$$

with C *independent of* ε, *and*

$$\mathcal{T}^b_\varepsilon([w_\varepsilon]) \rightharpoonup [w]\,, \qquad \textit{weakly in } L^2(\Omega \times \Gamma)\,, \tag{4.17}$$

where, with abuse of notation, we set $[w] = w^{out} - w^{int}$ *and we have identified* $w^{int}_\varepsilon, w^{out}_\varepsilon$ *with their traces on* Γ_ε. *Finally, assume that*

$$\|[w_\varepsilon]\|_{L^2(\Gamma_\varepsilon)} \le C\varepsilon^{-m/2}, \tag{4.18}$$

with C *independent of* ε. *Then, if* $m = -1$, 0, *we have* $w^{int} = w^{out}$ *and, if* $m = -1$, *we have*

$$\mathcal{T}^b_\varepsilon\Big(\frac{[w_\varepsilon]}{\varepsilon}\Big) \rightharpoonup [\hat{w}] \quad \textit{weakly in} \quad L^2(\Omega \times \Gamma). \tag{4.19}$$

Theorem 4.8. *Let $m = 0, \pm 1$ and $w_\varepsilon = (w_\varepsilon^{int}, w_\varepsilon^{out})$ be a bounded sequence in $\mathcal{H}^\varepsilon_{0,m}(\Omega)$. Then, there exist $\xi^{int}, \xi^{out} \in L^2(\Omega)$, such that, up to a subsequence (still denoted by ε), we have*

$$\frac{\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}} w_\varepsilon^{int}) - \mathcal{M}_\Gamma(\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}} w_\varepsilon^{int}))}{\varepsilon} \rightharpoonup y_\Gamma \cdot \nabla w^{int} + \hat{w}^{int} + \xi^{int} \quad \text{weakly in } L^2(\Omega \times E^{int}),$$
$$\frac{\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}} w_\varepsilon^{out}) - \mathcal{M}_\Gamma(\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}} w_\varepsilon^{out}))}{\varepsilon} \rightharpoonup y_\Gamma \cdot \nabla w^{out} + \hat{w}^{out} + \xi^{out} \quad \text{weakly in } L^2(\Omega \times E^{out}),$$

where $y_\Gamma = y - \mathcal{M}_\Gamma(y)$ and $w^{int}, w^{out}, \hat{w}^{int}, \hat{w}^{out}$ are the same functions appearing in Theorem 4.7.

4.2. **Compactness results for the connected-disconnected geometry.** In this subsection, we recall and focus on some convergence results for the connected-disconnected geometrical settings.

Theorem 4.9. *Let $m = 0, \pm 1$ and $w_\varepsilon = (w_\varepsilon^{int}, w_\varepsilon^{out})$ be a sequence in $\mathcal{H}^\varepsilon_{0,m}(\Omega)$. Assume that (4.11) holds. Then, there exist $w^{int} \in L^2(\Omega), w^{out} \in H^1_0(\Omega)$, $\hat{w}^{int} \in L^2(\Omega; H^1_{per}(E^{int}))$ and $\hat{w}^{out} \in L^2(\Omega; H^1_{per}(E^{out}))$ such that, up to subsequence, as $\varepsilon \to 0$, we have that (4.13),(4.15) and (4.17) hold, while in (4.12) the strong convergence must be replaced with the weak convergence and, instead of (4.14), we have*

$$\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}} \nabla w_\varepsilon) \rightharpoonup \chi_{E^{int}} \big(\nabla w^{out} + \nabla_y \hat{w}^{int}\big), \text{ weakly in } L^2(\Omega \times Y)\,; \tag{4.20}$$

Moreover, assuming again (4.18), we obtain that, if $m = -1$, 0, $w^{int} = w^{out}$ and, if $m = -1$, (4.19) holds.

Theorem 4.10. *Let $m = 0, \pm 1$ and $w_\varepsilon = (w_\varepsilon^{int}, w_\varepsilon^{out})$ be a bounded sequence in $\mathcal{H}^\varepsilon_{0,m}(\Omega)$. Then, there exist $\xi^{int}, \xi^{out} \in L^2(\Omega)$, such that, up to a subsequence (still denoted by ε), we have*

$$\frac{\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}} w_\varepsilon^{int}) - \mathcal{M}_\Gamma(\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}} w_\varepsilon^{int}))}{\varepsilon} \rightharpoonup y_\Gamma \cdot \nabla w^{out} + \hat{w}^{int} + \xi^{int} \quad \text{weakly in } L^2(\Omega \times E^{int}),$$
$$\frac{\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}} w_\varepsilon^{out}) - \mathcal{M}_\Gamma(\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}} w_\varepsilon^{out}))}{\varepsilon} \rightharpoonup y_\Gamma \cdot \nabla w^{out} + \hat{w}^{out} + \xi^{out} \quad \text{weakly in } L^2(\Omega \times E^{out}),$$

where $y_\Gamma = y - \mathcal{M}_\Gamma(y)$ and $w^{int}, w^{out}, \hat{w}^{int}, \hat{w}^{out}$ are the same functions appearing in Theorem 4.9.

It is worthwhile to remark that, in the connected-disconnected geometry, the space $H^1(E^{int})$ coincides with $H^1_{per}(E^{int})$.

5. Homogenization of the problem $\mathcal{A}^m_\varepsilon$

In this section, we apply the unfolding technique to rigorously describe the asymptotic behavior as $\varepsilon \to 0$ of the function u_ε, solution of the elliptic problem $\mathcal{A}^m_\varepsilon$ (3.1). We will consider both the connected/connected and the connected/disconnected geometrical settings.

5.1. **The case** $m = -1$. In this section, we consider problem (3.1) for the scaling $m = -1$.

Theorem 5.1. *For any $\varepsilon > 0$, let $u_\varepsilon = (u_\varepsilon^{int}, u_\varepsilon^{out}) \in \mathcal{H}^\varepsilon_{0,m}(\Omega)$ be the unique solution of problem (3.1), for $m = -1$. Then, there exist $u_0 \in H^1_0(\Omega)$ and $u_1 = (u_1^{int}, u_1^{out}) \in L^2(\Omega, \mathcal{H}_{per}(Y))$ with $M_Y(u_1) = 0$, such that, as $\varepsilon \to 0$, we have*

$$\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u_0\,, \qquad \text{weakly in } \ L^2(\Omega \times Y); \tag{5.1}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{int}} u_\varepsilon) \rightharpoonup u_0\,, \qquad \text{weakly in } \ L^2(\Omega; H^1_{per}(E^{int})); \tag{5.2}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^{out}} u_\varepsilon) \to u_0\,, \qquad \text{strongly in } \ L^2(\Omega; H^1_{per}(E^{out})); \tag{5.3}$$

$$\mathcal{T}_\varepsilon(\,\chi_{\Omega^{int}_\varepsilon} \nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y u_1^{int}\,, \qquad \text{weakly in } \ L^2(\Omega \times E^{int}); \tag{5.4}$$

$$\mathcal{T}_\varepsilon(\,\chi_{\Omega^{out}_\varepsilon} \nabla u_\varepsilon) \rightharpoonup \nabla u_0 + \nabla_y u_1^{out}\,, \qquad \text{weakly in } \ L^2(\Omega \times E^{out}); \tag{5.5}$$

$$\mathcal{T}^b_\varepsilon\Big(\frac{[u_\varepsilon]}{\varepsilon}\Big) \rightharpoonup [u_1]\,, \qquad \text{weakly in } \ L^2(\Omega \times \Gamma); \tag{5.6}$$

$$\mathcal{T}^b_\varepsilon\big(\nabla^B[u_\varepsilon]\big) \rightharpoonup \nabla^B_y[u_1] \qquad \text{weakly in } \ L^2(\Omega \times \Gamma). \tag{5.7}$$

Moreover, the pair (u_0, u_1) is the unique weak solution of the following homogenized two-scale problem

$$\begin{cases} -\mathrm{div}\Big(\displaystyle\int_{E^{int}\cup E^{out}} A\,(\nabla u_0 + \nabla_y u_1)\, dy\Big) = f & \text{in } \ \Omega, & (5.8a) \\ -\,\mathrm{div}_y\Big(\, A\,(\nabla u_0 + \nabla_y u_1^{int})\Big) = 0 & \text{in } \ \Omega \times E^{int}, & (5.8b) \\ -\,\mathrm{div}_y\Big(\, A\,(\nabla u_0 + \nabla_y u_1^{out})\Big) = 0 & \text{in } \ \Omega \times E^{out}, & (5.8c) \\ \big[\,A\,(\nabla u_0 + \nabla_y u_1)\cdot \nu\big] \,=\, 0 & \text{on } \ \Omega \times \Gamma, & (5.8d) \\ \alpha\,[u_1] \,-\, \beta \,\triangle^B_y\,[u_1] \,=\, A\,(\nabla u_0 + \nabla_y u_1^{out})\cdot \nu & \text{on } \ \Omega \times \Gamma. & (5.8e) \end{cases}$$

Proof.
Taking into account (3.8) with $m = -1$, up to a subsequence, the convergences in (5.1)–(5.6) are consequence of Theorem 4.7 and of Theorem 4.9 with $u_1^{int} \in L^2(\Omega; H^1_{per}(E^{int}))$ and $u_1^{out} \in L^2(\Omega; H^1_{per}(E^{out}))$. To prove that $u_1 = (u_1^{int}, u_1^{out}) \in L^2(\Omega, \mathcal{H}_{per}(Y))$ and that the convergence (5.7) holds, let us take into account again (3.8) in order to obtain

$$\int_\Omega \int_\Gamma \big|\mathcal{T}^b_\varepsilon\,(\nabla^B[u_\varepsilon])\big|^2\, d\sigma\, dx \,\le\, \varepsilon \int_{\Gamma_\varepsilon} |\,\nabla^B[u_\varepsilon]\,|^2\, d\sigma \,\le C, \tag{5.9}$$

where C is a positive constant, independent of ε. This implies that, up to a subsequence (still denoted by ε), there exists a vector function $\zeta^b \in L^2(\Omega \times \Gamma)$, such that as $\varepsilon \to 0$, we have

$$\mathcal{T}^b_\varepsilon\,(\nabla^B[u_\varepsilon]) \rightharpoonup \zeta^b \quad \text{weakly in} \quad L^2(\Omega \times \Gamma).$$

