



Quadratic repulsiveness for LTI systems: Definition, analysis and design

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ABSTRACT

This paper introduces and describes the notion of quadratic repulsiveness for linear time-invariant systems. Such novel concept arises from the goal of escaping from an undesirable region of the state space represented by the super level sets of an indefinite quadratic function. The analysis of the special eigenstructure of a system attaining quadratic repulsiveness is carried out first, revealing a number of interesting properties. Design algorithms based on LMIs are proposed for the synthesis of controllers enabling for the simultaneous stabilization and repulsiveness of the closed-loop system. The theoretical results in this paper are illustrated through examples that show the main features of quadratic repulsiveness.

1. Introduction

Quadratic stability and *quadratic boundedness* are fundamental concepts in systems theory. A dynamical system is said to be quadratically stable when its asymptotic stability is certified using a Lyapunov function expressed as a positive-definite quadratic form (Barmish, 1983). According to Lyapunov's direct method, the proof of stability is achieved by demonstrating that the system's dynamics consistently yield negative derivatives of the Lyapunov function, except at equilibrium. In the particular case of linear systems, this test takes the form of a set of linear matrix inequalities (LMIs), which are efficient to solve using available solvers. This has allowed the study of this concept across various classes of systems, including fuzzy (Kiriakidis, Grivas, & Tzes, 1998), linear parameter-varying (Xie, 2008), and switched systems (Zhao & Dimirovski, 2004). Quadratic stability remains an active area of research, particularly in the data-driven control community (Dai & Szafer, 2023).

When accounting for unknown disturbances, the broader concept of quadratic boundedness emerges (Brockman & Corless, 1998). In this scenario, instead of the Lyapunov function converging to zero, a less strict requirement is imposed: the existence of an upper bound for the Lyapunov function that is never exceeded despite disturbances. Consequently, the state converges to a confined set enclosing the equilibrium. Due to the quadratic form chosen for the Lyapunov function, this confined set is geometrically represented as an ellipsoid. This notion has been applied to several case studies, including the design of robust state (Alessandri, Baglietto, & Battistelli, 2006) and fault (Buciakowski et al., 2017) estimators, as well as synchronization problems in multi-agent systems (Vazquez Trejo, Rotondo, Theilliol, & Adam Medina, 2023).

This paper introduces a new concept closely related to the aforementioned ideas, which can be considered a generalization of quadratic stability to the case of indefinite quadratic forms. Specifically, we require the dynamical system to yield negative derivatives for state vector values that produce positive values of the quadratic form. This property is formally named *quadratic repulsiveness*, as it corresponds to the system repelling the state away from an undesirable region of the state-space, whose boundary is geometrically represented by a conical surface. Notably, a relationship between quadratic repulsiveness and quadratic stability can be established, as the former reduces to the latter for positive-definite quadratic forms. In this case, the exponential stability of the origin can be reinterpreted as the repulsiveness of the rest of the state-space.

The paper proceeds with a thorough analysis for continuous-time LTI systems, which reveals that the eigenstructure of a quadratically repulsive system must necessarily satisfy a certain relationship with the eigenstructure of the matrix associated to the indefinite quadratic form. The eigenstructure constraints brought by quadratic repulsiveness must be taken into account during the design process in order to achieve feasibility of the LMIs that produce a controller that makes the closed-loop system simultaneously quadratically repulsive and asymptotically stable. In this sense, an additional contribution of our paper is to connect some classical results about the inertia of matrices by Ostrowski and Schneider (Ostrowski & Schneider, 1962) with the recent S-variable approach originally proposed by Ebihara, Peaucelle, and Arzelier (2015).

The work presented in this paper is primarily theoretical and aims to provide further insight into the behavior of LTI systems relative

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to the characteristics of their eigenstructure. It can be considered a further step towards the completion of LTI systems theory, intended as an area of applied mathematics with possible applications in many technical fields, such as electrical circuit analysis and design, signal processing, filter design, control theory, mechanical engineering, and image processing. However, we would like to point out that in many real-world applications, some regions of the state-space correspond to undesirable operating conditions due to constraints imposed by modeling limitations, performance requirements, or safety concerns. For instance, in chemical processes, certain combinations of temperature and fractional conversion of reactant can initiate hazardous runaway reactions (Luo & Hu, 1998). In power systems, voltage fluctuations induced by excessive reactive power can lead to equipment damage or grid instability (Chaudhary & Rizwan, 2018). Inspired by this last scenario, we present a proof of concept using an electrical circuit. This example illustrates how quadratic repulsiveness can drive the load towards an operating condition where it consistently absorbs power from that point onward. In other words, during the remainder of the transient phase, the load will absorb power without ever returning it to the rest of the circuit. To further elucidate the concept, we conclude the paper with an example using a third-order numerical system, offering a detailed geometrical illustration of quadratic repulsiveness.

The remainder of the paper is structured as follows. The definition of quadratic repulsiveness is given and discussed in Section 2, which also identifies the main goals of our work. The main findings about the eigenstructure analysis are provided in Section 3, with the corresponding technical details provided in Appendices A–B. Control design is addressed in Section 4, where stability and repulsiveness conditions are framed in terms of suitable LMIs. Illustrative examples, covering systems in $\mathbb{R}^2$ and $\mathbb{R}^3$, are proposed in Section 5 to support the theoretical results and showcase their practical implications. Finally, the main conclusions are drawn in Section 6.

1.1. Notation

The notation used in this paper is fairly standard. Given a complex number λ , its conjugate is denoted by λ^* . The symbols $\mathbb{R}^n$ ($\mathbb{C}^n$), $\mathbb{R}^{n \times m}$ ($\mathbb{C}^{n \times m}$) and $\mathbb{S}^n$ ($\mathbb{H}^n$) are used to denote, respectively, the sets of n -dimensional real (complex) vectors, $n \times m$ -dimensional real (complex) matrices, and symmetric (Hermitian) $n \times n$ -dimensional real (complex) matrices. For a vector v , v^T denotes its (possibly conjugate) transpose. Similarly, for a matrix X , X^T denotes its (possibly conjugate) transpose, and $\text{He}\{X\}$ stands as a short notation for $X + X^T$. Terms that can be inferred from the symmetry (hermitianity) of symmetric (Hermitian) matrices are denoted by $\star$.

Given a matrix $A \in \mathbb{R}^{n \times n}$, we denote by a_{ij} the element located at row i and column j , by $\lambda_i \in \mathbb{C}$ its i th eigenvalue, and by $v_i \in \mathbb{C}^n$ the corresponding eigenvector. We assume that eigenvalues and eigenvectors are arranged from rightmost to leftmost, i.e., in descending order based on the real parts of the eigenvalues: $\text{Re}\{\lambda_1\} \geq \text{Re}\{\lambda_2\} \geq \dots \geq \text{Re}\{\lambda_n\}$ (the order assigned to eigenvalues with the same real part is irrelevant).

Following (Ostrowski & Schneider, 1962), for a matrix $X \in \mathbb{H}^n$ with π positive, ν negative, and δ null eigenvalues, we will call the ordered triple (π, ν, δ) the *inertia* of X , and denote it by $\text{In}(X)$. The shorthand notation $\text{In}(X) = (\pi, \nu)$ indicates that $\delta(X) = 0$. We will denote the indices as $\pi(X)$, $\nu(X)$, $\delta(X)$, respectively, where $\pi(X) + \nu(X) + \delta(X) = n$. If $\pi(X) = n$ ($\nu(X) = n$), the matrix X is said to be *positive* (*negative*) *definite*, and this will be denoted by $X > 0$ ($X < 0$). More generally, for $A \in \mathbb{R}^{n \times n}$, $\text{In}(A) = (\pi, \nu, \delta)$ is a shorthand notation for the matrix A having π eigenvalues with positive real part, ν eigenvalues with negative real part, and δ purely imaginary eigenvalues.

