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Symmetries on juggling varieties

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Abstract

This thesis aims to introduce and study an isotropic subvariety of a quiver Grassmannian for the equioriented cycle. That is, the subvariety of points consisting of collections of vector subspaces satisfying conditions derived from endowing the ambient space with a symplectic form. It seeks to merge and expand two articles that I have published during my PhD, one that I have co-authored and one under my sole name. It is the first step in this direction, as the symplectic approach has only ever been considered for acyclic quivers. Said quiver Grassmannian is obtained by linearly degenerating the action of the general linear group GL_{2n} on the Grassmannian $\mathrm{Gr}(k, 2n)$ in a way that it is compatible with a fixed symplectic form on $\mathbb{C}^{2n}$, so that the process restricts to both the symplectic group Sp_{2n} and the isotropic Grassmannian $\mathrm{Gr}(k, 2n)^{sp}$. This results in the degeneration of the former acting on that of the latter, and producing a cellular decomposition of the variety into its orbits. We then investigate the poset structure on the set of orbits, given by inclusion of the closures. We explicitly describe the group, compute the dimension of the cells, and develop the underlying combinatorics. Particular focus is then given to the Lagrangian case, as it is especially interesting for dimensional reasons. Many more results can be proven in this case, such as describing the irreducible components of the subvariety, labeling each cell with an element of an affine Coxeter group of type C whose length equals the dimension of the cell, and equipping the subvariety with a skeletal action of an algebraic torus. This torus allows us to show that the closure-inclusion order on the set of orbits for the action of the symplectic subgroup on the subvariety is inherited by the closure-inclusion order on the set of orbits for the action of the ambient group on the ambient variety.

Keywords: quiver Grassmannian, flag variety, linear degeneration, symplectic form, affine permutations.

Contents

Notation

$[n, m]$	The set of integers from n to m
$[n]$	The set of integers from 1 to n
$\binom{[N]}{k}$	The set of subsets of $\{1, 2, \ldots, N\}$ with cardinality k
Δ_n	The equioriented cyclic quiver with n vertices
$\mathbb{P}^n$	The complex n-dimensional projective space
$\vec{A}_n$	The equioriented line quiver with n vertices
e_i	The i-th standard basis vector for $\mathbb{C}^n$
g^{-t}	Short for $(g^{-1})^t = (g^t)^{-1}$
$S(V)$	The symmetric algebra of a vector space V
S_n	The symmetric group on n letters
$V^\perp$	The subspace orthogonal to V with respect to a fixed non-degenerate bilinear form
V_I	The coordinate vector subspace of $\mathbb{C}^n$ spanned by standard basis vectors indexed by elements in $I \subseteq [n]$

Introduction

The main object of study of this thesis is a linear degeneration $X(k, 2n)^{sp}$ of the isotropic Grassmannian $\mathrm{Gr}(k, 2n)^{sp}$, the locus of k-dimensional subspaces of $\mathbb{C}^{2n}$ on which a fixed symplectic form vanishes. We realize it as a subvariety of a quiver Grassmannian called the juggling variety, by defining a symmetry on the quiver which, together with a fixed symplectic form on the total space of the representation, allows us to define a notion of orthogonal point to a given point of the variety. We investigate the actions of some algebraic groups on the subvariety, a cellular decomposition given by group orbits, and the underlying combinatorics. This work was first started in collaboration with Evgeny Feigin, Martina Lanini and Alexander Pütz, resulting in the joint paper [FLMP25], and then continued in the article [Mic25] where I explored the Lagrangian case in further detail.

Quiver Grassmannians, first introduced by Schofield [Sch92], lie at the intersection of representation theory, algebraic geometry and combinatorics, providing a framework to study many projective varieties using results from several branches of mathematics. All projective varieties can be realized as quiver Grassmannians [Rei13, Rin18], many of which in a simple fashion for elementary quivers, like the widely studied projective spaces, Grassmannians and flag varieties. The juggling varieties $X(k, N)$ first appeared as local models of Shimura varieties for the affine Grassmannian [Gö01, Gai01, PRS13, Zho19]. The approach via quiver Grassmannians was developed in [FLP22]: here the juggling variety $X(k, N)$ is realized as a quiver Grassmannian for a nilpotent representation $U_{[N]}$ of the equioriented cycle Δ_N on N vertices, and is shown to be a flat, linear degeneration of the Grassmannian $\mathrm{Gr}(k, N)$. These papers are part of a larger trend, established in the last 10-15 years: there has been growing interest on quiver Grassmannians for the equioriented cycle, their combinatorics and their cohomology [Pü22, LP23a, LP23b, FLP24, FLP25], as well as the global study of degenerations of flag varieties [Fei12, FFL11, CIFR12, FFR17, CIFF$^+$17], whose goal is to understand under which hypothesis the degenerations keep some, if any, of the properties of the ambient variety.

By employing techniques from quiver representation theory, various characteristics of $X(k, N)$ are deduced in [FLP22], reminiscent of the geometry of flag varieties for GL_N; we list a few in this paragraph. The juggling variety admits a cellular decomposition in terms of the orbits for the automorphism group G_N of $U_{[N]}$, which itself is a degeneration of the general linear group GL_N. It is also shown that, with respect to the closure-inclusion order, the set of the G_N-orbits is isomorphic to the poset of (k, N)-*juggling patterns*. They are combinatorial objects, first introduced in [KLS13] to parametrize positroid varieties, a refinement of the Richardson decomposition of the Grassmannian [KL02, Ric92]. Equivalent posets are those of plabic graphs and Grassmann necklaces, used to label positroid varieties in the totally nonnegative Grassmannian [Pos06, Lam15]. There also exists an order-preserving bijection between the poset of G_N-orbits in $X(k, N)$ and a lower-order ideal of a

Coxeter group of affine type $A_{N-1}^{(1)}$, with respect to the Bruhat order; its elements are called affine permutations, and this bijection is such that the dimension of a G_N-orbit is precisely the length of the corresponding affine permutation. The Bruhat order is generated by reflections, which correspond in the variety to one-dimensional orbits for the action of an $(N+1)$-dimensional algebraic torus T. There exists a unique zero-dimensional T-orbit, i.e. fixed point, in each G_N-orbit, therefore they are also parametrized by juggling patterns. Each juggling pattern uniquely determines a particular subquiver of the coefficient quiver (Definition 2.1.7) for the representation $U_{[N]}$, and the counterpart on this level to a one-dimensional T-orbit is called a *mutation*: it consists of cutting and glueing the segments of the coefficient quiver, and these moves offer great help in visualizing the combinatorics. Thus, mutations generate the closure-inclusion order on the set of G_N-orbits, just like reflections do for the Bruhat order in a Coxeter group, and furthermore one can work out the dimension of an orbit by counting mutations. Lastly, in the same paper [FLP22], the T-action is also shown to be *skeletal*, i.e. it has finitely many fixed points and one-dimensional orbits, therefore the T-equivariant cohomology of $X(k, N)$ with rational coefficients can be computed via the theory of Goresky, Kottwitz and MacPherson, illustrated in [GKM98].

Shimura varieties for classical groups that are not of type A were studied in [Gö03, Pap18, PZ22], and while quiver symmetries have been utilized in [BF19, BF20, BCI21, BCIFF] to study orthogonal and symplectic subvarieties of quiver Grassmannians for type A Dynkin quivers, the work in this thesis opens the road to the extension of the cyclic quiver approach to the symplectic case.

In order to define the desired subvariety, we fix a symplectic form on $\mathbb{C}^{2n}$ and extend it to the total space of the representation $U_{[2n]}$, which then becomes a symmetric representation in the sense of Derksen and Weyman [DW02]. This means, among other things, that its dimension vector is compatible with the quiver symmetry and the two vector spaces on the vertices in any given orbit are dual to one another. This takes inspiration from the symmetry on the equioriented line quiver, which gives rise to an automorphism of the flag variety whose fixed points make up the symplectic flag variety. We obtain a notion of orthogonal subrepresentation τV to a subrepresentation V of $U_{[2n]}$ (Definition 3.2.4). Thus, for $k \leq n$, we are able to define $X(k, 2n)^{sp}$, the subvariety of $X(k, 2n)$ consisting of subrepresentations V contained in τV; for $n \leq k \leq 2n$ the inclusion is instead reversed.

Letting G_{2n}^{sp} be the subgroup of G_{2n} that preserves the symplectic form, we show that G_{2n}^{sp} and $X(k, 2n)^{sp}$ are degenerations of the symplectic group Sp_{2n} and of the isotropic Grassmannian $\mathrm{Gr}(k, 2n)^{sp}$ respectively (§3.2.1). We are able to study both the group and its Lie algebra, obtained as a subalgebra of $\mathrm{Lie}(G_{2n})$, by "folding" the ambient spaces in half with automorphisms of order two. This allows us to produce a set of complete and explicit equations that cut out the subgroup of G_{2n} (Lemma 3.2.8), as well as a basis of the Lie subalgebra (Definition 3.2.11). We also show that the G_{2n}^{sp}-action on $X(k, 2n)^{sp}$ has similar properties to the action of G_{2n} on $X(k, 2n)$. Each orbit coincides with the intersection of the appropriate G_{2n}-orbit with the subvariety, and they form a cellular decomposition (Proposition 3.3.1 and Corollary 3.3.1.1). We define *symplectic juggling patterns* in order to parametrize these orbits: in analogous fashion to the case of the isotropic Grassmannian, we first find the combinatorial counterpart to taking the orthogonal subrepresentation of a given subrepresentation, and then consider those juggling patterns satisfying a condition similar to isotropy for vector spaces. In the same vein, we observe that applying a mutation on a symplectic juggling pattern can produce a non-symplectic one, but this hurdle can be overcome because there is always another mutation that one can apply next which "fixes" the issue (Lemma

3.3.14). We define symplectic mutations to be either the composition of two such mutations, or one single mutation between symplectic juggling patterns. A consequence of this definition is that, analogously to the previous case, symplectic mutations allow one to compute the dimension of a G_{2n}^{sp}-orbit (Lemma 3.3.17). We also conjecture that the order inherited by the symplectic juggling patterns as a subset of the set of juggling patterns coincides with the G_{2n}^{sp}-orbit closure-inclusion order, as well as with the order generated by composing symplectic mutations.

Particular attention is then given to the Lagrangian case, the one where k is equal to n. This is the most interesting one, as $X(n,2n)^{sp}$ is now the fixed-point subvariety for an automorphism of the ambient variety of order two, just like the Lagrangian Grassmannian $\Lambda(2n)$ is fixed by an automorphism of $\mathrm{Gr}(n,2n)$. This added piece of structure allows us to prove a core lemma (Lemma 4.1.1), which paves the road to several other results. The key observation is that, for dimensional reasons, in this setting the condition of being symplectic is the strictest. This also makes it so that we can transfer the symmetry from the variety to the poset of juggling patterns, and create an order-two automorphism of the type A group of affine permutations. Its fixed-point subgroup is a Coxeter group of affine type C, whose elements are, unsurprisingly, also called *symplectic*; here, a reflection in type C is either a single reflection of type A, or the product of two reflections in the same orbit (Proposition 4.2.10). The symplectic counterparts to many of the posets mentioned above, such as plabic graphs, are covered in [Kar18], where they are used to parametrize the projected Richardson varieties in the Lagrangian Grassmannian $\Lambda(2n)$.

The definition of symplectic mutations as either a "correct" mutation or the composition of two paired "incorrect" ones hints at them behaving in the same way as reflections in the Coxeter group of type C. This is indeed the case: we show that the Bruhat order on the lower-order ideal of symplectic permutations which correspond to symplectic juggling patterns coincides with the geometric closure-inclusion order on the set of orbits (Theorem 4.4.3). In other words, the conjecture above holds in the Lagrangian case. We prove it by defining the action of an $(n+1)$-dimensional algebraic torus T^{sp} on $X(n,2n)^{sp}$ under which the G_{2n}^{sp}-orbits are stable, and whose one-dimensional orbits correspond to Bruhat order relations given by reflections, and therefore symplectic mutations. We show that this action is again skeletal (Theorem 4.3.3), and prove that the theory of Goresky, Kottwitz and MacPherson also applies to the T^{sp}-variety $X(n,2n)$, therefore the computation for its T^{sp}-equivariant cohomology group is straightforward.

Structure of the thesis

This thesis is structured as follows. In Chapter 1 we go over some basics on flag varieties and quiver Grassmannians, since flag varieties for GL_N can be realized as quiver Grassmannians for equioriented type A Dynkin quivers, and type C flag varieties are their fixed point subsets under an automorphism arising from the quiver symmetry. We also introduce the notion of degeneration, both for quiver representations and for projective varieties. We show how they are related, so that, after introducing the juggling variety $X(k,N)$ in Chapter 2, we are able to see how it arises as a degeneration of the Grassmannian $\mathrm{Gr}(k,N)$. In this chapter, based primarily on [FLP22], we examine the juggling variety in depth, so that we may better understand the tools we can use to study $X(k,2n)^{sp}$. Here we see that some of its properties resemble those of the flag variety, and the goal of the following chapters is to deduce similar results for the symplectic subvariety. Chapter 3 covers the results obtained in [FLMP25], discussed above, which hold for any k; here we also discuss the orthogonal version of this problem in both even and odd dimensions, and see why it is not nearly as interesting as the more classical setting [Mih07, Pro88] even though the combinatorics is identical to the symplectic version. Lastly, Chapter 4 focuses on the aforementioned Lagrangian case, following [Mic25].

Chapter 1

Flag varieties and quiver Grassmannians

We first go over some basics on flag varieties and quiver representation theory, and present several examples. We then devote our attention to flag varieties for the symplectic group, and lastly we end the chapter discussing linear degenerations of quiver representations and Grassmannians.

1.1 Overview of flag varieties

We will recall the theory in a more general setting than strictly needed, letting G be a connected, reductive, algebraic group over an algebraically closed field throughout this whole section. The unfamiliar reader is encouraged to picture G as the general linear group $GL_n(\mathbb{C})$, consisting of invertible $n \times n$ matrices with complex coefficients, which is the most classical and most eloquent case, and which will make up the bulk of our more concrete examples. Some classical texts on the subject include [Hum75, Bor91, Spr98, LB18, Bri05].

1.1.1 Borel and parabolic subgroups

Definition 1.1.1. A *Borel subgroup* of G is a maximal, connected, solvable subgroup. A *parabolic subgroup* is a subgroup of G which contains a Borel subgroup.

We will work only in the field of complex numbers, so we will simply write GL_n from this point on, in place of $GL_n(\mathbb{C})$. Its standard Borel subgroup is that of upper-triangular matrices, denoted by B_n. Consequently, parabolic subgroups of GL_n containing B_n are the so called "staircase" subgroups [AB95, §2.5], i.e. those with matrices that have the block form

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots \\ 0 & A_{22} & A_{23} & \dots \\ 0 & 0 & A_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the diagonal blocks A_{ii} are square [Hum75, §29.3].

Definition 1.1.2. Let B be a fixed Borel subgroup of G. The *flag variety* of G is defined as

$$G/B\,,$$

the set of left cosets of B.

This is a well-defined projective variety, independent up to isomorphism on the choice of B despite the apparent reliance on a given Borel subgroup. This is because all Borel subgroups are conjugate to one another, therefore isomorphic, and are self-normalizing [Spr98, §6.2]. More generally, the *partial flag variety* corresponding to a parabolic subgroup $P \supseteq B$ is G/P.

Now let us see what (partial) flag varieties for $G = GL_n$ look like. We do so by defining some projective varieties dependent on a sequence of integers, and proving that they are isomorphic to appropriate geometric quotients of GL_n. The *complete* flag variety of $\mathbb{C}^n$, denoted by $\mathfrak{Fl}(n)$, is the projective variety whose points, called *complete flags*, are chains $V_1 \subset V_2 \subset \cdots \subset V_{n-2} \subset V_{n-1}$ of $n-1$ vector subspaces of $\mathbb{C}^n$ such that $\dim V_i = i$. A *partial flag* is a chain of strictly nested subspaces where we allow some dimensions to be missing. For any increasing list of positive integers $k_1 < k_2 < \cdots < k_t < n$, the a variety whose points are partial flags consisting of t subspaces with the prescribed dimensions $(k_1, k_2, \ldots, k_t)$ will be denoted by $\mathfrak{Fl}(n; \underline{k})$; the list $\underline{k}$ of dimensions is called the *signature* of the partial flag variety.

Remark 1.1.3. The nomenclature was first set when dealing with the GL_n case, and then expanded to the general theory. The term flag was chosen because, for $n = 3$, a point in the complete flag variety is comprised of a line, i.e. a "pole", then of a 2-dimensional space, i.e. the flag swinging from it. Therefore we use the term flag variety to refer to both the geometric quotients of an algebraic group G, and these varieties comprised of "higher dimensional flags". In the rest of this section it will always be clear whether we are talking about the GL_n case or the general one.

The action of GL_n on $\mathbb{C}^n$ extends to any such flag variety, partial or complete, in the obvious way: the image of the flag $V = (V_i)_i$ under the group element g is $g \cdot V = (gV_i)_i$. This action is transitive, so all classical flag varieties $\mathfrak{Fl}(n; \underline{k})$ are homogeneous spaces. Consider the *standard* flag $E \coloneqq (E_i)_i$, where E_i is spanned by the first k_i vectors in the standard basis of $\mathbb{C}^n$. The stabilizer of the standard flag in GL_n is a parabolic subgroup containing B_n, therefore it is a staircase subgroup with diagonal blocks of size $k_{i+1} - k_i$, for $i = 1, \ldots, t$. Here $k_0 = 0$ and $k_{t+1} = n$. In other words, it is equal to the semi-direct product $U \rtimes L$ of its unipotent radical with its Levi subgroup. The former is the maximal subgroup of elements x satisfying $(x - \mathrm{Id})^m = 0$ for some $m \geq 0$, and the latter is the subgroup of diagonal block matrices with blocks the sizes given above. In addition, stabilizers of flags are conjugate subgroups by transitivity of the action of GL_n. Indeed, the stabilizer of a generic flag gE is

$$\mathrm{Stab}(gE) = g\,\mathrm{Stab}(E) g^{-1}\,.$$

Hence, we have found a projective variety equipped with a left, transitive action of GL_n for which any stabilizer is isomorphic to the parabolic subgroup $\mathrm{Stab}(E)$, therefore $\mathfrak{Fl}(n; \underline{k})$ must be isomorphic to the flag variety $G/\mathrm{Stab}(E)$. Moreover, this parabolic subgroup has dimension

$$\sum_{i=0}^{t} k_{i+1}(k_{i+1} - k_i)\,,$$

and as a result the (only) orbit has dimension

$$n^2 - \dim \mathrm{Stab}(E) = \sum_{i=0}^{t} (n - k_{i+1})(k_{i+1} - k_i).$$

In the case of the complete flag variety $\mathfrak{Fl}(n)$, this dimension is equal to

$$\binom{n}{2} = \frac{n(n-1)}{2} .$$

The most widely known examples of partial flag varieties are those with the shortest signature: for a signature with one element, say (k), we obtain the Grassmannian $\mathrm{Gr}(k, n)$, the variety of k-dimensional subspaces of $\mathbb{C}^n$. It has dimension $k(n-k)$.

Notation: we will denote the standard basis of $\mathbb{C}^n$ with $\{e_1, e_2, \dots, e_n\}$.

Example 1.1.4. Let us look at the partial flag variety for $n = 4$ and signature $(1, 3)$: the corresponding parabolic subgroup, which is the stabilizer of the standard flag

$$\langle e_1 \rangle \subset \langle e_1, e_2, e_3 \rangle ,$$

is given by matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

and its unipotent radical and Levi subgroup are respectively

$$U = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & 0 & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} , \quad L = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \right\} .$$

1.1.2 The Bruhat decomposition

Let us now go back to the general case. Fix a maximal algebraic torus $T \subset B$, and consider its normalizer $N(T)$ in G. The quotient of this normalizer by T itself is a discrete group. To the pair (G, T), one can associate a root datum, and therefore define a Coxeter system (W, S), where W coincides with $N(T)/T$ [Spr98, Chapter 7]. Plenty of literature discusses the theory of root systems, which connects several topics, from semisimple Lie algebras and highest weight theory, to Coxeter systems and group theory.

It is a well-known fact that the Coxeter group W parametrizes the double B-cosets that G is partitioned into [Spr98, §8.3].

Definition 1.1.5. The *Bruhat decomposition* of G is

$$G = \bigsqcup_{w \in W} B g_w B = BWB ,$$

where g_w is a representative in G of the element $w \in W$.

More generally, this decomposition descends to the geometric quotient G/P for any parabolic subgroup P [LB18, §10.9], therefore all (partial) flag varieties are stratified by the action of B on the left. The B-orbits are affine spaces called *Schubert cells*, parametrized by the cosets in W with respect to an appropriate subgroup of W dependent on P. They are denoted by C_w, with w any element of such a coset. In particular, for $P = B$ the subgroup is trivial so each coset consists of a single group element, and we have

$$\dim C_w = \ell(w)\,,$$

where $\ell(w)$ is the length of w in the Coxeter system (W, S) [Spr98, §8.5]. Their closures are the *Schubert varieties* of the flag variety, and $X_w \coloneqq \overline{C_w}$ is given by the union of the cell itself and all other Schubert cells whose corresponding group element w' is lower than w in the Bruhat order:

$$X_w = \bigsqcup_{w' \leq w} C_{w'}\,. \tag{1.1.1}$$

In the case where $G = GL_n$, the standard choice of maximal torus in B_n is the subgroup T_n of diagonal matrices. It follows that each Schubert cell in the complete flag variety $\mathfrak{Fl}(n)$ contains exactly one flag with coordinate vector spaces, i.e. spaces spanned by subsets of the standard basis [Bri05, §1.1]. Such a flag is obtained by permuting the indices of the basis vectors that make up the spaces of the standard complete flag E, therefore the Schubert cells and varieties are parametrized by the symmetric group S_n , a Weyl group of type A_{n-1}. The most natural representative in GL_n of the flag E_w, the one obtained by acting on E via the permutation $w \in S_n$, is the *permutation matrix* for w, which has a 1 in positions $(w(i), i)$ and 0 elsewhere.

Example 1.1.6. For $n = 4$, the complete flag

$$\langle e_2 \rangle \subset \langle e_2, e_3 \rangle \subset \langle e_1, e_2, e_3 \rangle$$

is obtained by acting on the standard flag in $\mathfrak{Fl}(4)$ with the permutation $(123) \in S_4$, i.e. with the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Schubert cell corresponding to $E_{(123)}$ consists of flags of the form

$$\langle e_2 + \lambda e_1 \rangle \subset \langle e_2 + \lambda e_1, e_3 + \mu e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle$$

for $\lambda, \mu \in \mathbb{C}$, and the coordinate flags in its closure are

$$\begin{aligned} E_{\mathrm{Id}} &= \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ E_{(12)} &= \langle e_2 \rangle \subset \langle e_2, e_3 \rangle \subset \langle e_1, e_2 e_3 \rangle \\ E_{(13)} &= \langle e_1 \rangle \subset \langle e_1, e_3 \rangle \subset \langle e_1, e_2, e_3 \rangle\,. \end{aligned}$$

For partial flag varieties, the stabilizer of the standard flag under the S_n-action on $\mathfrak{Fl}(n; \underline{k})$ is generated by the reflections s_i for all i that do not appear in $\underline{k}$. The set of right cosets with respect to this subgroup then parametrizes the coordinate flags of signature $\underline{k}$, since it acts transitively on it. We then deduce that, when considering a signature (k) with on entry, there are $\binom{n}{k}$ Schubert cells in the Grassmannian $\mathrm{Gr}(k, n)$, indexed by cardinality-k subsets of $[n] \coloneqq \{1, 2, \ldots, n\}$ because each coordinate flag is determined by k basis vectors, out of the total n.

Example 1.1.7. If we consider the partial flag variety with signature $(1,3)$, the stabilizer in S_4 of the standard flag is $H = \langle(2,3)\rangle$. The Schubert cell for

$$E_{(1,2)(3,4)H} = \langle e_2 \rangle \subset \langle e_1, e_2, e_4 \rangle$$

consists of points of the form

$$\langle e_2 + \lambda e_1 \rangle \subset \langle e_1, e_2, e_4 + \mu e_3 \rangle$$

and its closure contains the Schubert cells for

$$\begin{aligned} E_H &= \langle e_1 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ E_{(1,2)H} &= \langle e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \\ E_{(3,4)H} &= \langle e_1 \rangle \subset \langle e_1, e_2, e_4 \rangle . \end{aligned}$$

Lastly, we briefly go over a particular $\mathbb{C}^*$-action on the flag varieties. Observe that the action of $\mathbb{C}^*$ on $\mathbb{C}^n$ given by

$$z \cdot \left(\sum_{i=1}^{n} \lambda_i e_i \right) = \sum_{i=1}^{n} z^{n-i+1} \lambda_i e_i$$

extends to all flag varieties $\mathfrak{Fl}(n; \underline{k})$ in the natural way, i.e. by letting $z \in \mathbb{C}^*$ act like the matrix

$$\begin{pmatrix} z^n & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & z^2 & 0 \\ 0 & \dots & 0 & z \end{pmatrix}$$

in GL_n. The $\mathbb{C}^*$-fixed points are the coordinate flags E_w, and the attracting sets

$$\{F \mid \lim_{z \to 0} z \cdot F = E_w\}$$

for this action coincide with the Schubert cells.

1.1.3 Flag varieties for the symplectic group

Consider now the even-dimensional space $\mathbb{C}^{2n}$, which we can equip with a symplectic form, i.e. a skew-symmetric, non degenerate, bilinear form. We will denote it by $(-,-)$, and use the standard notation $V^\perp$ for the orthogonal subspace

$$\{w \in \mathbb{C}^{2n} \mid (w, v) = 0 \;\; \forall v \in V\} \tag{1.1.2}$$

to a given subspace $V \subseteq \mathbb{C}^{2n}$. Assume, without loss of generality, that the subspace orthogonal to any standard basis vector is a coordinate subspace. This means that the Gram matrix Ω for the symplectic form in the standard basis is anti-symmetric and has one nonzero entry in each row and column. A matrix $g \in GL_{2n}$ is called symplectic if it preserves the form, that is, if it satisfies

$$(g(v), g(w)) = (v, w)$$

for all $v, w \in \mathbb{C}^{2n}$. These matrices make up a connected and reductive subgroup of GL_{2n} of dimension $n(2n+1)$, denoted by Sp_{2n}. Its standard Borel subgroup SpB_{2n} is given by upper-triangular symplectic matrices [Spr98, Exercise 6.2.11(4)], and the corresponding complete flag variety can be realized as a subvariety of $\mathfrak{Fl}(2n)$. More explicitly, we define $\mathfrak{Fl}(2n)^{sp}$ to be the subvariety of complete flags fixed under the automorphism

$$(V_i)_i \longmapsto \left(V_{2n-i}^{\perp}\right)_i, \tag{1.1.3}$$

also called symplectic flags. The map is well-defined, since $\dim V^{\perp} = 2n - \dim V$ and $V^{\perp} \supseteq W^{\perp}$ for any pair of subspaces $V \subseteq W$. Furthermore, Sp_{2n} is the subgroup of fixed points in GL_{2n} under the group automorphism of order two

$$g \longmapsto \Omega \left(g^{-1}\right)^t \Omega^{-1}.$$

The subgroup acts transitively on this subvariety, and the stabilizer of the standard flag is again the Borel subgroup mentioned above. Analogously to the argument in the previous section, this implies that $\mathfrak{Fl}(2n)^{sp}$ is the geometric quotient of Sp_{2n} by SpB_{2n}.

To examine its Schubert cells, we observe that the choice of Ω produces an automorphism of the symmetric group S_{2n}, whose fixed-point subgroup is a Weyl group of type C. It is the Coxeter group of (Sp_{2n}, SpT_{2n}), where SpT_{2n} is the algebraic torus of symplectic diagonal matrices. Analogously to the previous case, the fixed-point subgroup of S_{2n} parametrizes the Schubert cells C_w^{sp} since the flags are complete. Moreover, the Schubert varieties X_w^{sp} are the union of cells $C_{w'}^{sp}$ with $w' \leq w$ with respect to the Bruhat order, and the dimension of C_w^{sp} is equal to the length of w in this Weyl group.

Partial flag varieties for Sp_{2n} are also fixed-point subvarieties. Consider $\mathfrak{Fl}(2n; \underline{k})$ for a symmetric signature $\underline{k}$, one such that whenever a number k appears in it, so does $2n-k$. Then we can apply (1.1.3) again, and obtain the corresponding fixed-point subvariety $\mathfrak{Fl}(2n; \underline{k})^{sp}$. Once again considering the case of a signature (k) with one entry, for $k \leq n$ the *isotropic* Grassmannian $\mathrm{Gr}(k, 2n)^{sp}$ is defined as the variety of k-dimensional subspaces V of $\mathbb{C}^{2n}$ contained in $V^{\perp}$, which means the symplectic form is zero when restricted to $V \times V$. If $2n - k \geq n$, we can instead define the Grassmannian of *coisotropic* vector spaces, those that contain their orthogonal subspace. These two Grassmannians are isomorphic via $V \longmapsto V^{\perp}$, and they are both isomorphic to the symplectic partial flag variety of signature $(k, 2n-k)$ via the projection on the left and right components respectively.

$$\begin{array}{ccc} & (V \subseteq V^{\perp}) & \\ \swarrow & & \searrow \\ V & & V^{\perp} \end{array}$$

All of these varieties have dimension $k(2n-k) - \frac{k(k-1)}{2}$, and the Schubert cells in the isotropic

Grassmannian are parametrized by subsets J of $[2n]$ with the property that $(e_j, e_{j'}) = 0$ for all $j, j' \in J$.

When $k = n$, the isotropic and coisotropic Grassmannians coincide, since for dimensional reasons the vector space inclusions become equalities. In this case we refer to this variety as the *Lagrangian* Grassmannian, and we say that a vector subspace $V \subseteq \mathbb{C}^{2n}$ of dimension n is Lagrangian if it coincides with $V^{\perp}$.

1.2 Quiver representations

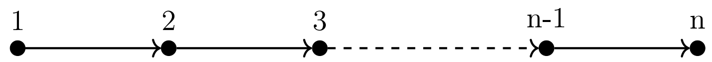
In this section we introduce quiver representations and go over some key notions. For more, see for example [KJ16].

Definition 1.2.1. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a finite oriented graph, where Q_0 is the vertex set, Q_1 is the edge set, and the maps $s, t\colon Q_1 \longrightarrow Q_0$ specify the orientation of the edges. An *arrow* is an oriented edge, often denoted with $\alpha\colon i \longrightarrow j$, where $i = s(\alpha)$ is its *source* and $j = t(\alpha)$ is its *target*.

Example 1.2.2. A few of the quivers that will be mentioned throughout this thesis are

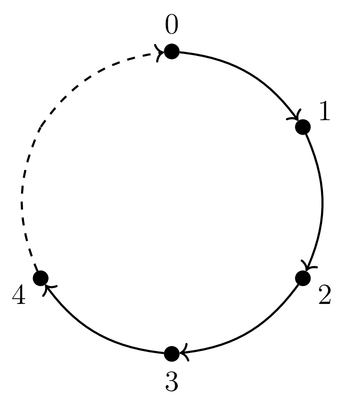
◇ the singleton:

◇ the equioriented type A quiver on n vertices, denoted by $\vec{A}_n$:

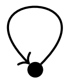


its set of vertices is $[n]$, the set of integers from 1 to n, and it has arrows $i \longrightarrow i+1$ for $i \in [n-1]$;

◇ the equioriented cycle on n vertices, denoted by Δ_n :



its vertex set equal to $\mathbb{Z}_n$, the ring of integers modulo n, and it has arrows $i \longrightarrow i+1$ for all $i \in \mathbb{Z}_n$. We will always number the vertices so that the topmost one is labeled 0. The quiver Δ_1



is called the *1-loop*, as the quiver with one vertex and n loops is aptly named the *n-loop*.

As stated already, we work over the field of complex numbers, but all the definitions in the rest of the chapter work for arbitrary fields.

Definition 1.2.3. A *representation* $M = (M_i, M_\alpha)$ of Q is a collection of finite-dimensional vector spaces $(M_i)_{i\in Q_0}$, one for each vertex, and linear maps $(M_\alpha\colon M_{s(\alpha)} \longrightarrow M_{t(\alpha)})_{\alpha\in Q_1}$, one for each edge. A *morphism* between two Q-representations M and N is a collection of linear maps $\varphi = (\varphi_i\colon M_i \longrightarrow N_i)_{i\in Q_0}$ between the vector spaces of M and N over each vertex, such that for any arrow $\alpha\colon i \longrightarrow j \in Q_1$ we have $N_\alpha \circ \varphi_i = \varphi_j \circ M_\alpha$, i.e. the following square

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_i} & N_i \\ {\scriptstyle M_\alpha}\downarrow & & \downarrow{\scriptstyle N_\alpha} \\ M_j & \xrightarrow{\varphi_j} & N_j \end{array}$$

is commutative. A *subrepresentation* $V \subseteq M$ is a collection of vector subspaces $(V_i \subseteq M_i)_{i\in Q_0}$ over the vertices such that, for any arrow $\alpha\colon i \longrightarrow j$, the inclusion $M_\alpha(V_i) \subseteq V_j$ is satisfied. The *dimension vector* of M is the element of $\mathbb{N}^{Q_0}$ given by $\underline{\dim}\, M \coloneqq (\dim M_i)_{i\in Q_0}$.

Representations of a given quiver form the abelian category rep_Q [GM03, §IV.4], so as usual an isomorphism is an invertible morphism, and an automorphism is an isomorphism from a representation to itself. In this category, direct sum and quotients are defined in the way one would expect, vertex by vertex. For instance, the direct sum of the following $\vec{A}_3$ representations

$$\mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C} \xrightarrow{0} 0 \bigoplus \mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C}$$

results in

$$\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \xrightarrow{\pi_2} \mathbb{C}\ ,$$

where π_2 denotes the projection onto the second coordinate. Additionally, notice that a subrepresentation $V \subseteq M$ is implicitly a representation with the maps given by the restrictions $M_\alpha|_{V_{s(\alpha)}}$.

Remark 1.2.4. Representations of the singleton quiver are just finite-dimensional vector spaces; thus, when applied to the singleton, quiver representation theory reduces to linear algebra.

Example 1.2.5. The $\vec{A}_3$-representation

$$\mathbb{C}^4 \xrightarrow{\mathrm{Id}} \mathbb{C}^4 \xrightarrow{\mathrm{Id}} \mathbb{C}^4$$

has dimension vector $(4, 4, 4)$, and is isomorphic to any other representation with the same dimension vector and with invertible maps g and h on the arrows, via the following morphism.

$$\begin{array}{ccccc} \mathbb{C}^4 & \xrightarrow{\mathrm{Id}} & \mathbb{C}^4 & \xrightarrow{\mathrm{Id}} & \mathbb{C}^4 \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle hg} \\ \mathbb{C}^4 & \xrightarrow{g} & \mathbb{C}^4 & \xrightarrow{h} & \mathbb{C}^4 \end{array}$$

One of its subrepresentations is, for instance, given by the spaces $\langle e_1\rangle, \langle e_1, e_2\rangle, \langle e_1, e_2, e_3\rangle$. The reader will notice that this subrepresentation is exactly the standard flag in $\mathfrak{Fl}(4)$.

