On chromatic symmetric homology and planarity of graphs

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Abstract

Sazdanovic and Yip (2018) defined a categorification of Stanley's chromatic symmetric function called the chromatic symmetric homology, given by a suitable family of representations of the symmetric group. In this paper we prove that, as conjectured by Chandler, Sazdanovic, Stella and Yip (2019), if a graph G is non-planar, then its chromatic symmetric homology in bidegree (1,0) contains $\mathbb{Z}_2$-torsion. Our proof follows a recursive argument based on Kuratowsky's theorem.

Mathematics Subject Classifications: 05C31, 05E05, 05C10, 20C30, 55U15

1 Introduction

The *chromatic symmetric function* of a graph, defined by Stanley in [1], is a remarkable combinatorial invariant which refines the chromatic polynomial. Recenty, in [2], Sazdanovic and Yip categorified this invariant by defining a new homological theory, called the *chromatic symmetric homology* of a graph G. This construction, inspired by Khovanov's categorification of the Jones polynomial [5], is obtained by assigning a graded representation of the symmetric group to every subgraph of G with the same vertices as G, and a differential to every cover relation in the Boolean poset of such subgraphs of G. The chromatic symmetric homology is then defined as the homology of this chain complex; its bigraded Frobenius series $Frob_G(q,t)$, when evaluated at $q = t = 1$, reduces to Stanley's chromatic symmetric function expressed in the Schur basis.

As proved in [3], this categorification produces a truly stronger invariant: in other words, chromatic symmetric homology can distinguish couples of graphs that have the same chromatic symmetric function. Furthermore, in the same paper, the properties of chromatic symmetric homology with integer coefficients have been investigated. The authors of [3] provided examples of graphs whose chromatic symmetric homology has torsion, leaving open the following conjecture:

Conjecture 1. A graph G is non-planar if and only if its chromatic symmetric homology in bidegree (1,0) contains $\mathbb{Z}_2$-torsion.

In this paper we prove one direction of this conjecture, namely:

Theorem 2. *Let G be a finite non-planar graph. Then its chromatic symmetric homology in bidegree (1,0) contains $\mathbb{Z}_2$-torsion.*

Our strategy is based on applying Kuratowsky's theorem: we show that the torsion elements in the homology of the complete graph K_5 and of the complete bipartite graph $K_{3,3}$ are mapped to torsion elements in the homology of the graphs that are obtained from them by the operations of edge subdivision and graph inclusion, i.e. all the non-planar graphs.

2 Computing q-degree zero homology

The interested reader can find a complete description of the construction of chromatic symmetric homology for a graph in the paper [2]. Here we limit ourselves to briefly recall how to compute homology in q-degree zero. For simplifying the notation, we will denote by C_i the i-th chain module and by H_i the i-th homology module. They correspond respectively to $C_{i,0}$ and $H_{i,0}$ in the notation of [2].

Let G be a graph with n vertices and m edges. We can assume without loss of generality that G is simple: indeed if G has a loop, then its chromatic symmetric homology is zero (Proposition 3.1 of [2]), while if G has two vertices connected by multiple edges, then we can replace them by a single edge without affecting the chromatic symmetric homology (Proposition 3.2 of [2]). Hence we denote by (p, q) the edge incident to the vertices p and q, and we order the set of edges $E(G)$ lexicographically.

Each subset S of $E(G)$ is naturally identified with a subgraph of G, having the same vertices as G and S as set of edges. As the authors of [3, 4], we call such a subgraph a "spanning subgraph", even if the word "spanning" is sometimes used with a different meaning in graph theory and matroid theory. The set of all spanning subgraphs G has a stucture of Boolean lattice $\mathbf{B}(G)$, ordered by reverse inclusion. In the Hasse diagram of $\mathbf{B}(G)$, we direct an edge $\epsilon(F, F')$ from a subgraph F to a subgraph F' if and only if F' can be obtained by removing an edge from F.

