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Deterministic KPZ-type equations with nonlocal "gradient terms"

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Abstract

The main goal of this paper is to prove existence and non-existence results for deterministic Kardar–Parisi–Zhang type equations involving non-local "gradient terms". More precisely, let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\Omega$ of class C^2. For $s \in (0, 1)$, we consider problems of the form

$$\begin{cases} (-\Delta)^s u = \mu(x)\,|\mathbb{D}(u)|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{KPZ}$$

where $q > 1$ and $\lambda > 0$ are real parameters, f belongs to a suitable Lebesgue space, $\mu \in L^\infty(\Omega)$ and $\mathbb{D}$ represents a nonlocal "gradient term". Depending on the size of $\lambda > 0$, we derive existence and non-existence results. In particular, we solve several open problems posed in [Abdellaoui in Nonlinearity 31(4): 1260-1298 (2018), Section 6] and [Abdellaoui in Proc Roy Soc Edinburgh Sect A 150(5): 2682-2718 (2020), Section 7]

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1 Introduction

In this paper we analyse the existence and non–existence of solutions for deterministic Kardar–Parisi–Zhang type equations involving non-local "gradient terms". For $s, t \in (0, 1)$, we consider problems of the form

$$\begin{cases} (-\Delta)^s u = \mu(x)\, |\mathbb{D}_t(u)|^q + \lambda f(x), & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{KPZ}$$

depending on a real parameter $\lambda > 0$. Here, $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with boundary $\partial\Omega$ of class C^2, f belongs to a suitable Lebesgue space, $\mu \in L^\infty(\Omega)$, $q \in (1, +\infty)$ and $\mathbb{D}_t$ represents one of the following nonlocal "gradient terms":

- $(-\Delta)^{\frac{t}{2}} u(x) := a_{N,\frac{t}{2}}\, \text{P.V.} \displaystyle\int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+t}} dy$ (Half t − Laplacian), (KPZ$_1$)
- $\nabla^t u(x) := \mu_{N,t} \displaystyle\int_{\mathbb{R}^N} \frac{(x-y)(u(x) - u(y))}{|x-y|^{N+t+1}} dy$ (Riesz t − Gradient), (KPZ$_2$)
- $\mathcal{D}_t u(x) := \left(\dfrac{a_{N,t}}{2} \displaystyle\int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2t}} dy. \right)^{\frac{1}{2}}$ (Stein t − Functional). (KPZ$_3$)

Note that the previous definitions make sense for any function $u \in C_c^\infty(\mathbb{R}^N)$. Also, let us point out that

$$a_{N,\sigma} := -\frac{2^{2\sigma}\Gamma\left(\frac{N}{2} + \sigma\right)}{\pi^{\frac{N}{2}}\Gamma(-\sigma)} \quad \text{and} \quad \mu_{N,\sigma} := \frac{2^\sigma \Gamma(\frac{N+\sigma+1}{2})}{\pi^{\frac{N}{2}}\Gamma(\frac{1-\sigma}{2})},$$

are normalization constants and "P.V." stands for "in the principal value sense". Since both these constants and the "principal value sense" will not play an important role in our work, we will omit them from now on.

Before going further, we would like to emphasize that the three different nonlocal "gradient terms" that we consider can be traced back many years ago. Since nowadays the *fractional Laplacian* does not need any further presentation, let us focus on the other two terms. As very well explained in [23, page 3], the origin of the *Riesz t–Gradient* seems to be [19]. Note also that this operator has been rediscovered several times since [19] and has received considerable attention in the last few years. See for instance [10, 15, 22, 27]. On the other hand, the *Stein t–Functional* can be at least traced back to [25]. Moreover, this operator naturally appears as the nonlocal equivalent to the gradient when considering the minimization of fractional Harmonic maps into the sphere. See for instance the recent papers [7, 16, 21].

In contrast with the local case

$$\begin{cases} -\Delta u = \mu(x)|\nabla u|^q + \lambda f(x), & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for which the literature is very extensive, there exist very few results dealing with equations of the form (KPZ). We refer to [2], by the first two authors, for a detailed introduction to the subject. However, since the publication of [2], some results have been published. For instance, we would like to mention the recent paper [1], where the authors establish existence and non–existence results for problems of the form (KPZ) with a local operator (the Laplacian instead of the fractional Laplacian) and a nonlocal nonlinearity. Also, let us mention [6], where the authors establish the equivalence between different notions of solution to problems of the form (KPZ). Finally, let us stress that solutions to equations with nonlocal diffusion and a nonlocal "gradient term" with nonsingular kernels have been studied in [13].

In [2], the first two authors analyse the existence and non–existence of solutions to (KPZ), under the additional assumption $t = s \in (1/2, 1)$. The main goal of this paper is to refine the approach of [2] in order to deal also with the cases where $s \in (0, 1/2]$ and/or $t \neq s$. Depending on the real parameter $\lambda > 0$, we analyse the existence and non–existence of weak solutions to (KPZ) under the assumptions

$$\begin{cases} q \in (1, +\infty), \\ 0 < t < \min\{1, s(1 + (qN)^{-1})\}, \\ f \in L^m(\Omega) \text{ for some } m > N/s \text{ and } \mu \in L^\infty(\Omega). \end{cases} \tag{A1}$$

Following [8, 9], we introduce the subsequent notion of weak solution to (KPZ) :

Definition 1.1 We say that u is a *weak solution* to (KPZ) if u and $|\mathbb{D}_t(u)|^q$ belong to $L^1(\Omega)$, $u \equiv 0$ (a.e.) in $\mathbb{R}^N \setminus \Omega$ and

$$\int_\Omega u(-\Delta)^s \phi \, dx = \int_\Omega \big(\mu(x)|\mathbb{D}_t(u)|^q + \lambda f(x)\big)\phi \, dx, \tag{1.2}$$

for all ϕ belonging to

$$\mathbb{X}^s(\Omega) := \Big\{\phi \in C^s(\mathbb{R}^N) \, : \, \phi(x) = 0 \text{ for all } x \in \mathbb{R}^N \setminus \Omega \text{ and } (-\Delta)^s \phi \in L^\infty(\Omega)\Big\}.$$

Remark 1.2 With a slight abuse of notation we use (KPZ$_i$), $i = 1, 2, 3$, to refer to (KPZ) with $\mathbb{D}_t = (-\Delta)^{\frac{t}{2}}$, $\mathbb{D}_t = \nabla^t$ and $\mathbb{D}_t = \mathcal{D}_t$ respectively. See (KPZ$_1$), (KPZ$_2$) and (KPZ$_3$) for the corresponding definitions.

Our main existence result can be informally stated as follows:

Theorem 1.3 *Assume that (A1) holds true. Then, there exists* $\lambda^\star > 0$ *such that, for all* $0 < \lambda \leq \lambda^\star$, (KPZ$_i$), $i = 1, 2, 3$, *has a weak solution* u. *Moreover,* $u \in W^{s,p}(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$ *for all* $1 < p < +\infty$.

Theorem 1.3 is a particular case of the more general existence results proved in Sect. 3. We refer directly to Sect. 3 for more general statements. In particular, let us emphasize that, in Sect. 3, we substantially weaken the regularity on the datum f. Furthermore, arguing as in the proof of [2, Theorem 1.3], it is possible to show that the regularity considered in Sect. 3 is almost optimal. Note also that our existence results solve several open problems posed in [2, 4].

The proofs of our existence results rely on the combination of fixed point arguments in the spirit of [14, 18] with *global* fractional Calderón–Zygmund regularity results for the fractional Poisson equation

$$\begin{cases} (-\Delta)^s u = h, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

This approach was already implemented in [2] by the first two authors. However, the required *global* fractional Calderón–Zygmund regularity theory was not available for $s \in (0, 1/2]$. Furthermore, the Calderón–Zygmund regularity used in [2] (cf. [2, Section 3]) contains several imprecisions. In the recent paper [3], we establish sharp *global* fractional Calderón–Zygmund regularity results for (1.3) in the full range $s \in (0, 1)$ (fixing in particular the issues of [2, Section 3]). Having at hand these regularity results, the proofs of our existence results follow from a refinement of the fixed point approach implemented in [2]. We refer to Sect. 3 for more details.

Taking into account Theorem 1.3, it is very natural to ask whether the smallness assumption on λ is necessary or not. Note that, in [2, Theorem 1.2], the first two authors established a non–existence result for (KPZ_3) with $t = s \in (0, 1)$. We focus here in proving a non-existence result to (KPZ_1). This was left as an open problem in [2, Section 7].