Let us choose a test vector function $\varphi(x,y)=\phi_1(x)\,\Psi_2(y)$ where the scalar function $\phi_1\in C_c^\infty(\Omega)$ and the vector function $\Psi_2\in C_{per}^\infty(\Gamma)$, and consider

$$\int\limits_{\Omega\times\Gamma}\zeta^b\cdot\Psi_2(y)\,\phi_1(x)\,d\sigma\,dx \leftharpoonup \int\limits_{\Omega\times\Gamma}\mathcal{T}_\varepsilon^b\left(\nabla^B[u_\varepsilon]\right)\cdot\Psi_2(y)\,\phi_1(x)\,d\sigma\,dx$$
$$=\int\limits_{\Omega\times\Gamma}\nabla_y^B\mathcal{T}_\varepsilon^b\Big(\frac{[u_\varepsilon]}{\varepsilon}\Big)\cdot\Psi_2(y)\,\phi_1(x)\,d\sigma\,dx=-\int\limits_{\Omega\times\Gamma}\mathcal{T}_\varepsilon^b\Big(\frac{[u_\varepsilon]}{\varepsilon}\Big)\mathrm{div}_y^B\Psi_2(y)\,\phi_1(x)\,d\sigma\,dx$$
$$\rightharpoonup -\int\limits_{\Omega\times\Gamma}[u_1]\mathrm{div}_y^B\Psi_2(y)\,\phi_1(x)\,d\sigma\,dx$$

where we have taken into account (4.10) and (5.6). This implies that $\zeta^b=\nabla_y^B[u_1]$ and, hence, $[u_1]\in L^2(\Omega;H^1(\Gamma))$, i.e. $u_1=(u_1^{int},u_1^{out})\in L^2(\Omega,\mathcal{H}_{per}(Y))$.
In order to prove that the pair (u_0,u_1) is the solution of the two-scale problem (5.8) we proceed as follow. First, we recall that, by Proposition 4.6, we get $\mathcal{T}_\varepsilon(A_\varepsilon)\to A$ strongly in $L^2(\Omega\times Y)$. Moreover, let us choose in the weak formulation (3.4) the test function $\varphi_\varepsilon(x)=\phi_1(x)+\varepsilon\,\phi_2(x,\frac{x}{\varepsilon})$, where $\phi_1\in C_c^\infty(\Omega)$, $\phi_2=(\phi_2^{int},\phi_2^{out})$ with $\phi_2\in C_c^\infty(\Omega;\mathcal{H}_{per}(Y))$, so that, by unfolding, we get

$$\int\limits_{\Omega\times E^{int}}\mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}}\nabla u_\varepsilon^{int})\cdot\mathcal{T}_\varepsilon(\nabla\phi_1)\,dxdy+\varepsilon\int\limits_{\Omega\times E^{int}}\mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}}\nabla u_\varepsilon^{int})\cdot\mathcal{T}_\varepsilon(\nabla_x\phi_2^{int})\,dydx$$
$$+\int\limits_{\Omega\times E^{int}}\mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}}\nabla u_\varepsilon^{int})\cdot\mathcal{T}_\varepsilon(\nabla_y\phi_2^{int})\,dydx+\int\limits_{\Omega\times E^{out}}\mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}}\nabla u_\varepsilon^{out})\cdot\mathcal{T}_\varepsilon(\nabla\phi_1)\,dydx$$
$$+\varepsilon\int\limits_{\Omega\times E^{out}}\mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}}\nabla u_\varepsilon^{out})\cdot\mathcal{T}_\varepsilon(\nabla_x\phi_2^{out})\,dydx+\int\limits_{\Omega\times E^{out}}\mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}}\nabla u_\varepsilon^{out})\cdot\mathcal{T}_\varepsilon(\nabla_y\,\phi_2^{out})\,dydx$$
$$+\frac{\alpha}{\varepsilon}\int\limits_{\Omega\times\Gamma}\mathcal{T}_\varepsilon^b([u_\varepsilon])\cdot\mathcal{T}_\varepsilon^b([\phi_2])\,d\sigma dx+\beta\,\varepsilon\int\limits_{\Omega\times\Gamma}\mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])\cdot\mathcal{T}_\varepsilon^b(\nabla_x^B[\phi_2])\,d\sigma dx$$
$$+\beta\int\limits_{\Omega\times\Gamma}\mathcal{T}_\varepsilon^b(\,\nabla^B[u_\varepsilon]\,)\cdot\mathcal{T}_\varepsilon^b(\nabla_y^B[\phi_2])\,d\sigma dx$$
$$=\int\limits_{\Omega\times Y}\mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_1)\,dydx+\varepsilon\int\limits_{\Omega\times Y}\mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_2)\,dydx+R_\varepsilon$$

where $R_\varepsilon = o\,(1)$ for $\varepsilon \to 0$.
Then, by passing to the limit $\varepsilon \to 0$, up to a subsequence, we arrive at

$$\begin{aligned} &\int_{\Omega\times E^{int}} A\,(\nabla u_0 + \nabla_y u_1^{int})\cdot(\nabla\phi_1 + \nabla_y \phi_2^{int})\,dydx \\ &+ \int_{\Omega\times E^{out}} A\,(\nabla u_0 + \nabla_y u_1^{out})\cdot(\nabla\phi_1 + \nabla_y \phi_2^{out})\,dydx + \alpha \int_{\Omega\times\Gamma} [u_1]\,[\,\phi_2\,]\,d\sigma dx \\ &+ \beta \int_{\Omega\times\Gamma} \nabla_y^B [u_1]\cdot\nabla_y^B[\,\phi_2\,]\,d\sigma\,dx = \int_{\Omega\times Y} f\,\phi_1\,dydx. \end{aligned} \tag{5.10}$$

By a standard localization procedure, taking first $\phi_2^{out} = \phi_2^{int} = 0$, then $\phi_1 = \phi_2^{int} = 0$ and, finally, $\phi_1 = \phi_2^{out} = 0$, respectively, we obtain

$$\int_{\Omega\times(E^{int}\cup E^{out})} A\,(\nabla u_0 + \nabla_y u_1)\cdot\nabla\phi_1\,dydx = \int_{\Omega\times Y} f\,\phi_1\,dydx, \tag{5.11}$$

$$\begin{aligned} &\int_{\Omega\times E^{out}} A\,(\nabla u_0 + \nabla_y u_1^{out})\cdot\nabla_y\phi_2^{out}\,dydx + \alpha\int_{\Omega\times\Gamma}[u_1]\,\phi_2^{out}\,d\sigma dx \\ &+ \beta\int_{\Omega\times\Gamma}\nabla_y^B[u_1]\cdot\nabla_y^B\,\phi_2^{out}\,d\sigma dx = 0 \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} &\int_{\Omega\times E^{int}} A\,(\nabla u_0 + \nabla_y u_1^{int})\cdot\nabla_y\phi_2^{int}\,dydx - \alpha\int_{\Omega\times\Gamma}[u_1]\,\phi_2^{int}\,d\sigma dx \\ &- \beta\int_{\Omega\times\Gamma}\nabla_y^B[u_1]\cdot\nabla_y^B\,\phi_2^{int}\,d\sigma dx = 0, \end{aligned} \tag{5.13}$$

which lead to (5.8a) and (5.8b)–(5.8e).
In order to prove the uniqueness of (u_0, u_1), assume by contradiction that the problem (5.8) has two distinct pairs of solutions (u_0, u_1) and $(\widetilde{u}_0, \widetilde{u}_1)$. Let us set $U_0 = u_0 - \widetilde{u}_0$ and $U_1 := (U_1^{int},\, U_1^{out}) = \left(u_1^{int} - \widetilde{u}_1^{int},\, u_1^{out} - \widetilde{u}_1^{out}\right)$. By following the idea presented in [6, Remark 4.2], we choose U_0 as a test function in (5.8a) written for u_0, and U_1 as a test function in (5.8b) and (5.8c) written for u_1. Next, we add the equations and integrate them by parts keeping in view the interface conditions (5.8e) and (5.8d). Then, we will repeat the same procedure for the pair $(\widetilde{u}_0,\, \widetilde{u}_1)$. Precisely, take U_0 as a test function in (5.8a) written for $\widetilde{u}_0$, and U_1 as a test function in (5.8b) and (5.8c); written for $\widetilde{u}_1$, add the obtained equations and integrate them by parts by using the interface conditions (5.8e) and (5.8d). By subtracting the resulting equality and

taking into account the coercivity of the matrix A (see (3.3)), we arrive at

$$\int\limits_{\Omega\times(E^{int}\cup E^{out})} \left|\nabla U_0 + \nabla_y U_1\right|^2 dxdy + \alpha \int\limits_{\Omega\times\Gamma} \left|\,[U_1]\,\right|^2 d\sigma dx + \beta \int\limits_{\Omega\times\Gamma} \left|\,\nabla_y^B [U_1]\,\right|^2 d\sigma dx$$

$$\leq \int\limits_{\Omega\times(E^{int}\cup E^{out})} A(\nabla U_0 + \nabla_y U_1)\cdot(\nabla U_0 + \nabla_y U_1)\, dxdy + \alpha \int\limits_{\Omega\times\Gamma} \left|\,[U_1]\,\right|^2 d\sigma dx$$

$$+\beta \int\limits_{\Omega\times\Gamma} \left|\,\nabla_y^B [U_1]\,\right|^2 d\sigma dx = 0, \tag{5.14}$$

which implies $[U_1] = 0$. Moreover,

$$0 \geq \int\limits_{\Omega\times(E^{int}\cup E^{out})} \left|\nabla U_0 + \nabla_y U_1\right|^2 dydx$$

$$= \int\limits_{\Omega}\int\limits_{Y} \left|\,\nabla U_0\,\right|^2 dydx + \int\limits_{\Omega}\int\limits_{Y} \left|\,\nabla_y U_1\,\right|^2 dydx + 2\int\limits_{\Omega} \nabla U_0 \cdot \Big(\int\limits_{Y} \nabla_y U_1\, dy\Big)\, dx$$

$$= \int\limits_{\Omega} \left|\,\nabla U_0\,\right|^2 dx + \int\limits_{\Omega}\int\limits_{Y} \left|\,\nabla_y U_1\,\right|^2 dydx,$$

where, in the last term of the second line, we have taken into account the periodicity of U_1 and the fact that $[U_1] = 0$. Clearly, from the obtained inequality, $\nabla U_0 = 0 = \nabla_y U_1$, which implies that U_0 is a constant in Ω. Thus, $U_0 = 0$, as U_0 satisfy the homogeneous boundary condition on $\partial\Omega$. Also, $U_1 = 0$, since it is constant and has null mean average over Y. Hence, the pair (u_0, u_1) is unique. Therefore, the whole sequence, and not only a subsequence, converges. This concludes the proof. □

Remark 5.2. We notice that the linear dependence of u_1 with respect to ∇u_0 in (5.8b)–(5.8e) leads to the usual factorization of u_1 in terms of ∇u_0. □

Proposition 5.3. *Let (u_0, u_1) be the unique weak solution of the two-scale problem (5.8). Then the function u_1 can be uniquely factorized as*

$$u_1(x,y) = -\mathcal{X}_{\mathcal{L}}(y)\cdot\nabla u_0(x), \tag{5.15}$$

where $\mathcal{X}_{\mathcal{L}} = (\mathcal{X}_{\mathcal{L}}^1, \dots, \mathcal{X}_{\mathcal{L}}^N) \in \mathcal{H}_{per}(Y)$ is the vector function with null mean average over Y, whose Y-periodic components, for $j = 1,\dots,N$, satisfy the cell problem

$$\begin{cases} -\mathrm{div}_y\Big(A\,\nabla_y\big(\chi_{\mathcal{L}}^{j,int} - y_j\big)\Big) = 0 & in \quad E^{int}, & (5.16a) \\ -\mathrm{div}_y\Big(A\,\nabla_y\big(\chi_{\mathcal{L}}^{j,out} - y_j\big)\Big) = 0 & in \quad E^{out}, & (5.16b) \\ \big[A\,\nabla_y(\chi_{\mathcal{L}}^{j} - y_j)\,\cdot\,\nu\big] = 0 & on \quad \Gamma, & (5.16c) \\ \alpha\,[\chi_{\mathcal{L}}^{j}] - \beta\,\triangle_y^B[\chi_{\mathcal{L}}^{j}] = A\,\nabla_y\big(\chi_{\mathcal{L}}^{j,out} - y_j\big)\cdot\nu & on \quad \Gamma. & (5.16d) \end{cases}$$

Moreover, $u_0 \in H^1_0(\Omega)$ is the unique solution of the homogenized problem

$$-\mathrm{div}(A_{\mathcal{L}} \nabla u_0) = f, \tag{5.17}$$

where $A_{\mathcal{L}}$ is the constant homogenized matrix defined by

$$A_{\mathcal{L}} := \int_{E^{int}\cup E^{out}} A\,(I - \nabla_y \mathcal{X}_{\mathcal{L}}(y))\,dy\,. \tag{5.18}$$

We remark that the well-posedness of problem (5.16) is a consequence of the Lax-Milgram Lemma applied in this periodic framework, in analogy to what has been done, for example, in [14, Lemma 2.1].