2. Quadratic repulsiveness: definition

Let us consider the following autonomous continuous-time LTI system:

$$\dot{x}(t) = Ax(t) \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, and $A \in \mathbb{R}^{n \times n}$ is the state matrix. We are interested in analyzing whether the state trajectories of the system (1) escape from a prescribed region of the state-space, which corresponds to undesirable conditions. This property is formally defined as *quadratic repulsiveness*, where the adjective *quadratic* refers to the specific structure of the function used to define the above-mentioned region of the state-space.

Definition 1. Given $H \in \mathbb{S}^n$, the system (1) is said to be *quadratically repulsive* for $\mathcal{R}(x) = x^T H x$ if $\dot{\mathcal{R}}(x) < 0$ for all $x \neq 0$ such that $\mathcal{R}(x) \geq 0$.

Notably, a relation between quadratic repulsiveness and quadratic stability can be established by noticing that the latter is obtained when $H > 0$. In this case, H becomes a Lyapunov matrix, and $\mathcal{R}(x) \geq 0$ describes the entire state-space. Hence, the exponential stability of the origin becomes equivalent to the repulsiveness of the rest of the state-space: indeed, the state escapes from $\mathbb{R}^n$ towards the origin of the state-space, and it does so by taking progressively decreasing values of $\mathcal{R}(x)$. Given that the case $H > 0$ has been covered quite extensively in the literature, the remaining of the paper will focus on the case where H is an indefinite matrix.

The analysis of quadratic repulsiveness of the system (1) for $\mathcal{R}(x)$ is a relatively simple task, which requires a well-known lemma commonly referred to as the *S-procedure* (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994).

Lemma 1. Consider $x \in \mathbb{R}^n$ and the quadratic functions $F(x)$, $G(x)$. The inequality $F(x) < 0$ is satisfied for all $x \neq 0$ such that $G(x) \geq 0$ if and only if there exists $\tau \geq 0$ such that $F(x) + \tau G(x) < 0$.

Proof. See for example (Derinkuyu & Pinar, 2006). $\square$

Hence, the following result holds.

Theorem 1. The system (1) is quadratically repulsive for $\mathcal{R}(x)$ if and only if there exists a scalar $\tau \geq 0$ so that:

$$A^T H + H A + \tau H < 0 \quad (2)$$

Proof. By differentiating $\mathcal{R}(x)$ while accounting for the dynamics in (1), we obtain:

$$\dot{\mathcal{R}}(x) = x^T (A^T H + H A) x \quad (3)$$

By applying Lemma 1, $\dot{\mathcal{R}}(x) < 0$ for all $x \neq 0$ such that $\mathcal{R}(x) \geq 0$, i.e., quadratic repulsiveness for $\mathcal{R}(x)$, if and only if there exists $\tau \geq 0$ such that (2) holds. $\square$

From now on, we will assume that $\delta(H) = 0$, as the case $\delta(H) > 0$ (implying that H has null eigenvalues) is incompatible with quadratic repulsiveness, as shown in the following corollary.

Corollary 1. Let $H \in \mathbb{S}^n$ with $\delta(H) \neq 0$. Then, the system (1) is not quadratically repulsive for $\mathcal{R}(x)$.

Proof. Let us assume, by contradiction, that the system (1) is quadratically repulsive for $\mathcal{R}(x)$ with H having at least one zero eigenvalue. Consider an eigenvector v corresponding to such zero eigenvalue, so that $Hv = 0$. Then, from (2), we know that $\tau \geq 0$ exists such that

$$v^T (A^T H + H A) v < -\tau v^T H v \quad (4)$$

which, using $Hv = 0$ and $v^T H = 0$, leads to the contradiction $0 < 0$. $\square$

In the remaining of the paper, we shall first analyze the constraints imposed on the eigenstructure of the matrix A by the quadratic repulsiveness. Later, we will include an exogenous input vector $u \in \mathbb{R}^m$ that affects the system's dynamics through the input matrix $B \in \mathbb{R}^{n \times m}$:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (5)$$

and design a state-feedback control law:

$$u(t) = Kx(t) \quad (6)$$

where $K \in \mathbb{R}^{m \times n}$ is the controller gain to be designed, so that for the closed-loop system obtained as the interconnection of (5) and (6):

$$\dot{x}(t) = (A + BK)x(t) \quad (7)$$

some requirements connected to quadratic repulsiveness hold. In particular, we consider sequentially:

- (i) the design of K so that (7) is quadratically repulsive¹ for $\mathcal{R}(x)$
- (ii) the design of K so that (7) is quadratically repulsive for $\mathcal{R}(x)$ and asymptotically stable
- (iii) the design of K so that (7) is quadratically repulsive for $\mathcal{R}(x)$ and asymptotically stable, with some prescribed guaranteed decay rates on $\mathcal{R}(x)$ and the Lyapunov function $V(x)$

3. Eigenstructure analysis

The quadratic repulsiveness property imposes constraints on the eigenvectors of A , requiring them to be positioned in a specific manner relative to the region of the state space where $\mathcal{R}(x) > 0$. This section presents theoretical results that clarify the peculiar eigenstructure of the matrix A in an autonomous LTI system that is quadratically repulsive for $\mathcal{R}(x)$. The constraints on the eigenstructure of a quadratically repulsive system must be considered in the design phase to ensure feasible design conditions, as explained in the following section.

Note that the discussion is limited to state matrices that are diagonalizable over $\mathbb{C}$. Future research will build upon the findings described in this section to extend the eigenstructure analysis so that it includes the case where some eigenvalues have a geometric multiplicity lower than their algebraic multiplicity, i.e., the case of Jordan forms.

We show in the following that for a system $\dot{x} = Ax$ quadratically repulsive for $\mathcal{R}(x) = x^T H x$, with $\text{In}(H) = (\pi, \nu)$:

- the sign of $\mathcal{R}(x)$ is invariant over eigenspaces generated by eigenvalues of A with the same real part (Theorem 2);
- the π leftmost eigenvectors of A lie in the region of the state-space where $\mathcal{R}(x) < 0$, and the remaining ν rightmost eigenvectors of A lie in the region of the state-space where $\mathcal{R}(x) < 0$ (Theorem 3).

Since the proofs of the theorems in this section are highly technical, we provide only an informal and intuitive sketch here, deferring the formal proofs to the appendices.

Theorem 2. Let the system (1) be quadratically repulsive for $\mathcal{R}(x)$ with A diagonalizable over $\mathbb{C}$. Denote as $\mathcal{V}_\sigma \subseteq \mathbb{C}^n$ the union of the eigenspaces associated to eigenvalues $\lambda_i \in \mathbb{C}$ of A with $\text{Re}(\lambda_i) = \sigma$. Then, either $\mathcal{R}(v) > 0$ for all $v \in \mathcal{V}_\sigma \setminus \{0\}$ or $\mathcal{R}(v) < 0$ for all $v \in \mathcal{V}_\sigma \setminus \{0\}$.