Let $F \subseteq E(G)$ be a spanning subgraph of G with connected components $B_1, \dots, B_r$ of sizes, i.e. the number of vertices in each connected component, $b_1, \dots, b_r$ respectively. Then the module associated to it in q-degree zero is the *permutation module*

$$M_F = Ind^{\mathfrak{S}_n}_{\mathfrak{S}_{B_1} \times \cdots \times \mathfrak{S}_{B_r}} (\mathbf{S}_{(b_1)} \otimes \cdots \otimes \mathbf{S}_{(b_r)}),$$

where $\mathfrak{S}_n$ is the permutation group on n elements and $\mathbf{S}_{(i)}$ is the Specht module related to the partition (i).

We define

$$C_i(G) = \bigoplus_{|F|=i} M_F,$$

where the sum is over the spanning subgraphs of G with i edges. Therefore the i-th chain module $C_i(G)$ of the graph is a direct sum of $\binom{m}{i}$ permutation modules of $\mathfrak{S}_n$. If $\lambda = (b_1, \dots, b_r)$ is the partition whose parts are the sizes of the connected components of F, then

$$M_F \cong M_\lambda = \mathbb{C}[\mathfrak{S}_n] \otimes_{\mathbb{C}[\mathfrak{S}_\lambda]} \mathbf{S}_{(n)},$$

where $\mathfrak{S}_\lambda$ is the subgroup of $\mathfrak{S}_n$ given by $\mathfrak{S}_{b_1} \times \cdots \times \mathfrak{S}_{b_r}$.

Let F and F' be spanning subgraphs of G where $F' = F - e$ with $e \in E(F)$.

There is an *edge map* $d_{\epsilon(F,F')} : M_F \to M_{F'}$, defined in our case as the inclusion (for the general definition see [2]). Moreover, the sign of $\epsilon = \epsilon(F, F')$, $sgn(\epsilon)$, is defined as $(-1)^k$, where k is the number of edges of F less than e.

Finally, the *i-th chain map* $d_i : C_i(G) \to C_{i-1}(G)$ is defined as

$$d_i = \sum_{\epsilon} sgn(\epsilon) d_\epsilon,$$

where the sum is over all the edges ϵ in $\mathbf{B}(G)$ which join a spanning subgraph of G with i edges to a spanning subgraph with $i-1$ edges. Sometimes we will use the notation d_i^G, where it may not be clear which graph we are referring to.

We need to recall the following definitions from [3].

Definition 3. Let F be a spanning subgraph of G, and let $\lambda \vdash n$ be the partition whose parts are the sizes of the connected components of F. The numbering $T(F)$ associated to F is the filling of a Young diagram of shape λ such that each row consists of the elements in a connected component of F arranged in increasing order, and rows of $T(F)$ having the same size are ordered so that the minimum element in each row is increasing down the first column.

Let $T = T(F)$. The *q-degree zero permutation module* M_T associated to the numbering T is cyclically generated by the Young symmetrizer

$$a_T = \sum_{\rho \in R(T)} \rho$$

where $R(T) \leqslant \mathfrak{S}_n$ is the subgroup of permutations that permute elements within each row of T. We have:

$$M_F \cong M_T = \mathbb{C}[\mathfrak{S}_n] \cdot a_T.$$

Definition 4. For any numberings S and T of shape λ, let

$$v_T^S = \sigma_{T,S} b_T a_T = b_S a_S \sigma_{T,S} \in M_T,$$

where a_T is as above,

$$b_T = \sum_{\zeta \in C(T)} sgn(\zeta)\zeta,$$

$C(T) \leqslant \mathfrak{S}_n$ is the subgroup of permutations that permute elements within each column of T, and $\sigma_{T,S} \in \mathfrak{S}_n$ is such that $\sigma_{T,S} \cdot T = S$.

Moreover, the *q-degree zero Specht module* $\mathbf{S}_T$ associated to the numbering T is cyclically generated by the Young symmetrizer $c_T = b_T a_T$:

$$\mathbf{S}_T = \mathbb{C}[\mathfrak{S}_n] \cdot c_T,$$

and $\mathbf{S}_T \cong \mathbf{S}_\lambda$.

We also recall the following result from [4], Section 7.2, Proposition 2.