Definition 1.2.6. The *path algebra* $\mathbb{C}Q$ of a quiver Q over the complex numbers is the associative algebra whose underlying vector space has a basis given by directed paths

$$\alpha = \alpha_m \circ \cdots \circ \alpha_2 \circ \alpha_1 \quad \text{with} \quad t(\alpha_i) = s(\alpha_{i+1})$$

in Q, and where the product is given by concatenation of paths:

$$\alpha \cdot \alpha' = \begin{cases} \alpha \circ \alpha' & \text{if } t(\alpha') = s(\alpha)\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Letting A be the ideal of $\mathbb{C}Q$ generated by all arrows, an ideal I is called *admissible* if $A^m \subseteq I \subseteq A^2$ for some $m \geq 2$.

This algebra is finite-dimensional if and only if Q contains no cycles. Otherwise, one can consider the quotient of $\mathbb{C}Q$ by an admissible ideal I of $\mathbb{C}Q$, which is finite-dimensional. It is a classical result that the category of Q-representations is equivalent to that of modules for the path algebra of Q, via

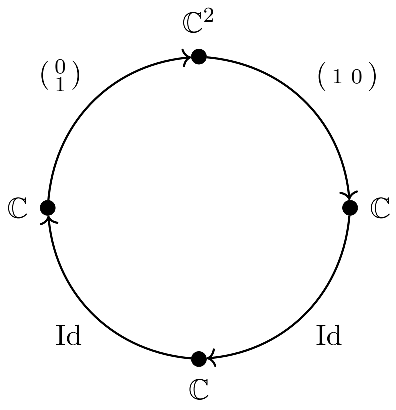
$$\begin{aligned} \mathrm{rep}_Q &\longrightarrow \mathbb{C}Q\text{-mod} \\ M = (M_i, M_\alpha) &\longmapsto \bigoplus_i M_i\,. \end{aligned}$$

The same holds for $(\mathbb{C}Q)/I$-mod and the category of Q-representation such that the compositions of the linear maps along paths in I vanishes. These categories are Krull-Schmidt, which means every representation is direct sum of indecomposables [KJ16, Theorem 1.11].

Example 1.2.7. The indecomposables in $\mathrm{rep}_{\vec{A}_n}$ are indexed by pairs of vertices $i \leq j$, and they have $\mathbb{C}$ on every vertex between i and j, identities on the arrows between them, and trivial spaces and maps elsewhere. Such a representation is denoted by $U(i, j)$. The indecomposable $U(2, 4)$ for $\vec{A}_6$ is

$$0 \longrightarrow \mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C} \xrightarrow{\mathrm{Id}} \mathbb{C} \longrightarrow 0 \longrightarrow 0\ .$$

Similarly, the indecomposable $U(i; l)$, with $i \in \mathbb{Z}_n$ and $l \geq 1$, for the equioriented cycle Δ_n is obtained by "wrapping" an indecomposable $U(i, i + l - 1)$ for an equioriented quiver of type A of sufficient length around the cycle, and performing direct sums of vector spaces whenever we loop around: we then obtain a representation such that the composition of the maps starting at vertex i vanishes exactly after l arrows. For instance the indecomposable Δ_4-representation $U(0; 5)$ looks like this.



1.3 Quiver Grassmannians

In this section we discuss quiver Grassmannians, since we will realize our main object of study as one, and then explain what a degeneration of a variety is, so that we will be able to properly introduce the main object and to give the reader all the information they need to visualize it. For an introductory overview, see [CI20].

Definition 1.3.1. Let let M be a representation of a quiver Q, and let $\underline{k} \in \mathbb{N}^{Q_0}$. The *quiver Grassmannian* $\mathrm{Gr}_{\underline{k}}(M)$ is the space of subrepresentations of M that have dimension vector $\underline{k}$:

$$\mathrm{Gr}_{\underline{k}}(M) \coloneqq \{U \subseteq M \mid \underline{\dim} U = \underline{k}\}.$$

Any quiver Grassmannians $\mathrm{Gr}_{\underline{k}}(M)$ is a projective variety: if we let $\underline{d} \coloneqq \underline{\dim} M$, since the inclusions $M_\alpha(U_i) \subseteq U_j$ are closed, the space embeds into the product

$$\prod_{i \in Q_0} \mathrm{Gr}(k_i, d_i),$$

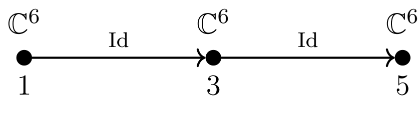
which in turn embeds into a projective space via the Plücker and Segre maps. In fact, it was shown by Markus Reineke in the aptly named paper [Rei13], that the converse also holds. Unfortunately this tells us that there can be no general tool or result applicable to all quiver Grassmannians; we must therefore study them in families.

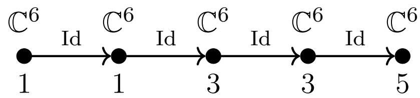
Example 1.3.2. As hinted at in Example 1.2.5, the complete flag variety $\mathfrak{Fl}(n)$ can be realized as the quiver Grassmannian of subrepresentations of the $\vec{A}_{n-1}$-representation

$$\mathbb{C}^n \xrightarrow{\mathrm{Id}} \mathbb{C}^n \xrightarrow{\mathrm{Id}} \mathbb{C}^n \dashrightarrow \mathbb{C}^n \xrightarrow{\mathrm{Id}} \mathbb{C}^n$$

with dimension vector $\underline{k} = (1, 2, 3, \dots, n-1)$.

Example 1.3.3. Analogously, one can realize in the same way any partial flag variety of GL_n of signature $(k_1, k_2, \dots, k_t)$, by either removing vertices and thus "shortening" the quiver from $\vec{A}_{n-1}$ to $\vec{A}_t$, or by repeating entries in the dimension vector, i.e. allowing repetition in the signature in order to reach $t = n-1$. We show this for the flag variety of GL_6 of signature $(1, 3, 5)$.





Definition 1.3.4. The *space of representations* of Q of dimension vector $\underline{d}$ is

$$R_{\underline{d}}(Q) \coloneqq \bigoplus_{\alpha\colon i\to j\in Q_1} \operatorname{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

and is equipped with the action of the algebraic group

$$G_{\underline{d}}(Q) \coloneqq \prod_{i\in Q_0} GL_{d_i}$$

given by $(g_i)_i \cdot (f_\alpha)_\alpha = \left(g_{h(\alpha)} \circ f_\alpha \circ g_{t(\alpha)}^{-1}\right)_\alpha$. A *basis* $\mathscr{B}$ of a Q-representation M is a collection of bases $(\mathscr{B}_i)_{i\in Q_0}$ of each of its vector spaces M_i.

Remark 1.3.5. Any Q-representation M over $\mathbb{C}$ in the sense of Definition 1.2.3 is isomorphic to a point in $R_{\underline{\dim}(M)}(Q)$ via choosing a basis to write the maps of M as matrices, and basis change corresponds to acting with $G_{\underline{\dim}(M)}(Q)$. The orbit of M is its isomorphism class, and its stabilizer is the group of its automorphisms $\operatorname{Aut}(M)$. Lastly, observe that $\operatorname{Aut}(M)$ acts on every quiver Grassmannian for M: for any arrow $\alpha\colon i \longrightarrow j$ and any group element $g \in \operatorname{Aut}(M)$, we have

$$M_\alpha(U_i) \subseteq U_j \implies g_j \circ M_\alpha(U_i) \subseteq g_j(U_j) \implies M_\alpha \circ g_i(U_i) = g_j \circ M_\alpha \circ g_i^{-1} \circ g_i(U_i) \subseteq g_j(U_j)$$

since the matrices g_i are invertible and commute with the linear maps on the arrows.

One can globally look at all quiver Grassmannians for a fixed dimension vector $\underline{k}$ and all representations in $R_{\underline{d}}(Q)$ together, by realizing them as fibers for the following proper $G_{\underline{d}}(Q)$-equivariant map.

Definition 1.3.6. Let $\operatorname{Gr}(\underline{k}, \underline{d})$ be the product of the classical Grassmannians $\operatorname{Gr}(k_i, d_i)$ over all vertices $i \in Q_0$, and let

$$\operatorname{Gr}_{\underline{k}}(\underline{d}) \coloneqq \{((V_i), (f_\alpha)) \mid f_\alpha(V_{s(\alpha)}) \subseteq V_{t(\alpha)} \;\forall\, \alpha \in Q_1\} \subseteq \operatorname{Gr}(\underline{k}, \underline{d}) \times R_{\underline{d}}(Q)\,.$$

The *universal quiver Grassmannian* $\pi_{\underline{k},\underline{d}}$ is the projection onto the second component

$$\pi_{\underline{k},\underline{d}}\colon \operatorname{Gr}_{\underline{k}}(\underline{d}) \longrightarrow R_{\underline{d}}(Q)\,.$$

Observe that by definition, the quiver Grassmannian $\operatorname{Gr}_{\underline{k}}(M)$ for a representation $M \in R_{\underline{d}}(Q)$ is the fiber of $\pi_{\underline{k},\underline{d}}$ over M. Moreover, fibers of isomorphic representations produce quiver Grassmannians that are isomorphic as projective varieties. Indeed, the fiber over $N = (g_j \circ f_\alpha \circ g_i^{-1}) \in R_{\underline{d}}(Q)$, for $g \in G_{\underline{d}}(Q)$ and $M = (f_\alpha) \in R_{\underline{d}}(Q)$, consists of points of the form

$$\left((g_i V_i), (g_j \circ f_\alpha \circ g_i^{-1})\right)\,,$$

where $((V_i), (f_\alpha))$ is a point in the fiber over M.

1.3.1 Degenerations

Fix a quiver Q and a dimension vector $\underline{d} \in \mathbb{N}^{Q_0}$.

Definition 1.3.7. Given two representations $M, N \in R_{\underline{d}}(Q)$, we say that M *degenerates* to N if N is contained in the closure of the $G_{\underline{d}}(Q)$-orbit of M. The degeneration process produces a partial order on $R_{\underline{d}}(Q)$, by setting $M \leq N$ whenever $N \in \overline{G_{\underline{d}}(Q) \cdot M}$.

Example 1.3.8. Consider the quiver

$$\vec{A}_2 \colon \bullet \longrightarrow \bullet \ .$$

The space $R_{(2,2)(\vec{A}_2)}$ is just $M_{2\times 2}(\mathbb{C})$, the space of 2×2 matrices, while the isomorphism class of the identity is GL_2, whose closure is the whole space. Thus the identity degenerates, for example, to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ via

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right) \underset{z \to 0}{\longrightarrow} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$$

or to the null map via

$$\left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right) \underset{z \to 0}{\longrightarrow} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) .$$

If M degenerates to N as representations, then for any dimension vector $\underline{k}$, $\mathrm{Gr}_{\underline{k}}(M)$ degenerates to $\mathrm{Gr}_{\underline{k}}(N)$ as projective varieties in the following sense:

Definition 1.3.9. A projective variety X *degenerates* to another projective variety Y if there exists a morphism

$$\pi \colon \mathcal{X} \longrightarrow A^1_{\mathbb{C}}$$

from a variety $\mathcal{X}$ to the affine line $A^1_{\mathbb{C}}$ such that the fiber over 0 is isomorphic to Y while every other fiber is isomorphic to X.

Example 1.3.10. Grassmannians degenerate to a host of different varieties, since they can be realized in many ways as quiver Grassmannians. Let us look at the complex sphere $\mathbb{P}^1 = \mathrm{Gr}(1, 2)$ as the quiver Grassmannian

$$\mathrm{Gr}_{(1,1)} \left(\mathbb{C}^2 \xrightarrow{\mathrm{Id}} \mathbb{C}^2 \right)$$

for the $\vec{A}_2$ quiver; it degenerates to the quiver Grassmannian for

$$\mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} \mathbb{C}^2 ,$$

the first representation in Example 1.3.8, which is a pair of one-dimensional projective spaces touching at one point. Explicitly, the two spheres are respectively made of points of the form

$$\left(V \, , \, \mathbb{C} \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \right)$$

and

$$\left(\mathbb{C} \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \, , \, W \right)$$

for V and W one-dimensional subspaces, and the meeting point is the subrepresentation

$$\left(\mathbb{C} \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \, , \, \mathbb{C} \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \right) .$$

Essentially, as z approaches 0, the quiver Grassmannian

$$\mathrm{Gr}_{(1,1)}\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right)\right),$$

isomorphic to $\mathbb{P}^1$ for all $z \in \mathbb{C}^*$, bends and shrinks in the middle, until it pinches at $z = 0$. Lastly, $\mathrm{Gr}(1, 2)$ also degenerates to the quiver Grassmannian for the null map, which is just $\mathbb{P}^1 \times \mathbb{P}^1$.

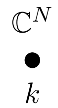
Chapter 2

The juggling variety

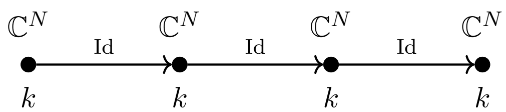
In this chapter we define and study a linear degeneration of the Grassmannian $\mathrm{Gr}(k, N)$, first introduced by Feigin, Lanini and Pütz in [FLP22]. We will report many results about this variety from the paper above, which we will use in the next chapters to derive properties of its symplectic subvarieties.

2.1 The juggling quiver Grassmannian

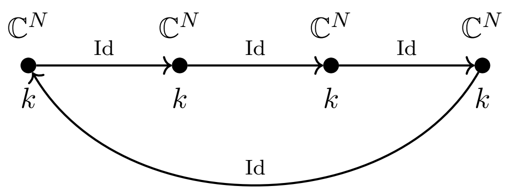
We fix two non-negative integers $k \leq N$. As mentioned previously, we want to realize $\mathrm{Gr}(k, N)$ as a quiver Grassmannian and then linearly degenerate it by degenerating the chosen representation. First, the most obvious choice of representation is the following one, for the singleton.



Here the dimension vector (k) is shown below the quiver. As hinted at in Example 1.3.10, we can use longer equioriented quivers of type A by putting $\mathbb{C}^N$ on every vertex and the identity on every map, and by choosing the dimension vector to be constant to k:



From now on, $\underline{k}$ will denote a dimension vector with all entries equal to k. The identities on the arrows and the constant dimension vector make sure that every subrepresentation has the same vector subspace of $\mathbb{C}^N$ on every vertex. Nothing changes if we close the loop by adding an arrow from the last vertex to the first one, provided that the corresponding linear map is also the identity.



The number of vertices in this equioriented cycle can be arbitrary, but for our purposes we set it to be exactly N, so we end up with a Δ_N representation with $\mathbb{C}^N$ and identities everywhere. Taking this one step further, we change the map on each arrow so that its matrix in the standard basis is the permutation matrix for the cycle

$$w = (1\,2\,3\,\cdots\,N) \in S_N$$

and call this representation $U_{[N]}(1)$. The only non-zero entries in this matrix are 1s in the first subdiagonal and in the upper right corner. We can "deform" $U_{[N]}(1)$ by replacing this last 1 with a complex number z, and obtain a matrix

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & z \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

which will be called $s(z)$. The resulting representation will be denoted with $U_{[N]}(z)$. Since the number of arrows is equal to the dimension of our vector spaces, the composition of $s(z)$ along the arrows of one loop around the cycle is $z \cdot \mathrm{Id}$, and the Grassmannian for this representation and dimension vector $\underline{k}$ is still isomorphic to $\mathrm{Gr}(k, N)$ via

$$\begin{aligned} \mathrm{Gr}(k, N) &\longrightarrow \mathrm{Gr}_{\underline{k}}(U_{[N]}(z)) \\ V \quad &\longmapsto (s(z)^i \cdot V)_i \end{aligned}$$

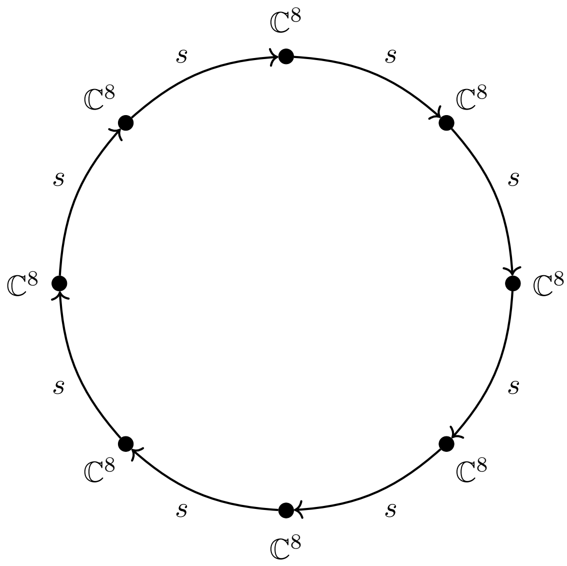
whenever $z \neq 0$. We chose to put V on vertex 0, but we could have done so on any other vertex. The next logical step is to degenerate $U_{[N]}(z)$ for $z \longrightarrow 0$, after which we obtain the representation needed to define the juggling variety.

Definition 2.1.1. Let $U_{[N]} := U_{[N]}(0)$ be the Δ_N-representation with $\mathbb{C}^N$ on every vertex and $s := s(0)$ on every arrow. We will interchangeably use s to denote both the linear map

$$s(e_j) = \begin{cases} e_{j+1} & j \neq N \\ 0 & j = N \end{cases} \tag{2.1.1}$$

and its matrix in the standard basis. Also let G_N be the automorphism group of $U_{[N]}$.

For example with $N = 8$, we obtain the following representation.



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Definition 2.1.2. The *juggling variety* is the quiver Grassmannian

$$X(k,N) \coloneqq \mathrm{Gr}_{\underline{k}}\left(U_{[N]}\right) .$$

A point of $X(k,N)$ consists therefore of a collection $(V_i)_{i\in\mathbb{Z}_N}$ of k-dimensional subspaces of $\mathbb{C}^N$, such that for every vertex i the inclusion $s(V_i) \subseteq V_{i+1}$ holds. The group $G_N = \mathrm{Aut}(U_{[N]})$ acts on the quiver Grassmannian in such a way that each of its orbits contains exactly one point consisting of coordinate subspaces [FLP22, Theorem 4.10]. Such a point is characterized solely by the collections of indices of basis vectors that span its subspaces. The coordinate points, and thus the orbits, are then parametrized by combinatorial objects known as juggling patterns [KLS13, §3].

Definition 2.1.3. A (k,N)*-juggling pattern* is a collection $\mathcal{J} = (J_i)_{i\in\mathbb{Z}_N}$ of cardinality k subsets of $[N]$, such that for any $i \in \mathbb{Z}_N$ and $j \in J_i \setminus \{N\}$, we have $j+1 \in J_{i+1}$.

The set of (k,N)-juggling patterns will be denoted by $JP(k,N)$. Conversely, the point in $X(k,N)$ corresponding to a $\mathcal{J} \in JP(k,N)$ will be denoted by $p_{\mathcal{J}}$ and is defined by

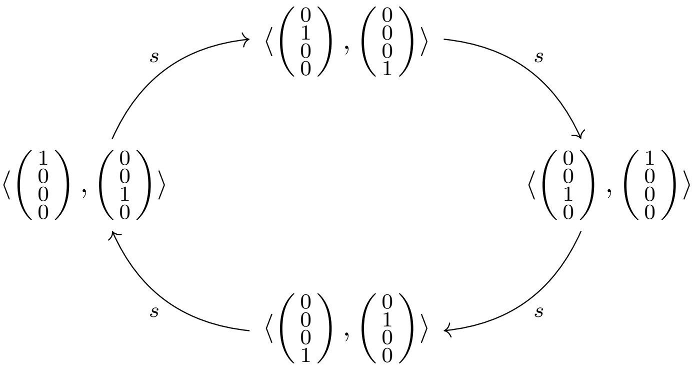
$$(p_{\mathcal{J}})_i = \mathrm{Span}\{e_j^{(i)} \mid j \in J_i\} ,$$

where $e_j^{(i)}$ is the j-th vector of the standard basis of $U_{[N]}^{(i)}$, the copy of $\mathbb{C}^N$ which sits on vertex i of Δ_N. These points $p_{\mathcal{J}}$ are called *juggling pattern points.*

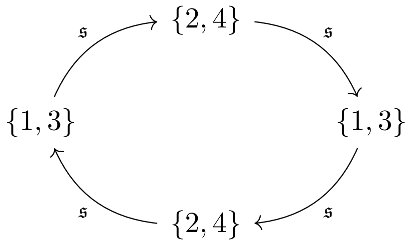
Notation: for a subset I of $[N]$, V_I will denote the subspace of $\mathbb{C}^N$ spanned the basis vectors with indices in I.

This means that $(p_{\mathcal{J}})_i = V_{J_i}$.

Example 2.1.4. For $N = 4$ and $k = 2$, the collection



is a point in the variety. The corresponding juggling pattern is then



which we will shorten to $13 \, {}^{24}_{24} \, 13$. The arrows $J_i \longrightarrow J_{i+1}$ are labeled by the map

$$\begin{aligned} \mathfrak{s}\colon [N-1] &\longrightarrow [2, N] \\ j &\longmapsto j+1 \end{aligned}$$

induced by how s acts on the standard basis, with the meaning that $\mathfrak{s}\left(J_i \setminus \{N\}\right) \subseteq J_{i+1}$. Here $[n, m]$ stands for the set of integers from n to m.

Remark 2.1.5. Throughout this thesis, the notation for residue classes of integers will not be very strict. More precisely, if i is an integer, we might use it as a subscript for a set of a juggling pattern even though i is not in $\mathbb{Z}_N$. Here it is understood that we are considering the residue class of i modulo N. Conversely, for $a \in \mathbb{Z}_N$, we might also use it to mean its representative in a given, suitable subset of the integers, such as $[N]$ or $[0, N-1]$.

2.1.1 The group of representation automorphisms

Recall from Section 1.2 that an element g of $G_N = \mathrm{Aut}(U_{[N]})$, with respect to the standard basis $\mathscr{SB} := \{e_j^{(i)} \mid j \in [N], i \in \mathbb{Z}_N\}$, is a tuple of invertible $N \times N$ matrices g_i, one for each vertex $i \in \mathbb{Z}_N$, such that

$$g_{i+1} \circ s = s \circ g_i$$

for all i. Explicitly, each g_i is lower-triangular, and if we denote by $g_j^{(i)}$ the j-th coefficient in the first column of g_i, then we obtain a description of the m-th column of g_i: its first $m-1$ entries are zeroes, followed by $g_1^{(i-m+1)}, g_2^{(i-m+1)} \ldots, g_{N-m+1}^{(i-m+1)}$, the first $N-m+1$ coefficients of the first column of g_{i-m+1} [FLP22, Proposition 4.5]. Observe that G_N has dimension N^2, and it is a linear degeneration of GL_N. This holds because the automorphism group of $U_{[N]}(z)$ consists of collections of invertible matrices $(h_i)_{i\in\mathbb{Z}_N}$ such that $h_{i+1}\circ s(z) = s(z)\circ h_i$, and the whole tuple is determined by a single one of them whenever $z \neq 0$. Therefore the group morphism

$$\begin{aligned}\mathrm{Aut}(U_{[N]}(z)) &\longrightarrow GL_N\\ (h_i)_i \quad &\longmapsto h_0\end{aligned}$$

is an isomorphism. When taking the limit for $z \longrightarrow 0$, which is compatible with the group structure, $\mathrm{Aut}(U_{[N]}(z))$ degenerates to G_N.

Example 2.1.6. An element of G_4 is a tuple of the following form,

$$g_0 = \begin{pmatrix} g_1^{(0)} & 0 & 0 & 0\\ g_2^{(0)} & g_1^{(3)} & 0 & 0\\ g_3^{(0)} & g_2^{(3)} & g_1^{(2)} & 0\\ g_4^{(0)} & g_3^{(3)} & g_2^{(2)} & g_1^{(1)} \end{pmatrix}$$

$$g_3 = \begin{pmatrix} g_1^{(3)} & 0 & 0 & 0\\ g_2^{(3)} & g_1^{(2)} & 0 & 0\\ g_3^{(3)} & g_2^{(2)} & g_1^{(1)} & 0\\ g_4^{(3)} & g_3^{(2)} & g_2^{(1)} & g_1^{(0)} \end{pmatrix} \qquad g_1 = \begin{pmatrix} g_1^{(1)} & 0 & 0 & 0\\ g_2^{(1)} & g_1^{(0)} & 0 & 0\\ g_3^{(1)} & g_2^{(0)} & g_1^{(3)} & 0\\ g_4^{(1)} & g_3^{(0)} & g_2^{(3)} & g_1^{(2)} \end{pmatrix}$$

$$g_2 = \begin{pmatrix} g_1^{(2)} & 0 & 0 & 0\\ g_2^{(2)} & g_1^{(1)} & 0 & 0\\ g_3^{(2)} & g_2^{(1)} & g_1^{(0)} & 0\\ g_4^{(2)} & g_3^{(1)} & g_2^{(0)} & g_1^{(3)} \end{pmatrix}$$

where $g_j^{(i)} \in \mathbb{C}$ for $j \neq 1$, and $g_1^{(i)} \in \mathbb{C}^*$.

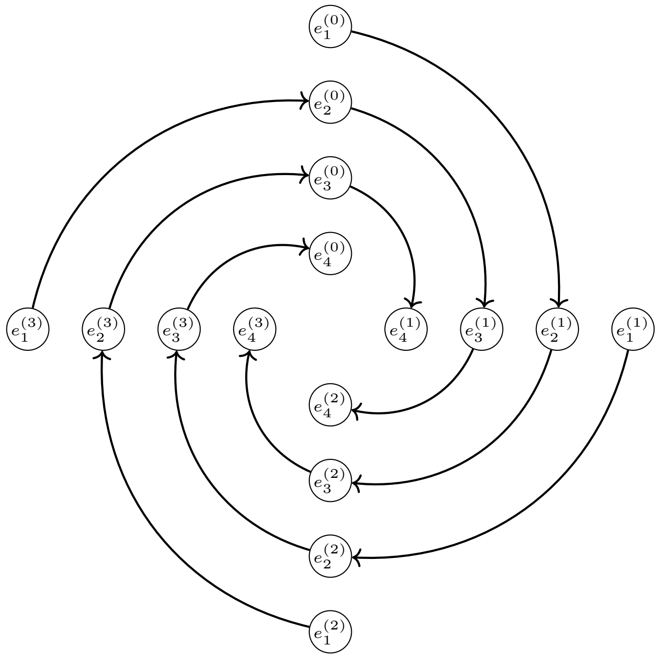
2.1.2 Diagrams

The reader can visualize juggling patterns with a diagram by considering them as *successor-closed subquivers* of the coefficient quiver of $U_{[N]}$ with respect to the standard basis $\mathscr{S}\mathscr{B}$. We recall [LP23b, Definition 1.3]:

Definition 2.1.7. The *coefficient quiver* for a Q-representation M with respect to a basis $\mathscr{B}$, denoted by $Q(M,\mathscr{B})$, is the quiver whose vertices are the elements of $\mathscr{B}$, with an arrow between basis elements $\beta \in \mathscr{B}_i$ and $\beta' \in \mathscr{B}_j$ whenever there exists an arrow $\alpha\colon i \longrightarrow j \in Q_1$ such that the coefficient of β' for the expression of $M_\alpha(\beta)$ in the basis $\mathscr{B}_j$ is nonzero.

The coefficient quiver for our representation in the standard basis looks like a spiral, when the basis vectors for each $U_{[N]}^{(i)}$ are lined up so that together they form N concentric circles.

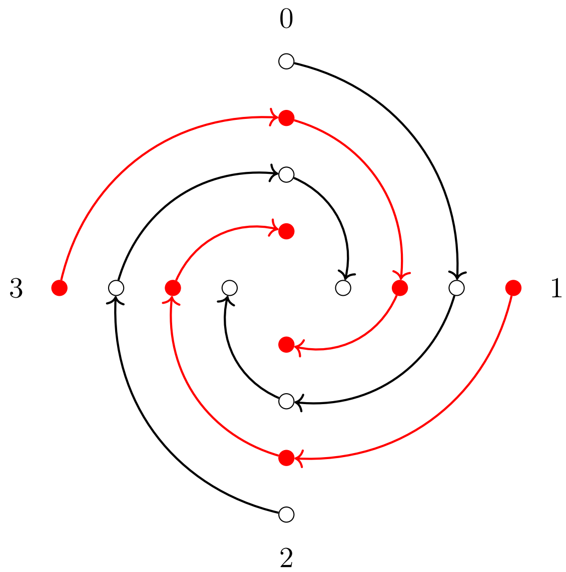
Example 2.1.8. Below is a picture of such a quiver for $N = 4$:



In the rest of this thesis the line of basis elements for $U_{[N]}^{(0)}$ will be at the top, ordered from the outside in and followed clockwise by those for the others vertices of Δ_N. We denote by $b_{j,p}$ the p-th element of the segment ending with $e_N^{(j)}$, that is, $b_{j,p} = e_p^{(j+p)}$. In this case we say that the segment ends at vertex j of Δ_N. Here $p \in [N]$ and we consider the representative of the class j that lies in $[0, N-1]$. To any juggling pattern $\mathcal{J}$, one can associate a subquiver $S_\mathcal{J}$ of $Q(U_{[N]}, \mathscr{S}\mathscr{B})$ in the obvious way, i.e. by considering the full subquiver with all vertices $e_j^{(i)}$ such that $j \in J_i$. The subquiver we obtain is successor-closed by Definition 2.1.3, and has exactly k vertices $e_j^{(i)}$ for each $i \in Z_N$. Conversely, the indices of the vertices of any successor-closed subquiver Q' with this property will give rise to a (k, N)-juggling pattern $\mathcal{J}$, where

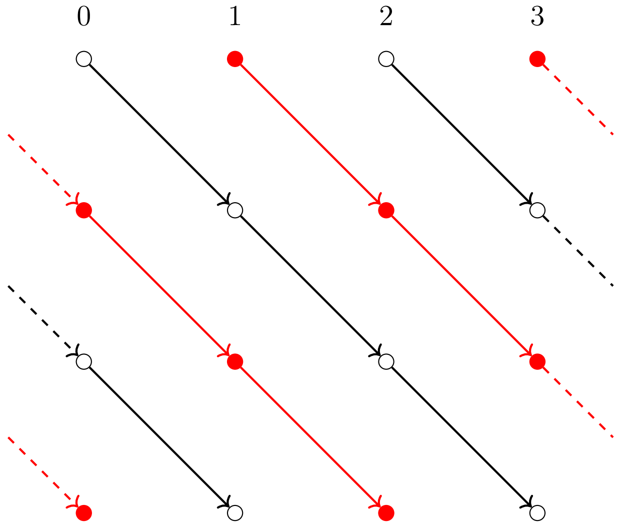
$$J_i = \{j \mid e_j^{(i)} \in Q'_0\} \,.$$

Example 2.1.9. The subquiver corresponding to the juggling pattern $13 \overset{24}{\underset{24}{}} 13$ from Example 2.1.4 is



and it consists of the segments ending at 0 and 2.

One can alternatively "cut" the arrows in the coefficient quiver between vertices $N-1$ and 0, and "unfurl" the spiral, thus depicting the basis vectors in vertical columns starting on the left with the basis for $U^{(0)}_{[N]}$, from top to bottom. Following is the same successor-closed subquiver as in Example 2.1.9, but as an "unfurled" square diagram.



When looking at such a picture, the reader should remember that there are actually arrows between the last and first columns. The square diagrams help highlight where the name "juggling pattern" comes from: the map s transforms a standard basis vector into the next one, and when in column form, the coloured dots are "falling" at each subsequent step like juggled balls in the air, and when they reach the last position at N, i.e. the juggler's hand, they are thrown back up to a higher position.

2.1.3 Orbit closure-inclusion order

The second type of diagram also helps visualize an ordering that can be given to $JP(k, N)$ starting from the closure-inclusion order of the G_N-orbits. By [FLP22, Theorems 1 and 2], we obtain:

Theorem 2.1.10. *The G_N-action provides a cellular decomposition of $X(k, N)$, meaning that each orbit is a complex affine cell.*

As already mentioned, each orbits contains exactly one juggling pattern point $p_{\mathcal{J}}$, therefore we denote by $C_{\mathcal{J}}$ the G_N-orbit of $p_{\mathcal{J}}$. To describe the generic point in a cell $C_{\mathcal{J}}$ we start from the coordinate vector spaces of $p_{\mathcal{J}}$. We then change some of the zeroes in the standard basis vectors to complex parameters. More precisely, consider a generator $e^{(a)}_j$ on a vertex a; for $N \geq i > j$ and $i \notin J_a$, swap the zero in position i for a parameter $u^{(a)}_{i,j}$. This creates N bases

$$\left\{ v^{(a)}_j := e^{(a)}_j + \sum_{\substack{i>j \\ i\notin J_a}} u^{(a)}_{i,j} e^{(a)}_i \right\}_{a\in\mathbb{Z}_N,\ j\in J_a} \tag{2.1.2}$$

of cardinality k for a collection of vector subspaces $(V_a)_a$, and in order for this collection to be a point in $X(k, N)$, these parameters have to satisfy $u^{(a)}_{i,j} = u^{(a+1)}_{i+1,j+1}$.

Definition 2.1.11. A *descending triple* for $\mathcal{J} \in JP(k, N)$ is a triple (a, i, j) where

$$a \in \mathbb{Z}_N,\ \ j \in J_a \text{ and } j < i \notin J_a\,.$$

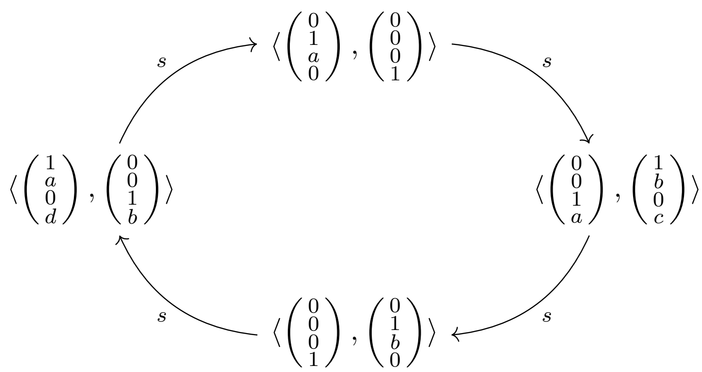
We say that (a, i, j) is *terminal* for $\mathcal{J}$ if $(a+1, i+1, j+1)$ is not a descending triple for $\mathcal{J}$. That is, if either $i = N$ or $i + 1 \in J_{a+1}$ [LP23a, Definition 6.1].

A collection of bases as in (2.1.2) exists for any point in the cell $C_{\mathcal{J}}$, so a set of affine coordinates for it is given by

$$\left(u^{(a)}_{i,j} \mid (a, i, j) \text{ is a descending triple for } \mathcal{J},\ u^{(a)}_{i,j} = u^{(a+1)}_{i+1,j+1} \right) .$$

We conclude that the dimension of a cell $C_{\mathcal{J}}$ is the number of equivalence classes of descending triples under the relation $(a, i, j) \sim (a+1, i+1, j+1)$, and thus is also equal to the number of terminal triples, since each $\sim$-equivalence class contains exactly one.

Example 2.1.12. The generic point in the cell corresponding to the juggling pattern of our running example $13\ {}^{24}_{24}\ 13$ is given by



for any $a, b, c, d \in \mathbb{C}$. Here

- $a = u^{(3)}_{2,1} = u^{(0)}_{3,2} = u^{(1)}_{4,3}$;
- $b = u^{(1)}_{2,1} = u^{(2)}_{3,2} = u^{(3)}_{4,3}$;
- $c = u^{(1)}_{4,1}$;
- $d = u^{(3)}_{4,1}$.

Looking at vertex 0 in the example above, we changed e_2 to $e_2 + ae_3$, and did not add any other parameter, since e_4 was a generator of $(p_{\mathcal{J}})_0$.

