UV asymptotics of n-point correlators of twist-2 operators in SU(N) Yang-Mills theory

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The generating functional $\mathcal{W}^E[J_\mathcal{O}]$ of Euclidean correlators of twist-2 operators in SU(N) Yang-Mills theory admits the 't Hooft large-N expansion: $\mathcal{W}^E[J_\mathcal{O}] = \mathcal{W}^E_{\text{sphere}}[J_\mathcal{O}] + \mathcal{W}^E_{\text{torus}}[J_\mathcal{O}] + \cdots$. Nonperturbatively, $\mathcal{W}^E_{\text{sphere}}[J_\mathcal{O}]$ is a sum of tree diagrams involving glueball propagators and vertices, while $\mathcal{W}^E_{\text{torus}}[J_\mathcal{O}]$ is a sum of glueball one-loop diagrams. Moreover, it has been predicted that $\mathcal{W}^E_{\text{torus}}[J_\mathcal{O}]$ should admit the structure of the logarithm of a functional determinant summing glueball one-loop diagrams. We work out in a closed form the ultraviolet (UV) asymptotics of $\mathcal{W}^E_{\text{sphere}}[J_\mathcal{O},\lambda] \sim \mathcal{W}^E_{\text{asym sphere}}[J_\mathcal{O},\lambda]$ and $\mathcal{W}^E_{\text{torus}}[J_\mathcal{O},\lambda] \sim \mathcal{W}^E_{\text{asym torus}}[J_\mathcal{O},\lambda]$ in the coordinate representation as all the coordinates of the correlators are uniformly rescaled by a factor $\lambda \to 0$. The calculation is performed in two steps. First, extending our previous work, we compute—directly from its functional-integral definition as a Gaussian integral—the generating functional of the conformal correlators $\mathcal{W}_{\text{conf}}[J_\mathcal{O}] = \mathcal{W}_{\text{conf sphere}}[J_\mathcal{O}] + \mathcal{W}_{\text{conf torus}}[J_\mathcal{O}]$ to the lowest perturbative order of all the twist-2 operators with maximal spin along the p_+ direction, in both Minkowskian and—by analytical continuation—Euclidean spacetimes. Thus, we provide a purely perturbative explanation as to why $\mathcal{W}_{\text{conf}}[J_\mathcal{O}]$ has the structure of the logarithm of a functional determinant. Second, by means of a careful choice of the renormalization scheme that reduces the mixing of the above operators to the multiplicatively renormalizable case to all orders of perturbation theory, we lift the generating functional of the Euclidean conformal correlators $\mathcal{W}^E_{\text{conf}}[J_\mathcal{O}]$ to the generating functional of the renormalization-group improved correlators $\mathcal{W}^E_{\text{asym}}[J_\mathcal{O},\lambda] = \mathcal{W}^E_{\text{asym sphere}}[J_\mathcal{O},\lambda] + \mathcal{W}^E_{\text{asym torus}}[J_\mathcal{O},\lambda]$ that inherits the very same structure of the logarithm of a functional determinant. Remarkably, we verify the above prediction that $\mathcal{W}^E_{\text{asym torus}}[J_\mathcal{O},\lambda]$—being asymptotic in the UV to $\mathcal{W}^E_{\text{torus}}[J_\mathcal{O},\lambda]$—admits the structure of the logarithm of a functional determinant as well. Hence, the computation above sets strong UV asymptotic constraints on the nonperturbative solution of large-N Yang-Mills theory, and it may be a pivotal guide for the search of such a solution.

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I. INTRODUCTION, PHYSICS MOTIVATIONS, AND CONCLUSIONS

The generating functional $\mathcal{W}^E[J_\mathcal{O}] = \mathcal{W}^E[J_\mathbb{O}, J_{\tilde{\mathbb{O}}}, J_\mathbb{S}, J_{\tilde{\mathbb{S}}}]$ of Euclidean connected correlators of single-trace gauge-invariant twist-2 operators in SU(N) Yang-Mills (YM) theory admits the 't Hooft large-N expansion [1] in powers of $\frac{1}{N}$:

$$\mathcal{W}^E[J_\mathcal{O}] = \mathcal{W}^E_{\text{sphere}}[J_\mathcal{O}] + \mathcal{W}^E_{\text{torus}}[J_\mathcal{O}] + \cdots. \tag{1}$$

Perturbatively, in terms of the 't Hooft gauge coupling $g^2 = N g^2_{\text{YM}}$ [1], $\mathcal{W}^E_{\text{sphere}}[J_\mathcal{O}]$ contains a sum of Feynman diagrams that—in the 't Hooft double-line representation—have the topology of a punctured sphere, while $\mathcal{W}^E_{\text{torus}}[J_\mathcal{O}]$ includes diagrams that have the topology of a punctured torus, the punctures arising in both cases from the insertion of the sources $J_\mathcal{O}$ dual to the twist-2 operators $\mathcal{O}$.

Alternatively, $\mathcal{W}^E_{\text{sphere}}[J_\mathcal{O}]$ may be defined as a sum of planar diagrams [1] and $\mathcal{W}^E_{\text{torus}}[J_\mathcal{O}]$ as a sum of leading-order (LO) nonplanar contributions.

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Nonperturbatively, in terms of the renormalization-group (RG) invariant scale $\Lambda_{\rm YM}$, $\mathcal{W}^E_{\rm sphere}[J_{\mathcal{O}}]$ is a sum of tree diagrams [2] involving glueball propagators and vertices, while $\mathcal{W}^E_{\rm torus}[J_{\mathcal{O}}]$ is a sum of glueball one-loop diagrams, both arising from the effective description of the—yet to come—nonperturbative solution of large-N YM theory as a theory of an infinite number of weakly coupled glueballs with coupling of order $\frac{1}{N}$ [1–3].

Moreover, on the basis of the existence of a nonperturbative glueball effective action, it has been predicted [4] that $\mathcal{W}^E_{\rm torus}[J_{\mathcal{O}}]$ should admit the structure of the logarithm of a functional determinant summing glueball one-loop diagrams.

Presently, nothing is known quantitatively on the structure of $\mathcal{W}^E_{\rm sphere}[J_{\mathcal{O}}]$ and $\mathcal{W}^E_{\rm torus}[J_{\mathcal{O}}]$ nonperturbatively. Hence, one of the aims of the present paper is to put strong quantitative constraints on the aforementioned nonperturbative structure by working out in a closed form the ultraviolet (UV) asymptotics of $\mathcal{W}^E_{\rm sphere}[J_{\mathcal{O}},\lambda]\sim\mathcal{W}^E_{\rm asym\ sphere}[J_{\mathcal{O}},\lambda]$ and $\mathcal{W}^E_{\rm torus}[J_{\mathcal{O}},\lambda]\sim\mathcal{W}^E_{\rm asym\ torus}[J_{\mathcal{O}},\lambda]$ in the coordinate representation as all the coordinates of the correlators are uniformly rescaled by a factor $\lambda\to 0$. This is achieved in two steps:

(I) The efficient computation in a closed form of the generating functional of conformal correlators $\mathcal{W}^E_{\rm conf}[J_{\mathcal{O}}]=\mathcal{W}^E_{\rm conf\ sphere}[J_{\mathcal{O}}]+\mathcal{W}^E_{\rm conf\ torus}[J_{\mathcal{O}}]$ of twist-2 operators to the lowest order of perturbation theory.

(II) The RG-improvement from $\mathcal{W}^E_{\rm conf}[J_{\mathcal{O}}]=\mathcal{W}^E_{\rm conf\ sphere}[J_{\mathcal{O}}]+\mathcal{W}^E_{\rm conf\ torus}[J_{\mathcal{O}}]$ to $\mathcal{W}^E_{\rm asym}[J_{\mathcal{O}},\lambda]=\mathcal{W}^E_{\rm asym\ sphere}[J_{\mathcal{O}},\lambda]+\mathcal{W}^E_{\rm asym\ torus}[J_{\mathcal{O}},\lambda]$, which is obtained by standard RG methods in combination with a new technique [5] that reduces the operator mixing of twist-2 operators to the multiplicatively renormalizable case to all orders of perturbation theory.

We describe the steps (I) and (II) in more detail as follows.

Recently, we have computed [6] to the lowest perturbative order in YM theory n-point conformal correlators in the coordinate $G^{(n)}_{\rm conf}(x_1,\ldots,x_n)$ and momentum $G^{(n)}_{\rm conf}(p_1,\ldots,p_n)$ representation of the gauge-invariant twist-2 operators with maximal spin along the p_+ direction, in both Minkowskian and—by analytic continuation—Euclidean spacetimes.

Specifically, we have calculated the n-point conformal correlators of the balanced $\mathbb{O},\tilde{\mathbb{O}}$ and unbalanced $\mathbb{S},\bar{\mathbb{S}}$ operators[1] with collinear twist 2 [6] in their separate sectors and the three-point correlators in the mixed sector as well.

We have also reconstructed [6] from the n-point conformal correlators in each separate sector the corresponding generating functionals, $\Gamma_{\rm conf}[j_{\mathbb{O}},j_{\tilde{\mathbb{O}}}]=\Gamma_{\rm conf\ sphere}[j_{\mathbb{O}},j_{\tilde{\mathbb{O}}}]+\Gamma_{\rm conf\ torus}[j_{\mathbb{O}},j_{\tilde{\mathbb{O}}}]$ and $\Gamma_{\rm conf}[j_{\mathbb{S}},j_{\bar{\mathbb{S}}}]=\Gamma_{\rm conf\ sphere}[j_{\mathbb{S}},j_{\bar{\mathbb{S}}}]+\Gamma_{\rm conf\ torus}[j_{\mathbb{S}},j_{\bar{\mathbb{S}}}]$, in both Minkowskian and Euclidean spacetimes and in both the coordinate and momentum representations that turn out [6] to have the structure of the logarithm of functional determinants.

We have pointed out [6] that the rationale behind the generating functionals $\Gamma_{\rm conf\ torus}[j_{\mathbb{O}},j_{\tilde{\mathbb{O}}}]$ and $\Gamma_{\rm conf\ torus}[j_{\mathbb{S}},j_{\bar{\mathbb{S}}}]$ being the logarithm of functional determinants is the prediction [4] that the LO nonplanar contribution to the nonperturbative effective action [4] should have the structure of the logarithm of a functional determinant that sums the glueball one-loop diagrams. Hence, in relation to the above computations, the aim of the present paper is fourfold.

First, as opposed to the aforementioned nonperturbative argument, since the computation of the n-point conformal correlators is by Feynman diagrams [6], we find a perturbative reason as to why the corresponding generating functionals $\Gamma_{\rm conf}[j_{\mathbb{O}},j_{\tilde{\mathbb{O}}}]$ and $\Gamma_{\rm conf}[j_{\mathbb{S}},j_{\bar{\mathbb{S}}}]$ are the logarithm of functional determinants [6]: As all the aforementioned collinear twist-2 operators are quadratic in the gauge field in the light-cone gauge to the lowest order of perturbation theory, the functional integral that defines their generating functional $\mathcal{W}_{\rm conf}[J_{\mathcal{O}}]$ is Gaussian and leads to a functional determinant involving the corresponding sources $J_{\mathcal{O}}$.

Then, once some technical difficulties are properly understood, a tedious but straightforward computation shows that the functional determinants $\Gamma_{\rm conf}[j_{\mathbb{O}},j_{\tilde{\mathbb{O}}}]$ and $\Gamma_{\rm conf}[j_{\mathbb{S}},j_{\bar{\mathbb{S}}}]$ worked out in [6] actually arise from their very definition as Gaussian functional integrals $\mathcal{W}_{\rm conf}[J_{\mathbb{O}},J_{\tilde{\mathbb{O}}}]$ and $\mathcal{W}_{\rm conf}[J_{\mathbb{S}},J_{\bar{\mathbb{S}}}]$ in the present paper.

Second, our new method to compute the generating functional $\mathcal{W}_{\rm conf}[J_{\mathcal{O}}]$ allows us to produce a vast generalization of our previous computations. For example, we calculate the generating functional $\mathcal{W}_{\rm conf}[J_{\mathbb{O}},J_{\tilde{\mathbb{O}}},J_{\mathbb{S}},J_{\bar{\mathbb{S}}}]$ in the mixed balanced/unbalanced sector[2] in YM theory, verify by means of it our previous computation of the mixed three-point correlators [6], and compute the mixed four-point correlators.

Besides, the same technique will allow us to compute in the future—as the logarithm of a functional (super) determinant—the generating functional of the lowest-order conformal correlators of the collinear twist-2 operators in $\mathcal{N}=1,2,4$ supersymmetric (SUSY) YM theory, QCD and $\mathcal{N}=1$ SUSY QCD—the superdeterminant arising in the supersymmetric cases.

Remarkably, the very same technique also applies to all the possibly higher-twist operators that are quadratic [7] in the elementary fields in the light-cone gauge to the lowest order of perturbation theory in all the above theories.

[1]In our terminology "unbalanced" and "balanced" refer to either the different or equal numbers of dotted and undotted indices that the aforementioned operators, respectively, possess in the spinorial representation. Unbalanced operators are referred to as "asymmetric" in [7] and "anisotropic" in [8].

[2]In [6] we could not compute the n-point correlators with $n>3$ in the mixed balanced/unbalanced sector by means of Feynman diagrams because of their complexity. As a consequence we could not reconstruct $\Gamma_{\rm conf}[j_{\mathbb{O}},j_{\tilde{\mathbb{O}}},j_{\mathbb{S}},j_{\bar{\mathbb{S}}}]$ from the correlators.

Third, we work out the UV asymptotics of the Euclidean nonpertubative n-point correlators in a certain renormalization scheme described below. The computation is based on both our lowest-order calculation of the Euclidean conformal correlators and its improvement by means of the RG.

Generally, collinear twist-2 operators $\mathcal{O}_{s_i}$ with maximal spin projection $s_i \geq 2$ mix under renormalization with total derivatives of the same operators with lower spins, so that an explicit evaluation of the RG-improved asymptotics of their Euclidean n-point correlators in the $\overline{\text{MS}}$ renormalization scheme would require the asymptotic estimate—that seems practically impossible—of the product of the sum of $O(s_i)$ terms for each $\mathcal{O}_{s_i}$ occurring in the n-point correlators because of the triangular nature of their renormalized mixing matrix $Z^{(s_i)}$, not to mention that in general $Z^{(s_i)}$ is actually only known to two loops [9–11] in perturbation theory and not in its RG-improved form.

Recently, a differential-geometric approach to operator mixing in massless QCD-like theories has been proposed [5] precisely to overcome the above difficulty. Specifically, it has been determined under which conditions a renormalization scheme exists where each $Z^{(s_i)}$ may be set in a diagonal canonical form that is one-loop exact to all perturbative orders—according to the nonresonant diagonalizable $\frac{\gamma_0}{\beta_0}$ case (I) [5]. Moreover, it has been verified [12] that the balanced collinear twist-2 operators belong to the case (I), and we show in the present paper that this is also the case for the unbalanced ones.[3]

Remarkably, in such a scheme the needed terms for the asymptotic evaluation of the n-point correlators reduce to just one, since in this scheme the operators are multiplicatively renormalizable and in general their correlators do not vanish to the lowest order of perturbation theory. The nonresonant diagonal scheme may also apply to the collinear twist-2 operators in QCD and its supersymmetric extensions above.

Fourth, we employ the UV asymptotics of the Euclidean n-point correlators in the nonresonant diagonal scheme to work out the corresponding generating functional $\mathcal{W}^E_{\text{asym}}[J_\mathcal{O},\lambda]$ that also turns out to be the logarithm of a functional determinant. In fact, after a suitable rescaling of the operators, it may be decomposed for large N into the sum of the generating functionals of the planar $\mathcal{W}^E_{\text{asym sphere}}[J_\mathcal{O},\lambda]$ and LO nonplanar asymptotic correlators $\mathcal{W}^E_{\text{asym torus}}[J_\mathcal{O},\lambda]$, which has the structure of the logarithm of a functional determinant as well.

This is our last—and perhaps most remarkable—step. Indeed, according to the prediction in [4] the nonperturbative glueball one-loop generating functional $\mathcal{W}^E_{\text{torus}}[J_\mathcal{O},\lambda]$ should have the structure of the logarithm of a functional determinant and as a consequence $\mathcal{W}^E_{\text{asym torus}}[J_\mathcal{O},\lambda]$ should inherit the very same structure as well.

In a forthcoming paper we will further investigate the above nonperturbative interpretation. In any case the generating functionals of planar $\mathcal{W}^E_{\text{asym sphere}}[J_\mathcal{O},\lambda]$ and LO nonplanar asymptotic correlators $\mathcal{W}^E_{\text{asym torus}}[J_\mathcal{O},\lambda]$ set strong UV asymptotic constraints on the nonperturbative solution of large-N YM theory and may be a pivotal guide for the search of such a solution.

[3]In both cases numerically up to $s = 10^4$.

II. PLAN OF THE PAPER

In Sec. III we compute the generating functional $\mathcal{W}_{\text{conf}}[J_\mathcal{O}] = \mathcal{W}_{\text{conf}}[J_\mathbb{O}, J_{\bar{\mathbb{O}}}, J_\mathbb{S}, J_{\bar{\mathbb{S}}}]$ in the coordinate representation of all the correlators of collinear twist-2 operators (Appendix B) to the lowest order of perturbation theory from the YM functional integral in Minkowskian spacetime.

In Sec. IV we restrict $\mathcal{W}_{\text{conf}}[J_\mathbb{O}, J_{\bar{\mathbb{O}}}, J_\mathbb{S}, J_{\bar{\mathbb{S}}}]$ to the generating functionals in the separate balanced and unbalanced sectors, $\mathcal{W}_{\text{conf}}[J_\mathbb{O}, J_{\bar{\mathbb{O}}}]$ and $\mathcal{W}_{\text{conf}}[J_\mathbb{S}, J_{\bar{\mathbb{S}}}]$, and connect them to the generating functionals, $\Gamma_{\text{conf}}[j_\mathbb{O}, j_{\bar{\mathbb{O}}}]$ and $\Gamma_{\text{conf}}[j_\mathbb{S}, j_{\bar{\mathbb{S}}}]$, previously computed in [6] directly from the corresponding correlators, finding perfect agreement.

In Sec. V we work out the generating functional $\mathcal{W}_{\text{conf}}[J_\mathcal{O}]$ in the momentum representation involving both its explicit expansion and a compact representation by means of the Gegenbauer polynomials.

In Sec. VI we calculate the three- and four-point correlators in the mixed sector from $\mathcal{W}_{\text{conf}}[J_\mathcal{O}]$. The result agrees with our previous computation [6] of the mixed three-point correlators.

In Sec. VII we recall some recent results [5] on operator mixing and apply them to work out the UV asymptotics in the coordinate representation of the Euclidean n-point correlators of collinear twist-2 operators in the aforementioned nonresonant diagonal scheme [5].

In Sec. VIII we compute the generating functional in the coordinate representation of the RG-improved correlators in Euclidean spacetime $\mathcal{W}^E_{\text{asym}}[J_\mathcal{O},\lambda] = \mathcal{W}^E_{\text{asym sphere}}[J_\mathcal{O},\lambda] + \mathcal{W}^E_{\text{asym torus}}[J_\mathcal{O},\lambda]$.

In Appendix A we compute the functional determinant that leads to the generating functional $\mathcal{W}_{\text{conf}}[J_\mathcal{O}]$.

In Appendix B we recall the definition of the standard basis [6] of collinear twist-2 operators and describe its conformal properties to the leading and next-to-leading perturbative order.

In Appendix C we compute the normalization of the two-point conformal correlators employing the properties of the Gegenbauer polynomials.

In Appendix D we verify that the three-point correlators up to $s = 4$ are—indeed—a linear combination of the structures implied by the conformal symmetry.

III. GENERATING FUNCTIONAL $\mathcal{W}_{\text{conf}}$ OF THE MINKOWSKIAN CONFORMAL CORRELATORS FROM THE YM FUNCTIONAL INTEGRAL

The Minkowskian YM action is

$$S_{\text{YM}} = -\frac{1}{2}\int d^4x \text{Tr} F_{\mu\nu}F^{\mu\nu}, \tag{2}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\frac{g}{\sqrt{N}}[A_\mu, A_\nu] \tag{3}$$

and

$$A_\mu = A_\mu^a T^a \tag{4}$$

with the Hermitian generators of the SU(N) Lie algebra

$$[T^a, T^b] = if^{abc}T^c \tag{5}$$

in the fundamental representation normalized as

$$\mathrm{Tr}(T^a T^b) = \frac{\delta^{ab}}{2} \tag{6}$$

and $g^2 = Ng_{\mathrm{YM}}^2$ the 't Hooft coupling [1]. We set

$$V_+ = \frac{V_0 + V_3}{\sqrt{2}}, \qquad V_- = \frac{V_0 - V_3}{\sqrt{2}},$$
$$V = \frac{V_1 + iV_2}{\sqrt{2}}, \qquad \bar{V} = \frac{V_1 - iV_2}{\sqrt{2}} \tag{7}$$

for a vector V_μ. We choose the light-cone gauge

$$A_+ = 0. \tag{8}$$

After integrating out A_-, the YM action in the light-cone gauge reads [13]

$$\begin{aligned} S_{\mathrm{YM}}(A,\bar{A}) = &-\int \bar{A}^a \Box A^a \\ &+ 2\frac{g}{\sqrt{N}} f^{abc}(A^a\partial_+\bar{A}^b\bar{\partial}\partial_+^{-1}A^c + \bar{A}^a\partial_+A^b\partial\partial_+^{-1}\bar{A}^c) \\ &+ 2\frac{g^2}{N} f^{abc}f^{ade}\partial_+^{-1}(A^b\partial_+\bar{A}^c)\partial_+^{-1}(\bar{A}^d\partial_+A^e)d^4x. \end{aligned} \tag{9}$$

The vacuum expectation value (VEV) of a product of local gauge-invariant operators $\mathcal{O}_i(A)$ that do not depend on A_- reads

$$\begin{aligned} &\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \\ &= \frac{1}{Z}\int \mathcal{D}A\mathcal{D}\bar{A}e^{iS_{\mathrm{YM}}(A,\bar{A})}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n). \end{aligned} \tag{10}$$

To the leading perturbative order it reduces to

$$\begin{aligned} &\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \\ &= \frac{1}{Z}\int \mathcal{D}A\mathcal{D}\bar{A}e^{-i\int d^4x\bar{A}^a\Box A^a}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n) \end{aligned} \tag{11}$$

with

$$\Box = g^{\mu\nu}\partial_\mu\partial_\nu = \partial_0^2 - \sum_{i=1}^{3}\partial_i^2, \tag{12}$$

where we employ the mostly minus metric $g^{\mu\nu}$ in Minkowskian spacetime [6]. The corresponding generating functional reads to the leading order

$$\begin{aligned} &\mathcal{Z}_{\mathrm{conf}}[J_{\mathcal{O}}] \\ &= \frac{1}{Z}\int \mathcal{D}A\mathcal{D}\bar{A}e^{-i\int d^4x\bar{A}^a\Box A^a}\exp\left(\int d^4x\sum_i J_{\mathcal{O}_i}\mathcal{O}_i\right). \end{aligned} \tag{13}$$