Proposition 5. *Let F be a spanning subgraph of G with associated numbering T of shape λ. Then*

$$\mathbf{S}_T = span\{b_S a_S \sigma_{T,S} | S \in SYT(\lambda)\} = span\{v_T^S | S \in SYT(\lambda)\},$$

where $SYT(\lambda)$ is the set of standard Young tableaux of shape λ.

2.1 Computation of $H_1(G)$

We will describe the chromatic homology in terms of Specht modules. Since each Specht module is cyclically generated, then our inclusion maps are completely determined by specifying the image of a cyclic generator for each Specht module. We now show how to achieve these computations systematically.

We restrict ourselves to Specht modules of type $\lambda = (2^k, 1^{n-2k})$ for $k \geqslant 1$, so we will be computing

$$C_2(G)_{|S_\lambda} \xrightarrow{d_2} C_1(G)_{|S_\lambda} \xrightarrow{d_1} C_0(G)_{|S_\lambda} \to 0.$$

We order the edges of G in lexicographic order and label these as $e_1, \dots, e_m$. In homological degree zero, there is only one subgraph without edges. The chain group $C_0(G) = M_{F_\emptyset} \cong M_{(1^n)}$ is the regular representation of $\mathfrak{S}_n$, where $F_\emptyset$ is the edgeless subgraph. By Corollary 1 in Section 7.3 of [4], the multiplicity of S_λ in $\mathbb{C}[S_n]$ is the number $f^\lambda = K_{\lambda,(1^n)}$ of standard Young tableaux of shape λ. We list the tableaux $Y_1(G), \dots, Y_{f^\lambda}(G) \in SYT(\lambda)$ with respect to the following total order: if T and S are numberings of shape λ such that the i-th row is the lowest row in which the numberings are different, the j-th column is the rightmost column in that row in which the numberings are different and $T(i,j) > S(i,j)$, then we say that $T > S$.

We have

$$C_0(G)|_{S_\lambda} = \bigoplus_{i=1}^{f^\lambda} \mathbb{C}[\mathfrak{S}_n] \cdot v_{Y_i}^{Y_1}.$$

Example 6. Let $G = K_5$. We order the edges of G in lexicographic order; that is,

$$(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5),$$

and label these as $e_1, \ldots, e_{10}$. The standard Young tableaux of shape $\lambda = (2^2, 1)$ listed with respect to the ordering defined earlier are

$$Y_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \; Y_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \; Y_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} \; Y_4 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \; Y_5 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Then

$$C_0(K_5)_{|\mathbf{S}_{(2^2,1)}} = \bigoplus_{i=1}^{5} (\mathbb{C}[S_5] \cdot v_{Y_i}^{Y_1}) \cong \mathbf{S}_{(2^2,1)}^{\oplus 5}.$$

In homological degree one, there are m spanning subgraphs with exactly one edge, thus

$$C_1(G) = \bigoplus_{i=1}^{m} M_{F_{e_i}}.$$

If F_{e_i} is the spanning subgraph containing the edge $e_i = (p,q)$, then the permutation module $M_{F_{e_i}}$ has the associated numbering $T(F_{e_i})$ of shape $\mu = (2, 1^{n-2})$, and $M_{F_{e_i}} = \mathbb{C}[\mathfrak{S}_n] \cdot (e + (pq)) \cong M_\mu$.
The multiplicity of S_λ in $M_{F_{e_i}}$ is the number $K_{\lambda,\mu}$ of semistandard Young tableaux of shape λ and weight μ. We next obtain numberings of shape λ that will index these $K_{\lambda,\mu}$ Specht modules $\mathbf{S}_\lambda$, by standardizing the set $SSYT(\lambda,\mu)$ of semistandard Young tableaux of shape λ and weight μ with respect to $T(F_{e_i})$ in the following way. For any numbering S, the word $w(S)$ of S is obtained by reading the entries of the rows of S from left to right, and from the top row to the bottom row (note that this is not the usual definition of a reading word for tableaux). So, given $Y \in SSYT(\lambda,\mu)$, let $w(Y) = y_1, \ldots, y_n$ be the word of Y, let $w(T) = t_1, \ldots, t_n$ be the word of $T = T(F_{e_i})$ and let σ be the permutation that orders $y_1, \ldots, y_n$ without exchanging y_i and y_j if $y_i = y_j$. From this we obtain a numbering X of shape λ by replacing the entry in Y that corresponds to y_k by $t_{\sigma(k)}$. We list the numberings $X_i^1(G), \ldots, X_i^{K_{\lambda,\mu}}(G)$ obtained using the procedure just described to $SSYT(\lambda,\mu)$ with respect to $T(F_{e_i})$. Observe that since $\mu = (2, 1^{n-2})$ and $\lambda = (2^k, 1^{n-2k})$ where $k \geqslant 1$, then this procedure guarantees that the first row of each numbering $X_i^j(G)$ is $\begin{array}{|c|c|}\hline p & q \\ \hline\end{array}$. So $v_{X_i^j(G)}^{Y_1} \in M_{F_{e_i}}$ and $\mathbb{C}[\mathfrak{S}_n] \cdot v_{X_i^j(G)}^{Y_1} \cong \mathbf{S}_\lambda$ for $j = 1, \ldots, K_{\lambda,\mu}$. Thus