We know that G_N is an algebraic group, therefore the closure of an orbit consists of the union of orbits. In Section 2.3 we will see that taking closures involves a Coxeter group, much in the same way that a Schubert variety in a flag variety is the union of Schubert cells. We can now fix the following partial order on the set of G_N-orbits:

$$C_{\mathcal{J}} \leq C_{\mathcal{J}'} \overset{\text{def}}{\iff} C_{\mathcal{J}} \subseteq \overline{C_{\mathcal{J}'}}\,,$$

which can then be transferred to the set $JP(k,N)$ via

$$\mathcal{J} \longleftrightarrow C_{\mathcal{J}} .$$

Definition 2.1.13. Given two juggling patterns $\mathcal{J}, \mathcal{J}' \in JP(k,N)$, we say that $\mathcal{J} \leq \mathcal{J}'$ if and only if $p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}}$.

The closure-inclusion order can also be expressed combinatorially. Consider two subsets $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$ of $[N]$, ordered increasingly; we say

$$A \leq B \overset{\text{def}}{\iff} a_i \leq b_i \;\forall\, i \in [k] . \tag{2.1.3}$$

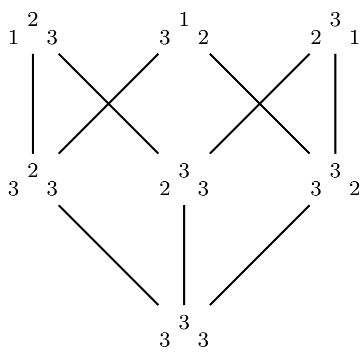
The following result is [FLP22, Proposition 7.3]:

Proposition 2.1.14. *For two juggling patterns $\mathcal{J}, \mathcal{J}' \in JP(k,N)$, the condition $p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}}$ is equivalent to $J_i \geq J_i'$ for all $i \in \mathbb{Z}_N$.*

Corollary 2.1.14.1. *If $\mathcal{J}' \geq \mathcal{J}$, then*

$$\{i \in \mathbb{Z}_N \mid N \in J_i'\} \subseteq \{i \in \mathbb{Z}_N \mid N \in J_i\} .$$

Example 2.1.15. The set $JP(1,3)$ has 7 elements, and its Hasse diagram with respect to the order given above is the following.



A consequence of Proposition 2.1.14 is the next result:

Lemma 2.1.16. *The following conditions for a (k,N)-juggling pattern $\mathcal{J}$ are all equivalent to maximality in the partial order.*

- *exactly k of its N sets contain a 1;*
- *whenever $N \in J_i$ for some i, then $1 \in J_{i+1}$;*
- *the successor-closed subquiver $S_{\mathcal{J}}$ corresponding to $\mathcal{J}$ consists of precisely k segments, which must therefore be of length N.*

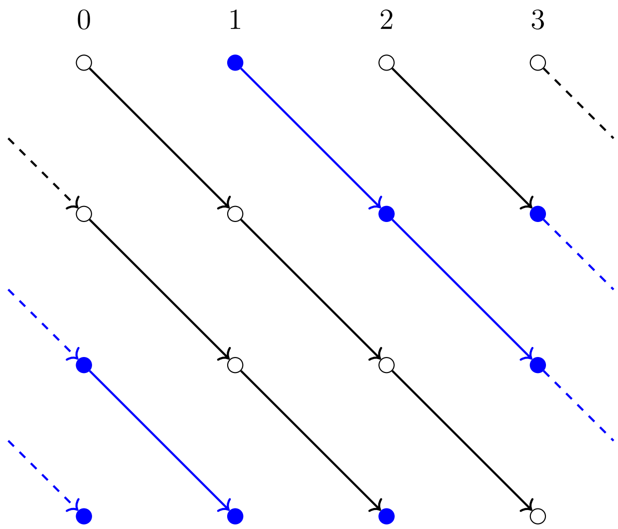
There are $\binom{N}{k}$ maximal juggling patterns and the closures of the corresponding cells are the irreducible components of $X(k,N)$.

Proof. The equivalence of these conditions follows from Definition 2.1.3, and they equate to maximality by [FLP22, Remark 4.12]. There are as many maximal juggling patterns as the number of k-subsets of $\mathbb{Z}_N$, and from the same remark we see that the closure of the maximal cells are the irreducible components of $X(k, N)$. $\square$

On the other hand, the unique minimal juggling pattern is such that every set is equal to

$$\{N-k+1, N-k+2, \dots, N\}.$$

Example 2.1.17. Below is the square diagram of the juggling pattern $\mathcal{J} = \begin{smallmatrix} & 34 & \\ 23 & & 14 \\ & 24 & \end{smallmatrix}$, lower in the partial order than the maximal $\mathcal{J}' = \begin{smallmatrix} & 24 & \\ 13 & & 13 \\ & 24 & \end{smallmatrix}$, which we have been using as a running example so far.



The point $p_{\mathcal{J}} \in X(2,4)$ is in the closure of the cell $C_{\mathcal{J}'}$: observe that a point

$$\begin{array}{ccc} & \langle e_2^{(0)} + a e_3^{(0)}, e_4^{(0)} \rangle & \\ \langle e_1^{(3)} + a e_2^{(3)}, e_3^{(3)} \rangle & & \langle e_1^{(1)}, e_3^{(1)} + a e_4^{(1)} \rangle \\ & \langle e_2^{(2)}, e_4^{(2)} \rangle & \end{array}$$

with $a = u_{2,1}^{(3)} = u_{3,2}^{(0)} = u_{4,3}^{(1)}$ as the only non-zero coordinate is the image of $p_{\mathcal{J}'}$ under the group element

$$\begin{array}{ccc} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a & 1 \end{pmatrix} \\ & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \end{array}$$

so we recover $p_{\mathcal{J}}$ when taking the limit for $a \longrightarrow \infty$.

If the reader is partial to one type of diagrams rather than the other, it is important to remember that moving vertices downwards in a square diagram translates to moving them inwards in the spiral diagram.

2.1.4 Grassmann necklaces

In [FLP22], which we have repeatedly cited, a different but isomorphic Δ_N-representation is studied. It necessarily has $\mathbb{C}^N$ on every vertex as well, but the linear map on the arrow $i \longrightarrow i+1$ is the projection p_i defined by

$$p_i(e_j) = (1 - \delta_{ij})e_j \,,$$

which preserves every basis vector except e_i. This representation $U'_{[N]}$ is still nilpotent, and is isomorphic to $U_{[N]}$ via the morphism $\varphi = (\varphi_i)_i$, where

$$\varphi_i(e_j) = e_{i-j} \,.$$

Indeed, we have $p_i = \varphi_{i+1} \circ s \circ \varphi_i$. It follows that both $U_{[N]}$ and $U'_{[N]}$ are the direct sum of the same indecomposables $U(i;N)$ for $i \in \mathbb{Z}_N$, and that φ extends to an isomorphism from $X(k,N)$ to the quiver Grassmannian

$$X'(k,N) \coloneqq \mathrm{Gr}_{\underline{k}}(U'_{[N]}) \,.$$

Points of $X'(k,N)$ are collections of k-dimensional subspaces W_i such that $p_i(W_i) \subseteq W_{i+1}$, and the combinatorial set parametrizing the coordinate subspaces in $X'(k,N)$ is that of (k,N)-*Grassmann necklaces* [Pos06, Definition 16.1]: a (k,N)-Grassmann necklace is a collection $\mathcal{I}$ of subsets of $[N]$ of cardinality k, indexed by $\mathbb{Z}_N$, such that

$$I_a \setminus \{a\} \subseteq I_{a+1}$$

for all $a \in [N]$. As the reader will expect, the sets of (k,N)-juggling patterns and (k,N)-Grassmann necklaces are in bijection, via

$$(J_i)_i \longmapsto (\{i-j \mid j \in J_i\})_i \,.$$

As mentioned, we have been, and will continue being, not completely rigorous with residue classes: here a indicated both the integer and its residue class modulo $[N]$. Likewise, by $i-j$ we mean the representative of its residue class which sits in $[N]$. For $a \in [N]$, one can define an ordering $\leq_a$ on $[N]$ given by

$$a \leq_a a+1 \leq_a \cdots \leq_a N \leq_a 1 \leq_a \cdots \leq_a a-1 \,.$$

Thus the order given to the set of Grassmann necklaces in [FLP22], inherited from inclusions of the orbit closures, is

$$\mathcal{I} \leq \mathcal{I}' \iff I_a \leq_a I'_a \;\forall\, a \in [N]$$

and the above bijection is a poset isomorphism. All the results from [FLP22] we refer to hold in the juggling pattern setting as well, and the authors of that paper remark that one point of view can be more suitable than the other, depending on what one wants to prove.

2.2 Torus actions

In this section we define the actions of two algbraic tori on $X(k, N)$, one of dimension 1 and the other of dimension $N + 1$. These actions provide a different interpretation of the cells $C_{\mathcal{J}}$ and will allow us to generate the partial order on $JP(k, N)$, similarly to how reflections generate the Bruhat order in a Coxeter group. From [LP23a, CI11, Pü22] we know that, fixed an integer-valued function **wt**, called a *weight* or *grading*, on a basis

$$\mathscr{B} = \{\beta_l^{(i)} \mid i \in Q_0,\ 1 \le l \le \dim M_i\}$$

of a nilpotent Δ_N-representation M, there exists a $\mathbb{C}^*$-action on the total space

$$\bigoplus_{i \in Q_0} M_i$$

of M defined on a basis element β by

$$z \cdot \beta := z^{\mathbf{wt}(\beta)} \beta\,.$$

Under appropriate assumptions on **wt** or on a dimension vector $\underline{d}$, it also extends to the quiver Grassmannian $\mathrm{Gr}_{\underline{d}}(M)$ [Pü25]. One condition under which the action extends to any quiver Grassmannian for M is that **wt** is *attractive* [LP23b, Definition 1.17], i.e. it satisfies

$$\mathbf{wt}(\beta_m^{(i)}) < \mathbf{wt}(\beta_{m'}^{(i)})$$

for $m < m'$, and there exists an integer D such that

$$\mathbf{wt}\left(\beta_m^{(i)}\right) = \mathbf{wt}\left(\beta_l^{(j)}\right) + D$$

whenever $\beta_l^{(i)} \longrightarrow \beta_m^{(j)}$ is an arrow in the coefficient quiver $Q(M, \mathscr{B})$ (recall Definition 2.1.7). Our grading of choice for the standard basis of $U_{[N]}$ is

$$\mathbf{wt}\colon b_{j,p} \longmapsto p\,.$$

Remember that $b_{j,p}$ denotes the p-th vertex of the segment of $Q(U_{[N]}, \mathscr{S}\mathscr{B})$ ending at j, which is $e_p^{(j+p)}$, so the only arrows in the coefficient quiver are $b_{j,p} \longrightarrow b_{j,p+1}$ whenever $p \ne N$. This grading is attractive since $\mathbf{wt}(e_m^{(i)}) < \mathbf{wt}(e_{m'}^{(i)})$ for $m < m'$ and $\mathbf{wt}(b_{j,p+1}) = \mathbf{wt}(b_{j,p}) + 1$. Therefore the $\mathbb{C}^*$-action

$$z \cdot b_{j,p} = z^p\, b_{j,p} \tag{2.2.1}$$

is well defined on $X(k, N)$. Its fixed points are the coordinate points $p_{\mathcal{J}}$ [CI11, Theorem 1], and the attracting set

$$\{V \in X(k, N) \mid \lim_{z \to 0} z \cdot V = p_{\mathcal{J}}\}$$

to a fixed point $p_{\mathcal{J}}$ is the G_N-orbit $C_{\mathcal{J}}$ [FLP22, Theorem 4.10]. As already stated, these orbits are affine cells.

Remark 2.2.1. This action is similar to the one shown in Subsection 1.1.2, but different in that the weights are now increasing in order, rather than decreasing. This choice is due to the fact that lower-dimensional cells in the flag variety $\mathfrak{Fl}(n)$ correspond to flags spanned by vector spaces with smaller indices, while in the juggling variety the smaller cells correspond to juggling patterns whose sets contain higher numbers.

Now let us move on to the second action. From [LP23a], we know that quiver Grassmannians for nilpotent representations of the equioriented cycle, such as the varieties $X(k, N)$, can be equipped with the action of a bigger algebraic torus. We will see how it encodes information about the G_N-orbit closure-inclusion order. Let $T := (\mathbb{C}^*)^{N+1}$, and let us denote its elements by $t = (z, \gamma_0, \gamma_1, \ldots, \gamma_{N-1})$. The T-action on the standard basis of $U_{[N]}$ is given by

$$t \cdot b_{j,p} := z^p \, \gamma_j \, b_{j,p} \tag{2.2.2}$$

and it also extends to the variety. Its fixed points are precisely the points $p_{\mathcal{J}}$, i.e. the cocharacter

$$\begin{aligned} \rho \colon \mathbb{C}^* &\longrightarrow \quad T \\ z &\longmapsto (z, 1, \ldots, 1) \end{aligned}$$

which recovers the previous $\mathbb{C}^*$-action (2.2.1), is *generic* [LP23a, Theorem 5.14]. See also [Pü22].

Proposition 2.2.2. *Every T-orbit is contained in an affine cell $C_{\mathcal{J}}$.*

Proof. For an element $t = (z, \gamma_0, \gamma_1, \ldots, \gamma_{N-1}) \in T$, let t_i be the matrix given by the action of t on the standard basis of $U_{[N]}^{(i)}$. Then t satisfies

$$t_{i+1} \circ s = z \cdot s \circ t_i \,.$$

So T is not a subgroup of G_N, but t differs from an element of G_N only by z and therefore acts on $X(k, N)$ like one, since multiplication by z does nothing on vector spaces. We deduce that the cells $C_{\mathcal{J}}$ are T-stable. □

Let us now go over what the action looks like coordinate-wise. Remember that the vector space V_a of a point $V \in C_{\mathcal{J}}$ with coordinates $(u_{i,j}^{(a)})$ is spanned by vectors of the form

$$v_j^{(a)} = e_j^{(a)} + \sum_{\substack{i>j \\ i \notin J_a}} u_{i,j}^{(a)} \, e_i^{(a)}$$

with $j \in J_a$. Therefore $t \in T$ acts on this basis of V_a as

$$t \cdot v_j^{(a)} = z^j \, \gamma_{a-j} \, e_j^{(a)} + \sum_{\substack{i>j \\ i \notin J_a}} z^i \, \gamma_{a-i} \, u_{i,j}^{(a)} \, e_i^{(a)} \,.$$

Since $b_{j,p} = e_p^{(j+p)}$, $a - i$ indexes the segment of the coefficient quiver that $e_i^{(a)}$ lies on, and it is intended as an integer in $[0, N-1]$, while $i \in [N]$ is its position in the segment. In other words, t acts on the affine cell $C_{\mathcal{J}}$, coordinate-wise, as follows:

$$t \cdot \left(u_{i,j}^{(a)}\right) = \left(u_{i,j}^{(a)} \, z^i \, z^{-j} \, \gamma_{a-i} \, \gamma_{a-j}^{-1}\right) = \left(u_{i,j}^{(a)} \, z^{i-j} \, \gamma_{a-i} \, \gamma_{a-j}^{-1}\right) \,. \tag{2.2.3}$$

The fact that the T-fixed points are the juggling pattern points could also be inferred from (2.2.3): for a point $V \in C_{\mathcal{J}}$ to be fixed, its affine coordinates must be all zero, i.e. it must be precisely $p_{\mathcal{J}}$.

2.2.1 Mutations

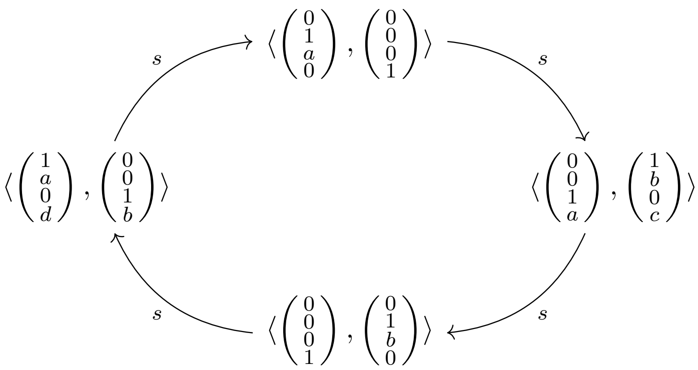
The T-action makes $X(k, N)$ into what is called a GKM variety; we give an equivalent [LP23a] definition to the one given in the seminal paper [GKM98].

Definition 2.2.3. A projective variety X, equipped with the action of an algebraic torus T and a T-equivariant embedding into a projective space, is *GKM* if

- ◇ the T-action on X is *skeletal*, i.e. it has finitely many fixed points and one-dimensional orbits;
- ◇ the rational cohomology of X vanishes in odd degree.

By [LP23a, Theorem 6.6], our quiver Grassmannian $X(k, N)$ is a GKM variety with respect to T, and we will now give an explicit description of the one-dimensional T-orbits. By [LP23a, Remark 1.5], the T-action on $X(k, N)$ is *locally linearizable*: for any one-dimensional orbit O there exists a T-action on $\mathbb{P}^1$ and a T-equivariant isomorphism $\overline{O} \cong \mathbb{P}^1$. Thus, the closure of each one-dimensional orbit is a sphere $\mathbb{P}^1$ and the boundary consists of two zero-dimensional orbits, i.e. fixed points. Since the affine cells are T-stable, the two fixed points $p_{\mathcal{J}}$ and $p_{\mathcal{J}'}$ that O "connects" must be comparable in the G_N-orbit closure-inclusion order and the orbit O must be contained in the cell corresponding to the higher juggling pattern.

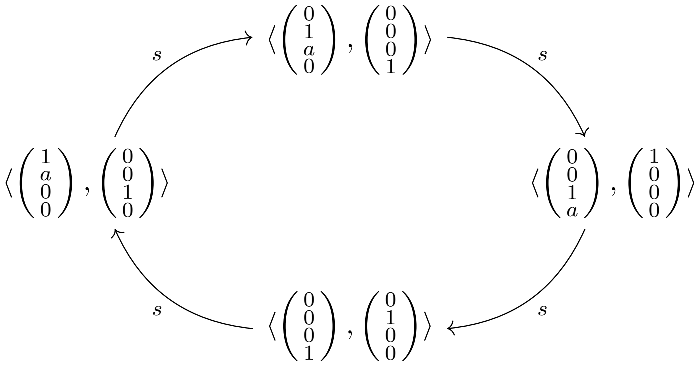
Example 2.2.4. In Example 2.1.17, we saw how $\mathcal{J} = 23 \, {}^{34}_{24} \, 14$ is lower than $\mathcal{J}' = 13 \, {}^{24}_{24} \, 13$ by finding a path contained in the cell $C_{\mathcal{J}'}$ connecting the two. Let us show that this path is a one-dimensional T-orbit. The torus element $t = (z, \gamma_0, \gamma_1, \gamma_2, \gamma_3)$, when acting on the point



of the cell $C_{\mathcal{J}'}$, which has coordinates (a, b, c, d), produces the point with coordinates

$$(a\, z\, \gamma_1\, \gamma_2^{-1},\ b\, z\, \gamma_3\, \gamma_0^{-1},\ c\, z^3\, \gamma_1\, \gamma_0^{-1},\ d\, z^3\, \gamma_3\, \gamma_2^{-1}).$$

When $b = c = d = 0$ and $a \neq 0$, we obtain a one-dimensional orbit whose points are of the form



for $a \in \mathbb{C}^*$. Taking the limit for $a \longrightarrow 0$ we get back $p_{\mathcal{J}'}$, while for $\frac{1}{a} \longrightarrow 0$ we get $p_{\mathcal{J}}$.

One dimensional T-orbits translate, on the combinatorial level, to the following [FLP22, Definition 6.2]:

Definition 2.2.5. We say that two (k, N)-juggling patterns $\mathcal{J}$ and $\mathcal{J}'$ are connected by a *mutation* $\mathcal{J}' \xrightarrow{\mu} \mathcal{J}$ if the successor closed subquiver $S_{\mathcal{J}}$ is obtained from $S_{\mathcal{J}'}$ by cutting the tail end of a segment in the subquiver and glueing it at the tail end of a lower segment, i.e. removing some elements

$$x \in J'_a\,, \quad x+1 \in J'_{a+1}\,, \quad \dots, \quad x+l \in J'_{a+l}$$

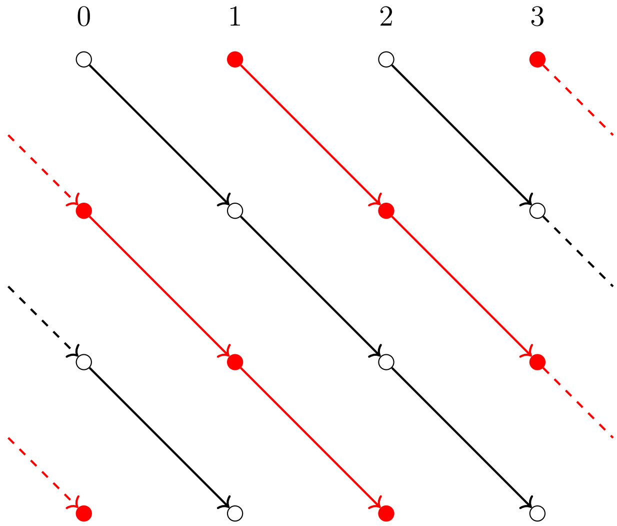
for $a \in \mathbb{Z}_N$ and $l \geq 0$, and replacing them with

$$x+m \in J_a\,, \quad x+m+1 \in J_{a+1}\,, \quad \dots, \quad x+m+l \in J_{a+l}$$

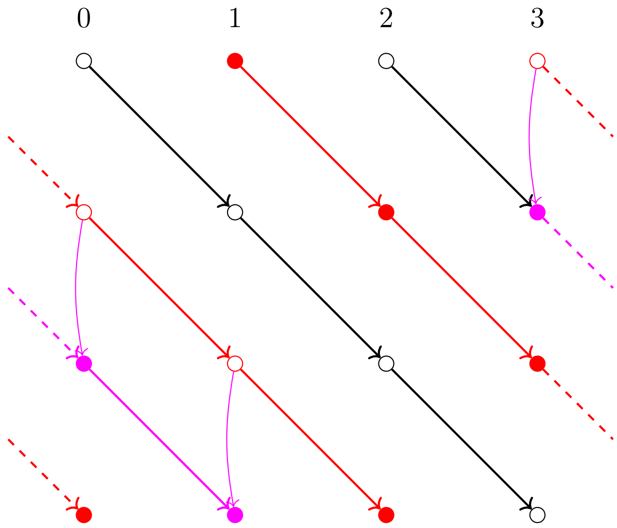
where $m \geq 1$ and $x+m+l+1$ is either equal to $N+1$, or it's in both J'_{a+l+1} and J_{a+l+1}. We call m the *breadth* of the mutation μ.

Immediately we notice that the existence of a mutation $\mathcal{J}' \xrightarrow{\mu} \mathcal{J}$ implies $\mathcal{J}' \geq \mathcal{J}$, by Proposition 2.1.14. As mentioned, by [FLP22, Proposition 7.3] such a mutation exists if and only if $p_{\mathcal{J}'}$ and $p_{\mathcal{J}}$ are the closure points of a one-dimensional T-orbit and $p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}}$.

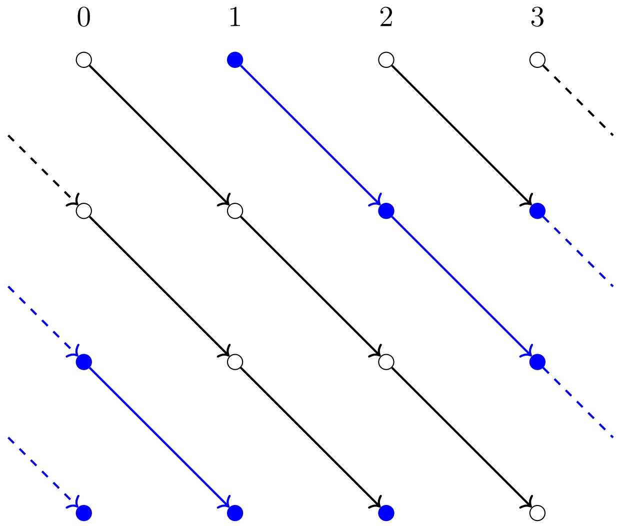
Example 2.2.6. The mutation corresponding to the one dimensional T-orbit from Example 2.2.4 is the following. We first show the square diagram for $\mathcal{J}' = 13 \substack{24 \\ \\ 24} 13$ again, in red.



The mutation we apply cuts three vertices from the tail of the red segment ending at 2, which are hollowed out in the following diagram. They are glued back one position lower, since the breadth of μ is 1, and are shown in pink.



If we then colour in blue the filled-in vertices and in black the hollow ones, we get back the diagram for $\mathcal{J} = {}_{23}\,{}^{34}_{24}\,{}_{14}$, shown previously in Example 2.1.17.



By Definitions 2.2.5 and 2.1.11, each mutation starting at a juggling pattern $\mathcal{J}$, i.e. of the form $\mathcal{J} \xrightarrow{\mu} \mathcal{J}^{(1)}$, corresponds to a $\sim$-equivalence class of descending triples for $\mathcal{J}$. Indeed, to the equivalence class of a terminal triple (a, i, j) corresponds the mutation which lowers $j \in J_a$ down to i, together with the part of segment of $S_{\mathcal{J}}$ that precedes it. Conversely, if μ is fixed, then the triples (a, i, j) where a is a vertex and μ replaces $j \in J_a$ with i form a $\sim$-equivalence class.

We also claim that the equivalence classes of descending triples for $\mathcal{J}^{(1)}$ are strictly less in number than those for $\mathcal{J}$. Let (a, i', j') be a terminal triple for $\mathcal{J}^{(1)}$. If it is not a triple for $\mathcal{J}$ as well, then μ affects vertex a, where it replaces some $j \in J_a$ with some $i \in J_a^{(1)}$. Thus, either $j' \notin J_a$ so it must be equal to i, or $i' \in J_a$ and it must coincide with j. The two things cannot happen simultaneously since a mutation changes at most one number on each vertex. Therefore there exists a descending triple for $\mathcal{J}$ given by (a, i', i) or (a, j, j'). Also notice that triples for $\mathcal{J}$ of the form (a, x, j) or (a, i, y) with $x \geq i$ and $y \leq j$ are "lost" after applying μ. We deduce the following.

Lemma 2.2.7. *The dimension of an affine cell $C_{\mathcal{J}}$ equals the number of mutations starting at $\mathcal{J}$. Moreover, given a mutation $\mathcal{J} \xrightarrow{\mu} \mathcal{J}^{(1)}$, we have* $\dim C_{\mathcal{J}} > \dim C_{\mathcal{J}^{(1)}}$.

Corollary 2.2.7.1. *The cells corresponding to maximal juggling patterns in $JP(k, N)$ are equidimensional of dimension $k(N-k)$. Therefore* $\dim X(k, N) = k(N-k)$.

Proof. Observe first that a mutation starting at $\mathcal{J}$ necessarily moves the tail vertex of a segment in $S_{\mathcal{J}}$, and then maybe its successors, depending of the position it is moved to. When $\mathcal{J}$ is maximal, $S_{\mathcal{J}}$ consists of k segments of length N, whose starting vertex is $e_1^{(i)}$ for some $i \in \mathbb{Z}_N$. If $e_1^{(i)}$ is moved to $e_j^{(i)}$ with $j > 1$, then the mutation moves $N - j + 1$ vertices in total. Also, there are $N - k$ basis elements of $U_{[N]}^{(i)}$ whose index is not in J_i, and $e_1^{(i)}$ can be moved to any of them. Therefore we have $N - k$ possible mutations for each of the k segments, and these mutation are all distinct. □

Example 2.2.8. The terminal triples for $\mathcal{J} = \begin{smallmatrix} & 34 & \\ 23 & & 14 \\ & 24 & \end{smallmatrix}$ are $(1, 3, 1)$, $(3, 4, 2)$ and $(3, 4, 3)$. The first one is terminal since $4 \in J_2$, while the others are because $i = 4$. The corresponding mutations produce respectively

$$\begin{matrix} & 34 & \\ 23 & & 34 \\ & 24 & \end{matrix}, \quad \begin{matrix} & 34 & \\ 34 & & 14 \\ & 24 & \end{matrix}, \quad \begin{matrix} & 34 & \\ 24 & & 24 \\ & 34 & \end{matrix}.$$

2.3 Combinatorics

The goal of this section is to introduce another poset, isomorphic to $JP(k, N)$, which will allow us to better describe the dimension of an affine cell $C_{\mathcal{J}}$ and which will be useful in later chapters when studying the symplectic subvarieties of the juggling varieties. These appear also in [Kar18, §2.7].

2.3.1 Affine permutations

Definition 2.3.1. A (k, N)-*affine permutation* is a bijection $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ of the integers that satisfies

$$f(i + N) = f(i) + N$$

for all $i \in \mathbb{Z}$, as well as

$$\sum_{i=1}^{N}(f(i) - i) = kN\,.$$

The set of (k, N)-affine permutations will be denoted by A_N^k.

Remark 2.3.2. The sum in the second condition, when performed over any N consecutive numbers, will still result in kN.

Example 2.3.3. The permutation f such that

$$\begin{aligned} f(4t) &= 4t+4 \\ f(4t+1) &= 4t+2 \\ f(4t+2) &= 4t+5 \\ f(4t+3) &= 4t+3 \end{aligned}$$

where $t \in \mathbb{Z}$, is an element of A_4^2.

The composition of two (k, N)-affine permutations is not another (k, N)-affine permutation, since the second condition fails, unless $k = 0$. Moreover, A_N^0 with the composition product forms a group, which is a Coxeter group of affine type $A_{N-1}^{(1)}$ [BB96]. The set S of its simple reflections consists of the affine permutations $s_0, s_1, \ldots, s_{N-1}$, where s_i switches $i + tN$ and $i + 1 + tN$ for all $t \in \mathbb{Z}$. For an integer j, we write s_j for the simple reflection s_i with $j \equiv_N i$. The relations they satisfy are of the form

$$s_i \circ s_{i+1} \circ s_i = s_{i+1} \circ s_i \circ s_{i+1}\,,$$

for all $i \in \mathbb{Z}$. Conjugates of simple reflections are called reflections or transpositions. The reflection that switches $i + tN$ and $j + tN$, for all $t \in \mathbb{Z}$ is denoted by (i, j).

While A_N^k is not a group, it is in bijection with A_N^0 via

$$\begin{aligned} A_N^0 &\longrightarrow A_N^k \\ g &\longmapsto g \circ \mathrm{id}_k\,, \end{aligned}$$

where id_k is the affine permutation that sends i to $i + k$. Since (A_N^0, S) is a Coxeter group, it is a poset with the *Bruhat* order; let us recall the definition [BB05, Definition 2.1.1]. In a Coxeter system (W, S), we say that two elements $w, w' \in W$ satisfy $w < w'$ if there exists a sequence of reflections, i.e. conjugates of simple reflections, $t_1, t_2, \ldots, t_m$ such that $w' = w \cdot t_1 \cdot t_2 \cdots t_m$ and $\ell(w) < \ell(w')$. Here $\ell(w)$ is defined as the length of the shortest possible word in the alphabet S representing w, and is named the *length* of w. See [BB05, §2] for more. When $m = 1$, such a relation will be called *elementary*.

The map above allows us to transfer the Bruhat order from A_N^0 to A_N^k in the expected way: we have $g \circ \mathrm{id}_k \leq g' \circ \mathrm{id}_k$ whenever $g \leq g'$ in A_N^0. For $f \in A_N^k$, its length $\ell(f)$ is defined as the cardinality of the set of *inversions* of f:

$$L(f) \coloneqq \{(x, y) \in [0, N-1] \times \mathbb{Z} \mid x < y \wedge f(x) > f(y)\}\,. \tag{2.3.1}$$

If $f = g \circ \mathrm{id}_k$, its length as a (k, N)-affine permutation coincides with the length of g as an element of the Coxeter group (A_N^0, S). Indeed, by [BB05, Proposition 8.3.1] the length of g as an element of (A_N^0, S) is precisely the number of inversions of g, and a pair (x, y) is in $L(f)$ if and only if, up to equivalence modulo N of the entries, $(x + k, y + k) \in L(g)$.

Definition 2.3.4. A (k, N)-affine permutation f is *bounded* if it satisfies

$$i \leq f(i) \leq i + N$$

for all $i \in \mathbb{Z}$.

Their set will be denoted by $\mathcal{B}_{(k,N)}$. It is a lower-order ideal of A_N^k [KL02, Lemma 3.6] and is in order-preserving bijection with the poset of (k, N)-juggling patterns [Lam15, Theorem 6.2]. In that paper, as well as in [FLP22], the correspondence with Grassmann necklaces is shown, which in our setting translates to the following: given a (k, N)-juggling pattern $\mathcal{J}$, the corresponding (k, N)-bounded affine permutation is defined by

$$f_{\mathcal{J}}(a) := \begin{cases} a & N \notin J_a\,; \\ a + N + 1 - b & J_{a+1} = \mathfrak{s}(J_a \setminus \{N\}) \cup \{b\}\,. \end{cases} \tag{2.3.2}$$

We will denote by $g_{\mathcal{J}}$ the permutation in A_N^0 such that $f_{\mathcal{J}} = g_{\mathcal{J}} \circ \mathrm{id}_k$. Conversely, to get a juggling pattern $\mathcal{J}_f$ from a bounded affine permutation f, we start with an integer a and consider the set

$$\{a - f(b) \mid b < a \wedge f(b) \geq a\} \subset \mathbb{Z}\,. \tag{2.3.3}$$

Then $(\mathcal{J}_f)_a$ is given by the representatives in $[N]$ of the residue classes modulo N of the elements in (2.3.3). Observe that $(\mathcal{J}_f)_a$ really only depends on the residue class of a modulo N, because f is bounded and affine. Since the bijection is order-preserving, it follows that the closure of a cell $C_{\mathcal{J}}$ is the union of cells corresponding to juggling patterns whose affine permutations are lower than $f_{\mathcal{J}}$ in the Bruhat order, similarly to Schubert cells in the complete flag variety (1.1.1).

Remark 2.3.5. Another equivalent condition for maximality of a juggling pattern $\mathcal{J} \in JP(k, N)$ is the following: there exists $J \subseteq [N]$ with $|J| = k$ such that $f_{\mathcal{J}}(i) = i + N$ for $i \in J$, and $f_{\mathcal{J}}(i) = i$ for $i \notin J$, in which case J is the set of vertices of Δ_N where the $\mathcal{J}$ contains N.

Notation: the set of cardinality-k subsets of $[N]$ will be denoted with $\binom{[N]}{k}$.

The next result, which is [FLP22, Theorem 7.5], explains why we introduced bounded affine permutations.

Proposition 2.3.6. *For $\mathcal{J} \in JP(k, N)$, the dimension of the affine cell $C_{\mathcal{J}}$ is equal to $\ell(f_{\mathcal{J}})$.*

Corollary 2.3.6.1. *Given a mutation $\mathcal{J}' \xrightarrow{\mu} \mathcal{J}$, we have $\ell(f_{\mathcal{J}'}) > \ell(f_{\mathcal{J}})$.*

Example 2.3.7. The bounded affine permutation corresponding to our running example $\mathcal{J}' = \begin{smallmatrix} & 24 & \\ 13 & & 13 \\ & 24 & \end{smallmatrix}$

is

$$\begin{aligned} f_{\mathcal{J}'}(4t) &= 4t+4 \\ f_{\mathcal{J}'}(4t+1) &= 4t+1 \\ f_{\mathcal{J}'}(4t+2) &= 4t+6 \\ f_{\mathcal{J}'}(4t+3) &= 4t+3 \end{aligned}$$

while the one corresponding to $\mathcal{J} = 23\,\substack{34\\24}\,14$ is the permutation shown in Example 2.3.3. The $(0,4)$-affine permutation $g_{\mathcal{J}'}$ is

$$\begin{aligned} g_{\mathcal{J}'}(4t) &= 4t+2 \\ g_{\mathcal{J}'}(4t+1) &= 4t-1 \\ g_{\mathcal{J}'}(4t+2) &= 4t+4 \\ g_{\mathcal{J}'}(4t+3) &= 4t+1\,. \end{aligned}$$

When written as a minimal length expression in terms of simple reflections, we have $g_{\mathcal{J}'} = s_1 \circ s_3 \circ s_2 \circ s_0$, coherently with the fact that $C_{\mathcal{J}'}$ has dimension 4.