Theorem 1.4 *Assume that (A1) holds true with $t = s$ and $q = 2$ and suppose that $\mu(x) \geq \mu_1 > 0$ and $f \gneqq 0$. Then, there exists $\lambda^{\star\star} > 0$ such that, for all $\lambda > \lambda^{\star\star}$,* ($\mathrm{KPZ}_1$) *has no weak solution in $W^{s,2}(\mathbb{R}^N)$.*

Remark 1.5 The non–existence for λ large in the case where $\mathbb{D}_t = \nabla^t$ remains completely open and a different approach is needed.

Organization of the paper

In the next section we introduce the main function spaces involved in our results and prove the continuity and compactness of the solution map for the fractional Poisson equation with L^1–data. In Sect. 3, we prove our main existence results to (KPZ), from which Theorem 1.3 immediately follows. Finally, in Sect. 4, we prove non-existence results for (KPZ_1) and (KPZ_3) when $\lambda > 0$ is large.

2 Function spaces and tools

We collect here the definitions of the main function spaces involved in our results and some other tools. First of all, recall that, for all $s \in (0, 1)$ and $1 \leq p < +\infty$, the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy < +\infty \right\}.$$

It is a Banach space endowed with the usual norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{\frac{1}{p}}.$$

Also, having at hand $W^{s,p}(\mathbb{R}^N)$, we define the space $W_0^{s,p}(\Omega)$ as

$$W_0^{s,p}(\Omega) := \left\{ u \in W^{s,p}(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\},$$

and recall that, thanks to the Sobolev inequality, it is a Banach space endowed with the norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\iint_{D_\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{1/p},$$

$$\text{where} \quad D_\Omega := (\Omega \times \mathbb{R}^N) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega).$$

Next, we remind that, for any $s \in (0,1)$ and $1 \leq p < +\infty$, the Bessel potential space is defined as

$$L^{s,p}(\mathbb{R}^N) := \overline{\{u \in C_c^\infty(\mathbb{R}^N)\}}^{|\cdot|_{L^{s,p}(\mathbb{R}^N)}},$$

where

$$|u|_{L^{s,p}(\mathbb{R}^N)} = \|(1-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)}$$
$$\text{and} \quad (1-\Delta)^{\frac{s}{2}} u = \mathcal{F}^{-1}((1+|\cdot|^2)^{\frac{s}{2}} \mathcal{F}u), \quad \text{for all } u \in C_c^\infty(\mathbb{R}^N).$$

Let us stress that, in the case where $s \in (0,1)$ and $1 < p < +\infty$,

$$\|u\|_{L^{s,p}(\mathbb{R}^N)} := \|u\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)}$$

is an equivalent norm for $L^{s,p}(\mathbb{R}^N)$ (see e.g. [25, Theorem 2]). By [25, Theorem 1.1], we also know that, if in addition $2N/(N+2s) < p < +\infty$, then $L^{s,p}(\mathbb{R}^N)$ can be equipped with the equivalent norm

$$|||u|||_{L^{s,p}(\mathbb{R}^N)} := \|u\|_{L^p(\mathbb{R}^N)} + \|\mathcal{D}_s u\|_{L^p(\mathbb{R}^N)}.$$

In analogy with $W_0^{s,p}(\Omega)$, let us define

$$L_0^{s,p}(\Omega) := \{u \in L^{s,p}(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega\},$$

and stress that, if $0 < s < 1$ and $1 < p < +\infty$, it is a Banach space endowed with the norm

$$\|u\|_{L_0^{s,p}(\Omega)} := \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^N)}.$$

If in addition $2N/(N+2s) < p < +\infty$, then $L_0^{s,p}(\Omega)$ can also be equipped with the equivalent norm

$$|||u|||_{L_0^{s,p}(\Omega)} := \|\mathcal{D}_s(u)\|_{L^p(\mathbb{R}^N)}.$$

Let us as well recall that, for all $0 < \epsilon < \sigma < 1$ and all $1 < p < +\infty$, by [5, Theorem 7.63, (g)], we have

$$L^{\sigma+\epsilon,p}(\mathbb{R}^N) \subset W^{\sigma,p}(\mathbb{R}^N) \subset L^{\sigma-\epsilon,p}(\mathbb{R}^N).$$

It is also well known that, for all $1 \leq p < +\infty$ and all $0 < \sigma \leq \sigma' < 1$,

$$W^{\sigma',p}(\mathbb{R}^N) \subset W^{\sigma,p}(\mathbb{R}^N),$$

and that (cf. [5, Theorem 7.63 (c)]), if in addition $1 < p < +\infty$,

$$L^{\sigma',p}(\mathbb{R}^N) \subset L^{\sigma,p}(\mathbb{R}^N).$$

Since the constants will be useful later on, let us emphasize there exists $\widetilde{k} := \widetilde{k}(\sigma, \sigma', p) \geq 1$ such that

$$\|u\|_{L^{\sigma,p}(\mathbb{R}^N)} \leq \widetilde{k} \|u\|_{L^{\sigma',p}(\mathbb{R}^N)}, \quad \text{for all } u \in L^{\sigma',p}(\mathbb{R}^N), \tag{2.1}$$

and

$$\|u\|_{W^{\sigma,p}(\mathbb{R}^N)} \leq \widetilde{k} \|u\|_{W^{\sigma',p}(\mathbb{R}^N)}, \quad \text{for all } u \in W^{\sigma',p}(\mathbb{R}^N), \tag{2.2}$$

Now, for $0 < s \le t < \min\{1, s(1+N^{-1})\}$, let us set[1]

$$\widetilde{p}\,(m,s,t) := \begin{cases} \dfrac{1}{(t-s)^+}, & \text{if } m > \dfrac{N}{2s-t}, \\ \min\left\{\dfrac{mN}{N-ms+mN(t-s)}, \dfrac{1}{(t-s)^+}\right\}, & \text{if } 1 \le m < \dfrac{N}{2s-t}. \end{cases}$$

In our next result, which will be very useful in the sequel, we describe the regularity of the (unique) solution to (1.3). Note that such a result is contained in [3, Theorems 1.3 and 5.2, Corollaries 5.3 – 5.6].

Proposition 2.1 *Let $0 < s \le t < \min\{1, s(1+N^{-1})\}$ and let u be the (unique) weak solution to (1.3) with $h \in L^m(\Omega)$ for some $m \ge 1$. Then, for all $1 < p < \widetilde{p}$, there exists $\widetilde{C}(N, s, p, m, \Omega) > 0$ such that*

$$\|u\|_{L^{t,p}(\mathbb{R}^N)} \le \widetilde{C}\, \|h\|_{L^m(\Omega)} \quad \text{and} \quad \|u\|_{W^{t,p}(\mathbb{R}^N)} \le \widetilde{C}\, \|h\|_{L^m(\Omega)}. \tag{2.3}$$

We also present here a technical but useful lemma proved in [3, Lemma 5.1] and a classical result from harmonic analysis.

Lemma 2.2 *Let $s \in (0,1)$, $s \le t < \min\{1, 2s\}$ and and let u be the (unique) weak solution to (1.3) with $h \in L^1(\Omega)$. Then, there exists $C := C(N, s, t, \Omega) > 0$ such that*

$$\left|(-\Delta)^{\frac{t}{2}} u(x)\right| \le C \left[g_1(x) + |\log \delta(x)| g_2(x) + \frac{1}{\delta^{t-s}(x)} g_3(x) \right], \quad \textit{for a.e. } x \in \Omega. \tag{2.4}$$

Here, $\delta(x) := \operatorname{dist}(x, \partial\Omega)$ and the functions g_i, $i = 1, 2, 3$, satisfy:

- *For all $0 < \lambda < 2s - t$, there exists $C := C(\lambda) > 0$ such that*
$$g_1(x) \le C \int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t-\lambda)}} dy, \tag{2.5}$$
- $$g_2(x) := \int_\Omega \frac{|f(y)|}{|x-y|^{N-(2s-t)}} dy, \tag{2.6}$$
- $$g_3(x) := (t-s) \int_\Omega \frac{|f(y)|}{|x-y|^{N-s}} dy. \tag{2.7}$$

Moreover, for any $R \ge \frac{1}{3} + \frac{4}{3}(\operatorname{diam}(\Omega) + \operatorname{dist}(0, \Omega))$, it follows that

$$\left|(-\Delta)^{\frac{t}{2}} u(x)\right| \le \int_\Omega \frac{|u(y)|}{\delta^s(y)} \frac{dy}{|x-y|^{N+(t-s)}}, \quad \textit{for a.e. } x \in B_R(0) \setminus \Omega, \tag{2.8}$$

and

$$\left|(-\Delta)^{\frac{t}{2}} u(x)\right| \le \frac{4^{N+t}}{(1+|x|)^{N+t}} \int_\Omega |u(y)| dy, \quad \textit{for a.e. } x \in \mathbb{R}^N \setminus B_R(0). \tag{2.9}$$