Proof.
It is not difficult to prove that the function u_1 given in (5.15) satisfies (5.8b) - (5.8e); moreover, since problems (5.8) and (5.16) are well-posed, it is the unique solution. Now, let insert the factorization of u_1 given in (5.15) in (5.8a) so that

$$\begin{aligned}&-\mathrm{div}\Big(\int_{E^{int}\cup E^{out}} A\,(\nabla u_0 - \nabla_y \mathcal{X}_{\mathcal{L}}\,\nabla u_0)\,dy\Big)\\ &\qquad = -\mathrm{div}\left(\Big(\int_{E^{int}\cup E^{out}} A\,(I - \nabla_y \mathcal{X}_{\mathcal{L}})\,dy\Big)\nabla u_0\right) = f \qquad in \quad \Omega.\end{aligned}$$

Then, (5.17) follows, by taking into account (5.18).
The uniqueness of the solution of equation (5.17) in $H^1_0(\Omega)$ is a standard consequence of the symmetry and positive-definiteness of the matrix $A_{\mathcal{L}}$, stated in the next proposition. □

Proposition 5.4. *The matrix $A_{\mathcal{L}}$ is symmetric and positive definite.*

Proof.
The components of the matrix $A_{\mathcal{L}}$ given in (5.18) can be written as

$$(A_{\mathcal{L}})_{ij} = -\int_{E^{int}\cup E^{out}} A\,\nabla_y(\chi^j_{\mathcal{L}} - y_j)\cdot\nabla_y y_i\,dy\,. \tag{5.19}$$

Using $\mathcal{X}^{i,int}_{\mathcal{L}}$ and $\mathcal{X}^{i,out}_{\mathcal{L}}$ as test function in (5.16a) and (5.16b), respectively, integrating by parts, summing the resulting equations and using (5.16c) and (5.16d), we get

$$0 = \int_{E^{int}\cup E^{out}} A\nabla_y(\chi^j_{\mathcal{L}} - y_j)\cdot\nabla_y(\chi^i_{\mathcal{L}})\,dy + \alpha\int_\Gamma [\chi^j_{\mathcal{L}}][\chi^i_{\mathcal{L}}]\,d\sigma + \beta\int_\Gamma \nabla^B_y[\chi^j_{\mathcal{L}}]\cdot\nabla^B_y[\chi^i_{\mathcal{L}}]\,d\sigma. \tag{5.20}$$

By adding the equations (5.19) and (5.20), we obtain

$$(A_{\mathcal{L}})_{ij} = \int_{E^{int}\cup E^{out}} A\nabla_y(\chi^j_{\mathcal{L}} - y_j)\cdot\nabla_y(\chi^i_{\mathcal{L}} - y_i)\,dy + \alpha\int_\Gamma [\chi^j_{\mathcal{L}}][\chi^i_{\mathcal{L}}]\,d\sigma + \beta\int_\Gamma \nabla^B_y[\chi^j_{\mathcal{L}}]\cdot\nabla^B_y[\chi^i_{\mathcal{L}}]\,d\sigma.$$

Therefore, the symmetry of the matrix $A_{\mathcal{L}}$ is achieved. In order to prove that $A_{\mathcal{L}}$ is also positive definite, for every $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$, consider

$$\sum_{i,j=1}^{N} (A_{\mathcal{L}})_{ij}\xi_i\xi_j = \int_{E^{int}\cup E^{out}} \sum_{i,j=1}^{N} A\nabla_y\big(\xi_i\chi^i_{\mathcal{L}} - \xi_i y_i\big)\cdot\nabla_y\big(\xi_j\chi^j_{\mathcal{L}} - \xi_j y_j\big)\,dy + \alpha\int_{\Gamma}\sum_{i,j=1}^{N}\big[\xi_i\chi^i_{\mathcal{L}}\big]\big[\xi_j\chi^j_{\mathcal{L}}\big]\,d\sigma$$
$$+\beta\int_{\Gamma}\sum_{i,j=1}^{N}\nabla^B_y\big[\xi_i\chi^i_{\mathcal{L}}\big]\cdot\nabla^B_y\big[\xi_j\chi^j_{\mathcal{L}}\big]\,d\sigma \geq \lambda\int_{E^{int}\cup E^{out}}\Big|\sum_{j=1}^{N}\big(\xi_j\chi^j_{\mathcal{L}} - \xi_j y_j\big)\Big|^2 dy \geq 0.$$

To prove that the last inequality is actually strict for $\xi \neq 0$, let us assume that, by contradiction,

$$\int_{E^{int}\cup E^{out}}\Big|\nabla_y\Big(\sum_{j=1}^{N}(\xi_j\,\mathcal{X}^j_{\mathcal{L}} - \xi_j\,y_j)\Big)\Big|^2 dy = 0$$

for some $\xi \neq 0$. This implies, in particular, that $\xi\cdot\mathcal{X}^{out}_{\mathcal{L}} - \xi\cdot y$ is independent of y in E^{out}, which is not possible, due to the Y-periodicity of the cell function. Hence, the thesis. □

5.2. **The case $m = 0$.** In this section, we consider problem (3.1) for the scaling $m = 0$.

Theorem 5.5. *For any $\varepsilon > 0$, let $u_\varepsilon = (u^{int}_\varepsilon, u^{out}_\varepsilon) \in \mathcal{H}^\varepsilon_{0,m}(\Omega)$ be the unique solution of problem (3.1), for $m = 0$. Then, there exist $u_0 \in H^1_0(\Omega)$ and $u_1 = (u^{int}_1, u^{out}_1)$, with $u^{int}_1 \in L^2(\Omega; H^1_{per}(E^{int}))$, $u^{out}_1 \in L^2(\Omega; H^1_{per}(E^{out}))$ and $M_{E^{int}}(u^{int}_1) = 0 = M_{E^{out}}(u^{out}_1)$, such that as $\varepsilon \to 0$ we have that (5.1)–(5.5) hold and*

$$\mathcal{T}^b_\varepsilon\big([u_\varepsilon]\big) \rightharpoonup 0, \qquad \textit{weakly in} \quad L^2(\Omega\times\Gamma); \tag{5.21}$$
$$\varepsilon\,\mathcal{T}^b_\varepsilon\big(\nabla^B[u_\varepsilon]\big) \to 0, \qquad \textit{strongly in} \quad L^2(\Omega\times\Gamma). \tag{5.22}$$

Moreover, the pair (u_0, u_1) is the unique weak solution of the following homogenized two-scale problem

$$\begin{cases} -\mathrm{div}\Big(\displaystyle\int_{E^{int}\cup E^{out}} A\,(\nabla u_0 + \nabla_y u_1)\,dy\Big) = f & \textit{in} \quad \Omega, & (5.23a)\\ -\mathrm{div}_y\big(A\,(\nabla u_0 + \nabla_y u_1)\big) = 0 & \textit{in} \quad \Omega\times E^{int}, & (5.23b)\\ -\mathrm{div}_y\big(A\,(\nabla u_0 + \nabla_y u_1)\big) = 0 & \textit{in} \quad \Omega\times E^{out}, & (5.23c)\\ A\,(\nabla u_0 + \nabla_y u^{int}_1)\cdot\nu = 0 & \textit{on} \quad \Omega\times\Gamma, & (5.23d)\\ A\,\nabla u_0 + \nabla_y u^{out}_1)\cdot\nu = 0 & \textit{on} \quad \Omega\times\Gamma. & (5.23e) \end{cases}$$

Proof.
Taking into account (3.8) with $m = 0$, up to a subsequence, the convergences (5.1)–(5.5) and (5.21) follow from Theorem 4.7, for the connected-connected geometrical settings, and from Theorem 4.9, for the connected-disconnected geometrical settings. In order to prove (5.22), still using (3.8) with $m = 0$, we have

$$\|\sqrt{\varepsilon}\mathcal{T}^b_\varepsilon(\nabla^B[u_\varepsilon])\|^2_{L^2(\Omega\times\Gamma)} = \varepsilon\|\mathcal{T}^b_\varepsilon(\nabla^B[u_\varepsilon])\|^2_{L^2(\Omega\times\Gamma)} \leq \varepsilon^2\|\nabla^B[u_\varepsilon]\|^2_{L^2(\Gamma_\varepsilon)} \leq C\,,$$

where C is a constant independent of ε. Thus, $\sqrt{\varepsilon}\mathcal{T}^b_\varepsilon(\nabla^B[u_\varepsilon])$ is bounded in $L^2(\Omega\times\Gamma)$ uniformly with respect to ε and, therefore, $\sqrt{\varepsilon}\big(\sqrt{\varepsilon}\mathcal{T}^b_\varepsilon(\nabla^B[u_\varepsilon])\big)\to 0$ strongly in $L^2(\Omega\times\Gamma)$, as stated in (5.22).
In order to prove that the pair (u_0,u_1) solves problem (5.23), we proceed as in the proof of Theorem 5.1 (with the same test function), arriving to

$$
\begin{aligned}
&\int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega^{int}_\varepsilon}\nabla u^{int}_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla\phi_1)\,dxdy+\varepsilon\int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega^{int}_\varepsilon}\nabla u^{int}_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla_x\phi^{int}_2)\,dydx\\
&+\int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega^{int}_\varepsilon}\nabla u^{int}_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla_y\phi^{int}_2)\,dydx+\int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega^{out}_\varepsilon}\nabla u^{out}_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla\phi_1)\,dydx\\
&+\varepsilon\int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega^{out}_\varepsilon}\nabla u^{out}_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla_x\phi^{out}_2)\,dydx+\int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega^{out}_\varepsilon}\nabla u^{out}_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla_y\,\phi^{out}_2)\,dydx\\
&\qquad+\alpha\int\limits_{\Omega\times\Gamma}\mathcal{T}^b_\varepsilon([u_\varepsilon])\cdot\mathcal{T}^b_\varepsilon([\phi_2])\,d\sigma dx+\beta\varepsilon^2\int\limits_{\Omega\times\Gamma}\mathcal{T}^b_\varepsilon(\nabla^B[u_\varepsilon])\cdot\mathcal{T}^b_\varepsilon(\nabla^B_x[\phi_2])\,d\sigma dx\\
&\qquad\qquad+\beta\varepsilon\int\limits_{\Omega\times\Gamma}\mathcal{T}^b_\varepsilon(\nabla^B[u_\varepsilon])\cdot\mathcal{T}^b_\varepsilon(\nabla^B_y[\phi_2])\,d\sigma dx\\
&\qquad\qquad=\int\limits_{\Omega\times Y}\mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_1)\,dydx+\varepsilon\int\limits_{\Omega\times Y}\mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_2)\,dy\,dx\;+\;R_\varepsilon
\end{aligned}
$$

where $R_\varepsilon=o(1)$ for $\varepsilon\to 0$. Then, passing to the limit, up to a subsequence, we obtain

$$
\begin{aligned}
&\int\limits_{\Omega\times E^{int}} A\,(\nabla u_0+\nabla_y u^{int}_1)\cdot(\nabla\phi_1+\nabla_y\phi^{int}_2)\,dydx\\
&+\int\limits_{\Omega\times E^{out}} A\,(\nabla u_0+\nabla_y u^{out}_1)\cdot(\nabla\phi_1+\nabla_y\phi^{out}_2)\,dydx\\
&\qquad=\int\limits_{\Omega\times Y} f\,\phi_1\,dydx.
\end{aligned}
\tag{5.24}
$$

which is the weak formulation of the homogenized limit problem (5.23). For the uniqueness, following the same idea as in the proof of Theorem 5.1, we get

$$
\int\limits_{\Omega\times E^{int}}\left|\nabla U_0+\nabla_y U^{int}_1\right|^2 dydx\;+\int\limits_{\Omega\times E^{out}}\left|\nabla U_0+\nabla_y U^{out}_1\right|^2 dydx\le 0. \tag{5.25}
$$