Proof. The proof follows by demonstrating that the following cases are incompatible with quadratic repulsiveness:

- (i) there are two linearly independent eigenvectors $v_1, v_2 \in \mathcal{V}_\sigma$ associated to $\lambda_i = \sigma$ such that $\mathcal{R}(v_1)\mathcal{R}(v_2) < 0$;

- (ii) there are two complex conjugate eigenvectors $v, v^* \in \mathcal{V}_\sigma$ associated to $\lambda_i = \sigma + j\omega$ and $\lambda_i^* = \sigma - j\omega$, respectively, such that $\mathcal{R}(v)\mathcal{R}(v^*) < 0$;
- (iii) there are two complex conjugate eigenvectors $v_1, v_1^* \in \mathcal{V}_\sigma$ associated to $\lambda_i = \sigma + j\omega$ and $\lambda_i^* = \sigma - j\omega$ and a real eigenvector $v_2 \in \mathcal{V}_\sigma$ associated to $\lambda_j = \sigma$ such that $\mathcal{R}(v_1)\mathcal{R}(v_2) < 0$;
- (iv) there are two pairs of complex conjugate eigenvectors $v_1, v_1^*, v_2, v_2^* \in \mathcal{V}_\sigma$ associated to $\lambda_i = \sigma + j\omega_a$, $\lambda_i^* = \sigma - j\omega_a$, $\lambda_j = \sigma + j\omega_b$, $\lambda_j^* = \sigma - j\omega_b$ such that $\mathcal{R}(v_1)\mathcal{R}(v_2) < 0$;

The details are provided in Appendix A. $\square$

Theorem 3. Let $H \in \mathbb{S}^n$ be indefinite with $\text{In}(H) = (\pi, \nu)$, and let the system $\dot{x} = Ax$ be quadratically repulsive for $\mathcal{R}(x) = x^T H x$. Then, the eigenvectors v_i of A , $i = 1, \dots, n$, are such that:

$$\mathcal{R}(v_i) < 0 \quad i = 1, \dots, \nu \quad (8)$$

$$\mathcal{R}(v_i) > 0 \quad i = \nu + 1, \dots, n \quad (9)$$

Proof. The proof starts by demonstrating that the following cases are incompatible with quadratic repulsiveness:

- (i) A has more than ν eigenvectors v_i such that $\mathcal{R}(v_i) < 0$;
- (ii) A has more than π eigenvectors v_i such that $\mathcal{R}(v_i) > 0$;
- (iii) A has at least one eigenvector v_i such that $\mathcal{R}(v_i) = 0$;

Since (i)-(iii) imply that A must have exactly ν eigenvectors with $\mathcal{R}(v_i) < 0$ and π eigenvectors with $\mathcal{R}(v_i) > 0$, the proof is completed by showing that the eigenvalues corresponding to the former eigenvectors cannot have smaller real parts than those corresponding to the latter eigenvectors. The details are provided in Appendix B. $\square$

4. LMI-based control design

This section contains LMI-based results that allow designing the controller gain K so that the desired requirements about the repulsiveness for a predefined quadratic function $\mathcal{R}(x) = x^T H x$ are satisfied by the closed-loop system (7). The rest of this section presents first some *auxiliary results* that are required for the subsequent discussion. Then, we discuss the simpler case where only repulsiveness is desired. Later, we obtain LMIs to be used for simultaneous repulsion and stabilization. Finally, we incorporate the possibility of requiring some guaranteed rate for how the repulsive function $\mathcal{R}(x)$ and the Lyapunov function $V(x)$, used to certify stability, decay over time.

4.1. Auxiliary results

The following lemma is based on the works by Ebihara et al. (2015), and hence belongs to the class of *S-variable approaches*.

Lemma 2. Given matrices $A \in \mathbb{R}^{n \times n}$, $A_o \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $F \in \mathbb{R}^{n \times n}$, $X \in \mathbb{S}^n$, $Y \in \mathbb{R}^{m \times n}$, and a scalar $\gamma \in \mathbb{R}$, let $K = YF^{-1}$. Then:

$$\begin{bmatrix} \text{He}\{F\} & \star \\ X - AF - BY - A_o F^T & \gamma X + \text{He}\{(AF + BY)A_o^T\} \end{bmatrix} < 0 \quad (10)$$

is equivalent to:

$$\text{He}\{(A + BK)X\} + \gamma X < 0 \quad (11)$$

and:

$$\text{He}\{A_o X\} + \gamma X < 0 \quad (12)$$

Proof. The equivalence between (10) and (11)–(12) is obtained by rewriting (10) as follows:

$$\begin{bmatrix} 0 & X \\ X & \gamma X \end{bmatrix} + \text{He}\left\{\begin{bmatrix} I \\ -A - BK \end{bmatrix} F \begin{bmatrix} I & -A_o^T \end{bmatrix}\right\} < 0 \quad (13)$$

¹ Unfortunately, the rules of English grammar make a control law (6) that achieves this goal *repulsing*.

and applying the Elimination Lemma (Lemma 1.2 in [Ebihara et al. \(2015\)](#)). $\square$

As described in [Ebihara et al. \(2015\)](#), if $X > 0$ and $\gamma = 0$, then (12) implies necessarily that A_o is Hurwitz. Hereafter, by exploiting some results about the inertia of matrices by Ostrowski and Schneider ([Ostrowski & Schneider, 1962](#)), we show that (12) induces a tight relation between the value of γ , the inertia of X and the eigenvalues of the matrix A_o , which is formally expressed by the following proposition.

Proposition 1. *Let (10) hold. Then, the matrix $\tilde{A}_o = A_o + \frac{\gamma}{2}I$ has no purely imaginary eigenvalue and:*

$$\text{In}(\tilde{A}_o) = \text{In}(-X) \quad (14)$$

Proof. By rewriting (12) in terms of $\tilde{A}_o$, and changing the sign of the inequality, we obtain:

$$\text{He}\{\tilde{A}_o(-X)\} > 0 \quad (15)$$

Using Theorem 1 in [Ostrowski and Schneider \(1962\)](#), a necessary and sufficient condition for (15) is that $\tilde{A}_o$ has no purely imaginary eigenvalues, and $\text{In}(\tilde{A}_o) = \text{In}(-X)$, thus completing the proof. $\square$

When $X > 0$, the constraint (14) corresponds to the requirement that the real parts of the eigenvalues of the matrix A_o are less than $-\gamma/2$, thus recovering the result described in ([Man, 1969](#)). In the more general case where X is indefinite, i.e., $\text{In}(X) = (\pi, \nu)$, the condition (14) states that if (10) holds, then the matrix A_o has necessarily π eigenvalues with real part less than $-\gamma/2$ and ν eigenvalues with real part bigger than $-\gamma/2$. In the following, the term A_o in Lemma 2 will lead to some matrices that need to be fixed a priori in order to obtain LMIs for the design of the controller gain K . In such context, the above results allow to restrict the search space for suitable pre-fixed matrices by excluding all the choices that would inevitably produce infeasible LMIs.

4.2. Design of a repulsing controller gain

The design of a controller gain K that makes (7) quadratically repulsive for $\mathcal{R}(x)$ is a relatively simple procedure, as described by the following theorem.

Theorem 4. *Given matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $H \in \mathbb{S}^n$, and $K \in \mathbb{R}^{m \times n}$, the system (7) is quadratically repulsive for $\mathcal{R}(x)$ if and only if there exists a scalar $\tau \geq 0$ so that:*

$$(A + BK)^T H + H(A + BK) + \tau H < 0 \quad (16)$$

Proof. This follows straightforwardly from Theorem 1, by replacing the open-loop matrix A with the closed-loop matrix $A + BK$. $\square$

Note that since H is preassigned, the conditions provided by Theorem 4 are linear in both K and τ .

4.3. Design of a stabilizing and repulsing controller gain

The design of a controller gain K that makes (7) simultaneously quadratically repulsive for $\mathcal{R}(x)$ and asymptotically stable is a more complicate procedure, which naturally leads to bilinear matrix inequalities due to the simultaneous presence of the terms KP and KH^{-1} , where K and P are decision variables. However, by means of the S-variable approach, it is possible to linearize the matrix inequalities by introducing a common slack variable F instead of P and H^{-1} , as stated in the following theorem.