Remark 2.3.8. Given an affine permutation f, we have $f \circ (i,j) = (f(i), f(j)) \circ f$. Moreover, if both f and $f' = f \circ (i,j)$ are bounded, then $|\, j-i\,| < N$. Otherwise, assuming f is bounded, the following chain of inequalities

$$i \leq f(i) = f'(j) \leq i+N < j$$

proves that f' cannot be bounded as well.

By [FLP22, Remark 6.4], a one-dimensional T-orbit in $X(k,N)$ connecting two juggling pattern points $p_{\mathcal{J}'}$ and $p_{\mathcal{J}}$ corresponds to an elementary Bruhat relation between the corresponding $(0,N)$-affine permutations $g_{\mathcal{J}'}$ and $g_{\mathcal{J}}$. Hence, mutations generate the closure-inclusion order on $JP(k,N)$.

Lemma 2.3.9. *Let $f \leq f'$ be two (k,N)-bounded affine permutations. Then f' and f differ by a reflection if and only if there is a mutation $\mathcal{J}_{f'} \xrightarrow{\mu} \mathcal{J}_f$. In particular, if we let $f = g \circ id_k$ and $f' = g \circ t \circ id_k$, for some $g \in A_N^0$ and $t = (i,j)$ with $i < j$, then the two juggling patterns differ only on the vertices from $i-k+1$ to $j-k$, and the difference is that $(\mathcal{J}_f)_{i-k+m}$ and $(\mathcal{J}_{f'})_{i-k+m}$ contain, up to equivalence modulo N, $i-k+m-f(i-k)$ and $i-k+m-f'(i-k)$ respectively.*

Proof. For $b \in \mathbb{Z}$, we have that $f(b) = f'(b)$ if and only if $b \not\equiv_N i-k,\, j-k$, and that $f(i-k) = f'(j-k)$, $f(j-k) = f'(i-k)$. In addition, since f and f' are bounded and affine we know that $f(i-k), f'(i-k) \geq j-k$, thus for $i-k+1 \leq a \leq j-k$, the sets in (2.3.3) for f and f' contain $a - f(i-k)$ and $a - f'(i-k)$ respectively. We observe that they have different residue class modulo N, once again because f and f' are affine. The argument is identical in the opposite direction. □

Example 2.3.10. The bounded affine permutations corresponding to

$$\mathcal{J} = 34 \quad \substack{24 \\ \\ \\ 34} \quad 34 \qquad \text{and} \qquad \mathcal{J}' = 34 \quad \substack{24 \\ \\ \\ 34} \quad 23$$

are determined respectively by

$$f_{\mathcal{J}} : 0 \longmapsto 1, \quad 1 \longmapsto 3, \quad 2 \longmapsto 4, \quad 3 \longmapsto 6;$$
$$f_{\mathcal{J}'} : 0 \longmapsto 3, \quad 1 \longmapsto 1, \quad 2 \longmapsto 4, \quad 3 \longmapsto 6,$$

and they satisfy $f_{\mathcal{J}'} = f_{\mathcal{J}} \circ (0, 1)$, therefore we have $f_{\mathcal{J}'} = g_{\mathcal{J}} \circ (2, 3) \circ \mathrm{id}_2$ where $f_{\mathcal{J}} = g_{\mathcal{J}} \circ \mathrm{id}_2$. The two juggling patterns differ only on vertex 1, where the difference is $4 \in J_1$ and $2 \in J_1'$. The only terminal triple for $\mathcal{J}$ is $(0, 3, 2)$, so $\dim C_{\mathcal{J}} = 1$, while those for $\mathcal{J}'$ are $(1, 4, 2)$ and $(1, 4, 3)$.

Chapter 3

The symplectic juggling variety

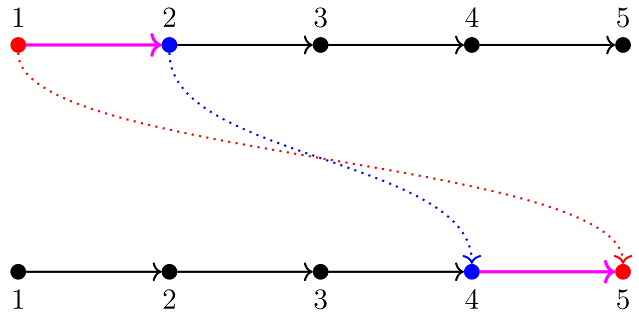
In Chapter 1 we mentioned the isotropic Grassmannian $\mathrm{Gr}(k, 2n)^{sp}$, and in Chapter 2 we introduced and examined $X(k, N)$, a linear degeneration of the Grassmannian $\mathrm{Gr}(k, N)$; the goal now is to define and study analogous "isotropic" subvarieties for $X(k, 2n)$. We primarily follow [FLMP25] in this chapter.

3.1 Quiver symmetries

We start by recalling that, chosen a symplectic form on $\mathbb{C}^{2n}$, the flag variety for Sp_{2n} is the fixed-point set for the automorphism

$$(V_i)_i \longmapsto \left(V_{2n-i}^{\perp}\right)_i$$

of $\mathfrak{Fl}(2n)$, shown in (1.1.3). We remind the reader that $V^{\perp}$ is the orthogonal subspace to V. This automorphism arises from a symmetry of the $\vec{A}_{2n-1}$ quiver which reverses the orientation of the arrows:



Observe that the pink arrow in the top quiver starts at the red vertex and ends at the blue one, while it does the opposite in the bottom quiver. The isotropic Grassmannian $\mathrm{Gr}(k, 2n)^{sp}$ is also the fixed point subvariety of $\mathfrak{Fl}(2n; (k, 2n-k))$ under $(V, W) \longmapsto (W^{\perp}, V^{\perp})$, or in other words it consists of subspaces on which the symplectic form vanishes.

In this chapter we will find a similar symmetry for the equioriented cycle Δ_{2n}. Derksen and Weyman [DW02] introduce the following few definitions to generalize this concept.

Definition 3.1.1. A *symmetric quiver* is a tuple (Q, τ, ς), consisting of a quiver Q, an involution

$$\tau \colon Q_0 \sqcup Q_1 \longrightarrow Q_0 \sqcup Q_1$$

on both the set of vertices and the set of edges that reverses the orientation of the arrows, i.e. satisfies

- $s\tau(\alpha) = \tau(t\alpha)$ for all $\alpha \in Q_1$;
- $t\tau(\alpha) = \tau(s\alpha)$ for all $\alpha \in Q_1$,

and a map $\varsigma \colon Q_0^\tau \cup Q_1^\tau \longrightarrow \{\pm 1\}$.

Usually it is also required that τ fixes loops, that is, if $s\alpha = t\alpha$ then $\tau(\alpha) = \alpha$. We will not need this hypothesis for our purposes, since Δ_{2n} has no loops. We choose one element for each τ-equivalence class of non-fixed vertices and edges, denote these sets with Q_0^+ and Q_1^+ respectively, and partition Q_0 into $Q_0^\tau \sqcup Q_0^+ \sqcup Q_0^-$, where $Q_0^- = \tau(Q_0^+)$ and $Q_1^- = \tau(Q_1^+)$.

Example 3.1.2. We have seen a symmetry for an oriented type A quiver. Each orientation of the same graph A_N can have either one or no symmetries at all, since there is only one possible graph automorphism. Type D Dynkin quivers, on the other hand, can have no symmetries, since the only automorphism of D_N does not allow a reversal of the arrows.

Definition 3.1.3. A dimension vector $\underline{d} \in \mathbb{N}^{Q_0}$ is *symmetric* if $d_i = d_{\tau(i)}$ for all $i \in Q_0$. A *symmetric representation* V for (Q, τ, ς) is a representation such that

I. $\underline{\dim}\, V$ is symmetric;

II. $V_i \cong V_{\tau(i)}^*$ for all $i \in Q_0^-$;

III. for $i \in Q_0^\tau$, V_i is equipped with a scalar product if $\varsigma(i) = 1$ or a (non-degenerate) symplectic form if $\varsigma(i) = -1$. Observe that either form provides a natural isomorphism $\psi_i \colon V_i \longrightarrow V_i^*$;

IV. If $\alpha \in Q_1^\tau$, then $V_\alpha = \varsigma(\alpha) \cdot V_\alpha^*$, where V^{**} is identified with V canonically;

V. If $\alpha \in Q_1^+$ and neither its source or target are τ-fixed, then $V_{\tau\alpha} = V_\alpha^*$;

VI. If $\alpha \in Q_1^+$ with $t\alpha \in Q_0^\tau$, then $V_{\tau(\alpha)} = V_\alpha^* \circ \psi_{t\alpha}$;

VII. If $\alpha \in Q_1^+$ with $s\alpha \in Q_0^\tau$, then $V_{\tau(\alpha)} = \psi_{s\alpha} \circ V_\alpha^*$.

We work in the field of complex numbers, but this definition is valid in characteristic other than 2.

Example 3.1.4. The following $\vec{A}_3$-representation is symmetric,

$$\mathbb{C} \xrightarrow{\binom{1}{0}} \mathbb{C}^2 \xrightarrow{(1\ 0)} \mathbb{C}$$

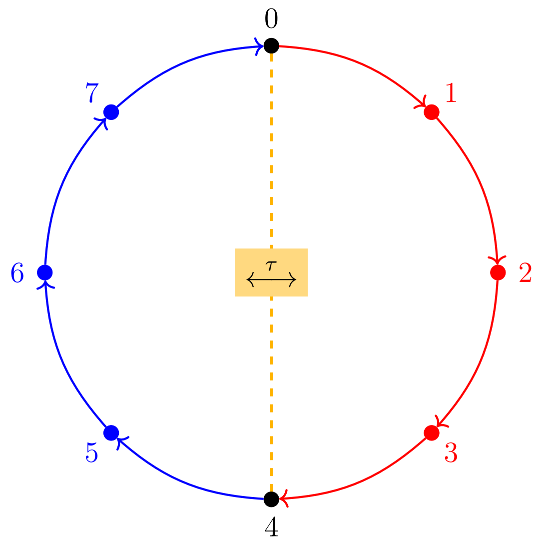
where the two-dimensional vector space is equipped with the standard scalar product, so $\varsigma(2) = +1$, and the one-dimensional vector spaces are paired by scalar multiplication.

3.2 The symplectic subvariety

The $\vec{A}_{2n-1}$ symmetry above is given, on the set of vertices, by $i \longmapsto 2n-i$, and therefore it sends the edge $i - i+1$ to $2n-i-1 - 2n-i$. We want to work with a symplectic form on the vector spaces of our representation $U_{[N]}$, so we set the dimension $N = 2n$ to be an even number. We are going to define a symmetry on the equioriented cycle on an even number of vertices Δ_{2n}, which will allow us to define the isotropic subvariety of $X(k, 2n)$ in the same fashion as the isotropic Grassmannian. Our quiver symmetry of choice is given by

$$\tau\colon i \longmapsto -i$$

for $i \in \mathbb{Z}_{2n}$.



The symmetry for $n = 4$.

Recall that vertex 0, which is fixed, is always depicted at the top of the cycle; therefore the other fixed vertex, n, is always on the bottom of the cycle. The symmetry then flips the quiver on the vertical axis between them. The image of the edge $i - i+1$ is $-i-1 - -i$, therefore this newly defined τ reverses the orientation of the arrows in Δ_{2n}; moreover, Q_1^τ is empty. The choice of ς is arbitrary, so we pick $\varsigma(0) = \varsigma(n) = -1$ since we want to work with symplectic forms. We set Q_0^+ to be the set of vertices on the right, i.e. $\{1, \dots, n-1\}$ and Q_1^+ to be the set of edges on the right, i.e. those with arrows $0 \longrightarrow 1, \dots, n-1 \longrightarrow n$. In the picture above Q_0^+ and Q_1^+ are shown in red, and their images under τ in blue.

Proposition 3.2.1. *The representation $U_{[2n]}$, for $(\Delta_{2n}, \tau, \varsigma)$, is symmetric.*

We first state a linear algebra result that will prove useful more than once.

Lemma 3.2.2. *Let n be a positive integer and Ω a $2n \times 2n$ invertible matrix such that*

$$\Omega^t = \Omega^{-1} = -\Omega \,, \tag{3.2.1}$$

so that it is the Gram matrix for a symplectic form $(-,-)$ on $\mathbb{C}^{2n}$; then for every $2n \times 2n$ matrix

A and every pair of vectors $v, w \in \mathbb{C}^{2n}$ we have

$$(Av, w) = \left(v, -\Omega A^t \Omega w\right)$$

Specifically, given any subspace W of $\mathbb{C}^{2n}$, we have

$$\Omega A^t \Omega \, (AW)^{\perp} \subseteq W^{\perp} ,$$

where $W^{\perp}$ is the subspace orthogonal to W with respect to Ω, and if A is invertible then the equality holds.

Proof. The following chain of equalities holds:

$$(Av, w) = (Av)^t \, \Omega w = v^t A^t \Omega w = v^t \left(-\Omega^2\right) A^t \Omega w = -v^t \Omega^2 A^t \Omega w = -\left(v, \Omega A^t \Omega w\right) \; .$$

□

Proof of Proposition 3.2.1. We fix a skew-symmetric $2n \times 2n$ matrix which will be the Gram matrix for our chosen symplectic form on $\mathbb{C}^{2n}$:

$$\Omega \coloneqq \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix} .$$

We choose this specific form because it satisfies

$$s^t = \Omega \circ s \circ \Omega \; . \tag{3.2.2}$$

Condition I in Definition 3.1.3 is trivially satisfied since $\underline{\dim}\, U_{[2n]}$ is constant. Condition III follows from equipping each vector space with the symplectic form Ω, which allows us to canonically identify $\mathbb{C}^{2n}$ with its dual, in particular on the vertices in Q_0^-; thus condition II also holds. The rest are fulfilled because of Lemma 3.2.2 and (3.2.2). □

Given two vectors $v = \sum_{i=1}^n v_i e_i$ and $w = \sum_{i=1}^n w_i e_i$, evaluating our symplectic form at v and w gives explicitly

$$(v, w) = \sum_{i=1}^{n} (-1)^{i+1} \cdot v_i \cdot w_{2n-i+1} \; .$$

Since n will be fixed throughout this text, we will write

$$\widetilde{i} \coloneqq 2n - i + 1 \tag{3.2.3}$$

for brevity without causing any confusion. This way we have $(e_i, e_j) = (-1)^{\widetilde{i}} \delta_{\widetilde{i}j}$.

Now fix k an integer from 1 to n. In order to emulate the isomorphism

$$\begin{aligned} \mathrm{Gr}(k,2n) &\longrightarrow \mathrm{Gr}(2n-k,2n) \\ V &\longmapsto V^{\perp} \end{aligned}$$

in the $X(k,2n)$ case, we equip every copy of $\mathbb{C}^{2n}$ on the vertices of Δ_{2n} with the above symplectic form, and take orthogonal subspaces while switching vertices under the quiver symmetry τ. With a slight notation abuse we also denote this new map by τ.

Proposition 3.2.3. *The image of the map*

$$\begin{aligned} \tau\colon X(k,2n) &\longrightarrow \prod_{i\in\mathbb{Z}_{2n}} Gr(2n-k,2n) \\ (V_i)_i &\longmapsto (V_{-i}^{\perp})_i \end{aligned} \tag{3.2.4}$$

is $X(2n-k,2n)$, and the composition

$$X(k,2n) \xrightarrow{\tau} X(2n-k,2n) \xrightarrow{\tau} X(k,2n)$$

is the identity. In particular, when $k=n$, τ is an automorphism of $X(n,2n)$ of order 2.

Proof. From Lemma 3.2.2 we know that given $W\subseteq\mathbb{C}^{2n}$ a subspace and A a $2n\times 2n$ matrix, one has $\Omega A^t\Omega\big((A\cdot W)^{\perp}\big)\subseteq W^{\perp}$. Now if $V=(V_i)_i$ is a point in $X(k,2n)$, its vector spaces satisfy $sV_i\subseteq V_{i+1}$ for all i, where s is the linear map defined in (2.1.1). Taking the orthogonal subspaces we get $V_{i+1}^{\perp}\subseteq(sV_i)^{\perp}$, then applying s to both sides we find

$$s(V_{i+1}^{\perp})\subseteq s\big((sV_i)^{\perp}\big)=\Omega s^t\Omega\big((sV_i)^{\perp}\big)\subseteq V_i^{\perp}\ .$$

That is, $s(\tau V)_{-i-1}\subseteq(\tau V)_{-i}$ for any $i\in\mathbb{Z}_{2n}$. Hence we obtain $\tau V\in X(2n-k,2n)$. □

Notation: given two juggling patterns $\mathcal{J}\in JP(k,N)$ and $\mathcal{J}'\in JP(m,N)$ with $k\le m$, we write $\mathcal{J}\subseteq\mathcal{J}'$ if $J_a\subseteq J'_a$ for all $a\in\mathbb{Z}_n$. Equivalently, for $V\in X(k,N)$ and $W\in X(m,N)$ we write $V\subseteq W$ if $V_i\subseteq W_i$ for all i. Therefore $\mathcal{J}\subseteq\mathcal{J}'$ if and only if $p_{\mathcal{J}}\subseteq p_{\mathcal{J}'}$. Observe that τ reverses inclusions, like taking the orthogonal subspace does for points in the classical Grassmannians.

Definition 3.2.4. We say that a subrepresentation $V=(V_i)_i\in X(k,2n)$ is *isotropic* if it satisfies

$$V_i\subseteq(V_{-i})^{\perp}$$

for all $i\in\mathbb{Z}_{2n}$, i.e. $V\subseteq\tau V$. We define

$$X(k,2n)^{sp}=\{V\in X(k,2n)\mid V\subseteq\tau V\}\,,$$

the locus of isotropic subrepresentations. It will be referred to as the *isotropic* or *symplectic subvariety*.

This subvariety was first introduced in [FLMP25], where we derived properties of $X(k,2n)^{sp}$ from those of $X(k,2n)$. In my next paper, [Mic25], the focus is specifically on the case $k=n$; here the fact that τ is now an automorphism of $X(n, 2n)$ provides additional insight. This case will be discussed in Chapter 4.

Remark 3.2.5. The subvariety of coisotropic points $X(2n-k, 2n)^{sp}$, defined analogously as

$$\{W \in X(2n-k, 2n) \mid W \supseteq \tau W\} ,$$

is the image of $X(k, 2n)^{sp}$ under τ, which is also and isomorphism between them.

3.2.1 The subgroup of symplectic automorphisms

Now we are going to define an automorphism of G_{2n} whose fixed-point subgroup acts on $X(n, 2n)^{sp}$ and which is a linear degeneration of Sp_{2n}, in the same way that G_{2n} is for GL_{2n}. Recall that a tuple of matrices $(g_i)_i \in G_{2n}$ satisfies $s \circ g_i = g_{i+1} \circ s$ for all vertices i, and that Sp_{2n} is the fixed-point subgroup of GL_{2n} under $g \longmapsto \Omega \circ (g^{-1})^t \circ \Omega^{-1} = -\Omega \circ (g^{-1})^t \circ \Omega$. For brevity, we write g^{-t} in place of $(g^{-1})^t$.

Proposition 3.2.6. *The image of*

$$\begin{aligned} \sigma\colon\ G_{2n} &\longrightarrow \prod_{i \in \mathbb{Z}_{2n}} GL_{2n} \\ (g_i)_i &\longmapsto (-\Omega \circ g_{-i}^{-t} \circ \Omega)_i \end{aligned} \tag{3.2.5}$$

is G_{2n}. Also, the equality

$$\tau g \tau(V) = \sigma(g)(V) \tag{3.2.6}$$

holds for all $g \in G_{2n}$ and $V \in X(k, 2n)$

Proof. For the first statement to hold, we need to prove that $s \circ \sigma(g)_i = \sigma(g)_{i+1} \circ s$ for all $i \in \mathbb{Z}_{2n}$. Taking the transpose of both sides in (3.2.2) we obtain $\Omega \circ s^t \circ \Omega = s$, thus

$$\begin{aligned} \sigma(g)_{i+1} \circ s = -\Omega \circ g_{-i-1}^{-t} \circ \Omega \circ s = \Omega \circ g_{-i-1}^{-t} \circ s^t \circ \Omega = \Omega \circ \left(s \circ (g_{-i-1}^{-1})\right)^t \circ \Omega = \\ = \Omega \circ \left((g_{-i}^{-1}) \circ s\right)^t \circ \Omega = \Omega \circ s^t \circ g_{-i}^{-t} \circ \Omega = -\Omega \circ s^t \circ \Omega^2 \circ g_{-i}^{-t} \circ \Omega = -s \circ \Omega \circ g_{-i}^{-t} \circ \Omega = s \circ \sigma(g)_i \end{aligned}$$

Lastly, we see that τ and σ satisfy (3.2.6) by Lemma 3.2.2. Indeed, if we set $A = g_{-i}^{-1}$ and $W = g_{-i}V_{-i}$ in the inclusion from said lemma we have the equality

$$\Omega \circ g_{-i}^{-t} \circ \Omega(V_{-i})^{\perp} = (g_{-i}V_{-i})^{\perp}$$

since A is invertible. We can add a negative sign to the left-hand side with no issue since we are dealing with vector spaces, and therefore get

$$\sigma(g)_i \left((\tau V)_i\right)^{\perp} = (\tau g V)_i \,,$$

which also equals $(\sigma(g)\tau V)_i$, so the claim is proven. □

Definition 3.2.7. An element $g = (g_i)_i$ of G_{2n} is called *symplectic* if it satisfies $(g_i v, g_{-i} w) = (v, w)$ for all $i \in \mathbb{Z}_{2n}$, $v \in U_{[2n]}^i$ and $w \in U_{[2n]}^{-i}$. The subgroup of symplectic elements is denoted by G_{2n}^{sp}.

Any two matrices A and B in GL_{2n} that satisfy $(A \cdot v, B \cdot w) = (v, w)$ for all $v, w \in \mathbb{C}^{2n}$ must also satisfy

$$B = -\Omega \circ (A^{-t}) \circ \Omega \,.$$

Hence G_{2n}^{σ}, the σ-fixed subgroup of G_{2n}, is precisely G_{2n}^{sp}.

The symplectic subvariety and subgroup are obtained as the linear degenerations of $\mathrm{Gr}(k, 2n)^{sp}$ and Sp_{2n} respectively, as they sit inside the degenerating $\mathrm{Gr}(k, 2n)$ and GL_{2n}. More precisely, the symplectic form Ω is compatible with $s(z)$ for $z \neq 0$ just like it is with $s = s(0)$, that is, it satisfies

$$s(z)^t = \Omega \circ s(z) \circ \Omega \,.$$

Therefore we find the isomorphism

$$\tau(z)\colon\ \mathrm{Gr}_{\underline{k}}\left(U_{[2n]}(z)\right) \longrightarrow \mathrm{Gr}_{\underline{2n-k}}\left(U_{[2n]}(z)\right)$$

and the automorphism

$$\sigma(z)\colon\ \mathrm{Aut}\left(U_{[2n]}(z)\right) \longrightarrow \mathrm{Aut}\left(U_{[2n]}(z)\right)$$

defined identically to τ and σ, together with a subvariety $X(k, 2n)(z)^{sp}$ and a subgroup $G_{2n}(z)^{sp}$ which acts on it. They are defined as expected:

$$X(k, 2n)(z)^{sp} := \{W \mid W_i \subseteq W_{-i}^{\perp}\}\,;$$
$$G_{2n}(z)^{sp} := \{h \mid h_i = -\Omega \circ h_{-i}^{-t} \circ \Omega\}\,.$$

For $z \neq 0$, they are isomorphic to $\mathrm{Gr}(k, 2n)^{sp}$ and Sp_{2n} respectively via the projection on a fixed vertex, and degenerate to $X(k, 2n)^{sp}$ and G_{2n}^{sp} as z approaches 0.

Let us now describe G_{2n}^{sp} explicitly. We know that for $g = (g_i)_{i \in \mathbb{Z}_{2n}} \in G_{2n}$, with respect to the standard basis, each g_i is a lower triangular matrix with nonzero diagonal entries and g is determined by the the entries $g_j^{(i)}$ of the first column of each matrix.

Lemma 3.2.8. *An element $g \in G_{2n}$ is symplectic if and only if the following conditions are satisfied:*

$$g_1^{(i)} g_1^{(j)} = 1, \quad \text{for } i, j \in \mathbb{Z}_{2n} \text{ such that } i + j = 1\,; \tag{3.2.7}$$

$$\sum_{l=0}^{r-1} (-1)^l g_{1+l}^{(i)}\, g_{r-l}^{(r-i)} = 0 \quad \text{for } i \in \mathbb{Z}_{2n} \text{ and } r = 2, \ldots, 2n\,. \tag{3.2.8}$$

Proof. By definition, $g \in G_{2n}^{sp}$ if and only if it satisfies

$$(g_i e_j^{(i)}, g_{-i} e_{j'}^{(-i)}) = (e_j^{(i)}, e_{j'}^{(-i)}) \tag{3.2.9}$$

for any vertex i and any $j, j' \in [2n]$. We use the explicit group element description given in §2.1.1, and we remind the reader that $g_i \cdot e_m^{(i)}$ is the m-th column of g_i, given by $m-1$ zeroes and then by $g_t^{(i-m+1)}$, with $t = 1 \ldots 2n-m+1$. First we see that by plugging $j = 1$ and $j' = 2n$ into (3.2.9) we recover $g_1^{(i)} g_1^{(-i+1)} = 1$, that is, the first equation. This actually exhausts all the possible cases where (3.2.9) equals 1, i.e. those where $j' = \widetilde{j}$. This is true because in those cases the left hand side equals $g_1^{(i-j+1)} g_1^{(-i+j)}$. To prove the second equation, we choose $j = 1$ and $j' = 2n - r + 1 < 2n$.

Notice that the column $g_i e_1^{(i)}$ has no zeroes, while $g_{-i} e_{2n-r+1}^{(-i)}$ has $2n - r$ zeroes. Thus, computing the left hand side we get

$$0 = \sum_{l=1}^{r} (-1)^{l+1} g_l^{(i)} g_{r-l+1}^{(r-i)} ,$$

which is exactly (3.2.8) with a shifted sum. All the other cases where the symplectic form equals zero are covered by this equation. We can assume $j < 2n - j' + 1 = \widetilde{j'}$, therefore $g_i e_j^{(i)}$ has $j - 1$ zeroes while $g_{-i} e_{j'}^{(-i)}$ has $g_{\widetilde{j'}}^{(-i-j'+1)}$ as the last coefficient. Computing the symplectic form between them yields

$$0 = \sum_{l=1}^{\widetilde{j'}-j+1} = (-1)^{l+j} g_l^{(i-j+1)} g_{\widetilde{j'}-j+2-l}^{(-i-j'+1)} ,$$

which, when setting $r = \widetilde{j'} - j + 1$, becomes exactly the equation we need, though evaluated at the vertex $i - j + 1$ instead of i and with a shifted sum. □

Remark 3.2.9. The equations from Lemma 3.2.8 satisfy the following properties:

- half of the $2n$ coefficients $g_1^{(i)}$ are determined by the other half;
- for any pair $i_1 \neq i_2 \in \mathbb{Z}_{2n}$ there is a single relation involving $g_j^{(i_1)}$ and $g_{j'}^{(i_2)}$ for $j, j' \in [2n]$;
- the relation (3.2.8) allows one to reconstruct $g_r^{(r-i)}$ starting from $g_{r'}^{(\bullet)}$ with $r' \leq r$.

Moreover, the (3.2.8) is trivial for $r \equiv_{2n} 2i$.

3.2.2 Lie algebra of the symplectic automorphism group

The goal now is computing the dimension of G_{2n}^{sp}, by finding a basis of its Lie algebra. Let $\mathfrak{g}_{2n}$ be the Lie algebra of G_{2n}, that is, $\mathrm{Lie}(G_{2n}) = \mathrm{End}_{\Delta_{2n}}(U_{[2n]})$. These endomorphisms are tuples of $2n \times 2n$ matrices $x = (x_i)_{i \in \mathbb{Z}_{2n}}$ such that

$$x_{i+1} \circ s = s \circ x_i$$

for all i [FLP22, Proposition 4.5]. We fix the following basis for $\mathfrak{g}_{2n}$: for $a \in [2n]$ and $b \in \mathbb{Z}_{2n}$, let $x(a, b)$ be the element of $\mathfrak{g}_{2n}$ acting as

$$x(a, b)\big(e_{1+j}^{(b+j)}\big) = e_{a+j}^{(b+j)} \quad \text{for } j = 0, \ldots, 2n - a$$

and as zero on other standard basis vectors. Essentially $x(a, b)$ lowers the segments starting at vertex a so that is starts not at $e_1^{(a)}$ but at $e_b^{(a)}$. As a matrix tuple, the entry in position (l, m) of $x(a, b)_{b+j}$, for $j \in [0, 2n - 1]$ is

$$\begin{cases} 1 & \text{if } l - j = a \text{ and } m - j = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $x(a, b)_{b+j}$ is the null matrix as soon as $a + j > 2n$. The automorphism σ of G_{2n} induces the following Lie algebra automorphism of $\mathfrak{g}_{2n}$ of order 2:

$$d\sigma(x_i) := (\Omega \circ x_{-i}^t \circ \Omega)_i \,.$$

Lemma 3.2.10. *The automorphism $d\sigma$ acts on the above basis of $\mathfrak{g}_{2n}$ as follows:*

$$d\sigma\big(x(a,b)\big) = (-1)^a\, x(a, a-b).$$

Proof. As mentioned, we will not be precise with residue classes or their representatives, since they will be unequivocally determined. Fix a and $i \in [2n]$, $b \in \mathbb{Z}_{2n}$ and let $-i \equiv_{2n} b + j$ for $j \in [0, 2n-1]$. Suppose first $j \leq 2n - a$; then the only non-zero entry of $x(a,b)_{-i}$ is a 1 in position $(a+j, 1+j)$. Therefore its transpose has a 1 in position $(1+j, a+j)$ and zeroes elsewhere. Recall that $\Omega_{l,m} = (-1)^{l+1}\, \delta_{l+m,2n+1}$; then, for any matrix $A = (A_{l,m})_{l,m}$ we have

$$(\Omega A \Omega)_{l,m} = (-1)^{l+m+1}\, A_{2n-l+1, 2n-m+1}.$$

Thus $\Omega \circ x^t_{-i} \circ \Omega$ has a $(-1)^{(1+j)+(a+j)+1} = (-1)^a$ in position $(2n-j, 2n-a-j+1)$ and zeroes elsewhere. Now let $q = 2n - a - j \in [0, 2n-a]$. Then $i = a - b + q$, and therefore $d\sigma\big(x(a,b)\big)_i = (-1)^a\, x(a, a-b)_i$. Now we suppose instead that $j \geq 2n - a + 1$. In this more trivial case $x(a,b)_{-i}$ is the zero matrix, and $q + 2n \geq 2n - a + 1$. Thus $d\sigma\big(x(a,b)\big)_i$ and $(-1)^a\, x(a, a-b)_i$ are both zero. ☐

Corollary 3.2.10.1. *The Lie algebra of G^{sp}_{2n} is the fixed-point subalgebra $\mathfrak{g}^{d\sigma}_{2n}$ of $\mathfrak{g}_{2n}$.*

We then consider the collection of the averages of basis element with their image, in order to obtain a basis of the fixed-point subalgebra.

Definition 3.2.11. For $a \in [2n]$ and $b \in \mathbb{Z}_{2n}$, let $y(a,b) \coloneqq \frac{1}{2}\big[x(a,b) + (-1)^a\, x(a, a-b)\big]$.

Proposition 3.2.12. *The dimension of G^{sp}_{2n} is equal to $n(2n+1) = \dim Sp_{2n}$.*

Proof. The elements from Definition 3.2.11 span $\mathfrak{g}^{d\sigma}_{2n}$ since the $x(a,b)$ form a basis of $\mathfrak{g}_{2n}$. Notice that $y(a,b) = (-1)^a\, y(a, a-b)$ and that out of the $4n^2$ possible pairs (a,b), the $2n$ such that $a \equiv_{2n} 2b$ are fixed under $(a,b) \longmapsto (a, a-b)$. The remaining $2n(2n-1)$ are then paired into $n(2n-1)$ pairs. Therefore a non-redundant set of generators of $\mathfrak{g}^{d\sigma}_{2n}$ consisting of elements $y(a,b)$ has cardinality $n(2n+1)$. ☐

3.3 Symplectic cells

We are interested in studying the decomposition of $X(k,2n)^{sp}$ into strata, i.e. the G^{sp}_{2n}-orbits, since the cellular decomposition of $X(k,2n)$ given by the G_{2n}-orbits provides a lot of insight.

Proposition 3.3.1. *The G^{sp}_{2n}-orbit of a point V in $X(k,2n)^{sp}$ is the intersection of the G_{2n}-orbit of V in $X(k,2n)$ with the subvariety.*

Proof. We want to prove the following equality:

$$G^{sp}_{2n} \cdot V = (G_{2n} \cdot V) \cap X(n,2n)^{sp}\ . \tag{3.3.1}$$

The inclusion $\subseteq$ is trivial since G^{sp}_{2n} acts on $X(n,2n)^{sp}$. In the opposite direction, we know that there exists $h \in G_{2n}$ and $\mathcal{J} \in JP(k,2n)$ such that $V = h \cdot p_{\mathcal{J}}$, and we want to find $g \in G^{sp}_{2n}$ such that $V = g \cdot p_{\mathcal{J}}$. The point $p_{\mathcal{J}}$ can be seen as a Δ_{2n}-representation, and can be broken down into the direct sum of several indecomposables $p_{\mathcal{J},1} \oplus \cdots \oplus p_{\mathcal{J},m}$. These are precisely the segments that make up the successor closed subquiver $S_{\mathcal{J}}$. We denote with $e_{j_c}^{(i_c)}$ the basis vector corresponding to the starting point of the indecomposable segment $p_{\mathcal{J},c}$, for $c \in [m]$. As a subrepresentation of $U_{[2n]}$, it would be isomorphic to $U(i_c; 2n - j_c + 1)$, where the indecomposable $U(i; l)$ is described in Example 1.2.7. Notice that these starting points completely determine $p_{\mathcal{J}}$, therefore their images under h determine V. These are vectors of the form

$$v_{i_c} = \sum_{j=j_c}^{2n} h_{j-j_c+1}^{(i_c-j_c+1)} e_j^{(i_c)} \in V_{i_c}$$

where $h_1^{(i_c-j_c+1)} \neq 0$. To satisfy $g \cdot p_J = V$, we require m out of the $2n$ first columns of g to start with the numbers $h_{j-j_c+1}^{(i_c-j_c+1)}$. Additionally, V is a symplectic point, so these several first entries of first columns of the matrices of g satisfy the orthogonality conditions. We want to complete these few coefficients of h to a symplectic group element g, possibly different from h. We proceed by applying Lemma 3.2.8 inductively. With Remark 3.2.9 in mind, we see that the coefficients $g_1^{(i)}$ can be either recovered or simply chosen in $\mathbb{C}^*$; indeed, $g_1^{(i)}$ is either a coefficient in h, its inverse, or it is free. We move on to the $g_2^{(i)}$ and so on. At each step, some coefficients are known from $V = h \cdot p_{\mathcal{J}}$ while the others are either free or recovered from (3.2.8). Observe that the number of free parameters depends on the dimension of the G_{2n}^{sp}-stabilizer of $p_{\mathcal{J}}$. □

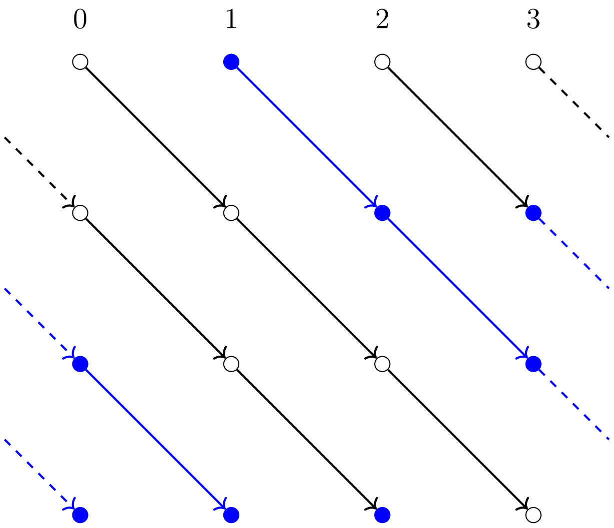
Since every equation defining the symplectic group elements is linear in the variables of the previous one, we obtain the next result.