A. Standard basis

The collinear twist-2 operators in the standard basis [6] are (Appendix B)

$$\begin{aligned} \mathbb{O}_s &= \frac{1}{2}\bar{A}^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)A^a(x), \\ \tilde{\mathbb{O}}_s &= \frac{1}{2}\bar{A}^a(x)\mathcal{H}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)A^a(x), \\ \mathbb{S}_s &= \frac{1}{2\sqrt{2}}\bar{A}^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)\bar{A}^a(x), \\ \bar{\mathbb{S}}_s &= \frac{1}{2\sqrt{2}}A^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)A^a(x), \end{aligned} \tag{14}$$

where $s = 2, 4, 6, \ldots$ for $\mathbb{O}_s$, $\mathbb{S}_s$, and $\bar{\mathbb{S}}_s$, $s = 3, 5, 7, \ldots$ for $\tilde{\mathbb{O}}_s$, the sum over repeated color indices is understood, and [6]

$$\begin{aligned} &\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+) \\ &= \overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+ \\ &= \frac{\Gamma(3)\Gamma(s+3)}{\Gamma(5)\Gamma(s+1)}i^{s-2}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2} \\ &\quad\times(-1)^{s-k}\overleftarrow{\partial}_+^{s-k-1}\overrightarrow{\partial}_+^{k+1} \end{aligned} \tag{15}$$

for even spin, and

$$\begin{aligned} &\mathcal{H}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+) \\ &= \overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+ \\ &= \frac{\Gamma(3)\Gamma(s+3)}{\Gamma(5)\Gamma(s+1)}i^{s-2}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2} \\ &\quad\times(-1)^{s-k}\overleftarrow{\partial}_+^{s-k-1}\overrightarrow{\partial}_+^{k+1} \end{aligned} \tag{16}$$

for odd spin. The corresponding generating functional of conformal correlators to the leading order reads

$$\mathcal{Z}_{\rm conf}[J_{\mathbb{O}},J_{\tilde{\mathbb{O}}},J_{\mathbb{S}},J_{\bar{\mathbb{S}}}] = \frac{1}{Z}\int \mathcal{D}A\mathcal{D}\bar{A}e^{-i\int d^4x\bar{A}^a\Box A^a} \times \exp\left(\int d^4x\sum_s J_{\mathbb{O}_s}\mathbb{O}_s + J_{\tilde{\mathbb{O}}_s}\tilde{\mathbb{O}}_s + J_{\mathbb{S}_s}\mathbb{S}_s + J_{\bar{\mathbb{S}}_s}\bar{\mathbb{S}}_s\right), \tag{17}$$

where the currents $J_{\mathbb{O}_s}, J_{\tilde{\mathbb{O}}_s}, J_{\mathbb{S}_s}, J_{\bar{\mathbb{S}}_s}$ are defined to be zero for s different from the spin of the corresponding operator. Explicitly,

$$\begin{aligned}\mathcal{Z}_{\rm conf}[J_{\mathbb{O}},J_{\tilde{\mathbb{O}}},J_{\mathbb{S}},J_{\bar{\mathbb{S}}}] &= \frac{1}{Z}\int \mathcal{D}A\mathcal{D}\bar{A}e^{-i\int d^4x\bar{A}^a\Box A^a} \times \exp\left(\frac{1}{2}\int d^4x\sum_s J_{\mathbb{O}_s}\bar{A}^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+,\overleftarrow{\partial}_+)A^a(x)\right.\\ &+ J_{\tilde{\mathbb{O}}_s}\bar{A}^a(x)\mathcal{H}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+,\overleftarrow{\partial}_+)A^a(x)\\ &+ J_{\mathbb{S}_s}\frac{1}{\sqrt{2}}\bar{A}^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+,\overleftarrow{\partial}_+)\bar{A}^a(x)\\ &\left.+ J_{\bar{\mathbb{S}}_s}\frac{1}{\sqrt{2}}A^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+,\overleftarrow{\partial}_+)A^a(x)\right).\end{aligned} \tag{18}$$

The above functional integral is quadratic in the elementary fields, and therefore it may be computed exactly. Employing the symmetry properties of the Gegenbauer polynomials (Appendix B)

$$\begin{aligned}\mathbb{O}_s &= \frac{1}{4}\left(\bar{A}^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+,\overleftarrow{\partial}_+)A^a(x)\right.\\ &\quad\left.+ A^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+,\overleftarrow{\partial}_+)\bar{A}^a(x)\right),\\ \tilde{\mathbb{O}}_s &= \frac{1}{4}\left(\bar{A}^a(x)\mathcal{H}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+,\overleftarrow{\partial}_+)A^a(x)\right.\\ &\quad\left.- A^a(x)\mathcal{H}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+,\overleftarrow{\partial}_+)\bar{A}^a(x)\right),\end{aligned} \tag{19}$$

we obtain

$$\mathcal{Z}_{\rm conf}[J_{\mathbb{O}},J_{\tilde{\mathbb{O}}},J_{\mathbb{S}},J_{\bar{\mathbb{S}}}] = \frac{1}{Z}\int \mathcal{D}A\mathcal{D}\bar{A}e^{-\frac{1}{2}\int d^4x\begin{pmatrix}\bar{A}^a(x) & A^a(x)\end{pmatrix}\mathcal{M}^{ab}\begin{pmatrix}A^b(x)\\ \bar{A}^b(x)\end{pmatrix}} \tag{20}$$

with

$$\mathcal{M}^{ab} = \delta^{ab}\begin{pmatrix} i\Box - \frac{1}{2}\sum_s J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2} - \frac{1}{2}\sum_s J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2} & -\frac{1}{\sqrt{2}}\sum_s J_{\mathbb{S}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}\\ -\frac{1}{\sqrt{2}}\sum_s J_{\bar{\mathbb{S}}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2} & i\Box - \frac{1}{2}\sum_s J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2} + \frac{1}{2}\sum_s J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\end{pmatrix}, \tag{21}$$

where we have introduced the symbol $\otimes$ to imply that the right and left derivatives do not act on the sources J.

Hence, performing the Gaussian integral we get

$$\mathcal{Z}_{\rm conf}[J_{\mathbb{O}},J_{\tilde{\mathbb{O}}},J_{\mathbb{S}},J_{\bar{\mathbb{S}}}] = \mathrm{Det}^{-\frac{1}{2}}(\mathcal{M}), \tag{22}$$

where the above determinant—up to a trivial normalization factor—is computed in Eq. (A7). Hence, the generating functional of the connected correlators reads

$$\begin{aligned}\mathcal{W}_{\rm conf}[J_{\mathbb{O}},J_{\tilde{\mathbb{O}}},J_{\mathbb{S}},J_{\bar{\mathbb{S}}}] &= \log\mathcal{Z}_{\rm conf}[J_{\mathbb{O}},J_{\tilde{\mathbb{O}}},J_{\mathbb{S}},J_{\bar{\mathbb{S}}}]\\ &= -\frac{1}{2}\log\mathrm{Det}\left(\mathcal{I} + \frac{1}{2}i\Box^{-1}J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2} + \frac{1}{2}i\Box^{-1}J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\right)\\ &\quad - \frac{1}{2}\log\mathrm{Det}\left(\mathcal{I} + \frac{1}{2}i\Box^{-1}J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2} - \frac{1}{2}i\Box^{-1}J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\right)\\ &\quad - \frac{1}{2}\log\mathrm{Det}\left[\mathcal{I} - \frac{1}{2}\left(\mathcal{I} + \frac{1}{2}i\Box^{-1}J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2} - \frac{1}{2}i\Box^{-1}J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\right)^{-1}\right.\\ &\quad\left.\times i\Box^{-1}J_{\bar{\mathbb{S}}_{s_1}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_1-2}\left(\mathcal{I} + \frac{1}{2}i\Box^{-1}J_{\mathbb{O}_{s_2}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_2-2} + \frac{1}{2}i\Box^{-1}J_{\tilde{\mathbb{O}}_{s_2}}\otimes\mathcal{H}^{\frac{5}{2}}_{s_2-2}\right)^{-1}i\Box^{-1}J_{\mathbb{S}_{s_3}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_3-2}\right],\end{aligned} \tag{23}$$

where $\mathcal{I}$ is the identity in both color and spacetime and the sum over repeated spin indices is understood.

After rescaling the operators

$$\mathcal{O}'_s(x) = \frac{1}{N}\frac{2\Gamma(5)\Gamma(s+1)}{\Gamma(3)\Gamma(s+3)}\mathcal{O}_s(x) \tag{24}$$

so that their two-point correlators are of order 1 for large N, we obtain more explicitly

$$\begin{aligned}
\mathcal{W}_{\rm conf}[J_{\mathbb{O}'}, J_{\tilde{\mathbb{O}}'}, J_{\mathbb{S}'}, J_{\tilde{\mathbb{S}}'}] = &-\frac{N^2-1}{2}\log\mathrm{Det}\left(I + \frac{1}{N}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(i\overrightarrow{\partial}_+)^{s-k-1} i\Box^{-1}(J_{\mathbb{O}'_s} + J_{\tilde{\mathbb{O}}'_s})(i\overrightarrow{\partial}_+)^{k+1}\right)\\
&-\frac{N^2-1}{2}\log\mathrm{Det}\left(I + \frac{1}{N}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(i\overrightarrow{\partial}_+)^{s-k-1} i\Box^{-1}(J_{\mathbb{O}'_s} - J_{\tilde{\mathbb{O}}'_s})(i\overrightarrow{\partial}_+)^{k+1}\right)\\
&-\frac{N^2-1}{2}\log\mathrm{Det}\Bigg[I - \frac{2}{N^2}\left(I + \frac{1}{N}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(i\overrightarrow{\partial}_+)^{s-k-1} i\Box^{-1}(J_{\mathbb{O}'_s} - J_{\tilde{\mathbb{O}}'_s})(i\overrightarrow{\partial}_+)^{k+1}\right)^{-1}\\
&\times\sum_{k_1=0}^{s_1-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}(i\overrightarrow{\partial}_+)^{s_1-k_1-1} i\Box^{-1} J_{\tilde{\mathbb{S}}'_{s_1}}(i\overrightarrow{\partial}_+)^{k_1+1}\\
&\times\left(I + \frac{1}{N}\sum_{k_2=0}^{s_2-2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}(i\overrightarrow{\partial}_+)^{s_2-k_2-1} i\Box^{-1}(J_{\mathbb{O}'_{s_2}} + J_{\tilde{\mathbb{O}}'_{s_2}})(i\overrightarrow{\partial}_+)^{k_2+1}\right)^{-1}\\
&\times\sum_{k_3=0}^{s_3-2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}(i\overrightarrow{\partial}_+)^{s_3-k_3-1} i\Box^{-1} J_{\mathbb{S}'_{s_3}}(i\overrightarrow{\partial}_+)^{k_3+1}\Bigg],
\end{aligned} \tag{25}$$

where I is the identity in spacetime and we have already performed the color trace. To obtain Eq. (25) we have employed the definitions in Eqs. (15) and (16), and

$$i\Box^{-1}\overleftarrow{\partial}_+^{s-k-1} = (-1)^{s-k-1}\overrightarrow{\partial}_+^{s-k-1} i\Box^{-1} \tag{26}$$

that follows from (minus) the propagator in the coordinate representation [6]

$$\frac{1}{4\pi^2}\frac{1}{|x-y|^2 - i\epsilon} = i\Box^{-1}(x-y). \tag{27}$$

IV. CONNECTION OF $\mathcal{W}_{\rm conf}$ WITH THE GENERATING FUNCTIONALS $\Gamma_{\rm conf}$

Remarkably, $\mathcal{W}_{\rm conf}[J_{\mathbb{O}}, J_{\tilde{\mathbb{O}}}, J_{\mathbb{S}}, J_{\tilde{\mathbb{S}}}]$ is the generating functional of all the Minkowskian conformal correlators that extends the generating functionals in the separate balanced and unbalanced sectors obtained [6] by guessing their structure from the perturbative computation of the corresponding n-point correlators. However, even restricting to the separate balanced and unbalanced sectors, the connection with the approach in [6] is not immediately visible. Indeed, the connected generating functionals in the separate sectors read [6]

$$\begin{aligned}
\Gamma_{\rm conf}[j_{\mathbb{O}}, j_{\tilde{\mathbb{O}}}] = &-\frac{N^2-1}{2}\log\mathrm{Det}(\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} + \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}})\\
&-\frac{N^2-1}{2}\log\mathrm{Det}(\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} - \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}}),\\
\Gamma_{\rm conf}[j_{\mathbb{S}}, j_{\tilde{\mathbb{S}}}] = &-\frac{N^2-1}{2}\log\mathrm{Det}(\mathbb{I} - 2\mathcal{D}^{-1} j_{\tilde{\mathbb{S}}}\mathcal{D}^{-1} j_{\mathbb{S}})
\end{aligned} \tag{28}$$

in Minkowskian spacetime, where $\mathbb{I}$ is the identity in both spacetime and the discrete indices defined below. By making manifest the spacetime and discrete-indices dependence in the kernels of the integral operators above, Eq. (28) reads in the coordinate representation [6]

$$\begin{aligned}
\Gamma_{\rm conf}[j_{\mathbb{O}}, j_{\tilde{\mathbb{O}}}] = &-\frac{N^2-1}{2}\log\mathrm{Det}\left(\delta_{s_1k_1,s_2k_2}\delta^{(4)}(x-y) + \mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)(j_{\mathbb{O}_{s_2k_2}}(y) + j_{\tilde{\mathbb{O}}_{s_2k_2}}(y))\right)\\
&-\frac{N^2-1}{2}\log\mathrm{Det}\left(\delta_{s_1k_1,s_2k_2}\delta^{(4)}(x-y) + \mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)(j_{\mathbb{O}_{s_2k_2}}(y) - j_{\tilde{\mathbb{O}}_{s_2k_2}}(y))\right),\\
\Gamma_{\rm conf}[j_{\mathbb{S}}, j_{\tilde{\mathbb{S}}}] = &-\frac{N^2-1}{2}\log\mathrm{Det}\left(\delta_{s_1k_1,s_2k_2}\delta^{(4)}(x-y) - 2\int d^4z\sum_{sk}\mathcal{D}^{-1}_{s_1k_1,sk}(x-z) j_{\tilde{\mathbb{S}}_{sk}}(z)\mathcal{D}^{-1}_{sk,s_2k_2}(z-y) j_{\mathbb{S}_{s_2k_2}}(y)\right),
\end{aligned} \tag{29}$$

where, by a slight abuse of notation, we have displayed as arguments of the determinants the corresponding integral kernels, with [6]

$$\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)=\frac{i^{s_1}}{8\pi^2}\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\binom{s_1}{k_1}\binom{s_2}{k_2+2}\times(-\partial_+)^{s_1-k_1+k_2}\frac{1}{|x-y|^2-i\epsilon}. \tag{30}$$

Equivalently, we may employ the modified kernel

$$\mathcal{D}'^{-1}_{s_1k_1,s_2k_2}(x-y)=\frac{1}{8\pi^2}\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\binom{s_1}{k_1}\binom{s_2}{k_2+2}\times(-i\partial_+)^{s_1-k_1+k_2}\frac{1}{|x-y|^2-i\epsilon}. \tag{31}$$

It gives origin to the same correlators, since it differs from $\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)$ by the factor $i^{-k_1+k_2}$ that cancels in the loops.

The currents $j_{\mathcal{O}_{sk}}$ are dual to the component operators $\mathcal{O}_{sk}$ that are employed to construct the conformal operators $\mathcal{O}_s$ [6]:

$$\mathcal{O}_s=\sum_{k=0}^{l}\mathcal{O}_{sk} \tag{32}$$

with $l=s-2$ for the standard basis. Consequently, the n-point correlators satisfy [6]

$$\langle\mathcal{O}_{s_1}(x_1)\cdots\mathcal{O}_{s_n}(x_n)\rangle=\sum_{k_1=0}^{l_1}\frac{\delta}{\delta j_{\mathcal{O}_{s_1k_1}}(x_1)}\cdots\sum_{k_n=0}^{l_n}\frac{\delta}{\delta j_{\mathcal{O}_{s_nk_n}}(x_n)}\Gamma_{\rm conf}[j_{\mathcal{O}}]. \tag{33}$$

Hence, we should demonstrate that

$$\sum_{k_1=0}^{l_1}\frac{\delta}{\delta j_{\mathcal{O}_{s_1k_1}}(x_1)}\cdots\sum_{k_n=0}^{l_n}\frac{\delta}{\delta j_{\mathcal{O}_{s_nk_n}}(x_n)}\Gamma_{\rm conf}[j_{\mathcal{O}}]=\frac{\delta}{\delta J_{\mathcal{O}_{s_1}}(x_1)}\cdots\frac{\delta}{\delta J_{\mathcal{O}_{s_n}}(x_n)}\mathcal{W}_{\rm conf}[J_{\mathcal{O}}] \tag{34}$$

with the functional derivatives computed at $j_{\mathcal{O}_{sk}}=0$ and $J_{\mathcal{O}_s}=0$, respectively. While $\Gamma_{\rm conf}[j_{\mathcal{O}}]$ involves determinants of integral operators formally of Fredholm type, the functional-integral computation of $\mathcal{W}_{\rm conf}[J_{\mathcal{O}}]$ involves determinants of quadratic forms that originally arise from their very definition in Eq. (23) by employing both left and right derivatives in Eqs. (15) and (16). This is the source of some technical complications that we resolve momentarily creating a dictionary that relates $\mathcal{W}_{\rm conf}[J_{\mathcal{O}}]$ to $\Gamma_{\rm conf}[j_{\mathcal{O}}]$.

A. An example of the dictionary

We choose as an example the generating functional $\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0]$ restricted to the balanced operators with even spin in the standard basis. We formally expand the logarithm of the functional determinant:

$$\begin{aligned}\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0]=(N^2-1)\sum_{l=1}^{\infty}\frac{(-1)^l}{2^l l}\int d^4x_1\cdots d^4x_l\sum_{s_1k_1}\cdots\sum_{s_lk_l}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_l}{k_l}\binom{s_l}{k_l+2}\\ \times\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\cdots\frac{\Gamma(3)\Gamma(s_l+3)}{\Gamma(5)\Gamma(s_l+1)}(i\overrightarrow{\partial}_{x_1^+})^{s_1-k_1-1}i\Box^{-1}(x_1-x_2)J_{\mathbb{O}_{s_1}}(x_2)(i\overrightarrow{\partial}_{x_2^+})^{k_1+1}\\ \times(i\overrightarrow{\partial}_{x_2^+})^{s_2-k_2-1}i\Box^{-1}(x_2-x_3)J_{\mathbb{O}_{s_2}}(x_3)(i\overrightarrow{\partial}_{x_3^+})^{k_2+1}\cdots\times(i\overrightarrow{\partial}_{x_l^+})^{s_l-k_l-1}i\Box^{-1}(x_l-x_1)J_{\mathbb{O}_{s_l}}(x_1)(i\overrightarrow{\partial}_{x_1^+})^{k_l+1},\end{aligned} \tag{35}$$

where, by a slight abuse of notation, k_i is actually a short notation for k_{s_i}. By combining together the above derivatives with respect to the same coordinate, we get

$$\begin{aligned}\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0]=(N^2-1)\sum_{l=1}^{\infty}\frac{(-1)^l}{2^l l}\int d^4x_1\cdots d^4x_l\sum_{s_1k_1}\cdots\sum_{s_lk_l}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_l}{k_l}\binom{s_l}{k_l+2}\\ \times\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\cdots\frac{\Gamma(3)\Gamma(s_l+3)}{\Gamma(5)\Gamma(s_l+1)}(i\overrightarrow{\partial}_{x_1^+})^{s_1-k_1+k_l}i\Box^{-1}(x_1-x_2)J_{\mathbb{O}_{s_1}}(x_2)\\ \times(i\overrightarrow{\partial}_{x_2^+})^{s_2-k_2+k_1}i\Box^{-1}(x_2-x_3)J_{\mathbb{O}_{s_2}}(x_3)\cdots\times(i\overrightarrow{\partial}_{x_l^+})^{s_l-k_l+k_{l-1}}i\Box^{-1}(x_l-x_1)J_{\mathbb{O}_{s_l}}(x_1).\end{aligned} \tag{36}$$

We redefine the spin labels that we sum over, $s_1k_1\to s_2k_2$, $s_2k_2\to s_3k_3$, …, $s_{l-1}k_{l-1}\to s_lk_l$, $s_lk_l\to s_1k_1$, to get

$$\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0] = (N^2-1)\sum_{l=1}^{\infty}\frac{(-1)^l}{2^l l}\int d^4x_1\cdots d^4x_l \sum_{s_1k_1}\cdots\sum_{s_lk_l}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_l}{k_l}\binom{s_l}{k_l+2}$$
$$\times\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\cdots\frac{\Gamma(3)\Gamma(s_l+3)}{\Gamma(5)\Gamma(s_l+1)}(i\overrightarrow{\partial}_{x_1^+})^{s_2-k_2+k_1} i\Box^{-1}(x_1-x_2)J_{\mathbb{O}_{s_2}}(x_2)$$
$$\times(i\overrightarrow{\partial}_{x_2^+})^{s_3-k_3+k_2} i\Box^{-1}(x_2-x_3)J_{\mathbb{O}_{s_3}}(x_3)\cdots\times(i\overrightarrow{\partial}_{x_l^+})^{s_1-k_1+k_l} i\Box^{-1}(x_l-x_1)J_{\mathbb{O}_{s_1}}(x_1). \tag{37}$$

Further redefining the labels of both the spin and the coordinate, $x_i s_i k_i \to x_{l-i+2}s_{l-i+2}k_{l-i+2}$ for $2\le i\le l$, keeping $x_1s_1k_1$ fixed, we obtain

$$\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0] = (N^2-1)\sum_{l=1}^{\infty}\frac{(-1)^l}{2^l l}\int d^4x_1\cdots d^4x_l \sum_{s_1k_1}\cdots\sum_{s_lk_l}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_l}{k_l}\binom{s_l}{k_l+2}$$
$$\times\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\cdots\frac{\Gamma(3)\Gamma(s_l+3)}{\Gamma(5)\Gamma(s_l+1)}(i\overrightarrow{\partial}_{x_1^+})^{s_l-k_l+k_1} i\Box^{-1}(x_1-x_l)J_{\mathbb{O}_{s_l}}(x_l)$$
$$\times(i\overrightarrow{\partial}_{x_l^+})^{s_{l-1}-k_{l-1}+k_l} i\Box^{-1}(x_l-x_{l-1})J_{\mathbb{O}_{s_{l-1}}}(x_{l-1})\cdots\times(i\overrightarrow{\partial}_{x_3^+})^{s_2-k_2+k_3} i\Box^{-1}(x_3-x_2)J_{\mathbb{O}_{s_2}}(x_2)$$
$$\times(i\overrightarrow{\partial}_{x_2^+})^{s_1-k_1+k_2} i\Box^{-1}(x_2-x_1)J_{\mathbb{O}_{s_1}}(x_1). \tag{38}$$

Employing

$$\partial^{s_a-k_a+k_b}_{x_a^+} i\Box^{-1}(x_a-x_b) = (-1)^{s_a-k_a+k_b}\partial^{s_a-k_a+k_b}_{x_b^+} i\Box^{-1}(x_a-x_b) = (-\partial_{x_b^+})^{s_a-k_a+k_b} i\Box^{-1}(x_b-x_a) \tag{39}$$

we get

$$\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0] = (N^2-1)\sum_{l=1}^{\infty}\frac{(-1)^l}{2^l l}\int d^4x_1\cdots d^4x_l \sum_{s_1k_1}\cdots\sum_{s_lk_l}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_l}{k_l}\binom{s_l}{k_l+2}$$
$$\times\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\cdots\frac{\Gamma(3)\Gamma(s_l+3)}{\Gamma(5)\Gamma(s_l+1)}(-i\overrightarrow{\partial}_{x_l^+})^{s_l-k_l+k_1} i\Box^{-1}(x_l-x_1)J_{\mathbb{O}_{s_l}}(x_l)$$
$$\times(-i\overrightarrow{\partial}_{x_{l-1}^+})^{s_{l-1}-k_{l-1}+k_l} i\Box^{-1}(x_{l-1}-x_l)J_{\mathbb{O}_{s_{l-1}}}(x_{l-1})\cdots\times(-i\overrightarrow{\partial}_{x_2^+})^{s_2-k_2+k_3} i\Box^{-1}(x_2-x_3)J_{\mathbb{O}_{s_2}}(x_2)$$
$$\times(-i\overrightarrow{\partial}_{x_1^+})^{s_1-k_1+k_2} i\Box^{-1}(x_1-x_2)J_{\mathbb{O}_{s_1}}(x_1). \tag{40}$$

Rearranging the position of the various factors, we finally obtain

$$\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0] = (N^2-1)\sum_{l=1}^{\infty}\frac{(-1)^l}{2^l l}\int d^4x_1\cdots d^4x_l \sum_{s_1k_1}\cdots\sum_{s_lk_l}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_l}{k_l}\binom{s_l}{k_l+2}$$
$$\times\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\cdots\frac{\Gamma(3)\Gamma(s_l+3)}{\Gamma(5)\Gamma(s_l+1)}(-i\overrightarrow{\partial}_{x_1^+})^{s_1-k_1+k_2} i\Box^{-1}(x_1-x_2)J_{\mathbb{O}_{s_2}}(x_2)$$
$$\times(-i\overrightarrow{\partial}_{x_2^+})^{s_2-k_2+k_3} i\Box^{-1}(x_2-x_3)J_{\mathbb{O}_{s_3}}(x_3)\cdots\times(-i\overrightarrow{\partial}_{x_{l-1}^+})^{s_{l-1}-k_{l-1}+k_l} i\Box^{-1}(x_{l-1}-x_l)J_{\mathbb{O}_{s_l}}(x_l)$$
$$\times(-i\overrightarrow{\partial}_{x_l^+})^{s_l-k_l+k_1} i\Box^{-1}(x_l-x_1)J_{\mathbb{O}_{s_1}}(x_1). \tag{41}$$

Given the corresponding expansion for $\Gamma_{\rm conf}[j_{\mathbb{O}},0]$

$$\Gamma_{\rm conf}[j_{\mathbb{O}},0] = (N^2-1)\sum_{l=1}^{\infty}\frac{(-1)^l}{2^l l}\int d^4x_1\cdots d^4x_l \sum_{s_1k_1}\cdots\sum_{s_lk_l}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_l}{k_l}\binom{s_l}{k_l+2}$$
$$\times\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\cdots\frac{\Gamma(3)\Gamma(s_l+3)}{\Gamma(5)\Gamma(s_l+1)}(-i\partial_{x_1^+})^{s_1-k_1+k_2} i\Box^{-1}(x_1-x_2)j_{\mathbb{O}_{s_2k_2}}(x_2)$$
$$\times(-i\partial_{x_2^+})^{s_2-k_2+k_3} i\Box^{-1}(x_2-x_3)j_{\mathbb{O}_{s_3k_3}}(x_3)\cdots\times(-i\partial_{x_{l-1}^+})^{s_{l-1}-k_{l-1}+k_l} i\Box^{-1}(x_{l-1}-x_l)j_{\mathbb{O}_{s_lk_l}}(x_l)$$
$$\times(-i\partial_{x_l^+})^{s_l-k_l+k_1} i\Box^{-1}(x_l-x_1)j_{\mathbb{O}_{s_1k_1}}(x_1), \tag{42}$$

it is clear that the functional derivatives of the two objects in Eqs. (41) and (42) yield the very same result, so that Eq. (34) is proved.