Theorem 5. *Given matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $H \in \mathbb{S}^n$, let $A_s \in \mathbb{R}^{n \times n}$ be Hurwitz and $A_r \in \mathbb{R}^{n \times n}$ be such that:*

$$\text{He}\{A_r N\} + \tau N < 0 \quad (17)$$

for some $\tau \geq 0$, where $N \triangleq H^{-1}$. If there exist matrices $P \in \mathbb{S}^n$, $F \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{m \times n}$ such that:

$$P > 0 \quad (18)$$

$$\begin{bmatrix} \text{He}\{F\} & \star \\ P - AF - BY - A_s F^T & \text{He}\{(AF + BY)A_s^T\} \end{bmatrix} < 0 \quad (19)$$

$$\begin{bmatrix} \text{He}\{F\} & \star \\ N - AF - BY - A_r F^T & \tau N + \text{He}\{(AF + BY)A_r^T\} \end{bmatrix} < 0 \quad (20)$$

then the closed-loop system (7) with $K = YF^{-1}$ is quadratically repulsive for $\mathcal{R}(x)$ and asymptotically stable.

Proof. Quadratic repulsiveness for $\mathcal{R}(x)$ is obtained by setting that $\dot{\mathcal{R}}(x) < 0$, i.e.:

$$x^T \text{He}\{H(A + BK)\}x < 0 \quad (21)$$

for all $x \neq 0$ such that $\mathcal{R}(x) \geq 0$. By using Lemma 1, this happens if and only if there exists $\tau \geq 0$ such that:

$$\text{He}\{H(A + BK)\} + \tau H < 0 \quad (22)$$

By pre- and post-multiplying (22) by $N \triangleq H^{-1}$, we obtain:

$$\text{He}\{(A + BK)N\} + \tau N < 0 \quad (23)$$

which, upon application of Lemma 2 with $X = N$ and $\gamma = \tau$ leads to (20).

To fulfill the asymptotic stability requirement, let us consider the Lyapunov function $V(x(t)) = x(t)^T P^{-1} x(t)$, with P satisfying (18). Simple calculations lead to:

$$\text{He}\{(A + BK)P\} < 0 \quad (24)$$

By applying Lemma 2 to (24) with $X = P$ and $\gamma = 0$, we obtain (19), which completes the proof. $\square$

The requirements on the matrices A_s and A_r in Theorem 5 come from the application of Proposition 1 to (19)–(20). In particular, (17) corresponds to a requirement on A_r being quadratically repulsive for $\mathcal{R}(x)$. Based on previous discussion, if $\text{In}(H) = (\pi, \nu)$, then a suitable matrix A_r should have ν eigenvectors v_i , $i = 1, \dots, \nu$, associated to eigenvalues with real part bigger than $-\tau/2$ and such that $\mathcal{R}(v_i) < 0$, and the remaining π eigenvectors v_i , $i = \nu + 1, \dots, n$, associated to eigenvalues with real part smaller than $-\tau/2$ and such that $\mathcal{R}(v_i) > 0$.

4.4. Incorporation of guaranteed decay rates into the design

This section briefly describes how the theoretical results described in Section 4.3 can be modified to incorporate some prescribed guaranteed decay rates on $\mathcal{R}(x)$ and $V(x)$, which means ensuring that:

- for a given $\alpha > 0$, $\dot{\mathcal{R}}(x) < -\alpha \mathcal{R}(x)$ for all $x \neq 0$ such that $\mathcal{R}(x) \geq 0$;
- for a given $\beta > 0$, $\dot{V}(x) < -\beta V(x)$ for all $x \neq 0$ such that $V(x) \neq 0$.

Theorem 6. *Given matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $H \in \mathbb{S}^n$ and some prescribed guaranteed decay rates $\alpha > 0$ and $\beta > 0$, let $A_s \in \mathbb{R}^{n \times n}$ have all its eigenvalues with real part smaller than $-\beta/2$ and let $A_r \in \mathbb{R}^{n \times n}$ be such that (17) holds for some $\tau \geq \alpha$, where $N \triangleq H^{-1}$. If there exist matrices $P \in \mathbb{S}^n$, $F \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{m \times n}$ such that (18), (20) and:*

$$\begin{bmatrix} \text{He}\{F\} & \star \\ P - AF - BY - A_s F^T & \beta P + \text{He}\{(AF + BY)A_s^T\} \end{bmatrix} < 0 \quad (25)$$

hold. Then, the closed-loop system (7) with $K = YF^{-1}$ is quadratically repulsive for $\mathcal{R}(x)$ and asymptotically stable, with $\dot{\mathcal{R}}(x) < -\alpha \mathcal{R}(x)$ for all $x \neq 0$ such that $\mathcal{R}(x) \geq 0$ and $\dot{V}(x) < -\beta V(x)$ for all $x \neq 0$ such that $V(x) \neq 0$.

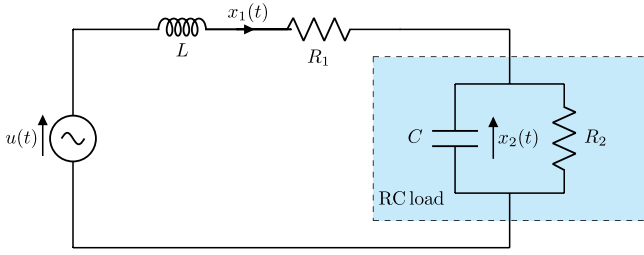


Fig. 1. Example 1 — Electrical circuit.

Proof. The proof is similar to that of Theorem 5, so only the main differences are outlined. By setting the condition $\dot{\mathcal{R}}(x) < -\alpha\mathcal{R}(x)$ and using Lemma 1, we obtain a condition on the existence of some $\tau' \geq 0$ such that:

$$\text{He}\{(A+BK)N\} + \alpha N + \tau' N < 0 \quad (26)$$

which corresponds to (23) with $\tau = \alpha + \tau'$, so that the requirement $\tau' \geq 0$ becomes $\tau \geq \alpha$. On the other hand, $\dot{V}(x) < -\beta V(x)$ with $V(x) = x(t)^T P^{-1} x(t)$ leads to:

$$\text{He}\{(A+BK)P\} + \beta P < 0 \quad (27)$$

and, by applying Lemma 2 with $X = P$ and $\gamma = \beta$, we obtain (25) where, according to Proposition 1, the matrix A_s must have all its eigenvalues with real part smaller than $-\beta/2$. $\square$

It is worth noting that Proposition 1 imposes the constraint $\alpha > \beta$. In fact, the closed-loop matrix $A+BK$ must satisfy simultaneously:

$$\text{In}(A+BK + \frac{\alpha}{2}I) = \text{In}(-H) \quad (28)$$

$$\text{In}(A+BK + \frac{\beta}{2}I) = \text{In}(-P) \quad (29)$$

which means that all its eigenvalues must have real part smaller than $-\beta/2$ and as many eigenvalues as the number of negative eigenvalues of H must have real part bigger than $-\alpha/2$, which is possible only if $\alpha > \beta$.

5. Illustrative examples

This section describes three examples that illustrate the main features of the theory proposed in this paper. The LMIs have been solved using the YALMIP (Löfberg, 2004) toolbox in combination with the SeDuMi (Sturm, 1999) solver.