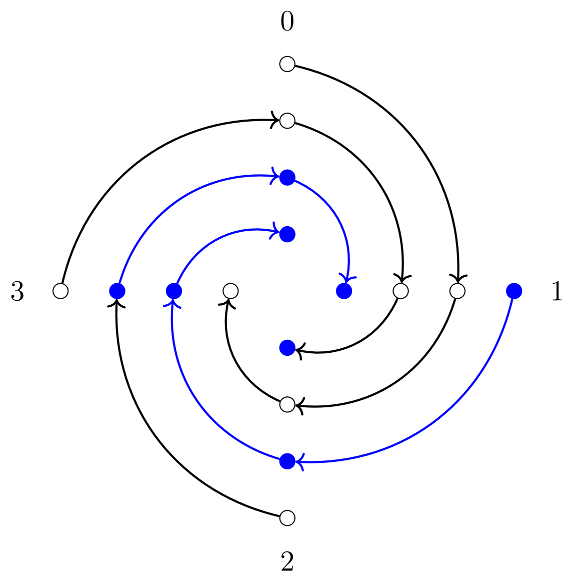
Corollary 3.3.1.1. *Analogously to the non-symplectic case, the G_{2n}^{sp}-orbits are affine spaces.*

Example 3.3.2. The indecomposable segments for

$$\mathcal{J} = \begin{matrix} & 34 & \\ 23 & & 14 \\ & 24 & \end{matrix}$$

are 3 in number, and can be identified by looking at either the square diagram or the spiral diagram.

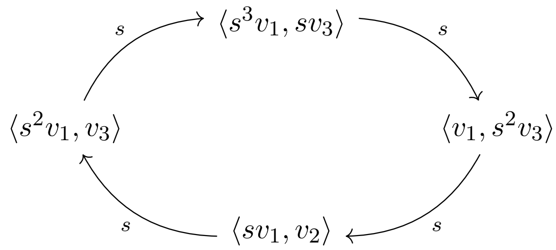




First notice that $p_{\mathcal{J}}$ itself is a symplectic point. The pairs (i_c, j_c) are $(1,1)$, $(2,4)$ and $(3,2)$, so $m = 3$, and they mark the starting points of the segments. A point $V = (V_i)_i \in C_{\mathcal{J}} \cap X(k,2n)^{sp}$ is completely determined by the following vectors

$$\begin{aligned} v_1 =& h_1^{(1)} e_1^{(1)} + h_2^{(1)} e_2^{(1)} + h_3^{(1)} e_3^{(1)} + h_4^{(1)} e_4^{(1)} \in V_1\,, \\ v_2 =& h_1^{(3)} e_4^{(2)} \in V_2\,, \\ v_3 =& h_1^{(2)} e_2^{(3)} + h_2^{(2)} e_3^{(3)} + h_3^{(2)} e_4^{(3)} \in V_3 \end{aligned}$$

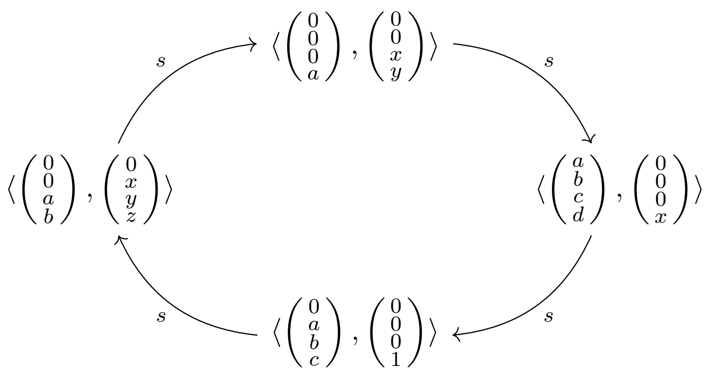
meaning that V is of the form



Here the coefficients $h_1^{(1)}$, $h_1^{(2)}$ and $h_1^{(3)}$ are nonzero since $h \in G_{2n}$. They also need to satisfy $(v_1, v_3) = h_1^{(1)} h_3^{(2)} - h_2^{(1)} h_2^{(2)} + h_3^{(1)} h_1^{(2)} = 0$. The coefficient $h_1^{(3)}$ only matters up to nonzero scalar, and we can rename all the other coefficients to lighten up the notation:

$$\begin{aligned} v_1 =& a e_1^{(1)} + b e_2^{(1)} + c e_3^{(1)} + d e_4^{(1)} \in V_1\,, \\ v_2 \in& \mathbb{C} \cdot e_4^{(2)} \subset V_2\,, \\ v_3 =& x e_2^{(3)} + y e_3^{(3)} + z e_4^{(3)} \in V_3\,. \end{aligned}$$

So V is given by



and we rescale and can take linear combinations of these basis vectors without altering the spaces

V_i they span: if we let $b' = ba^{-1}$, $c' = ca^{-1}$, $y' = yx^{-1}$ and $z' = zx^{-1}$ then the point

$$\begin{array}{ccccc}
 & \nearrow^{s} & \left\langle \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix} \right\rangle & \searrow^{s} & \\
\left\langle \begin{pmatrix} 0\\0\\1\\b' \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\z'-y'b' \end{pmatrix} \right\rangle & & & & \left\langle \begin{pmatrix} 1\\b'\\c'\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \right\rangle \\
 & \nwarrow_{s} & \left\langle \begin{pmatrix} 0\\1\\b'\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \right\rangle & \swarrow_{s} &
\end{array}$$

is also V. By the symplectic conditions, $z' - y'b'$ must be equal to c'. We want to find $g \in G_4^{sp}$ such that $V = g \cdot p_{\mathcal{J}}$, so we can set $g_1^{(1)} = 1 = g_1^{(2)}$ and therefore, by (3.2.7), $g_1^{(3)} = 1 = g_1^{(0)} = 1$. Following is a possible completion of these equations to a tuple $g \in G_4^{sp}$.

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ c' & 0 & 1 & 0\\ 0 & -c' & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -c' & 0 & 1 & 0\\ -b'c' & -c' & b' & 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 0 & 0 & 0\\ b' & 1 & 0 & 0\\ c' & 0 & 1 & 0\\ 0 & c' & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -c' & b' & 1 & 0\\ 0 & c' & 0 & 1 \end{pmatrix}$$

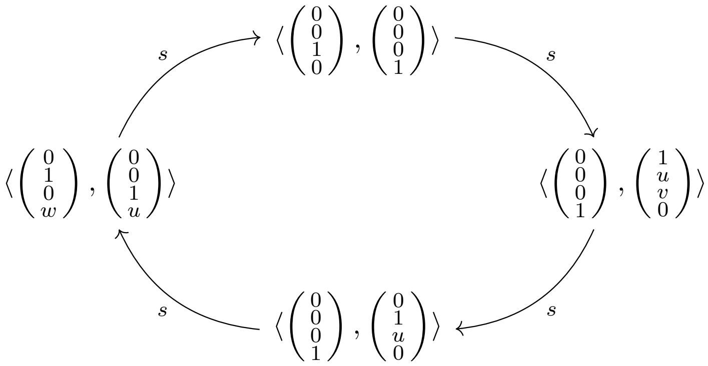
This also tells us that the generic symplectic point in $C_{\mathcal{J}}$ is of the form

$$\begin{array}{ccccc}
 & \nearrow^{s} & \left\langle \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \right\rangle & \searrow^{s} & \\
\left\langle \begin{pmatrix} 0\\1\\0\\-v \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\u \end{pmatrix} \right\rangle & & & & \left\langle \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\u\\v\\0 \end{pmatrix} \right\rangle \\
 & \nwarrow_{s} & \left\langle \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\u\\0 \end{pmatrix} \right\rangle & \swarrow_{s} &
\end{array}$$

obtained by applying the condition

$$0 = \left(\begin{pmatrix} 0\\1\\0\\w \end{pmatrix}, \begin{pmatrix} 1\\u\\v\\0 \end{pmatrix} \right) = v + w$$

to the generic point in $C_{\mathcal{J}}$



as described in (2.1.2). So the G_4^{sp}-orbit of $p_{\mathcal{J}}$ has dimension 2.

3.3.1 Symplectic juggling patterns

The reader may have observed that the orthogonal subspace under Ω for a coordinate subspace is again a coordinate subspace. For a subset $J \in \binom{[2n]}{k}$, let $RJ \in \binom{[2n]}{2n-k}$ be such that $V_{RJ} = V_J^{\perp}$. Explicitly $RJ = [2n] \setminus \{\widetilde{j} \mid j \in I\}$. If $J \subseteq RJ$ then $V_J \in \mathrm{Gr}(k,2n)^{sp}$, and we say that J is isotropic while V_{RJ} is coisotropic; in either case we call them symplectic. In the same fashion, the vector spaces that make up $\tau(p_{\mathcal{J}})$ are also coordinate subspaces, so we can extend

$$R\colon \binom{[2n]}{k} \longrightarrow \binom{[2n]}{2n-k}$$

to the poset of juggling patterns.

Definition 3.3.3. For $\mathcal{J} = (J_i) \in JP(k,2n)$, we define $R\mathcal{J}$ to be the $(2n-k,2n)$-juggling pattern such that $p_{R\mathcal{J}} = \tau(p_{\mathcal{J}})$, i.e. the one with

$$(R\mathcal{J})_i = R(J_{-i})\,. \tag{3.3.2}$$

Similarly to Proposition 3.2.3, R is a bijection and the composition

$$JP(k,2n) \xrightarrow{R} JP(2n-k,2n) \xrightarrow{R} JP(k,2n)$$

is the identity. We could also prove by hand that $R\mathcal{J}$, if defined only by (3.3.2), is a juggling pattern: first observe that the map R, for sets, satisfies the following.

- $R(I \cup J) = R(I) \cap R(J)$;
- $R(I \cap J) = R(I) \cup R(J)$;
- $I \subseteq J \Longrightarrow R(J) \subseteq R(I)$.

For a set $J \subset [2n]$ of cardinality k that does not contain $2n$, it must be that $1 \in R(J)$ and, by definition of $\mathfrak{s}$, that $1 \notin \mathfrak{s}J$. As a consequence, $2n \in R(\mathfrak{s}J)$. So $R(\mathfrak{s}J) = [2n] \setminus \{\widetilde{j+1} \mid j \in J\}$ can be rewritten as

$$[2n] \setminus \{\widetilde{j} - 1 \mid j \in J\} = \{a \in [2n] \mid a + 1 \neq \widetilde{j}\ \forall j \in J\}\,.$$

We remove $2n$ and apply $\mathfrak{s}$:

$$\mathfrak{s}\big(R(\mathfrak{s}J) \setminus \{2n\}\big) = \{a+1 \mid a \in R(\mathfrak{s}J) \wedge a \neq 2n\} = R(J) \setminus \{1\} \ . \tag{3.3.3}$$

Back to $\mathcal{J}$, we want to show that $\mathfrak{s}\big((R\mathcal{J})_i \setminus \{2n\}\big) \subseteq (R\mathcal{J})_{i+1}$. By hypothesis $\mathfrak{s}(J_{-i-1} \setminus \{2n\}) \subseteq J_{-i}$, therefore

$$R(\mathcal{J})_i = R(J_{-i}) \subseteq R\big(\mathfrak{s}(J_{-i-1} \setminus \{2n\})\big) \, .$$

We conclude with the following chain, applying (3.3.3) to $J = J_{-i-1} \setminus \{2n\}$:

$$\begin{gathered} \mathfrak{s}\big((R\mathcal{J})_i \setminus \{2n\}\big) = \mathfrak{s}\big(R(J_{-i}) \setminus \{2n\}\big) \subseteq \\ \subseteq \mathfrak{s}\Big(R\big(\mathfrak{s}(J_{-i-1} \setminus \{2n\})\big) \setminus \{2n\}\Big) \overset{(3.3.3)}{=} R(J_{-i-1} \setminus \{2n\}) \setminus \{1\} \subseteq \\ \subseteq R(J_{-i-1}) = (R\mathcal{J})_{i+1}. \end{gathered}$$

The last inclusion is an equality when $2n \in J_{-i-1}$, and is strict otherwise.

Definition 3.3.4. A $(k, 2n)$-juggling pattern $\mathcal{J}$ is *isotropic*, or *symplectic*, if it satisfies the following equivalent conditions:

- ◇ $p_{\mathcal{J}} \in X(k, 2n)^{sp}$;
- ◇ $\mathcal{J} \subseteq R\mathcal{J}$;
- ◇ $j \in J_i \implies \widetilde{j} \notin J_{-i}$.

We denote the set of $(k, 2n)$-symplectic juggling patterns by $JP(k, 2n)^{sp}$.

The three conditions are equivalent since the first one coincides with $p_{\mathcal{J}} \subseteq \tau(p_{\mathcal{J}})$, and by Definition 3.3.3.

Example 3.3.5. As we saw in Example 3.3.2, the juggling pattern

$$\mathcal{J} = \begin{array}{ccc} & 34 & \\ 23 & & 14 \\ & 24 & \end{array}$$

is symplectic, while

$$\mathcal{J}' = \begin{array}{ccc} & 24 & \\ 34 & & 23 \\ & 34 & \end{array}$$

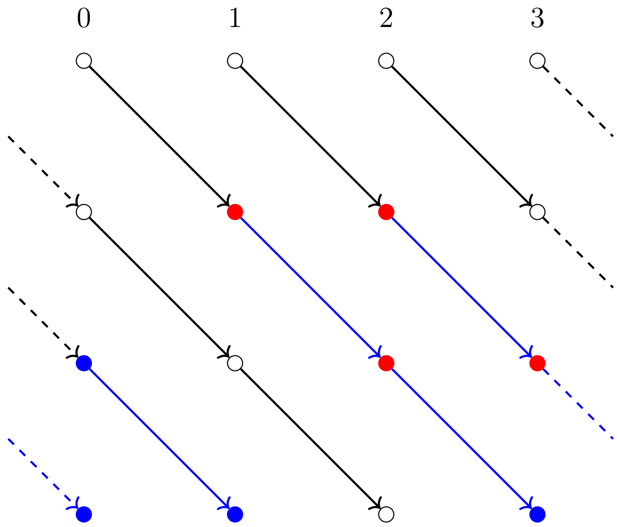
is not, since $2 \in J'_1$ and $3 \in J'_3$. Therefore $R(\mathcal{J}) = \mathcal{J}$ and

$$R(\mathcal{J}') = \begin{array}{ccc} & 24 & \\ 14 & & 34 \\ & 34 & \end{array} \neq \mathcal{J}' \, .$$

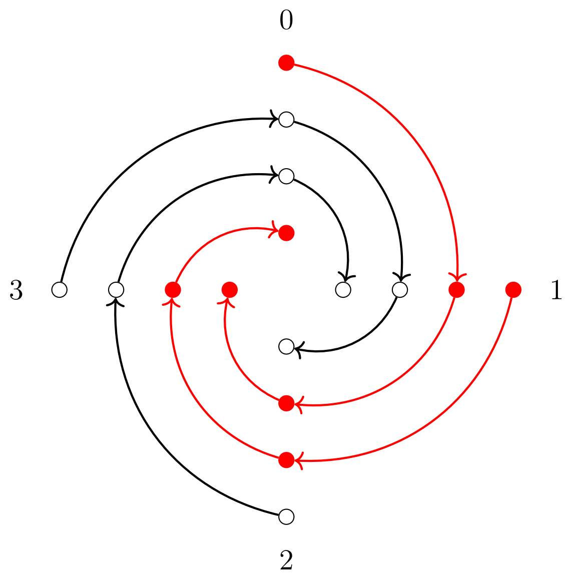
Lemma 3.3.6. *If (a, i, j) is a descending triple for $\mathcal{J}$ then $(-a, \widetilde{j}, \widetilde{i})$ is a descending triple for $R\mathcal{J}$.*

Proof. Since $j \in J_a$ and $i \notin J_a$, by definition of $R\mathcal{J}$ we have $\widetilde{i} \in R(J_a) = (R\mathcal{J})_{-a}$ and $\widetilde{j} \notin R(J_a) = (R\mathcal{J})_{-a}$. Also $i > j$ implies $\widetilde{j} > \widetilde{i}$. □

If $\mathcal{J} \in JP(k, 2n)$ is not symplectic then at least one element $x \in J_a$ is paired to another element $\widetilde{x} \in J_{-a}$, and the vertices of the coefficient quiver $Q(U_{[2n]}, \mathscr{SB})$ corresponding to any such pair are at the tail end of the respective segments in $S_{\mathcal{J}}$. Indeed, if $1 \leq x - 1 \in J_{a-1}$ then $2n \geq \widetilde{x-1} = \widetilde{x} + 1 \in J_{-a+1}$. Therefore the vertex paired to the tail end of a segment is the "last" of its segment to be paired to another vertex, i.e. the rightmost one in the square diagram. This is visible in the following diagram for the $(2,4)$-juggling pattern $34 \, {}^{34}_{23} \, 24$, where the vertices that make it non-symplectic are in red.



This implies that the segment of the coefficient quiver ending at $j \in \mathbb{Z}_{2n}$ is paired with the segment ending at $-j-1 \in \mathbb{Z}_{2n}$. In the following diagram, the paired segments ending at 0 and 3 are highlighted in red.



Remark 3.3.7. Notice that any segment of the coefficient quiver cannot contain both x on a vertex a and $\widetilde{x}$ on vertex $-a$, for parity reasons.

Proposition 3.3.8. *If $C_{\mathcal{J}} \cap X(k, 2n)^{sp} \neq \varnothing$ then $p_{\mathcal{J}} \in X(k, 2n)^{sp}$.*

Proof. Let $\mathcal{J}$ be a non-symplectic $(k, 2n)$-juggling pattern, i.e. $\mathcal{J} \not\subseteq R\mathcal{J}$. Thus there exists $i \in \mathbb{Z}_{2n}$ and $j \in [2n]$ with $j \in J_i$ and $\widetilde{j} \in J_{-i}$. By the description of the G_{2n}-orbits in $X(k, 2n)$ given in

§2.1.3, every point $V \in C_{\mathcal{J}}$ is such that V_i and V_{-i} respectively contain vectors of the form

$$v = e_j^{(i)} + \sum_{m=j+1}^{2n} x_m e_m^{(i)} \qquad \text{and} \qquad w = e_{\widetilde{j}}^{(-i)} + \sum_{m=\widetilde{j}+1}^{2n} y_m e_m^{(-i)}$$

for some $x_m, y_m \in \mathbb{C}$. Therefore $(v, w) \neq 0$ and $V \notin X(k, 2n)^{sp}$. $\square$

Since only the G_{2n}-orbits $C_{\mathcal{J}}$ for $\mathcal{J}$ symplectic have non-empty intersection with $X(k, 2n)^{sp}$ and this intersection is $G_{2n}^{sp} \cdot p_{\mathcal{J}}$, as a consequence of Proposition 3.3.8 we have the next result.

Corollary 3.3.8.1. *The G_{2n}^{sp}-orbits in $X(k, 2n)$ are parametrized by $JP(k, 2n)^{sp}$.*

Definition 3.3.9. The *symplectic cell $C_{\mathcal{J}}^{sp}$* corresponding to a juggling pattern $\mathcal{J} \in JP(k, 2n)^{sp}$ is the orbit of $p_{\mathcal{J}}$ under the symplectic subgroup G_{2n}^{sp}.

Therefore, Proposition 3.3.1 can be stated as

$$C_{\mathcal{J}}^{sp} = C_{\mathcal{J}} \cap X(k, 2n)^{sp} \tag{3.3.4}$$

and can be extended to hold for non-isotropic points thanks to Proposition 3.3.8, by defining the symplectic cell for such a juggling pattern to be empty.

Proposition 3.3.10. *The map $R\colon JP(k, 2n) \longrightarrow JP(2n-k, 2n)$ is order-preserving.*

Proof. First, notice that R sends the minimal $(k, 2n)$-juggling pattern, whose sets are all equal to $\{2n-k+1, \ldots, 2n\}$ to the minimal $(2n-k, 2n)$-juggling pattern, which is constantly equal to $\{k+1, \ldots, 2n\}$. Now let $V = g \cdot p_{\mathcal{J}'}$ for some $g \in G_{2n}$ and $\mathcal{J}'$ some $(k, 2n)$-juggling pattern. By Proposition 3.2.6 we obtain,

$$\tau V = \tau(g p_{\mathcal{J}'}) = \tau(g \tau(p_{R\mathcal{J}'})) = \sigma(g) p_{R\mathcal{J}'}.$$

Hence τ sends isomorphically the cell $C_{\mathcal{J}'}$ into the cell $C_{R\mathcal{J}'}$, since they are G_{2n}-orbits. Recall that, for another juggling pattern $\mathcal{J}$, the condition $\mathcal{J} \leq \mathcal{J}'$, is equivalent to $p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}}$. Since τ is an isomorphism of varieties, we get $p_{R\mathcal{J}} \in \overline{C_{R\mathcal{J}'}}$ $\square$

This could also be proven as a corollary of the next result and of Proposition 2.1.14, observing that $x \geq y$ implies $\widetilde{y} \geq \widetilde{x}$. We recall the order given in (2.1.3) on subsets of $[2n]$ of the same cardinality.

Lemma 3.3.11. *Let $A \leq B$ be subsets of $[N]$ of cardinality d. Then $B^c \leq A^c$.*

Proof. We proceed by induction on d, and the base case $d = 0$ is trivial. For $d \geq 1$, let us denote the elements of A and B respectively with $a_1 < a_2 < \cdots < a_d$ and $b_1 < b_2 < \cdots < b_d$. By assumption $A \leq B$, therefore $A \setminus \{a_d\} \leq B \setminus \{b_d\}$. These sets have cardinality $d-1$, so we deduce

$$X := A^c \cup \{a_d\} \geq B^c \cup \{b_d\} =: Y$$

by inductive hypothesis. Let x_i and y_i be their elements, listed increasingly, where $i \in [N-d+1]$. The elements $a_d \in X$ and $b_d \in Y$ are of the form x_t and y_s for some integers $t \leq s$, since $a_d \leq b_d$.

Let x'_i and y'_i denote the ordered elements of $A^c = X \setminus \{a_d\}$ and $B^c = Y \setminus \{b_d\}$. By definition they are

$$\begin{cases} x'_i = x_i & 1 \leq i < t \\ x'_i = x_{i+1} & t \leq i \leq N-d \end{cases} \qquad \begin{cases} y'_i = y_i & 1 \leq i < s \\ y'_i = y_{i+1} & s \leq i \leq N-d \end{cases}.$$

Comparing them, if $t = s$ we can simply remove them from X and Y, otherwise we have

$$\begin{cases} y'_i = y_i \leq x_i = x'_i & 1 \leq i < t \\ y'_i = y_i \leq x_i < x_{i+1} = x'_i & t \leq i < s \\ y'_i = y_{i+1} \leq x_{i+1} = x'_i & s \leq i \leq N-d\,. \end{cases}$$

□

Remark 3.3.12. The minimal juggling pattern is always symplectic.

3.3.2 Symplectic mutations

Now we would like to define a notion of "symplectic" mutation, so that the number of those that start at a symplectic juggling pattern is the dimension of the corresponding symplectic cell, and thus prove a result analogous to Lemma 2.2.7.

We still follow [FLMP25], but unfortunately a mistake was found in one of the proofs in that article, and thus some results only hold for $k = n$ and not in general. Namely, we stated that the symplectic juggling patterns that are maximal with respect to the G_{2n}^{sp}-closure inclusion order are all and only the ones that are symplectic and maximal in $JP(k, 2n)$, and that the closures of the corresponding symplectic cells are the irreducible components of $X(k, 2n)^{sp}$. One step in the proof of these results was faulty, thus we are not able to compute the dimension of the subvariety nor to describe its irreducible components when $k < n$. Even though the lemma we used to prove these results is not correct in general, as will be shown at the beginning of the next chapter, the coauthors and I believe that these results hold nonetheless and will work to patch their proofs. The rest of this section will contain only the results from that paper that are valid for all k, and in the next chapter we will include proofs for the aforementioned results.

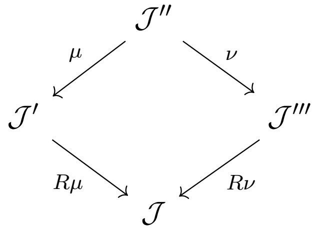
Remark 3.3.13. Consider a mutation $\mathcal{J}'' \xrightarrow{\mu} \mathcal{J}$ with $\mathcal{J}, \mathcal{J}'' \in JP(k, 2n)^{sp}$. Then for any i added on vertex a in place of a smaller number j, we have that either $\widetilde{i}$ was not in J''_{-a}, or it was but μ removes it. If so, it is replaced with $\widetilde{j}$ since $i - j = \widetilde{j} - \widetilde{i}$. On the other hand, any mutation that simultaneously removes some j on a vertex a and adds $\widetilde{j}$ on vertex $-a$ is such that if either of its starting or ending juggling pattern is symplectic, the other one is as well.

Lemma 3.3.14. *Consider $\mathcal{J}'' \in JP(k, 2n)^{sp}$ and a mutation $\mathcal{J}'' \xrightarrow{\mu} \mathcal{J}'$ of breadth m. If $\mathcal{J}'$ is not symplectic then there exists $\mathcal{J} \in JP(k, 2n)^{sp}$ and a mutation $\mathcal{J}' \xrightarrow{R\mu} \mathcal{J}$ of breadth m which removes all elements $\widetilde{y} \in J'_{-b}$ paired to any $y \in J'_b$ added by μ, as well as mutations $\mathcal{J}'' \xrightarrow{\nu} \mathcal{J}''' \xrightarrow{R\nu} \mathcal{J}$ of breadth m, with $\mathcal{J}'''$ not symplectic, which remove the same elements as $R\mu$ and μ respectively.*

Proof. Suppose (a, i, j) is the descending triple for μ such that $j \in J''_a$ is the tail end of its segment in $S_{\mathcal{J}''}$. Since $\mathcal{J}'$ is not symplectic, $\widetilde{i}$ is in J'_{-a} and it belongs to a different segment of $Q(U_{[2n]}, \mathscr{SB})$ than $j \in J''_a$. If it did then $\widetilde{i} \in J''_{-a}$, from which follow two impossibilities: either it is removed by μ, which would make $\mathcal{J}'$ symplectic, or it is not removed because $\widetilde{i} + m = \widetilde{j}$ is already in J''_{-a}, which would make $\mathcal{J}''$ not symplectic. Therefore $\widetilde{i}$ is also in J''_{-a} and is not removed by μ. By assumption $\widetilde{j}$ is not in J''_{-a}, and it also is not in J'_{-a} because otherwise it would need to be added by μ in place of $\widetilde{i}$, which is in a different segment than the one affected by the mutation.

We have found that $(-a, \widetilde{j}, \widetilde{i})$ is a descending triple for $\mathcal{J}''$ not in the same equivalence class as (a, i, j), and also a descending triple for $\mathcal{J}'$. The corresponding mutation $\mathcal{J}' \xrightarrow{R\mu} \mathcal{J}$ removes all predecessors of $\widetilde{i} \in J'_{-a}$, which are all the elements that caused $\mathcal{J}''$ to be non-symplectic. On the other hand, we also obtain a mutation ν starting at $\mathcal{J}''$ with triple $(-a, \widetilde{j}, \widetilde{i})$ which produces $\mathcal{J}'''$, non-symplectic since it contains both j on vertex a and $\widetilde{j}$ on vertex $-a$.

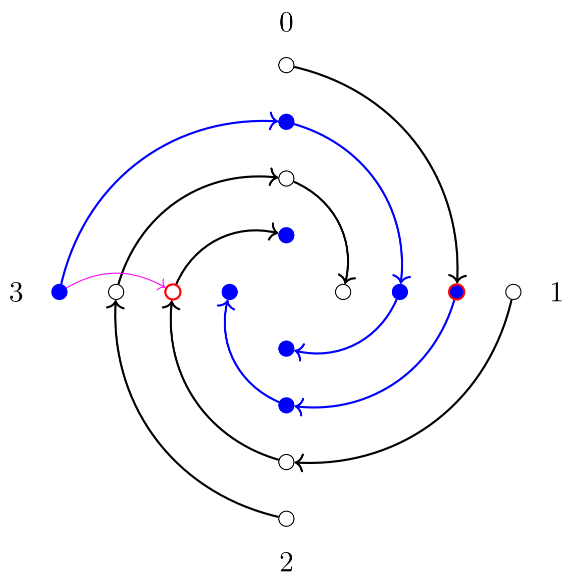
We conclude by applying the same reasoning to ν instead of μ to find $R\nu$, and this produces the desired diagram.



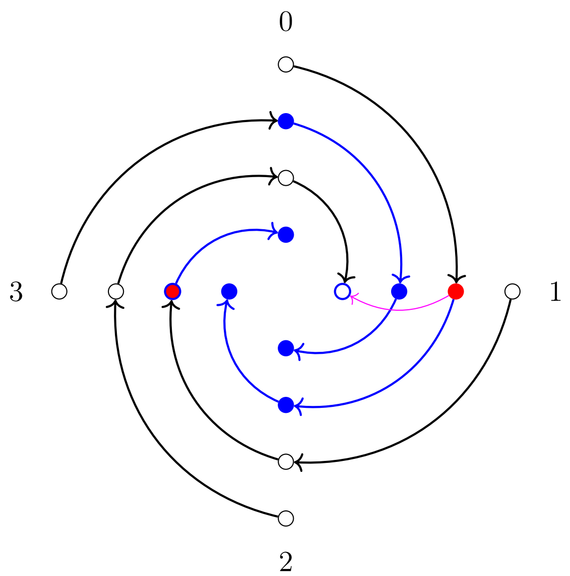
□

Definition 3.3.15. The mutation $R\mu$ as in Lemma 3.3.14 will be called *correction* of μ. A *symplectic mutation* $\mathcal{J}'' \xrightarrow[sp]{\mu} \mathcal{J}$ is either a single mutation $\mathcal{J}'' \xrightarrow{\mu} \mathcal{J}$ such that both $\mathcal{J}''$ and $\mathcal{J}$ are symplectic, or the composition of a mutation and its correction $\mathcal{J}'' \xrightarrow{\mu} \mathcal{J}' \xrightarrow{R\mu} \mathcal{J}$.

Example 3.3.16. Consider the mutation μ between the $(2,4)$-juggling patterns $\mathcal{J}'' = 14 \overset{24}{\underset{34}{}} 23$ and $\mathcal{J}' = 34 \overset{24}{\underset{34}{}} 23$, shown in the following diagram.



The former is symplectic while the latter is not, as it contains 3 on vertex 3 and 2 on vertex 1. By Lemma 3.3.14, we get a mutation $R\mu$ starting at $\mathcal{J}'$ which replaces the 2 in J'_1 with a 4, producing $\mathcal{J} = 34 \overset{24}{\underset{34}{}} 34$.



We conclude by also finding that $\mathcal{J}''' = 14 \begin{matrix}24\\ \\ 34\end{matrix} 34$.

Lemma 3.3.17. *The dimension of a symplectic cell $C^{sp}_{\mathcal{J}}$ is equal to the number of symplectic mutations starting at $\mathcal{J}$.*

Proof. Symplectic conditions on $C_{\mathcal{J}}$ are such that the coefficient $u^{(a)}_{i,j}$ for the $\sim$-equivalence class of a descending triple (a, i, j) is tied to that for $(-a, \widetilde{j}, \widetilde{i})$, if it identifies an equivalence class. So these coefficients are only independent if they belong to different symplectic mutations. Thus the number of free parameters for $C^{sp}_{\mathcal{J}}$ is equal to the number of such pairs of equivalence classes. □

It might happen that a descending triple (a, i, j) for $\mathcal{J}$ is such that $(-a, \widetilde{j}, \widetilde{i})$ is not a descending triple for $\mathcal{J}$. In this case the mutation produces a symplectic juggling pattern. This is the case for every mutation when $k = 1$, as shown in the next result. A direct consequence of skew-symmetry of simplectic forms is that one-dimensional subspaces are always isotropic, that is, $\mathrm{Gr}(1, 2n)^{sp} = \mathrm{Gr}(1, 2n)$. This holds for their linear degenerations as well.

Recall from Lemma 2.1.16 that a juggling pattern $\mathcal{J}$ is maximal in the partial order if $2n \in J_i$ implies $1 \in J_{i+1}$ for all $i \in \mathbb{Z}_{2n}$. In particular it is completely determined by just one of its $2n$ sets J_i: for any vertex a, given $p \in \mathbb{Z}$ such that $a = i + p$, J_a consists of the representatives in $[2n]$ of the residue classes of $\{j + p \mid j \in J_i\}$.