Lemma 2.3 *Let $\omega \subset \mathbb{R}^N$, $N \ge 2$, be an open bounded domain and let $0 < \alpha < N$ and $1 \le p < \ell < \infty$ be such that $\dfrac{1}{\ell} = \dfrac{1}{p} - \dfrac{\alpha}{N}$. Moreover, for $g \in L^p(\omega)$, let*

$$J_\lambda(g)(x) := \int_\omega \frac{g(y)}{|x-y|^{N-\alpha}} dy\,.$$

It follows that there exists $C = C(N, \alpha, p, \sigma > 0, \ell, \omega) > 0$ such that

[1] We use, here and in the sequel, the convention $1/a^+ = +\infty$ if $a \le 0$.

a) J_α *is well defined (in the sense that the integral converges absolutely for a.e.* $x \in \omega$*).*
b) $[J_\alpha(g)]_{M^\ell(\omega)} \le C\,\|g\|_{L^1(\omega)}$*. In particular,* $\|J_\alpha(g)\|_{L^\sigma(\omega)} \le C\,\|g\|_{L^1(\omega)}$ *for all* $1 \le \sigma < \ell$*.*
c) If $1 < p < \frac{N}{\alpha}$*, then* $\|J_\alpha(g)\|_{L^\ell(\omega)} \le C\,\|g\|_{L^p(\omega)}$*.*
d) If $p = \frac{N}{\alpha}$*, then* $\|J_\alpha(g)\|_{L^\sigma(\omega)} \le C\,\|g\|_{L^p(\omega)}$ *for all* $1 \le \sigma < +\infty$*.*
e) If $p > \frac{N}{\alpha}$*, then* $\|J_\alpha(g)\|_{L^\infty(\omega)} \le C\,\|g\|_{L^p(\omega)}$*.*

Proof Parts a), b), c) and d) follow from [26, Theorem I, Section 1.2, Chapter V, page 119]. Part e) is contained in [12, Lemma 7.12] (see also [17, Theorem 2.2]). □

We conclude this section proving a compactness result for the fractional Poisson equation (1.3) that will be key in the proof of our main existence results. Here, we denote by $G_s : \mathbb{R}^{2N}_* \to \mathbb{R}$ the Green function associated to $(-\Delta)^s$ in Ω with homogeneous Dirichlet boundary conditions. Note that $\mathbb{R}^{2N}_* := \{(x, y) \in \mathbb{R}^{2N} : x \ne y\}$.

Proposition 2.4 *Let* $0 < t < \min\{1, s(1 + N^{-1})\}$ *and* $1 < p < N/(N(1 + t - s) - s)$*. The solution map*

$$\mathbb{G}_s : L^1(\Omega) \to L^{t,p}_0(\Omega), \qquad h \mapsto \mathbb{G}_s[h] := \int_\Omega G_s(x, y)h(y)\,dy,$$

is well-defined, continuous and compact.

Proof First of all, note that without loss of generality we can assume that $s \le t < \min\{1, s(1+N^{-1})\}$. Indeed, once we have the result in this case, we can infer the result for $t \in (0, s)$ interpolating as in the proof of [2, Proposition 3.10]. Let us also stress that, by Proposition 2.1, the solution map $\mathbb{G}_s$ is well-defined and continuous for all $1 < p < N/(N(1+t-s)-s)$. Hence, we just have to show that $\mathbb{G}_s$ is compact for the same range of p.

Let $(h_n)_n \subset L^1(\Omega)$ be a sequence such that $\|h_n\|_{L^1(\Omega)} \le 1$ for all $n \in \mathbb{N}$ and let $u_n = \mathbb{G}_s(h_n)$ for all $n \in \mathbb{N}$. By [8, Proposition 2.6] we know that, up to a subsequence, $(u_n)_n$ is strongly convergent in $L^q(\Omega)$ for all $1 \le q < N/(N - 2s)$. Moreover, combining this strong convergence with Vitali's convergence theorem, we deduce that, up to a subsequence, $(u_n/\delta^s)_n$ is strongly convergent in $L^r(\Omega)$ for all $1 \le r < N/(N - s)$. Having at hand the (up to a subsequence) strong convergence of $(u_n)_n$ in $L^q(\Omega)$ for all $1 \le q < N/(N - 2s)$, to end the proof, we just have to show that, up to a subsequence, $((-\Delta)^{\frac{t}{2}} u_n)_n$ is strongly convergent in $L^p(\mathbb{R}^N)$ for all $1 < p < N/(N(1 + t - s) - s)$.

First, for $0 < \alpha < 2$, let us consider the integral operator

$$T_\alpha : L^1(\Omega) \to L^\gamma(\Omega), \qquad g \mapsto \int_{\mathbb{R}^N} \frac{g(y)\mathbb{1}_\Omega(y)}{|x - y|^{N-\alpha}}\,dy.$$

Note that, by [12, Lemma 7.12], T_α is well-defined and continuous for all $1 \le \gamma < N/(N - \alpha)$. Moreover, following step by step the proof of [24, Theorem 2.2], one can prove that T_α is compact for all $1 \le \gamma < N/(N - \alpha)$. We only sketch the proof. Let $(g_n)_n \subset L^1(\Omega)$ be a sequence such that $\|g_n\|_{L^1(\Omega)} \le 1$ for all $n \in \mathbb{N}$, $v_n := T_\alpha(g_n)$ for all $n \in \mathbb{N}$ and $1 \le \gamma < N/(N - \alpha)$. If we prove the existence of a (not relabelled) subsequence $(v_n)_n$ that is Cauchy in $L^\gamma(\Omega)$, the compactness immediately follows. Let η_ϵ be a standard mollifier and let $v_n^\epsilon := v_n \star \eta_\epsilon$ for all $\epsilon > 0$ and all $n \in \mathbb{N}$. Following [24, Theorem 2.2], we obtain that, for all $1 \le \gamma < N/(N - \alpha)$, there exist constants $C > 0$ and $\sigma > 0$ (independent of ϵ and n) such that

$$\|v_n^\epsilon - v_n\|_{L^\gamma(\Omega)} \le C\epsilon^\sigma. \tag{2.10}$$

On the other hand, using the standard properties of the mollifiers η_ϵ, we get that, for any $\epsilon > 0$, the sequence $(v_n^\epsilon)_n$ is bounded and equicontinuous in $C(\mathbb{R}^N)$. Thus, Arzelà-Ascoli Theorem implies the existence of a subsequence uniformly convergent in $\overline{\Omega}$. Combining (2.10) with the uniform convergence in $\overline{\Omega}$, a standard diagonal argument shows the existence of a (not relabelled) Cauchy subsequence $(v_n)_n$ in $L^\gamma(\Omega)$, as desired.

Having at hand the compactness of T_α and Lemma 2.2 we now prove that, up to a subsequence, $((-\Delta)^{\frac{t}{2}} u_n)_n$ is strongly convergent in $L^p(\mathbb{R}^N)$ for all $1 \le p < N/\big(N(1+t-s)-s\big)$. To that end, we fix $R \ge \frac{1}{3} + \frac{4}{3}\big(\operatorname{diam}(\Omega) + \operatorname{dist}(0, \Omega)\big)$ as in Lemma 2.2 and split $\mathbb{R}^N$ in three regions: Ω, $B_R(0) \setminus \Omega$ and $\mathbb{R}^N \setminus B_R(0)$.

Combining (2.4) with the linearity of the problem (1.3), the compactness of T_α and Hölder inequality, we get that, up to a subsequence, $((-\Delta)^{\frac{t}{2}} u_n)_n$ is strongly convergent in $L^p(\Omega)$ for all $1 \le p < N/\big(N(1+t-s)-s\big)$.

Next, we deal with the strong convergence of $((-\Delta)^{\frac{t}{2}} u_n)_n$ in $L^p(B_R(0) \setminus \Omega)$ for all $1 \le p < N/(N(1+t-s)-s)$. Since $|x-y| \ge \max\{\delta(y), \delta(x)\}$ for all $y \in \Omega$ and $x \in B_R(0) \setminus \Omega$, by (2.8) we have that, for all $\epsilon > 0$,

$$\begin{aligned}\big|(-\Delta)^{\frac{t}{2}} u_n(x)\big| &\le \int_\Omega \frac{|u_n(y)|}{|x-y|^{N+t}} dy \\ &\le \frac{1}{\delta^{t-s+\epsilon}(x)} \int_\Omega \frac{|u_n(y)|}{\delta^s(y)} \frac{dy}{|x-y|^{N-\epsilon}} \quad \text{for a.e. } x \in \mathbb{R}^N \setminus B_R(0)\,.\end{aligned}$$

Combining this inequality with Lemma 2.3, Hölder inequality and the (up to a subsequence) strong convergence of $(u_n/\delta^s)_n$ in $L^r(\Omega)$ for all $1 \le r < N/(N-s)$, we get the desired convergence in $L^p(B_R(0) \setminus \Omega)$.