This implies,

$$
\nabla U_0+\nabla_y U^{out}_1\;=\;0\qquad\Longleftrightarrow\qquad\nabla_y\,(\,y\cdot\nabla U_0+U^{out}_1)\;=\;0.
$$

Hence, the function $y\cdot\nabla U_0+U^{out}_1$ is a constant with respect to y and, exploiting the Y-periodicity of U_1, this leads to $\nabla U_0=0$ and U^{out}_1 independent of y. By recalling the homogeneous boundary condition, we arrive to $U_0=0$ and, by taking into account that U_1^{out} has null mean average over E^{out}, it follows that also $U_1^{out} = 0$. Now, using the fact that $U_0 = 0$, from (5.25) and taking into account that U_1^{int} has null mean average over E^{int}, we have that also $U_1^{int} = 0$. Hence, the pair (u_0, u_1) is unique. Therefore, the whole sequence, and not only a subsequence, converges. This concludes the thesis. $\square$

Proposition 5.6. *Let (u_0, u_1) be the unique weak solution of the two-scale problem (5.23), then the function u_1 can be uniquely factorize as*

$$u_1(x, y) = -\chi_{\mathcal{M}}(y) \cdot \nabla u_0(x), \tag{5.26}$$

where $\chi_{\mathcal{M}} = (\chi_{\mathcal{M}}^{int}, \chi_{\mathcal{M}}^{out}) : Y \to \mathbb{R}^N$, with $\chi_{\mathcal{M}}^{int} \in H^1_{per}(E^{int})$ and $\chi_{\mathcal{M}}^{out} \in H^1_{per}(E^{out})$, is the vector function with null mean average over E^{int} and E^{out}, separately, whose Y-periodic components satisfy the cell problem

$$\begin{cases} -\mathrm{div}_y\Big(A\,\nabla_y\big(\chi_{\mathcal{M}}^{j,int} - y_j\big)\Big) = 0 & \text{in} \quad E^{int}, & (5.27a) \\ -\mathrm{div}_y\Big(A\,\nabla_y\big(\chi_{\mathcal{M}}^{j,out} - y_j\big)\Big) = 0 & \text{in} \quad E^{out}, & (5.27b) \\ A\,\nabla_y\big(\chi_{\mathcal{M}}^{j,int} - y_j\big) \cdot \nu = 0 & \text{on} \quad \Gamma, & (5.27c) \\ A\,\nabla_y\big(\chi_{\mathcal{M}}^{j,out} - y_j\big) \cdot \nu = 0 & \text{on} \quad \Gamma. & (5.27d) \end{cases}$$

Moreover, u_0 is the unique solution of the homogenized problem

$$-\mathrm{div}\big(A_{\mathcal{M}}\,\nabla_x u_0\big) = f, \tag{5.28}$$

where $A_{\mathcal{M}}$ is the constant homogenized matrix, defined by

$$A_{\mathcal{M}} = A_{\mathcal{M}}^{int} + A_{\mathcal{M}}^{out} := \int_{E^{int}} A\,\big(I - \nabla_y \chi_{\mathcal{M}}^{int}(y)\big)\,dy + \int_{E^{out}} A\,\big(I - \nabla_y \chi_{\mathcal{M}}^{out}(y)\big)\,dy\,. \tag{5.29}$$

Notice that (5.27) is a system of two decoupled Neumann problems in E^{int} and E^{out}, respectively; therefore, the well-posedness is a classical matter. Moreover, it is well-known that the homogenized matrix $A_{\mathcal{M}}$ can be rewritten in the more meaningful form

$$A_{\mathcal{M}} = \int_{E^{int} \cup E^{out}} A\,\nabla_y\big(\chi_{\mathcal{M}}(y) - y\big)\,\nabla_y\big(\chi_{\mathcal{M}}(y) - y\big)\,dy\,.$$

Proof.
It is not difficult to prove that the function u_1 given in (5.26) satisfies (5.23b)–(5.23e); moreover, since problems (5.23) and (5.27) are well-posed, it is the unique solution. Now, let insert the factorization of u_1 given in (5.26) in (5.23a), so that

$$-\mathrm{div}\left(\Big(\int_{E^{int} \cup E^{out}} A\,(I - \nabla_y \chi_{\mathcal{M}}\,)\,dy\Big)\nabla u_0 \right) = f,$$

which corresponds to (5.28), once we take into account (5.29). The uniqueness of the solution of equation (5.28) is a consequence of the symmetry and positive definiteness of the matrix $A_{\mathcal{M}}$, which is a standard matter (see, for instance, [9, Remark 4.8]). $\square$

Remark 5.7.
We notice that, in the connected-disconnected geometrical setting, from (5.27a) and (5.27c), it is not difficult to prove that $\chi_{\mathcal{M}}^{j,int} = y_j$ (up to an additive constant), so that we obtain $A_{\mathcal{M}}^{int} = 0$ and the homogenized matrix reduces to

$$A_{\mathcal{M}} = \int_{E^{out}} A\,(I - \nabla_y \chi_{\mathcal{M}}^{out})\,dy = \int_{E^{out}} A\,\nabla_y\big(\chi_{\mathcal{M}}^{out}(y) - y\big)\,\nabla_y\big(\chi_{\mathcal{M}}^{out}(y) - y\big)\,dy = A_{\mathcal{M}}^{out}\,.$$

□

5.3. **The case** $m = 1$**.** In this section, we consider problem (3.1) for $m = 1$.

Theorem 5.8. *Assume to be in the connected-connected geometrical setting. For every* $\varepsilon > 0$*, let* $u_\varepsilon = (u_\varepsilon^{int}, u_\varepsilon^{out}) \in \mathcal{H}_{0,m}^{\varepsilon}(\Omega)$ *be the unique solution of problem* (3.1)*, for* $m = 1$*. Then, there exist* $u_0 = (u_0^{int}, u_0^{out}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and* $u_1 = (u_1^{int}, u_1^{out})$*, with* $u_1^{int} \in L^2(\Omega; H_{per}^1(E^{int}))$*,* $u_1^{out} \in L^2(\Omega; H_{per}^1(E^{out}))$ *and* $M_{E^{int}}(u_1) = 0 = M_{E^{out}}(u_1)$*, such that as* $\varepsilon \to 0$ *we have that*

$$\mathcal{T}_\varepsilon\big(\chi_{\Omega_\varepsilon^{int}}\,u_\varepsilon^{int}\big) \to u_0^{int} \qquad \textit{strongly in} \quad L^2(\Omega; H_{per}^1(E^{int})), \tag{5.30}$$

$$\mathcal{T}_\varepsilon\big(\chi_{\Omega_\varepsilon^{out}}\,u_\varepsilon^{out}\big) \to u_0^{out} \qquad \textit{strongly in} \quad L^2(\Omega; H_{per}^1(E^{out})), \tag{5.31}$$

$$\mathcal{T}_\varepsilon\big(\chi_{\Omega_\varepsilon^{int}}\,\nabla u_\varepsilon^{int}\big) \rightharpoonup \nabla u_0^{int} + \nabla_y u_1^{int} \qquad \textit{weakly in} \quad L^2(\Omega \times E^{int}), \tag{5.32}$$

$$\mathcal{T}_\varepsilon\big(\chi_{\Omega_\varepsilon^{out}}\,\nabla u_\varepsilon^{out}\big) \rightharpoonup \nabla u_0^{out} + \nabla_y u_1^{out} \qquad \textit{weakly in} \quad L^2(\Omega \times E^{out}), \tag{5.33}$$

$$\mathcal{T}_\varepsilon^b\big([u_\varepsilon]\big) \rightharpoonup [u_0] \qquad \textit{weakly in} \quad L^2(\Omega \times \Gamma), \tag{5.34}$$

$$\varepsilon^2\,\mathcal{T}_\varepsilon^b\big(\nabla^B[u_\varepsilon]\big) \to 0 \qquad \textit{strongly in} \quad L^2(\Omega \times \Gamma). \tag{5.35}$$

Moreover, the pair $(u_0,\, u_1)$ *is the unique weak solution of the following homogenized two-scale problem*

$$\begin{cases} -\mathrm{div}\Big(\displaystyle\int_{E^{int}} A(\nabla u_0^{int} + \nabla_y u_1^{int})dy\Big) - \alpha|\Gamma|[u_0] = f|E^{int}| & \textit{in } \Omega, & (5.36a)\\ -\mathrm{div}\Big(\displaystyle\int_{E^{out}} A(\nabla u_0^{out} + \nabla_y u_1^{out})dy\Big) + \alpha|\Gamma|[u_0] = f|E^{out}| & \textit{in } \Omega, & (5.36b)\\ -\mathrm{div}_y\Big(A(\nabla u_0^{int} + \nabla_y u_1^{int})\Big) = 0 & \textit{in } \Omega \times E^{int}, & (5.36c)\\ -\mathrm{div}_y\Big(A(\nabla u_0^{out} + \nabla_y u_1^{out})\Big) = 0 & \textit{in } \Omega \times E^{out}, & (5.36d)\\ A(\nabla u_0^{int} + \nabla_y u_1^{int}) \cdot \nu = 0 & \textit{on } \Omega \times \Gamma, & (5.36e)\\ A(\nabla u_0^{out} + \nabla_y u_1^{out}) \cdot \nu = 0 & \textit{on } \Omega \times \Gamma. & (5.36f) \end{cases}$$

Proof.
Taking into account (3.8) with $m = 1$, up to a subsequence, the convergences (5.30)–(5.34) follow from Theorem 4.7. For the convergence (5.35), we still take into account (3.8), obtaining

$$\|\varepsilon\mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])\|_{L^2(\Omega\times\Gamma)}^2 = \varepsilon^2\|\mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])\|_{L^2(\Omega\times\Gamma)}^2 \le \varepsilon^3\|\nabla^B[u_\varepsilon]\|_{L^2(\Gamma_\varepsilon)}^2 \le C,$$

where C is independent of ε. Thus, $\varepsilon \mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])$ is bounded in $L^2(\Omega \times \Gamma)$ uniformly with respect to ε and, therefore, $\varepsilon\big(\varepsilon \mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])\big) \to 0$ strongly in $L^2(\Omega\times\Gamma)$, as stated in (5.35).

