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## Advanced Fiber Beam Finite Element Model for Neural Network Training in Vibration-Based Bridge Monitoring

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### Abstract

Recent advancements in civil infrastructure monitoring have witnessed the increasingly high-performance sensor technologies and data-driven algorithms, opening up new possibilities for assessing structural conditions. In recent years, there has been a growing interest in leveraging the potential of Artificial Intelligence for civil infrastructure monitoring. One promising approach is the use of computational models to train and test data-driven algorithms aiming to tackle damage detection problems. To enhance the effectiveness of such procedures based on simulated data, this study proposes a high-performance beam finite element model for training a neural network model able to predict the dynamic response of the structure and for generating various damage scenarios. Compared to 2D and 3D finite element models, the advanced fiber beam model offers superior computational efficiency while accurately capturing the nonlinear behavior of structural elements. Specifically, a force-based beam finite element based on a damage-plasticity model is implemented to describe damage and degradation of materials in reinforced concrete girders. Through the simulation of the dynamic structural response under white noise excitation, a neural network model representing the structure in the undamaged conditions is obtained. The prediction error of such network model is investigated as a suitable measure for the definition of a damage indicator able to detect the presence of damage (concrete cracks and reinforcement yielding). The integration of an advanced fiber beam model, accurate constitutive law and neural network models shows promising potential in the monitoring of existing bridges.

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**Keywords:** damage-plastic model; time series prediction, machine learning; damage sensitive features.

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## 1. Introduction

Vibration-based structural monitoring plays a relevant role in the management of existing infrastructures, as it provides reliable methods for the safety evaluation of such assets and a valuable support for visual inspections. Even if visible damages (e.g. corrosion, delamination, cracks or spalling) can be efficiently detected and quantified through inspection operations assisted by the use of autonomous platforms and image processing algorithms (Crognale et al., 2023), the corresponding reliability of the inspected structure can be difficult to assess (Catbas et al., 2002). Therefore, for a global condition assessment, vibration-based procedures are more suitable to extract features sensitive to changes in the structural behavior and unaffected by environmental conditions. Such procedures require the identification of the dynamic properties of structures and their monitoring over time (Rehman et al., 2024). Furthermore, damage sensitive features can be extracted from both time series analysis (Tee, 2018, Gul et al., 2011) and dynamic response in the frequency domain (Salawu, 1997, Ndambi et al., 2002). These methods proved to be very efficient in detecting the damages and their accuracy in its locating the damages can also be improved by increasing the number of the measurement points, as for example demonstrated by Rinaldi et al. (2022) using high-speed camera images for dynamic displacement measurements. Vibration-based techniques are usually classified in model-based and data-driven methods; the former exploit structural identification and model updating procedures to calibrate physical models according to experimental measurements (Teughels et al., 2004), while the second are based on training statistical models using supervised or unsupervised learning algorithms (Figueiredo et al., 2018).

From a model-based perspective, it is crucial that Finite Element (FE) models accurately describe the nonlinear static and dynamic structural response. In case of static loading and unloading cycles, the dynamic response is significantly influenced by several mechanical phenomena such as concrete tensile damage and the partial closure of cracks induced by the presence of concrete aggregate (Pranno et al., 2022). In data-driven approaches, supervised methods perform the training on labeled data regarding both undamaged and damaged states, while unsupervised methods train models using only undamaged conditions and performing damaged detection as a novelty one (Worden et al., 2000). Unsupervised learning proved to be a practical approach that can detect any deviation from the predicted behavior without the need of the knowledge of the damaged state data (Eltouny et al. 2023).

In this work, an unsupervised approach has been implemented to obtain a neural network time series model able to predict the dynamic response of reinforced concrete girders in undamaged conditions and provide a damage indicator based on the evaluation of the prediction error. To test the procedure, several damage scenarios have been simulated through an efficient fiber beam model that can accurately represent the nonlinear behavior of the reinforced concrete including the partial closure of cracks. Although the nonlinear structural response of the bridges can be efficiently obtained through 2D and 3D FE models, having a computationally efficient fiber FE model accompanied by an accurate constitutive law allows to perform fast and reliable analysis that are a valuable support to define the threshold levels for the outlier analysis which is a critical aspect in the unsupervised method adopted for damage detection tasks (Eltouny et al. 2023). The paper is organized as follows. Section 2 describes the fiber beam element formulation and the damage-plasticity constitutive model implemented to accurately represent the nonlinear behavior of the material and simulate the data to train the neural network model. Section 3 introduces the adopted neural network model and damage detection strategy, and Section 4 shows a case study application where the trend of the damage indicator according to the damage states is provided.