Finally, the (up to a subsequence) strong convergence of $((-\Delta)^{\frac{t}{2}} u_n)_n$ in $L^p(\mathbb{R}^N \setminus B_R(0))$ for all $1 \le p < N/(N(1+t-s)-s)$ follows from (2.9). Indeed, combining (2.9) with the (up to a subsequence) strong convergence of $(u_n)_n$ in $L^q(\Omega)$ for all $1 \le q < N/(N-2s)$, we get the desired convergence in $L^p(\mathbb{R}^N \setminus B_R(0))$. □

Corollary 2.5 *Let* $0 < t < \min\{1, s(1+N^{-1})\}$ *and* $1 < p < N/(N(1+t-s)-s)$. *The solution map*

$$\mathbb{G}_s : L^1(\Omega) \to W_0^{t,p}(\Omega), \qquad h \mapsto \mathbb{G}_s[h] := \int_\Omega G_s(x,y) h(y)\, dy,$$

is well-defined, continuous and compact.

Proof Having at hand Proposition 2.4 the result follows arguing as in the proof of [3, Corollary 5.6] □

Remark 2.6
- In the case where $1/2 < t < \min\{1, s(1+N^{-1})\}$, Proposition 2.4 and Corollary 2.5 can be proved arguing as in the proof of [2, Proposition 3.10].
- We believe Proposition 2.4 and Corollary 2.5 are of independent interest and will be useful elsewhere.

3 Existence results

This section is devoted to prove existence results for (KPZ) with the different choices of $\mathbb{D}_t$ present in the introduction. We will analyse in parallel $\mathbb{D}_t = (-\Delta)^{\frac{t}{2}}$ and $\mathbb{D}_t = \nabla^t$ and separately $\mathbb{D}_t = \mathcal{D}_t$. Let us emphasize that the main existence result stated in the introduction, namely Theorem 1.3, immediately follows from the results of this section.

We first analyse the case $\mathbb{D}_t = (-\Delta)^{\frac{t}{2}}$ (the case $\mathbb{D}_t = \nabla^t$ follows arguing on the exact same way, as we will detail later on). More precisely, for $q \in (1, +\infty)$, we analyse the existence of weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)|(-\Delta)^{\frac{t}{2}} u|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{KPZ$_1$}$$

Let us impose $0 < t < \min\{1, s(1 + N^{-1})\}$ and set

$$\overline{q}(m, s, t) := \begin{cases} +\infty, & \text{if } t \le s \text{ and } m \ge N/s, \\ s/(N(t-s)), & \text{if } t > s \text{ and } m > N/s, \\ N/(N - ms), & \text{if } t \le s \text{ and } 1 \le m < N/s, \\ N/(N - sm + mN(t-s)), & \text{if } t > s \text{ and } 1 \le m \le N/s. \end{cases} \tag{3.1}$$

Having at hand $\overline{q} \in (1, +\infty]$, our main result concerning (KPZ$_1$) reads as follows:

Theorem 3.1 *Assume that $0 < t < \min\{1, s(1 + N^{-1})\}$, $f \in L^m(\Omega)$ for some $m \ge 1$ and $\mu \in L^\infty(\Omega)$. Then, for all $1 < q < \overline{q}$, there exists $\lambda_\star > 0$ such that, for all $0 < \lambda \le \lambda_\star$, (KPZ$_1$) has a weak solution u. Moreover:*

- *If $m \ge N/s$, then $u \in W^{s,p}(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$ for all $1 < p < +\infty$.*
- *If $1 \le m < N/s$, then $u \in W^{s,p}(\mathbb{R}^N)$ for all $1 < p < mN/(N - ms)$.*

Proof of Theorem 3.1 We use some ideas of [2, Sections 4 and 6] and consider separately the cases $m > N/s$ and $1 \le m \le N/s$.

Case 1: $m > N/s$.

First of all, observe that, without loss of generality, we can assume that $N/s < m < 1/(q(t-s)^+)$. Then, let us fix $r = r(m, s, t, q) > 0$ such that $1 < qm < r < \frac{1}{(t-s)^+}$ and define

$$\lambda_\star := \frac{q-1}{q\|f\|_{L^m(\Omega)}} \left(q(\widetilde{C}\widetilde{k})^q \|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}} \right)^{-\frac{1}{q-1}}, \tag{3.2}$$

with $\widetilde{C} > 0$ as in Proposition 2.1 and $\widetilde{k}$ as in (2.1). Since $q > 1$, we know (cf. [2, Lemma 4.1]) there exists a unique $\ell \in (0, \infty)$ such that

$$\widetilde{C}\left(\|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}} \widetilde{k}^q \ell + \lambda_\star \|f\|_{L^m(\Omega)} \right) = \ell^{\frac{1}{q}}. \tag{3.3}$$

Having at hand $\lambda_\star$ and ℓ, we define

$$E_\eta := \left\{ v \in L_0^{\gamma, 1+\eta}(\Omega) : \|v\|_{L_0^{\gamma, r}(\Omega)} \le \ell^{\frac{1}{q}} \right\}, \tag{3.4}$$

with

$$\gamma := \max\{t, s\}, \quad \text{and } 0 < \eta < \min\left\{ q - 1, \frac{s - N(\gamma - s)}{N(1 + (\gamma - s)) - s} \right\}, \tag{3.5}$$

and point out that E_η is a closed and convex subset of $L_0^{\gamma, 1+\eta}(\Omega)$. Moreover E_η is also bounded in $L_0^{\gamma, 1+\eta}(\Omega)$. Indeed, for any $R > 0$, we have that

$$\left\|(-\Delta)^{\frac{\gamma}{2}} u\right\|_{L^{1+\eta}(\mathbb{R}^N)} \le \left\|(-\Delta)^{\frac{\gamma}{2}} u\right\|_{L^{1+\eta}(B_R(0))} + \left\|(-\Delta)^{\frac{\gamma}{2}} u\right\|_{L^{1+\eta}(\mathbb{R}^N \setminus B_R(0))}. \tag{3.6}$$

Then, observe that, since $r > 1 + \eta$,

$$\big\|(-\Delta)^{\frac{\gamma}{2}} u\big\|_{L^{1+\eta}(B_R(0))} \leq C_R \big\|(-\Delta)^{\frac{\gamma}{2}} u\big\|_{L^r(B_R(0))} \leq C_R \big\|(-\Delta)^{\frac{\gamma}{2}} u\big\|_{L^r(\mathbb{R}^N)} \leq C_R\, \ell^{\frac{1}{q}}. \tag{3.7}$$

On the other hand, choosing $R \geq \frac{1}{3} + \frac{4}{3}\big(\operatorname{diam}(\Omega) + \operatorname{dist}(0, \Omega)\big)$, we have that $|x - y| \geq \frac{1}{4}\big(1 + |x|\big)$ for all $y \in \Omega$ and all $x \in \mathbb{R}^N \setminus B_R(0)$ and thus, we get that

$$\begin{aligned} \big\|(-\Delta)^{\frac{\gamma}{2}} u\big\|^{1+\eta}_{L^{1+\eta}(\mathbb{R}^N\setminus B_R(0))} &\leq \int_{\mathbb{R}^N\setminus B_R(0)} \left|\int_\Omega \frac{|u(y)|}{|x-y|^{N+\gamma}} dy\right|^{1+\eta} dx \\ &\leq C\|u\|^{1+\eta}_{L^{1+\eta}(\Omega)} \int_{\mathbb{R}^N\setminus B_R(0)} \frac{dx}{(1+|x|)^{(N+\gamma)(1+\eta)}} \leq \overline{C}\, \|u\|^{1+\eta}_{L^{1+\eta}(\Omega)} \leq \widetilde{C}\, \ell^{\frac{1+\eta}{q}}. \end{aligned} \tag{3.8}$$

Note that the last inequality follows from the fractional Sobolev inequality (see for instance [23, Theorem 1.8]) and the definition of E_η. Gathering (3.6)–(3.8) the boundedness of E_η in $L^{\gamma,1+\eta}_0(\Omega)$ follows.