B. Dictionary in the general case

The dictionary in the general case is constructed as follows. We expand the traces of the logarithms in Eq. (23)

$$\begin{aligned}\mathcal{W}_{\rm conf}[J_{\mathbb{O}},J_{\tilde{\mathbb{O}}},J_{\mathbb{S}},J_{\tilde{\mathbb{S}}}] =& \frac{N^2-1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Tr}\left(\frac{1}{2}i\Box^{-1}\left(J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}+J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\right)\right)^n\\ &+\frac{N^2-1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Tr}\left(\frac{1}{2}i\Box^{-1}\left(J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}-J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\right)\right)^n\\ &+\frac{N^2-1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\left[\frac{1}{2}\left(I+\frac{1}{2}i\Box^{-1}\left(J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}-J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\right)\right)^{-1}\right.\\ &\left.\times\, i\Box^{-1}J_{\tilde{\mathbb{S}}_{s_1}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_1-2}\left(I+\frac{1}{2}i\Box^{-1}\left(J_{\mathbb{O}_{s_2}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_2-2}+J_{\tilde{\mathbb{O}}_{s_2}}\otimes\mathcal{H}^{\frac{5}{2}}_{s_2-2}\right)\right)^{-1}i\Box^{-1}J_{\mathbb{S}_{s_3}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_3-2}\right]^n. \end{aligned}\tag{43}$$

Further expanding

$$\left(I+\frac{1}{2}i\Box^{-1}(J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}\pm J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2})\right)^{-1}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}i\Box^{-1}(J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}\pm J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2})\right)^n,\tag{44}$$

we obtain

$$\begin{aligned}\mathcal{W}_{\rm conf}[J_{\mathbb{O}},J_{\tilde{\mathbb{O}}},J_{\mathbb{S}},J_{\tilde{\mathbb{S}}}] =& \frac{N^2-1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Tr}\left(\frac{1}{2}i\Box^{-1}(J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}+J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2})\right)^n\\ &+\frac{N^2-1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Tr}\left(\frac{1}{2}i\Box^{-1}(J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}-J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2})\right)^n\\ &+\frac{N^2-1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\left[\frac{1}{2}\sum_{m_1=0}^{\infty}(-1)^{m_1}\left(\frac{1}{2}i\Box^{-1}(J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}-J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2})\right)^{m_1}\right.\\ &\left.\times\, i\Box^{-1}J_{\tilde{\mathbb{S}}_{s_1}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_1-2}\sum_{m_2=0}^{\infty}(-1)^{m_2}\left(\frac{1}{2}i\Box^{-1}(J_{\mathbb{O}_{s_2}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_2-2}+J_{\tilde{\mathbb{O}}_{s_2}}\otimes\mathcal{H}^{\frac{5}{2}}_{s_2-2})\right)^{m_2}i\Box^{-1}J_{\mathbb{S}_{s_3}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_3-2}\right]^n. \end{aligned}\tag{45}$$

Though the above expansion looks complicated, it suffices to observe that a generic term of the expansion has the structure

$$\begin{aligned}&\mathrm{Tr}i\Box^{-1}J\mathcal{Y}i\Box^{-1}J\mathcal{Y}i\Box^{-1}J\mathcal{Y}i\Box^{-1}\cdots\\ &\quad\times i\Box^{-1}J\mathcal{H}i\Box^{-1}J\mathcal{H}i\Box^{-1}J\mathcal{H}i\Box^{-1}\cdots i\Box^{-1}J\mathcal{Y}i\Box^{-1}J\mathcal{Y},\end{aligned}\tag{46}$$

where the trace Tr includes all the objects to its right. Therefore, the left derivatives in the products of $i\Box^{-1}\mathcal{Y}i\Box^{-1}$ or $i\Box^{-1}\mathcal{H}i\Box^{-1}$ may be rearranged thanks to the cyclicity of the trace to build the corresponding kernels of integral operators that only involve the right derivatives acting on the same $i\Box^{-1}$, as in the previous example of the dictionary. Analogously, after a suitable relabeling of the indices, the contribution of each trace to the generating functional may be rewritten in a form that matches the corresponding structure in terms of the kernel $\mathcal{D}^{-1}$ and the currents $j_{\mathcal{O}_{sk}}$ dual to the component operators $\mathcal{O}_{sk}$.

C. Generating functional $\Gamma_{\rm conf}$

Hence, the construction above may be condensed into the single formula

$$\Gamma_{\rm conf}[j_{\mathbb{O}},j_{\tilde{\mathbb{O}}},j_{\mathbb{S}},j_{\tilde{\mathbb{S}}}]=-\frac{N^2-1}{2}\log\mathrm{Det}\boldsymbol{M}\tag{47}$$

with

$$\boldsymbol{M} = \begin{pmatrix} \mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} + \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}} & \sqrt{2}\mathcal{D}^{-1} j_{\mathbb{S}} \\ \sqrt{2}\mathcal{D}^{-1} j_{\tilde{\mathbb{S}}} & \mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} - \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}} \end{pmatrix} \quad (48)$$

and

$$\mathcal{D}^{-1}_{s_1 k_1, s_2 k_2} = \frac{i^{s_1}}{2} \frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)} \binom{s_1}{k_1} \binom{s_2}{k_2+2} \times (-\partial_+)^{s_1-k_1+k_2} i\Box^{-1}. \quad (49)$$

Equivalently, we may employ the modified kernel in Eq. (31). The above determinant reads (Appendix A)

$$\begin{aligned} &\Gamma_{\mathrm{conf}}[j_{\mathbb{O}}, j_{\tilde{\mathbb{O}}}, j_{\mathbb{S}}, j_{\tilde{\mathbb{S}}}] \\ &= -\frac{N^2-1}{2} \mathrm{Tr} \log (\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} + \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}}) \\ &\quad - \frac{N^2-1}{2} \mathrm{Tr} \log (\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} - \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}}) \\ &\quad - \frac{N^2-1}{2} \mathrm{Tr} \log [\mathbb{I} - 2(\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} - \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}})^{-1} \\ &\quad \times \mathcal{D}^{-1} j_{\tilde{\mathbb{S}}} (\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} + \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}})^{-1} \mathcal{D}^{-1} j_{\mathbb{S}}]. \end{aligned} \quad (50)$$

The generating functional immediately reduces—by setting the corresponding sources to zero—to the generating functionals in the separate balanced and unbalanced sectors [6]

$$\begin{aligned} \Gamma_{\mathrm{conf}}[j_{\mathbb{O}}, j_{\tilde{\mathbb{O}}}] &= \Gamma_{\mathrm{conf}}[j_{\mathbb{O}}, j_{\tilde{\mathbb{O}}}, 0, 0] \\ &= -\frac{N^2-1}{2} \mathrm{Tr} \log (\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} - \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}}) \\ &\quad - \frac{N^2-1}{2} \mathrm{Tr} \log (\mathbb{I} + \mathcal{D}^{-1} j_{\mathbb{O}} + \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}}) \end{aligned} \quad (51)$$

and

$$\begin{aligned} \Gamma_{\mathrm{conf}}[j_{\mathbb{S}}, j_{\tilde{\mathbb{S}}}] &= \Gamma_{\mathrm{conf}}[0, 0, j_{\mathbb{S}}, j_{\tilde{\mathbb{S}}}] \\ &= -\frac{N^2-1}{2} \mathrm{Tr} \log (\mathbb{I} - 2\mathcal{D}^{-1} j_{\tilde{\mathbb{S}}} \mathcal{D}^{-1} j_{\mathbb{S}}). \end{aligned} \quad (52)$$

Moreover, the generating functional of the Euclidean correlators reads

$$\begin{aligned} &\Gamma^E_{\mathrm{conf}}[j_{\mathbb{O}^E}, j_{\tilde{\mathbb{O}}^E}, j_{\mathbb{S}^E}, j_{\tilde{\mathbb{S}}^E}] \\ &= -\frac{N^2-1}{2} \mathrm{Tr} \log (\mathbb{I} + \mathcal{D}_E^{-1} j_{\mathbb{O}^E} + \mathcal{D}_E^{-1} j_{\tilde{\mathbb{O}}^E}) \\ &\quad - \frac{N^2-1}{2} \mathrm{Tr} \log (\mathbb{I} + \mathcal{D}_E^{-1} j_{\mathbb{O}^E} - \mathcal{D}_E^{-1} j_{\tilde{\mathbb{O}}^E}) \\ &\quad - \frac{N^2-1}{2} \mathrm{Tr} \log [\mathbb{I} - 2(\mathbb{I} + \mathcal{D}_E^{-1} j_{\mathbb{O}^E} - \mathcal{D}^{-1} j_{\tilde{\mathbb{O}}^E})^{-1} \\ &\quad \times \mathcal{D}_E^{-1} j_{\tilde{\mathbb{S}}^E} (\mathbb{I} + \mathcal{D}_E^{-1} j_{\mathbb{O}^E} + \mathcal{D}_E^{-1} j_{\tilde{\mathbb{O}}^E})^{-1} \mathcal{D}_E^{-1} j_{\mathbb{S}^E}] \end{aligned} \quad (53)$$

with [6]

$$\mathcal{D}^{-1}_{E\, s_1 k_1, s_2 k_2} = \frac{(-i)^{k_1-k_2}}{2} \frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)} \binom{s_1}{k_1} \binom{s_2}{k_2+2} \partial_z^{s_1-k_1+k_2} \triangle^{-1} \quad (54)$$

obtained by Wick rotation [6]

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}} \to -ix^z = -i\frac{x^4 + ix^3}{\sqrt{2}}, \quad (55)$$

where

$$\triangle = \delta_{\mu\nu} \partial_\mu \partial_\nu = \partial_4^2 + \sum_{i=1}^{3} \partial_i^2 \quad (56)$$

and [6]

$$\triangle^{-1}(x-y) = -\frac{1}{4\pi^2} \frac{1}{(x-y)^2}. \quad (57)$$

Equivalently, we may employ the modified kernel:

$$\mathcal{D}'^{-1}_{E\, s_1 k_1, s_2 k_2} = \frac{1\Gamma(3)\Gamma(s_1+3)}{2\Gamma(5)\Gamma(s_1+1)} \binom{s_1}{k_1} \binom{s_2}{k_2+2} \partial_z^{s_1-k_1+k_2} \triangle^{-1}. \quad (58)$$

V. GENERATING FUNCTIONAL IN THE MOMENTUM REPRESENTATION

The generating functional in the momentum representation is defined by the functional integral

$$\begin{aligned} \mathcal{Z}_{\mathrm{conf}}[J_{\mathcal{O}}] = \int \mathcal{D}A \mathcal{D}\bar{A} e^{-i \int d^4x \bar{A}^a \Box A^a} \\ \times \exp\left(\int \frac{d^4p}{(2\pi)^4} \sum_i J_{\mathcal{O}_i}(-p) \mathcal{O}_i(p) \right). \end{aligned} \quad (59)$$

Correspondingly, the correlators read

$$\begin{aligned} &\langle \mathcal{O}_{s_1}(p_1) \cdots \mathcal{O}_{s_n}(p_n) \rangle \\ &= (2\pi)^4 \frac{\delta}{\delta J_{\mathcal{O}_{s_1}}(-p_1)} \cdots (2\pi)^4 \frac{\delta}{\delta J_{\mathcal{O}_{s_n}}(-p_n)} \mathcal{W}_{\mathrm{conf}}[J_{\mathcal{O}}]. \end{aligned} \quad (60)$$

Employing the dictionary we get as well

$$\langle \mathcal{O}_{s_1}(p_1)\cdots\mathcal{O}_{s_n}(p_n)\rangle = \sum_{k_1=0}^{l_1}(2\pi)^4\frac{\delta}{\delta j_{\mathcal{O}_{s_1k_1}}(-p_1)} \cdots\sum_{k_n=0}^{l_n}(2\pi)^4\frac{\delta}{\delta j_{\mathcal{O}_{s_nk_n}}(-p_n)}\Gamma_{\rm conf}[j_{\mathcal{O}}]. \tag{61}$$

A. Generating functional as a Fredholm determinant

To find the explicit form of the generating functional in the momentum representation we choose again as example

$$\Gamma_{\rm conf}[j_{\mathbb{O}}] = -(N^2-1)\log{\rm Det}\Big(\delta_{s_1k_1,s_2k_2}\delta^{(4)}(x-y) + \mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)j_{\mathbb{O}_{s_2k_2}}(y)\Big), \tag{62}$$

where the argument of the determinant is the kernel

$$K_{s_1k_1,s_2k_2}(x,y) = \delta_{s_1k_1,s_2k_2}\delta^{(4)}(x-y)+\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)j_{\mathbb{O}_{s_2k_2}}(y) \tag{63}$$

of the integral operator

$$\psi_{s_1k_1}(x) = \sum_{s_2k_2}\int K_{s_1k_1,s_2k_2}(x,y)\phi_{s_2k_2}(y)d^4y. \tag{64}$$

To obtain the kernel in the momentum representation [6] we perform the Fourier transform of the left-hand side (LHS)

$$\psi_{s_1k_1}(q) = \int\psi_{s_1k_1}(x)e^{-iq\cdot x}d^4x = \sum_{s_2k_2}\int K_{s_1k_1,s_2k_2}(x,y)\phi_{s_2k_2}(y)e^{-iq\cdot x}d^4xd^4y \tag{65}$$

and write the right-hand side (RHS) in terms of the Fourier-transformed fields

$$\psi_{s_1k_1}(q) = \sum_{s_2k_2}\int K_{s_1k_1,s_2k_2}(x,y)\phi_{s_2k_2}(p)e^{ip\cdot y}e^{-iq\cdot x}\times d^4xd^4y\frac{d^4p}{(2\pi)^4}. \tag{66}$$

Substituting the kernel in Eq. (63)

$$\psi_{s_1k_1}(q) = \sum_{s_2k_2}\int \delta_{s_1k_1,s_2k_2}\delta^{(4)}(x-y)\phi_{s_2k_2}(p)e^{ipy}e^{-iqx} + \mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)j_{\mathbb{O}_{s_2k_2}}(y)\phi_{s_2k_2}(p)e^{ipy}e^{-iqx}\times d^4xd^4y\frac{d^4p}{(2\pi)^4}, \tag{67}$$

we get

$$\psi_{s_1k_1}(q) = \sum_{s_2k_2}\int \delta_{s_1k_1,s_2k_2}(2\pi)^4\delta^{(4)}(p-q)\phi_{s_2k_2}(p)\frac{d^4p}{(2\pi)^4} + \sum_{s_2k_2}\int\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)j_{\mathbb{O}_{s_2k_2}}(y)\phi_{s_2k_2}(p)\times e^{ip\cdot y}e^{-iq\cdot x}d^4xd^4y\frac{d^4p}{(2\pi)^4}. \tag{68}$$

The second line in the above equation becomes

$$\sum_{s_2k_2}\int\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(k_1)j_{\mathbb{O}_{s_2k_2}}(k_2)\phi_{s_2k_2}(p)\times e^{ik_1\cdot(x-y)}e^{ik_2\cdot y}e^{ip\cdot y}e^{-iq\cdot x}d^4xd^4y\frac{d^4p}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} \tag{69}$$

that reduces to

$$\sum_{s_2k_2}\int\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(q)j_{\mathbb{O}_{s_2k_2}}(q-p)\phi_{s_2k_2}(p)\frac{d^4p}{(2\pi)^4}. \tag{70}$$

Therefore, the kernel in the momentum representation is

$$K_{s_1k_1,s_2k_2}(q_1,q_2) = \delta_{s_1k_1,s_2k_2}(2\pi)^4\delta^{(4)}(q_1-q_2) + \mathcal{D}^{-1}_{s_1k_1,s_2k_2}(q_1)j_{\mathbb{O}_{s_2k_2}}(q_1-q_2) \tag{71}$$

that defines the integral operator

$$\psi_{s_1k_1}(q_1) = \sum_{s_2k_2}\int K_{s_1k_1,s_2k_2}(q_1,q_2)\phi_{s_2k_2}(q_2)\frac{d^4q_2}{(2\pi)^4} \tag{72}$$

with

$$\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(p) = \frac{i^{s_1}}{2}\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\binom{s_1}{k_1}\binom{s_2}{k_2+2}\times(-ip_+)^{s_1-k_1+k_2}\frac{-i}{|p|^2+i\epsilon}. \tag{73}$$

Equivalently, we may employ the kernel

$$\mathcal{D}'^{-1}_{s_1k_1,s_2k_2}(p) = \frac{1}{2}\frac{\Gamma(3)\Gamma(s_1+3)}{\Gamma(5)\Gamma(s_1+1)}\binom{s_1}{k_1}\binom{s_2}{k_2+2}\times p_+^{s_1-k_1+k_2}\frac{-i}{|p|^2+i\epsilon}. \tag{74}$$

Hence, the generating functional in the momentum representation reads [6]

$$\Gamma_{\rm conf}[j_{\mathbb{O}}] = -(N^2-1)\log{\rm Det}\Big(\delta_{s_1k_1,s_2k_2}(2\pi)^4\delta^{(4)}(q_1-q_2) + \mathcal{D}^{-1}_{s_1k_1,s_2k_2}(q_1)j_{\mathbb{O}_{s_2k_2}}(q_1-q_2)\Big). \tag{75}$$

To obtain the correlators we expand the generating functional

$$\Gamma_{\rm conf}[j_{\mathbb{O}}] = -(N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int\frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_n}{(2\pi)^4}\sum_{s_1k_1}\cdots\sum_{s_nk_n}\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(q_1)j_{\mathbb{O}_{s_2k_2}}(q_1-q_2)$$
$$\times\,\mathcal{D}^{-1}_{s_2k_2,s_3k_3}(q_2)j_{\mathbb{O}_{s_3k_3}}(q_2-q_3)\cdots\times\mathcal{D}^{-1}_{s_nk_n,s_1k_1}(q_n)j_{\mathbb{O}_{s_1k_1}}(q_n-q_1). \tag{76}$$

We change variables defining $p_i = q_i - q_{i+1}$ with $q_{n+1}\equiv q_1$ that automatically ensures $\sum_{i=1}^{n} p_i = 0$

$$\begin{aligned}
q_1 &= q_n - p_n,\\
q_2 &= q_1 - p_1 = q_n - p_n - p_1,\\
q_3 &= q_2 - p_2 = q_n - p_n - p_2 - p_1,\\
&\vdots\\
q_{n-1} &= q_n - p_n - \sum_{i=1}^{n-2} p_i,\\
q_n &= q_{n-1} - p_{n-1},
\end{aligned} \tag{77}$$

so that, setting $q_n\equiv q$ and inserting in the RHS of Eq. (76) the integral $\int\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2+\cdots+p_n)$ to keep p_n while enforcing $\sum_{i=1}^{n} p_i = 0$, we obtain

$$\Gamma_{\rm conf}[j_{\mathbb{O}}] = -(N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2+\cdots+p_n)\int\frac{d^4q}{(2\pi)^4}\sum_{s_1k_1}\cdots$$
$$\sum_{s_nk_n}\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(q-p_n)j_{\mathbb{O}_{s_2k_2}}(p_1)\mathcal{D}^{-1}_{s_2k_2,s_3k_3}(q-p_1-p_n)j_{\mathbb{O}_{s_3k_3}}(p_2)\cdots\times\mathcal{D}^{-1}_{s_nk_n,s_1k_1}(q)j_{\mathbb{O}_{s_1k_1}}(p_n). \tag{78}$$

Moreover, making the substitution $p_i\to -p_i$, we get

$$\Gamma_{\rm conf}[j_{\mathbb{O}}] = -(N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2+\cdots+p_n)\int\frac{d^4q}{(2\pi)^4}\sum_{s_1k_1}\cdots$$
$$\sum_{s_nk_n}\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(q+p_n)j_{\mathbb{O}_{s_2k_2}}(-p_1)\mathcal{D}^{-1}_{s_2k_2,s_3k_3}(q+p_1+p_n)j_{\mathbb{O}_{s_3k_3}}(-p_2)\cdots\times\mathcal{D}^{-1}_{s_nk_n,s_1k_1}(q)j_{\mathbb{O}_{s_1k_1}}(-p_n). \tag{79}$$

Explicitly, by Eq. (74)

$$\Gamma_{\rm conf}[j_{\mathbb{O}}] = (N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^{n}}{2^n n}\sum_{s_1\ldots s_n}\int\frac{d^4q}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2+\cdots+p_n)$$
$$\times\frac{\Gamma(s_1+3)\Gamma(3)}{\Gamma(5)\Gamma(s_1+1)}\cdots\frac{\Gamma(s_n+3)\Gamma(3)}{\Gamma(5)\Gamma(s_n+1)}\sum_{k_1=0}^{s_1-2}\cdots\sum_{k_n=0}^{s_n-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_n}{k_n}\binom{s_n}{k_n+2}$$
$$\times\, j_{\mathbb{O}_{s_1k_1}}(-p_n)\frac{-i(q+p_n)_+^{s_1-k_1+k_2}}{|q+p_n|^2+i\epsilon}j_{\mathbb{O}_{s_2k_2}}(-p_1)\frac{-i(q+p_1+p_n)_+^{s_2-k_2+k_3}}{|q+p_1+p_n|^2+i\epsilon}\cdots\times j_{\mathbb{O}_{s_nk_n}}(-p_{n-1})\frac{-iq_+^{s_n-k_n+k_1}}{|q|^2+i\epsilon}. \tag{80}$$

We now relabel $p_n\to p_1$, $p_1\to p_2$, …, $p_{n-1}\to p_n$ so that

$$\Gamma_{\rm conf}[j_{\mathbb{O}}] = (N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^{n}}{2^n n}\sum_{s_1\ldots s_n}\int\frac{d^4q}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2+\cdots+p_n)$$
$$\times\frac{\Gamma(s_1+3)\Gamma(3)}{\Gamma(5)\Gamma(s_1+1)}\cdots\frac{\Gamma(s_n+3)\Gamma(3)}{\Gamma(5)\Gamma(s_n+1)}\sum_{k_1=0}^{s_1-2}\cdots\sum_{k_n=0}^{s_n-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_n}{k_n}\binom{s_n}{k_n+2}$$
$$\times\, j_{\mathbb{O}_{s_1k_1}}(-p_1)\frac{-i(q+p_1)_+^{s_1-k_1+k_2}}{|q+p_1|^2+i\epsilon}j_{\mathbb{O}_{s_2k_2}}(-p_2)\frac{-i(q+p_1+p_2)_+^{s_2-k_2+k_3}}{|q+p_1+p_2|^2+i\epsilon}\cdots\times j_{\mathbb{O}_{s_nk_n}}(-p_n)\frac{-iq_+^{s_n-k_n+k_1}}{|q|^2+i\epsilon}. \tag{81}$$

We employ Eq. (61) to obtain the correlators

$$\left(\frac{2\Gamma(5)\Gamma(s_1+1)}{\Gamma(3)\Gamma(s_1+3)}\right)\cdots\left(\frac{2\Gamma(5)\Gamma(s_n+1)}{\Gamma(3)\Gamma(s_n+3)}\right)\langle\mathbb{O}_{s_1}(p_1)\cdots\mathbb{O}_{s_n}(p_n)\rangle$$
$$=(N^2-1)(2\pi)^4 i^n\delta^{(4)}(p_1+\cdots+p_n)\sum_{k_1=0}^{s_1-2}\cdots\sum_{k_n=0}^{s_n-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_n}{k_n}\binom{s_n}{k_n+2}$$
$$\times\frac{1}{n}\sum_{\sigma\in P_n}\int\frac{d^4q}{(2\pi)^4}\frac{(p_{\sigma(1)}+q)_+^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}}}{|p_{\sigma(1)}+q|^2}\frac{(p_{\sigma(1)}+p_{\sigma(2)}+q)_+^{s_{\sigma(2)}-k_{\sigma(2)}+k_{\sigma(3)}}}{|p_{\sigma(1)}+p_{\sigma(2)}+q|^2}\cdots\times\frac{(\sum_{l=1}^{n-1}p_{\sigma(l)}+q)_+^{s_{\sigma(n-1)}-k_{\sigma(n-1)}+k_{\sigma(n)}}}{|\sum_{l=1}^{n-1}p_{\sigma(l)}+q|^2}$$
$$\times\frac{(q)_+^{s_{\sigma(n)}-k_{\sigma(n)}+k_{\sigma(1)}}}{|q|^2} \tag{82}$$

that coincide with the computation in [6]. More generally, the generating functional in the momentum representation reads from Eq. (50) with the kernel in the momentum representation defined by Eq. (73) or Eq. (74).