$$C_1(G)_{|S_\lambda} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{K_{\lambda,\mu}} \mathbb{C}[S_n] \cdot v_{X_i^j(G)}^{Y_1}$$

Example 7. Let $G = K_5$. There are 10 spanning subgraphs with exactly one edge. Furthermore, there are two semistandard Young tableaux of shape $\lambda = (2^2, 1)$ and weight $(2, 1^3)$,

$$Z_1 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad Z_2 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\ ,$$

so the multiplicity of $\mathbf{S}_{(2^2,1)}$ in each $M_{F_{e_i}} \cong M_{(2,1^3)}$ is 2. Let $X_i^1(K_5)$ and $X_i^2(K_5)$ denote the numberings which index the two copies of $\mathbf{S}_{(2^2,1)}$ in each $M_{F_{e_i}}$, again listed with respect to the same ordering. So

$$C_1(K_5)|_{\mathbf{S}_{(2^2,1)}} = \bigoplus_{i=1}^{10} (\mathbb{C}[S_5] \cdot v^{Y_1}_{X_i^1(K_5)} \oplus \mathbb{C}[S_5] \cdot v^{Y_1}_{X_i^2(K_5)}) \cong \mathbf{S}^{\oplus 20}_{(2^2,1)}.$$

Consider for instance the spanning subgraph F_{e_1} of K_5 with the edge $e_1 = (1,2)$ only. The numbering associated to it is

$$T(F_{e_1}) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}.$$

Let $z_1^i, \dots, z_5^i$ be the word of Z_i, $i = 1, 2$. The permutation that orders $z_1^1, \dots, z_5^1$ is the identity and the one that orders $z_1^2, \dots, z_5^2$ is (45). Therefore we have

$$X_1^1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} = Y_1 \qquad \text{and} \qquad X_1^2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} = Y_2,$$

then

$$v^{Y_1}_{X_1^1} = v^{Y_1}_{Y_1} \qquad \text{and} \qquad v^{Y_1}_{X_1^2} = v^{Y_1}_{Y_2}.$$

Lastly, we consider the chain module in homological degree two. The spanning subgraphs of G with exactly two edges have connected components of partition type $(2^2, 1^{n-4})$ or $(3, 1^{n-3})$. We are only concerned with Specht modules of type $\lambda = (2^k, 1^{n-2k})$ with $k \geqslant 2$ necessarily and, since $\lambda \not\trianglerighteq (3, 1^{n-3})$, then, by Corollary 1 in Section 7 of [4], S_λ does not appear as a summand in a permutation module isomorphic to $M_{(3,1^{n-3})}$. Hence, we only need to consider the spanning subgraphs with connected components of partition type $(2^2, 1^{n-4})$.