5.1. Example 1 — second-order electrical circuit

Let us consider the electrical circuit shown in Fig. 1, with $R_1 = 20 \Omega$, $R_2 = 10 \Omega$, $L = 100 \text{ mH}$, and $C = 1 \text{ mF}$. By applying the well known Kirchhoff's voltage law together with the characteristic equation of the components, we obtain:

$$\begin{cases} \dot{x}_1(t) = -\frac{R_1}{L}x_1(t) - \frac{1}{L}x_2(t) + \frac{1}{L}u(t) \\ \dot{x}_2(t) = \frac{1}{C}x_1(t) - \frac{1}{R_2C}x_2(t) \end{cases} \quad (30)$$

which corresponds to a system described by matrices:

$$A = \begin{bmatrix} -200 & -10 \\ 1000 & -100 \end{bmatrix} \quad B = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

The instantaneous power absorbed (if positive) or transferred (if negative) by the RC load highlighted in arctic blue color in Fig. 1 is computed as $P(t) = x_1(t)x_2(t)$. We aim to apply the newly introduced concept of quadratic repulsiveness to influence $u(t)$, such that, if the initial conditions $x_1(0)$ and $x_2(0)$ reflect power being transferred by the load, the circuit is guided towards a state where it will subsequently only absorb power for the remainder of the transient. The

system will approach this state by progressively reducing the amount of power transferred. To achieve this, we define $\mathcal{R}(x) = -x_1(t)x_2(t)$, corresponding to the following matrix H :

$$H = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}$$

Clearly, the open-loop system is not repulsive for $\mathcal{R}(x)$, since the eigenvalues of A are complex conjugate: $\lambda_{1,2} = (-150 \pm 86.6j)$. The LMI (16) is solved obtaining $\tau = 967.5$ and $K = [-66.7517 \quad 92.8974]$. The resulting closed-loop state matrix has a dominant (unstable) eigenvalue $\lambda_1 = 548.8$ with associated eigenvector $v_1 = [-0.5443 \quad -0.8389]^T$, whereas the other (stable) eigenvalue is $\lambda_2 = -1516.3$ with eigenvector $v_2 = [-0.8169 \quad 0.5768]^T$. In agreement with Theorem 3, one has that $\mathcal{R}(v_1) = -0.4566 < 0$ and $\mathcal{R}(v_2) = 0.4712 > 0$. On the other hand, an asymptotically stable design is achieved by solving the LMIs (18)–(20) with pre-fixed matrices:

$$A_s = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad A_r = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -10 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -5.5 & 4.5 \\ 4.5 & -5.5 \end{bmatrix}$$

obtaining $\tau = 12.2675$ and $K = [5.6872 \quad 2.3975]$. The resulting closed-loop state matrix has a dominant (slow) eigenvalue $\lambda_1 = -1.3973$ with associated eigenvector $v_1 = [-0.0981 \quad -0.9952]^T$ ($\mathcal{R}(v_1) = -0.0977 < 0$) whereas the other (fast) eigenvalue is $\lambda_2 = -241.7307$ with eigenvector $v_2 = [-0.1403 \quad 0.9901]^T$ ($\mathcal{R}(v_2) = 0.1389 > 0$). Finally, a guaranteed decay rate design is achieved by solving LMIs (18), (20) and (25). With $\alpha = 50$, $\beta = 10$, we choose:

$$A_s = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} \quad A_r = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -50 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -25.5 & 24.5 \\ 24.5 & -25.5 \end{bmatrix}$$

obtaining $\tau = 74.7146$ and $K = [-2.0333 \quad 2.8107]$. The resulting closed-loop state matrix has a dominant (slow) eigenvalue $\lambda_1 = -12.7660 \in [-\alpha/2, -\beta/2] = [-25, -5]$ with associated eigenvector $v_1 = [-0.0869 \quad -0.9962]^T$ ($\mathcal{R}(v_1) = -0.0866 < 0$) whereas the other (fast) eigenvalue is $\lambda_2 = -307.5673 < -\alpha/2 = -25$ with eigenvector $v_2 = [-0.2032 \quad 0.9791]^T$ ($\mathcal{R}(v_2) = 0.1990 > 0$). The phase plane plots of the open-loop and the obtained closed-loop systems are shown in Figs. 2–5. It can be seen that in the open-loop system, some arrows point towards increasing values of the function $\mathcal{R}(x)$ (highlighted using red color in Fig. 2), corresponding to lack of repulsiveness. On the other hand, for the closed-loop systems all the arrows inside the yellow region point towards decreasing values of the function $\mathcal{R}(x)$, thus confirming the repulsiveness.

5.2. Example 2 — three dimensional system with two-sheeted repulsive hyperboloids

Let us now consider a system described by matrices:

$$A = \begin{bmatrix} 9 & 7 & 9 \\ 5 & 6 & 0 \\ 9 & 2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 8 \\ 7 & 9 \\ 2 & 1 \end{bmatrix}$$

and the function $\mathcal{R}(x) = x_1^2 - x_2^2 - x_3^2$, described by the matrix $H = \text{diag}(1, -1, -1)$ and whose positive level sets correspond to two-sheeted hyperboloids. This system is not repulsive for $\mathcal{R}(x)$, in fact it is such that $\mathcal{R}([1 \ 0 \ 0]^T) = 18$ even though $\mathcal{R}([1 \ 0 \ 0]^T) = 1$. The repulsive-only design is obtained with $\tau = 20.5419$ and:

$$K = \begin{bmatrix} -0.6997 & 9.4156 & -1.7361 \\ -3.3159 & -8.0958 & -0.0000 \end{bmatrix}$$

which leads to closed-loop eigenvalues $\lambda_1 = 15.5330$, $\lambda_2 = 2.6499$, $\lambda_3 = -40.2342$, and corresponding eigenvectors:

$$v_1 = \begin{bmatrix} -0.5639 \\ 0.7229 \\ -0.5639 \end{bmatrix} \quad v_2 = \begin{bmatrix} -0.3878 \\ 0.3994 \\ 0.8307 \end{bmatrix} \quad v_3 = \begin{bmatrix} -0.8066 \\ -0.5293 \\ 0.2631 \end{bmatrix}$$

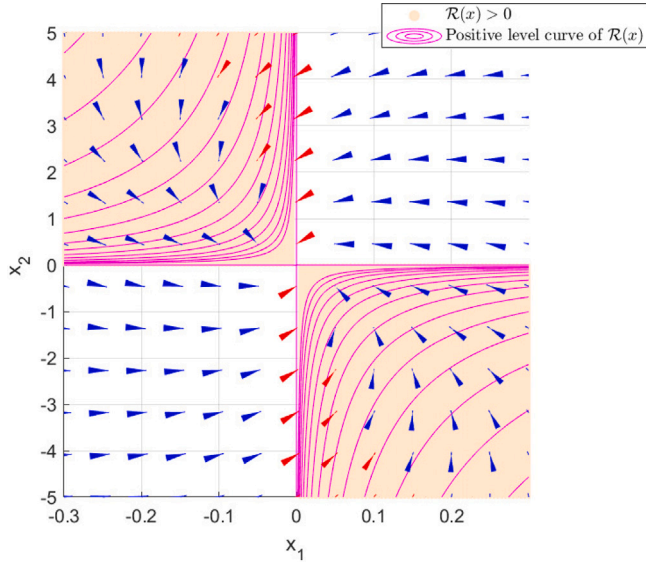


Fig. 2. Example 1 — Open-loop phase plane.