B. Generating functional as determinant of a quadratic form in terms of the Gegenbauer polynomials

Alternatively, we write the generating functional employing the functional-integral definition in Eq. (25) with (minus) the propagator in the momentum representation

$$i\Box^{-1}(p)=\frac{-i}{|p|^2+i\epsilon} \tag{83}$$

and correspondingly

$$(i\partial_+)^{s-k_1+k_2}i\Box^{-1}(p)=(-p_+)^{s-k_1+k_2}i\Box^{-1}(p). \tag{84}$$

In fact, especially in the conformal setting (Appendix B), it may be convenient to express the generating functional in a more compact—but more implicit—form in terms of the Gegenbauer polynomials. For example, the analog of Eq. (62) reads

$$\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0]=-\log\mathrm{Det}\left(\mathcal{I}+\frac{1}{2}i\Box^{-1}J_{\mathbb{O}_s}\otimes\mathcal{Y}_{s-2}^{\frac{5}{2}}\right) \tag{85}$$

with

$$\mathcal{Y}_{s-2}^{\frac{5}{2}}(\overrightarrow{\partial}_+,\overleftarrow{\partial}_+)=\overleftarrow{\partial}_+(i\overrightarrow{\partial}_++i\overleftarrow{\partial}_+)^{s-2}\times C_{s-2}^{\frac{5}{2}}\left(\frac{\overrightarrow{\partial}_+-\overleftarrow{\partial}_+}{\overrightarrow{\partial}_++\overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+. \tag{86}$$

Expanding Eq. (85) we get

$$\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0]$$
$$=(N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n}\sum_{s_1\ldots s_n}\int d^4x_1\cdots d^4x_n$$
$$\times i\Box^{-1}(x_1-x_2)J_{\mathbb{O}_{s_2}}(x_2)\otimes\mathcal{Y}_{s_2-2}^{\frac{5}{2}}(\overrightarrow{\partial}_{x_2^+},\overleftarrow{\partial}_{x_2^+})$$
$$\cdots\times i\Box^{-1}(x_n-x_1)J_{\mathbb{O}_{s_1}}(x_1)\otimes\mathcal{Y}_{s_1-2}^{\frac{5}{2}}(\overrightarrow{\partial}_{x_1^+},\overleftarrow{\partial}_{x_1^+}). \tag{87}$$

Employing the Fourier transformed sources and propagators

$$i\Box^{-1}(x_i-x_j)=\int\frac{d^4q}{(2\pi)^4}e^{iq\cdot(x_i-x_j)}\frac{-i}{|q|^2+i\epsilon},$$
$$J_{\mathbb{O}_{s_i}}(x_i)=\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x_i}J_{\mathbb{O}_{s_i}}(p_i), \tag{88}$$

we obtain

$$\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0]=(N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n}\sum_{s_1\ldots s_n}\int d^4x_1\cdots d^4x_n\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_n}{(2\pi)^4}$$
$$\times e^{-ix_1\cdot(q_n-q_1-p_1)}e^{-ix_2\cdot(q_1-q_2-p_2)}e^{-ix_3\cdot(q_2-q_3-p_3)}\cdots\times e^{-ix_n\cdot(q_{n-1}-q_n-p_n)}\frac{-i}{|q_1|^2+i\epsilon}J_{\mathbb{O}_{s_1}}(p_1)$$
$$\times q_{1+}(q_{2+}-q_{1+})^{s_2-2}C_{s_2-2}^{\frac{5}{2}}\left(\frac{q_{2+}+q_{1+}}{q_{2+}-q_{1+}}\right)q_{2+}\frac{-i}{|q_2|^2+i\epsilon}J_{\mathbb{O}_{s_2}}(p_2)q_{2+}(q_{3+}-q_{2+})^{s_3-2}C_{s_3-2}^{\frac{5}{2}}\left(\frac{q_{2+}+q_{3+}}{q_{3+}-q_{2+}}\right)q_{3+}$$
$$\cdots\times\frac{-i}{|q_{n-1}|^2+i\epsilon}J_{\mathbb{O}_{s_{n-1}}}(p_{n-1})q_{n-1+}(q_{n+}-q_{n-1+})^{s_n-2}C_{s_n-2}^{\frac{5}{2}}\left(\frac{q_{n+}+q_{n-1+}}{q_{n+}-q_{n-1+}}\right)q_{n+}\frac{-i}{|q_n|^2+i\epsilon}J_{\mathbb{O}_{s_n}}(p_n)$$
$$\times q_{n+}(q_{1+}-q_{n+})^{s_1-2}C_{s_1-2}^{\frac{5}{2}}\left(\frac{q_{1+}+q_{n+}}{q_{1+}-q_{n+}}\right)q_{1+} \tag{89}$$

that, grouping together the q_{i+} on the left and right of the Gegenbauer polynomials, we rewrite as

$$\begin{aligned}\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0] = (N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n}\sum_{s_1\ldots s_n}\int d^4x_1\cdots d^4x_n\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_n}{(2\pi)^4}\\ \times e^{-ix_1\cdot(q_n-q_1-p_1)}e^{-ix_2\cdot(q_1-q_2-p_2)}e^{-ix_3\cdot(q_2-q_3-p_3)}\cdots\times e^{-ix_n\cdot(q_{n-1}-q_n-p_n)}\frac{-iq_{1+}^2}{|q_1|^2+i\epsilon}J_{\mathbb{O}_{s_1}}(p_1)\\ \times(q_{2+}-q_{1+})^{s_2-2}C^{\frac{5}{2}}_{s_2-2}\left(\frac{q_{2+}+q_{1+}}{q_{2+}-q_{1+}}\right)\frac{-iq_{2+}^2}{|q_2|^2+i\epsilon}J_{\mathbb{O}_{s_2}}(p_2)(q_{3+}-q_{2+})^{s_3-2}C^{\frac{5}{2}}_{s_3-2}\left(\frac{q_{2+}+q_{3+}}{q_{3+}-q_{2+}}\right)\\ \cdots\times\frac{-iq_{n-1+}^2}{|q_{n-1}|^2+i\epsilon}J_{\mathbb{O}_{s_{n-1}}}(p_{n-1})(q_{n+}-q_{n-1+})^{s_n-2}C^{\frac{5}{2}}_{s_n-2}\left(\frac{q_{n+}+q_{n-1+}}{q_{n+}-q_{n-1+}}\right)\frac{-iq_{n+}^2}{|q_n|^2+i\epsilon}J_{\mathbb{O}_{s_n}}(p_n)\\ \times(q_{1+}-q_{n+})^{s_1-2}C^{\frac{5}{2}}_{s_1-2}\left(\frac{q_{1+}+q_{n+}}{q_{1+}-q_{n+}}\right).\end{aligned}\tag{90}$$

Integrating over the coordinates and momenta we get

$$\begin{aligned}&q_1 = q_n - p_1,\\ &q_2 = q_1 - p_2 = q_n - p_1 - p_2,\\ &q_3 = q_2 - p_3 = q_n - p_1 - p_2 - p_3,\\ &\vdots\\ &q_{n-2} = q_{n-3} - p_{n-2} = q_n - \sum_{i=1}^{n-2} p_i,\\ &q_{n-1} = q_{n-2} - p_{n-1} = q_n - \sum_{i=1}^{n-1} p_i.\end{aligned}\tag{91}$$

The last integration enforces $\sum_{i=1}^{n} p_i = 0$, so that, inserting in the RHS of Eq. (90) the integral $\int\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2+\cdots+p_n)$ to keep p_n, we obtain

$$\begin{aligned}\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0] = (N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n}\sum_{s_1,\ldots,s_n}\int\frac{d^4q}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2+\cdots+p_n)\\ \times\frac{-i(q-p_1)_+^2}{|q-p_1|^2+i\epsilon}J_{\mathbb{O}_{s_1}}(p_1)(-p_{1+})^{s_1-2}C^{\frac{5}{2}}_{s_1-2}\left(\frac{2q_+-p_{1+}}{-p_{1+}}\right)\frac{-i(q-p_1-p_2)_+^2}{|q-p_1-p_2|^2+i\epsilon}J_{\mathbb{O}_{s_2}}(p_2)\\ \times(-p_{2+})^{s_2-2}C^{\frac{5}{2}}_{s_2-2}\left(\frac{2q_+-2p_{1+}-p_{2+}}{-p_{2+}}\right)\frac{-i(q-p_1-p_2-p_3)_+^2}{|q-p_1-p_2-p_3|^2+i\epsilon}J_{\mathbb{O}_{s_3}}(p_3)\\ \times(-p_{3+})^{s_3-2}C^{\frac{5}{2}}_{s_3-2}\left(\frac{2q_+-2p_{1+}-2p_{2+}-p_{3+}}{-p_{3+}}\right)\cdots\times\frac{-iq_+^2}{|q|^2+i\epsilon}J_{\mathbb{O}_{s_n}}(p_n)\\ \times(-p_{n+})^{s_n-2}C^{\frac{5}{2}}_{s_n-2}\left(\frac{2q_+-2\sum_{i=1}^{n-1}p_{i+}-p_{n+}}{-p_{n+}}\right)\end{aligned}\tag{92}$$

with $q_n = q$. Finally, substituting $p_i \to -p_i$ we get

$$\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0]=(N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n}\sum_{s_1\ldots s_n}\int\frac{d^4q}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2+\cdots+p_n)$$
$$\times\frac{-i(q+p_1)_+^2}{|q+p_1|^2+i\epsilon}J_{\mathbb{O}_{s_1}}(-p_1)(p_{1+})^{s_1-2}C^{\frac{5}{2}}_{s_1-2}\left(\frac{2q_++p_{1+}}{p_{1+}}\right)\frac{-i(q+p_1+p_2)_+^2}{|q+p_1+p_2|^2+i\epsilon}J_{\mathbb{O}_{s_2}}(-p_2)$$
$$\times(p_{2+})^{s_2-2}C^{\frac{5}{2}}_{s_2-2}\left(\frac{2q_++2p_{1+}+p_{2+}}{p_{2+}}\right)\frac{-i(q+p_1+p_2+p_3)_+^2}{|q+p_1+p_2+p_3|^2+i\epsilon}J_{\mathbb{O}_{s_3}}(-p_3)$$
$$\times(p_{3+})^{s_3-2}C^{\frac{5}{2}}_{s_3-2}\left(\frac{2q_++2p_{1+}+2p_{2+}+p_{3+}}{p_{3+}}\right)\cdots\times\frac{-iq_+^2}{|q|^2+i\epsilon}J_{\mathbb{O}_{s_n}}(-p_n)$$
$$\times(p_{n+})^{s_n-2}C^{\frac{5}{2}}_{s_n-2}\left(\frac{2q_++2\sum_{i=1}^{n-1}p_{i+}+p_{n+}}{p_{n+}}\right).\tag{93}$$

We verify the equivalence with the dictionary expanding the Gegenbauer polynomials

$$p_+^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{2l_++p_+}{p_+}\right)=\frac{\Gamma(s+3)\Gamma(3)}{\Gamma(5)\Gamma(s+1)}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}l_+^k(l+p)_+^{s-k-2}\tag{94}$$

according to their definition (Appendix B)

$$C^{\frac{5}{2}}_{s-2}(x)=\frac{\Gamma(s+3)\Gamma(3)}{\Gamma(5)\Gamma(s+1)}\sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{s-k-2}\tag{95}$$

and substituting back in Eq. (93)

$$\mathcal{W}_{\rm conf}[J_{\mathbb{O}},0,0,0]=(N^2-1)\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n}\sum_{s_1\ldots s_n}\int\frac{d^4q}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2+\cdots+p_n)$$
$$\times\frac{\Gamma(s_1+3)\Gamma(3)}{\Gamma(5)\Gamma(s_1+1)}\cdots\frac{\Gamma(s_n+3)\Gamma(3)}{\Gamma(5)\Gamma(s_n+1)}\sum_{k_1=0}^{s_1-2}\cdots\sum_{k_n=0}^{s_n-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_n}{k_n}\binom{s_n}{k_n+2}$$
$$\times J_{\mathbb{O}_{s_1}}(-p_1)\frac{-i(q+p_1)_+^{s_1-k_1+k_2}}{|q+p_1|^2+i\epsilon}J_{\mathbb{O}_{s_2}}(-p_2)\frac{-i(q+p_1+p_2)_+^{s_2-k_2+k_3}}{|q+p_1+p_2|^2+i\epsilon}\cdots\times J_{\mathbb{O}_{s_n}}(-p_n)\frac{-iq_+^{s_n-k_n+k_1}}{|q|^2+i\epsilon}\tag{96}$$

that essentially coincides with Eq. (81).

VI. MIXED THREE- AND FOUR-POINT CONFORMAL CORRELATORS

We employ the generating functional to calculate the three- and four-point correlators in the mixed balanced/unbalanced sector.

A. Three-point correlators in Minkowskian spacetime

The generating functional of the mixed $\mathbb{O}$, $\mathbb{S}$, and $\bar{\mathbb{S}}$ correlators reads

$$\Gamma_{\rm conf}[j_{\mathbb{O}},0,j_{\mathbb{S}},j_{\bar{\mathbb{S}}}]=-\frac{N^2-1}{2}\mathrm{Tr}\log[\mathbb{I}-2(\mathbb{I}+\mathcal{D}^{-1}j_{\mathbb{O}})^{-1}\mathcal{D}^{-1}j_{\bar{\mathbb{S}}}(\mathbb{I}+\mathcal{D}^{-1}j_{\mathbb{O}})^{-1}\mathcal{D}^{-1}j_{\mathbb{S}}].\tag{97}$$

Only two terms contribute to the mixed three-point correlator $\langle\mathbb{O}_{s_1}(x)\mathbb{S}_{s_2}(y)\bar{\mathbb{S}}_{s_3}(z)\rangle$

$$\begin{aligned}\Gamma_{\rm conf}[j_{\mathbb{O}},0,j_{\mathbb{S}},j_{\bar{\mathbb{S}}}] &= \frac{N^2-1}{2}2\mathrm{Tr}[(\mathbb{I}+\mathcal{D}^{-1}j_{\mathbb{O}})^{-1}\mathcal{D}^{-1}j_{\bar{\mathbb{S}}}(\mathbb{I}+\mathcal{D}^{-1}j_{\mathbb{O}})^{-1}\mathcal{D}^{-1}j_{\mathbb{S}}]+\cdots\\ &= \frac{N^2-1}{2}2\mathrm{Tr}[(\mathbb{I}-\mathcal{D}^{-1}j_{\mathbb{O}})\mathcal{D}^{-1}j_{\bar{\mathbb{S}}}(\mathbb{I}-\mathcal{D}^{-1}j_{\mathbb{O}})\mathcal{D}^{-1}j_{\mathbb{S}}]+\cdots\\ &= -\frac{N^2-1}{2}2\mathrm{Tr}(\mathcal{D}^{-1}j_{\mathbb{O}}\mathcal{D}^{-1}j_{\bar{\mathbb{S}}}\mathcal{D}^{-1}j_{\mathbb{S}})-\frac{N^2-1}{2}2\mathrm{Tr}(\mathcal{D}^{-1}j_{\bar{\mathbb{S}}}\mathcal{D}^{-1}j_{\mathbb{O}}\mathcal{D}^{-1}j_{\mathbb{S}})+\cdots. \end{aligned} \tag{98}$$

Hence,

$$\begin{aligned}\langle \mathbb{O}_{s_1}(x)\mathbb{S}_{s_2}(y)\bar{\mathbb{S}}_{s_3}(z)\rangle &= \sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\frac{\delta}{\delta j_{\mathbb{O}_{s_1k_1}}(x)}\frac{\delta}{\delta j_{\mathbb{S}_{s_2k_2}}(y)}\frac{\delta}{\delta j_{\bar{\mathbb{S}}_{s_3k_3}}(z)}\Gamma_{\rm conf}[j_{\mathbb{O}},j_{\tilde{\mathbb{O}}},j_{\mathbb{S}},j_{\bar{\mathbb{S}}}]\\ &= -\frac{N^2-1}{2}2\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\frac{\delta}{\delta j_{\mathbb{O}_{s_1k_1}}(x)}\frac{\delta}{\delta j_{\mathbb{S}_{s_2k_2}}(y)}\frac{\delta}{\delta j_{\bar{\mathbb{S}}_{s_3k_3}}(z)}\int d^4x_1d^4x_2d^4x_3\\ &\quad\times\sum_{k_1's_1'}\sum_{k_2's_2'}\sum_{k_3's_3'}\Big[\mathcal{D}^{-1}_{s_1'k_1',s_2'k_2'}(x_1-x_2)j_{\mathbb{O}_{s_2'k_2'}}(x_2)\mathcal{D}^{-1}_{s_2'k_2',s_3'k_3'}(x_2-x_3)j_{\mathbb{S}_{s_3'k_3'}}(x_3)\mathcal{D}^{-1}_{s_3'k_3',s_1'k_1'}(x_3-x_1)j_{\bar{\mathbb{S}}_{s_1'k_1'}}(x_1)\\ &\quad+\mathcal{D}^{-1}_{s_1'k_1',s_2'k_2'}(x_1-x_2)j_{\bar{\mathbb{S}}_{s_2'k_2'}}(x_2)\mathcal{D}^{-1}_{s_2'k_2',s_3'k_3'}(x_2-x_3)j_{\mathbb{O}_{s_3'k_3'}}(x_3)\mathcal{D}^{-1}_{s_3'k_3',s_1'k_1'}(x_3-x_1)j_{\mathbb{S}_{s_1'k_1'}}(x_1)\Big]\\ &= -(N^2-1)\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\Big[\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)\mathcal{D}^{-1}_{s_2k_2,s_3k_3}(y-z)\mathcal{D}^{-1}_{s_3k_3,s_1k_1}(z-x)\\ &\quad+\mathcal{D}^{-1}_{s_2k_2,s_3k_3}(y-z)\mathcal{D}^{-1}_{s_3k_3,s_1k_1}(z-x)\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)\Big]\\ &= -2(N^2-1)\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\mathcal{D}^{-1}_{s_1k_1,s_2k_2}(x-y)\mathcal{D}^{-1}_{s_2k_2,s_3k_3}(y-z)\mathcal{D}^{-1}_{s_3k_3,s_1k_1}(z-x). \end{aligned} \tag{99}$$

Employing Eq. (30), we obtain

$$\begin{aligned}\langle \mathbb{O}_{s_1}(x)\mathbb{S}_{s_2}(y)\bar{\mathbb{S}}_{s_3}(z)\rangle &= -\frac{1}{(4\pi^2)^3}2\left(\frac{2}{4!}\right)^3\frac{N^2-1}{8}i^{s_1+s_2+s_3}2^{s_1+s_2+s_3}(s_1+1)(s_1+2)(s_2+1)(s_2+2)(s_3+1)(s_3+2)\\ &\quad\times\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}\\ &\quad\times(s_1-k_1+k_2)!(s_2-k_2+k_3)!(s_3-k_3+k_1)!\frac{(x-y)_+^{s_1-k_1+k_2}}{(|x-y|^2)^{s_1+1-k_1+k_2}}\frac{(y-z)_+^{s_2-k_2+k_3}}{(|y-z|^2)^{s_2+1-k_2+k_3}}\\ &\quad\times\frac{(z-x)_+^{s_3-k_3+k_1}}{(|z-x|^2)^{s_3+1-k_3+k_1}} \end{aligned} \tag{100}$$

that agrees with [6]. We also infer from the generating functional that [6]

$$\langle\tilde{\mathbb{O}}_{s_1}(x)\mathbb{S}_{s_2}(y)\bar{\mathbb{S}}_{s_3}(z)\rangle = 0. \tag{101}$$

Indeed,

$$\Gamma_{\rm conf}[0,j_{\tilde{\mathbb{O}}},j_{\mathbb{S}},j_{\bar{\mathbb{S}}}] = -\frac{N^2-1}{2}\mathrm{Tr}\log[\mathbb{I}-2(\mathbb{I}-\mathcal{D}^{-1}j_{\tilde{\mathbb{O}}})^{-1}\mathcal{D}^{-1}j_{\bar{\mathbb{S}}}(\mathbb{I}+\mathcal{D}^{-1}j_{\tilde{\mathbb{O}}})^{-1}\mathcal{D}^{-1}j_{\mathbb{S}}] \tag{102}$$

contributes two terms linear in $j_{\tilde{\mathbb{O}}}$ with opposite signs that cancel each other.

B. Four-point correlators in Minkowskian spacetime

Only three terms contribute to the mixed four-point correlator $\langle \mathbb{O}_{s_1}(x_1)\mathbb{O}_{s_2}(x_2)\mathbb{S}_{s_3}(x_3)\bar{\mathbb{S}}_{s_4}(x_4)\rangle$

$$\begin{aligned}\Gamma_{\mathrm{conf}}[j_{\mathbb{O}},0,j_{\mathbb{S}},j_{\bar{\mathbb{S}}}] &= \frac{N^2-1}{2}2\mathrm{Tr}[(\mathbb{I}-\mathcal{D}^{-1}j_{\mathbb{O}}+\mathcal{D}^{-1}j_{\mathbb{O}}\mathcal{D}^{-1}j_{\mathbb{O}}))\mathcal{D}^{-1}j_{\bar{\mathbb{S}}}(\mathbb{I}-\mathcal{D}^{-1}j_{\mathbb{O}}+\mathcal{D}^{-1}j_{\mathbb{O}}\mathcal{D}^{-1}j_{\mathbb{O}})\mathcal{D}^{-1}j_{\mathbb{S}}]+\cdots \\ &= \frac{N^2-1}{2}2\mathrm{Tr}(\mathcal{D}^{-1}j_{\mathbb{O}}\mathcal{D}^{-1}j_{\bar{\mathbb{S}}}\mathcal{D}^{-1}j_{\mathbb{O}}\mathcal{D}^{-1}j_{\mathbb{S}})+\frac{N^2-1}{2}2\mathrm{Tr}(\mathcal{D}^{-1}j_{\mathbb{O}}\mathcal{D}^{-1}j_{\mathbb{O}}\mathcal{D}^{-1}j_{\bar{\mathbb{S}}}\mathcal{D}^{-1}j_{\mathbb{S}}) \\ &\quad+\frac{N^2-1}{2}2\mathrm{Tr}(\mathcal{D}^{-1}j_{\bar{\mathbb{S}}}\mathcal{D}^{-1}j_{\mathbb{O}}\mathcal{D}^{-1}j_{\mathbb{O}}\mathcal{D}^{-1}j_{\mathbb{S}})+\cdots. \end{aligned} \tag{103}$$

Hence,

$$\begin{aligned}\langle \mathbb{O}_{s_1}(x_1)\mathbb{O}_{s_2}(x_2)\mathbb{S}_{s_3}(x_3)\bar{\mathbb{S}}_{s_4}(x_4)\rangle &= \sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\frac{\delta}{\delta j_{\mathbb{O}_{s_1k_1}}(x_1)}\frac{\delta}{\delta j_{\mathbb{O}_{s_2k_2}}(x_2)}\frac{\delta}{\delta j_{\mathbb{S}_{s_3k_3}}(x_3)}\frac{\delta}{\delta j_{\bar{\mathbb{S}}_{s_4k_4}}(x_4)}\Gamma_{\mathrm{conf}}[j_{\mathbb{O}},j_{\bar{\mathbb{O}}},j_{\mathbb{S}},j_{\bar{\mathbb{S}}}] \\ &= \frac{N^2-1}{2}2\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\frac{\delta}{\delta j_{\mathbb{O}_{s_1k_1}}(x_1)}\frac{\delta}{\delta j_{\mathbb{O}_{s_2k_2}}(x_2)}\frac{\delta}{\delta j_{\mathbb{S}_{s_3k_3}}(x_3)}\frac{\delta}{\delta j_{\bar{\mathbb{S}}_{s_4k_4}}(x_4)} \\ &\quad\int d^4y_1d^4y_2d^4y_3d^4y_4\sum_{k_1's_1'}\sum_{k_2's_2'}\sum_{k_3's_3'}\sum_{k_4's_4'} \\ &\quad\Big[\mathcal{D}^{-1}_{s_1'k_1',s_2'k_2'}(y_1-y_2)j_{\mathbb{O}_{s_2'k_2'}}(y_2)\mathcal{D}^{-1}_{s_2'k_2',s_3'k_3'}(y_2-y_3)j_{\bar{\mathbb{S}}_{s_3'k_3'}}(y_3) \\ &\quad\times\mathcal{D}^{-1}_{s_3'k_3',s_4'k_4'}(y_3-y_4)j_{\mathbb{O}_{s_4'k_4'}}(y_4)\mathcal{D}^{-1}_{s_4'k_4',s_1'k_1'}(y_4-y_1)j_{\mathbb{S}_{s_1'k_1'}}(y_1) \\ &\quad+\mathcal{D}^{-1}_{s_1'k_1',s_2'k_2'}(y_1-y_2)j_{\mathbb{O}_{s_2'k_2'}}(y_2)\mathcal{D}^{-1}_{s_2'k_2',s_3'k_3'}(y_2-y_3)j_{\mathbb{O}_{s_3'k_3'}}(y_3) \\ &\quad\times\mathcal{D}^{-1}_{s_3'k_3',s_4'k_4'}(y_3-y_4)j_{\bar{\mathbb{S}}_{s_4'k_4'}}(y_4)\mathcal{D}^{-1}_{s_4'k_4',s_1'k_1'}(y_4-y_1)j_{\mathbb{S}_{s_1'k_1'}}(y_1) \\ &\quad+\mathcal{D}^{-1}_{s_1'k_1',s_2'k_2'}(y_1-y_2)j_{\mathbb{O}_{s_2'k_2'}}(y_2)\mathcal{D}^{-1}_{s_2'k_2',s_3'k_3'}(y_2-y_3)j_{\mathbb{O}_{s_3'k_3'}}(y_3) \\ &\quad\times\mathcal{D}^{-1}_{s_3'k_3',s_4'k_4'}(y_3-y_4)j_{\mathbb{S}_{s_4'k_4'}}(y_4)\mathcal{D}^{-1}_{s_4'k_4',s_1'k_1'}(y_4-y_1)j_{\bar{\mathbb{S}}_{s_1'k_1'}}(y_1)\Big]. \end{aligned} \tag{104}$$