In order to prove that the pair (u_0, u_1) solves problem (5.36), we choose in the weak formulation (3.4) the test function $\varphi_\varepsilon(x) = \phi_1(x)+\varepsilon\, \phi_2(x, \frac{x}{\varepsilon})$, where $\phi_1 = (\phi_1^{int},\, \phi_1^{out})$, with $\phi_1^{int}, \phi_1^{out} \in C_c^\infty(\Omega)$ and $\phi_2 = (\phi_2^{int},\, \phi_2^{out})$, with $\phi_2 \in C_c^\infty(\Omega; \mathcal{H}_{per}(Y))$. By unfolding, we get

$$
\begin{aligned}
&\int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}}\nabla u_\varepsilon^{int})\cdot\mathcal{T}_\varepsilon(\nabla\phi_1^{int})\,dydx+\varepsilon\int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}}\nabla u_\varepsilon^{int})\cdot\mathcal{T}_\varepsilon(\nabla_x\phi_2^{int})\,dydx\\
&+\int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}}\nabla u_\varepsilon^{int})\cdot\mathcal{T}_\varepsilon(\nabla_y\phi_2^{int})\,dydx+\int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}}\nabla u_\varepsilon^{out})\cdot\mathcal{T}_\varepsilon(\nabla\phi_1^{out})\,dydx\\
&+\varepsilon\int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}}\nabla u_\varepsilon^{out})\cdot\mathcal{T}_\varepsilon(\nabla_x\phi_2^{out})\,dydx+\int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\mathcal{T}_\varepsilon^{out}(\chi_{\Omega_\varepsilon^{out}}\nabla u_\varepsilon^{out})\cdot\mathcal{T}_\varepsilon(\nabla_y\phi_2^{out})\,dydx\\
&\qquad+\alpha\int\limits_{\Omega\times\Gamma}\mathcal{T}_\varepsilon^b([u_\varepsilon])\mathcal{T}_\varepsilon^b([\phi_1])\,d\sigma dx+\alpha\varepsilon\int\limits_{\Omega\times\Gamma}\mathcal{T}_\varepsilon^b([u_\varepsilon])\mathcal{T}_\varepsilon^b([\phi_2])d\sigma dx\\
&\quad+\beta\varepsilon^2\int\limits_{\Omega\times\Gamma}\mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])\cdot\mathcal{T}_\varepsilon^b(\nabla_x^B[\phi_1])\,d\sigma dx+\beta\varepsilon^3\int\limits_{\Omega\times\Gamma}\mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])\cdot\mathcal{T}_\varepsilon^b(\nabla_x^B[\phi_2])\,d\sigma dx\\
&\qquad+\beta\varepsilon^2\int\limits_{\Omega\times\Gamma}\mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])\cdot\mathcal{T}_\varepsilon^b(\nabla_y^B[\phi_2])\,d\sigma dx=\int\limits_{\Omega\times E^{int}}\mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_1^{int})\,dydx\\
&\qquad\quad+\varepsilon\int\limits_{\Omega\times E^{int}}\mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_2^{int})\,dydx+\int\limits_{\Omega\times E^{out}}\mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_1^{out})\,dydx\\
&\qquad\qquad\qquad+\varepsilon\int\limits_{\Omega\times E^{out}}\mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_2^{out})\,dydx+R_\varepsilon
\end{aligned}
$$

where $R_\varepsilon = o\,(1)$ for $\varepsilon \to 0$.

By passing to the limit, up to a subsequence, we arrive at

$$
\begin{aligned}
&\int\limits_{\Omega\times E^{int}} A(\nabla u_0^{int}+\nabla_y u_1^{int})\cdot(\nabla\phi_1^{int}+\nabla_y\phi_2^{int})\,dydx\\
&+\int\limits_{\Omega\times E^{out}} A(\nabla u_0^{out}+\nabla_y u_1^{out})\cdot(\nabla\phi_1^{out}+\nabla_y\phi_2^{out})\,dydx\\
&+\alpha\int\limits_{\Omega\times\Gamma}[u_0]\,[\phi_1]\,d\sigma\,dx=\int\limits_{\Omega\times E^{int}} f\phi_1^{int}\,dydx+\int\limits_{\Omega\times E^{int}} f\phi_1^{out}\,dydx,
\end{aligned}
\tag{5.37}
$$

which is the weak formulation of the homogenized limit problem (5.36). For the uniqueness, following the same idea as in the proof of Theorems 5.1 and 5.5, we get

$$\int_{\Omega\times E^{int}} |\nabla U_0^{int}+\nabla_y U_1^{int}|^2\,dydx+\int_{\Omega\times E^{out}} |\nabla U_0^{out}+\nabla_y U_1^{out}|^2\,dydx+\int_{\Omega\times\Gamma} |[U_0]|^2\,d\sigma\,dx\le 0.$$

This leads to $[U_0]=0$, i.e. $U_0^{int}=U_0^{out}$, and then, as in the proof of Theorem 5.5, we obtain $U_0=0=U_1$. Therefore, the whole sequence, and not only a subsequence, converges and, hence, the proof is accomplished. □

Proposition 5.9.
Let (u_0,u_1) be the unique weak solution of the two-scale problem (5.36), then the function u_1 can be uniquely factorize as

$$u_1(x,y)=\begin{cases}-\chi_{\mathcal{M}}^{int}(y)\,\cdot\,\nabla u_0^{int}(x) & in\quad E^{int},\\ -\chi_{\mathcal{M}}^{out}(y)\,\cdot\,\nabla u_0^{out}(x) & in\quad E^{out},\end{cases}\tag{5.38}$$

where $\chi_{\mathcal{M}}=(\chi_{\mathcal{M}}^{int},\chi_{\mathcal{M}}^{out}):Y\to\mathbb{R}^N$ is the same vector function appearing in Proposition 5.6. Moreover, u_0 satisfy the homogenized problem

$$\begin{cases}-\mathrm{div}\big(\,A_{\mathcal{M}}^{int}\nabla u_0^{int}\,\big)-\alpha\,|\Gamma|\,[u_0]\;=f\,|E^{int}| & in\quad\Omega,\\ -\mathrm{div}\big(A_{\mathcal{M}}^{out}\nabla u_0^{out}\,\big)+\,\alpha\,|\Gamma|\,[u_0]\;=\;f\,|E^{out}| & in\quad\Omega,\end{cases}\tag{5.39}$$

where the homogenized matrices $A_{\mathcal{M}}^{int}$ and $A_{\mathcal{M}}^{out}$, are defined in (5.29).

Proof.
It is not difficult to prove that the function u_1 defined in (5.38) is the unique solution of problem (5.36c)–(5.36f). Moreover, inserting the factorization of u_1 given in (5.38) in (5.36a) and (5.36b), we easily obtain (5.39). □

Remark 5.10. In the connected/disconnected geometrical setting, following Theorem 4.9, we have that the convergences stated in Theorem 5.8 remain true, except for (5.32), which is replaced by

$$\mathcal{T}_\varepsilon\big(\chi_{\Omega_\varepsilon^{int}}\nabla u_\varepsilon^{int}\big)\rightharpoonup\nabla u_0^{out}+\nabla_y u_1^{int}\quad weakly\ in\quad L^2(\Omega\times E^{int}).\tag{5.40}$$

and the fact that now $u_0^{int}\in L^2(\Omega)$ (instead of $u_0^{int}\in H_0^1(\Omega)$). Thus, we obtain the homogenized two-scale limit problem (5.36), with u_0^{int} replaced by u_0^{out}. Moreover, as above, we have that $u_1^{int}=-\,y\cdot\nabla u_0^{out}$, up to a normalization function independent of y, which implies that (5.36a) becomes

$$-\alpha[u_0]|\Gamma|=f|E^{int}|\quad\text{or, equivalently,}\quad u_0^{int}=\frac{|E^{int}|}{\alpha|\Gamma|}f+u_0^{out}.\tag{5.41}$$

Hence, the bidomain limit problem (5.39) reduces to

$$\begin{cases}-\mathrm{div}\big(A_{\mathcal{M}}^{out}\cdot\nabla u_0^{out}\,\big)=f & in\quad\Omega,\\ u_0^{int}=\frac{|E^{int}|}{\alpha|\Gamma|}f+u_0^{out} & in\quad\Omega,\end{cases}\tag{5.42}$$

with $A_{\mathcal{M}}^{out}$ given in (5.29).

6. Homogenization of the problem $\mathcal{B}_\varepsilon^m$

This section is devoted to analyze problem $\mathcal{B}_\varepsilon^m$ (3.2), introduced in Section 3. We will consider both the connected/connected and the connected/disconnected geometrical settings, and we prove in a rigorous way what we formally obtained in the previous paper [8].

6.1. The case $m = -1$.

In this section, we consider the problem (3.2) for the scaling $m = -1$.

Theorem 6.1. *For every $\varepsilon > 0$, let $u_\varepsilon = (u_\varepsilon^{int}, u_\varepsilon^{out}) \in \widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$ be the unique solution of problem (3.2), for $m = -1$. Then, there exist $u_0 \in H^1_0(\Omega)$ and $u_1 = (u_1^{int}, u_1^{out}) \in L^2(\Omega;\, \widehat{\mathcal{H}}_{per}(Y))$ with $M_Y(u_1) = 0$, such that, as $\varepsilon \to 0$, we have that (5.1)–(5.7) hold. Moreover,*

$$\mathcal{T}_\varepsilon^b\big(\nabla^B\{u_\varepsilon\}\big) \rightharpoonup \nabla^B\{u_0\} + \nabla^B_y\{u_1\} \qquad \text{weakly in} \quad L^2(\Omega\times\Gamma). \tag{6.1}$$

Finally, the pair $(u_0,\, u_1)$ is the unique weak solution of the following homogenized two-scale problem

$$\begin{cases} -\mathrm{div}\Big(\displaystyle\int_{E^{int}\cup E^{out}} A(\nabla u_0 + \nabla_y u_1)\, dy + \gamma \int_\Gamma (\nabla^B\{u_0\} + \nabla^B_y\{u_1\})\, d\sigma\Big) = f & \text{in } \Omega, \quad (6.2a)\\ -\,\mathrm{div}_y\Big(\, A\,(\nabla u_0 + \nabla_y u_1^{int})\,\Big) = 0 & \text{in } \Omega\times E^{int}, \quad (6.2b)\\ -\,\mathrm{div}_y\Big(\, A\,(\nabla u_0 + \nabla_y u_1^{out})\,\Big) = 0 & \text{in } \Omega\times E^{out}, \quad (6.2c)\\ -\,\gamma\,\mathrm{div}^B_y(\nabla^B\{u_0\} + \nabla^B_y\{u_1\}) = [\big(\nabla u_0 + \nabla_y u_1\big)\cdot\nu] & \text{on } \Omega\times\Gamma, \quad (6.2d)\\ \alpha\,[u_1] \,-\, \beta\,\triangle^B_y\,[u_1] \,=\, \{\,A\,(\nabla u_0 + \nabla_y u_1)\cdot\nu\} & \text{on } \Omega\times\Gamma. \quad (6.2e) \end{cases}$$

Remark 6.2. We recall that

$$\nabla^B\{u_0\} = 2(I - \nu\otimes\nu)\nabla u_0 = \nabla^B_y\{y\}\nabla u_0\,.$$

□

Proof.
Taking into account (3.8) with $m = -1$, the convergences in (5.1)–(5.6) are consequence of Theorem 4.7 in the connected/connected case and of Theorem 4.9 in the connected/disconnected case, while (5.7) is proven in Theorem 5.1. In order to prove (6.1), again by (3.8) jointly with (3.9), we get

$$\int_\Omega\int_\Gamma \big|\mathcal{T}_\varepsilon^b\,(\nabla^B[u_\varepsilon])\,\big|^2\, d\sigma dx \,\le\, \varepsilon\int_{\Gamma_\varepsilon} |\,\nabla^B[u_\varepsilon]\,|^2\, d\sigma \,\le C,$$

and

$$\int_\Omega\int_\Gamma \big|\mathcal{T}_\varepsilon^b\,(\nabla^B\{u_\varepsilon\})\,\big|^2\, d\sigma dx \,\le\, \varepsilon\int_{\Gamma_\varepsilon} |\,\nabla^B\{u_\varepsilon\}\,|^2\, d\sigma \,\le C,$$

which implies

$$\int_\Omega \int_\Gamma \left|\mathcal{T}_\varepsilon^b(\nabla^B u_\varepsilon^{int})\right|^2 dy\,dx \le C \qquad \text{and} \qquad \int_\Omega \int_\Gamma \left|\mathcal{T}_\varepsilon^b(\nabla^B u_\varepsilon^{out})\right|^2 dy\,dx \le C,$$

where C is a constant independent of ε. Hence, up to a subsequence (still denoted by ε), there exist two vector functions ζ_b^{int}, $\zeta_b^{out} \in L^2(\Omega \times \Gamma)$, such that, as $\varepsilon \to 0$, we have

$$\mathcal{T}_\varepsilon^b(\nabla^B u_\varepsilon^{int}) \rightharpoonup \zeta_b^{int}, \quad \mathcal{T}_\varepsilon^b(\nabla^B u_\varepsilon^{out}) \rightharpoonup \zeta_b^{out} \quad \textit{weakly in } L^2(\Omega \times \Gamma).$$