To prove the existence of a weak solution to (KPZ_1) belonging to E_η, we use Schauder's fixed point Theorem. Let us consider

$$T_1 : E_\eta \to L^{\gamma,1+\eta}_0(\Omega), \quad \varphi \mapsto u, \tag{3.9}$$

where u is the unique weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)|(-\Delta)^{\frac{t}{2}}\varphi|^q + \lambda f(x)\,, & \text{in } \Omega, \\ \qquad u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.10}$$

and observe that, if we prove that T_1 has a fixed point in E_η, the existence part immediately follows. Note that, by Proposition 2.4, the operator T_1 is well defined. Hence, to end the proof in this case, we just have to prove that T_1 is continuous and compact and that $T_1(E_\eta) \subset E_\eta$.

We start proving that $T_1(E_\eta) \subset E_\eta$. Let $\varphi \in E_\eta$ and $u = T_1(\varphi)$. Using Proposition 2.1, it is immediate to see that $u \in L^{\gamma,1+\eta}_0(\Omega)$ and that

$$\begin{aligned} \|u\|_{L^{\gamma,r}_0(\Omega)} &\leq \widetilde{C}\Big(\lambda_\star\|f\|_{L^m(\Omega)} + \|\mu\|_{L^\infty(\Omega)}\||(-\Delta)^{\frac{t}{2}}\varphi|^q\|_{L^m(\Omega)}\Big) \\ &\leq \widetilde{C}\Big(\lambda_\star\|f\|_{L^m(\Omega)} + \|\mu\|_{L^\infty(\Omega)}|\Omega|^{\frac{r-qm}{rm}}\,\|(-\Delta)^{\frac{t}{2}}\varphi\|^q_{L^r(\Omega)}\Big) \\ &\leq \widetilde{C}\Big(\lambda_\star\|f\|_{L^m(\Omega)} + \|\mu\|_{L^\infty(\Omega)}|\Omega|^{\frac{r-qm}{rm}}\,\|\varphi\|^q_{L^{t,r}_0(\Omega)}\Big) \\ &\leq \widetilde{C}\Big(\lambda_\star\|f\|_{L^m(\Omega)} + \|\mu\|_{L^\infty(\Omega)}|\Omega|^{\frac{r-qm}{rm}}\,\widetilde{k}^q\,\|\varphi\|^q_{L^{\gamma,r}_0(\Omega)}\Big) \leq \ell^{\frac{1}{q}}\,. \end{aligned} \tag{3.11}$$

Hence, it follows that $T_1(E_\eta) \subset E_\eta$.

Next, we prove that T_1 is compact. Let $(\varphi_n)_n \subset E_\eta$ be such that $\|\varphi_n\|_{L^{\gamma,1+\eta}_0(\Omega)} \leq 1$ for all $n \in \mathbb{N}$. Also, let $h_n := \mu(x)|(-\Delta)^{\frac{t}{2}}\varphi_n|^q + \lambda f(x)$ for all $n \in \mathbb{N}$. Arguing as in (3.11), it is immediate to check that $(h_n)_n$ is bounded in $L^1(\Omega)$ and thus, the compactness of T_1 immediately follows from Proposition 2.4.

Finally, we prove that T_1 is continuous. Let $(\varphi_n)_n \subset E_\eta$ be a sequence such that $\varphi_n \to \varphi$ in $L_0^{\gamma,1+\eta}(\Omega)$ and let $u_n = T_1(\varphi_n)$ for all $n \in \mathbb{N}$ and $u = T_1(\varphi)$. Note that

$$\begin{cases} (-\Delta)^s(u_n - u) = \mu(x)\Big(|(-\Delta)^{\frac{t}{2}}\varphi_n|^q - |(-\Delta)^{\frac{t}{2}}\varphi|^q\Big), & \text{in } \Omega, \\ \qquad u_n - u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.12}$$

If we show that the L^1–norm of the right hand side in the (3.12) goes to 0 as $n \to \infty$, the continuity of T_1 immediately follows from Proposition 2.4. By direct computations (using Hölder inequality and the Mean value Theorem), it follows that

$$\Big\|\mu(x)\Big(|(-\Delta)^{\frac{t}{2}}\varphi_n|^q - |(-\Delta)^{\frac{t}{2}}\varphi|^q\Big)\Big\|_{L^1(\Omega)} \le C\,\|(-\Delta)^{\frac{t}{2}}(\varphi_n - \varphi)\|_{L^q(\Omega)}, \tag{3.13}$$

for some $C > 0$ depending only on $q, r, \|\mu\|_{L^\infty(\Omega)}, |\Omega|$ and ℓ. On the other hand, using (2.1) and Littlewood's inequality (or interpolation in L^p–spaces), we infer that

$$\begin{aligned} \|(-\Delta)^{\frac{t}{2}}(\varphi_n - \varphi)\|_{L^q(\Omega)} &\le \|\varphi_n - \varphi\|_{L_0^{t,q}(\Omega)} \\ &\le \widetilde{k}\|\varphi_n - \varphi\|_{L_0^{\gamma,q}(\Omega)} = \widetilde{k}\|(-\Delta)^{\frac{\gamma}{2}}(\varphi_n - \varphi)\|_{L^q(\mathbb{R}^N)} \\ &\le \widetilde{k}\|(-\Delta)^{\frac{\gamma}{2}}(\varphi_n - \varphi)\|^\theta_{L^{1+\eta}(\mathbb{R}^N)}\|(-\Delta)^{\frac{\gamma}{2}}(\varphi_n - \varphi)\|^{1-\theta}_{L^r(\mathbb{R}^N)} \\ &= \widetilde{k}\|\varphi_n - \varphi\|^\theta_{L_0^{\gamma,1+\eta}(\Omega)}\|\varphi_n - \varphi\|^{1-\theta}_{L_0^{\gamma,r}(\Omega)} \\ &\le (2\ell^{\frac{1}{q}})^{1-\theta}\widetilde{k}\|\varphi_n - \varphi\|^\theta_{L_0^{\gamma,1+\eta}(\Omega)}, \end{aligned} \tag{3.14}$$

with $\frac{1}{q} = \frac{\theta}{1+\eta} + \frac{1-\theta}{r}$. Since $\varphi_n \to \varphi$ in $L_0^{\gamma,1+\eta}(\Omega)$, combining (3.13) and (3.14), we conclude that

$$\lim_{n\to\infty}\Big\|\mu\Big(|(-\Delta)^{\frac{t}{2}}\varphi_n|^q - |(-\Delta)^{\frac{t}{2}}\varphi|^q\Big)\Big\|_{L^1(\Omega)} = 0,$$

as desired. The proof of the existence in the case where $m > N/s$ is thus finished. Once we have the existence of a weak solution $u \in E_\eta$, the claimed regularity immediately follows from the definition of E_η, our choice of r, Proposition 2.1 and [20, Proposition 1.4 (iii)].

Case 2: $1 \le m \le N/s$.

First of all, let us fix $r = r(m, s, t, q) > 0$ such that $1 < qm < r < \frac{mN}{(N-ms+mN(\gamma-s))^+}$ and consider $\lambda_\star$ and ℓ as in (3.2) and (3.3) respectively. Then, we define

$$\widetilde{E}_\eta := \Big\{v \in L_0^{\gamma,1+\eta}(\Omega) : \|v\|_{L_0^{\gamma,1+\eta}(\Omega)} \le M \text{ and } \|v\|_{L_0^{\gamma,r}(\Omega)} \le \ell^{\frac{1}{q}}\Big\}, \tag{3.15}$$

with γ and η as in (3.5) and

$$M := \tilde{C}_1\Big(\|\mu\|_{L^\infty(\Omega)}|\Omega|^{\frac{r-q}{r}}\widetilde{k}^q\ell + \lambda_\star\|f\|_{L^1(\Omega)}\Big),$$

where $\tilde{C}_1 > 0$ is the constant that appears in Proposition 2.1 for $m = 1$. It is immediate to see that $\widetilde{E}_\eta$ is a bounded, closed and convex set of $L_0^{\gamma,1+\eta}(\Omega)$. Moreover, arguing as in the first case, one can prove that

$$\widetilde{T}_1 : \widetilde{E}_\eta \to L_0^{\gamma,1+\eta}(\Omega), \quad \varphi \mapsto u, \tag{3.16}$$

where u is the unique weak solution to (3.10), is well-defined, continuous, compact and satisfies $\widetilde{T}_1(\widetilde{E}_\eta) \subset \widetilde{E}_\eta$. Hence, applying again Schauder's fixed point Theorem, the existence follows also in this case. Having at hand the existence, the claimed regularity follows again from Proposition 2.1. □

Next, we analyse the existence of weak solution to (KPZ) in the case where $\mathbb{D}_t = \nabla^t$. More precisely, we deal with the existence of weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)|\nabla^t u|^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{KPZ$_2$}$$

Our main result concerning (KPZ$_2$) can be formulated as follows:

Theorem 3.2 *Assume that* $0 < t < \min\{1, s(1 + N^{-1})\}$*,* $f \in L^m(\Omega)$ *for some* $m \geq 1$ *and* $\mu \in L^\infty(\Omega)$*. Then, for all* $1 < q < \overline{q}$*, there exists* $\lambda_\star > 0$ *such that, for all* $0 < \lambda \leq \lambda_\star$*, (KPZ$_2$) has a weak solution u. Moreover:*

- *If* $m \geq N/s$*, then* $u \in W^{s,p}(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$ *for all* $1 < p < +\infty$*.*
- *If* $1 \leq m < N/s$*, then* $u \in W^{s,p}(\mathbb{R}^N)$ *for all* $1 < p < mN/(N - ms)$*.*

Remark 3.3 Let us emphasize that, having at hand [23, Theorem 1.7], the proof of Theorem 3.2 follows arguing exactly as in the proof of Theorem 3.1.