Performing the functional derivatives and employing the symmetry properties of the Gegenbauer polynomials (Appendix B), we get

$$\begin{aligned}\langle \mathbb{O}_{s_1}(x_1)\mathbb{O}_{s_2}(x_2)\mathbb{S}_{s_3}(x_3)\bar{\mathbb{S}}_{s_4}(x_4)\rangle &= \sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\frac{\delta}{\delta j_{\mathbb{O}_{s_1k_1}}(x_1)}\frac{\delta}{\delta j_{\mathbb{O}_{s_2k_2}}(x_2)}\frac{\delta}{\delta j_{\mathbb{S}_{s_3k_3}}(x_3)}\frac{\delta}{\delta j_{\bar{\mathbb{S}}_{s_4k_4}}(x_4)}\Gamma_{\mathrm{conf}}[j_{\mathbb{O}},j_{\bar{\mathbb{O}}},j_{\mathbb{S}},j_{\bar{\mathbb{S}}}] \\ &= (N^2-1)\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\sum_{\sigma\in P_2}\Big[\mathcal{D}^{-1}_{s_3k_3,s_{\sigma(1)}k_{\sigma(1)}}(x_3-x_{\sigma(1)})\mathcal{D}^{-1}_{s_{\sigma(1)}k_{\sigma(1)},s_4k_4}(x_{\sigma(1)}-x_4) \\ &\quad\times\mathcal{D}^{-1}_{s_4k_4,s_{\sigma(2)}k_{\sigma(2)}}(x_4-x_{\sigma(2)})\mathcal{D}^{-1}_{s_{\sigma(2)}k_{\sigma(2)},s_3k_3}(x_{\sigma(2)}-x_3)+2\mathcal{D}^{-1}_{s_3k_3,s_{\sigma(1)}k_{\sigma(1)}}(x_3-x_{\sigma(1)}) \\ &\quad\times\mathcal{D}^{-1}_{s_{\sigma(1)}k_{\sigma(1)},s_{\sigma(2)}k_{\sigma(2)}}(x_{\sigma(1)}-x_{\sigma(2)})\mathcal{D}^{-1}_{s_{\sigma(2)}k_{\sigma(2)},s_4k_4}(x_{\sigma(2)}-x_4)\mathcal{D}^{-1}_{s_4k_4,s_3k_3}(x_4-x_3)\Big]. \end{aligned} \tag{105}$$

Explicitly,

$$
\begin{aligned}
&2^4\frac{\Gamma(5)\Gamma(s_1+1)}{\Gamma(3)\Gamma(s_1+3)}\frac{\Gamma(5)\Gamma(s_2+1)}{\Gamma(3)\Gamma(s_2+3)}\frac{\Gamma(5)\Gamma(s_3+1)}{\Gamma(3)\Gamma(s_3+3)}\frac{\Gamma(5)\Gamma(s_4+1)}{\Gamma(3)\Gamma(s_4+3)}\langle\mathbb{O}_{s_1}(x_1)\mathbb{O}_{s_2}(x_2)\mathbb{S}_{s_3}(x_3)\bar{\mathbb{S}}_{s_4}(x_4)\rangle\\
&=(N^2-1)\frac{1}{(4\pi^2)^4}2^{\sum_{l=1}^4 s_l}i^{\sum_{l=1}^4 s_l}\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}\binom{s_4}{k_4}\binom{s_4}{k_4+2}\\
&\quad\times\sum_{\sigma\in P_2}\Bigg[(s_3-k_3+k_{\sigma(1)})!(s_{\sigma(1)}-k_{\sigma(1)}+k_4)!(s_4-k_4+k_{\sigma(2)})!(s_{\sigma(2)}-k_{\sigma(2)}+k_3)!\frac{(x_3-x_{\sigma(1)})_+^{s_3-k_3+k_{\sigma(1)}}}{(|x_3-x_{\sigma(1)}|^2)^{s_3-k_3+k_{\sigma(1)}+1}}\\
&\quad\times\frac{(x_{\sigma(1)}-x_4)_+^{s_{\sigma(1)}-k_{\sigma(1)}+k_4}}{(|x_{\sigma(1)}-x_4|^2)^{s_{\sigma(1)}-k_{\sigma(1)}+k_4+1}}\frac{(x_4-x_{\sigma(2)})_+^{s_4-k_4+k_{\sigma(2)}}}{(|x_4-x_{\sigma(2)}|^2)^{s_4-k_4+k_{\sigma(2)}+1}}\frac{(x_{\sigma(2)}-x_3)_+^{s_{\sigma(2)}-k_{\sigma(2)}+k_3}}{(|x_4-x_1|^2)^{s_4-k_4+k_1+1}}\\
&\quad+2(s_3-k_3+k_{\sigma(1)})!(s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)})!(s_{\sigma(2)}-k_{\sigma(2)}+k_4)!(s_4-k_4+k_3)!\\
&\quad\times\frac{(x_3-x_{\sigma(1)})_+^{s_3-k_3+k_{\sigma(1)}}}{(|x_3-x_{\sigma(1)}|^2)^{s_3-k_3+k_{\sigma(1)}+1}}\frac{(x_{\sigma(1)}-x_{\sigma(2)})_+^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}}}{(|x_{\sigma(1)}-x_{\sigma(2)}|^2)^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}+1}}\frac{(x_{\sigma(2)}-x_4)_+^{s_{\sigma(2)}-k_{\sigma(2)}+k_4}}{(|x_{\sigma(2)}-x_4|^2)^{s_{\sigma(2)}-k_{\sigma(2)}+k_4+1}}\frac{(x_4-x_3)_+^{s_4-k_4+k_3}}{(|x_4-x_3|^2)^{s_4-k_4+k_3+1}}\Bigg]
\end{aligned}
\tag{106}
$$

that in the momentum representation reads

$$
\begin{aligned}
&2^4\frac{\Gamma(5)\Gamma(s_1+1)}{\Gamma(3)\Gamma(s_1+3)}\frac{\Gamma(5)\Gamma(s_2+1)}{\Gamma(3)\Gamma(s_2+3)}\frac{\Gamma(5)\Gamma(s_3+1)}{\Gamma(3)\Gamma(s_3+3)}\frac{\Gamma(5)\Gamma(s_4+1)}{\Gamma(3)\Gamma(s_4+3)}\langle\mathbb{O}_{s_1}(p_1)\mathbb{O}_{s_2}(p_2)\mathbb{S}_{s_3}(p_3)\bar{\mathbb{S}}_{s_4}(p_4)\rangle\\
&=(N^2-1)(2\pi)^4i^4\delta^{(4)}(p_1+p_2+p_3+p_4)\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}\\
&\quad\times\binom{s_4}{k_4}\binom{s_4}{k_4+2}\sum_{\sigma\in P_2}\int\frac{d^4q}{(2\pi)^4}\Bigg[\frac{(p_3+q)_+^{s_3-k_3+k_{\sigma(1)}}}{|p_3+q|^2+i\epsilon}\frac{(p_3+p_{\sigma(1)}+q)_+^{s_{\sigma(1)}-k_{\sigma(1)}+k_4}}{|p_3+p_{\sigma(1)}+q|^2+i\epsilon}\frac{(p_3+p_{\sigma(1)}+p_4+q)_+^{s_{\sigma(4)}-k_{\sigma(4)}+k_{\sigma(2)}}}{|p_3+p_{\sigma(1)}+p_4+q|^2}\\
&\quad\times\frac{(q)_+^{s_{\sigma(2)}-k_{\sigma(2)}+k_3}}{|q|^2+i\epsilon}+2\frac{(p_3+q)_+^{s_3-k_3+k_{\sigma(1)}}}{|p_3+q|^2+i\epsilon}\frac{(p_3+p_{\sigma(1)}+q)_+^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}}}{|p_3+p_{\sigma(1)}+q|^2+i\epsilon}\frac{(p_3+p_{\sigma(1)}+p_{\sigma(2)}+q)_+^{s_{\sigma(2)}-k_{\sigma(2)}+k_4}}{|p_3+p_{\sigma(1)}+p_{\sigma(2)}+q|^2+i\epsilon}\frac{(q)_+^{s_4-k_4+k_3}}{|q|^2+i\epsilon}\Bigg].
\end{aligned}
\tag{107}
$$

Similarly, we obtain

$$
\begin{aligned}
\langle\tilde{\mathbb{O}}_{s_1}(x_1)\tilde{\mathbb{O}}_{s_2}(x_2)\mathbb{S}_{s_3}(x_3)\bar{\mathbb{S}}_{s_4}(x_4)\rangle&=\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\frac{\delta}{\delta j_{\tilde{\mathbb{O}}_{s_1k_1}}(x_1)}\frac{\delta}{\delta j_{\tilde{\mathbb{O}}_{s_2k_2}}(x_2)}\frac{\delta}{\delta j_{\mathbb{S}_{s_3k_3}}(x_3)}\frac{\delta}{\delta j_{\bar{\mathbb{S}}_{s_4k_4}}(x_4)}\Gamma_{\mathrm{conf}}[j_{\mathbb{O}},j_{\tilde{\mathbb{O}}},j_{\mathbb{S}},j_{\bar{\mathbb{S}}}]\\
&=(N^2-1)\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\sum_{\sigma\in P_2}\Bigg[-\mathcal{D}^{-1}_{s_3k_3,s_{\sigma(1)}k_{\sigma(1)}}(x_3-x_{\sigma(1)})\mathcal{D}^{-1}_{s_{\sigma(1)}k_{\sigma(1)},s_4k_4}(x_{\sigma(1)}-x_4)\\
&\quad\times\mathcal{D}^{-1}_{s_4k_4,s_{\sigma(2)}k_{\sigma(2)}}(x_4-x_{\sigma(2)})\mathcal{D}^{-1}_{s_{\sigma(2)}k_{\sigma(2)},s_3k_3}(x_{\sigma(2)}-x_3)+2\mathcal{D}^{-1}_{s_3k_3,s_{\sigma(1)}k_{\sigma(1)}}(x_3-x_{\sigma(1)})\\
&\quad\times\mathcal{D}^{-1}_{s_{\sigma(1)}k_{\sigma(1)},s_{\sigma(2)}k_{\sigma(2)}}(x_{\sigma(1)}-x_{\sigma(2)})\mathcal{D}^{-1}_{s_{\sigma(2)}k_{\sigma(2)},s_4k_4}(x_{\sigma(2)}-x_4)\mathcal{D}^{-1}_{s_4k_4,s_3k_3}(x_4-x_3)\Bigg].
\end{aligned}
\tag{108}
$$

Explicitly, in the coordinate and momentum representation, respectively,

$$
\begin{aligned}
&2^4\frac{\Gamma(5)\Gamma(s_1+1)}{\Gamma(3)\Gamma(s_1+3)}\frac{\Gamma(5)\Gamma(s_2+1)}{\Gamma(3)\Gamma(s_2+3)}\frac{\Gamma(5)\Gamma(s_3+1)}{\Gamma(3)\Gamma(s_3+3)}\frac{\Gamma(5)\Gamma(s_4+1)}{\Gamma(3)\Gamma(s_4+3)}\langle\tilde{\mathbb{O}}_{s_1}(x_1)\tilde{\mathbb{O}}_{s_2}(x_2)\mathbb{S}_{s_3}(x_3)\bar{\mathbb{S}}_{s_4}(x_4)\rangle\\
&=(N^2-1)\frac{1}{(4\pi^2)^4}2^{\sum_{l=1}^4 s_l}i^{\sum_{l=1}^4 s_l}\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}\\
&\quad\times\binom{s_4}{k_4}\binom{s_4}{k_4+2}\sum_{\sigma\in P_2}\Bigg[-(s_3-k_3+k_{\sigma(1)})!(s_{\sigma(1)}-k_{\sigma(1)}+k_4)!(s_4-k_4+k_{\sigma(2)})!(s_{\sigma(2)}-k_{\sigma(2)}+k_3)!\\
&\quad\times\frac{(x_3-x_{\sigma(1)})_+^{s_3-k_3+k_{\sigma(1)}}}{(|x_3-x_{\sigma(1)}|^2)^{s_3-k_3+k_{\sigma(1)}+1}}\frac{(x_{\sigma(1)}-x_4)_+^{s_{\sigma(1)}-k_{\sigma(1)}+k_4}}{(|x_{\sigma(1)}-x_4|^2)^{s_{\sigma(1)}-k_{\sigma(1)}+k_4+1}}\frac{(x_4-x_{\sigma(2)})_+^{s_4-k_4+k_{\sigma(2)}}}{(|x_4-x_{\sigma(2)}|^2)^{s_4-k_4+k_{\sigma(2)}+1}}\frac{(x_{\sigma(2)}-x_3)_+^{s_{\sigma(2)}-k_{\sigma(2)}+k_3}}{(|x_4-x_1|^2)^{s_4-k_4+k_1+1}}\\
&\quad+2(s_3-k_3+k_{\sigma(1)})!(s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)})!(s_{\sigma(2)}-k_{\sigma(2)}+k_4)!(s_4-k_4+k_3)!\frac{(x_3-x_{\sigma(1)})_+^{s_3-k_3+k_{\sigma(1)}}}{(|x_3-x_{\sigma(1)}|^2)^{s_3-k_3+k_{\sigma(1)}+1}}\\
&\quad\times\frac{(x_{\sigma(1)}-x_{\sigma(2)})_+^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}}}{(|x_{\sigma(1)}-x_{\sigma(2)}|^2)^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}+1}}\frac{(x_{\sigma(2)}-x_4)_+^{s_{\sigma(2)}-k_{\sigma(2)}+k_4}}{(|x_{\sigma(2)}-x_4|^2)^{s_{\sigma(2)}-k_{\sigma(2)}+k_4+1}}\frac{(x_4-x_3)_+^{s_4-k_4+k_3}}{(|x_4-x_3|^2)^{s_4-k_4+k_3+1}}\Bigg],
\end{aligned}
\tag{109}
$$

$$
\begin{aligned}
&2^4\frac{\Gamma(5)\Gamma(s_1+1)}{\Gamma(3)\Gamma(s_1+3)}\frac{\Gamma(5)\Gamma(s_2+1)}{\Gamma(3)\Gamma(s_2+3)}\frac{\Gamma(5)\Gamma(s_3+1)}{\Gamma(3)\Gamma(s_3+3)}\frac{\Gamma(5)\Gamma(s_4+1)}{\Gamma(3)\Gamma(s_4+3)}\langle\!\langle\tilde{\mathbb{O}}_{s_1}(p_1)\tilde{\mathbb{O}}_{s_2}(p_2)\mathbb{S}_{s_3}(p_3)\bar{\mathbb{S}}_{s_4}(p_4)\rangle\!\rangle\\
&=(N^2-1)(2\pi)^4i^4\delta^{(4)}(p_1+p_2+p_3+p_4)\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}\\
&\quad\times\binom{s_4}{k_4}\binom{s_4}{k_4+2}\sum_{\sigma\in P_2}\int\frac{d^4q}{(2\pi)^4}\Bigg[-\frac{(p_3+q)_+^{s_3-k_3+k_{\sigma(1)}}}{|p_3+q|^2+i\epsilon}\frac{(p_3+p_{\sigma(1)}+q)_+^{s_{\sigma(1)}-k_{\sigma(1)}+k_4}}{|p_3+p_{\sigma(1)}+q|^2+i\epsilon}\frac{(p_3+p_{\sigma(1)}+p_4+q)_+^{s_{\sigma(4)}-k_{\sigma(4)}+k_{\sigma(2)}}}{|p_3+p_{\sigma(1)}+p_4+q|^2}\\
&\quad\times\frac{(q)_+^{s_{\sigma(2)}-k_{\sigma(2)}+k_3}}{|q|^2+i\epsilon}+2\frac{(p_3+q)_+^{s_3-k_3+k_{\sigma(1)}}}{|p_3+q|^2+i\epsilon}\frac{(p_3+p_{\sigma(1)}+q)_+^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}}}{|p_3+p_{\sigma(1)}+q|^2+i\epsilon}\frac{(p_3+p_{\sigma(1)}+p_{\sigma(2)}+q)_+^{s_{\sigma(2)}-k_{\sigma(2)}+k_4}}{|p_3+p_{\sigma(1)}+p_{\sigma(2)}+q|^2+i\epsilon}\frac{(q)_+^{s_4-k_4+k_3}}{|q|^2+i\epsilon}\Bigg].
\end{aligned}
\tag{110}
$$

C. Four-point correlators in Euclidean spacetime

In Euclidean spacetime we obtain as well

$$
\begin{aligned}
&2^4\frac{\Gamma(5)\Gamma(s_1+1)}{\Gamma(3)\Gamma(s_1+3)}\frac{\Gamma(5)\Gamma(s_2+1)}{\Gamma(3)\Gamma(s_2+3)}\frac{\Gamma(5)\Gamma(s_3+1)}{\Gamma(3)\Gamma(s_3+3)}\frac{\Gamma(5)\Gamma(s_4+1)}{\Gamma(3)\Gamma(s_4+3)}\langle\mathbb{O}^E_{s_1}(x_1)\mathbb{O}^E_{s_2}(x_2)\mathbb{S}^E_{s_3}(x_3)\bar{\mathbb{S}}^E_{s_4}(x_4)\rangle\\
&=(N^2-1)\frac{1}{(4\pi^2)^4}2^{\sum_{l=1}^4 s_l}(-1)^{\sum_{l=1}^4 s_l}\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}\\
&\quad\times\binom{s_4}{k_4}\binom{s_4}{k_4+2}\sum_{\sigma\in P_2}\Bigg[(s_3-k_3+k_{\sigma(1)})!(s_{\sigma(1)}-k_{\sigma(1)}+k_4)!(s_4-k_4+k_{\sigma(2)})!(s_{\sigma(2)}-k_{\sigma(2)}+k_3)!\\
&\quad\times\frac{(x_3-x_{\sigma(1)})_z^{s_3-k_3+k_{\sigma(1)}}}{((x_3-x_{\sigma(1)})^2)^{s_3-k_3+k_{\sigma(1)}+1}}\frac{(x_{\sigma(1)}-x_4)_z^{s_{\sigma(1)}-k_{\sigma(1)}+k_4}}{((x_{\sigma(1)}-x_4)^2)^{s_{\sigma(1)}-k_{\sigma(1)}+k_4+1}}\frac{(x_4-x_{\sigma(2)})_z^{s_4-k_4+k_{\sigma(2)}}}{((x_4-x_{\sigma(2)})^2)^{s_4-k_4+k_{\sigma(2)}+1}}\frac{(x_{\sigma(2)}-x_3)_z^{s_{\sigma(2)}-k_{\sigma(2)}+k_3}}{((x_4-x_1)^2)^{s_4-k_4+k_1+1}}\\
&\quad+2(s_3-k_3+k_{\sigma(1)})!(s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)})!(s_{\sigma(2)}-k_{\sigma(2)}+k_4)!(s_4-k_4+k_3)!\frac{(x_3-x_{\sigma(1)})_z^{s_3-k_3+k_{\sigma(1)}}}{((x_3-x_{\sigma(1)})^2)^{s_3-k_3+k_{\sigma(1)}+1}}\\
&\quad\times\frac{(x_{\sigma(1)}-x_{\sigma(2)})_z^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}}}{((x_{\sigma(1)}-x_{\sigma(2)})^2)^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}+1}}\frac{(x_{\sigma(2)}-x_4)_z^{s_{\sigma(2)}-k_{\sigma(2)}+k_4}}{((x_{\sigma(2)}-x_4)^2)^{s_{\sigma(2)}-k_{\sigma(2)}+k_4+1}}\frac{(x_4-x_3)_z^{s_4-k_4+k_3}}{((x_4-x_3)^2)^{s_4-k_4+k_3+1}}\Bigg]
\end{aligned}
\tag{111}
$$

and

$$
\begin{aligned}
&2^4\frac{\Gamma(5)\Gamma(s_1+1)}{\Gamma(3)\Gamma(s_1+3)}\frac{\Gamma(5)\Gamma(s_2+1)}{\Gamma(3)\Gamma(s_2+3)}\frac{\Gamma(5)\Gamma(s_3+1)}{\Gamma(3)\Gamma(s_3+3)}\frac{\Gamma(5)\Gamma(s_4+1)}{\Gamma(3)\Gamma(s_4+3)}\langle\tilde{\mathbb{O}}^E_{s_1}(x_1)\tilde{\mathbb{O}}^E_{s_2}(x_2)\mathbb{S}^E_{s_3}(x_3)\bar{\mathbb{S}}^E_{s_4}(x_4)\rangle\\
&\quad=(N^2-1)\frac{1}{(4\pi^2)^4}2^{\sum_{l=1}^4 s_l}(-1)^{\sum_{l=1}^4 s_l}\sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2}\sum_{k_4=0}^{s_4-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}\\
&\quad\times\binom{s_4}{k_4}\binom{s_4}{k_4+2}\sum_{\sigma\in P_2}\Bigg[-(s_3-k_3+k_{\sigma(1)})!(s_{\sigma(1)}-k_{\sigma(1)}+k_4)!(s_4-k_4+k_{\sigma(2)})!(s_{\sigma(2)}-k_{\sigma(2)}+k_3)!\\
&\quad\times\frac{(x_3-x_{\sigma(1)})_z^{s_3-k_3+k_{\sigma(1)}}}{((x_3-x_{\sigma(1)})^2)^{s_3-k_3+k_{\sigma(1)}+1}}\frac{(x_{\sigma(1)}-x_4)_z^{s_{\sigma(1)}-k_{\sigma(1)}+k_4}}{((x_{\sigma(1)}-x_4)^2)^{s_{\sigma(1)}-k_{\sigma(1)}+k_4+1}}\frac{(x_4-x_{\sigma(2)})_z^{s_4-k_4+k_{\sigma(2)}}}{((x_4-x_{\sigma(2)})^2)^{s_4-k_4+k_{\sigma(2)}+1}}\frac{(x_{\sigma(2)}-x_3)_z^{s_{\sigma(2)}-k_{\sigma(2)}+k_3}}{((x_4-x_1)^2)^{s_4-k_4+k_1+1}}\\
&\quad+2(s_3-k_3+k_{\sigma(1)})!(s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)})!(s_{\sigma(2)}-k_{\sigma(2)}+k_4)!(s_4-k_4+k_3)!\frac{(x_3-x_{\sigma(1)})_z^{s_3-k_3+k_{\sigma(1)}}}{((x_3-x_{\sigma(1)})^2)^{s_3-k_3+k_{\sigma(1)}+1}}\\
&\quad\times\frac{(x_{\sigma(1)}-x_{\sigma(2)})_z^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}}}{((x_{\sigma(1)}-x_{\sigma(2)})^2)^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}+1}}\frac{(x_{\sigma(2)}-x_4)_z^{s_{\sigma(2)}-k_{\sigma(2)}+k_4}}{((x_{\sigma(2)}-x_4)^2)^{s_{\sigma(2)}-k_{\sigma(2)}+k_4+1}}\frac{(x_4-x_3)_z^{s_4-k_4+k_3}}{((x_4-x_3)^2)^{s_4-k_4+k_3+1}}\Bigg].
\end{aligned}
\tag{112}
$$

VII. UV ASYMPTOTICS OF EUCLIDEAN n-POINT CORRELATORS

A. Operator mixing

The Euclidean n-point correlator at distinct points of local renormalized operators $\mathcal{O}_i(x)$ that mix under renormalization

$$
\langle\mathcal{O}_{k_1}(x_1)\cdots\mathcal{O}_{k_n}(x_n)\rangle=G^{(n)}_{k_1\ldots k_n}(x_1,\ldots,x_n;\mu,g(\mu)) \tag{113}
$$

satisfies the Callan-Symanzik equation

$$
\begin{aligned}
&\left(\sum_{\alpha=1}^n x_\alpha\cdot\frac{\partial}{\partial x_\alpha}+\beta(g)\frac{\partial}{\partial g}+\sum_{\alpha=1}^n D_{\mathcal{O}_\alpha}\right)G^{(n)}_{k_1\cdots k_n}\\
&\quad+\sum_a\Big(\gamma_{k_1a}(g)G^{(n)}_{ak_2\cdots k_n}+\gamma_{k_2a}(g)G^{(n)}_{k_1ak_3\cdots k_n}\\
&\quad\cdots+\gamma_{k_na}(g)G^{(n)}_{k_1\cdots a}\Big)=0,
\end{aligned}
\tag{114}
$$

where $D_{\mathcal{O}_i}$ is the canonical dimension of $\mathcal{O}_i(x)$ and $\gamma_{ij}(g)$ the matrix of the anomalous dimensions. Its solution reads

$$
\begin{aligned}
&G^{(n)}_{k_1\cdots k_n}(\lambda x_1,\ldots,\lambda x_n;\mu,g(\mu))\\
&\quad=\sum_{j_1\cdots j_n}Z_{k_1j_1}(\lambda)\cdots Z_{k_nj_n}(\lambda)\lambda^{-\sum_{i=1}^n D_{\mathcal{O}_i}}\\
&\qquad\times G^{(n)}_{j_1\ldots j_n}(x_1,\ldots,x_n;\mu,g(\tfrac{\mu}{\lambda})),
\end{aligned}
\tag{115}
$$

where in matrix notation

$$
\left(\frac{\partial}{\partial g}+\frac{\gamma(g)}{\beta(g)}\right)Z(\lambda)=0 \tag{116}
$$

with $g\equiv g(\mu)$ and

$$
Z(\lambda)=P\exp\left(\int_{g(\mu)}^{g(\frac{\mu}{\lambda})}\frac{\gamma(g')}{\beta(g')}dg'\right). \tag{117}
$$

Hence, the corresponding UV asymptotics as $\lambda\to 0$ reads

$$
\begin{aligned}
&G^{(n)}_{k_1\cdots k_n}(\lambda x_1,\ldots,\lambda x_n;\mu,g(\mu))\\
&\sim\sum_{j_1\cdots j_n}Z_{k_1j_1}(\lambda)\cdots Z_{k_nj_n}(\lambda)\lambda^{-\sum_{i=1}^n D_{\mathcal{O}_i}}G^{(n)}_{\mathrm{conf}\,j_1\ldots j_n}(x_1,\ldots,x_n)
\end{aligned}
\tag{118}
$$

provided that $G^{(n)}_{\mathrm{conf}\,j_1\cdots j_n}(x_1,\ldots,x_n)$—the conformal correlators to the lowest order of perturbation theory—do not vanish.