Proposition 3.3.18. *For any $n \geq 1$, $X(1, 2n)^{sp} = X(1, 2n)$.*

Proof. We start by proving that $JP(1, 2n)^{sp} = JP(1, 2n)$. Consider a $(1, 2n)$-juggling pattern $\mathcal{J}$ such that $J_a = \{x\}$ and $J_{-a} = \{\widetilde{x}\}$ for some $x \in [2n]$ and $a \in \mathbb{Z}_{2n}$. Let m be the number of arrows needed to reach a starting from $-a$, which must be even since it satisfies $m \equiv_{2n} 2a$. But x and $\widetilde{x}$ have different parity, therefore between a and $-a-1$, as well as between $-a$ and $a-1$, there must be a vertex b such that $J_b = \{2n\}$. This implies the inequalities $x + m > 2n$ and $\widetilde{x} + 2n - m > 2n$, which are incompatible and therefore such a $\mathcal{J}$ cannot exist. Next we show that $C^{sp}_{\mathcal{J}} = C_{\mathcal{J}}$ for a maximal $(1, 2n)$-juggling pattern $\mathcal{J}$. Let a be the vertex with $J_a = \{1\}$, so we have $J_{a+m} = \{1+m\}$ for $m \in [0, 2n-1]$. Given $V \in C_{\mathcal{J}}$, there exist coefficients $y_2, \ldots, y_{2n}$ such that V_{a+m} is spanned by the vector

$$v^{(a+m)} := \sum_{j=1}^{2n-m} y_j e^{(a+m)}_{m+j} ,$$

where $y_1 \neq 0$. Now let $m' \in [0, 2n-1]$ be such that $a + m \equiv_{2n} -(a + m')$. If $m + m' \geq 2n$, the symplectic form between $v^{(a+m)}$ and $v^{(a+m')}$ trivially vanishes. So, assuming $m' + 1 \leq 2n - m$, we compute

$$(v^{(a+m')}, v^{(a+m)}) = \sum_{j=m'+1}^{2n-m} (-1)^{j+1} y_{j-m'} \cdot y_{2n-m+1-j} \,.$$

The number of summands is even since m and m' have the same parity, so $m' + 1$ and $2n - m$ have different parity. Every summand appears twice with opposite signs, so that this sum vanishes too. Finally, we observe that the union of all maximal cells is dense in $X(1, 2n)$ and that $X(1, 2n)^{sp}$ is a closed subset of $X(1, 2n)$. □

Proposition 3.3.19. *If there exists a symplectic mutation* $\mathcal{J}'' \xrightarrow[sp]{\mu} \mathcal{J}$*, then* $p_{\mathcal{J}}$ *is in the closure of the* G_{2n}^{sp}*-orbit of* $p_{\mathcal{J}''}$ *in* $X(k, 2n)^{sp}$*.*

Proof. We define the path $V(t), t \in \mathbb{C}$, as follows: for $a \in \mathbb{Z}_{2n}$ we set

$$V(t)_a := \operatorname{Span}\Big(\{e_j^{(a)} \mid j \in J_a \cap J''_a\} \cup \{e_{j-m}^{(a)} + t \cdot e_j^{(a)} \mid j \in J_a \backslash J''_a\}\Big) \,,$$

where m is the breadth of μ. It follows from the explicit description of the mutations that $V(t)$ is a point in the cell $C_{\mathcal{J}''}$ of $X(k, 2n)$ for all $t \in \mathbb{C}$, since it is spanned by vectors of the form given in (2.1.2). We show that $V(t)$ is contained in $X(k, 2n)^{sp}$. For any $j \in J_a \cap J''_a$, the vector $e_j^{(a)}$ pairs trivially with any other element in $V(t)_{-a}$, so we can take $j \in J_a \setminus J''_a$ and $y \in J_{-a} \setminus J''_{-a}$. Then

$$(t \cdot e_j^{(a)} + e_{j-m}^{(a)}, t \cdot e_y^{(-a)} + e_{y-m}^{(-a)}) = (t \cdot e_j^{(a)}, e_{y-m}^{(-a)}) + (e_{j-m}^{(a)}, t \cdot e_y^{(-a)}) = 0 \,.$$

The expression is zero because the two summands are opposites of each other, since $j - m$ and $y - m$ are respectively in $J''_a \setminus J_a$ and in $J''_{-a} \setminus J_{-a}$. Finally, we observe that $V(0) = p_{\mathcal{J}''}$ and the boundary point is $p_{\mathcal{J}}$. □

We saw in Section 2.3 that sequences of mutations define the partial order $\leq$ on $JP(k, 2n)$ given by closure inclusion of the affine cells, and by Proposition 2.1.14 it is also expressed combinatorially. The subset $JP(k, 2n)^{sp}$ naturally inherits this combinatorial order, and it is logical to compare it to the geometric order given by closure inclusion of G_{2n}^{sp}-orbits in $X(k, 2n)^{sp}$, as well as with the partial order given by sequences of symplectic mutations.

Definition 3.3.20. Given two symplectic juggling patterns $\mathcal{J}$ and $\mathcal{J}''$, we will write $\mathcal{J} \leq_{sp}^{m} \mathcal{J}'$ if there exists a sequence of symplectic mutations from $\mathcal{J}'$ to $\mathcal{J}$, and we will write $\mathcal{J} \leq_{sp} \mathcal{J}''$ if $p_{\mathcal{J}} \in \overline{C_{\mathcal{J}''}^{sp}}$.

The combinatorial order $\leq$ on $JP(k, 2n)^{sp}$ is at least finer than the geometric order $\leq_{sp}$, since

$$p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}^{sp}} \implies p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}} \,,$$

and the latter is, in turn, finer than $\leq_{sp}^{m}$, by Proposition 3.3.19.

Example 3.3.21. The set $JP(2,4)^{sp}$ consists of the following 13 elements

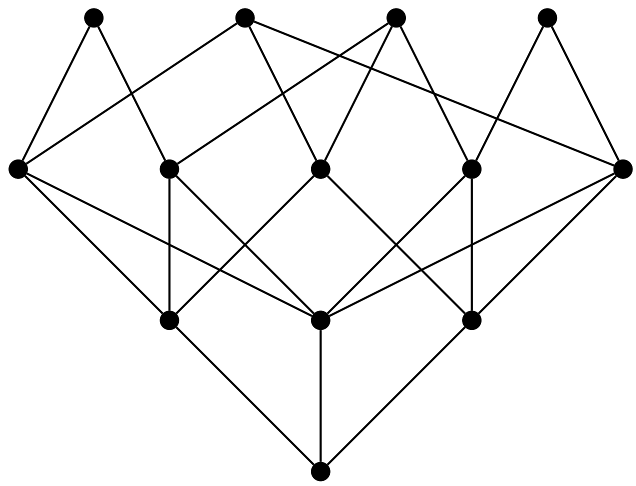
$$\left|\begin{array}{ccc} & 12 & \\ 14 & & 23 \\ & 34 & \end{array}\right| \left|\begin{array}{ccc} & 24 & \\ 13 & & 13 \\ & 24 & \end{array}\right| \left|\begin{array}{ccc} & 13 & \\ 24 & & 24 \\ & 13 & \end{array}\right| \left|\begin{array}{ccc} & 34 & \\ 23 & & 14 \\ & 12 & \end{array}\right|$$

$$\left|\begin{array}{ccc} & 24 & \\ 14 & & 23 \\ & 34 & \end{array}\right| \left|\begin{array}{ccc} & 13 & \\ 24 & & 24 \\ & 34 & \end{array}\right| \left|\begin{array}{ccc} & 24 & \\ 34 & & 34 \\ & 24 & \end{array}\right| \left|\begin{array}{ccc} & 34 & \\ 24 & & 24 \\ & 13 & \end{array}\right| \left|\begin{array}{ccc} & 34 & \\ 23 & & 14 \\ & 24 & \end{array}\right|$$

$$\left|\begin{array}{ccc} & 24 & \\ 34 & & 34 \\ & 34 & \end{array}\right| \left|\begin{array}{ccc} & 34 & \\ 24 & & 24 \\ & 34 & \end{array}\right| \left|\begin{array}{ccc} & 34 & \\ 34 & & 34 \\ & 24 & \end{array}\right|$$

$$\left|\begin{array}{ccc} & 34 & \\ 34 & & 34 \\ & 34 & \end{array}\right|$$

grouped into horizontal tiers by the dimension of their G^{sp}-orbit, from 3 to 0, top to bottom. The Hasse diagram of $JP(2,4)^{sp}$ is the following



In this case, the three orders coincide: the closure of a cell $C_{\mathcal{J}}^{sp}$ coincides with the union of symplectic cells for lower symplectic juggling patterns, and there exist symplectic mutations that connect $\mathcal{J}$ with any of them.

In [FLMP25], we conjectured the following:

Conjecture 3.1. Let $\mathcal{J}, \mathcal{J}' \in JP(k,2n)^{sp}$ be such that $p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}} \subset X(k,2n)$. Then $p_{\mathcal{J}} \in \overline{C_{\mathcal{J}'}^{sp}} \subset X(k,2n)^{sp}$.

We will see in the next chapter that the conjecture is always true for $JP(n,2n)^{sp}$. Moreover, in that case the three orders coincide. That is, we will show that if $\mathcal{J}'$ and $\mathcal{J} \in JP(k,2n)^{sp}$ are connected by a sequence of mutations, we can find such a sequence that consists only of symplectic mutations.

3.3.3 Maximal juggling patterns

The next results tell us information about the symplectic juggling patterns that are maximal with respect to the combinatorial order.

Proposition 3.3.22. *A maximal $(k, 2n)$-juggling pattern $\mathcal{J}$ is symplectic if and only if there exists a vertex i such that $J_i \subseteq R(J_{-i})$. There are $2^k \cdot \binom{n}{k}$ of them.*

Proof. One implication is trivial by Definition 3.3.4. With a slight notation abuse, for this proof we are blurring the line between integers and their residue classes. Let $j \in J_a$ for some vertex a. We want to prove that $\widetilde{j} \notin J_{-a}$. Suppose $a = i + p$ for some p. Then

$$j - p \in J_i \implies 2n - j + p + 1 \notin J_{-i} \implies 2n - j + 1 \notin J_{-a}\,.$$

Each maximal $(k, 2n)$-juggling pattern that is also symplectic is labeled by a subset J of $[2n]$ of cardinality k that does not contain two numbers of the form j and $\widetilde{j}$. There are n such pairs, only k of which have an element in J, and there are two elements per pair. Therefore there exist $2^k \cdot \binom{n}{k} = \frac{(2n)!!}{k!(2n-2k)!!}$ such juggling patterns. □

Proposition 3.3.23. *The dimension of a symplectic cell in $X(k, 2n)^{sp}$ corresponding to a juggling pattern that is maximal in $JP(k, 2n)$ is $k(2n - k) - \frac{k(k-1)}{2}$.*

Proof. Fix $\mathcal{J} \in JP(k, 2n)^{sp}$ maximal in $JP(k, 2n)$. In order to apply Lemma 3.3.17, we recall from Lemma 2.2.7 that there exist $2n - k$ mutations that can start at each of the k-many segments in $S_{\mathcal{J}}$. But $k - 1$ of them for each segment make $\mathcal{J}$ non-symplectic and must be paired with their corrections: if i is the starting vertex of a segment, we can lower $e_1^{(i)}$ to any position not paired to an element of J_{-i} that is not in the same segment. In total there are $k(2n - k) - k(k - 1)$ single mutations starting at $\mathcal{J}$ which are symplectic, and $\frac{k(k-1)}{2}$ pairs. The two numbers sum up to $k(2n - k) - \frac{k(k-1)}{2}$. □

3.4 Additional comments

3.4.1 Grassmann necklaces setting

In Chapter 2 it was mentioned that the juggling variety $X(k, N)$ was first studied with a different representation of the equioriented cycle, where juggling patterns were replaced by Grassmann necklaces, and the linear map s was replaced by the projection p_i, with $i \in [N]$ defined by

$$p_i(e_j) = (1 - \delta_{ij})e_j\,,$$

on the arrow $i \longrightarrow i + 1$. It is no different for the symplectic subvarieties $X(k, 2n)^{sp}$. Observe that the matrices for the projections in the standard basis satisfy

$$-\Omega p_i \Omega = p_{\widetilde{i}} = p_{\widetilde{i}}^t\,.$$

Therefore one can equip the same symplectic form on the vector spaces of this other representation and define, analogously to the main setting,

$$R(\mathcal{I}) \coloneqq (R(I_{-i}))_i \ ,$$

together with *symplectic Grassmann Necklaces* and a symplectic subvariety

$$X'(k,2n)^{sp} \coloneqq \{(V_i)_i \mid p_i(V_i) \subseteq V_{i+1} \ \wedge \ V_i \subseteq V_{-i}^{\perp}\} \, .$$

The isomorphism $\varphi\colon X(k,2n) \longrightarrow X'(k,2n)$ restricts to an isomorphism $X(k,2n)^{sp} \longrightarrow X'(k,2n)^{sp}$: considering two vectors

$$v_i = \sum_{j=1}^{2n} x_j e_j^{(i)} \in U_{[2n]}^{(i)} \quad \text{and} \quad v_{-i} = \sum_{j=1}^{2n} y_{\tilde{j}} e_{\tilde{j}}^{(-i)} \in U_{[2n]}^{(-i)} \, ,$$

their images, under φ_i and φ_{-i} respectively, are

$$\sum_{j=1}^{2n} x_{-j+i} e_{-j+i}^{(i)} \quad \text{and} \quad v_{-i} = \sum_{j=1}^{2n} y_{j-i+1} e_{j-i+1}^{(-i)}$$

where the subscripts are considered to be their representatives in $[2n]$ modulo $2n$. In conclusion we find

$$(v_i, v_{-i}) = \sum_{j=1}^{2n} (-1)^{j+1} x_j y_{\tilde{j}} = (\varphi_i(v_i), \varphi_{-i}(v_{-i})) \ .$$

3.4.2 The orthogonal case

When equipping the symmetry τ on the equioriented cycle, we could very well have chosen ς to be 1 instead of -1, and therefore studied orthogonal points of the juggling variety. We conclude the chapter by showing that the orthogonal case is not as well-behaved as the symplectic case, neither in odd nor even dimension, and therefore we refrain from delving more deeply into it.

We pick a non-degenerate symmetric bilinear form compatible with s, the same way we did with Ω. Our form of choice is

$$\left(\sum_{j=1}^{N} v_j e_j, \sum_{j=1}^{N} w_j e_j \right) = \sum_{j=1}^{N} v_j w_{N-j+1}$$

so that it has

$$O \coloneqq \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

as its Gram matrix in the standard basis. It satisfies

$$O \circ s \circ O = s^t$$

and the quiver symmetry $i \longmapsto -i$ is obviously defined even with N odd, in which case it would have one fixed vertex, 0, and one fixed arrow, $\frac{N+1}{2} \longrightarrow \frac{N+1}{2}+1$. The combinatorics are similar to that of the symplectic case, and can be extended to the odd-dimensional case with no issue: we define

$$\begin{aligned} \tau_o \colon X(k,N) &\longrightarrow X(N-k,N) \\ (V_i)_i &\longmapsto (V_{-i}^{\perp}) \end{aligned}$$

where $V^{\perp}$ is the subspace orthogonal to V with respect to the symmetric form, as well as

$$X(k,N)^o \coloneqq \{V \mid V_i \subseteq V_{-i}^{\perp} \ \forall\, i \in \mathbb{Z}_N\}\,.$$

The subspace orthogonal to a coordinate subspace is also coordinate, and the recipe is identical to the one we are familiar with by now. We can therefore define

$$\begin{aligned} R\colon JP(k,N) &\longrightarrow \quad JP(N-k,N) \\ \mathcal{J} = (J_i)_i &\longmapsto ([N] \setminus \{N_j + 1 \mid j \in J_{-i}\})_i \end{aligned}$$

with any N, as well as *orthogonal* juggling patterns. These, as expected, are those that satisfy $\mathcal{J} \subseteq R(\mathcal{J})$, and they coincide with symplectic juggling patterns when N is even. Unfortunately, the orthogonal case is not particularly interesting, since Conjecture 3.1 fails even in the smallest cases and the variety is disconnected, so there is no hope that the affine cells are, as a poset, isomorphic to a lower order ideal of a Coxeter group.

We present the $(1,3)$ case and, most importantly, the $(2,4)$ case, which in the symplectic setting has instead been our running example.

Example 3.4.1. There are 5 orthogonal $(1,3)$-juggling patterns out of the 7 total, namely

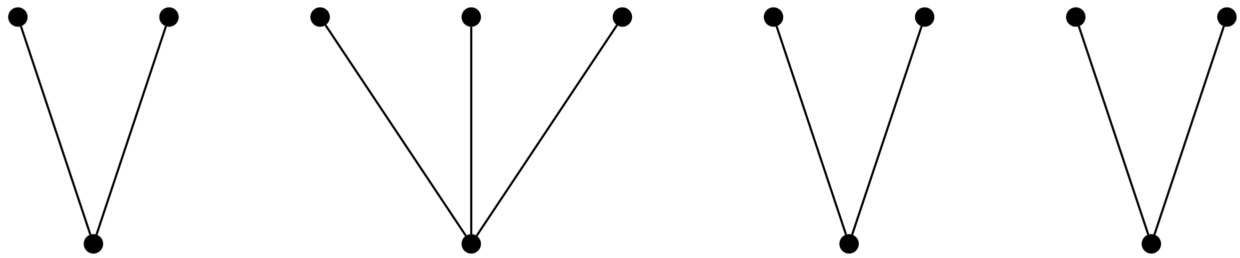
$${}_{3}{}^{1}{}_{2} \qquad {}_{3}{}^{3}{}_{2} \qquad {}_{2}{}^{3}{}_{3} \qquad {}_{2}{}^{3}{}_{1} \qquad {}_{3}{}^{3}{}_{3}\,.$$

The intersections $C_{\mathcal{J}} \cap X(1,3)^o$ have all dimension 1 except for the last one, the minimal juggling pattern with respect to the combinatorial order. The problem is that the closure of the cell for $\mathcal{J}' = {}_{2}{}^{3}{}_{1}$ contains ${}_{2}{}^{3}{}_{3}$ and not ${}_{3}{}^{3}{}_{3}$, which previously was in the closure of the full cell $C_{\mathcal{J}'}$.

Example 3.4.2. In the $(2,4)$ case the conjecture fails even more spectacularly, since we now find four juggling patterns such that $C_{\mathcal{J}} \cap X(2,4)^o$ is just a point:

$$\begin{smallmatrix} & 24 & \\ 34 & & 34 \\ & 34 & \end{smallmatrix} \qquad \begin{smallmatrix} & 34 & \\ 34 & & 34 \\ & 34 & \end{smallmatrix} \qquad \begin{smallmatrix} & 24 & \\ 34 & & 34 \\ & 24 & \end{smallmatrix} \qquad \begin{smallmatrix} & 34 & \\ 34 & & 34 \\ & 24 & \end{smallmatrix}\,.$$

Furthermore, $C_{\mathcal{J}} \cap X(2,4)^o$ for the remaining 9 juggling patterns has dimension 1, and the subvariety is very disconnected. Therefore the Hasse diagram with respect to the closure-inclusion order looks like this



in stark contrast to Example 3.3.21.

Chapter 4

The Lagrangian case

This chapter focuses on the study of $X(n,2n)^{sp}$, a degeneration of the Lagrangian Grassmannian $\mathrm{Gr}(n,2n)^{sp}$ as illustrated in Sections 2.1 and 3.2. The case where $k=n$ is particularly interesting, since now the symplectic subvariety is the fixed-point set under τ, from Proposition 3.2.3. This added piece of structure results in simpler proofs for some of the results that were shown in the previous chapter, as well as proofs for new statements that would not hold for generic k. In particular, we are able to prove a result analogous to Lemma 3.3.14 which allows us to produce a symplectic juggling pattern higher in the partial order starting from a given non-maximal one. Results regarding the irreducible components and the dimension of $X(n,2n)^{sp}$ will follow. Later we will develop some combinatorics, analogously to what was done in Chapter 2, and will answer the Conjecture 3.1.

We begin this chapter with a less computational proof of both Proposition 3.3.1 and Proposition 3.3.8, applicable in the Lagrangian case. Recall that these two results together combine into the following.

Proposition 4.0.1. *The equality*

$$C_{\mathcal{J}}^{sp} = C_{\mathcal{J}} \cap X(n,2n)^{sp}$$

holds for any $\mathcal{J} \in JP(n,2n)$.

Proof. First we show that the right side is non-empty only if $\mathcal{J}$ is symplectic. Let $g \cdot p_{\mathcal{J}} \in X(n,2n)^{sp}$ for some $g \in G_{2n}$. Then we have

$$g \cdot p_{\mathcal{J}} = \tau(g \cdot p_{\mathcal{J}}) = \sigma(g) \cdot (\tau p_{\mathcal{J}}) = \sigma(g) \cdot (p_{R\mathcal{J}})\,.$$

The two G_{2n}-orbits $C_{\mathcal{J}}$ and $C_{R\mathcal{J}}$ intersect, therefore the corresponding juggling patterns must coincide. Suppose therefore that $\mathcal{J} \in JP(n,2n)^{sp}$. By [MWZ98, §2.1], to show

$$G_{2n}^{sp} \cdot p_{\mathcal{J}} = (G_{2n} \cdot p_{\mathcal{J}}) \cap X(n,2n)^{sp}\,,$$

we only need to prove that

1. G_{2n} is a subgroup of the group of invertible elements E^* of a finite-dimensional associative $\mathbb{C}$-algebra E;

2. the anti-involution of G_{2n} given by $g \longmapsto \sigma(g^{-1})$ extends to a $\mathbb{C}$–linear anti-involution on E;
3. for every τ-fixed point $V \in X(n, 2n)^{sp}$, its stabilizer $H = \mathrm{Stab}_{G_{2n}}(V)$ is the group of invertible elements of its linear span in E.

The right choice for E is the vector space underlying $\mathfrak{g}_{2n} = \mathrm{End}(U_{[2n]})$, with associative product given by matrix multiplication vertex by vertex, since $G_{2n} = \mathrm{Aut}(U_{[2n]}) = E^*$. Next, the anti-involution

$$g = (g_i)_i \longmapsto \sigma(g^{-1}) = (-\Omega \circ g^t_{-i} \circ \Omega)_i$$

is well defined for non-invertible elements of E too. Lastly, we want to show how any invertible element of $\mathrm{Span}_{\mathbb{C}}(H)$ is in H. Let $a^1, \dots, a^m \in \mathbb{C}^*$, $h^1, \dots, h^m \in H$, and $V = (V_i)_i \in X(n, 2n)^{sp}$. Each h^j consists of $2n$ matrices h^j_i. Then let

$$A = \sum_{j=1}^{m} a^j h^j \,.$$

Its component A_i on vertex i is

$$\sum_{j=1}^{m} a^j h^j_i$$

By assumption, A is invertible in E, so it is an element of G_{2n}. Then

$$A_i \cdot V_i = \sum_{j=1}^{m} a^j h^j_i V_i = V_i$$

since $h^j \in H$ and the V_i are vector spaces, so $A \in H$. □

4.1 Irreducible components

One advantage of the Lagrangian case is the following: since the inclusions $V \subseteq V^\perp$ are now equalities for dimensional reasons, a point in $X(n, 2n)^{sp}$ is determined only by the spaces on the vertices in $Q_0^\tau \cup Q_0^+$, or equivalently in $Q_0^\tau \cup Q_0^-$. In particular, given a set J_i for a symplectic juggling pattern $\mathcal{J}$, we not only know that

$$j \in J_i \implies \widetilde{j} \notin J_{-i} \,,$$

but also that

$$j \notin J_i \implies \widetilde{j} \in J_{-i} \,.$$

A consequence of this is a crucial result that allows us to describe the irreducible components of $X(n, 2n)^{sp}$.

Lemma 4.1.1. *For any $\mathcal{J} \in JP(n, 2n)^{sp}$ and any mutation $\mathcal{J}' \xrightarrow{\mu} \mathcal{J}$, with $\mathcal{J}'$ is not symplectic, there exists $\mathcal{J}'' \in JP(n, 2n)^{sp}$ and a mutation $\mathcal{J}'' \xrightarrow{\mu'} \mathcal{J}'$ such that $\mu = R\mu'$.*

Proof. Let

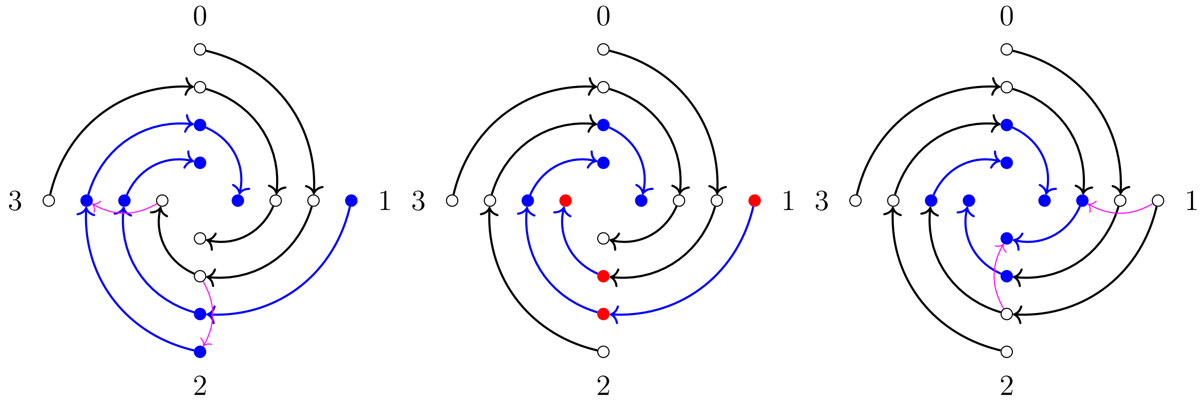
$$x \in J'_a \,, \quad x + 1 \in J'_{a+1} \,, \quad \cdots \,, \quad x + l \in J'_{a+l}$$

be all and only the elements that μ replaces, and let

$$x + m \in J_a\,, \quad x + m + 1 \in J_{a+1}\,, \quad \cdots, \quad x + m + l \in J_{a+l}$$

be the elements of $\mathcal{J}$ introduced by μ. Then J'_{-a} contains $\widetilde{x}$ since $x \in J'_a$ is the tail end of a segment of $S_{\mathcal{J}'}$ and μ is the only difference from $\mathcal{J}'$ and the symplectic $\mathcal{J}$. Then we claim that all and only the elements of $\mathcal{J}'$ removed my μ were paired to some other element of $\mathcal{J}'$, that is, $\widetilde{x} - l \in J'_{-a-l}$ and $\widetilde{x} - l - 1 \notin J'_{-a-l-1}$. Indeed, by Definition 2.2.5 we have $x + l \notin J_{a+l}$ and $x + l + 1 \in J_{a+l+1}$, which imply $\widetilde{x} - l \in J_{-a-l}$ and $\widetilde{x} - l - 1 \notin J_{-a-l-1}$. By Remarks 3.3.13 and 3.3.7, we deduce that $\widetilde{x} - l \in J'_{-a-l}$ and $\widetilde{x} - l - 1 \notin J'_{-a-l-1}$ respectively. So there is a segment of $S_{\mathcal{J}'}$ starting with $\widetilde{x} - l \in J'_{-a-l}$. Furthermore, $S_{\mathcal{J}'}$ does not contain $\widetilde{x} - m$ on vertex $-a$ nor its predecessors in its segment, because of Remark 3.3.13. But it does contain $\widetilde{x} - m + 1$ on vertex $-a + 1$, since $S_{\mathcal{J}}$ does because $\mathcal{J}$ is symplectic, and by Remark 3.3.7. Therefore we can cut from $S_{\mathcal{J}'}$ the tail end of the segment consisting of $\widetilde{x} \in J'_{-a}$ and it predecessors, and instead glue $\widetilde{x} - m$ on vertex $-a$ together with its predecessors, since $\widetilde{x} - m + 1$ is already in $S_{\mathcal{J}'}$. In other words, we have produced a juggling pattern $\mathcal{J}''$, symplectic since we removed the elements that made $\mathcal{J}'$ non-symplectic and added numbers not paired to any other, together with a mutation $\xrightarrow{\mu'} \mathcal{J}'$ which adds the appropriate elements, i.e. such that $R\mu' = \mu$. □

Example 4.1.2. Essentially, whenever we lower a segment to remove some "obstacles" to $\mathcal{J}'$ being symplectic, we can instead raise the paired obstacles. For $k = n = 2$, consider $\mathcal{J}' = \begin{smallmatrix} & 34 & \\ 34 & & 14 \\ & 23 & \end{smallmatrix}$ and $\mathcal{J} = \begin{smallmatrix} & 34 & \\ 34 & & 34 \\ & 34 & \end{smallmatrix}$. The following diagrams represent $S_{\mathcal{J}''}$, $S_{\mathcal{J}'}$ and $S_{\mathcal{J}}$ respectively. The red vertices are the obstacles in $\mathcal{J}'$, and the pink arrows represent the cutting/glueing operation we perform on $S_{\mathcal{J}'}$.

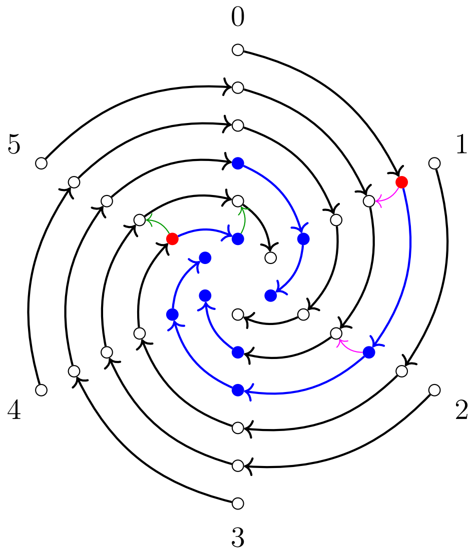


However, Lemma 4.1.1 does not work for $1 < k < n$, as shown in the next counterexample.

Example 4.1.3. The smallest case we can consider is $n = 3$ and $k = 2$. We choose

$$\mathcal{J}' = \begin{array}{ccc} & 46 & \\ 56 & & 25 \\ & & \\ 56 & & 36 \\ & 45 & \end{array} \quad \text{and} \quad \mathcal{J} = \begin{array}{ccc} & 46 & \\ 56 & & 35 \\ & & \\ 56 & & 46 \\ & 45 & \end{array}$$

and depict $S_{\mathcal{J}'}$ in the following diagram.



The first one is not symplectic because $2 \in J_1'$ and $5 \in J_5'$ (vertices in red), the former of which is replaced by 3 in J_1 by μ (shown in pink). To produce a $\mathcal{J}''$ from $\mathcal{J}'$ following the recipe from Lemma 4.1.1, we would need to replace $5 \in J_5'$ with a 4, and therefore the $6 \in J_0'$ with a 5 (shown in green). But J_1' does not contain 6, so the result is not a juggling pattern.

By applying Propositions 3.3.22 and 3.3.23 to the Lagrangian case, we deduce the next result.

Corollary. *The symplectic $(n, 2n)$-juggling patterns that are maximal with respect to the G_{2n}^{sp}-orbit closure-inclusion order are all and only those that are maximal in $JP(n, 2n)$. The irreducible components of $X(n, 2n)$ are the closures of the corresponding symplectic cells, are 2^n in number, and have all dimension equal to*

$$\frac{n(n+1)}{2},$$

the dimension of the Lagrangian Grassmannian $\Lambda(2n)$.

Example 4.1.4. With this formula we find that $X(2,4)^{sp}$ has dimension 3, in line with Example 3.3.21.

4.2 Symplectic bounded affine permutations

In this section we transfer the symmetry R from the set of juggling patterns to the set of bounded affine permutations. We draw the following diagram of poset isomorphisms:

$$\begin{array}{ccc} JP(k,2n) & \longleftrightarrow & \mathcal{B}_{(k,2n)} \\ \updownarrow R & & \\ JP(2n-k,2n) & \longleftrightarrow & \mathcal{B}_{(2n-k,2n)} \end{array}$$

which we want to complete to a square. We will give an explicit map $\mathcal{B}_{(k,2n)} \longrightarrow \mathcal{B}_{(2n-k,2n)}$, once again called R, and then prove that the notation is justified, i.e. that it makes the square commutative. This would imply that it too is order-preserving. Taking a page from [Kar18, §3], we define Rf in the following way:

$$Rf(i) \coloneqq 2n - f(-i-1) - 1 = 4n - f(2n-i-1) - 1 \,. \tag{4.2.1}$$

Proposition 4.2.1. *The image of a $(k, 2n)$-bounded affine permutation under R is a $(2n-k, 2n)$-bounded affine permutation.*

Proof. Recall Definitions 2.3.1 and 2.3.4. Given $f \in \mathcal{B}_{(k,2n)}$, let us first show that $Rf \in A_{2n}^{2n-k}$:

$$Rf(i+2n) = 2n - f(-i-2n-1) - 1 = 4n - f(-i-1) - 1 = Rf(i) + 2n;$$

$$\sum_{i=1}^{2n}(Rf(i)-i) = \sum_{i=1}^{2n}(2n - f(-i-1) - i - 1) = 4n^2 - \sum_{i=1}^{2n}(f(-i-1)+i+1) =$$
$$= 4n^2 - \sum_{j=-2n-1}^{-2}(f(j)-j) = 4n^2 - 2kn = 2n(2n-k)\ .$$

Now we prove that Rf is bounded: for all $i \in \mathbb{Z}$, we have

$$-i-1 \leq f(-i-1) \leq 2n-i-1$$

since f is bounded. We change signs and add $2n-1$ to obtain

$$i \leq Rf(i) \leq i+2n \quad \forall\, i \in \mathbb{Z}$$

which concludes the proof. □

Proposition 4.2.2. *For any $(k, 2n)$-juggling pattern $\mathcal{J}$, the equality $Rf_{\mathcal{J}} = f_{R\mathcal{J}}$ holds.*

Proof. Suppose first that $2n \in J_a$ for some vertex a, and that $J_{a+1} = \mathfrak{s}(J_a \setminus \{2n\}) \cup \{b\}$ for some $b > 1$. Then $2n \in (R\mathcal{J})_{-a-1}$ and $(R\mathcal{J})_{-a} = \mathfrak{s}((R\mathcal{J})_{-a-1}\setminus\{2n\}) \cup \{\widetilde{b}+1\}$ with $\widetilde{b}+1 > 1$. This holds because this set has cardinality $2n-k$, and none of its elements are paired to an element of J_a. We are going to compute the images of the elements in $[2n]$ under $f_{R\mathcal{J}}$ and show that they coincide with the images of the same numbers under $Rf_{\mathcal{J}}$. The reader is reminded that the bounded affine permutation corresponding to a juggling pattern is defined in (2.3.2). The proof consists of three chains of equivalent statements, which all make use of

$$1 \in J_{a+1} \iff 2n \in J_a \wedge J_{a+1} = \mathfrak{s}(J_a \setminus \{2n\}) \cup \{1\}$$

or its contrapositive.

I

$$f_{R\mathcal{J}}(a) = a \iff 2n \notin (R\mathcal{J})_a \iff 1 \in J_{-a} \iff$$
$$\iff 2n \in J_{-a-1} \wedge J_{-a} = \mathfrak{s}(J_{-a-1} \setminus \{2n\}) \cup \{1\} \iff$$
$$\iff f_{\mathcal{J}}(-a-1) = -a-1+2n \iff Rf_{\mathcal{J}}(a) = a\,.$$

II

$$\begin{gathered} f_{R\mathcal{J}}(a) = a + 2n \iff 2n \in (R\mathcal{J})_a \wedge (RJ)_{a+1} = \mathfrak{s}\big((R\mathcal{J})_a \setminus \{2n\}\big) \cup \{1\} \iff \\ \iff 1 \in (R\mathcal{J})_{a+1} \iff 2n \notin J_{-a-1} \iff \\ \iff f_{\mathcal{J}}(-a-1) = -a-1 \iff Rf_{\mathcal{J}}(a) = a + 2n\,. \end{gathered}$$

III

$$\begin{gathered} f_{R\mathcal{J}}(a) = 2n + a + 1 - b,\ b \in [2, 2n] \iff \\ \iff 2n \in (R\mathcal{J})_a \wedge (R\mathcal{J})_{a+1} = \mathfrak{s}\big((R\mathcal{J})_a \setminus \{2n\}\big) \cup \{b\},\ b \in [2, 2n] \iff \\ \iff 2n \in J_{-a-1} \wedge J_{-a} = \mathfrak{s}(J_{-a-1} \setminus \{2n\}) \cup \{\widetilde{b} + 1\},\ b \in [2, 2n] \iff \\ \iff f_{\mathcal{J}}(-a-1) = 2n + (-a-1) + 1 - (\widetilde{b} + 1),\ \widetilde{b} + 1 \in [2, 2n] \iff \\ \iff Rf_{\mathcal{J}}(a) = 2n + a + 1 - b\,. \end{gathered}$$

□

Example 4.2.3. In Example 2.3.10, we saw the bounded affine permutations

$$\begin{aligned} f_{\mathcal{J}} &: 0 \longmapsto 1, \quad 1 \longmapsto 3, \quad 2 \longmapsto 4, \quad 3 \longmapsto 6; \\ f_{\mathcal{J}'} &: 0 \longmapsto 3, \quad 1 \longmapsto 1, \quad 2 \longmapsto 4, \quad 3 \longmapsto 6, \end{aligned}$$

corresponding to $\mathcal{J} = 34\genfrac{}{}{0pt}{}{24}{34}34$ and $\mathcal{J}' = 34\genfrac{}{}{0pt}{}{24}{34}23$. We see that $R\mathcal{J} = \mathcal{J}$ while $R\mathcal{J}' = 14\genfrac{}{}{0pt}{}{24}{34}34 \neq \mathcal{J}'$, and we also see that $Rf_{\mathcal{J}} = f_{\mathcal{J}}$. Lastly, we can compute $f_{R\mathcal{J}'} = Rf_{\mathcal{J}'}$ to be

$$0 \longmapsto 1, \quad 1 \longmapsto 3, \quad 2 \longmapsto 6, \quad 3 \longmapsto 4.$$

Remark 4.2.4. The map $R\colon \mathcal{B}_{(k,2n)} \longrightarrow \mathcal{B}_{(2n-k,2n)}$ was introduced in this chapter because it is not clear what condition, for two bounded affine permutations $f_{\mathcal{J}}$ and $f_{\mathcal{J}'}$, would be equivalent to $\mathcal{J} \subset \mathcal{J}'$. Therefore, for the next definition, we require $k = n$ so that the inclusion is an equality and its equivalent condition on permutations is also an equality. In addition, many of the interesting results that we use to prove Conjecture 3.1 work only when $k = n$.