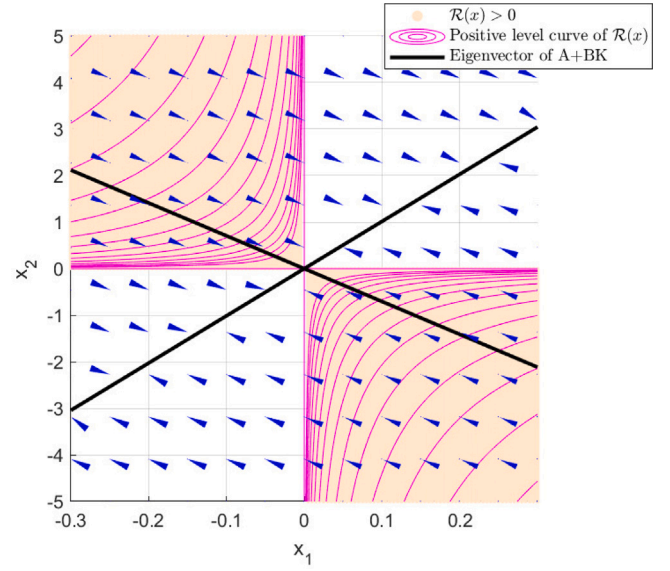


Fig. 4. Example 1 — Closed-loop phase plane (as. stable design).

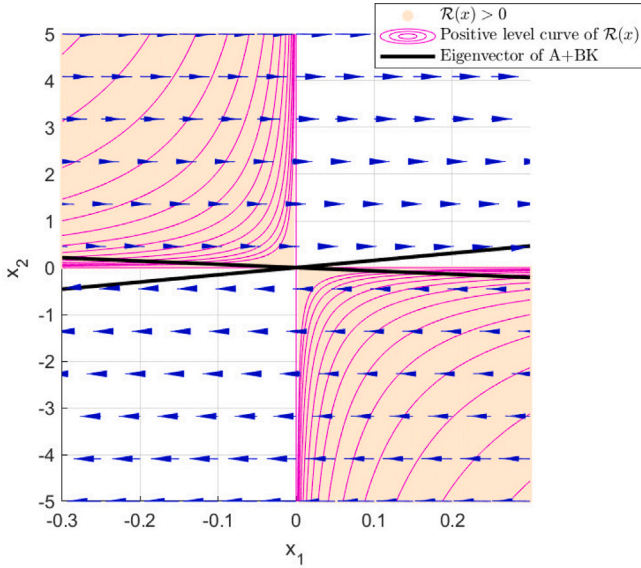


Fig. 3. Example 1 — Closed-loop phase plane (repulsive design).

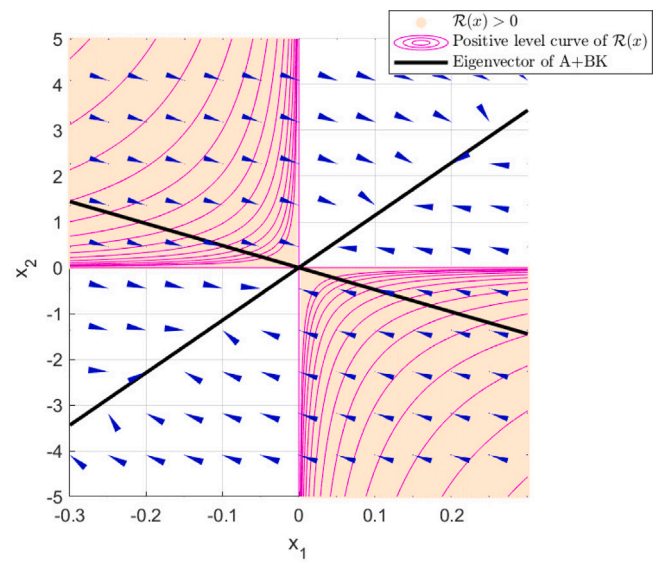


Fig. 5. Example 1 — Closed-loop phase plane (guar. decay rate design).

According to Theorem 3, a system which is repulsive for $\mathcal{R}(x)$ should have $\mathcal{R}(v_1) < 0$, $\mathcal{R}(v_2) < 0$ and $\mathcal{R}(v_3) > 0$, which is indeed the case, since $\mathcal{R}(v_1) = -0.6992$, $\mathcal{R}(v_2) = -0.3631$ and $\mathcal{R}(v_3) = 0.3012$.

On the other hand, an asymptotically stable design is achieved by solving the LMIs (18)–(20) with pre-fixed matrices:

$$A_s = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad A_r = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

obtaining $\tau = 11.5350$ and:

$$K = \begin{bmatrix} 7.2587 & 0.9549 & 5.0950 \\ -7.1106 & -1.6510 & -4.6160 \end{bmatrix}$$

which leads to a closed-loop matrix with eigenvalues $\lambda_1 = \lambda_2^* = -0.9256 + 0.3744j$, $\lambda_3 = -18.8587$ and eigenvectors:

$$v_1 = v_2^* = \begin{bmatrix} -0.0783 + 0.1063j \\ 0.9641 \\ -0.0959 - 0.2093j \end{bmatrix} \quad v_3 = \begin{bmatrix} -0.8210 \\ -0.2167 \\ 0.5281 \end{bmatrix}$$

with $\mathcal{R}(v_1) = \mathcal{R}(v_2) = -0.9652$ and $\mathcal{R}(v_3) = 0.3482$.

By using guaranteed decay rate parameters $\alpha = 50$ and $\beta = 10$, with pre-fixed matrices:

$$A_s = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -10 \end{bmatrix} \quad A_r = \begin{bmatrix} -50 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

we obtain $\tau = 71.1669$ and:

$$K = \begin{bmatrix} 15.0856 & -1.7589 & 0.4690 \\ -15.5060 & 0.3302 & -4.2399 \end{bmatrix}$$

with closed-loop eigenvalues $\lambda_1 = \lambda_2^* = -7.6491 \pm 2.5589j$ (note that $\text{Re}(\lambda_1) \in [-\alpha/2, -\beta/2]$), $\lambda_3 = -59.1355 < -\alpha/2$, and corresponding eigenvectors:

$$v_1 = v_2^* = \begin{bmatrix} -0.0362 + 0.0379j \\ 0.9889 \\ 0.0922 - 0.1040j \end{bmatrix} \quad v_3 = \begin{bmatrix} 0.9002 \\ 0.2400 \\ -0.3634 \end{bmatrix}$$

such that $\mathcal{R}(v_1) = \mathcal{R}(v_2) = -0.9945$ and $\mathcal{R}(v_3) = 0.6207$.

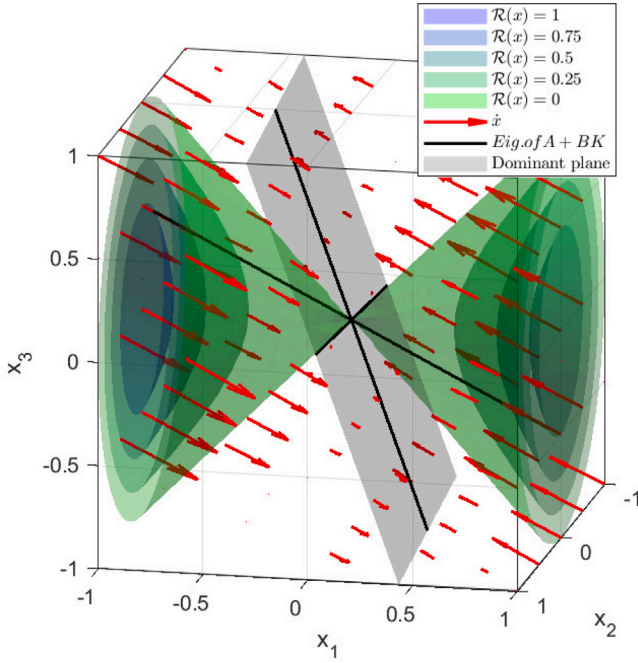


Fig. 6. Example 2 — Closed-loop phase plane (guar. decay rate design).

For the sake of illustration, Fig. 6 shows the phase space portrait of the closed-loop system obtained with $\alpha = 50$ and $\beta = 10$. It can be seen that for states corresponding to positive values of $\mathcal{R}(x)$, the corresponding vector field points towards decreasing values of $\mathcal{R}(x)$, which corresponds to the repulsiveness condition.