## 2. Advanced fiber beam finite element model

Fiber beam models are commonly used due to their computational efficiency in nonlinear analyses, with a significative reduction of the number of elements and computational effort, if compared to models using 2D or 3D finite elements. Among the approaches proposed in literature for finite element beam models, the displacement-based (DB) formulation is widely used; following this formulation, compatible displacement and strain fields along the element are assumed and the equilibrium is satisfied in a weak form (Zienkiewicz et al., 1994). In nonlinear analyses, as is the case with this work, such approach is not convenient because it requires a fine discretization, and the force-based (FB) formulation turns out to be more advantageous (Spacone et al., 1996, Addessi et al., 2007).

Under the plane section hypothesis, in the fiber beam models the cross-section, located at the Gauss point of the beam element, is subdivided into fibers. The constitutive response and the stiffness of the section is evaluated by

integration of the response of the fibers, which is derived by the constitutive law of the material. For prestressed beams, the constitutive behaviour of the section is given by the response of the concrete and steel fibers corresponding to tendons and reinforcement. In this work, the 3D damage-plastic model for concrete-like materials proposed in Addessi et al. (2002) is considered for concrete fibers, properly modified to consider the unilateral effect due to the re-closure in compression of tensile cracks. The stress-strain relation is defined as:

$$\boldsymbol{\sigma} = (1 - D)^2 \mathbf{C}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^p) \quad (1)$$

where  $\boldsymbol{\sigma}$  is the stress vector,  $\boldsymbol{\epsilon}$  and  $\boldsymbol{\epsilon}^p$  are the total and plastic strain vector, respectively,  $D$  is the damage variable ( $D = 0$  for undamaged material,  $D = 1$  for a completely damaged state),  $\mathbf{C}$  is the constitutive matrix of the undamaged material, which depends on the Young's modulus and Poisson ratio.

To consider the unilateral effect due to the re-closure in compression of tensile cracks, the model proposed in Addessi et al. (2002), Gatta et al. (2018), Di Re et al. (2018) and Fusco et al. (2023) introduces a damage variable for tension,  $D_t$ , and one for compression,  $D_c$  whose combination provide the overall damage variable  $D$ :

$$D = \alpha_t D_t + \alpha_c D_c \quad (2)$$

where  $\alpha_t$  and  $\alpha_c$  are the following weighting factors:

$$\alpha_t = \frac{\eta_t^2}{\eta_t^2 + \eta_c^2}, \quad \alpha_c = 1 - \alpha_t \quad (3)$$

$$\eta_t = \frac{Y_t^e}{Y_{t0} + (\alpha_t Y_t^e + b_t) D}, \quad \eta_c = \frac{Y_c^e}{Y_{c0} + (\alpha_t Y_c^e + b_c) D} \quad (4)$$

The evolution of damage is governed by associated variables defined as equivalent strain measures as:

$$Y_t = \sqrt{\sum_{i=1}^3 \langle e_i \rangle_+^2}, \quad Y_c = \sqrt{\sum_{i=1}^3 \langle e_i \rangle_-^2 - k \sum_{j \neq i=1}^3 \langle e_i \rangle_- \langle e_j \rangle_-} \quad (5)$$

The material parameter  $k$  determines the shape of the limit function in compression and  $e_i$  is related to the principal total strains  $\hat{\epsilon}_i$  by the following relation:

$$e_i = (1 - 2\nu)\hat{\epsilon}_i + \nu \sum_{j=1}^3 \hat{\epsilon}_j \quad (6)$$

Regarding to expression (4),  $Y_{c0}$  and  $Y_{t0}$  are the damage strain threshold;  $Y_t^e$  and  $Y_c^e$  are based on the principal elastic strains, adopting the same definition as that in equations (5) and (6).