The evaluation of the asymptotics above involves the estimate of each term in Eq. (118) that in turn involves the computation of the path-ordered exponential defining $Z_{ij}(\lambda)$ in Eq. (117). Both the computations are technically challenging, even in the special case where $\gamma(g)$ is triangular so that the expansion of the path-ordered exponential in Eq. (117) terminates to a finite order [14].

Therefore, it is of the utmost importance to establish whether a renormalization scheme exists where $Z(\lambda)$ is diagonalizable to all perturbative orders. Indeed, in such a scheme the sum in Eq. (118) would reduce to just one term and the path-ordered exponential to just the ordinary exponential, as in the multiplicatively renormalizable case.

B. Nonresonant diagonal renormalization scheme

The question above has been addressed on the basis of the differential-geometric interpretation [5] of the operator mixing in massless QCD-like theories to all orders of perturbation theory that we summarize in the following. We interpret a finite change of the basis of renormalized operators in matrix notation

$$\mathcal{O}'(x) = S(g)\mathcal{O}(x) \tag{119}$$

as a formal real-analytic invertible gauge transformation $S(g)$ that depends on g [5]. Under the action of the aforementioned gauge transformation, the matrix

$$A(g) = -\frac{\gamma(g)}{\beta(g)} = \frac{1}{g}\left(\frac{\gamma_0}{\beta_0} + \cdots\right), \tag{120}$$

associated with the differential equation for $Z(\lambda)$

$$\left(\frac{\partial}{\partial g} - A(g)\right) Z(\lambda) = 0, \tag{121}$$

defines a connection $A(g)$

$$A(g) = \frac{1}{g}\left(A_0 + \sum_{n=1}^{\infty} A_{2n} g^{2n}\right) \tag{122}$$

with a regular singularity at $g = 0$ that transforms as

$$A'(g) = S(g)A(g)S^{-1}(g) + \frac{\partial S(g)}{\partial g} S^{-1}(g) \tag{123}$$

with

$$\mathcal{D} = \frac{\partial}{\partial g} - A(g) \tag{124}$$

the corresponding covariant derivative. Consequently, $Z(\lambda)$ is interpreted as a Wilson line that transforms as

$$Z'(\lambda) = S(g(\mu))Z(\lambda)S^{-1}(g(\frac{\mu}{\lambda})) \tag{125}$$

for the gauge transformation $S(g)$. Moreover, everything that we have mentioned also applies by allowing the coupling g to be complex valued. It follows from the Poincaré-Dulac theorem that [5], if any two eigenvalues $\lambda_1, \lambda_2, \ldots$ of the matrix $\frac{\gamma_0}{\beta_0}$, in nonincreasing order $\lambda_1 \geq \lambda_2 \geq \cdots$, do not differ by a positive even integer

$$\lambda_i - \lambda_j - 2k \neq 0 \tag{126}$$

for $i \leq j$ and k a positive integer, then a formal holomorphic gauge transformation exists that sets $A(g)$ in the canonical nonresonant form [5]

$$A'(g) = \frac{\gamma_0}{\beta_0}\frac{1}{g} \tag{127}$$

that is one-loop exact to all orders of perturbation theory. As a consequence, if in addition $\frac{\gamma_0}{\beta_0}$ is diagonalizable, $Z(\lambda)$ is diagonalizable as well and its eigenvalues $Z_{\mathcal{O}_i}(\lambda)$ are computed by [5]

$$Z_{\mathcal{O}_i}(\lambda) = \left(\frac{g(\mu)}{g(\frac{\mu}{\lambda})}\right)^{\frac{\gamma_{0\mathcal{O}_i}}{\beta_0}} \tag{128}$$

with $\gamma_{0\mathcal{O}_i}$ the eigenvalues of γ_0. Correspondingly, in the nonresonant diagonal scheme

$$G^{(n)}_{k_1\cdots k_n}(\lambda x_1, \ldots, \lambda x_n; \mu, g(\mu)) \sim \frac{Z_{\mathcal{O}_{j_1}}(\lambda)\cdots Z_{\mathcal{O}_{j_n}}(\lambda)}{\lambda^{D_{\mathcal{O}_1}+\cdots+D_{\mathcal{O}_n}}} G^{(n)}_{\mathrm{conf}\, j_1\cdots j_n}(x_1, \ldots, x_n) \tag{129}$$

provided that $G^{(n)}_{\mathrm{conf}\, j_1\cdots j_n}(x_1, \ldots, x_n)$ does not vanish, with

$$g^2\left(\frac{\mu}{\lambda}\right) \sim \frac{1}{\beta_0 \log\left(\frac{\mu^2}{\lambda^2\Lambda^2_{\mathrm{YM}}}\right)}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log\left(\frac{\mu^2}{\lambda^2\Lambda^2_{\mathrm{YM}}}\right)}{\log\left(\frac{\mu^2}{\lambda^2\Lambda^2_{\mathrm{YM}}}\right)}\right) \sim \frac{1}{\beta_0 \log\left(\frac{1}{\lambda^2}\right)}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\log\log\left(\frac{1}{\lambda^2}\right)}{\log\left(\frac{1}{\lambda^2}\right)}\right), \tag{130}$$

where the second asymptotic equality holds within the universal—i.e., scheme independent—leading and next-to-leading logarithmic accuracy as $\lambda \to 0$.

To make the present paper self-contained we provide the construction [5] of the nonresonant diagonal scheme to all orders of perturbation theory under the above assumptions starting from a generic renormalization scheme—for example, the $\overline{\mathrm{MS}}$ scheme.

The construction proceeds by induction on $k = 1, 2, \ldots$ by demonstrating that, once A_0 and the first $k-1$ matrix coefficients $A_2, \ldots, A_{2(k-1)}$ in Eq. (122) have been set in the canonical nonresonant form in Eq. (127)—i.e., A_0 diagonal and $A_2, \ldots, A_{2(k-1)} = 0$—a formal holomorphic gauge transformation exists that leaves them invariant and also sets the kth coefficient A_{2k} to 0.

The 0 step of the induction consists just in setting A_0 in diagonal form—with the eigenvalues in nonincreasing order—by a global (i.e., constant) gauge transformation.

At the kth step we choose the holomorphic gauge transformation in the form

$$S_k(g) = 1 + g^{2k}H_{2k} \tag{131}$$

with H_{2k} a matrix to be found momentarily. Its inverse is

$$S_k^{-1}(g) = (1 + g^{2k}H_{2k})^{-1} = 1 - g^{2k}H_{2k} + \cdots, \tag{132}$$

where the dots represent terms of order higher than g^{2k}. The gauge action of $S_k(g)$ on the connection $A(g)$ furnishes

$$
\begin{aligned}
A'(g) &= 2kg^{2k-1}H_{2k}(1-g^{2k}H_{2k})^{-1} + (1+g^{2k}H_{2k})A(g)(1-g^{2k}H_{2k})^{-1} \\
&= 2kg^{2k-1}H_{2k}(1-g^{2k}H_{2k})^{-1} + (1+g^{2k}H_{2k})\frac{1}{g}\left(A_0 + \sum_{n=1}^{\infty} A_{2n}g^{2n}\right)(1-g^{2k}H_{2k})^{-1} \\
&= 2kg^{2k-1}H_{2k}(1-\cdots) + (1+g^{2k}H_{2k})\frac{1}{g}\left(A_0 + \sum_{n=1}^{\infty} A_{2n}g^{2n}\right)(1-g^{2k}H_{2k}+\cdots) \\
&= 2kg^{2k-1}H_{2k} + \frac{1}{g}\left(A_0 + \sum_{n=1}^{k} A_{2n}g^{2n}\right) + g^{2k-1}(H_{2k}A_0 - A_0H_{2k}) + \cdots \\
&= g^{2k-1}(2kH_{2k} + H_{2k}A_0 - A_0H_{2k}) + A_{2(k-1)}(g) + g^{2k-1}A_{2k} + \cdots,
\end{aligned} \tag{133}
$$

where we have skipped in the dots all the terms that contribute to an order higher than g^{2k-1}, and we have set

$$
A_{2(k-1)}(g) = \frac{1}{g}\left(A_0 + \sum_{n=1}^{k-1} A_{2n}g^{2n}\right) \tag{134}
$$

that is the part of $A(g)$ that is not affected by the gauge transformation $S_k(g)$, and thus verifies the hypotheses of the induction—i.e., that $A_2, \ldots, A_{2(k-1)}$ vanish.

Therefore, by Eq. (133) the kth matrix coefficient A_{2k} may be eliminated from the expansion of $A'(g)$ to the order g^{2k-1} provided that an H_{2k} exists such that

$$
A_{2k} + (2kH_{2k} + H_{2k}A_0 - A_0H_{2k}) = A_{2k} + (2k - ad_{A_0})H_{2k} = 0 \tag{135}
$$

with $ad_{A_0}Y = [A_0, Y]$.

If the inverse of $ad_{A_0} - 2k$ exists, the unique solution for H_{2k} is

$$
H_{2k} = (ad_{A_0} - 2k)^{-1}A_{2k}. \tag{136}
$$

Hence, to complete the induction, we should demonstrate that, under the above assumptions, $ad_{A_0} - 2k$ is invertible, i.e., its kernel is trivial.

Now $ad_\Lambda - 2k$, as a linear operator that acts on matrices, is diagonal, with eigenvalues $\lambda_i - \lambda_j - 2k$ and the matrices E_{ij}, whose only nonvanishing entries are $(E_{ij})_{ij}$, as eigenvectors.

The eigenvectors E_{ij}, normalized in such a way that $(E_{ij})_{ij} = 1$, form an orthonormal basis for the matrices. Thus, E_{ij} belongs to the kernel of $ad_\Lambda - 2k$ if and only if $\lambda_i - \lambda_j - 2k = 0$. As a consequence, since $\lambda_i - \lambda_j - 2k \neq 0$ for every i, j by the above assumptions, the kernel of $ad_\Lambda - 2k$ only contains 0, and the construction is complete.

C. Anomalous dimensions of twist-2 operators

We define the bare operators with $s \geq 2$ and $k \geq 0$ [9]

$$
\mathcal{O}_{Bs}^{(k)} = (-i\partial_+)^k \mathcal{O}_{Bs} \tag{137}
$$

that, to the leading order of perturbation theory, for $k > 0$ are conformal descendants [15] of the corresponding primary conformal operator $\mathcal{O}_{Bs}^{(0)} = \mathcal{O}_{Bs}$ in the standard basis. As a consequence of the operator mixing we obtain [9,15] for the renormalized operators

$$
\mathcal{O}_s^{(k)} = \sum_i^s Z_{si}\mathcal{O}_{Bi}^{(k+s-i)} \tag{138}
$$

with the mixing matrix Z and the matrix of the anomalous dimensions

$$
\gamma(g) = -\frac{\partial Z}{\partial \log \mu}Z^{-1} = \sum_{j=0}^{\infty} \gamma_j g^{2j+2} \tag{139}
$$

lower triangular and γ_0 diagonal in the $\overline{\text{MS}}$ scheme [9,15]. In our notation the eigenvalues of γ_0 are given by [9,13]

$$
\begin{aligned}
\gamma_{0s} = \frac{2}{(4\pi)^2}\Bigg(&4\psi(s+1) - 4\psi(1) - \frac{11}{3} \\
&- 8\frac{s^2+s+1}{(s-1)s(s+1)(s+2)}\Bigg)
\end{aligned} \tag{140}
$$

for $\mathbb{O}_s^{(k)}$ with even $s \geq 2$, and

$$
\begin{aligned}
\gamma_{0s} = \frac{2}{(4\pi)^2}\Bigg(&4\psi(s+1) - 4\psi(1) - \frac{11}{3} \\
&- 8\frac{s^2+s-2}{(s-1)s(s+1)(s+2)}\Bigg)
\end{aligned} \tag{141}
$$

for $\tilde{\mathbb{O}}_s^{(k)}$ with odd $s \geq 3$. $\gamma_{02} = 0$ consistently with the conservation of the stress-energy tensor. The eigenvalues of γ_0 for $\mathbb{S}_s^{(k)}$ and $\bar{\mathbb{S}}_s^{(k)}$ are given by [13]

$$\gamma_{0s} = \frac{2}{(4\pi)^2}\left(4\psi(s+1) - 4\psi(1) - \frac{11}{3}\right) \tag{142}$$

with even $s \geq 2$.

Interestingly, from the above equations it follows that in SU(N) YM theory γ_0 is independent of N and thus has no nonplanar contribution [12].

Moreover, we have verified numerically that the nonresonant condition in Eq. (126) is satisfied for all the collinear twist-2 operators in the standard basis up to $s = 10^4$.

Hence, the nonresonant diagonal basis exists for the collinear twist-2 operators and restricts to the lowest order of perturbation theory to the standard basis because γ_0 is diagonal in this basis.

Therefore, the standard basis may be employed to compute the conformal correlators in Eq. (129).

VIII. GENERATING FUNCTIONAL OF EUCLIDEAN RG-IMPROVED CORRELATORS

From the vantage point of view of the 't Hooft large-N expansion it is convenient to employ the rescaled operators in Eq. (24) in Euclidean spacetime, so that the two-point correlators are of order 1 for large N. The corresponding asymptotic connected correlators acquire the simpler form in the balanced

$$\begin{aligned}
&\langle \mathbb{O}'^E_{s_1}(\lambda x_1)\cdots\mathbb{O}'^E_{s_n}(\lambda x_n)\tilde{\mathbb{O}}'^E_{s_{n+1}}(\lambda x_{n+1})\cdots\tilde{\mathbb{O}}'^E_{s_{n+2m}}(\lambda x_{n+2m})\rangle \\
&\sim \frac{Z_{\mathbb{O}_{s_1}}(\lambda)}{\lambda^{2+s_1}}\cdots\frac{Z_{\mathbb{O}_{s_n}}(\lambda)}{\lambda^{2+s_n}}\frac{Z_{\tilde{\mathbb{O}}_{s_{n+1}}}(\lambda)}{\lambda^{2+s_{n+1}}}\cdots\frac{Z_{\tilde{\mathbb{O}}_{s_{n+2m}}}(\lambda)}{\lambda^{2+s_{n+2m}}}(N^{2-n-2m}-N^{-n-2m})\frac{1}{(4\pi^2)^{n+2m}}2^{\sum_{l=1}^{n+2m}s_l}(-1)^{\sum_{l=1}^{n+2m}s_l} \\
&\times\sum_{k_1=0}^{s_1-2}\cdots\sum_{k_{n+2m}=0}^{s_{n+2m}-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}\cdots\binom{s_{n+2m}}{k_{n+2m}}\binom{s_{n+2m}}{k_{n+2m}+2} \\
&\times\frac{1}{n+2m}\sum_{\sigma\in P_{n+2m}}(s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)})!\cdots(s_{\sigma(n+2m)}-k_{\sigma(n+2m)}+k_{\sigma(1)})! \\
&\times\frac{(x_{\sigma(1)}-x_{\sigma(2)})_z^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}}}{((x_{\sigma(1)}-x_{\sigma(2)})^2)^{s_{\sigma(1)}-k_{\sigma(1)}+k_{\sigma(2)}+1}}\cdots\frac{(x_{\sigma(n+2m)}-x_{\sigma(1)})_z^{s_{\sigma(n+2m)}-k_{\sigma(n+2m)}+k_{\sigma(1)}}}{((x_{\sigma(n+2m)}-x_{\sigma(1)})^2)^{s_{\sigma(n+2m)}-k_{\sigma(n+2m)}+k_{\sigma(1)}+1}}
\end{aligned} \tag{143}$$

and unbalanced sectors

$$\begin{aligned}
&\langle \mathbb{S}'^E_{s_1}(\lambda x_1)\cdots\mathbb{S}'^E_{s_n}(\lambda x_n)\bar{\mathbb{S}}'^E_{s'_1}(\lambda y_1)\cdots\bar{\mathbb{S}}'^E_{s'_n}(\lambda y_n)\rangle \\
&\sim \frac{Z_{\mathbb{S}_{s_1}}(\lambda)}{\lambda^{2+s_1}}\cdots\frac{Z_{\mathbb{S}_{s_n}}(\lambda)}{\lambda^{2+s_n}}\frac{Z_{\bar{\mathbb{S}}_{s'_1}}(\lambda)}{\lambda^{2+s'_1}}\cdots\frac{Z_{\bar{\mathbb{S}}_{s'_n}}(\lambda)}{\lambda^{2+s'_n}}(N^{2-2n}-N^{-2n})\frac{1}{(4\pi^2)^{2n}}2^{\sum_{l=1}^{n}s_l+s'_l}(-1)^{\sum_{l=1}^{n}s_l+s'_l} \\
&\times\sum_{k'_1=0}^{s'_1-2}\cdots\sum_{k'_n=0}^{s'_n-2}\binom{s'_1}{k'_1}\binom{s'_1}{k'_1+2}\cdots\binom{s'_n}{k'_n}\binom{s'_n}{k'_n+2} \\
&\times\frac{2^{n-1}}{n}\sum_{\sigma\in P_n}\sum_{\rho\in P_n}(s_{\sigma(1)}-k_{\sigma(1)}+k'_{\rho(1)})!(s'_{\rho(1)}-k'_{\rho(1)}+k_{\sigma(2)})! \\
&\cdots\times(s_{\sigma(n)}-k_{\sigma(n)}+k'_{\rho(n)})!(s'_{\rho(n)}-k'_{\rho(n)}+k_{\sigma(1)})! \\
&\times\frac{(x_{\sigma(1)}-y_{\rho(1)})_z^{s_{\sigma(1)}-k_{\sigma(1)}+k'_{\rho(1)}}}{((x_{\sigma(1)}-y_{\rho(1)})^2)^{s_{\sigma(1)}-k_{\sigma(1)}+k'_{\rho(1)}+1}}\frac{(y_{\rho(1)}-x_{\sigma(2)})_z^{s'_{\rho(1)}-k'_{\rho(1)}+k_{\sigma(2)}}}{((y_{\rho(1)}-x_{\sigma(2)})^2)^{s'_{\rho(1)}-k'_{\rho(1)}+k_{\sigma(2)}+1}} \\
&\cdots\times\frac{(x_{\sigma(n)}-y_{\rho(n)})_z^{s_{\sigma(n)}-k_{\sigma(n)}+k'_{\rho(n)}}}{((x_{\sigma(n)}-y_{\rho(n)})^2)^{s_{\sigma(n)}-k_{\sigma(n)}+k'_{\rho(n)}+1}}\frac{(y_{\rho(n)}-x_{\sigma(1)})_z^{s'_{\rho(n)}-k'_{\rho(n)}+k_{\sigma(1)}}}{((y_{\rho(n)}-x_{\sigma(1)})^2)^{s'_{\rho(n)}-k'_{\rho(n)}+k_{\sigma(1)}+1}}
\end{aligned} \tag{144}$$

according to Eqs. (25), (55), and (129). Correspondingly, the generating functional of the Euclidean n-point correlators of twist-2 operators decomposes into its planar $\mathcal{W}^E_{\text{sphere}}$ and LO-nonplanar $\mathcal{W}^E_{\text{torus}}$ contributions.

Perturbatively, according to general principles of the 't Hooft large-N expansion [1], $\mathcal{W}^E_{\text{torus}}$ involves a sum of Feynman diagrams that have the topology of a punctured torus in the 't Hooft double-line representation.

Nonperturbatively, $\mathcal{W}^E_{\text{torus}}$ must involve the sum of the glueball one-loop diagrams, and it has been predicted in [4]—on the basis of the existence of a large-N nonperturbative effective action for the glueballs—that it should have the structure of the logarithm of a functional determinant.

The UV asymptotics as $\lambda \to 0$ of $\mathcal{W}^E_{\text{sphere}}$ and $\mathcal{W}^E_{\text{torus}}$ for twist-2 operators in the nonresonant diagonal scheme follows from the computation of the RG-improved correlators (Sec. VII)

$$\mathcal{W}^E_{\text{sphere}}[J_{\mathcal{O}'^E},\lambda] \sim \mathcal{W}^E_{\text{asym sphere}}[J_{\mathcal{O}'^E},\lambda] \tag{145}$$

and

$$\mathcal{W}^E_{\text{torus}}[J_{\mathcal{O}'^E},\lambda] \sim \mathcal{W}^E_{\text{asym torus}}[J_{\mathcal{O}'^E},\lambda] \tag{146}$$

with

$$\mathcal{W}^E_{\text{asym sphere}}[J_{\mathcal{O}'^E},\lambda] = -N^2\mathcal{W}^E_{\text{asym torus}}[J_{\mathcal{O}'^E},\lambda] \tag{147}$$

according to Eq. (25), where

$$\begin{aligned}
&\mathcal{W}^E_{\text{asym torus}}[J_{\mathbb{O}'^E}, J_{\tilde{\mathbb{O}}'^E}, J_{\mathbb{S}'^E}, J_{\tilde{\mathbb{S}}'^E}, \lambda] \\
&= +\frac{1}{2}\log\mathrm{Det}\left(I + \sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(-\overrightarrow{\partial}_z)^{s-k-1}\triangle^{-1}\frac{(Z_{\mathbb{O}'_s}(\lambda)J_{\mathbb{O}'^E_s} + Z_{\tilde{\mathbb{O}}'_s}(\lambda)J_{\tilde{\mathbb{O}}'^E_s})}{N\lambda^{2+s}}(-\overrightarrow{\partial}_z)^{k+1}\right) \\
&\quad + \frac{1}{2}\log\mathrm{Det}\left(I + \sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(-\overrightarrow{\partial}_z)^{s-k-1}\triangle^{-1}\frac{(Z_{\mathbb{O}'_s}(\lambda)J_{\mathbb{O}'^E_s} - Z_{\tilde{\mathbb{O}}'_s}(\lambda)J_{\tilde{\mathbb{O}}'^E_s})}{N\lambda^{2+s}}(-\overrightarrow{\partial}_z)^{k+1}\right) \\
&\quad + \frac{1}{2}\log\mathrm{Det}\left[I - 2\left(I + \sum_{k=0}^{s-2}\binom{s}{k}\binom{s}{k+2}(-\overrightarrow{\partial}_z)^{s-k-1}\triangle^{-1}\frac{(Z_{\mathbb{O}'_s}(\lambda)J_{\mathbb{O}'^E_s} - Z_{\tilde{\mathbb{O}}'_s}(\lambda)J_{\tilde{\mathbb{O}}'^E_s})}{N\lambda^{2+s}}(-\overrightarrow{\partial}_z)^{k+1}\right)^{-1}\right. \\
&\quad \times \sum_{k_1=0}^{s_1-2}\binom{s_1}{k_1}\binom{s_1}{k_1+2}(-\overrightarrow{\partial}_z)^{s_1-k_1-1}\triangle^{-1}\frac{Z_{\tilde{\mathbb{S}}'_{s_1}}(\lambda)}{N\lambda^{2+s_1}}J_{\tilde{\mathbb{S}}'^E_{s_1}}(-\overrightarrow{\partial}_z)^{k_1+1} \\
&\quad \times \left(I + \sum_{k_2=0}^{s_2-2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}(-\overrightarrow{\partial}_z)^{s_2-k_2-1}\triangle^{-1}\frac{(Z_{\mathbb{O}'_{s_2}}(\lambda)J_{\mathbb{O}'^E_{s_2}} + Z_{\tilde{\mathbb{O}}'_{s_2}}(\lambda)J_{\tilde{\mathbb{O}}'^E_{s_2}})}{N\lambda^{2+s_2}}(-\overrightarrow{\partial}_z)^{k_2+1}\right)^{-1} \\
&\quad \left.\times \sum_{k_3=0}^{s_3-2}\binom{s_3}{k_3}\binom{s_3}{k_3+2}(-\overrightarrow{\partial}_z)^{s_3-k_3-1}\triangle^{-1}\frac{Z_{\mathbb{S}'_{s_3}}(\lambda)}{N\lambda^{2+s_3}}J_{\mathbb{S}'^E_{s_3}}(-\overrightarrow{\partial}_z)^{k_3+1}\right]
\end{aligned} \tag{148}$$

that follows from Eqs. (25), (27), (55), (57), and (129). By the above computation we verify the aforementioned prediction [4] that the generating functional $\mathcal{W}^E_{\text{asym torus}}$ of the LO nonplanar Euclidean RG-improved collinear twist-2 correlators should inherit the very same loop structure—specifically, the one of the logarithm of a functional determinant—of the corresponding nonperturbative object that involves the sum of glueball one-loop diagrams.