Finally, we analyse the existence of weak solution in the slightly more involved case where $\mathbb{D}_t = \mathcal{D}_t$. More precisely, we analyse the existence of weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)(\mathcal{D}_t u)^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{KPZ$_3$}$$

Setting $\gamma := \max\{t, s\}$ and

$$\overline{m}(s,t) := \frac{2N}{N + 2s - 2N(t-s)^+}$$

our main result concerning (KPZ$_3$) reads as follows:

Theorem 3.4 *Assume that* $0 < t < \min\{1, s(1 + N^{-1})\}$*,* $f \in L^m(\Omega)$ *for some* $m > \overline{m}$ *and* $\mu \in L^\infty(\Omega)$*. Then, for all* $1 < q < \overline{q}$*, there exists* $\lambda_\star > 0$ *such that, for all* $0 < \lambda \leq \lambda_\star$*, (KPZ$_3$) has a weak solution u. Moreover:*

- *If* $m \geq N/s$*, then* $u \in W^{s,p}(\mathbb{R}^N) \cap C^{0,s}(\mathbb{R}^N)$ *for all* $1 < p < +\infty$*.*
- *If* $\overline{m} < m < N/s$*, then* $u \in W^{s,p}(\mathbb{R}^N)$ *for all* $1 < p < mN/(N - ms)$*.*

Proof We consider separately the cases $m > N/s$ and $\overline{m} < m \leq N/s$.

Case 1: $m > N/s$.

First of all, observe that, without loss of generality, we can assume that $N/s < m < 1/(q(t-s)^+)$. Then, let us fix $r = r(m,s,t,q) > 0$ such that $\max\{qm, 2\} < r < \frac{1}{(t-s)^+}$ and define

$$\lambda_\star := \frac{q-1}{q\|f\|_{L^m(\Omega)}} \left(q(\widetilde{C}\widetilde{k})^q \|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}} \right)^{-\frac{1}{q-1}}, \tag{3.17}$$

with $\widetilde{C} > 0$ as in Proposition 2.1 and $\widetilde{k}$ as in (2.1). Since $q > 1$, we know (cf. [2, Lemma 4.1]) there exists a unique $\ell \in (0, \infty)$ such that

$$\widetilde{C} \left(\|\mu\|_{L^\infty(\Omega)} |\Omega|^{\frac{r-qm}{mr}} \widetilde{k}^q \ell + \lambda_\star \|f\|_{L^m(\Omega)} \right) = \ell^{\frac{1}{q}}. \tag{3.18}$$

Having at hand $\lambda_\star$ and ℓ, we define

$$E_\eta := \left\{ v \in L_0^{\gamma,1+\eta}(\Omega) : |||v|||_{L_0^{\gamma,r}(\Omega)} \le \ell^{\frac{1}{q}} \right\}, \tag{3.19}$$

with

$$\gamma = \max\{t, s\} \quad \text{and} \quad 0 < \eta < \min\left\{ q-1, \frac{s - N(\gamma - s)}{N(1+(\gamma-s)) - s} \right\}, \tag{3.20}$$

and we point out that E_η is a closed, bounded and convex subset of $L_0^{\gamma,1+\eta}(\Omega)$. To prove the existence of a weak solution to (KPZ_3) belonging to E_η, we use again Schauder's fixed point Theorem. Let us define

$$T_2 : E_\eta \to L_0^{\gamma,1+\eta}(\Omega), \quad \varphi \mapsto u, \tag{3.21}$$

where u the unique weak solution to

$$\begin{cases} (-\Delta)^s u = \mu(x)(\mathcal{D}_t\varphi)^q + \lambda f(x), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.22}$$

Note that, by Proposition 2.4, the operator T_2 is well defined. Hence, to conclude the proof in this case, we just have to prove that T_2 is continuous and compact and that $T_2(E_\eta) \subset E_\eta$. The compactness of T_2 and the fact that $T_2(E_\eta) \subset E_\eta$ can be proved arguing exactly as in the proof of Theorem 3.1. However, to prove that T_2 is continuous we have to argue in a different way. Let $(\varphi_n)_n \subset E_\eta$ be a sequence such that $\varphi_n \to \varphi$ in $L_0^{\gamma,1+\eta}(\Omega)$ and let $u_n = T_2(\varphi_n)$ for all $n \in \mathbb{N}$ and $u = T_2(\varphi)$. Note that

$$\begin{cases} (-\Delta)^s (u_n - u) = \mu(x)\Big((\mathcal{D}_t\varphi_n)^q - (\mathcal{D}_t\varphi)^q\Big), & \text{in } \Omega, \\ u_n - u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.23}$$

If we show that the L^1–norm of the right hand side in (3.23) goes to 0 as $n \to \infty$, the continuity of T_2 immediately follows from Proposition 2.4. We actually prove something more general from which the existence part of the result in the case $m > N/s$ immediately follows.

Claim *For all $1 < \alpha < r$, it follows that*

$$\lim_{n\to\infty} \int_\Omega \left|(\mathcal{D}_t\varphi_n(x))^\alpha - (\mathcal{D}_t\varphi(x))^\alpha\right| dx = 0. \tag{3.24}$$

Proof of the claim. First of all, let $2 \le \beta < r$ be fixed but arbirary. Using the Mean value Theorem and Hölder and triangular inequalities, one can easily get that

$$\begin{aligned} \int_\Omega |(\mathcal{D}_t\varphi_n(x))^\beta - (\mathcal{D}_t\varphi(x))^\beta| dx &= \int_\Omega \left|\big((\mathcal{D}_t\varphi_n(x))^2\big)^{\frac{\beta}{2}} - \big((\mathcal{D}_t\varphi(x))^2\big)^{\frac{\beta}{2}}\right| dx \\ &\le \frac{\beta}{2} \int_\Omega \left|(\mathcal{D}_t\varphi_n(x))^2 - (\mathcal{D}_t\varphi(x))^2\right| \big((\mathcal{D}_t\varphi_n(x))^2 + (\mathcal{D}_t\varphi(x))^2\big)^{\frac{\beta-2}{2}} dx \\ &\le \frac{\beta}{2} \int_\Omega (\mathcal{D}_t(\varphi_n - \varphi)(x))(\mathcal{D}_t(\varphi_n + \varphi)(x))\big((\mathcal{D}_t\varphi_n(x))^2 + (\mathcal{D}_t\varphi(x))^2\big)^{\frac{\beta-2}{2}} dx \\ &\le \frac{\sqrt{2}}{2}\beta \int_\Omega (\mathcal{D}_t(\varphi_n - \varphi)(x))\big(\mathcal{D}_t\varphi_n(x) + \mathcal{D}_t\varphi(x)\big)^{\beta-1} dx \\ &\le \frac{\sqrt{2}}{2}\beta \|\mathcal{D}_t(\varphi_n - \varphi)\|_{L^\beta(\Omega)} \|\mathcal{D}_t\varphi_n + \mathcal{D}_t\varphi\|_{L^\beta(\Omega)}^{\beta-1} \end{aligned}$$

$$\leq C\big(\|\mathcal{D}_t\varphi_n\|^{\beta-1}_{L^\beta(\Omega)} + \|\mathcal{D}_t\varphi\|^{\beta-1}_{L^\beta(\Omega)}\big)\|\mathcal{D}_t(\varphi_n-\varphi)\|_{L^\beta(\Omega)}$$
$$\leq C\big(|||\varphi_n|||^{\beta-1}_{L_0^{t,\beta}(\Omega)} + |||\varphi|||^{\beta-1}_{L_0^{t,\beta}(\Omega)}\big)|||\varphi_n-\varphi|||_{L_0^{t,\beta}(\Omega)},$$

for some $C > 0$ depending only on β. Then, arguing exactly as in the proof of (3.14), we conclude that

$$\lim_{n\to\infty}\int_\Omega \big|(\mathcal{D}_t\varphi_n(x))^\beta - (\mathcal{D}_t\varphi(x))^\beta\big|dx = 0.$$

Since $\beta \in [2, r)$ was fixed but arbitrary, we have proved (3.24) for all $2 \leq \alpha < r$.