The evolution of the two damage variables is controlled by the following damage limit functions for tension and compression,

$$f_t(Y_t, D_t) = Y_t - Y_{t0} - (a_t Y_t + b_t) D_t, \quad f_c(Y_c, D_c) = Y_c - Y_{c0} - (a_c Y_c + b_c) D_c \quad (7)$$

and ruled by the classical Kuhn–Tucker and consistency conditions. In (7) the parameters  $b_c$  and  $b_t$  influence the rate of damage growth in tension and compression and govern the maximum strength of the material. The parameters  $a_c$  and  $a_t$  control the gradient of the degradation curves in the post-peak softening regime.

As for the plastic mechanisms in concrete fibers, the Drucker-Prager plasticity model with isotropic and kinematic hardening is used in this work. This is capable of accurately representing the asymmetric plastic behaviour of concrete

under compression and tension. The adopted constitutive model considers damage and plasticity mechanisms activating both in compression and tension.

This study proposes a modified version of the described 3D damage-plastic model to account for the partial closure of cracks in compression of concrete cracks opened in tension. As previously mentioned, this mechanical phenomenon significantly influences the dynamic parameters of reinforced concrete beams subjected to incremental loading and unloading paths (Pranno et al., 2022). The proposed damage-plastic model considers that only a portion of the tensile damage, equal to  $\beta D_t$ , affects the stiffness matrix coefficients during the compression phase, corresponding to  $\alpha_c > 0$ . In particular, the constitutive law of concrete is still defined as in Eq. (1) and the compressive damage variable is defined as  $D = \alpha_t D_t + \alpha_c D_c$ , but  $\beta D_t < D_c \leq 1$ . Fig. 1a shows an example of the uniaxial constitutive response considering both total and partial closure of cracks, corresponding to  $\beta = 0$  and  $\beta = 0.5$ , respectively. This comparison illustrates that, when partial crack closure is considered, the compressive stiffness in reloading phase decreases as tensile damage increases. In case of total closure,  $\beta = 0$ , the compressive behavior of the material is unaffected by the history of tensile damage, consequently, the compressive stiffness in the reloading phase remains elastic. Fig. 1b illustrates the imposed strain history and the evolution of damage corresponding to the case  $\beta = 0.5$ . The damage evolution clearly shows that, during the compressive phase, the total damage of the material is equal to a portion of the accumulated tensile damage.

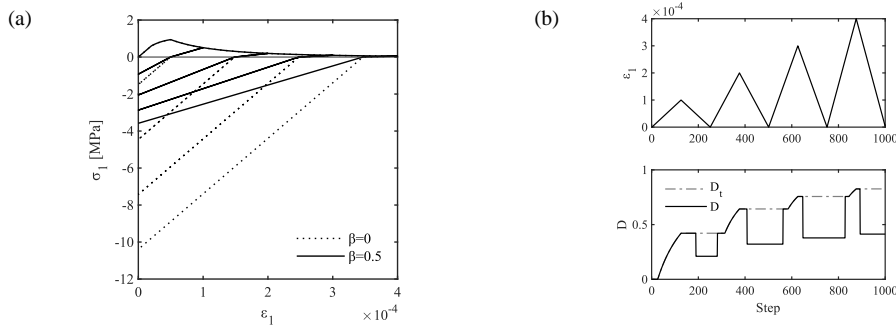


Fig. 1. Damage-plastic constitutive model: (a) Uniaxial stress-strain law with total ( $\beta = 0$ ) and partial ( $\beta = 0.5$ ) closure of cracks; (b) Strain history and damage evolution for partial closure of cracks  $\beta = 0.5$ .

### 3. Neural network time series prediction and damage detection

The presented fiber beam model represents an efficient tool to simulate the structural response of concrete and generate large amount of data to train machine learning algorithms for Structural Health Monitoring tasks. In this regard, a neural network model is considered to predict future values of a time series only from the knowledge of its past values. For this work, the Nonlinear AutoRegressive (NAR) network model is trained on the dynamic response data obtained through FE simulations in undamaged conditions. The updating of weight and bias values (training) of the neural network is performed by the Levenberg-Marquardt optimization algorithm (Yu et al., 2018) and the network architecture is defined by choosing the number of hidden layers and the time delay  $d$ , which is the number of the past values considered as inputs of the network to predict the value at the  $d+1$  time step. For the applications described in the following section, the hidden layers are set to 10 and the time delay to 6. The performance of the NAR model can be evaluated through the Root Mean Squared Error (RMSE) and the Normalized Root Mean Squared Error (NRMSE) between the target  $y(t)$  (numerical response used to train the network) and the output of the network  $\hat{y}(t)$  (prediction):