Hence, the aforementioned asymptotic results strongly constrain the nonperturbative solution of the large-N YM theory and may actually provide an essential guide to find it.

APPENDIX A: DETERMINANTS OF BLOCK MATRICES

The matrix in Eq. (21) has a block structure with noncommutative entries

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{A1}$$

where A, B, C, D are operators. Correspondingly, the determinant of M reads [16]

$$\mathrm{Det}\, M = \mathrm{Det}(A)\mathrm{Det}(D - CA^{-1}B) \tag{A2}$$

or [17]

$$\mathrm{Det}\, M = \mathrm{Det}(D)\mathrm{Det}(A - BD^{-1}C) \tag{A3}$$

provided that A^{-1} or D^{-1} exist, respectively. The above formulas may be rewritten as

$$\mathrm{Det}\, M = \mathrm{Det}(A)\mathrm{Det}(D)\mathrm{Det}(1 - D^{-1}CA^{-1}B) \tag{A4}$$

or

$$\mathrm{Det}\, M = \mathrm{Det}(D)\mathrm{Det}(A)\mathrm{Det}(1 - A^{-1}BD^{-1}C) \tag{A5}$$

provided that both A^{-1} and D^{-1} exist. In our case

$$\mathcal{M}^{ab}=\delta^{ab}\begin{pmatrix} i\Box-\frac{1}{2}\sum_s J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}-\frac{1}{2}\sum_s J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2} & -\frac{1}{\sqrt{2}}\sum_s J_{\mathbb{S}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2} \\ -\frac{1}{\sqrt{2}}\sum_s J_{\bar{\mathbb{S}}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2} & i\Box-\frac{1}{2}\sum_s J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}+\frac{1}{2}\sum_s J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\end{pmatrix}, \tag{A6}$$

and we employ Eq. (A4). Hence, up to a trivial constant normalization, we get

$$\begin{aligned}\mathrm{Det}\,\mathcal{M}=&\,\mathrm{Det}\left(\mathcal{I}+\frac{1}{2}i\Box^{-1}J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}+\frac{1}{2}i\Box^{-1}J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\right)\mathrm{Det}\left(\mathcal{I}+\frac{1}{2}i\Box^{-1}J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}-\frac{1}{2}i\Box^{-1}J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\right)\\&\times\mathrm{Det}\left[\mathcal{I}-\frac{1}{2}\left(\mathcal{I}+\frac{1}{2}i\Box^{-1}J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}-\frac{1}{2}i\Box^{-1}J_{\tilde{\mathbb{O}}_s}\otimes\mathcal{H}^{\frac{5}{2}}_{s-2}\right)^{-1}i\Box^{-1}J_{\bar{\mathbb{S}}_{s_1}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_1-2}\right.\\&\left.\times\left(\mathcal{I}+\frac{1}{2}i\Box^{-1}J_{\mathbb{O}_{s_2}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_2-2}+\frac{1}{2}i\Box^{-1}J_{\tilde{\mathbb{O}}_{s_2}}\otimes\mathcal{H}^{\frac{5}{2}}_{s_2-2}\right)^{-1}i\Box^{-1}J_{\mathbb{S}_{s_3}}\otimes\mathcal{Y}^{\frac{5}{2}}_{s_3-2}\right],\end{aligned} \tag{A7}$$

where the sum over repeated spin indices is understood.

APPENDIX B: CONFORMAL PROPERTIES OF THE STANDARD BASIS

The gauge-invariant collinear twist-2 operators in the light-cone gauge in the standard basis read [6]

$$\mathbb{O}_s=\mathrm{Tr}\partial_+\bar{A}(x)(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}\times C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+-\overleftarrow{\partial}_+}{\overrightarrow{\partial}_++\overleftarrow{\partial}_+}\right)\partial_+A(x), \tag{B1}$$

$$\tilde{\mathbb{O}}_s=\mathrm{Tr}\partial_+\bar{A}(x)(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}\times C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+-\overleftarrow{\partial}_+}{\overrightarrow{\partial}_++\overleftarrow{\partial}_+}\right)\partial_+A(x), \tag{B2}$$

$$\mathbb{S}_s=\frac{1}{\sqrt{2}}\mathrm{Tr}\partial_+\bar{A}(x)(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}\times C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+-\overleftarrow{\partial}_+}{\overrightarrow{\partial}_++\overleftarrow{\partial}_+}\right)\partial_+\bar{A}(x), \tag{B3}$$

$$\bar{\mathbb{S}}_s=\frac{1}{\sqrt{2}}\mathrm{Tr}\partial_+A(x)(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}\times C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+-\overleftarrow{\partial}_+}{\overrightarrow{\partial}_++\overleftarrow{\partial}_+}\right)\partial_+A(x), \tag{B4}$$

where $C^{\alpha'}_l(x)$ are the Gegenbauer polynomials that are a special case of the Jacobi polynomials [6]

$$C^{\alpha'}_l(x)=\frac{\Gamma(l+2\alpha')\Gamma\left(\alpha'+\frac{1}{2}\right)}{\Gamma(2\alpha')\Gamma\left(l+\alpha'+\frac{1}{2}\right)}P^{(\alpha'-\frac{1}{2},\alpha'-\frac{1}{2})}_l(x) \tag{B5}$$

with the symmetry properties

$$C^{\alpha'}_l(-x)=(-1)^lC^{\alpha'}_l(x). \tag{B6}$$

They are the restriction, up to perhaps normalization and linear combinations [6], to the component with maximal-spin projection s along the p_+ direction of the balanced, $\mathbb{O}^{\mathcal{T}=2}_s$, $\tilde{\mathbb{O}}^{\mathcal{T}=2}_s$, and unbalanced, $\mathbb{S}^{\mathcal{T}=2}_s$, twist-2 operators that to the leading order of perturbation theory transform as primary operators with respect to the conformal group [7]

$$\begin{aligned}\mathbb{O}^{\mathcal{T}=2}_s&=\mathrm{Tr}F^{\mu}_{(\rho_1}\overleftrightarrow{D}_{\rho_2}\cdots\overleftrightarrow{D}_{\rho_{s-1}}F_{\rho_s)\mu}-\text{traces},\\ \tilde{\mathbb{O}}^{\mathcal{T}=2}_s&=\mathrm{Tr}\tilde{F}^{\mu}_{(\rho_1}\overleftrightarrow{D}_{\rho_2}\cdots\overleftrightarrow{D}_{\rho_{s-1}}F_{\rho_s)\mu}-\text{traces},\\ \mathbb{S}^{\mathcal{T}=2}_s&=\mathrm{Tr}(F_{\mu(\nu}+i\tilde{F}_{\mu(\nu)})\overleftrightarrow{D}_{\rho_1}\ldots\overleftrightarrow{D}_{\rho_{s-2}}(F_{\lambda)\sigma}+i\tilde{F}_{\lambda)\sigma})\\&\quad-\text{traces},\end{aligned} \tag{B7}$$

where the parentheses stand for symmetrization of all the indices in between and the subtraction of the traces ensures that the contraction of any two indices is zero.

The above statement about the conformal properties needs some refinements. In a conformal field theory the propagator of a vector field $V_\mu(x)$ with conformal dimension 1 is purely longitudinal [18]

$$\langle V_\mu(x)V_\nu(0)\rangle=\mathcal{C}_2\left(g_{\mu\nu}-2\frac{x_\mu x_\nu}{|x|^2}\right)\frac{1}{|x|^2}=\frac{\mathcal{C}_2}{2}\partial_\mu\partial_\nu\log|x|^2. \tag{B8}$$

Therefore, there is no Lorentz-covariant gauge where the field A_μ may be primary with respect to the conformal group to the lowest order of perturbation theory. Yet, $F_{\mu\nu}$ decomposes into the $(1,0)\oplus(0,1)$ representations of the Lorentz group [6]

$$F_{a\dot{a}b\dot{b}}=2(f_{ab}\epsilon_{\dot{a}\dot{b}}-\epsilon_{ab}f_{\dot{a}\dot{b}}), \tag{B9}$$

where [6]

$$F_{a\dot{a}b\dot{b}} = \sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}}F_{\mu\nu} \tag{B10}$$

and

$$f_{ab} = \frac{i}{2}(\sigma^{\mu\nu})_{ab}F_{\mu\nu},$$
$$f_{\dot{a}\dot{b}} = -\frac{i}{2}(\bar{\sigma}^{\mu\nu})_{\dot{a}\dot{b}}F_{\mu\nu} \tag{B11}$$

with

$$\bar{f}_{ab} = f_{\dot{a}\dot{b}}. \tag{B12}$$

It turns out that to the leading order of perturbation theory f_{ab} and $f_{\dot{a}\dot{b}}$ are local primary fields with respect to the conformal group [7]. Consequently, it follows the construction [7] in Eq. (B7) of the local gauge-invariant operators that are conformal primaries to the leading order, with the caveat above that the corresponding representation of the conformal group does not extend outside the local gauge-invariant sector by including the field A_μ in a covariant gauge.

Besides, by projecting on the maximal-spin component along the p_+ direction the aforementioned gauge-invariant operators restrict to [6]

$$\mathbb{O}_s = \mathrm{Tr} f_{11}(x)(i\vec{D}_+ + i\bar{D}_+)^{s-2} \times C^{\frac{5}{2}}_{s-2}\left(\frac{\vec{D}_+ - \bar{D}_+}{\vec{D}_+ + \bar{D}_+}\right) f_{\dot{1}\dot{1}}(x), \tag{B13}$$

$$\tilde{\mathbb{O}}_s = \mathrm{Tr} f_{11}(x)(i\vec{D}_+ + i\bar{D}_+)^{s-2} \times C^{\frac{5}{2}}_{s-2}\left(\frac{\vec{D}_+ - \bar{D}_+}{\vec{D}_+ + \bar{D}_+}\right) f_{\dot{1}\dot{1}}(x), \tag{B14}$$

$$\mathbb{S}_s = \frac{1}{\sqrt{2}}\mathrm{Tr} f_{11}(x)(i\vec{D}_+ + i\bar{D}_+)^{s-2} \times C^{\frac{5}{2}}_{s-2}\left(\frac{\vec{D}_+ - \bar{D}_+}{\vec{D}_+ + \bar{D}_+}\right) f_{11}(x), \tag{B15}$$

$$\bar{\mathbb{S}}_s = \frac{1}{\sqrt{2}}\mathrm{Tr} f_{\dot{1}\dot{1}}(x)(i\vec{D}_+ + i\bar{D}_+)^{s-2} \times C^{\frac{5}{2}}_{s-2}\left(\frac{\vec{D}_+ - \bar{D}_+}{\vec{D}_+ + \bar{D}_+}\right) f_{\dot{1}\dot{1}}(x) \tag{B16}$$

that are primaries with respect to the collinear conformal subgroup $SL(2,R)$ [9] and reduce in the light-cone gauge to the operators in Eq. (B1) with $f_{11} = -\partial_+\bar{A}$. By allowing operator mixing with derivatives of operators with lower spin they may be extended to primary conformal operators to the next-to-leading order in the conformal renormalization scheme [15] that differs from the $\overline{\mathrm{MS}}$ scheme by a finite renormalization.

In fact, the fields $f \equiv f_{11}$ and $\bar{f} \equiv f_{\dot{1}\dot{1}}$ may be realized as primary operators for the collinear conformal subgroup to the leading order of perturbation theory in a noncanonical path-integral quantization of the YM theory in the light-cone gauge by a suitable change of variables [13]

$$S_{\mathrm{YM}}(f,\bar{f}) = \int \bar{f}^a\partial_+^{-2}\Box f^a + 2\frac{g}{\sqrt{N}}f^{abc}(\partial_+^{-1}\bar{f}^a f^b\overline{\partial}\partial_+^{-2}\bar{f}^c + \partial_+^{-1}f^a\bar{f}^b\partial\partial_+^{-2}f^c) - 2\frac{g^2}{N}f^{abc}f^{ade}\partial_+^{-1}(\partial_+^{-1}\bar{f}^b f^c)\partial_+^{-1}(\partial_+^{-1}f^d\bar{f}^e)d^4x, \tag{B17}$$

where

$$\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}f\mathcal{D}\bar{f}e^{iS_{\mathrm{YM}}(f,\bar{f})}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n). \tag{B18}$$

APPENDIX C: NORMALIZATION OF TWO-POINT CORRELATORS

The normalization of the two-point correlators has been computed [6] by means of a technique [19] involving the orthogonality of the Gegenbauer polynomials that makes apparent the vanishing of two-point correlators with different spins. In the standard basis the twist-2 operators $\mathcal{O}_s$ are normalized so that [6]

$$\begin{aligned}\langle\mathcal{O}_{s_1}(x)\mathcal{O}_{s_2}(y)\rangle &= \langle\mathbb{O}_{s_1}(x)\mathbb{O}_{s_2}(y)\rangle \\ &= \langle\mathbb{S}_{s_1}(x)\bar{\mathbb{S}}_{s_2}(y)\rangle, \quad s_1, s_2 \text{ even}, \\ \langle\mathcal{O}_{s_1}(x)\mathcal{O}_{s_2}(y)\rangle &= \langle\tilde{\mathbb{O}}_{s_1}(x)\tilde{\mathbb{O}}_{s_2}(y)\rangle, \quad s_1, s_2 \text{ odd}.\end{aligned} \tag{C1}$$

Hence, for example, it suffices to evaluate [6]

$$\langle\mathbb{O}_{s_1}(x)\mathbb{O}_{s_2}(y)\rangle = (N^2-1)\mathcal{Y}^{\frac{5}{2}}_{s_1-2}(\partial_{x_1^+},\partial_{x_2^+})\mathcal{Y}^{\frac{5}{2}}_{s_2-2}(\partial_{y_1^+},\partial_{y_2^+}) \times i\Box^{-1}(x_1-y_2)i\Box^{-1}(y_1-x_2)|^{y_1=y_2=y}_{x_1=x_2=x}, \tag{C2}$$

where

$$i\Box^{-1}(x-y) = \frac{1}{4\pi^2}\frac{1}{|x-y|^2 - i\epsilon} \tag{C3}$$

and

$$\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+) = \overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+. \tag{C4}$$

We restrict the correlators to $(x-y)_\perp = 0$, so that $|x-y|^2 = 2(x-y)_+(x-y)_-$. For $x_- > y_-$ we obtain

$$\frac{\Gamma(k)}{(x-y)_-^k}=\int_0^\infty d\tau\tau^{k-1}e^{-\tau(x-y)_-}. \tag{C5}$$

By the above formula we convert derivatives into multiplications

$$\begin{aligned}&\partial_{x_1^+}\to-\tau_1, \qquad \partial_{x_2^+}\to-\tau_2,\\ &\partial_{y_1^+}\to\tau_2, \qquad \partial_{y_2^+}\to\tau_1.\end{aligned} \tag{C6}$$

By the symmetry property

$$C_l^\alpha(-x)=(-1)^l C_l^\alpha(x), \tag{C7}$$

it follows

$$\begin{aligned}C^{\frac{5}{2}}_{s_1-2}\left(-\frac{\tau_2-\tau_1}{\tau_1+\tau_2}\right)&=(-1)^{s_1}C^{\frac{5}{2}}_{s_1-2}\left(\frac{\tau_2-\tau_1}{\tau_1+\tau_2}\right),\\ C^{\frac{5}{2}}_{s_2-2}\left(\frac{\tau_1-\tau_2}{\tau_2+\tau_1}\right)&=(-1)^{s_2}C^{\frac{5}{2}}_{s_2-2}\left(\frac{\tau_2-\tau_1}{\tau_1+\tau_2}\right).\end{aligned} \tag{C8}$$

Hence, we obtain

$$\begin{aligned}&\langle\mathbb{O}_{s_1}(x)\mathbb{O}_{s_2}(y)\rangle\\ &=\frac{1}{(4\pi^2)^2}\frac{N^2-1}{4}i^{s_1+s_2-4}\frac{1}{4(x-y)_+^2}(-1)^{s_1+s_2}\\ &\times\int_0^\infty d\tau_1 d\tau_2(\tau_1+\tau_2)^{s_1+s_2-4}\tau_1^2\tau_2^2e^{-(\tau_1+\tau_2)(x-y)_-}\\ &\times C^{\frac{5}{2}}_{s_1-2}\left(\frac{\tau_2-\tau_1}{\tau_1+\tau_2}\right)C^{\frac{5}{2}}_{s_2-2}\left(\frac{\tau_2-\tau_1}{\tau_1+\tau_2}\right).\end{aligned} \tag{C9}$$

From the substitution

$$\tau_1=\tau\gamma, \qquad \tau_2=\tau(1-\gamma), \tag{C10}$$

it follows

$$\begin{aligned}&\langle\mathbb{O}_{s_1}(x)\mathbb{O}_{s_2}(y)\rangle\\ &=\frac{1}{(4\pi^2)^2}\frac{N^2-1}{4}i^{s_1+s_2-4}\frac{1}{4(x-y)_+^2}(-1)^{s_1+s_2}\\ &\times\int_0^\infty d\tau\int_0^1 d\gamma\tau^{s_1+s_2+1}\gamma^2(1-\gamma)^2e^{-\tau(x-y)_-}\\ &\times C^{\frac{5}{2}}_{s_1-2}(1-2\gamma)C^{\frac{5}{2}}_{s_2-2}(1-2\gamma).\end{aligned} \tag{C11}$$

The τ integral yields

$$\int_0^\infty d\tau\tau^{s_1+s_2+1}e^{-\tau(x-y)_-}=\frac{\Gamma(s_1+s_2+2)}{(x-y)_-^{s_1+s_2+2}}. \tag{C12}$$

We rewrite

$$\int_0^1 d\gamma\gamma^2(1-\gamma)^2C^{\frac{5}{2}}_{s_1-2}(1-2\gamma)C^{\frac{5}{2}}_{s_2-2}(1-2\gamma) \tag{C13}$$

as

$$\int_{-1}^1\frac{du}{2}\left(\frac{1-u^2}{4}\right)^2C^{\frac{5}{2}}_{s_1-2}(u)C^{\frac{5}{2}}_{s_2-2}(u) \tag{C14}$$

with $u=1$–2γ. The orthogonality of the Gegenbauer polynomials reads

$$\begin{aligned}&\int_{-1}^1 dz(1-z^2)^{\alpha'-\frac{1}{2}}C^{\alpha'}_{n_1}(z)C^{\alpha'}_{n_2}(z)\\ &=\delta_{n_1n_2}\frac{\pi 2^{1-2\alpha'}\Gamma(n_1+2\alpha')}{n_1!(n_1+\alpha')\Gamma(\alpha')^2}.\end{aligned} \tag{C15}$$

Hence, for $\alpha'=\frac{5}{2}$

$$\begin{aligned}&\int_{-1}^1 du(1-u^2)^2C^{\frac{5}{2}}_{s_1-2}(u)C^{\frac{5}{2}}_{s_2-2}(u)\\ &=\delta_{s_1s_2}2^{-4}\frac{16(s_1+2)!}{9(s_1-2)!(2s_1+1)}.\end{aligned} \tag{C16}$$

Collecting all the above factors we get

$$\begin{aligned}&\langle\mathbb{O}_{s_1}(x)\mathbb{O}_{s_2}(y)\rangle\\ &=\delta_{s_1s_2}\frac{1}{(4\pi^2)^2}\frac{N^2-1}{4}i^{s_1+s_2-4}(-1)^{s_1+s_2}\frac{1}{2^4}\frac{1(s_1+2)!}{3^2(s_1-2)!}\\ &\times\frac{\Gamma(s_1+s_2+2)}{2s_1+1}\frac{1}{4(x-y)_+^2(x-y)_-^{s_1+s_2+2}}.\end{aligned} \tag{C17}$$

Going back outside the plane $(x-y)_\perp=0$,

$$\frac{1}{4(x-y)_+^2(x-y)_-^{s_1+s_2+2}}\to 2^{s_1+s_2}\frac{(x-y)_+^{s_1+s_2}}{(|x-y|^2)^{s_1+s_2+2}}, \tag{C18}$$

we obtain

$$\begin{aligned}\langle\mathbb{O}_{s_1}(x)\mathbb{O}_{s_2}(y)\rangle&=\delta_{s_1s_2}\frac{1}{(4\pi^2)^2}\frac{N^2-1}{4}i^{s_1+s_2-4}2^{s_1+s_2}(-1)^{s_1+s_2}\frac{1}{2^43^2}\frac{(s_1+2)!}{(s_1-2)!}\frac{\Gamma(s_1+s_2+2)}{2s_1+1}\frac{(x-y)_+^{s_1+s_2}}{(|x-y|^2)^{s_1+s_2+2}}\\ &=\delta_{s_1s_2}\frac{1}{(4\pi^2)^2}\frac{N^2-1}{4}\frac{2^{2s_1+2}}{(4!)^2}(-1)^{s_1}(s_1-1)s_1(s_1+1)(s_1+2)(2s_1)!\frac{(x-y)_+^{2s_1}}{(|x-y|^2)^{2s_1+2}}.\end{aligned} \tag{C19}$$

Analogously, the above equation extends to the balanced operators $\tilde{\mathbb{O}}_s$ with odd s.

APPENDIX D: CONFORMAL STRUCTURE OF THREE-POINT CORRELATORS

We work out the various conformal structures that contribute to the three-point correlators up to $s = 4$.