In order to identify ζ_b^{out}, we consider a test function $\varphi(x,y) = \phi_1(x)\Psi_2(y)$, where $\phi_1 \in C_c^\infty(\Omega)$ and the vector functon $\Psi_2 \in C_{per}^\infty(Y)$. Hence, by Theorem 4.8 in the connected/connected case and Theorem 4.10 in the connected/disconnected case, we get

$$\begin{aligned}
\int_{\Omega\times\Gamma} \zeta_b^{out} \cdot \Psi_2(y)\phi_1(x) d\sigma dx &\leftharpoondown \int_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B u_\varepsilon^{out}) \cdot \Psi_2(y)\phi_1(x)\, d\sigma dx \\
&= \int_{\Omega\times\Gamma} \nabla_y^B \left(\frac{\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}} u_\varepsilon^{out}) - M_\Gamma(\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}} u_\varepsilon^{out}))}{\varepsilon} \right) \cdot \Psi_2(y)\phi_1(x)\, d\sigma dx \\
&= -\int_{\Omega\times\Gamma} \frac{\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}} u_\varepsilon^{out}) - M_\Gamma(\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}} u_\varepsilon^{out}))}{\varepsilon} \operatorname{div}_y^B \Psi_2(y)\phi_1(x)\, d\sigma dx \\
&\rightharpoonup -\int_{\Omega\times\Gamma} (y_\Gamma \cdot \nabla u_0 + u_1^{out} + \xi^{out}) \operatorname{div}_y^B \Psi_2(y)\phi_1(x)\, d\sigma dx \\
&= \int_{\Omega\times\Gamma} \nabla_y^B (y_\Gamma \cdot \nabla u_0 + u_1^{out} + \xi^{out}) \cdot \Psi_2(y)\phi_1(x)\, d\sigma dx \\
&= \int_{\Omega\times\Gamma} (\nabla^B u_0 + \nabla_y^B u_1^{out}) \cdot \Psi_2(y)\phi_1(x)\, d\sigma dx.
\end{aligned}$$

This implies $\zeta_b^{out} = \nabla^B u_0 + \nabla_y^B u_1^{out}$. Similarly, we get $\zeta_b^{int} = \nabla^B u_0 + \nabla_y^B u_1^{int}$; hence, from the linearity of $\mathcal{T}_\varepsilon^b$, (6.1) follows and $u_1 = (u_1^{int}, u_1^{out}) \in L^2(\Omega; \widehat{\mathcal{H}}_{per}(Y))$.
In order to prove that the pair (u_0, u_1) is the solution of the two-scale problem (6.2), let us choose, in the weak formulation (3.5), the test function $\varphi_\varepsilon(x) = \phi_1(x) + \varepsilon\, \phi_2(x, \frac{x}{\varepsilon})$, where $\phi_1 \in C_c^\infty(\Omega)$ and $\phi_2 = (\phi_2^{int}, \phi_2^{out})$ with $\phi_2 \in C_c^\infty(\Omega; \widehat{\mathcal{H}}_{per}(Y))$. By unfolding,

we get

$$
\begin{aligned}
&\int_{\Omega}\int_{E^{int}} \mathcal{T}_{\varepsilon}^{int}(A_{\varepsilon})\,\mathcal{T}_{\varepsilon}^{int}(\nabla u_{\varepsilon}^{int})\cdot\mathcal{T}_{\varepsilon}^{int}(\nabla\phi_1)dy\,dx \\
&\qquad + \varepsilon\int_{\Omega}\int_{E^{int}} \mathcal{T}_{\varepsilon}^{int}(A_{\varepsilon})\mathcal{T}_{\varepsilon}^{int}(\nabla u_{\varepsilon}^{int})\cdot\mathcal{T}_{\varepsilon}^{int}(\nabla_x\phi_2^{int})dy\,dx \\
&\qquad + \int_{\Omega}\int_{E^{int}} \mathcal{T}_{\varepsilon}^{int}(A_{\varepsilon})\mathcal{T}_{\varepsilon}^{int}(\nabla u_{\varepsilon}^{int})\cdot\mathcal{T}_{\varepsilon}^{int}(\nabla_y\phi_2^{int})\,dy\,dx \\
&\qquad + \int_{\Omega}\int_{E^{out}} \mathcal{T}_{\varepsilon}^{out}(A_{\varepsilon})\mathcal{T}_{\varepsilon}^{out}(\nabla u_{\varepsilon}^{out})\cdot\mathcal{T}_{\varepsilon}^{out}(\nabla\phi_1)\,dy\,dx \\
&\qquad + \varepsilon\int_{\Omega}\int_{E^{out}} \mathcal{T}_{\varepsilon}^{out}(A_{\varepsilon})\mathcal{T}_{\varepsilon}^{out}(\nabla u_{\varepsilon}^{out})\cdot\mathcal{T}_{\varepsilon}^{out}(\nabla_x\phi_2^{out})\,dy\,dx \\
&\qquad + \int_{\Omega}\int_{E^{out}} \mathcal{T}_{\varepsilon}^{out}(A_{\varepsilon})\mathcal{T}_{\varepsilon}^{out}(\nabla u_{\varepsilon}^{out})\cdot\mathcal{T}_{\varepsilon}^{out}(\nabla_y\phi_2^{out})\,dy\,dx \\
&+ \frac{\alpha}{2\,\varepsilon}\int_{\Omega}\int_{\Gamma} \mathcal{T}_{\varepsilon}^{b}([u_{\varepsilon}])\cdot\mathcal{T}_{\varepsilon}^{b}([\,\phi_2\,])\,dx\,d\sigma + \frac{\beta}{2}\,\varepsilon\int_{\Omega}\int_{\Gamma} \mathcal{T}_{\varepsilon}^{b}(\nabla^{B}[u_{\varepsilon}])\cdot\mathcal{T}_{\varepsilon}^{b}(\nabla_x^{B}[\phi_2])\,d\sigma\,dx \\
&+ \frac{\beta}{2}\int_{\Omega}\int_{\Gamma} \mathcal{T}_{\varepsilon}^{b}(\nabla^{B}[u_{\varepsilon}])\cdot\mathcal{T}_{\varepsilon}^{b}(\nabla_y^{B}[\,\phi_2\,])\,d\sigma\,dx + \frac{\gamma}{2}\int_{\Omega}\int_{\Gamma} \mathcal{T}_{\varepsilon}^{b}(\nabla^{B}\{u_{\varepsilon}\})\cdot\mathcal{T}_{\varepsilon}^{b}(\nabla^{B}\,\{\phi_1\})\,d\sigma\,dx \\
&+ \frac{\gamma}{2}\,\varepsilon\int_{\Omega}\int_{\Gamma} \mathcal{T}_{\varepsilon}^{b}(\nabla^{B}\{u_{\varepsilon}\})\cdot\mathcal{T}_{\varepsilon}^{b}(\nabla_x^{B}\{\phi_2\})\,d\sigma\,dx + \frac{\gamma}{2}\int_{\Omega}\int_{\Gamma} \mathcal{T}_{\varepsilon}^{b}(\nabla^{B}\{u_{\varepsilon}\})\cdot\mathcal{T}_{\varepsilon}^{b}(\nabla_y^{B}\{\,\phi_2\,\})\,d\sigma\,dx \\
&\qquad\qquad = \int_{\Omega}\int_{\Gamma} \mathcal{T}_{\varepsilon}(f)\,(\mathcal{T}_{\varepsilon}(\phi_1) + \varepsilon\,\mathcal{T}_{\varepsilon}(\phi_2)\,)\,d\sigma\,dx \;+\; R_{\varepsilon},
\end{aligned}
$$

where $R_{\varepsilon} = o\,(1)$ for $\varepsilon \to 0$.
Then, by passing to the limit $\varepsilon \to 0$ and taking into account (5.1)–(5.7) and (6.1),

we arrive at

$$
\begin{aligned}
&\int_{\Omega\times E^{int}} A\,(\nabla u_0+\nabla_y u_1^{int})\cdot(\nabla\phi_1+\nabla_y\phi_2^{int})dy\,dx\\
&+\int_{\Omega\times E^{out}} A(\nabla u_0+\nabla_y u_1^{out})\cdot(\nabla\phi_1+\nabla_y\phi_2^{out})\,dy\,dx\\
&+\frac{\alpha}{2}\int_{\Omega\times\Gamma}[u_1]\,[\phi_2]\,dx\,d\sigma+\frac{\beta}{2}\int_{\Omega\times\Gamma}\nabla_y^B[u_1]\cdot\nabla_y^B[\phi_2]\,d\sigma\,dx\\
&+\frac{\gamma}{2}\int_{\Omega\times\Gamma}(\nabla^B\{u_0\}+\nabla_y^B\{u_1\})\cdot(\nabla^B\{\phi_1\}+\nabla_y^B\{\phi_2\})\,d\sigma\,dx\\
&=\int_{\Omega\times Y} f\phi_1\,d\sigma\,dx.
\end{aligned}
\tag{6.3}
$$

Clearly, (6.3) is the weak formulation of the homogenized limit problem (6.2). Indeed, by taking $\phi_2:=(\phi_2^{int},\,\phi_2^{out})\equiv 0$, it easily follows (6.2a). On the other hand, by taking first, $\phi_1\equiv 0$ with $\phi_2^{out}\equiv 0$, and then $\phi_1\equiv 0$ with $\phi_2^{int}\equiv 0$ in (6.3), we obtain

$$
\begin{aligned}
&\int_{\Omega\times E^{int}} A\,(\nabla u_0+\nabla_y u_1^{int})\cdot\nabla_y\phi_2^{int}dy\,dx\\
&-\frac{\alpha}{2}\int_{\Omega\times\Gamma}[u_1]\,\phi_2^{int}\,dx\,d\sigma-\frac{\beta}{2}\int_{\Omega\times\Gamma}\nabla_y^B[u_1]\cdot\nabla_y^B\,\phi_2^{int}\,d\sigma\,dx\\
&+\frac{\gamma}{2}\int_{\Omega\times\Gamma}(\nabla^B\{u_0\}+\nabla_y^B\{u_1\})\cdot\,\nabla_y^B\phi_2^{int}\,d\sigma\,dx=0,
\end{aligned}
\tag{6.4}
$$

$$
\begin{aligned}
&\int_{\Omega\times E^{out}} A\,(\nabla u_0+\nabla_y u_1^{out})\cdot\nabla_y\phi_2^{out}dy\,dx\\
&+\frac{\alpha}{2}\int_{\Omega\times\Gamma}[u_1]\,\phi_2^{out}\,dx\,d\sigma+\frac{\beta}{2}\int_{\Omega\times\Gamma}\nabla_y^B[u_1]\cdot\nabla_y^B\,\phi_2^{out}\,d\sigma\,dx\\
&+\frac{\gamma}{2}\int_{\Omega\times\Gamma}(\nabla^B\{u_0\}+\nabla_y^B\{u_1\})\cdot\,\nabla_y^B\phi_2^{out}\,d\sigma\,dx=0.
\end{aligned}
\tag{6.5}
$$

Clearly, (6.4) and (6.5) are the weak formulation of (6.2b)–(6.2e). Finally, the uniqueness of the pair u_0, u_1 can be obtained as in the proof of Theorem 5.1, so that the whole sequence, and not only a subsequence, converges. □

Proposition 6.3.
Let (u_0, u_1) be the unique weak solution of the two-scale problem (6.2). Then the function u_1 can be uniquely factorize as

$$u_1(x,y)=-\chi_{\mathcal{Q}}(y)\cdot\nabla u_0(x) \tag{6.6}$$

where $\chi_Q = (\chi_Q^1, \dots, \chi_Q^N) \in \widehat{\mathcal{H}}_{per}(Y)$ *is the vector function with null mean average over* Y*, whose* Y*-periodic components satisfy the cell problem*

$$\begin{cases} -\mathrm{div}_y\Big(A\,\nabla_y\left(\chi_Q^{j,int} - y_j\right)\Big) = 0 & in \quad E^{int}, & (6.7a) \\ -\mathrm{div}_y\Big(A\,\nabla_y\left(\chi_Q^{j,out} - y_j\right)\Big) = 0 & in \quad E^{out}, & (6.7b) \\ -\gamma\triangle_y^B\{\chi_Q^j - y_j\} = \big[A\nabla_y\big(\chi_Q^j - y_j\big)\cdot\nu\big] & on \quad \Gamma, & (6.7c) \\ \alpha[\chi_Q^j] - \beta\,\triangle_y^B[\chi_Q^j] = \{A\,\nabla_y\big(\chi_Q^j - y_j\big)\cdot\nu\} & on \quad \Gamma. & (6.7d) \end{cases}$$

Moreover, $u_0 \in H_0^1(\Omega)$ *is the unique solution of the homogenized problem*

$$-\mathrm{div}\big(A_Q\,\nabla u_0 \big) = f \qquad in \quad \Omega, \tag{6.8}$$

where A_Q *is the symmetric and positive definite constant homogenized matrix, defined by*

$$A_Q := \int\limits_{E^{int}\cup E^{out}} A(I - \nabla_y\chi_Q(y))\,dy + \gamma\int\limits_\Gamma \nabla_y^B\{y - \chi_Q\}\,d\sigma\,. \tag{6.9}$$

We remark that the well-posedness of problem (6.7) is a consequence of the Lax-Milgram Lemma, applied in the periodic framework as in Section 5.1.