$$\text{RMSE} = \frac{\sum_{k=d+1}^f [\hat{y}(t_k) - y(t_k)]^2}{f - d}, \quad \text{NRMSE} = \frac{\text{RMSE}}{y^{\max} - y^{\min}}, \quad (8)$$

where  $f$  denotes the final time step. As the network is trained in undamaged conditions, the increasing of the prediction error under the same load conditions can be related to a change in structural behavior due to the occurrence of damage. Therefore, in this work, the evaluation of such performance is investigated as damage indicator to detect

anomalies in the structural response as depicted in Fig. 2. This damage detection procedure is an unsupervised method that does not need the preparing of undamaged and damaged labelled data, as is the case of supervised methods; only the NAR model related to the healthy state is obtained and its failure in predicting the structural response is considered as damage indicator. The performance of the NAR model has been already investigated by De Iuliis et al. (2023) for the prediction of the dynamic response of a cable-stayed bridge induced by ambient vibrations. Therefore, for the aims of this work, the numerical time-series data are generated under white noise excitation as described in the following section.

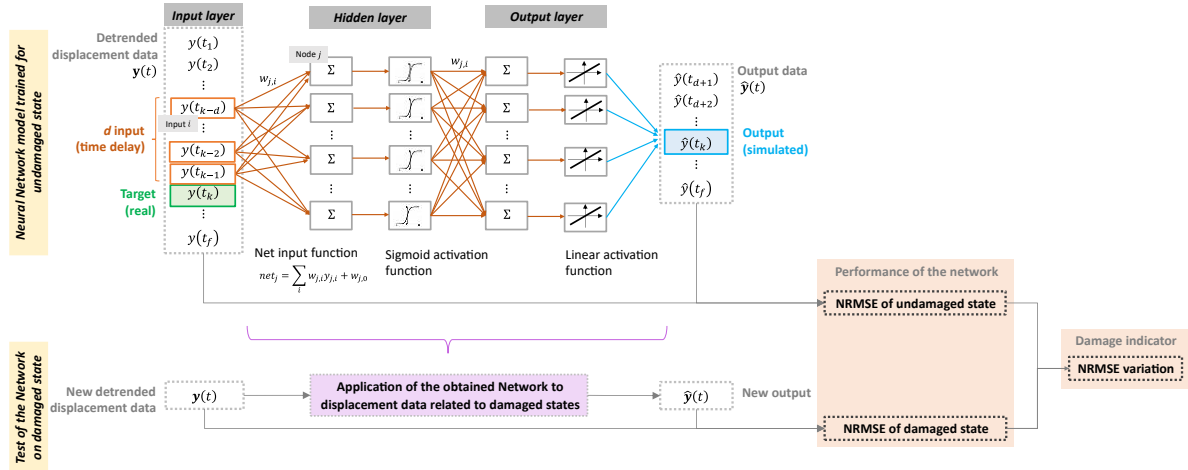


Fig. 2. Neural network model for time series prediction and unsupervised approach for damage detection.