5.3. Example 3 — three dimensional system with one-sheeted repulsive hyperboloids

Let us now consider the same system as in Example 2, but using the function $\mathcal{R}(x) = x_1^2 + x_2^2 - x_3^2$, described by the matrix $H = \text{diag}(1, 1, -1)$, and whose positive level sets are one-sheeted hyperboloids. This system is not repulsive for $\mathcal{R}(x)$, in fact it is such that $\dot{\mathcal{R}}([1 \ 0 \ 0])^T = 18$ even though $\mathcal{R}([1 \ 0 \ 0])^T = 1$. The repulsive-only design is obtained with $\tau = 28.6545$ and:

$$K = \begin{bmatrix} -2.4744 & -19.7482 & -3.3727 \\ -3.4309 & 11.8330 & 0.0000 \end{bmatrix}$$

which leads to closed-loop eigenvalues $\lambda_1 = 0.8863$, $\lambda_2 = \lambda_3^* = -28.6210 + 37.0094j$, and corresponding eigenvectors:

$$v_1 = \begin{bmatrix} -0.3674 \\ -0.2078 \\ 0.9066 \end{bmatrix} \quad v_2 = v_3^* = \begin{bmatrix} -0.6996 \\ 0.0539 - 0.6146j \\ 0.3233 - 0.1595j \end{bmatrix}$$

According to Theorem 3, a system which is repulsive for $\mathcal{R}(x)$ should have $\mathcal{R}(v_1) < 0$, $\mathcal{R}(v_2) > 0$ and $\mathcal{R}(v_3) > 0$, which is indeed the case, since $\mathcal{R}(v_1) = -0.6437$, and $\mathcal{R}(v_2) = \mathcal{R}(v_3) = 0.7401$.

By using guaranteed decay rate parameters $\alpha = 50$ and $\beta = 10$, with pre-fixed matrices:

$$A_s = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -10 \end{bmatrix} \quad A_r = \begin{bmatrix} -50 & 0 & 0 \\ 0 & -50 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

we obtain $\tau = 69.8954$ and:

$$K = \begin{bmatrix} 31.4397 & -27.1027 & 30.6603 \\ -25.4272 & 12.9704 & -23.6367 \end{bmatrix}$$

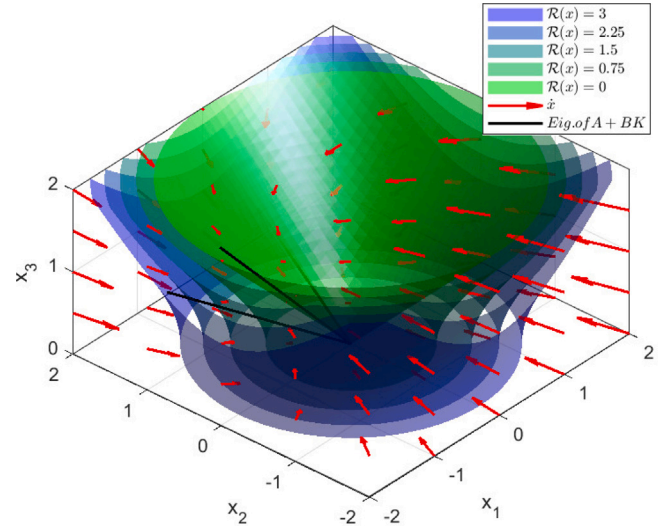
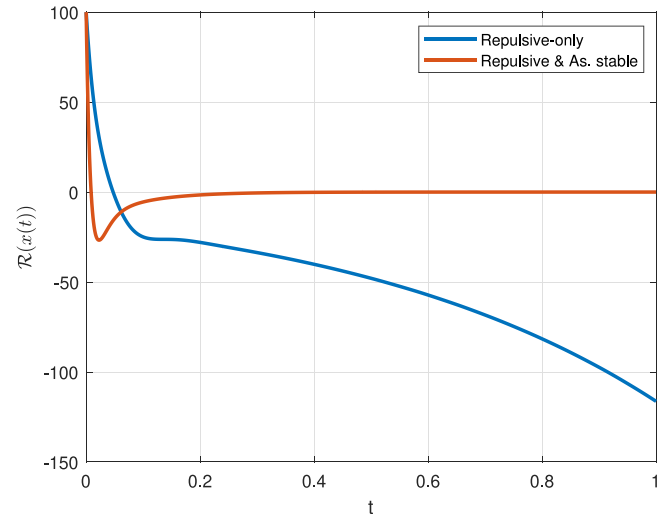


Fig. 7. Example 3 — Closed-loop phase plane (guar. decay rate design).

Fig. 8. Example 3 — Simulated trajectory of $\mathcal{R}(x)$.

with closed-loop eigenvalues $\lambda_1 = -5.9324 \in [-\alpha/2, -\beta/2]$, $\lambda_2 = -53.2247 < -\alpha/2$, $\lambda_3 = -68.2434 < -\alpha/2$, and corresponding eigenvectors:

$$v_1 = \begin{bmatrix} -0.6710 \\ 0.0643 \\ 0.7386 \end{bmatrix} \quad v_2 = \begin{bmatrix} -0.8008 \\ 0.2912 \\ 0.5234 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0.3856 \\ 0.9080 \\ 0.1641 \end{bmatrix}$$

such that $\mathcal{R}(v_1) = -0.0912$, $\mathcal{R}(v_2) = 0.4522$ and $\mathcal{R}(v_3) = 0.9461$.

The phase space portrait of the closed-loop system is shown in Fig. 7, which shows that all the trajectories of the system would converge towards the cone $x^T H x < 0$ by passing through progressively decreasing values of $\mathcal{R}(x)$.

Finally, Fig. 8 shows the trajectory of the function $\mathcal{R}(x)$ for an initial state $x(0) = [10, 0, 0]^T$. In both obtained closed-loop systems, the function $\mathcal{R}(x)$ decreases towards negative values. In the asymptotically stable case, the function $\mathcal{R}(x)$ undershoots and then converges towards $\mathcal{R}(x) = 0$ which corresponds to the convergence of the state to the origin of the state-space. On the other hand, the repulsive-only design has produced an unstable closed-loop system, for which the $\mathcal{R}(x)$ grows unbounded, but negatively.

6. Conclusions

This paper introduced the novel concept of quadratic repulsiveness and discussed the analysis and design for linear time-invariant systems. Quadratic repulsiveness can be perceived as a generalization of the notion of quadratic stability to the case of quadratic functions defined through sign-indefinite matrices. The name *repulsiveness* comes from the idea of dealing with specific regions of the state space identified as undesirable and which the system trajectories are requested to escape from. Defining these undesirable sets as the superlevel sets of sign-indefinite quadratic forms automatically characterizes them as (hyper)-hyperboloids, whose actual shape is determined by the inertia of the matrix H associated to the quadratic form. Investigation of the properties of linear time-invariant systems exhibiting quadratic repulsiveness shed the light on an interesting relationship between the eigenstructure of the system state matrix A and that of the matrix H , which are indeed forced to have a certain relative positioning. Embedding such requirement in the constraints, the design of controllers yielding quadratic repulsiveness of the closed-loop is then developed by using linear matrix inequalities. The synthesis is further enhanced by considering additional constraints, such as asymptotic stability and even guaranteed decay rate. Three examples, concerning second-order and third-order systems, have been used to illustrate the main features of the proposed quadratic repulsiveness-based control design. Future research will address the problem of robust control design within the quadratic repulsiveness framework, as well as the extension of the notion of quadratic repulsiveness to other types of systems.