Proof.
It is not difficult to see that the function u_1 given in (6.6) satisfies (6.2b)–(6.2e); moreover, since problems (6.2) and (6.7) are well-posed, it is the unique solution. Now, let us insert the factorization of u_1 given in (6.6) in (6.2a), so that

$$-\mathrm{div}\Big(\int\limits_{E^{int}\cup E^{out}} A(\nabla u_0 - \nabla_y\chi_Q\,\nabla u_0)\,dy + \gamma\int\limits_\Gamma(\nabla^B\{u_0\} - \nabla_y^B\{\chi_Q\}\,\nabla u_0)d\sigma\Big) = f \quad in\ \Omega,$$

Then, (6.8) follows, by taking into account (6.9). The uniqueness of the solution of equation (6.8) is a consequence of the symmetry and the positive-definiteness of the matrix A_Q (see [8, Theorem 5.1]). □

For the reader convenience, we recall that, as proven in [8, Theorem 5.1], the matrix A_Q can be written in the more meaningful form

$$\begin{aligned} (A_Q)_{ij} = &\int\limits_{E^{int}\cup E^{out}} A\,\nabla_y(\,\chi_Q^j - y_j)\cdot\nabla_y(\,\chi_Q^i - y_i)\,dy + \frac{\alpha}{2}\int\limits_\Gamma \big[\chi_Q^j\big]\,\big[\chi_Q^i\,\big]\,d\sigma \\ &+\frac{\beta}{2}\int\limits_\Gamma \nabla_y^B\big[\chi_Q^j\big]\cdot\nabla_y^B[\chi_Q^i]\,d\sigma + \frac{\gamma}{2}\int\limits_\Gamma \nabla_y^B\{\chi_Q^j - y_j\}\cdot\nabla_y^B\{\chi_Q^i - y_i\}\,d\sigma\,. \end{aligned} \tag{6.10}$$

6.2. **The case** $m = 0$**.** In this section, we consider problem (3.2) for the scaling $m = 0$.

Theorem 6.4. *For any* $\varepsilon > 0$*, let* $u_\varepsilon = (u_\varepsilon^{int}, u_\varepsilon^{out}) \in \widehat{\mathcal{H}}_{0,m}^\varepsilon(\Omega)$ *be the unique solution of problem (3.2), for* $m = 0$*. Then, there exist* $u_0 \in H_0^1(\Omega)$ *and* $u_1 = (u_1^{int}, u_1^{out})$*, with* $u_1^{int} \in L^2(\Omega; H_{per}^1(E^{int}))$*,* $u_1^{out} \in L^2(\Omega; H_{per}^1(E^{out}))$ *and* $M_{E^{int}}(u_1^{int}) = 0 =$

$M_{E^{out}}(u_1^{out})$, such that as $\varepsilon \to 0$ (5.1)–(5.5) and (5.21), (5.22) hold. Moreover, we have

$$\varepsilon\, \mathcal{T}_\varepsilon^b\big(\nabla^B\{u_\varepsilon\}\big) \to 0 \qquad \textit{strongly in} \quad L^2(\Omega\times\Gamma). \tag{6.11}$$

Finally, the pair (u_0, u_1) is the unique solution of the homogenized two-scale problem (5.23).

Proof.
Taking into account (3.8) with $m = 0$, up to a subsequence, the convergences (5.1)–(5.5) and (5.21) follow from Theorem 4.7, for the connected-connected geometrical settings, and from Theorem 4.9, for the connected-disconnected geometrical settings, while (5.22) has been proven in Theorem 5.5. Moreover, by (3.9), we obtain

$$||\varepsilon\mathcal{T}_\varepsilon^b(\nabla^B\{u_\varepsilon\})||^2_{L^2(\Omega\times\Gamma)} = \varepsilon^2||\mathcal{T}_\varepsilon^b(\nabla^B\{u_\varepsilon\})||^2_{L^2(\Omega\times\Gamma)} \le \varepsilon^3||\nabla^B\{u_\varepsilon\}||^2_{L^2(\Gamma_\varepsilon)} \le C\varepsilon,$$

which immediately implies (6.11).
In order to prove that the pair (u_0, u_1) is the solution of the two-scale problem (5.23), let us choose, in the weak formulation (3.5) the test function $\varphi_\varepsilon(x) = \phi_1(x) + \varepsilon\,\phi_2(x, \frac{x}{\varepsilon})$, where $\phi_1 \in C_c^\infty(\Omega)$ and $\phi_2 = (\phi_2^{int}, \phi_2^{out})$ with $\phi_2 \in C_c^\infty(\Omega; \widehat{\mathcal{H}}_{per}(Y))$, so that, by unfolding, we get

$$\begin{aligned}
&\int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}}\nabla u_\varepsilon^{int})\cdot\mathcal{T}_\varepsilon(\nabla\phi_1)\,dxdy + \varepsilon\int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}}\nabla u_\varepsilon^{int})\cdot\mathcal{T}_\varepsilon(\nabla_x\phi_2^{int})\,dydx\\
&+\int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}}\nabla u_\varepsilon^{int})\cdot\mathcal{T}_\varepsilon(\nabla_y\phi_2^{int})\,dydx + \int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}}\nabla u_\varepsilon^{out})\cdot\mathcal{T}_\varepsilon(\nabla\phi_1)\,dydx\\
&+\varepsilon\int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}}\nabla u_\varepsilon^{out})\cdot\mathcal{T}_\varepsilon(\nabla_x\phi_2^{out})\,dydx + \int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\,\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}}\nabla u_\varepsilon^{out})\cdot\mathcal{T}_\varepsilon(\nabla_y\,\phi_2^{out})\,dydx\\
&\qquad+\frac{\alpha}{2}\int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b([u_\varepsilon])\,\mathcal{T}_\varepsilon^b([\phi_2])\,d\sigma dx + \frac{\beta}{2}\varepsilon^2\int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])\cdot\mathcal{T}_\varepsilon^b(\nabla_x^B[\phi_2])\,d\sigma dx\\
&+\frac{\beta}{2}\varepsilon\int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])\cdot\mathcal{T}_\varepsilon^b(\nabla_y^B[\phi_2])\,d\sigma dx + \frac{\gamma}{2}\varepsilon\int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B\{u_\varepsilon\})\cdot\mathcal{T}_\varepsilon^b(\nabla_x^B\{\phi_1+\varepsilon\phi_2\})\,d\sigma dx\\
&+\frac{\gamma}{2}\varepsilon\int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B\{u_\varepsilon\})\cdot\mathcal{T}_\varepsilon^b(\nabla_y^B\{\phi_2\})\,d\sigma dx = \int\limits_{\Omega\times Y} \mathcal{T}_\varepsilon(f)\big(\mathcal{T}_\varepsilon(\phi_1)+\varepsilon\mathcal{T}_\varepsilon(\phi_2)\big)\,dydx + R_\varepsilon,
\end{aligned}$$

where $R_\varepsilon = o(1)$ for $\varepsilon \to 0$.
Then, by passing to the limit $\varepsilon \to 0$, up to a subsequence, we arrive at (5.24), which is the weak formulation of two-scale homogenized problem (5.23), whose uniqueness (proven in Theorem 5.5) guarantees the convergence of the whole sequence. □

Remark 6.5. It is worthwhile noting that, when $m = 0$, the physical quantities keeping into account the interface diffusivities disappear. In particular, the tangential diffusivity represented by γ does not affect the limit problem, so that, at the end, the two different microscopic problems (3.1) and (3.2), for $m = 0$, give rise to the same homogenized model. Therefore, the factorization (5.26) is still in force and, then, the limit problem can be written as in (5.28), with the matrix $A_{\mathcal{M}}$ defined in (5.29). Moreover, Remark 5.7 still holds. □

6.3. **The case** $m = 1$. In this section, we consider the problem (3.2) for the scaling $m = 1$.

Theorem 6.6. *Assume to be in the connected-connected geometrical setting. For every* $\varepsilon > 0$, *let* $u_\varepsilon = (u_\varepsilon^{int}, u_\varepsilon^{out}) \in \widehat{\mathcal{H}}^\varepsilon_{0,m}(\Omega)$ *be the unique solution of problem* (3.2), *for* $m = 1$. *Then, there exist* $u_0 = (u_0^{int}, u_0^{out}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and* $u_1 = (u_1^{int}, u_1^{out})$, *with* $u_1^{int} \in L^2(\Omega; H^1_{per}(E^{int}))$, $u_1^{out} \in L^2(\Omega; H^1_{per}(E^{out}))$ *and* $M_{E^{int}}(u_1) = 0 = M_{E^{out}}(u_1)$, *such that, as* $\varepsilon \to 0$, *we have that (5.30)–(5.35) are in force. Moreover, we have*

$$\varepsilon^2\, \mathcal{T}_\varepsilon^b\big(\nabla^B\{u_\varepsilon\}\big) \to 0 \qquad \textit{strongly in } L^2(\Omega \times \Gamma). \tag{6.12}$$

Finally, the pair (u_0, u_1) *is the unique weak solution of the homogenized two-scale problem* (5.36) *with* α *replaced by* $\alpha/2$.