It remains to deal with the case $1 < \alpha < 2$. To that end, note that $(\mathcal{D}_t\varphi_n)_n \subset L^2(\Omega)$ is a bounded non-negative sequence such that $\|\mathcal{D}_t\varphi_n\|_{L^2(\Omega)} \to \|\mathcal{D}_t\varphi\|_{L^2(\Omega)}$ as $n \to \infty$. Hence, it follows that $\mathcal{D}_t\varphi_n \to \mathcal{D}_t\varphi$ in $L^2(\Omega)$ as $n \to \infty$. Also, observe that, for any $\alpha \in (1, 2)$,

$$\int_\Omega \big|(\mathcal{D}_t\varphi_n(x))^\alpha - (\mathcal{D}_t\varphi(x))^\alpha\big|dx$$
$$\leq C\,\|\mathcal{D}_t(\varphi_n) - \mathcal{D}_t(\varphi)\|_{L^\alpha(\Omega)}\big(\|\mathcal{D}_t\varphi_n\|^{\alpha-1}_{L^\alpha(\Omega)} + \|\mathcal{D}_t\varphi\|^{\alpha-1}_{L^\alpha(\Omega)}\big) \leq \widetilde{C}\,\|\mathcal{D}_t\varphi_n - \mathcal{D}_t\varphi\|_{L^2(\Omega)},$$

with $C > 0$ depending only on $\alpha > 0$ and $\widetilde{C} > 0$ depending only on α, r, $|\Omega|$ and ℓ. Combining this chain of inequalities with the fact that $\mathcal{D}_t\varphi_n \to \mathcal{D}_t\varphi$ in $L^2(\Omega)$ as $n \to \infty$ we conclude that

$$\lim_{n\to\infty}\int_\Omega \big|(\mathcal{D}_t\varphi_n(x))^\alpha - (\mathcal{D}_t\varphi(x))^\alpha\big|dx = 0.$$

Once the claim is proved, to conclude the proof in the case where $m > N/s$, it just remains to prove the claimed regularity, that follows thanks to Proposition 2.1, [20, Proposition 1.4 (iii)] and the definition of E_η.

□

Case 2: $\overline{m} < m \leq N/s$.

First of all, let us fix $r = r(m, s, t, q) > 0$ such that $\max\{2, qm\} < r < \frac{mN}{(N-ms+mN(\gamma-s))^+}$ and consider $\lambda_\star$ and ℓ as in (3.17) and (3.18) respectively. Then, let us define

$$\widetilde{E}_\eta := \Big\{v \in L_0^{\gamma,1+\eta}(\Omega) : |||v|||_{L_0^{\gamma,1+\eta}(\Omega)} \leq M \text{ and } |||v|||_{L_0^{\gamma,r}(\Omega)} \leq \ell^{\frac{1}{q}}\Big\}, \tag{3.25}$$

with γ and η as in (3.20) and

$$M := C\Big(\|\mu\|_{L^\infty(\Omega)}|\Omega|^{\frac{r-q}{r}}\widetilde{k}^q\ell + \lambda_\star\|f\|_{L^1(\Omega)}\Big),$$

with $C > 0$ depending only on Ω, N, s and t (cf. Proposition 2.1). It is easy to check that $\widetilde{E}_\eta$ is a bounded, closed and convex set of $L_0^{\gamma,1+\eta}(\Omega)$. Moreover, arguing as we did in the first case, one can prove that

$$\widetilde{T}_2 : \widetilde{E}_\eta \to L_0^{\gamma,1+\eta}(\Omega), \quad \varphi \mapsto u, \tag{3.26}$$

where u is the unique weak solution to (3.22), is well-defined, continuous, compact and satisfies $\widetilde{T}_2(\widetilde{E}_\eta) \subset \widetilde{E}_\eta$. Hence, applying again Schauder's fixed point Theorem, the existence follows also in this case. Having at hand the existence, the claimed regularity follows again from Proposition 2.1. □

Remark 3.5 The presence of the upper bound $\overline{q} \in (1, +\infty]$ in Theorems 3.1 and 3.4 is natural in this kind of existence results. Even in the local case (1.1), one finds an upper bound on q depending on the regularity of the data f and on the dimension N. In our proofs, the upper bound $\overline{q}$ naturally appears when applying Proposition 2.1 and depends on the regularity of the datum f, on the dimension N and on the order of the "nonlocal gradient". The lower bound $\overline{m}$ is again related to the use of Proposition 2.1 but also to the fact that we need $r > 2$ in the proof of Theorem 3.4.

4 Non–existence results

This section is devoted to prove non-existence results for (KPZ_1) and (KPZ_3). Let us start analysing (KPZ_1). As already mentioned in the introduction, the proof of the non-existence result for (KPZ_1) is completely different from its counterpart for (KPZ_3). Here, we prove a generalization of Theorem 1.4.

Theorem 4.1 *Let $0 < t < \min\{1, 2s\}$, $\mu_2 \geq \mu(x) \geq \mu_1 > 0$, $f \in L^1(\Omega)$ with $f \gneqq 0$ and $q > \frac{2(s+1)}{t+2}$. Then, there exists $\lambda^{\star\star} > 0$ such that, for all $\lambda > \lambda^{\star\star}$, ($KPZ_1$) has no weak solution in $W_0^{s,2}(\Omega)$.*

Proof Let $u \in W_0^{s,2}(\Omega)$ be a weak solution to (KPZ_1) and let $\phi \in W_0^{s,2}(\Omega) \cap C^s(\mathbb{R}^N)$ be the unique (energy) solution to

$$\begin{cases} (-\Delta)^s \phi = 1, & \text{in } \Omega, \\ \phi = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

First of all, note that $\phi \in \mathbb{X}^s(\Omega)$, so that we can use ϕ as test function in (KPZ_1) and get that

$$\begin{aligned} \int_\Omega u\,dx &= \int_\Omega \mu(x)|(-\Delta)^{\frac{t}{2}} u|^q \phi\,dx + \lambda \int_\Omega f\phi\,dx \\ &\geq \mu_1 \int_\Omega |(-\Delta)^{\frac{t}{2}} u|^q \phi\,dx + \lambda \int_\Omega f\phi\,dx. \end{aligned} \tag{4.1}$$

On the other hand, let $\psi \in W_0^{\frac{t}{2},2}(\Omega) \cap C^{\frac{t}{2}}(\mathbb{R}^N)$ be the unique (energy) solution to the problem

$$\begin{cases} (-\Delta)^{\frac{t}{2}} \psi = 1, & \text{in } \Omega, \\ \psi = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.2}$$

Since $W_0^{s,2}(\Omega) \subset W_0^{\frac{t}{2},2}(\Omega)$ we can test (4.2) with u and integrate by parts, so that, using (4.1), we obtain

$$\int_\Omega \psi(-\Delta)^{\frac{t}{2}} u\,dx = \int_\Omega u\,dx \geq \mu_1 \int_\Omega |(-\Delta)^{\frac{t}{2}} u|^q \phi\,dx + \lambda \int_\Omega f\phi\,dx. \tag{4.3}$$

Moreover, using Young's inequality, we easily see that

$$\int_\Omega \psi(-\Delta)^{\frac{t}{2}} u\,dx \leq \mu_1 \int_\Omega |(-\Delta)^{\frac{t}{2}} u|^q \phi\,dx + C_q \mu_1^{-\frac{1}{q-1}} \int_\Omega \frac{\psi^{\frac{q}{q-1}}}{\phi^{\frac{1}{q-1}}}\,dx. \tag{4.4}$$

Thus, combining (4.3) and (4.4), we obtain that

$$\lambda \int_\Omega f\phi\, dx \le C_q \mu_1^{-\frac{1}{q-1}} \int_\Omega \frac{\psi^{\frac{q}{q-1}}}{\phi^{\frac{1}{q-1}}} dx. \tag{4.5}$$

Finally, note that (see e.g. [11, Eq. (1.15)]) there exists $C_0 > 0$ (depending only on Ω, s, t and N) such that

$$C_0^{-1}\delta^s(x) \le \phi(x) \le C_0\delta^s(x) \quad \text{and} \quad C_0^{-1}\delta^{\frac{t}{2}}(x) \le \psi(x) \le C_0\delta^{\frac{t}{2}}(x), \quad \text{in } \overline{\Omega}. \tag{4.6}$$

Hence, since $q > \frac{2(s+1)}{t+2}$, the right hand side in (4.5) is bounded, an thus necessarily

$$\lambda \le \frac{C_q \displaystyle\int_\Omega \frac{\psi^{\frac{q}{q-1}}}{\phi^{\frac{1}{q-1}}} dx.}{\mu_1^{\frac{1}{q-1}} \displaystyle\int_\Omega f\phi\, dx} =: \lambda^{\star\star}. \tag{4.7}$$

□

Remark 4.2 The regularity imposed on f can be slightly weakened. Indeed, to prove our non–existence result, namely Theorem 4.1, we only need $f \in L^1(\Omega, \delta^s(x)dx)$. Moreover, observe that in the proof we only used that u is a (weak) supersolution to (KPZ_1). On the negative side, observe that the bound appearing on the power q seems to be technical.