So suppose G has h spanning subgraphs whose connected components has partition type $\nu = (2^2, 1^{n-4})$. List these subgraphs with respect to the lexicographic order of their edge sets. Suppose F_{e_i,e_j} is the spanning subgraph that contains the edges $e_i = (p,q)$ and $e_j = (r,s)$ with $p < r$. The permutation module $M_{F_{e_i,e_j}}$ has the associated numbering $T(F_{e_i,e_j})$ of shape ν and

$$M_{F_{e_i,e_j}} = \mathbb{C}[\mathfrak{S}_n] \cdot (e + (pq))(e + (rs)) \cong M_\nu.$$

Similar to the previous case for $C_1(G)$, the multiplicity of S_λ in $M_{F_{e_i,e_j}}$ is $K_{\lambda,\nu}$. We list the numberings $W^1_{i,j}(G), \dots, W^{K_{\lambda,\nu}}_{i,j}(G)$ obtained using the procedure described above to $SSYT(\lambda,\nu)$ with respect to $T(F_{e_i,e_j})$. The procedure guarantees that the top two rows of each numbering W are $\begin{array}{|c|c|}\hline p & q \\ \hline r & s \\ \hline \end{array}$, so $v_W^{Y_1} \in M_{F_{e_i,e_j}}$ and

$$\mathbb{C}[S_n] \cdot v^{Y_1}_{W^l_{i,j}(G)} \cong \mathbf{S}_\lambda \text{ for } l = 1, \ldots, K_{\lambda,\nu}.$$

Thus

$$C_2(G)_{|S_\lambda} = \bigoplus_{i,j} \bigoplus_{l=1}^{K_{\lambda,\nu}} \mathbb{C}[S_n] \cdot v^{Y_1}_{W^l_{i,j}(G)},$$

where the direct sum is over the values of i and j corresponding to the couples of edges which form spanning subgraphs of type ν, i.e. the non-consecutive edges.

Example 8. Let $G = K_5$. There are 15 spanning subgraphs whose connected components have partition type $(2^2, 1)$. There is only one semistandard Young tableau of shape $\lambda = (2^2, 1)$ and weight $(2^2, 1)$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

so the multiplicity of $\mathbf{S}_{(2^2,1)}$ in each $M_{F_{e_i,e_j}} \cong M_{(2^2,1)}$ is 1, and we let $W_{i,j}$ denote the numbering which indexes the copy of $\mathbf{S}_{(2^2,1)}$ in $M_{F_{e_i,e_j}}$. Therefore,

$$C_2(K_5)_{|\mathbf{S}_{(2^2,1)}} = \bigoplus (\mathbb{C}[S_5] \cdot v^{Y_1}_{W_{i,j}}) \cong \mathbf{S}^{\oplus 15}_{(2^2,1)},$$

where the direct sum is over the values of i and j corresponding to the couples of non-consecutive edges.

To compute the edge maps we will need the following theorem (Corollary 2.18 of [3]).

Theorem 9. *For any numberings S of shape λ,*

$$v_S^T = (-1)^j \sum_{U \in \Xi_{i,j}(S)} v_U^T,$$

where $\Xi_{i,j}(S)$ is the set of all numberings U obtained from S by exchanging the first j entries in the $(i+1)$-th row of S with j entries in the i-th row of S, preserving the order of each subset of elements.

We let $\pi_{i,j}$ denote the operator on numberings such that

$$\pi_{i,j}(S) = (-1)^j \sum_{U \in \Xi_{i,j}(S)} U.$$

Example 10. Let $G = K_5$. We compute $d_1(v^{Y_1}_{X^1_3})$ and $d_1(v^{Y_1}_{X^2_3})$, so we consider the spanning subgraph F_{e_3} of K_5 with the edge $e_3 = (1,4)$ only. The numbering associated to it is

$$T(F_{e_3}) = \begin{ytableau} 1 & 4 \\ 2 \\ 3 \\ 5 \end{ytableau}\,.$$

We have

$$X_3^1 = \begin{ytableau} 1 & 4 \\ 2 & 3 \\ 5 \end{ytableau} \qquad \text{and} \qquad X_3^2 = \begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau} = Y_5.$$