#### 4. Case study application

In this section, an application of the proposed finite element model to simulations on beams is shown, and the numerical results are compared with those experimentally derived by Cerri et al. (2003). The experimental test involved both static and dynamic tests on two reinforced concrete beams, each measuring 2.45 meters in length, with a cross-section of 100 x 150 mm<sup>2</sup>. Further details regarding the reinforcements and materials used are provided in Cerri et al. (2003). In the static test, seven load-unload steps were performed. Subsequently, for each load step, upon removing the load, a dynamic test was conducted to assess the frequencies of the main vibration modes. The experimental test was numerically modeled through OpenSees software as solver and STKO as pre- and post-processor. The modified damage-plastic constitutive model with partial closure of cracks, described in the previous section was implemented as a new material in OpenSees (*nDMaterial*); the version of Opensees with the new constitutive model has not yet been published and distributed by the authors. The three-dimensional damage-plastic law was assigned to the concrete fibers; instead, a classical plastic model was considered for steel. To simulate the experimental test, a simply supported beam with a span equal to 2.25 m was modeled. A load-controlled analysis was performed by applying a vertical force at the midspan of beam, corresponding to the points of application of the experimental load. Fig. 3 shows the nonlinear response curves of the beam comparing the numerical and experimental results. In both, the behaviour is governed by the diffuse cracking of the beam and the yielding of the reinforcements. The results show a good agreement between the curves both in the concrete cracking phase and in the yielding phase. Thanks to the capability of modeling the tensile plasticity of concrete and the partial closure of cracks, it has been possible to capture the residual plastic displacement during the crack diffusion phase in the beam. The yielding of the reinforcements in the numerical model is predicted to occur at earlier displacement value than in the experimental test. This discrepancy may be attributed to a possible slip of the reinforcement bars during the experimental test, particularly since smoothed bars were used. The numerical model does not consider the bond slip between the reinforcement bars and the surrounding concrete. This phenomenon, while important, falls outside the scope of this study.

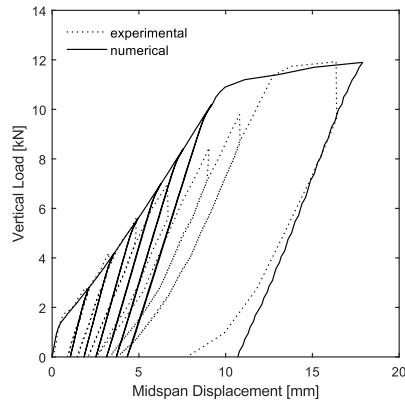


Fig. 3. Comparison between numerical and experimental nonlinear response of fiber beam elements.

The dynamic response of the beam under white noise excitation has been simulated in different states: three responses corresponding to undamaged conditions (U1-U3) in the elastic phase, six responses in six different damaged conditions (D1-D6) in the concrete cracking phase, and one response in the yielding phase (P1). For each load step considered, a nonlinear dynamic analysis was conducted by applying a low-amplitude force with a White Noise time variation. A different white noise signal has been generated for each scenario, using a sampling frequency of 2500 Hz which allows to obtain a frequency content in the load time series able to excite the significative modes. The analysis of the several damage scenarios revealed a significant frequency variation from around 10% to approximately 35% for the first flexural mode, which can be attributed to the concrete cracking and strand yielding that occurred during the loading process. The displacement time series response of the undamaged state U1 is exploited to train the NAR model, which is then tested for the several scenarios considered (U2, U3, D1-D6, P1). It is worth highlighting that, to avoid overfitting issues, the data of U1 state have been prepared by including different magnitude orders of the vibration amplitudes, as it can be noted in Fig. 4a. Both training and testing displacement data need to be detrended to obtain accurate prediction results, as the neural network model is not able to provide good results when the static displacement of testing signals is higher than the displacement of training signals. Fig. 4 shows the displacement numerical response (target) and network prediction (output) at mid-span and the corresponding frequency content for U1 state. It can be noted that the prediction of the response is accurate both in time and frequency domain. The NAR model trained in U1 conditions is then tested on white noise response related to U1-U3, D1-D6 and P1 conditions. The results in Fig. 5 show that the accuracy of the neural network model decreases with the occurrence of damages. By analyzing the prediction error through the evaluation of the NRMSE in different beam conditions, it is possible to establish a relationship between the prediction error and the state of the beam, as reported in Fig. 6a, where NRMSE Variation, defined as:

$$\text{NRMSE Variation} = \frac{\text{NRMSE} - \text{NRMSE}(U1)}{\text{NRMSE}(U1)}, \quad (9)$$

is plotted for the considered scenarios. The plot in Fig. 6a allows to establish the threshold level (the red line depicted) allowing to identify the starting of the concrete cracking. The comparison with the index based on the frequency variation (Fig. 6b) shows the reliability of the damage indicator based on the adopted unsupervised method.

## 5. Conclusions

This work presents the application of a fiber beam model based on a damage-plastic model for reinforced concrete girders accounting for the partial closure of cracks in concrete. The adopted force-based formulation allowed for efficient analysis of nonlinear structural responses of bridges which proved to be suitable to investigate on a vibration-based damage detection procedure using an unsupervised data-driven method.