The nonvanishing three-point correlators of twist-2 operators $\mathcal{O}_s$ in the standard basis all have the same structure [6]

$$\begin{aligned}\langle \mathcal{O}_{s_1}(x_1)\mathcal{O}_{s_2}(x_2)\mathcal{O}_{s_3}(x_3)\rangle &= \langle \mathbb{O}_{s_1}(x_1)\mathbb{O}_{s_2}(x_2)\mathbb{O}_{s_3}(x_3)\rangle \\ &= \langle \mathbb{O}_{s_1}(x_1)\mathbb{S}_{s_2}(x_2)\bar{\mathbb{S}}_{s_3}(x_3)\rangle, \quad s_1, s_2, s_3 \text{ even}, \\ \langle \mathcal{O}_{s_1}(x_1)\mathcal{O}_{s_2}(x_2)\mathcal{O}_{s_3}(x_3)\rangle &= \langle \mathbb{O}_{s_1}(x_1)\tilde{\mathbb{O}}_{s_2}(x_2)\tilde{\mathbb{O}}_{s_3}(x_3)\rangle, \quad s_1 \text{ even and } s_2, s_3 \text{ odd}, \end{aligned} \tag{D1}$$

where [6]

$$\begin{aligned}\langle \mathcal{O}_{s_1}(x_1)\mathcal{O}_{s_2}(x_2)\mathcal{O}_{s_3}(x_3)\rangle = &-\frac{1}{(4\pi^2)^3} 2\left(\frac{2}{4!}\right)^3 \frac{N^2-1}{8}\, i^{s_1+s_2+s_3} 2^{s_1+s_2+s_3} \\ &\times (s_1+1)(s_1+2)(s_2+1)(s_2+2)(s_3+1)(s_3+2) \\ &\times \sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2} \binom{s_1}{k_1}\binom{s_1}{k_1+2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}\binom{s_3}{k_3}\binom{s_3}{k_3+2} \\ &\times (s_1-k_1+k_2)!(s_2-k_2+k_3)!(s_3-k_3+k_1)! \\ &\times \frac{(x_1-x_2)_+^{s_1-k_1+k_2}}{(|x_1-x_2|^2)^{s_1+1-k_1+k_2}} \frac{(x_2-x_3)_+^{s_2-k_2+k_3}}{(|x_2-x_3|^2)^{s_2+1-k_2+k_3}} \frac{(x_3-x_1)_+^{s_3-k_3+k_1}}{(|x_3-x_1|^2)^{s_3+1-k_3+k_1}}. \end{aligned} \tag{D2}$$

We set $x_i - x_j = x_{ij}$ and—with an abuse of notation[4]—$x_{ij}^2 \equiv |x_{ij}|^2$. Hence,

$$\begin{aligned}\langle \mathcal{O}_{s_1}(x_1)\mathcal{O}_{s_2}(x_2)\mathcal{O}_{s_3}(x_3)\rangle = &\, C_{s_1s_2s_3} \sum_{k_1=0}^{s_1-2}\sum_{k_2=0}^{s_2-2}\sum_{k_3=0}^{s_3-2} \binom{s_1}{k_1}\binom{s_1}{k_1+2}\binom{s_2}{k_2}\binom{s_2}{k_2+2}\binom{s_3}{k_3}\binom{s_3}{k_3+2} \\ &\times (s_1-k_1+k_2)!(s_2-k_2+k_3)!(s_3-k_3+k_1)! \\ &\times \frac{x_{12+}^{s_1-k_1+k_2}}{(x_{12}^2)^{s_1+1-k_1+k_2}} \frac{x_{23+}^{s_2-k_2+k_3}}{(x_{23}^2)^{s_2+1-k_2+k_3}} \frac{x_{31+}^{s_3-k_3+k_1}}{(x_{31}^2)^{s_3+1-k_3+k_1}} \end{aligned} \tag{D3}$$

with

$$C_{s_1s_2s_3} = -\frac{1}{(4\pi^2)^3} 2\left(\frac{2}{4!}\right)^3 \frac{N^2-1}{8}\, i^{s_1+s_2+s_3} 2^{s_1+s_2+s_3} (s_1+1)(s_1+2)(s_2+1)(s_2+2)(s_3+1)(s_3+2). \tag{D4}$$

A generic three-point correlator of primary conformal operators $\mathcal{O}_s$ admits the structure [19]

$$\langle \mathcal{O}_{s_1}(x_1)\mathcal{O}_{s_2}(x_2)\mathcal{O}_{s_3}(x_3)\rangle = \sum_{n_{12},n_{13},n_{23}\geq 0} \lambda_{n_{12},n_{13},n_{23}} \frac{V_1^{s_1-n_{12}-n_{13}} V_2^{s_2-n_{12}-n_{23}} V_3^{s_3-n_{13}-n_{23}} H_{12}^{n_{12}} H_{13}^{n_{13}} H_{23}^{n_{23}}}{P_{12}^{\frac{1}{2}(q_1+q_2-q_3)} P_{13}^{\frac{1}{2}(q_1+q_3-q_2)} P_{23}^{\frac{1}{2}(q_2+q_3-q_1)}} \tag{D5}$$

with $s_1 - n_{12} - n_{13} \geq 0$, $s_2 - n_{12} - n_{23} \geq 0$, $s_3 - n_{13} - n_{23} \geq 0$, and $q_i = d_i + s_i$, where d_i is the conformal dimension of the operators and

[4]In the main body of the paper $|x|^2$ and x^2 are the squares of the Minkowskian and Euclidean lengths, respectively.

$$
\begin{aligned}
P_{ij} &= x_{ij}^2, \\
H_{ij} &= -2x_{ij+}^2, \\
V_1 &= \frac{x_{21+}x_{13}^2 - x_{31+}x_{12}^2}{x_{23}^2}, \\
V_2 &= \frac{x_{32+}x_{21}^2 - x_{12+}x_{23}^2}{x_{13}^2}, \\
V_3 &= \frac{x_{13+}x_{23}^2 - x_{23+}x_{13}^2}{x_{12}^2}.
\end{aligned} \tag{D6}
$$

For twist-2 operators $q_i = 2s_i + 2$, so that

$$
\langle \mathcal{O}_{s_1}(x_1)\mathcal{O}_{s_2}(x_2)\mathcal{O}_{s_3}(x_3)\rangle = \sum_{n_{12},n_{13},n_{23}\geq 0} \lambda_{n_{12},n_{13},n_{23}} \frac{V_1^{s_1-n_{12}-n_{13}} V_2^{s_2-n_{12}-n_{23}} V_3^{s_3-n_{13}-n_{23}} H_{12}^{n_{12}} H_{13}^{n_{13}} H_{23}^{n_{23}}}{(x_{12}^2)^{s_1+s_2-s_3+1}(x_{13}^2)^{s_1+s_3-s_2+1}(x_{23}^2)^{s_2+s_3-s_1+1}}. \tag{D7}
$$

As a preliminary step we work out the useful identity

$$
x_{12+}^n x_{13+}^n x_{23+}^n = \frac{1}{2^n}\left(V_1V_2V_3 + \frac{1}{2}(V_1H_{23} + V_2H_{13} + V_3H_{12})\right)^n \tag{D8}
$$

and the more trivial one

$$
x_{12+}^{2n} x_{13+}^{2n} x_{23+}^{2n} = \left(-\frac{1}{8}\right)^n H_{12}^n H_{13}^n H_{23}^n \tag{D9}
$$

so that we have two ways of writing the product $x_{12+}^n x_{13+}^n x_{23+}^n$. We are now in the position to identify the conformal tensor structures.

For $s_1 = s_2 = s_3 = 2$ we get

$$
\begin{aligned}
\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\rangle &= C_{222}\frac{x_{12+}^2}{x_{12}^6}\frac{x_{23+}^2}{x_{23}^6}\frac{x_{31+}^2}{x_{31}^6} \\
&= \frac{1}{8}C_{2,2,2}\frac{(V_1V_2V_3 + \frac{1}{2}(V_1H_{23} + V_2H_{13} + V_3H_{12}))^2 - \frac{1}{2}H_{12}H_{13}H_{23}}{x_{12}^6 x_{13}^6 x_{23}^6}.
\end{aligned} \tag{D10}
$$

Indeed, there are precisely $1 + 3 + 6$ terms by expanding the square, plus 1 extra term, so that there are exactly 11 distinct structures, as predicted by Eq. (D7).

For $s_1 = s_2 = 2$ and $s_3 = 4$,

$$
\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_4(x_3)\rangle = 576C_{224}\left(\frac{x_{12+}^2 x_{23+}^4 x_{13+}^2}{x_{12}^6 x_{23}^{10} x_{31}^6} - 2\frac{x_{12+}^2 x_{23+}^3 x_{13+}^3}{x_{12}^6 x_{23}^8 x_{31}^8} + \frac{x_{12+}^2 x_{23+}^2 x_{13+}^4}{x_{12}^6 x_{23}^6 x_{31}^{10}}\right). \tag{D11}
$$

To identify the conformal tensor structures we factorize the maximal common divisor of the numerators and the denominator in Eq. (D7)

$$
\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_4(x_3)\rangle = 576C_{224}\frac{x_{12+}^2 x_{23+}^2 x_{13}^2}{x_{12}^2 x_{13}^{10} x_{23}^{10}}\left(\frac{x_{23+}^2 x_{31}^4}{x_{12}^4} - 2\frac{x_{23+}x_{13+}x_{23}^2 x_{31}^2}{x_{12}^4} + \frac{x_{13+}^2 x_{23}^4}{x_{12}^4}\right). \tag{D12}
$$

The sum of terms inside the parentheses is exactly V_3^2

$$
\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_4(x_3)\rangle = 576C_{224}\frac{x_{12+}^2 x_{23+}^2 x_{13}^2}{x_{12}^2 x_{13}^{10} x_{23}^{10}} V_3^2 \tag{D13}
$$

so that

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_4(x_3)\rangle = 288 C_{224} \frac{(V_1V_2V_3 + \frac{1}{2}(V_1H_{23} + V_2H_{13} + V_3H_{12}))^2 - \frac{1}{2}H_{12}H_{13}H_{23}}{x_{12}^2 x_{13}^{10} x_{23}^{10}} V_3^2. \tag{D14}$$

For $s_1 = 2$ and $s_2 = s_3 = 3$ we get

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_3(x_2)\mathcal{O}_3(x_3)\rangle = 216 C_{233} \left(3\frac{x_{12+}^3 x_{23+}^3 x_{13+}^2}{x_{12}^8 x_{23}^8 x_{31}^6} + 4\frac{x_{12+}^2 x_{23+}^4 x_{13+}^2}{x_{12}^6 x_{23}^{10} x_{31}^6} - 3\frac{x_{12+}^3 x_{23+}^2 x_{13+}^3}{x_{12}^8 x_{23}^6 x_{31}^8} - 3\frac{x_{12+}^2 x_{23+}^3 x_{13+}^3}{x_{12}^6 x_{23}^8 x_{31}^8} \right). \tag{D15}$$

Similarly, after the above factorization we obtain

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_3(x_2)\mathcal{O}_3(x_3)\rangle = 216 C_{233} \frac{x_{12+}^2 x_{23+}^2 x_{13+}^2}{x_{12}^6 x_{23}^{10} x_{31}^6} \left(3\frac{x_{12+} x_{23+} x_{23}^2}{x_{12}^2} + 4x_{23+}^2 - 3\frac{x_{12+} x_{13+} x_{23}^4}{x_{12}^2 x_{31}^2} - 3\frac{x_{23+} x_{13+} x_{23}^2}{x_{31}^2} \right). \tag{D16}$$

The terms inside the parentheses read

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_3(x_2)\mathcal{O}_3(x_3)\rangle = 216 C_{233} \frac{x_{12+}^2 x_{23+}^2 x_{13+}^2}{x_{12}^6 x_{23}^{10} x_{31}^6} \left(3V_2V_3 - \frac{1}{2}H_{23} \right). \tag{D17}$$

Then

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_3(x_2)\mathcal{O}_3(x_3)\rangle = 108 C_{233} \frac{(V_1V_2V_3 + \frac{1}{2}(V_1H_{23} + V_2H_{13} + V_3H_{12}))^2 - \frac{1}{2}H_{12}H_{13}H_{23}}{x_{12}^6 x_{23}^{10} x_{31}^6} \left(3V_2V_3 - \frac{1}{2}H_{23} \right). \tag{D18}$$

For $s_1 = 2$, $s_2 = s_3 = 4$ we get

$$\begin{aligned} \langle \mathcal{O}_2(x_1)\mathcal{O}_4(x_2)\mathcal{O}_4(x_3)\rangle = 6912 C_{244} \Bigg(& 6\frac{x_{12+}^4 x_{23+}^4 x_{13+}^2}{x_{12}^{10} x_{23}^{10} x_{31}^6} + 20\frac{x_{12+}^3 x_{23+}^5 x_{13+}^2}{x_{12}^8 x_{23}^{12} x_{31}^6} + 15\frac{x_{12+}^2 x_{23+}^6 x_{13+}^2}{x_{12}^6 x_{23}^{14} x_{31}^6} \\ & - 12\frac{x_{12+}^4 x_{23+}^3 x_{13+}^3}{x_{12}^{10} x_{23}^8 x_{31}^8} - 32\frac{x_{12+}^3 x_{23+}^4 x_{13+}^3}{x_{12}^8 x_{23}^{10} x_{31}^8} - 20\frac{x_{12+}^2 x_{23+}^5 x_{13+}^3}{x_{12}^6 x_{23}^{12} x_{31}^8} + 6\frac{x_{12+}^4 x_{23+}^2 x_{13+}^4}{x_{12}^{10} x_{23}^6 x_{31}^{10}} \\ & + 12\frac{x_{12+}^3 x_{23+}^3 x_{13+}^4}{x_{12}^8 x_{23}^8 x_{31}^{10}} + 6\frac{x_{12+}^2 x_{23+}^4 x_{13+}^4}{x_{12}^6 x_{23}^{10} x_{31}^{10}} \Bigg). \end{aligned} \tag{D19}$$

Again factorizing we obtain

$$\begin{aligned} \langle \mathcal{O}_2(x_1)\mathcal{O}_4(x_2)\mathcal{O}_4(x_3)\rangle = 6912 C_{244} \frac{x_{12+}^2 x_{23+}^2 x_{13+}^2}{x_{12}^6 x_{23}^{14} x_{31}^6} \Bigg(& 6\frac{x_{12+}^2 x_{23+}^2 x_{23}^4}{x_{12}^4} + 20\frac{x_{12+} x_{23+}^3 x_{23}^2}{x_{12}^2} + 15 x_{23+}^4 \\ & - 12\frac{x_{12+}^2 x_{23+} x_{13+} x_{23}^6}{x_{12}^4 x_{31}^2} - 32\frac{x_{12+} x_{23+}^2 x_{13+} x_{23}^4}{x_{12}^2 x_{31}^2} - 20\frac{x_{23+}^3 x_{13+} x_{23}^2}{x_{31}^2} + 6\frac{x_{12+}^2 x_{13+}^2 x_{23}^8}{x_{12}^4 x_{31}^4} \\ & + 12\frac{x_{12+} x_{23+} x_{13+}^2 x_{23}^6}{x_{12}^2 x_{31}^4} + 6\frac{x_{23+}^2 x_{13+}^2 x_{23}^4}{x_{31}^4} \Bigg). \end{aligned} \tag{D20}$$

Hence,

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_4(x_2)\mathcal{O}_4(x_3)\rangle = 6912 C_{244} \frac{x_{12+}^2 x_{23+}^2 x_{13+}^2}{x_{12}^6 x_{23}^{14} x_{31}^6} \left(6(V_2V_3)^2 - 4(V_2V_3)H_{23} + \frac{1}{4}H_{23}^2 \right) \tag{D21}$$

so that

$$\begin{aligned} \langle \mathcal{O}_2(x_1)\mathcal{O}_4(x_2)\mathcal{O}_4(x_3)\rangle = & 3456 C_{244} \frac{(V_1V_2V_3 + \frac{1}{2}(V_1H_{23} + V_2H_{13} + V_3H_{12}))^2 - \frac{1}{2}H_{12}H_{13}H_{23}}{x_{12}^6 x_{23}^{14} x_{31}^6} \\ & \times \left(6(V_2V_3)^2 - 4(V_2V_3)H_{23} + \frac{1}{4}H_{23}^2 \right). \end{aligned} \tag{D22}$$

For $s_1 = 4$, $s_2 = s_3 = 3$ we get

$$\begin{aligned}\langle \mathcal{O}_4(x_1)\mathcal{O}_3(x_2)\mathcal{O}_3(x_3)\rangle = 5184 C_{433}\Bigg(&15\frac{x_{12+}^5 x_{23+}^3 x_{13+}^2}{x_{12}^{12}x_{23}^8 x_{31}^6} + 12\frac{x_{12+}^4 x_{23+}^4 x_{13+}^2}{x_{12}^{10}x_{23}^{10}x_{31}^6} - 15\frac{x_{12+}^5 x_{23+}^2 x_{13+}^3}{x_{12}^{12}x_{23}^6 x_{31}^8}\\ &- 33\frac{x_{12+}^4 x_{23+}^3 x_{13+}^3}{x_{12}^{10}x_{23}^8 x_{31}^8} - 24\frac{x_{12+}^3 x_{23+}^4 x_{13+}^3}{x_{12}^8 x_{23}^{10}x_{31}^8} + 32\frac{x_{12+}^4 x_{23+}^2 x_{13+}^4}{x_{12}^{10}x_{23}^6 x_{31}^{10}} + 33\frac{x_{12+}^3 x_{23+}^3 x_{13+}^4}{x_{12}^8 x_{23}^8 x_{31}^{10}}\\ &+ 12\frac{x_{12+}^2 x_{23+}^4 x_{13+}^4}{x_{12}^6 x_{23}^{10}x_{31}^{10}} - 15\frac{x_{12+}^3 x_{23+}^2 x_{13+}^5}{x_{12}^8 x_{23}^6 x_{31}^{12}} - 15\frac{x_{12+}^2 x_{23+}^3 x_{13+}^5}{x_6^6 x_{23}^8 x_{31}^{12}}\Bigg).\end{aligned} \tag{D23}$$

Factorizing as above,

$$\begin{aligned}\langle \mathcal{O}_4(x_1)\mathcal{O}_3(x_2)\mathcal{O}_3(x_3)\rangle = 5184 C_{433}\frac{x_{12+}^2 x_{23+}^2 x_{13+}^2}{x_{12}^{10}x_{23}^6 x_{31}^{10}}\Bigg(&15\frac{x_{12+}^3 x_{23+} x_{31}^4}{x_{12}^2 x_{23}^2} + 12\frac{x_{12+}^2 x_{23+}^2 x_{31}^4}{x_{23}^4} - 15\frac{x_{12+}^3 x_{13+} x_{31}^2}{x_{12}^2}\\ &- 33\frac{x_{12+}^2 x_{23+} x_{13+} x_{31}^2}{x_{23}^2} - 24\frac{x_{12+} x_{23+}^2 x_{13+} x_{12}^2 x_{31}^2}{x_{23}^4} + 32 x_{12+}^2 x_{13+}^2 + 33\frac{x_{12+} x_{23+} x_{13+}^2 x_{12}^2}{x_{23}^2}\\ &+ 12\frac{x_{23+}^2 x_{13+}^2 x_{12}^4}{x_{23}^4} - 15\frac{x_{12+} x_{13+}^3 x_{12}^2}{x_{31}^2} - 15\frac{x_{23+} x_{13+}^3 x_6^4}{x_{23}^2 x_{31}^2}\Bigg).\end{aligned} \tag{D24}$$

The terms in the parentheses above arise from the following combination of conformal tensor structures:

$$\begin{aligned}\langle \mathcal{O}_4(x_1)\mathcal{O}_3(x_2)\mathcal{O}_3(x_3)\rangle = 10368 C_{433}&\frac{(V_1V_2V_3 + \frac{1}{2}(V_1H_{23} + V_2H_{13} + V_3H_{12}))^2 - \frac{1}{2}H_{12}H_{13}H_{23}}{x_{12}^2 x_{13}^{10}x_{23}^{10}}\\ &\times (18V_1^2V_2V_3 - 6V_1V_3H_{12} - 6V_1V_2H_{13} - 3V_1^2H_{23} + H_{12}H_{13}).\end{aligned} \tag{D25}$$

For $s_1 = s_2 = s_3 = 4$ we get

$$\begin{aligned}\langle \mathcal{O}_4(x_1)\mathcal{O}_4(x_2)\mathcal{O}_4(x_3)\rangle = 55296 C_{444}\Bigg(&135\frac{x_{12+}^6 x_{23+}^4 x_{13+}^2}{x_{12}^{14}x_{23}^{10}x_{31}^6} + 300\frac{x_{12+}^5 x_{23+}^5 x_{13+}^2}{x_{12}^{12}x_{23}^{12}x_{31}^6} + 135\frac{x_{12+}^4 x_{23+}^6 x_{13+}^2}{x_{12}^{10}x_{23}^{14}x_{31}^6}\\ &- 270\frac{x_{12+}^6 x_{23+}^3 x_{13+}^3}{x_{12}^{14}x_{23}^8 x_{31}^8} - 660\frac{x_{12+}^5 x_{23+}^4 x_{13+}^3}{x_{12}^{12}x_{23}^{10}x_{31}^8} - 660\frac{x_{12+}^4 x_{23+}^5 x_{13+}^3}{x_{12}^{10}x_{23}^{12}x_{31}^8}\\ &- 270\frac{x_{12+}^3 x_{23+}^6 x_{13+}^3}{x_{12}^8 x_{23}^{14}x_{31}^8} + 135\frac{x_{12+}^6 x_{23+}^2 x_{13+}^4}{x_{12}^{14}x_{23}^6 x_{31}^{10}} + 660\frac{x_{12+}^5 x_{23+}^3 x_{13+}^4}{x_{12}^{12}x_{23}^8 x_{31}^{10}}\\ &+ 1132\frac{x_{12+}^4 x_{23+}^4 x_{13+}^4}{x_{12}^{10}x_{23}^{10}x_{31}^{10}} + 660\frac{x_{12+}^3 x_{23+}^5 x_{13+}^4}{x_{12}^8 x_{23}^{12}x_{31}^{10}} + 135\frac{x_{12+}^2 x_{23+}^6 x_{13+}^4}{x_{12}^6 x_{23}^{14}x_{31}^{10}}\\ &- 300\frac{x_{12+}^5 x_{23+}^2 x_{13+}^5}{x_{12}^{12}x_{23}^6 x_{31}^{12}} - 660\frac{x_{12+}^4 x_{23+}^3 x_{13+}^5}{x_{12}^{10}x_{23}^8 x_{31}^{12}} - 660\frac{x_{12+}^3 x_{23+}^4 x_{13+}^5}{x_{12}^8 x_{23}^{10}x_{31}^{12}}\\ &- 300\frac{x_{12+}^2 x_{23+}^5 x_{13+}^5}{x_{12}^6 x_{23}^{12}x_{31}^{12}} + 135\frac{x_{12+}^4 x_{23+}^2 x_{13+}^6}{x_{12}^{10}x_{23}^6 x_{31}^{14}} + 270\frac{x_{12+}^3 x_{23+}^3 x_{13+}^6}{x_{12}^8 x_{23}^8 x_{31}^{14}} + 135\frac{x_{12+}^2 x_{23+}^4 x_{13+}^6}{x_{12}^6 x_{23}^{10}x_{31}^{14}}\Bigg).\end{aligned} \tag{D26}$$

As above

$$\begin{aligned}\langle \mathcal{O}_4(x_1)\mathcal{O}_4(x_2)\mathcal{O}_4(x_3)\rangle = 55296 C_{444}\frac{x_{12+}^2 x_{23+}^2 x_{13+}^2}{x_{12}^{10}x_{23}^{10}x_{31}^{10}}\Bigg(&135\frac{x_{12+}^4 x_{23+}^2 x_{31}^4}{x_{12}^4} + 300\frac{x_{12+}^3 x_{23+}^3 x_{31}^4}{x_{12}^2 x_{23}^2} + 135\frac{x_{12+}^2 x_{23+}^4 x_{31}^4}{x_{23}^4}\\ &- 270\frac{x_{12+}^4 x_{23+} x_{13+} x_{23}^2 x_{31}^2}{x_{12}^4} - 660\frac{x_{12+}^3 x_{23+}^2 x_{13+} x_{31}^2}{x_{12}^2} - 660\frac{x_{12+}^2 x_{23+}^3 x_{13+} x_{31}^2}{x_{23}^2}\\ &- 270\frac{x_{12+} x_{23+}^4 x_{13+} x_{12}^2 x_{31}^2}{x_{23}^4} + 135\frac{x_{12+}^4 x_{13+}^2 x_{23}^4}{x_{12}^4} + 660\frac{x_{12+}^3 x_{23+} x_{13+}^2 x_{23}^2}{x_{12}^2}\end{aligned}$$

$$+1132x_{12+}^2x_{23+}^2x_{13+}^2+660\frac{x_{12+}x_{23+}^3x_{13+}^2x_{12}^2}{x_{23}^2}+135\frac{x_{23+}^4x_{13+}^2x_{12}^4}{x_{23}^4}$$
$$-300\frac{x_{12+}^3x_{13+}^2x_{23}^4}{x_{12}^2x_{31}^2}-660\frac{x_{12+}^2x_{23+}x_{13+}^3x_{23}^2}{x_{31}^2}-660\frac{x_{12+}x_{23+}^2x_{13+}^3x_{12}^2}{x_{31}^2}$$
$$-300\frac{x_{23+}^3x_{13+}^3x_{12}^4}{x_{23}^2x_{31}^2}+135\frac{x_{12+}^2x_{13+}^4x_{23}^4}{x_{31}^4}+270\frac{x_{12+}x_{23+}x_{13+}^4x_{12}^2x_{23}^2}{x_{31}^4}+135\frac{x_{23+}^2x_{13+}^4x_{12}^4}{x_{31}^4}\Big). \tag{D27}$$

The terms in the parentheses above arise from the following combination of conformal tensor structures:

$$\langle\mathcal{O}_4(x_1)\mathcal{O}_4(x_2)\mathcal{O}_4(x_3)\rangle = 13824C_{444}\frac{(V_1V_2V_3+\frac{1}{2}(V_1H_{23}+V_2H_{13}+V_3H_{12}))^2-\frac{1}{2}H_{12}H_{13}H_{23}}{x_{12}^{10}x_{23}^{10}x_{31}^{10}}$$
$$\times(90(V_1V_2V_3)^2-90V_1V_2^2V_3H_{13}-90V_1^2V_2V_3H_{23}-90V_1V_2V_3^2H_{12}$$
$$-13H_{12}H_{23}H_{13}+15V_1V_3H_{12}H_{23}+15V_2V_3H_{12}H_{23}+15V_1V_2H_{23}H_{13}). \tag{D28}$$

The algorithm to extract the conformal tensor structures from the three-point correlators is clear enough—according to our examples—though cumbersome in practice for increasing spins.

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