We compute

$$\pi_{1,1}(X_3^1) = -\begin{ytableau} 1 & 2 \\ 4 & 3 \\ 5 \end{ytableau} - \begin{ytableau} 2 & 4 \\ 1 & 3 \\ 5 \end{ytableau}$$

and

$$\pi_{1,2}\begin{ytableau} 2 & 4 \\ 1 & 3 \\ 5 \end{ytableau} = \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 \end{ytableau} = Y_3.$$

Then, by Theorem 9, we have

$$v_{X_3^1}^{Y_1} = -v_{Y_1}^{Y_1} - v_{Y_3}^{Y_1} \qquad \text{and} \qquad v_{X_3^2}^{Y_1} = v_{Y_5}^{Y_1},$$

and d_1 sends

$$v_{X_3^1}^{Y_1} \mapsto -v_{Y_1}^{Y_1} - v_{Y_3}^{Y_1} \qquad \text{and} \qquad v_{X_3^2}^{Y_1} \mapsto v_{Y_5}^{Y_1}.$$

Now we compute $d_2(v_{W_{1,8}}^{Y_1})$, so we have to consider the spanning subgraph F_{e_1,e_8} of K_5 with the edges $e_1 = (1,2)$ and $e_8 = (3,4)$. The numbering associated to it is

$$\begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 \end{ytableau}\,.$$

There are two spanning subgraphs of K_5 with only one edge that can be obtained removing one edge from F_{e_1,e_8}, i.e. F_{e_1} and F_{e_8}. The per-edge map $d_{\epsilon(F_{e_1,e_8},F_{e_1})}$ appears as a summand in d_2 with a minus sign; instead $d_{\epsilon(F_{e_1,e_8},F_{e_8})}$ appears in d_2 with a plus sign. Therefore, d_2 sends $v_{W_{1,8}}^{Y_1} \mapsto -v_{X_1^1}^{Y_1} + v_{X_8^1}^{Y_1}$.

3 The case of non-planar graphs

In this section we will prove that if G is a non-planar graph, then the chromatic symmetric homology $H_1(G;\mathbb{Z})$ contains $\mathbb{Z}_2$-torsion. We first recall two results from [3]:

Lemma 11. *The chromatic symmetric homology $H_1(K_5;\mathbb{Z})$ contains $\mathbb{Z}_2$-torsion.*

Proof. We compute

$$C_2(K_5)_{|\mathbf{S}_{(2^2,1)}} \xrightarrow{d_2} C_1(K_5)_{|\mathbf{S}_{(2^2,1)}} \xrightarrow{d_1} C_0(K_5)_{|\mathbf{S}_{(2^2,1)}} \to 0,$$

restricted to the $\mathbf{S}_{(2^2,1)}$ modules.

Following the notation introduced in 2.1, let $g = W_{1,8} + W_{1,9} + W_{1,10} + W_{2,6} - W_{2,7} - W_{2,10} + W_{3,5} + W_{3,7} + W_{3,9} + W_{4,5} + W_{4,6} + W_{4,8} - W_{5,10} - W_{6,9} + W_{7,8} \in C_2(K_5)$ and $h = X_9^1 + X_{10}^1 - X_2^1 + X_7^2 + X_9^2 \in C_1(K_5)$. We have that $h \notin \operatorname{im} d_2$, $d_2(g) = 2h$ and $d_1(h) = 0$, so h generates $\mathbb{Z}_2$-torsion in $H_1(K_5; \mathbb{Z})$. For more details see [3], Theorem 4.1. □

Lemma 12. *The chromatic symmetric homology $H_1(K_{3,3}; \mathbb{Z})$ contains $\mathbb{Z}_2$-torsion.*

Proof. We compute

$$C_2(K_{3,3})|_{\mathbf{S}_{(2^2,1^2)}} \xrightarrow{d_2} C_1(K_{3,3})|_{\mathbf{S}_{(2^2,1^2)}} \xrightarrow{d_1} C_0(K_{3,3})|_{\mathbf{S}_{(2^2,1^2)}} \to 0,$$

restricted to the $\mathbf{S}_{(2^2,1^2)}$ modules.