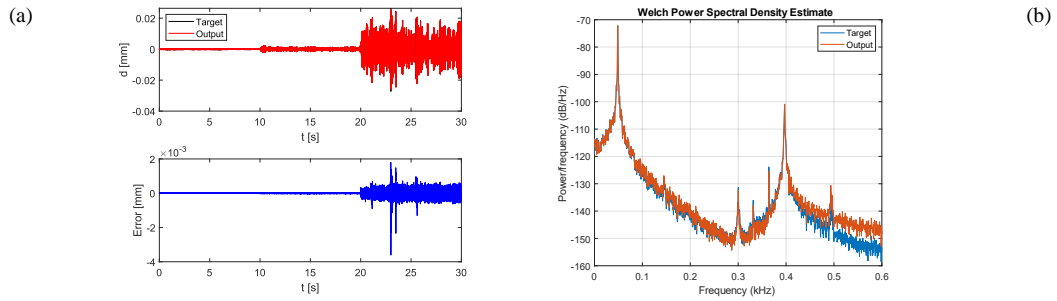


Fig. 4. Dynamic response prediction under white noise excitation through NAR model in training phase (U1): comparison between numerical response (target) and network prediction (output) of displacement time series (a) and frequency content (b).

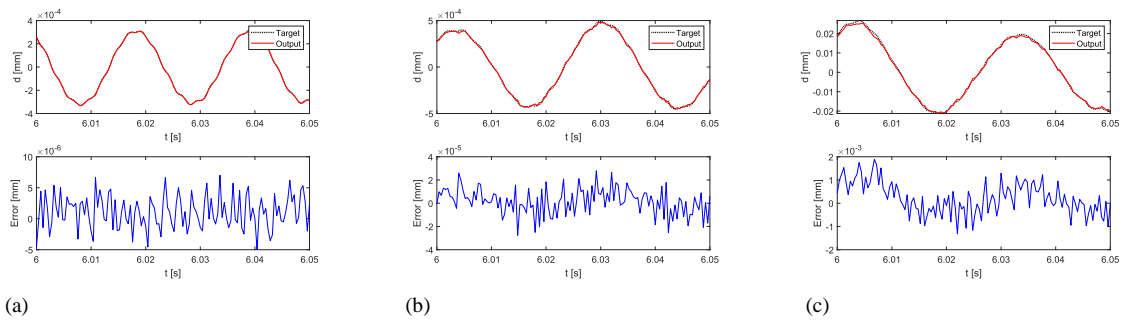


Fig. 5. Time series prediction of displacements and prediction error of the NAR model trained in U1 conditions and tested on response in U2 (a), D6 (b) and P1 (c) scenarios under white noise excitation.

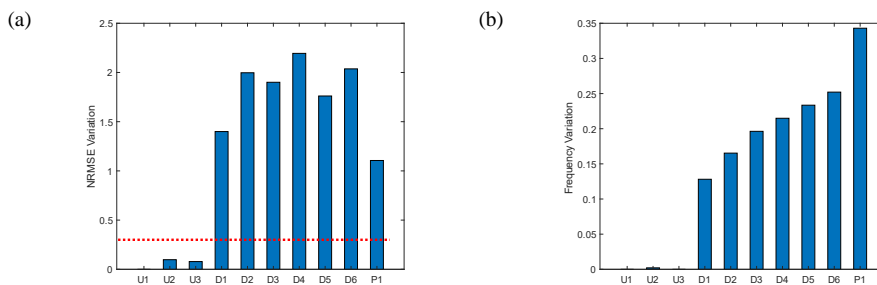


Fig. 6. NRMSE Variation (a) and Frequency Variation (b) for the considered scenarios.

Through the simulation of the dynamic response under white noise excitation, only a neural network model related to the healthy state of the structure has been obtained and its failure in predicting the structural response of the damaged beam has been considered as damage indicator. The prediction error of such network turned out to be a suitable measure for the definition of a damage indicator able to detect the presence of the concrete cracks and reinforcement yielding. Finally, the fiber beam model and the refined constitutive law considered allowed to perform fast and reliable analyses and to accurately define the threshold level for the outlier analysis, which is a critical aspect in the unsupervised data-driven methods adopted for damage detection tasks.

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