Definition 4.2.5. A bounded affine permutation $f \in \mathcal{B}_{(n,2n)}$ is called *symplectic* if $Rf = f$, or equivalently if it satisfies

$$f(2n-i-1) - (2n-i-1) + f(i) - i = 2n \quad \forall\, i \in [0, 2n-1] \tag{4.2.2}$$

for all $i \in \mathbb{Z}$. Their set will be denoted by $\mathcal{B}^{sp}_{(n,2n)}$.

Observe that an immediate consequence of this definition is that a juggling pattern $\mathcal{J} \in JP(n, 2n)$ is symplectic if and only if $f_{\mathcal{J}}$ is symplectic. Combining Propositions 3.3.10 and 4.2.2 one obtains:

Corollary 4.2.2.1. *The map $R\colon \mathcal{B}_{(k,2n)} \longrightarrow \mathcal{B}_{(2n-k,2n)}$ is order-preserving.*

Our aim now is describing the dimension of the symplectic cells in $X(n,2n)^{sp}$ in terms of bounded affine permutations, which was not done in the previous chapter for $k<n$. We rephrase Lemma 3.3.17 for the Lagrangian case.

Lemma 4.2.6. *If $\mathcal{J} \in JP(n,2n)$ is symplectic, then the affine coordinates for $C^{sp}_{\mathcal{J}}$ are parametrized by equivalence classes of descending triples*

$$(a,i,j) \text{ with } a \in \mathbb{Z}_{2n},\ j \in J_a,\ j < i \notin J_a$$

for $\mathcal{J}$, under the equivalence relation $\overset{sp}{\sim}$ given by

$$(a,i,j) \overset{sp}{\sim} (a+1,i+1,j+1) \quad \wedge \quad (a,i,j) \overset{sp}{\sim} (-a,\widetilde{j},\widetilde{i})\,.$$

Proof. Suppose for now that $\mathcal{J}$ is any $(n,2n)$-juggling pattern, and consider a point $V=(V_a)_a \in C_{\mathcal{J}}$ with affine coordinates $\big(u^{(a)}_{i,j} \mid a \in \mathbb{Z}_{2n},\ j \in J_a,\ j<i \notin J_a\big)$. Then the affine coordinates

$$\Big((-1)^{x+y+1}\cdot u^{(-b)}_{\widetilde{y},\widetilde{x}} \mid b \in \mathbb{Z}_{2n}, y \in (R\mathcal{J})_b, y < x \notin (R\mathcal{J})_b\Big).$$

define a point $W \in C_{R\mathcal{J}} \subset X(n,2n)$, as per (2.1.2). Recall that τ maps $C_{\mathcal{J}}$ to $C_{R\mathcal{J}}$ and viceversa. We claim that $W=\tau V$, and plan to prove it by showing that the generators for V and for W pair trivially via Ω. Fix a vertex a and two elements $j \in J_a, i \in (R\mathcal{J})_{-a}$, and consider $v^{(a)}_j$, the j-th generator of V_a, and in $w^{(-a)}_i$, the i-th generator of W_{-a}. We evaluate the symplectic form at them and find

$$\begin{aligned}(v^{(a)}_j, w^{(-a)}_i)_\Omega &= \Big(e^{(a)}_j + \sum_{\substack{x>j\\ x\notin J_a}} u^{(a)}_{x,j} e^{(a)}_x \,,\, e^{(-a)}_i + \sum_{\substack{y>i\\ y\notin (R\mathcal{J})_{-a}}} (-1)^{i+y+1}\cdot u^{(a)}_{\widetilde{i},\widetilde{y}} e^{(-a)}_y \Big)_\Omega = \\ &= (e^{(a)}_j + r_j, e^{(-a)}_i + r_i)_\Omega\end{aligned}$$

where r_j is short for the sum $v^{(a)}_j - e^{(a)}_j$ and r_i is short for $w^{(-a)}_i - e^{(-a)}_i$. Simplifying accordingly, we get

$$(e^{(a)}_j, e^{(-a)}_i)_\Omega + (e^{(a)}_j, r_i)_\Omega + (r_j, e^{(-a)}_i)_\Omega + (r_j, r_i)_\Omega\,.$$

The first term vanishes because $j \in J_a$ and $i \in R(J_a)$; the second and third summands can be nonzero respectively only if $\widetilde{j}$ appears as one of the indices y and $\widetilde{i}$ appears a one of the indices x. Both happen if and only if $\widetilde{j} > i$, in which case they are respectively equal to $(-1)^{j+1}(-1)^{i+j}\cdot u^{(a)}_{\widetilde{i},j} = (-1)^{i+1}\cdot u^{(a)}_{\widetilde{i},j}$ and $(-1)^{\widetilde{i}+1}\cdot u^{(a)}_{\widetilde{i},j} = (-1)^{i}\cdot u^{(a)}_{\widetilde{i},j}$. So they are opposites in all cases, and cancel out. Lastly, the fourth term vanishes because the basis vectors $e^{(a)}_x$ with $x \notin J_a$ pair trivially with the vectors $e^{(-a)}_y$ for $y \notin R(J_a)$. We conclude by observing that if $\mathcal{J}$ is now symplectic, then $V=\tau V$ if and only if

$$u^{(a)}_{i,j} = (-1)^{i+j+1} u^{(-a)}_{\widetilde{j},\widetilde{i}}$$

for all a, i and j. □

One can see from the definition that $\sim$-equivalence classes are subsets of $\overset{sp}{\sim}$-classes.

Example 4.2.7. As stated in Example 2.2.8, the terminal triples for $\mathcal{J} = 23\,{}^{34}_{24}\,14$ are are $(1,3,1)$, (3, 4, 2) and (3, 4, 3). The first two are equivalent modulo $\overset{sp}{\sim}$, while the last one is in a separate equivalence class. Therefore we have additional confirmation that the symplectic cell $C^{sp}_{\mathcal{J}}$ has dimension equal to 2, as we already found in Example 3.3.2.

Lemma 4.2.8. *The equality $\ell(f) = \ell(Rf)$ holds for all $f \in \mathcal{B}_{(k,2n)}$.*

Proof. The reader will recall from (2.3.1) that $\ell(f)$ is the cardinality of

$$L(f) := \{(x, y) \in [0, N-1] \times \mathbb{Z} \mid x < y \wedge f(x) > f(y)\}\,.$$

Let us prove that Φ, defined by

$$\Phi(x, y) \longmapsto \begin{cases} (2n - y - 1, 2n - x - 1) & \text{if } y \in [0, 2n-1]\,, \\ (4n - y - 1, 4n - x - 1) & \text{if } y \in [2n, 4n-1]\,, \end{cases}$$

is a bijection between $L(f)$ and $L(Rf)$. To start with, notice that the first entry of the pair $\Phi(x, y)$ is always in $[0, 2n-1]$ and is strictly lower than the second entry. Given a pair $(x, y) \in L(f)$, by Definition 2.3.4 we have

$$0 \le x < y \le f(y) < f(x) \le x + 2n \le 4n - 1\,, \tag{4.2.3}$$

changing sign and adding $2n - 1$ to which produces

$$-2n \le -x - 1 \le 2n - f(x) - 1 < 2n - f(y) - 1 \le 2n - y - 1 < 2n - x - 1 \le 2n - 1\,.$$

Next we apply Rf to the pair $\Phi(x, y)$:

$$\begin{aligned} Rf(2n - y - 1) &= 4n - f(y) - 1 > 4n - f(x) - 1 = Rf(2n - x - 1) && \text{if } y \in [0, 2n-1]\,, \\ Rf(4n - y - 1) &= 2n - f(y) - 1 > 2n - f(x) - 1 = Rf(4n - x - 1) && \text{if } y \in [2n, 4n-1]\,. \end{aligned}$$

Observing that Φ is bijective since $\Phi \circ \Phi(x, y) = (x, y)$, we conclude that $\Phi(x, y) \in L(Rf)$ if and only if $(x, y) \in L(f)$. □

Even though this result would easily follow from Lemma 3.3.6, we proved it this way to introduce the map Φ.

The first part of the proof of Proposition 4.2.1 tells us that R is well defined even for non-bounded affine permutations, so when we extend it to A^k_{2n} it has image equal to A^{2n-k}_{2n}. To produce a map $A^0_{2n} \longrightarrow A^0_{2n}$ we precompose R with id_k and postcompose it with id_{k-2n}. What we find is a group automorphism of order 2, which we will call R^0. It is defined by

$$(R^0 g)(i) = -g(-i - 1) - 1\,.$$

It satisfies

$$R^0(i, j) = (-j - 1, -i - 1)\,,$$

so in particular

$$R^0(s_i) = s_{-i-2}\,.$$

The simple reflections fixed under R^0 are s_{-1} and s_{n-1}. Remember from §2.3.1 that A^0_{2n} is a Coxeter group of affine type A with simple reflections $s_0, \ldots, s_{2n}$. We will now see that the R^0-fixed subgroup is also an affine Coxeter group, this time of type C.

Definition 4.2.9. We define

$$C^0_{n+1} \coloneqq \{g \in A^0_{2n} \mid R^0 g = g\}$$

as well as

$$S_C \coloneqq \{r_{-1}, r_0, \ldots, r_{n-1}\},$$

where:

- $r_{-1} \coloneqq s_{-1}$;
- $r_i \coloneqq s_i \circ s_{-i-2}$ for $i \in [0, n-2]$;
- $r_{n-1} \coloneqq s_{n-1}$.

Proposition 4.2.10. *$\left(C^0_{n+1}, S_C\right)$ is a Coxeter group of affine type $C^{(1)}_n$.*

Proof. We first need to show that S_C is actually a set of generators for C^0_{n+1}. By induction on the length $\ell(g)$ of one of its elements g, we show that g has an expression in terms of the r_i. Here ℓ is the length of g as an element of the Coxeter group $\left(A^0_{2n}, S\right)$. Let us go over some base cases:

- if $\ell(g) = 0$, g must be the identity;
- if instead $\ell(g) = 1$, it must be a fixed simple reflection, so either $s_{-1} = r_{-1}$ or $s_{n-1} = r_{n-1}$;
- lastly, if $\ell(g) = 2$ then g must be of the form $s_i s_j$ for a pair of indices $i, j \in [0, 2n-1]$. By assumption, $g = R^0 g$, so it must also equal $s_{-i-2} s_{-j-2}$; thus either $s_i = s_{-i-2}$ and $s_j = s_{-j-2}$, or $s_j = s_{-i-2}$. They respectively imply $g = s_{-1} s_{n-1} = r_{-1} r_{n-1}$ and $g = r_i$.

For the inductive step, let $u \in A^0_{2n}$ be such that $\ell(u) = \ell(g) - 1$ and $g = u \circ s_i$ for some i. Then $R^0 u$ also has length $\ell(g) - 1$, and $g = R^0 u \circ s_{-i-2}$. Suppose first that i is either -1 or $n-1$; then $s_i = r_i$ and $u = R^0 u$. By inductive hypothesis, u can be written as a word in the alphabet S_C, so g can as well. Suppose otherwise that $s_{-i-2} \neq s_i$. We can apply the exchange condition to $u \circ s_i = R^0 u \circ s_{-i-2}$, and deduce that there exists a word v of length $\ell(g) - 2$ such that $g = v \circ s_i \circ s_{-i-2}$. We conclude by applying the inductive hypothesis to $v \in C^0_{n+1}$. Observe that any minimal length expression for g in A^0_{2n} must contain both reflections s_i and s_{-i-2} since two minimal length expressions are composed by the same simple reflections [BB05, Corollary 1.4.8].

What is left is to show that the elements of S_C satisfy the defining relations of a Coxeter group of affine type C. The order of $r_{-1} \circ r_0 = s_{-1} \circ s_0 \circ s_{-2} = (-1, 0)(0, 1)(-2, -1)$ is 4, as is the order of $r_{n-2} \circ r_{n-1} = s_{n-2} \circ s_n \circ s_{n-1}$, since they can be written respectively as the cycles

$$(0, 1, -1, -2) \quad \text{and} \quad (n, n-2, n-1, n+1),$$

which have length 4. For $i \in [0, n-3]$, the order of

$$r_i \circ r_{i+1} = s_i \circ s_{-i-2} \circ s_{i+1} \circ s_{-i-3} = (s_i \circ s_{i+1}) \circ (s_{-i-3} \circ s_{-i-2})$$

is 3, the least common multiple of the orders of the bracketed factors. For $|i-j| \geq 2$, s_i, s_{-i-2}, s_j and s_{-j-2} all commute with one another by the Coxeter relations among elements of S, so $r_i \circ r_j = r_j \circ r_i$. There is no other relation between the generators of C^0_{n+1} of S_C because there is no non-Coxeter relation between the generators of A^0_{2n} in S. □

Remark 4.2.11. At some point in this proof we assumed that $n \geq 2$. Not a very limiting constraint, since R^0 is the trivial automorphism when $n = 1$, and also $X(1,2)^{sp} = X(1,2)$.

Corollary 4.2.10.1. *Reflections in C^0_{n+1}, i.e. conjugates of elements of S_C, are either reflections of the form $(i, -i-1)$ or pairs of reflections of the form $(i,j)(-j-1,-i-1)$.*

4.3 Symplectic torus actions

The conclusion to the previous section might remind the reader of the definition of symplectic mutation, Definition 3.3.15: either a mutation that is already symplectic, or a specific pair of mutations. To illustrate the connection, we will first study symplectic mutations more in depth in the Lagrangian case. In this section we will define the action of an algebraic torus of dimension $n+1$ on $X(n,2n)^{sp}$, realized as a subtorus of T (see Section 2.2), whose one-dimensional orbits correspond to symplectic mutations in the same way that 1 dimensional T-orbits in $X(n,2n)$ correspond to "regular" mutations, as explained in §2.2.1. In order to do so, we will slightly alter the $\mathbb{C}^*$-action defined in (2.2.1) and the T-action defined in (2.2.2). The weight function defining the new $\mathbb{C}^*$-action is

$$\widehat{\mathbf{wt}}\colon b_{j,p} \longmapsto p - \widetilde{p} = -2n + 2p - 1 \ ,$$

so $z \in \mathbb{C}^*$ acts on the standard basis as

$$z \cdot b_{j,p} = z^{-2n+2p-1}\, b_{j,p} = z^{p-\widetilde{p}}\, b_{j,p} \ . \tag{4.3.1}$$

Recall that $p \in [2n]$ and $j \in [0, 2n-1]$. For instance, when $n = 2$ the previous weight of the basis elements $\{e^{(i)}_1, e^{(i)}_2, e^{(i)}_3, e^{(i)}_4\}$ was $1,2,3,4$, while the new one is $-3,-1,1,3$. This new grading is again attractive (as seen in Section 2.2), with an increase on each arrow equal to 2, and is such that the corresponding $\mathbb{C}^*$-action restricts to $X(n,2n)^{sp}$, since it preserves the symplectic form: consider $v = \alpha_1 e^{(i)}_1 + \ldots \alpha_{2n} e^{(i)}_{2n} \in U^{(i)}_{[2n]}$, $w = \beta_1 e^{(-i)}_1 + \ldots \beta_{2n} e^{(-i)}_{2n} \in U^{(-i)}_{[2n]}$ and $z \in \mathbb{C}^*$, and recall that $b_{j,p} = e^{(j+p)}_p$. We then compute

$$(z \cdot v, z \cdot w)_\Omega = \sum_{p=1}^{2n} (-1)^{p+1}\, z^{p-\widetilde{p}}\, \alpha_p\, z^{\widetilde{p}-p}\, \beta_{\widetilde{p}}\, (e^{(i)}_p, e^{(-i)}_{\widetilde{p}})_\Omega = \sum_{p=1}^{2n} (-1)^{p+1}\, \alpha_p\, \beta_{\widetilde{p}}\, (e^{(i)}_p, e^{(-i)}_{\widetilde{p}})_\Omega = (v,w)_\Omega \,.$$

Proposition 4.3.1. *The fixed points for the $\mathbb{C}^*$-action given by $\widehat{\mathbf{wt}}$ are the symplectic juggling pattern points, and its attracting loci are the symplectic cells, which are therefore stable under this action.*

Proof. On every copy $U^{(i)}_{[2n]}$ of $\mathbb{C}^{2n}$ on the vertices of Δ_{2n}, the action is diagonal with strictly increasing eigenvalues. By [CI11, Theorem 1], we know that its fixed points are given by coordinate subspaces. Notice that the differences between the two $\mathbb{C}^*$-actions (2.2.1) and (4.3.1) are the group morphism

$$z \longmapsto z^2$$

and scalar multiplication by z^{-2n-1}, which acts trivially on vector spaces. Therefore they have the same attracting sets. For any symplectic juggling pattern $\mathcal{J}$ we have

$$\begin{aligned}\{V \in X(n,2n)^\tau \mid z \cdot V \xrightarrow{z\to 0} p_{\mathcal{J}}\} = \{V \in X(n,2n) \mid z \cdot V \xrightarrow{z\to 0} p_{\mathcal{J}}\} \cap X(n,2n)^\tau = \\ = C_{\mathcal{J}} \cap X(n,2n)^\tau = C^{sp}_{\mathcal{J}}\,,\end{aligned}$$

since the second action preserves the symplectic subvariety. □

Analogously to the non-symplectic setting, this $\mathbb{C}^*$-action can be recovered from the action of $T' \coloneqq (\mathbb{C}^*)^{2n+1}$ on $X(n,2n)$ defined on the basis by

$$(z,\gamma_0,\ldots,\gamma_{2n-1}) \cdot b_{j,p} = z^{-2n+2p-1}\,\gamma_j\,b_{j,p} = z^{p-\widetilde{p}}\,\gamma_j\,b_{j,p}\,. \tag{4.3.2}$$

via the generic cocharacter

$$\begin{aligned}\rho\colon \mathbb{C}^* &\longrightarrow \quad T' \\ z &\longmapsto (z,1,\ldots,1)\,.\end{aligned}$$

Notice that the two tori T' and T are the same group, but are denoted differently to avoid confusion. Nonetheless, the two actions are similar enough to have the same fixed points and one-dimensional orbits, since they both have different eigenvalues on each $U^{(i)}_{[2n]}$. Also, the T'-action commutes with the linear map s modulo a factor equal to z^2: if we denote $(z,\gamma_0,\ldots,\gamma_{2n-1})$ by t, then we have

$$\begin{aligned}s(t \cdot b_{j,p}) = s(z^{-2n+2p-1}\gamma_j b_{j,p}) = z^{-2n+2p-1}\gamma_j b_{j,p+1} = \\ = z^{-2n+2(p+1)-1} z^{-2}\gamma_j b_{j,p+1} = z^{-2} t \cdot (s(b_{j,p}))\,.\end{aligned}$$

Notice that the action does not preserve $X(n,2n)^{sp}$, so we must restrict it to a smaller subtorus of T'.

Definition 4.3.2. We define T^{sp} to be the maximal subgroup of T' that preserves the symplectic form. That is,

$$T^{sp} \coloneqq \left\{ t \in T' \mid (t \cdot v, t \cdot w) = (v,w) \quad \forall\, i \in \mathbb{Z}_{2n}, v \in U^{(i)}_{[2n]}, w \in U^{(-i)}_{[2n]} \right\}\,.$$

The symplectic form evaluated at two vectors $v = \alpha_1 e^{(i)}_1 + \cdots + \alpha_{2n} e^{(i)}_{2n} \in U^{(i)}_{[2n]}$ and $w = \beta_1 e^{(-i)}_1 + \cdots + \beta_{2n} e^{(-i)}_{2n} \in U^{(-i)}_{[2n]}$ is

$$(v,w)_\Omega = \sum_{p=1}^{2n} (-1)^{p+1}\,\alpha_p\,\beta_{\widetilde{p}}\,.$$

Their images under t are respectively

$$x \cdot v = \sum_{p=1}^{2n} \alpha_p \, z^{-2n+2p-1} \, \gamma_{i-p} \, e_p^{(i)} \quad \text{and}$$
$$x \cdot w = \sum_{p=1}^{2n} \beta_{\widetilde{p}} \, z^{2n-2p+1} \, \gamma_{-i+p-1} \, e_{\widetilde{p}}^{(-i)} \, ,$$

therefore

$$(t \cdot v, t \cdot w)_\Omega = \sum_{p=1}^{2n} (-1)^{p+1} \, \alpha_p \, \beta_{\widetilde{p}} \, z^0 \, \gamma_{i-p} \, \gamma_{-i+p-1} \; .$$

We conclude that

$$T^{sp} = \left\{ (z, \gamma_0, \dots, \gamma_{2n-1}) \mid \gamma_{-j-1} = \gamma_j^{-1} \quad \forall \, j \right\} \, ,$$

where the subscripts well defined, up to modulo $2n$. It is isomorphic to an algebraic torus of dimension $n+1$ via the monomorphism

$$\begin{array}{ccc} (\mathbb{C}^*)^{n+1} & \longrightarrow & T' \\ (z, \gamma_0, \dots, \gamma_{n-1}) & \longmapsto & (z, \gamma_0, \dots, \gamma_{n-1}, \gamma_{n-1}^{-1}, \dots, \gamma_0^{-1}) \end{array}$$

whose image is precisely T^{sp}. The action of $(\mathbb{C}^*)^{n+1}$ on $X(n, 2n)$ is given on the basis by

$$(z, \gamma_0, \dots, \gamma_{n-1}) \cdot b_{j,p} = \begin{cases} z^{p-\widetilde{p}} \, \gamma_j \, b_{j,p} & \text{if } j \in [0, n-1] \, , \\ z^{p-\widetilde{p}} \, \gamma_{2n-j-1}^{-1} \, b_{j,p} & \text{otherwise,} \end{cases}$$

and it preserves $X(n, 2n)^{sp}$, where the symplectic juggling pattern points are the only T^{sp}-fixed points. This is the beginning of the proof of the next result.

Theorem 4.3.3. *The variety $X(n, 2n)^{sp}$, equipped with the T^{sp}-action, is a GKM variety.*

Proof. We know that there are finitely many zero-dimensional T^{sp}-orbits, so let us try to understand the one-dimensional ones. Observe first that the cocharacter ρ factors through $T^{sp} \hookrightarrow T'$. Any

$$t = (z, \gamma_0, \dots, \gamma_{n-1}, \gamma_{n-1}^{-1}, \dots, \gamma_0^{-1}) \in T^{sp} \, ,$$

when realized as diagonal matrices in the standard basis on each vertex, factors as $\rho(z) \circ g$, with $g \in G_{2n}^{sp}$. Since $\rho(z)$ acts like $z \in \mathbb{C}^*$ does, and since the symplectic cells $C_{\mathcal{J}}^{sp}$ are stable under both $\mathbb{C}^*$ and G_{2n}^{sp}, they are stable under T^{sp} as well. More precisely, the image under t of a point $V \in C_{\mathcal{J}}^{sp}$ with coordinates $(u_{i,j}^{(a)})$ is

$$t \cdot \left(u_{i,j}^{(a)} \right) = \left(u_{i,j}^{(a)} \, z^{-2n+2i-1} \, z^{2n-2j+1} \, \gamma_{a-i} \, \gamma_{a-j}^{-1} \right) = \left(u_{i,j}^{(a)} \, z^{2i-2j} \, \gamma_{a-i} \, \gamma_{a-j}^{-1} \right) \, .$$

Here $a - i$ (or rather, the representative in $[0, 2n-1]$ of its residue class) determines the segment of the coefficient quiver that $e_i^{(a)}$ lies on, which is precisely the one ending at vertex $a - i$ of Δ_{2n}. Since $t \in T^{sp}$, we have $\gamma_c^{-1} = \gamma_{-c-1}$ for all c. We know that V is fixed if and only if all of its coordinates are zero, so we require at least one coordinate not to vanish. Fix a descending triple (a, i, j) for $\mathcal{J}$ such that $u_{i,j}^{(a)} \neq 0$. For $T^{sp} \cdot V$ to be one-dimensional, since the group acts on each parameter via multiplication, any other nonzero coordinate $u^{(b)}_{l,m}$ must be such that $2i-2j=2l-2m$, and that either $a-i=b-l$ and $a-j=b-m$, or $a-i=-(b-m)-1$ and $a-j=-(b-l)-1$. The segments $b-l$ and $b-m$ cannot coincide since $lem<l\leq 2n$, therefore $\gamma_{b-l}\cdot\gamma^{-1}_{b-m}$ and $\gamma_{a-i}\cdot\gamma^{-1}_{a-j}$ are different from 1. Moreover, they are equal either when $a-i=b-l$ and $a-j=b-m$, or when $a-i=m-b-1$ and $a-j=l-b-1$. In the first case, the T^{sp}-orbit coincides with a one-dimensional T'-orbit, while in the second case it is the intersection of $X(n,2n)^{sp}$ with a two-dimensional T'-orbit in $X(n,2n)$. We conclude that there is a finite number of one-dimensional T^{sp}-orbits in $X(n,2n)^{sp}$. Lastly, we need to check that the rational cohomology of $X(n,2n)^{sp}$ vanishes in odd degree, but this follows from Corollary 3.3.1.1 since the G^{sp}_{2n}-orbits that make up the cellular decomposition of $X(n,2n)^{sp}$ are complex affine cells. □

Remark 4.3.4. As a consequence of this result, the T^{sp}-action is locally linearizable [GKM98, §1.2]. Therefore the closure of any one-dimensional T^{sp}-orbit contains two fixed points. The existence of a one-dimensional T^{sp}-orbit with closure points $p_{\mathcal{J}''}$ and $p_{\mathcal{J}}$, with $\mathcal{J}''\geq\mathcal{J}$, implies the existence of either a single mutation $\mathcal{J}''\xrightarrow{\mu}\mathcal{J}$, or the composition of a mutation $\mathcal{J}''\xrightarrow{\mu}\mathcal{J}'$ with its correction $\mathcal{J}'\xrightarrow{R\mu}\mathcal{J}$. Therefore, for any such orbit, there exists a symplectic mutation $\mathcal{J}''\xrightarrow[sp]{\mu}\mathcal{J}$.

Proposition 4.3.5. *Two symplectic $(n,2n)$-juggling patterns $\mathcal{J}''\geq\mathcal{J}$ are connected by a symplectic mutation if and only if $p_{\mathcal{J}''}$ and $p_{\mathcal{J}}$ are the closure points of a one-dimensional T^{sp}-orbit.*

Proof. The path

$$V(t)_a \coloneqq \mathrm{Span}\Big(\{e^{(a)}_j \mid j\in J_a\cap J''_a\}\cup\{e^{(a)}_{j-m}+t\cdot e^{(a)}_j \mid j\in J_a\setminus J''_a\}\Big),$$

from Proposition 3.3.19 is a one-dimensional T^{sp}-orbit. □

4.4 Closure-inclusion order

The goal of this section is to prove that, in the Lagrangian case, the three orders $\leq$, $\leq_{sp}$ and $\leq^m_{sp}$ coincide. We will show that the one-dimensional T^{sp}-orbits encode the information necessary to recover the G^{sp}_{2n}-orbit closure-inclusion order on $JP(n,2n)^{sp}$, analogously to the setting described in Chapter 2. We will refer to it as the "type A" case, while we will refer to the symplectic setting as "type C".

Remark 4.4.1. For dimensional reasons, given a symplectic $(n,2n)$-juggling pattern $\mathcal{J}''$ and a mutation $\mu\colon\mathcal{J}''\longrightarrow\mathcal{J}'$ with $\mathcal{J}'$ non-symplectic, there is only one symplectic juggling pattern $\mathcal{J}$ obtained by mutating $\mathcal{J}'$ on the vertices opposite to those that μ changes. This mutation is $R\mu$.

Proposition 4.4.2. *There exists a symplectic mutation between $\mathcal{J}''$ and $\mathcal{J}\in JP(n,2n)^{sp}$ if and only if $g_{\mathcal{J}''}$ and $g_{\mathcal{J}}$ differ by a type C reflection, i.e. are in an elementary Bruhat order relation.*

Proof. Remember that $g_{\mathcal{J}''}$ and $g_{\mathcal{J}}$ are the images of $f_{\mathcal{J}''}$ and $f_{\mathcal{J}}$ under

$$\begin{aligned}\mathcal{B}^{sp}_{(n,2n)} &\longrightarrow C^0_{n+1}\\ f &\longmapsto f\circ\mathrm{id}_{-n}\,.\end{aligned}$$

Let $\mathcal{J}'' \xrightarrow[sp]{\mu} \mathcal{J}$ be a symplectic mutation. If μ is also a mutation in type A, then there exists a reflection $(i,j) \in A^0_{sn}$ such that $g_{\mathcal{J}''} = g_{\mathcal{J}} \circ (i,j)$. We assume $i < j$. Since both $g_{\mathcal{J}}$ and $g_{\mathcal{J}''}$ are in C^0_{n+1}, (i,j) must as well. Thus it must be of the form $(i, -i-1)$. This is also evident by the fact μ changes the sets on the vertices from $i-n+1$ to $j-n$, and they must satisfy $i-n+1 \equiv_{2n} -(j-n)$ by Remark 3.3.13. Therefore the transposition (i,j) is of the form $(i, -i-1)$.

If instead μ is a composition $\mathcal{J}'' \xrightarrow{\nu} \mathcal{J}' \xrightarrow{R\nu} \mathcal{J}$, then

$$g_{\mathcal{J}''} = g_{\mathcal{J}} \circ (i,j) \circ (i',j') \quad \text{and} \quad g_{\mathcal{J}'} = g_{\mathcal{J}} \circ (i,j)$$

for some transpositions (i,j), $(i',j') \in A^0_{2n}$. Remember that ν and $R\nu$ affect opposite vertices. By Lemma 2.3.9 the first and last affected vertices by ν are $i-n+1$ and $j-n$, while those for $R\nu$ are $i'-n+1$ and $j'-n$. It follows that $i-n+1 \equiv_{2n} j'+n$ and $j-n \equiv_{2n} -i'+n-1$, and since the three affine permutations are bounded, this implies that the transposition (i',j') coincides with $(-j-1, -i-1)$. One implication is proven.

Consider now two symplectic $(n,2n)$-juggling patterns $\mathcal{J} \leq \mathcal{J}''$ such that $g_{\mathcal{J}}$ and $g_{\mathcal{J}''}$ differ by a reflection t as in Corollary 4.2.10.1. Suppose first that $t = (i, -i-1)$ for some i, which is a reflection in type A as well; so there exists a simple mutation between $\mathcal{J}''$ and $\mathcal{J}$, symplectic by definition. Otherwise, t must be a product of reflections $t_1 \circ t_2 := (i,j)(-j-1, -i-1)$ for some $i < j$. Again by Lemma 2.3.9, the juggling patterns corresponding to $g \leq g \circ (i,j)$, for some $g \in A^0_{2n}$, differ only on the vertices $i-n+1, i-n+2, \ldots,, j-n$. Without loss of generality, we can assume $t_1 \neq t_2$. Then t_1 corresponds to a mutation on vertices $(i-n+1, \ldots, j-n)$ and t_2 corresponds to a mutation on vertices $(-j-n, \ldots, -i-n-1) = (-j+n, \ldots, -i+n-1)$, which are opposite segments under the Δ_{2n}-symmetry $\tau \colon a \longmapsto -a$. The mutation for t_2 produces a symplectic juggling pattern starting from the one corresponding to $g_{\mathcal{J}} \circ t_1$, non-symplectic, so by Remark 4.4.1 it must be the correction of the mutation for t_1. □

We are now ready to prove the result we have been after:

Theorem 4.4.3. *For any two symplectic $(n,2n)$-juggling patterns $\mathcal{J}$ and $\mathcal{J}''$, we have*

$$\mathcal{J} \leq \mathcal{J}'' \iff \mathcal{J} \leq_{sp} \mathcal{J}'' \iff \mathcal{J} \leq^m_{sp} \mathcal{J}' .$$

Proof. It is enough to prove that the order $\leq^m_{sp}$ is finer than $\leq$. If $\mathcal{J} \leq \mathcal{J}'$, then there exists a sequence of mutations $\mathcal{J}'' = \mathcal{J}^{(0)} \xrightarrow{\mu_1} \mathcal{J}^{(1)}, \ldots, \mathcal{J}^{(m-1)} \xrightarrow{\mu_m} \mathcal{J}^{(m)} = \mathcal{J}$, as well as reflections $t_1, \ldots, t_m \in (A^0_{2n}, S)$ such that $g_{\mathcal{J}^{(i-1)}} = g_{\mathcal{J}^{(i)}} \circ t_i$ for all $i \in [m]$ and that $g_{\mathcal{J}^{(i-1)}} \geq g_{\mathcal{J}^{(i)}}$. The crucial point is that the Bruhat order in (C^0_{n+1}, S_C) coincides with the inherited Bruhat order in (A^0_{2n}, S) via the inclusion $C^0_{n+1} \subseteq A^0_{2n}$ [Kar18, Proposition 3.1]. Therefore there exist type C reflections $t^C_1, t^C_2, \ldots, t^C_{m'} \in C^0_{n+1}$ whose product is the same as the product of the t_i's. The elements

$$g_{\mathcal{J}''} \geq g_{\mathcal{J}''} \circ t^C_1 \geq g_{\mathcal{J}''} \circ t^C_1 \circ t^C_2 \geq \ldots g_{\mathcal{J}}$$

produce, by Propositions 4.4.2 and 4.3.5, symplectic juggling patterns

$$\mathcal{J}'' \geq_{sp} \mathcal{J}^{(1)}_C \geq_{sp} \mathcal{J}^{(2)}_C \geq_{sp} \cdots \geq_{sp} \mathcal{J} ,$$

connected by symplectic mutations, and therefore the corresponding points in $X(n,2n)^{sp}$ are connected by one-dimensional T^{sp}-orbits, which are contained in the G^{sp}_{2n}-orbits. This implies $p_{\mathcal{J}} \in \overline{C^{sp}_{\mathcal{J}''}}$. □

Corollary 4.4.3.1. *If two $(n, 2n)$-symplectic juggling patterns $\mathcal{J} \leq \mathcal{J}''$ are such that there is no $\mathcal{J}' \in JP(n, 2n)^{sp}$ with $\mathcal{J} \leq \mathcal{J}' \leq \mathcal{J}''$, then they differ by at most two mutations in $JP(n, 2n)$.*

4.4.1 Dimension of the cells

In this last section, we want to show that in the Lagrangian case one can easily compute the dimension of the symplectic cells, analogously to the type A case.