Following the notation introduced in Section 2.1, let $g' = W_{1,6} - W_{1,7} + W_{1,8} + W_{1,9} - W_{2,4} - W_{2,5} + W_{2,7} + W_{2,9} + W_{3,4} - W_{3,5} + W_{3,6} + W_{3,8} + W_{4,8} + W_{4,9} + W_{5,6} + W_{5,7} - W_{6,9} + W_{7,8} \in C_2(K_{3,3})$ and $h' = X_6^3 - X_7^3 + X_8^3 - X_9^2 \in C_1(K_{3,3})$. We have that $h' \notin \operatorname{im} d_2$, $d_2(g') = 2h'$ and $d_1(h') = 0$, so h' generates $\mathbb{Z}_2$-torsion in $H_1(K_{3,3}; \mathbb{Z})$. For more details see [3], Theorem 4.2.

□

Proposition 13. *Let G be a graph with n vertices, $n \geqslant 2$, λ a partition of n of type $(2^k, 1^{n-2k})$ and $h \in C_1(G)_{|\mathbf{S}_\lambda}$ a generator of $\mathbb{Z}_p$-torsion in $H_1(G; \mathbb{Z})$. Let G' be a subdivision of G with n' vertices, $n' > n$, i.e. a graph obtained from G by inserting $n' - n$ vertices into the edges of G. Then there exists $h' \in C_1(G')_{|\mathbf{S}_{\lambda'}}$, with $\lambda' = (2^k, 1^{n'-2k})$, that generates $\mathbb{Z}_p$-torsion in $H_1(G', \mathbb{Z})$.*

Proof. it is enough to show that the statement holds for $n' = n + 1$.

Let G be a graph with n vertices and G' be the graph obtained from G by inserting a vertex into an edge of G. By hypothesis, there exists $h \in C_1(G)_{|\mathbf{S}_\lambda}$, with $\lambda = (2^k, 1^{n-2k})$, that generates $\mathbb{Z}_p$-torsion in $H_1(G; \mathbb{Z})$.

We number the vertices of G' from 1 to $n + 1$, so that the vertex added is $n + 1$.

We prove that h is mapped to a $\mathbb{Z}_p$-torsion generator in $H_1(G'; \mathbb{Z})$.

We have that all the edges of G are also edges of G', except for the edge that has been broken, let it be (a, b), which is no longer an edge of G', but it has been replaced by two edges, $(a, n + 1)$ and $(n + 1, b)$. We consider the partition $\lambda' = (2^k, 1^{n+1-2k})$, i.e. the partition λ with an extra box at the bottom. We divide the standard Young tableaux of shape λ' into two groups: those that are obtained simply by adding the box containing $n+1$ at the bottom of the standard Young tableaux of shape λ and those that don't have $n + 1$ in the last row. We do the same for the semistandard Young tableaux of shape λ' and weight $(2, 1^{n-1})$. If i does not correspond to the two new edges, the $X_i^\ell(G')$'s are identical to the $X_i^\ell(G)$'s with the box containing $n + 1$ added at the bottom. Therefore, the differential $d_1^{G'}$ acts on them in exactly the same way as d_1^G, since the π-operations don't concern the last row. Since $h \in \ker d_1^G$, we also have that $h \in \ker d_1^{G'}$.

We pause the proof to observe that we think of h as a 1-chain in G'. This can always be done using the π-operations and Theorem 9.

For example, consider K_5 and the graph G' obtained by adding the vertex 6 into the edge $(1,5)$. $X_4^1(K_5)$ with a box containing 6 added at the bottom, i.e. $\begin{array}{|c|c|}\hline 1&5\\\hline 2&3\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array}$, a priori is not a 1-chain in G' since the edge (1,5) is not an edge of G'; but we have

$$\begin{array}{|c|c|}\hline 1&5\\\hline 2&3\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array} = \pi_{1,1}\begin{array}{|c|c|}\hline 1&5\\\hline 2&3\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array} = -\begin{array}{|c|c|}\hline 2&5\\\hline 1&3\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1}\multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array} = -X_2^2(G') - X_1^2(G') \in C_1(G').$$

Remark 14. It cannot happen that there is cancellation among 1-cycles in G' that give the cycle h in G. In fact, the π-operations on numberings correspond, by Theorem 9, to equalities among the elements of $\mathbf{S}_{(2^k,1^{n-2k})}$ indexed by such numberings. Therefore, if there was cancellation among 1-cycles in G' that give h in G, it would also be among the 1-cycles of G that form h, so h would be the trivial 1-cycle of G, which is not true.