We also prove here a generalization of [2, Theorem 1.2] covering the cases where $q \ne 2$ and/or $t \ne s$.

Theorem 4.3 *Let $0 < t < \min\{1, 2s\}$, $\mu_2 \ge \mu(x) \ge \mu_1 > 0$, $f \in L^1(\Omega)$ with $f^+ \not\equiv 0$ and $q > 1$. Then, there exists $\lambda^{\star\star} > 0$ such that, for all $\lambda > \lambda^{\star\star}$, ($\text{KPZ}_3$) has no weak solution in $W_0^{s,2}(\Omega)$.*

Proof Let $u \in W_0^{s,2}(\Omega)$ be a weak solution to (KPZ_3) and let $\phi \in C_c^\infty(\Omega)$ be an arbitrary non-negative function such that

$$\int_\Omega f\phi^{\frac{q}{q-1}} dx > 0 \quad \text{and} \quad \int_\Omega \left(\mathcal{D}_{2s-t}\phi\right)^{\frac{q}{q-1}} dx < +\infty. \tag{4.8}$$

Using $\phi^{\frac{q}{q-1}}$ as test function in (KPZ_3), we get

$$\int_\Omega u(-\Delta)^s(\phi^{\frac{q}{q-1}})\, dx \ge \mu_1 \int_\Omega (\mathcal{D}_t u)^q \phi^{\frac{q}{q-1}} dx + \lambda \int_\Omega f\phi^{\frac{q}{q-1}} dx. \tag{4.9}$$

On the other hand, using the Mean value Theorem and Hölder's and Young's inequalities, we infer that

$$\begin{aligned}
&\int_\Omega u(-\Delta)^s(\phi^{\frac{q}{q-1}})\, dx = \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\phi^{\frac{q}{q-1}}(x) - \phi^{\frac{q}{q-1}}(y))}{|x-y|^{N+2s}} dydx \\
&\le C_q \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)||\phi(x)-\phi(y)|}{|x-y|^{N+2s}} (\phi^{\frac{1}{q-1}}(x) + \phi^{\frac{1}{q-1}}(y))\, dydx \\
&= 2C_q \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)||\phi(x)-\phi(y)|}{|x-y|^{N+2s}} \phi^{\frac{1}{q-1}}(x)\, dydx \le 2C_q \int_{\mathbb{R}^N} \left(\mathcal{D}_t u\right)\left(\mathcal{D}_{2s-t}\phi\right)\phi^{\frac{1}{q-1}}\, dx \\
&= 2C_q \int_\Omega \left(\mathcal{D}_t u\right)\left(\mathcal{D}_{2s-t}\phi\right)\phi^{\frac{1}{q-1}}\, dx \le \mu_1 \int_\Omega (\mathcal{D}_t u)^q \phi^{\frac{q}{q-1}} dx + \widetilde{C}_{q,\mu_1} \int_\Omega (\mathcal{D}_{2s-t}\phi)^{\frac{q}{q-1}} dx\,.
\end{aligned}$$

Combining the above chain of inequalities with (4.9), we get that necessarily

$$\lambda \int_\Omega f\phi^{\frac{q}{q-1}}\,dx \le \widetilde{C}_{q,\mu_1} \int_\Omega (\mathcal{D}_{2s-t}\phi)^{\frac{q}{q-1}}\,dx.$$

Hence, defining

$$\lambda^{\star\star} := \inf \left\{ \frac{\widetilde{C}_{q,\mu_1} \displaystyle\int_\Omega (\mathcal{D}_{2s-t}\phi)^{\frac{q}{q-1}}\,dx}{\displaystyle\int_\Omega f\phi^{\frac{q}{q-1}}\,dx} : \phi \in C_c^\infty(\Omega) \text{ is non-negative and satisfies (4.8)} \right\},$$

the result immediately follows. □

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References

1. Abatangelo, N., Cozzi, M.: An elliptic boundary value problem with fractional nonlinearity. SIAM J. Math. Anal. **53**(3), 3577–3601 (2021)
2. Abdellaoui, B., Fernández, A.J.: Nonlinear fractional Laplacian problems with nonlocal 'gradient terms'. Proc. Roy. Soc. Edinburgh Sect. A **150**(5), 2682–2718 (2020)
3. Abdellaoui, B., Fernández, A.J., Leonori, T., Younes, A. Global fractional Calderón-Zygmund type regularity. *Preprint*. https://arxiv.org/abs/2107.06535v2. (2021)
4. Abdellaoui, B., Peral, I.: Towards a deterministic KPZ equation with fractional diffusion: the stationary problem. Nonlinearity **31**(4), 1260–1298 (2018)
5. Adams, R.A.: Sobolev spaces. Academic Press, New York (1975)
6. Barrios, B., Medina, M.: Equivalence of weak and viscosity solutions in fractional non-homogeneous problems. Math. Ann. **381**(3–4), 1979–2012 (2021)
7. Caffarelli, L., Dávila, G.: Interior regularity for fractional systems. Ann. Inst. H. Poincaré Anal. Non Linéaire **36**(1), 165–180 (2019)
8. Chen, H., Véron, L.: Semilinear fractional elliptic equations involving measures. J. Differ. Equ. **257**(5), 1457–1486 (2014)
9. Chen, H., Véron, L.: Semilinear fractional elliptic equations with gradient nonlinearity involving measures. J. Funct. Anal. **266**(8), 5467–5492 (2014)
10. Comi, G.E., Stefani, G.: A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up. J. Funct. Anal. **277**(10), 3373–3435 (2019)
11. Fall, M.M., Jarohs, S.: Gradient estimates in fractional Dirichlet problems. Potential Anal. **54**(4), 627–636 (2021)
12. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Springer-Verlag, Berlin (2001)
13. Leonori, T., Molino, A., Segura de León, S.: Parabolic equations with natural growth approximated by nonlocal equations. Commun. Contemp. Math. **23**(1), 32 (2021)
14. Mengesha, T., Phuc, N.C.: Quasilinear Riccati type equations with distributional data in Morrey space framework. J. Differ. Equ. **260**(6), 5421–5449 (2016)
15. Mengesha, T., Spector, D.: Localization of nonlocal gradients in various topologies. Calc. Var. Partial Differ. Equ. **52**(1–2), 253–279 (2015)
16. Millot, V., Sire, Y.: On a fractional Ginzburg-Landau equation and 1/2-harmonic maps into spheres. Arch. Ration. Mech. Anal. **215**(1), 125–210 (2015)
17. Mizuta, Y.: Potential theory in Euclidean spaces. GakkBotosho Co. Ltd, Tokyo (1996)
18. Phuc, N.C.: Morrey global bounds and quasilinear Riccati type equations below the natural exponent. J. Math. Pures. Appl. **102**(1), 99–123 (2014)

19. Riesz, M.: L'intégrale de Riemann-Liouville et le problème de Cauchy. Acta Math. **81**, 1–223 (1949)
20. Ros-Oton, X., Serra, J.: The extremal solution for the fractional Laplacian. Calc. Var. Partial Differ. Equ. **50**(3–4), 723–750 (2014)
21. Schikorra, A.: Integro-differential harmonic maps into spheres. Comm. Partial Differ. Equ. **40**(3), 506–539 (2015)
22. Schikorra, A., Spector, D., Van Schaftingen, J.: An L^1-type estimate for Riesz potentials. Rev. Mat. Iberoam. **33**(1), 291–303 (2017)
23. Shieh, T., Spector, D.: On a new class of fractional partial differential equations. Adv. Calc. Var. **8**(4), 321–336 (2015)
24. Shieh, T., Spector, D.: On a new class of fractional partial differential equations II. Adv. Calc. Var. **11**(3), 289–307 (2018)
25. Stein, E.M.: The characterization of functions arising as potentials. Bull. Am. Math. Soc. **67**, 102–104 (1961)
26. Stein, E.M.: Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, NJ (1970)
27. Šilhavý, M.: Fractional vector analysis based on invariance requirements (critique of coordinate approaches). Contin. Mech. Thermodyn. **32**(1), 207–228 (2020)