Definition 4.4.4. For $g \in C^0_{n+1}$, we denote with $\ell^{sp}(g)$ its length as an element of the Coxeter group (C^0_{n+1}, S_C). For $f = g \circ \mathrm{id}_n \in \mathcal{B}_{(n,2n)}$ symplectic, we define $\ell^{sp}(f) \coloneqq \ell^{sp}(g)$.

For the next result, observe that for any two pairs $(x, y), (x', y')$ in the set of inversions (2.3.1)

$$L(g) = \{(x, y) \in [0, N-1] \times \mathbb{Z} \mid x < y \wedge f(x) > f(y)\}$$

of g, one has $x \equiv_{2n} x'$ if and only if $x = x'$. If additionally $y \equiv_{2n} y'$, then $y = y'$.

Lemma 4.4.5. *Let $g, s_i \in A^0_{2k}$ be an affine permutation and a simple reflection such that $\ell(g \circ s_i) = \ell(g) + 1$, and suppose $i \in [0, 2n-1]$; then $(i, i+1) \in L(g \circ s_i) \setminus L(g)$, and the map*

$$\begin{aligned} \iota_i \colon L(g) &\longrightarrow L(g \circ s_i) \\ (x, y) &\longmapsto (s_i(x), s_i(y)) \end{aligned}$$

is well defined (up to equivalence modulo $2n$ of the components and subscripts) and injective. In addition, the diagram

$$\begin{array}{ccc} L(g) & \xrightarrow{\quad\Phi\quad} & L(Rg) \\ \big\downarrow{\scriptstyle \iota_i} & & \big\downarrow{\scriptstyle \iota_{-i-2}} \\ L(g \circ s_i) & \xrightarrow{\ \Phi\ } & L(Rg \circ s_{-i-2}) \end{array}$$

is commutative.

Proof. The first part of the statement is [BB05, Proposition 8.3.1]; observe that $\ell(g \circ s_i) = \ell(g) + 1$ is equivalent to $g(i) < g(i+1)$. We just need to check the commutativity; it is a simple computation, the crucial point of which is that $R(s_i) = s_{-i-2}$. If $x, y \notin i, i+1$, then $\iota_i(x, y) = (x, y)$ so there is nothing to check. Therefore we can assume, for example, that $x = i$ and that $y \leq 2n-1$. By assumption, y must differ from $i+1$. We have

$$\Phi(i, y) = (2n - y - 1, 2n - i - 1) \quad \text{and} \quad \iota_i(i, y) = (i+1, y)$$

so

$$\iota_{-i-2} \circ \Phi(i, y) = (2n - y - 1, 2n - i - 2) \quad \text{and} \quad \Phi \circ \iota_i(i, y) = (2n - y - 1, 2n - i - 2)\,.$$

All the other nontrivial cases are proved analogously:

◇ if $x = i, y \geq 2n$, then

$$\Phi(i, y) = (4n - y - 1, 4n - i - 1)\,, \quad \iota_i(i, y) = (i + 1, y)$$

and so

$$\iota_{-i-2} \circ \Phi(i, y) = (4n - y - 1, 4n - i - 2) = \Phi \circ \iota_i(i, y)\,;$$

◇ if $x = i + 1, y \leq 2n - 1$, then

$$\Phi(i + 1, y) = (2n - y - 1, 2n - i - 2)\,, \quad \iota_i(i + 1, y) = (i, y)$$

and so

$$\iota_{-i-2} \circ \Phi(i + 1, y) = (2n - y - 1, 2n - i - 1) = \Phi \circ \iota_i(i + 1, y)\,;$$

◇ if $x = i + 1, y \geq 2n$, then

$$\Phi(i + 1, y) = (4n - y - 1, 4n - i - 2)\,, \quad \iota_i(i + 1, y) = (i + 1, y)$$

and so

$$\iota_{-i-2} \circ \Phi(i + 1, y) = (4n - y - 1, 4n - i - 1) = \Phi \circ \iota_i(i + 1, y)\,;$$

◇ if $y = i$, then

$$\Phi(x, i) = (2n - i - 1, 2n - x - 1)\,, \quad \iota_i(x, i) = (x, i + 1)$$

and so

$$\iota_{-i-2} \circ \Phi(x, i) = (2n - i - 2, 2n - x - 1) = \Phi \circ \iota_i(x, i)\,;$$

◇ if $i < 2n - 1, y = i + 1$, then

$$\Phi(x, i + 1) = (2n - i - 2, 2n - x - 1)\,, \quad \iota_i(x, i + 1) = (x, i)$$

and so

$$\iota_{-i-2} \circ \Phi(x, i + 1) = (2n - i - 1, 2n - x - 1) = \Phi \circ \iota_i(x, i + 1)\,;$$

◇ if $y = i + 1 = 2n$, observe that $s_{2n-1} = s_{-1}$, so the images of a pair (x', y') under ι_{2n-1} and ι_{-1} coincide. Thus

$$\Phi(x, 2n) = (2n - 1, 4n - x - 1)\,, \quad \iota_{2n-1}(x, 2n) = (x, 2n - 1)$$

and so

$$\iota_{-1} \circ \Phi(x, 2n) = (0, 2n - x - 1) = \Phi \circ \iota_{2n-1}(x, 2n)\,.$$

□

Remark 4.4.6. For a symplectic $(n, 2n)$-bounded affine permutation f, the map $\Phi\colon L(f) \longrightarrow L(f)$ provides $L(f)$ with the equivalence relation $\sim_\Phi$, given by

$$(i, j) \sim_\Phi (i', j') \iff (i', j') \in \{(i, j), \Phi(i, j)\}\;.$$

Lemma 4.4.7. *For $f \in \mathcal{B}_{(n,2n)}$, the number of $\sim_\Phi$-equivalence classes in $L(f)$ is $\ell^{sp}(f)$.*

Proof. Since we know that $f = g \circ \mathrm{id}_n$ for some $g \in C^0_{n+1}$, we can alternatively prove the statement for g. We do so by induction on $\ell^{sp}(g)$. If it is 0 there is nothing to prove. If instead $\ell^{sp}(g) = 1$ then there are two cases:

- ◇ g is also a simple reflection in A^0_{2n}, i.e. g is either $r_{-1} = s_{-1}$ or $r_{n-1} = s_{n-1}$, which implies $L(g)$ has one element and thus one equivalence class;
- ◇ $g = r_i = s_i \circ s_{-i-2}$ for $i \in [0, n-2]$, so $L(g) = \{(i, i+1), (2n-i-2, 2n-i-1)\}$ and the two pairs make one equivalence class.

For the inductive step there is also more than one case. First suppose $\ell^{sp}(g \circ r_{-1}) = \ell^{sp}(g) + 1$. This implies that $\ell(g \circ r_{-1}) = \ell(g) + 1$ and that $(2n-1, 2n) \notin L(g)$, so $L(g \circ r_{-1}) = \iota_{-1} L(g) \cup \{(2n-1, 2n)\}$ by Lemma 4.4.5. But Φ preserves this singleton, so by induction the number of classes in $L(g \circ r_{-1})$ is $\ell^{sp}(g) + 1$. For $g \circ r_{n-1}$ the argument is analogous. Now fix $i \in [0, n-2]$ and suppose $\ell^{sp}(g \circ r_i) = \ell^{sp}(g) + 1$. This means that

$$\ell(g \circ r_i) = \ell(g \circ s_i) + 1 = \ell(g \circ s_{-i}) + 1 = \ell(g) + 2\,,$$

so, by applying Lemma 4.4.5 twice, we get the following diagram.

$$\begin{array}{ccc} L(g) & \xrightarrow{\Phi} & L(g) \\ \big\downarrow{\scriptstyle \iota_i} & & \big\downarrow{\scriptstyle \iota_{-i-2}} \\ L(g \circ s_i) & \xrightarrow{\Phi} & L(g \circ s_{-i-2}) \\ \big\downarrow{\scriptstyle \iota_{-i-2}} & & \big\downarrow{\scriptstyle \iota_i} \\ L(g \circ r_i) & \xrightarrow{\Phi} & L(g \circ r_i) \end{array}$$

The two elements in $L(g \circ r_i)$ missing from the image of $L(g)$ are $(i, i+1)$ and $(2n-i-2, 2n-i-1) = \Phi(i, i+1)$. Lastly, observe that since $i \in [0, n-2]$, $i, i+1, 2n-i-2$ and $2n-i-1$ are necessarily four different numbers. ☐

With $L(f)^\Phi$ we indicate the set of Φ-fixed elements of $L(f)$.

Lemma 4.4.8. *For $f \in \mathcal{B}^{sp}_{(n,2n)}$, the set $L(f)^\Phi$ is in bijection with*

$$\{i \in [0, 2n-1] \mid f(i) - i > n\}$$

and its cardinality is equal to

$$2\,\ell^{sp}(f) - \ell(f)\,,$$

which is at most n.

Proof. Fix a pair $(i, j) \in L(f)^\Phi$ and let $m_i = f(i) - i > 0$, which satisfies $m_i \leq 2n$ since f is bounded. First, suppose that $j \leq 2n-1$, so $\Phi(i, j) = (2n-j-1, 2n-i-1)$ is equal to (i, j) if and only if $j = 2n-i-1$, and i must be in $[0, n-1]$ because $2n-i-1 = j > i$. By (4.2.2) we have $f(2n-i-1) - (2n-i-1) = 2n - m_i$, from which it follows that

$$4n - i - m_i - 1 = f(2n-i-1) < f(i) = i + m_i < 2n - i - 1 + m_i$$

and therefore that $m_i > n$. If instead $j \geq 2n$, it must be equal to $4n - i - 1$ and i must be in $[n, 2n-1]$, since $j \leq f(j) < f(i) \leq i + 2n$. This time, from (4.2.2) we get

$$6n - i - m_i - 1 = f(2n - i - 1) + 2n = f(4n - i - 1) < f(i) = i + m_i < 4n - i - 1 + m_i$$

which implies once again $m_i > n$. The bijection is now obvious: (i, j) in $L(f)$ is mapped to i, and in the opposite direction i is mapped to $(i, 2n - i - 1)$ if $i \in [0, n-1]$ and to $(i, 4n - i - 1)$ if $i \in [n, 2n - 1]$. Lastly, the computation for the cardinality is a consequence of Lemma 4.4.7, and the bound follows from (4.2.2). □

Remark 4.4.9. Let $\mathcal{J} \in JP(n, 2n)$ be maximal; this means that $J := \{i \in [0, 2n-1] \mid 2n \in J_i\}$ has cardinality n by Lemma 2.1.16, and $f_{\mathcal{J}}(i) = i + 2n$ if and only if $i \in J$, while $f_{\mathcal{J}}(i) = i$ otherwise. Computing $\ell(f_{\mathcal{J}})$ is quite straightforward:

$$\left|\{(i, j) \in [0, 2n-1] \times \mathbb{Z} \mid i \in J \wedge i < j < i + 2n \ \wedge \ j \notin J \bmod 2n\}\right| = n^2 .$$

Now suppose that $\mathcal{J}$ is also symplectic. This happens if and only if J does not contain both i and $2n - i - 1$ for some $i \in [0, 2n-1]$. The Φ-fixed pairs are in bijection with J by Lemma 4.4.8, so we deduce that $\ell^{sp}(f_{\mathcal{J}}) = \frac{n(n+1)}{2} = \dim X(n, 2n)^{sp}$, in line with what was proven so far.

The next and final result is the counterpart of Proposition 2.3.6 in the Lagrangian case.

Theorem 4.4.10. *For $\mathcal{J} \in JP(n, 2n)^{sp}$, the dimension of $C^{sp}_{\mathcal{J}}$ is equal to $\ell^{sp}(f_{\mathcal{J}})$.*

Proof. Given a mutation $\mathcal{J}' \xrightarrow{\mu} \mathcal{J}$ such that $g_{\mathcal{J}'} = g_{\mathcal{J}} \circ s_i$ for a simple reflection $s_i \in A^0_{2n}$, from [FLP22, Lemma 7.4] we know that $\dim C_{\mathcal{J}'} = \dim C_{\mathcal{J}} + 1$. Let $a \in \mathbb{Z}_{2n}$, $j \in J'_a$ and $j < i \notin J'_a$ such that (a, i, j) is the terminal triple for $\mathcal{J}'$ corresponding to μ. Then by Lemma 2.2.7 the other terminal triples for $\mathcal{J}'$ are in bijection those for $\mathcal{J}$. Also, from Lemma 4.2.6 we know that the affine coordinates for $C^{sp}_{\mathcal{J}}$ are parametrized by the $\overset{sp}{\sim}$-equivalence classes of descending triples, where we recall that

$$(-b, 2n - m + 1, 2n - l + 1) \overset{sp}{\sim} (b, l, m) \overset{sp}{\sim} (b + 1, l + 1, m + 1) .$$

We prove the claim by induction on $\ell^{sp}(f_{\mathcal{J}})$. The base case $\ell^{sp}(f_{\mathcal{J}}) = 0$ is trivial, since $\mathcal{J}$ is minimal and its cell is just the point $p_{\mathcal{J}}$. Now let $\mathcal{J}'' \underset{sp}{\xrightarrow{\mu}} \mathcal{J}$ be a symplectic mutation. We want to show that the claim holds for $\mathcal{J}''$, assuming that it holds for $\mathcal{J}$. We can assume that μ factors through no other symplectic mutation, i.e. its corresponding reflection in C^0_{n+1} is simple by Proposition 4.4.2. So there exists $i \in [-1, n-1]$ such that

$$f_{\mathcal{J}''} = r_i \circ f_{\mathcal{J}} \quad \text{and} \quad \ell^{sp}(f_{\mathcal{J}''}) = \ell^{sp}(f_{\mathcal{J}}) + 1 , \tag{4.4.1}$$

which implies either $\ell(f_{\mathcal{J}''}) = \ell(f_{\mathcal{J}}) + 1$ if $i \in \{-1, n-1\}$, or $\ell(f_{\mathcal{J}''}) = \ell(f_{\mathcal{J}}) + 2$ otherwise. In the first case μ is a single mutation; thus, for cardinality reasons and by Remark 3.3.13, the set

$$\{a \in \mathbb{Z}_{2n} \mid J''_a \neq J_a\}$$

must contain either the vertex 0 or the vertex n, and on these vertices μ must replace some j with $\widetilde{j} > j$. In addition, if μ replaces x with $x + m$ on a vertex b, then it must replace $\widetilde{x} - m$ with $\widetilde{x}$ on the vertex $-b$, and m must be equal to $\widetilde{j} - j$. Therefore the $\overset{sp}{\sim}$-classes of triples for $\mathcal{J}''$ are the same as those for J, plus the class of $(a, \widetilde{j}, j)$.

In the second case, $\mathcal{J}'' \xrightarrow[sp]{\mu} \mathcal{J}$ is the composition of two corrections $\mathcal{J}'' \xrightarrow[sp]{\nu} \mathcal{J}' \xrightarrow[sp]{R\nu} \mathcal{J}$. By definition, if ν replaces some $y \in \mathcal{J}''_b$ with x, then $R\nu$ replaces $\widetilde{x} \in \mathcal{J}'_{-b}$ with $\widetilde{y}$. The $\sim$-classes of descending triples for $\mathcal{J}''$ are exactly those for $C_{\mathcal{J}}$ plus (b, x, y) and $(-b, \widetilde{y}, \widetilde{x})$. Therefore, there is one more $\overset{sp}{\sim}$-equivalence class of descending triples for $\mathcal{J}''$ than for $\mathcal{J}$. In conclusion, in either case

$$\dim C^{sp}_{\mathcal{J}''} = \dim C^{sp}_{\mathcal{J}} + 1$$

By inductive hypothesis the statement holds for $\mathcal{J}$, and therefore for it does $\mathcal{J}''$. □

4.5 Equivariant cohomology

We end this work by computing the T^{sp}-equivariant cohomology ring for $X(2,4)^{sp}$, for it is the smallest nontrivial example of the Lagrangian juggling variety $X(n,2n)^{sp}$. First, let us give a brief overview on equivariant cohomology, or Borel cohomology; we follow [Tym05] at the start of this section, and some other references include [AB84, GZ18, And11, Tu13, Mei06].

Fix a topological space X equipped with the action of a group T. Consider a contractible space ET that has a free T-action, and let BT be the corresponding geometric quotient, called the *classifying space* of T. This means that the natural projection π gives rise to a fiber bundle

$$\begin{array}{c} ET \\ \downarrow \pi \\ BT \end{array}$$

called the *universal bundle* over BT. For instance, the space $E\mathbb{C}^*$ is $(\mathbb{C}^*)^\infty = \mathbb{C}^\infty \setminus \{0\}$, the direct limit of the sequence

$$\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^2 \hookrightarrow \cdots (\mathbb{C}^*)^n \hookrightarrow (\mathbb{C}^*)^{n+1} \hookrightarrow \cdots,$$

therefore the classifying space $B\mathbb{C}^*$ is $\mathbb{P}^\infty$, the infinite union of projective spaces.

The diagonal T-action on the product $X \times ET$, i.e. the one given by

$$t \cdot (x, y) = (t \cdot x, t \cdot y),$$

is free, because by definition it is free on the second factor. Its geometric quotient is often denoted with X_T, or $X \times_T ET$, and is called the *homotopy quotient* of X by T. It is the total space of the fiber bundle

$$\begin{array}{ccc} X & \longrightarrow & X_T \\ & & \downarrow \\ & & BT \end{array}$$

called the *Borel fibration*, which has fiber X. One can then define the T-equivariant cohomology of

X as the singular cohomology of the homotopy quotient X_T:

$$H^\bullet_T(X) := H^\bullet(X_T)\,.$$

If T is an algebraic torus and X is an algebraic variety, then we say that X is a T-variety. A telling example is given by considering the (trivial) action of $\mathbb{C}^*$ on a space $\{p\}$ consisting of a single point. The resulting homotopy quotient is $B\mathbb{C}^*$ itself, thus the Borel fibration is an isomorphism. We deduce that the $\mathbb{C}^*$-equivariant cohomology group of the point is the usual cohomology group of the infinite dimensional projective space, which we see, from [Tym05, §2.2], equals the polynomial ring $\mathbb{C}[z]$ in one variable. If we instead considered the action of $(\mathbb{C}^*)^n$ on a point, the spaces $E(\mathbb{C}^*)^n$ and $B(\mathbb{C}^*)^n$ would respectively be the products of n copies of $E\mathbb{C}^* = (\mathbb{C}^*)^\infty$ and $B\mathbb{C}^* = \mathbb{P}^\infty$. The corresponding equivariant cohomology of the point would then be a polynomial algebra in n variables. Lastly, we can observe that the T-equivariant cohomology of the union $\{p_1, \ldots, p_n\}$ of n points with a trivial T-action is the direct sum of the n individual groups:

$$\bigoplus_{i=1}^{n} H^\bullet_T(p_i)\,.$$

Lastly, let us introduce the *moment graph* for a projective variety X equipped with the skeletal action of an algebraic torus T. It is a graph which stores information on the 1-skeleton of the T-action on X. Its vertex set is the fixed-point set X^T, the 0-skeleton, while the edge set is constructed as follows: for every 1-dimensional T-orbit O, there is an edge x_o — y_o between its closure points x_o and y_o, which are T-fixed. This edge is labeled by a character $\chi_o\colon T \longrightarrow \mathbb{C}^*$, or alternatively by its derivative $\eta_o\colon \mathfrak{t} \longrightarrow \mathbb{C}$. Here $\mathfrak{t}$ denotes the Lie algebra of T. The character on each edge is unique up to a sign, which is irrelevant for the purposes of computing equivariant cohomology, and provides a T-equivariant isomorphism between the closure of O and the projective sphere $\mathbb{P}^1$. This was shown for the juggling variety in §2.2.1; see also [Tym05, Proposition 3.1].

We denote by $S(\mathfrak{t}^*)$ the symmetric algebra over the dual vector space $\mathfrak{t}^*$. i.e. the algebra of polynomials over $\mathfrak{t}$, of which η_o is an element of degree 1. Then the inclusion

$$X^T \hookrightarrow X$$

gives rise to the pullback map

$$H^\bullet_T(X) \longrightarrow H^\bullet_T(X^T) = \bigoplus_{x\in X^T} H^\bullet_T(\mathrm{pt}) \cong \bigoplus_{x\in X^T} S(\mathfrak{t}^*)\,,$$

which, by [GKM98, Theorem 1.2.2], is injective with image given by those tuples of polynomials $(f_x)_{x\in X^T}$ that satisfy

$$f_{x_o} = f_{y_o} \mod \eta_o$$

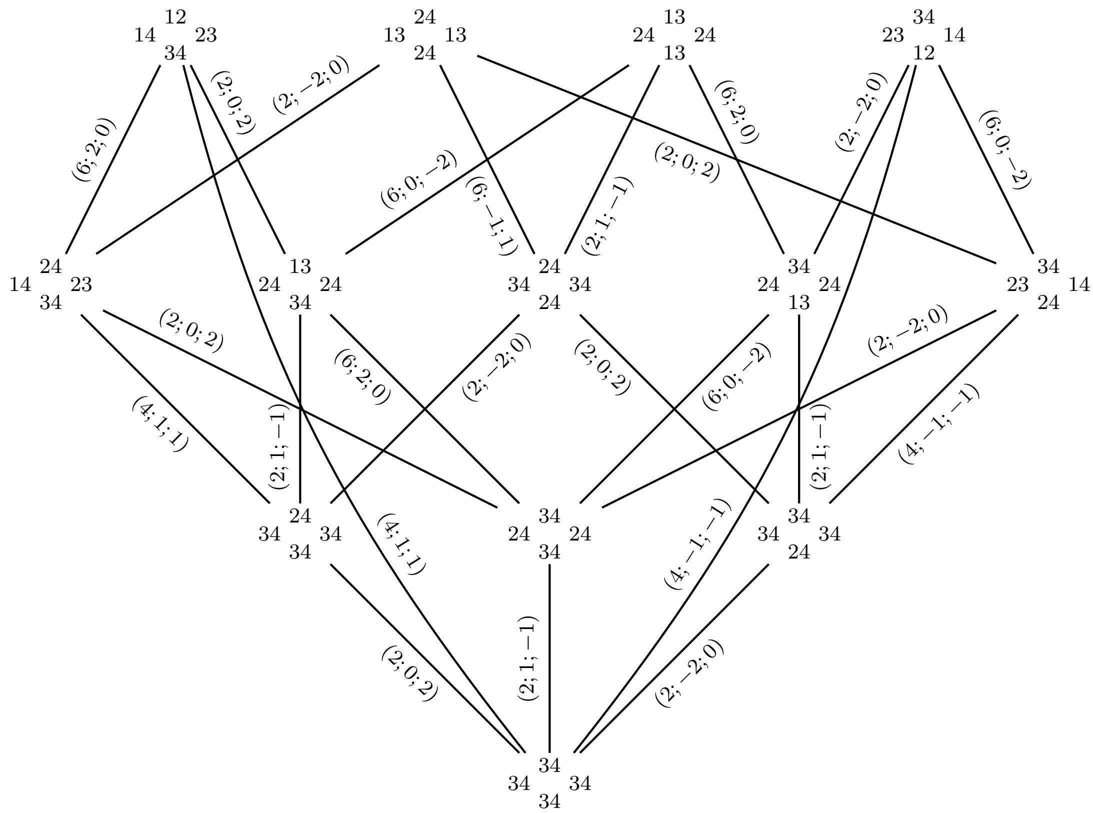
for every 1-dimensional orbit O with closure points x_o and y_o.

Let us now go back to the Lagrangian juggling variety, recall that we are considering its cohomology group with rational coefficients. Since $(X(n,2n)^{sp}, T^{sp})$ is a GKM variety by Theorem 4.3.3, computing the equivariant cohomology of $X(2,4)$ is quite straightforward. We will use $\mathfrak{t}$ to denote the Lie algebra of $T^{sp} \cong (\mathbb{C}^*)^3$, keeping the notation light. Remember that the generic element of

T^{sp} was of the form $(\alpha, \beta, \gamma, \gamma^{-1}, \beta^{-1})$, thus the natural basis for $\mathfrak{t} = \{(a, b, c, -c, -b) \mid a, c, b \in \mathbb{C}\}$ is

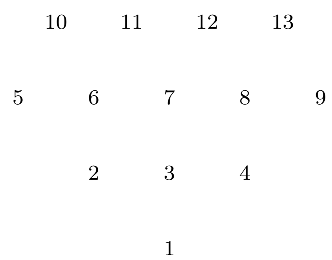
$$(1, 0, 0, 0, 0)\,, \quad (0, 1, 0, 0, -1)\,, \quad (0, 0, 1, -1, 0)\,.$$

If we let x, y, z be the basis of $\mathfrak{t}^*$ dual to it, then $S(\mathfrak{t}^*) \cong \mathbb{C}[x, y, z]$, so the edges of the moment graph for $(X(2,4)^{sp}, T^{sp})$ will be labeled by polynomials in three variables. The moment graph is shown below: its vertices are the symplectic $(2,4)$-juggling patterns, by Proposition 4.3.1, and two of them share an edge if and only if they are connected by a symplectic mutation, by Proposition 4.3.5.



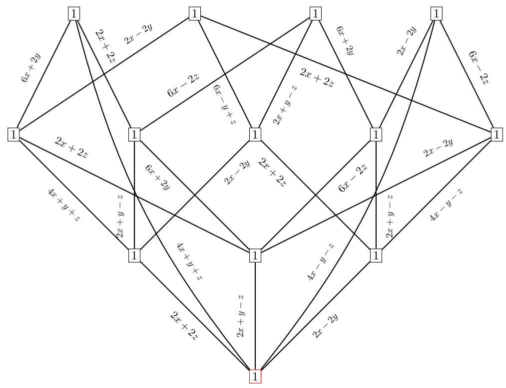
The label (a, b, c) on an edge O stands for the degree-one polynomial $\eta_o = ax + by + cz \in \mathfrak{t}^* \subset S(\mathfrak{t}^*)$. We compute the equivariant cohomology group $H^\bullet_{T^{sp}}(X(2,4)^{sp})$ by producing a basis for it. The elements of the basis are tuples of polynomials, one per symplectic juggling pattern. We present each basis element by arranging the polynomials that make it up on the vertex of the moment graph, so the reader can easily check the equivalences modulo the labels on the edges.

Furthermore, the basis elements themselves are labeled by the symplectic $(2,4)$-juggling patterns, and notice that a basis element corresponding to some $\mathcal{J}$ has zero components on any vertex $\mathcal{J}' \not\geq \mathcal{J}$. We number the vertices from the bottom up and from left to right, as in the following picture.

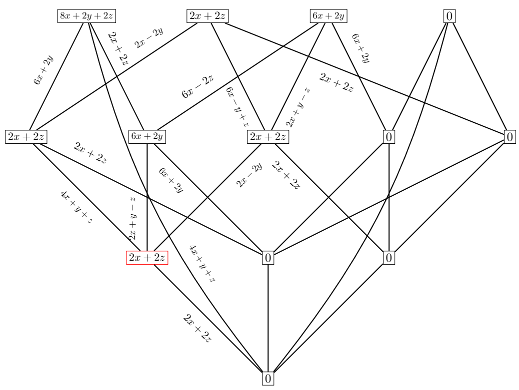


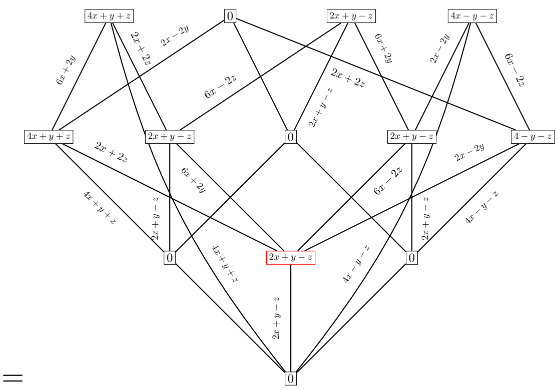
Therefore, as an example, the basis element ξ_6 corresponds to the juggling pattern $24\,{}^{13}_{34}\,24$, whose vertex will be highlighted in red, and it has null components on all vertices aside from 10 and 12. The basis is displayed in groups, divided by degree of the polynomial on the vertex in red, which coincides with the height of the juggling pattern in the poset $JP(n, 2n)^{sp}$, i.e. the length in type C of the corresponding permutation.

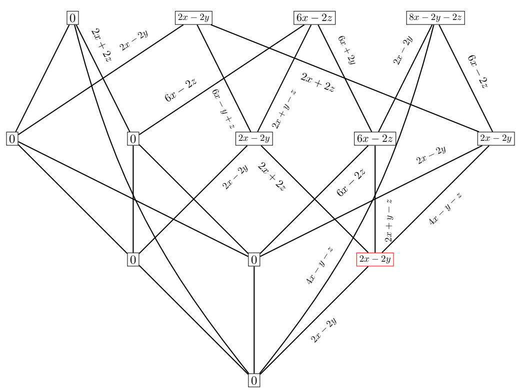
Degree 0

$\xi_1 =$ 

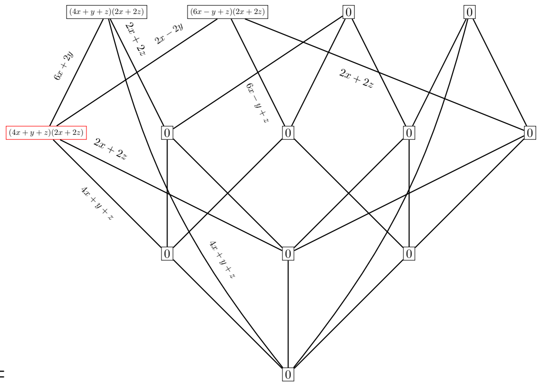
Degree 1

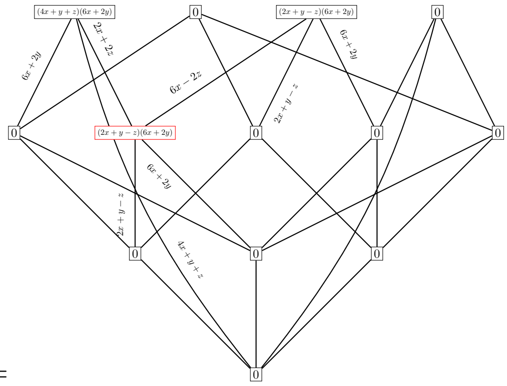
$\xi_2 =$ 

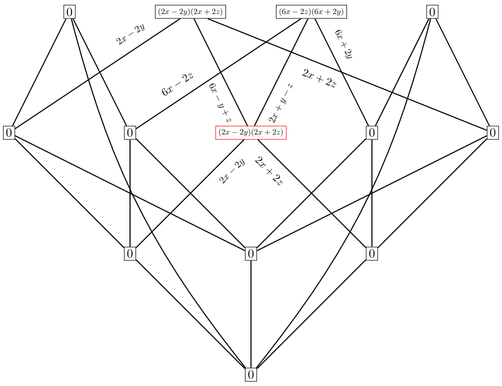
$\xi_3 =$ 

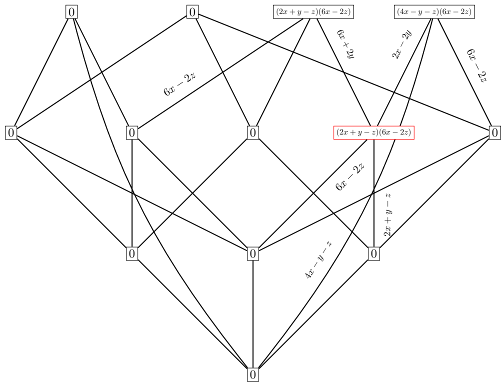
$\xi_4 =$ 

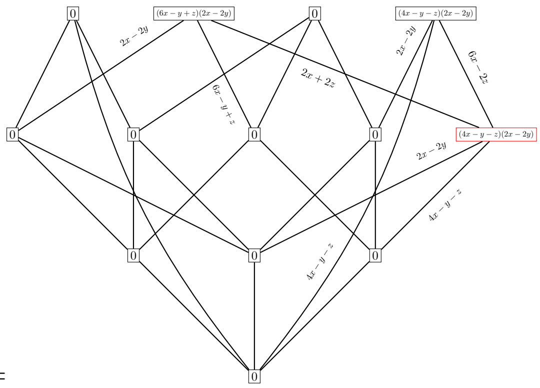
Degree 2

$\xi_5 =$ 

$\xi_6 =$ 

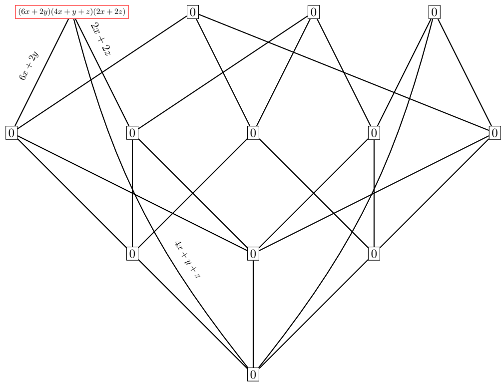
$\xi_7 =$ 

$\xi_8 =$ 

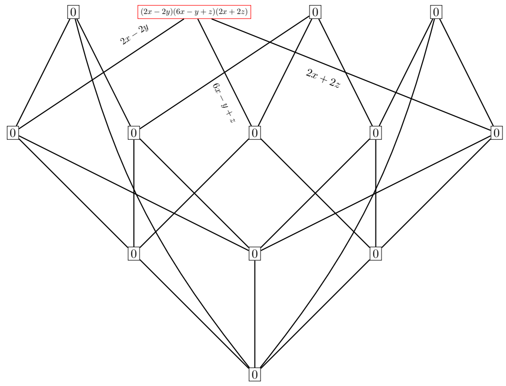
$\xi_9 =$ 

Degree 3

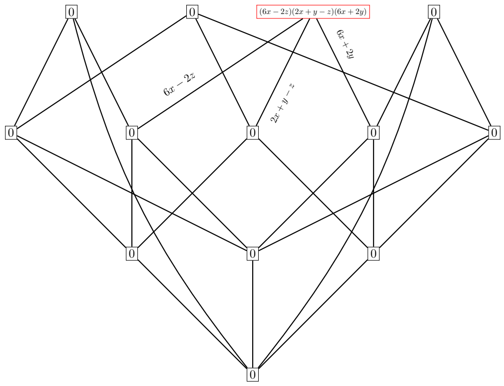
$\xi_{10} =$



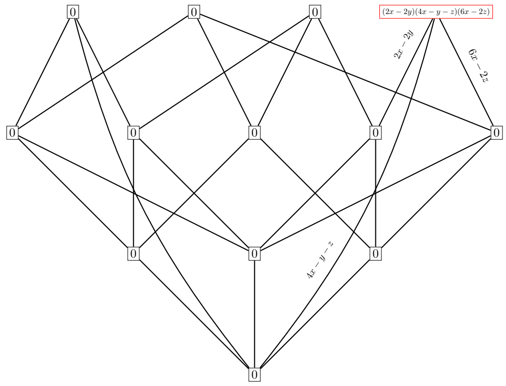
$\xi_{11} =$



$\xi_{12} =$



$\xi_{13} =$



The only equivalence that is not readily apparent is from ξ_7. The following calculation shows that $(6x - 2z)(6x + 2y)$ is equivalent to $(2x - 2y)(2x + 2z)$ modulo $2x + y - z$:

$$(6x - 2z)(6x + 2y) - 2(2x + y - z)(8x + 2y + 2z) =$$
$$(6x - 2z)(6x + 2y) - (4x + 2y - 2z)(6x + 2y) - (2x + 2z)(4x + 2y - 2z) = (2x - 2y)(2x + 2z)\,.$$

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