A similar argument applies to the $W^s_{h,k}$'s. Since, in G, there exists a 2-cycle g such that $d_2^G(g) = 2h$ and g can be written as 2-chain in G' with the π-operations, without becoming trivial as observed for h in Remark 14, we also have that $d_2^{G'}(g) = 2h$.
It remains to prove that $h \notin \operatorname{im} d_2^{G'}$. If h belonged to $\operatorname{im} d_2^{G'}$, since we know that $h \notin \operatorname{im} d_2^G$, it would be linear combination of the columns of $d_2^{G'}$ which are not columns of d_2^G, but it is not possible because h is a 1-chain in G.
Therefore, we have a $\mathbb{Z}_p$-torsion generator in $H_1(G',\mathbb{Z})$. □

Proposition 15. *Let G' be a graph with n' vertices and let G be a subgraph of G' with n vertices, $n \leqslant n'$. Assume that $h \in C_1(G)_{|\mathbf{S}_\lambda}$ is a generator of $\mathbb{Z}_p$-torsion in $H_1(G;\mathbb{Z})$. Then there exists $h' \in C_1(G')_{|\mathbf{S}_{\lambda'}}$, with $\lambda' = (2^k, 1^{n'-2k})$, that generates $\mathbb{Z}_p$-torsion in $H_1(G',\mathbb{Z})$.*

Proof. We have that all the edges of G are also edges of G'. We consider the partition $\lambda' = (2^k, 1^{n'-2k})$, i.e. the partition λ with $n'-n$ extra boxes at the bottom. We number the vertices of G' from 1 to n', so that the vertices eventually added are $n+1,\dots,n'$. We divide the standard Young tableaux of shape λ' into two groups: those that are obtained simply by adding the boxes containing $n+1,\dots,n'$ at the bottom of the standard Young tableaux of shape λ and those that don't have $n+1,\dots,n'$ in the last rows. We do the same for the semistandard Young tableaux of shape λ' and weight $(2,1^{n'-2})$. All the $X_i^\ell(G)$'s with extra boxes containing $n+1,\dots,n'$ at the bottom are among the $X_i^{\ell'}(G')$'s. Therefore, the differential $d_1^{G'}$ acts on them in exactly the same way as d_1^G, since the π-operations don't concern the last rows. Since $h \in \ker d_1^G$, we also have that $h \in \ker d_1^{G'}$.

A similar argument applies to the $W^s_{h,k}$'s. Since, in G, there exists a 2-cycle g such that $d_2^G(g) = 2h$, we also have that $d_2^{G'}(g) = 2h$.
It remains to prove that $h \notin \operatorname{im} d_2^{G'}$. If h belonged to $\operatorname{im} d_2^{G'}$, since we know that $h \notin \operatorname{im} d_2^G$, it would be linear combination of the columns of $d_2^{G'}$ which are not columns of d_2^G, but it

is not possible because h is a 1-chain in G.
Therefore, we have a $\mathbb{Z}_p$-torsion generator in $H_1(G', \mathbb{Z})$. $\square$

Theorem 16. *Let G be a finite non-planar graph. Then $H_1(G; \mathbb{Z})$ contains $\mathbb{Z}_2$-torsion.*

Proof. Since G is non-planar, by Kuratowsky's theorem it contains a subgraph G' which is a subdivision of K_5 or $K_{3,3}$. By Lemma 11 and Lemma 12, both $H_1(K_5; \mathbb{Z})$ and $H_1(K_{3,3}; \mathbb{Z})$ have a generator of $\mathbb{Z}_2$-torsion of type $(2^k, 1^{n-2k})$. Hence, by Proposition 13, also $H_1(G'; \mathbb{Z})$ contains such a $\mathbb{Z}_2$-torsion element; thus by Proposition 15 also $H_1(G; \mathbb{Z})$ does. $\square$

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