Proof.
Taking into account (3.8) with $m = 1$, the convergences (5.30)–(5.34) follow from Theorem 4.8, while (5.35) is proven in Theorem 5.8. Moreover, by (3.9), we obtain

$$\begin{aligned}\|\varepsilon^2 \mathcal{T}_\varepsilon^b(\nabla^B\{u_\varepsilon\})\|^2_{L^2(\Omega\times\Gamma)} &= \varepsilon^4 \|\mathcal{T}_\varepsilon^b(\nabla^B\{u_\varepsilon\})\|^2_{L^2(\Omega\times\Gamma)} \\ &\le \varepsilon^5 \|\nabla^B\{u_\varepsilon\}\|^2_{L^2(\Gamma_\varepsilon)} \le C\varepsilon^2,\end{aligned}$$

which immediately implies (6.12).
Now, in order to prove that the pair (u_0, u_1) solves problem (5.36), we choose, in the weak formulation (3.5) the test function $\varphi_\varepsilon(x, y) = \phi_1(x) + \varepsilon\, \phi_2(x, \frac{x}{\varepsilon})$, where $\phi_1 = (\phi_1^{int}, \phi_1^{out}) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$ and $\phi_2 = (\phi_2^{int}, \phi_2^{out})$ with $\phi_2 \in C_c^\infty(\Omega; \widehat{\mathcal{H}}_{per}(Y))$. By

unfolding, we get

$$\int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\, \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}} \nabla u_\varepsilon^{int})\cdot \mathcal{T}_\varepsilon(\nabla \phi_1^{int})\, dxdy + \varepsilon \int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\, \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}} \nabla u_\varepsilon^{int})\cdot \mathcal{T}_\varepsilon(\nabla_x \phi_2^{int})\, dydx$$
$$+ \int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(A_\varepsilon)\, \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{int}} \nabla u_\varepsilon^{int})\cdot \mathcal{T}_\varepsilon(\nabla_y \phi_2^{int})\, dydx + \int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\, \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}} \nabla u_\varepsilon^{out})\cdot \mathcal{T}_\varepsilon(\nabla \phi_1^{out})\, dydx$$
$$+\varepsilon \int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\, \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}} \nabla u_\varepsilon^{out})\cdot \mathcal{T}_\varepsilon(\nabla_x \phi_2^{out})\, dydx + \int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(A_\varepsilon)\, \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^{out}} \nabla u_\varepsilon^{out})\cdot \mathcal{T}_\varepsilon(\nabla_y\, \phi_2^{out})\, dydx$$
$$+\frac{\alpha}{2} \int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b[u_\varepsilon]\, \mathcal{T}_\varepsilon^b[\phi_1]\, d\sigma dx + \frac{\alpha}{2}\varepsilon \int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b([u_\varepsilon])\, \mathcal{T}_\varepsilon^b([\phi_2])\, d\sigma dx + \frac{\beta}{2}\,\varepsilon^2 \int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon])\cdot \mathcal{T}_\varepsilon^b(\nabla^B[\phi_1])\, d\sigma dx$$
$$+\frac{\beta}{2}\,\varepsilon^3 \int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon]) \cdot \mathcal{T}_\varepsilon^b(\nabla_x^B[\phi_2])\, d\sigma dx + \frac{\beta}{2}\,\varepsilon^2 \int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B[u_\varepsilon]) \cdot \mathcal{T}_\varepsilon^b(\nabla_y^B[\phi_2])\, d\sigma dx$$
$$+\frac{\gamma}{2}\,\varepsilon^2 \int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B\{u_\varepsilon\})\cdot \mathcal{T}_\varepsilon^b(\nabla^B\{\phi_1\})\, d\sigma dx + \frac{\gamma}{2}\,\varepsilon^3 \int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B\{u_\varepsilon\})\cdot \mathcal{T}_\varepsilon^b(\nabla_x^B\{\phi_2\})\, d\sigma dx$$
$$+\frac{\gamma}{2}\,\varepsilon^2 \int\limits_{\Omega\times\Gamma} \mathcal{T}_\varepsilon^b(\nabla^B\{u_\varepsilon\}) \cdot \mathcal{T}_\varepsilon^b(\nabla_y^B\{\phi_2\})\, d\sigma dx$$
$$= \int\limits_{\Omega\times E^{int}} \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_1^{int} + \varepsilon\phi_2^{int})\, dydx + \int\limits_{\Omega\times E^{out}} \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_1^{out} + \varepsilon\phi_2^{out})\, dydx + R_\varepsilon$$

where $R_\varepsilon = o\,(1)$ for $\varepsilon \to 0$.
By passing to the limit $\varepsilon \to 0$, up to a subsequence, we arrive at

$$\begin{aligned} &\int\limits_{\Omega\times E^{int}} A\,(\nabla u_0^{int} + \nabla_y u_1^{int}) \cdot (\nabla \phi_1^{int} + \nabla_y \phi_2^{int})\, dydx \\ &+ \int\limits_{\Omega\times E^{out}} A\,(\nabla u_0^{out} + \nabla_y u_1^{out}) \cdot (\nabla \phi_1^{out} + \nabla_y \phi_2^{out})\, dydx \\ &+\frac{\alpha}{2} \int\limits_{\Omega\times\Gamma} [u_0]\,[\,\phi_1\,]\, d\sigma\, dx = \int\limits_{\Omega\times E^{int}} f\phi_1^{int}\, dydx + \int\limits_{\Omega\times E^{out}} f\phi_1^{out}\, dydx, \end{aligned} \tag{6.13}$$

which gives the weak formulation of two-scale problem (5.36). Again, the uniqueness for such a problem, proved in Theorem 5.8, guarantees the convergence of the whole sequence. □

Remark 6.7. Notice that, due to the previous result, the factorization given in (5.38) is still in force, so that we can obtain the same bidomain problem (5.39), with α replaced by $\alpha/2$. □

Remark 6.8. As in Section 5.3, in the connected-disconnected geometrical setting, following Theorem 4.9, we have that the convergences stated in Theorem 6.6 remain true, except for (5.32), which should be replaced by (5.40) and the fact that $u_0^{int} \in L^2(\Omega)$ (instead of $u_0^{int} \in H_0^1(\Omega)$). Therefore, we still arrive to the bidomain limit problem (5.42), with α replaced by $\alpha/2$. $\square$

References

[1] G. Allaire. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*, 23(6):1482–1518, 1992.

[2] G. Allaire and F. Murat. Homogenization of the Neumann problem with nonisolated holes. *Asymptotic Analysis*, 7(2):81–95, 1993.

[3] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni. On a hierarchy of models for electrical conduction in biological tissues. *Mathematical Methods in the Applied Sciences*, 29(7):767–787, 2006.

[4] M. Amar, D. Andreucci, P. Bisegna, R. Gianni, et al. A hierarchy of models for the electrical conduction in biological tissues via two-scale convergence: The nonlinear case. *Differential and Integral Equations*, 26(9/10):885–912, 2013.

[5] M. Amar, D. Andreucci, R. Gianni, and P. Bisegna. Evolution and memory effects in the homogenization limit for electrical conduction in biological tissues. *Mathematical Models and Methods in Applied Sciences*, 14(09):1261–1295, 2004.

[6] M. Amar, D. Andreucci, and C. Timofte. Homogenization of a modified bidomain model involving imperfect transmission. *Comm. Pure Appl. Anal.* 20(5):1755–1782, 2021.

[7] M. Amar, D. Andreucci, and C. Timofte. Interface potential in composites with general imperfect transmission conditions. *Zeitschrift für angewandte Mathematik und Physik* 74(5): 200, 2023.

[8] M. Amar, A. Ayub, and R. Gianni. Double layer concentration and homogenization for the heat diffusion in a composite. *Mathematics and Mechanics of Complex Systems* 12(1):19–45, 2024.

[9] M. Amar, I. De Bonis, and G. Riey. Homogenization of elliptic problems involving interfaces and singular data. *Nonlinear Analysis*, 189:111562, 2019.

[10] M. Amar, I. De Bonis, and G. Riey. Corrigendum to homogenization of elliptic problems involving interfaces and singular data. *Nonlinear Anal.*, 203:112192, 2021.

[11] M. Amar and R. Gianni. Laplace-Beltrami operator for the heat conduction in polymer coating of electronic devices. *Discrete & Continuous Dynamical Systems-B*, 23(4):1739, 2018.

[12] J.-L. Auriault and H. I. Ene. Macroscopic modelling of heat transfer in composites with interfacial thermal barrier. *International journal of heat and mass transfer*, 37(18):2885–2892, 1994.

[13] A. Ayub. Homogenization of heat conduction with imperfect interface involving Laplace-Beltrami operator. *PhD Thesis, Dept. of Basic and Applied Sciences for Engineering, Sapienza - University of Rome*, 2022.

[14] A. Bensoussan, J. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures*, volume 374. American Mathematical Soc., 2011.

[15] M. Bohm, M. Peter. Different choices of scaling in homogenization of diffusion and interfacial exchange in a porous medium. *Math. Meth. Appl. Sci.*, (11)31:1257–1282, 2008.

[16] R. Bunoiu and C. Timofte. Homogenization of a thermal problem with flux jump. *Networks and Heterogeneous Media*, (4)11:545–562, 2016.

[17] D. Cioranescu, A. Damlamian, P. Donato, G. Griso, and R. Zaki. The periodic unfolding method in domains with holes. *SIAM Journal on Mathematical Analysis*, 44(2):718–760, 2012.

[18] D. Cioranescu, A. Damlamian, and G. Griso. Periodic unfolding and homogenization. *Comptes Rendus Mathématique*, 335(1):99–104, 2002.

[19] D. Cioranescu, A. Damlamian, and G. Griso. The periodic unfolding method in homogenization. *SIAM Journal on Mathematical Analysis*, 40(4):1585–1620, 2008.

[20] D. Cioranescu, A. Damlamian, and G. Griso. *The periodic unfolding method. Theory and Applications to Partial Differential Problems*, volume 3 of *Series in Contemporary Mathematics*. Springer, Singapore, 2018.

[21] D. Cioranescu, P. Donato, and R. Zaki. The periodic unfolding method in perforated domains. *Portugaliae Mathematica. Nova Série*, 63(4):467–496, 2006.

[22] P. Donato and K. Le Nguyen. Homogenization for diffusion problems with a nonlinear interfacial resistance. *Nonlinear Differ. Equ. Appl.*, 22:1345–1380, 2015.

[23] P. Donato, K. Le Nguyen, and R. Tardieu. The periodic unfolding method for a class of imperfect transmission problems. *Journal of Mathematical Sciences*, 176(6):891–927, 21.

[24] P. Donato and S. Monsurrò. Homogenization of two heat conductors with an interfacial contact resistance. *Analysis and Applications*, (3) 2:247–273, 2004.

[25] M. Gahn, P. Knabner and M. Neuss-Radu. Homogenization of reaction-diffusion processes in a two-component porous medium with a nonlinear flux condition at the interface, and application to metabolic processes in cells. *SIAM J. Appl. Math.*, 76:1819–1843, 2016.

[26] M. Gahn, M. Neuss-Radu and P. Knabner. Derivation of effective transmission conditions for domains separated by a membrane for different scaling of membrane diffusivity. *Discrete and Continuous Dynamical System - Series B*, (4)10:773–797, 2017.

[27] M. Gahn. Multi-scale modeling of processes in porous media-coupling reaction-diffusion processes in the solid and the fluid phase and on the separating interfaces. *Discrete and Continuous Dynamical System - Series B*, (12)24:6511–6531, 2019.

[28] X. Geng. Effective boundary conditions arising from the heat equation with three-dimensional interior inclusion. *Comm. Pure Appl. Anal.*, (4)22, 1394–1419, 2023.

[29] S. Gu and Q. He. Interfacial discontinuity relations for coupled multifield phenomena and their application to the modeling of thin interphases as imperfect interfaces. *Journal of the Mechanics and Physics of Solids*, 59:1413–1426, 2011.

[30] F. Lene and D. Leguillon. Étude de l'influence d'un glissement entre les constituants d'un matériau composite sur ses coefficients de comportement effectifs. *J. Mécanique*, 20(3):509–536, 1981.

[31] J. Liu, S. Gu, E. Monteiro, and Q. He. A versatile interface model for thermal conduction phenomena and its numerical implementation by xfem. *Computational Mechanics*, 53:825–843, 2014.

[32] E. Sánchez-Palencia. Non-homogeneous media and vibration theory. *Lecture Notes in Physics*, 127, 1980.

[33] M. Serpilli, R. Rizzoni, F. Lebon, and S. Dumont. An asymptotic derivation of a general imperfect interface law for linear multiphysics composites. *International Journal of Solids and Structures*, 180–181:97–107, 2019.