CRedit authorship contribution statement

Damiano Rotondo: Writing – original draft, Software, Formal analysis, Investigation, Visualization, Methodology, Conceptualization. **Andrea Cristofaro:** Writing – review & editing, Investigation, Conceptualization, Validation, Methodology, Formal analysis.

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Appendix A

Proof of Theorem 2. (continued) The following lemmas are instrumental in completing the proof of Theorem 2. They show that quadratic repulsiveness for an indefinite quadratic form is incompatible with the following situations:

- a second-order system with repeated real eigenvalues (Lemma 3);
- a second-order system with complex conjugate eigenvalues (Lemma 4);
- a third-order system with a pair of complex conjugate eigenvalues and a real eigenvalue that share the same real part, and for which the eigenvectors do not produce the same sign of the quadratic form (Lemma 5);
- a fourth-order system with two pairs of complex conjugate eigenvalues that share the same real part, and for which the eigenvectors do not produce the same sign of the quadratic form (Lemma 6).

Lemma 3. Let $\tilde{H} \in \mathbb{S}^2$ be indefinite. Then, the system:

$$\dot{z} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} z \quad (31)$$

with $\sigma \in \mathbb{R}$ is not quadratically repulsive for $\mathcal{R}(z) = z^T \tilde{H} z$.

Proof. By choosing any $z^* \neq 0$ so that $\mathcal{R}(z^*) = 0$, one obtains that $\dot{\mathcal{R}}(z^*) = 2\sigma\mathcal{R}(z^*) = 0$. $\square$

Lemma 4. Let $H \in \mathbb{S}^2$ be indefinite and the matrix $A \in \mathbb{R}^{2 \times 2}$ have complex conjugate eigenvalues $\lambda = \sigma + j\omega$, $\lambda^* = \sigma - j\omega$, with $\sigma, \omega \in \mathbb{R}$. Then, the system $\dot{x} = Ax$ is not quadratically repulsive for $\mathcal{R}(x) = x^T H x$.

Proof. Let us apply the change of coordinates $x = Tz$, where T is the matrix containing the real and imaginary part of the eigenvector associated to λ . Thus, we obtain:

$$\begin{cases} \dot{z}_1(t) = \sigma z_1(t) + \omega z_2(t) \\ \dot{z}_2(t) = -\omega z_1(t) + \sigma z_2(t) \end{cases} \quad (32)$$

with $\mathcal{R}(z) = z^T \tilde{H} z$, and $\tilde{H} = T^T H T$. Let us rewrite $\mathcal{R}(z)$ as follows:

$$\mathcal{R}(z_1, z_2) = \tilde{h}_{11} z_1^2 + 2\tilde{h}_{12} z_1 z_2 + \tilde{h}_{22} z_2^2 \quad (33)$$

where $\tilde{h}_{11}$, $\tilde{h}_{12}$, $\tilde{h}_{22}$ are necessarily such that $\tilde{h}_{12}^2 - \tilde{h}_{11}\tilde{h}_{22} > 0$ due to $\text{In}(\tilde{H}) = \text{In}(H) = (1, 1)$. Given a value $z_1 \neq 0$, there are two values z_2^+ and z_2^- that lead to $\mathcal{R}(z_1, z_2^+) = \mathcal{R}(z_1, z_2^-) = 0$:

$$z_2^\pm = \frac{-\tilde{h}_{12} \pm \sqrt{\tilde{h}_{12}^2 - \tilde{h}_{11}\tilde{h}_{22}}}{\tilde{h}_{22}} z_1 \quad (34)$$

Simple calculations lead to:

$$\dot{\mathcal{R}}(z_1, z_2^\pm) = \omega \left[\tilde{h}_{12} (z_2^{\pm 2} - z_1^2) + (\tilde{h}_{11} - \tilde{h}_{22}) z_1 z_2^\pm \right] \quad (35)$$

where the fact that $\mathcal{R}(z_1, z_2^\pm) = 0$ has been used to get rid of the term that depends on σ .

By accounting for (34), one obtains:

$$\dot{\mathcal{R}}(z_1, z_2^\pm) = \frac{2\omega z_1^2}{\tilde{h}_{22}^2} \left[2\tilde{h}_{12}(\tilde{h}_{12}^2 - \tilde{h}_{11}\tilde{h}_{22}) \pm (\tilde{h}_{11}\tilde{h}_{22} - 2\tilde{h}_{12}^2 - \tilde{h}_{22}^2) \sqrt{\tilde{h}_{12}^2 - \tilde{h}_{11}\tilde{h}_{22}} \right] \quad (36)$$

Then:

$$\dot{\mathcal{R}}(z_1, z_2^+) \dot{\mathcal{R}}(z_1, z_2^-) = \frac{4\omega^2 z_1^4}{\tilde{h}_{22}^2} [(h_{11} - h_{22})^2 + 4h_{12}^2] (h_{11}h_{22} - h_{12}^2) < 0$$

which means that either $\dot{\mathcal{R}}(z_1, z_2^+) > 0$ or $\dot{\mathcal{R}}(z_1, z_2^-) > 0$. Hence, we have proven the existence of a state $z^* \in \mathbb{R}^2 \setminus \{0\}$ such that $\mathcal{R}(z^*) = 0$ and $\dot{\mathcal{R}}(z^*) \geq 0$. Consequently, $x^* = Tz^*$ is such that $\mathcal{R}(x^*) = 0$ and $\dot{\mathcal{R}}(x^*) \geq 0$, which completes the proof. $\square$

Lemma 5. Let $H \in \mathbb{S}^3$ be indefinite and the matrix $A \in \mathbb{R}^{3 \times 3}$ have eigenvalues $\lambda_1 = \sigma + j\omega$, $\lambda_1^* = \sigma - j\omega$, $\lambda_2 = \sigma$ with $\sigma, \omega \in \mathbb{R}$. Assume that the eigenvectors v_1 and v_2 associated to λ_1 and λ_2 are such that:

$$\mathcal{R}(v_1)\mathcal{R}(v_2) < 0 \quad (37)$$

Then, the system $\dot{x} = Ax$ is not quadratically repulsive for $\mathcal{R}(x) = x^T H x$.

Proof. Let us apply the change of coordinates $x = Tz$ where $T = [\text{Re}(v_1) \quad \text{Im}(v_1) \quad v_2]$, thus obtaining:

$$\begin{cases} \dot{z}_1(t) = \sigma z_1(t) + \omega z_2(t) \\ \dot{z}_2(t) = -\omega z_1(t) + \sigma z_2(t) \\ \dot{z}_3(t) = \sigma z_3(t) \end{cases} \quad (38)$$

with $\mathcal{R}(z) = z^T \tilde{H} z$, and $\tilde{H} = T^T H T$. Let us partition the matrix $\tilde{H}$ as follows:

$$\tilde{H} = \begin{bmatrix} \tilde{H}_a & \tilde{H}_{a3} \\ \tilde{H}_{a3}^T & \tilde{h}_{33} \end{bmatrix} \quad (39)$$

with $\tilde{H}_a \in \mathbb{S}^2$, $\tilde{H}_{a3} \in \mathbb{R}^2$ and $\tilde{h}_{33} \in \mathbb{R}$. If $\tilde{H}_a$ is indefinite, then by virtue of Lemma 4 we can conclude that $\dot{x} = Ax$ is not quadratically repulsive for $\mathcal{R}(x)$. Hence, we will discuss in detail the non-trivial case where $\tilde{H}_a$ is definite, in which case (37) implies that $\tilde{H}_a \tilde{h}_{33} < 0$. Let

us now consider the change of basis $z = \tilde{T}y$ where $\tilde{T}$ has the following structure:

$$\tilde{T} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (40)$$

with θ chosen as follows:

$$\theta = \frac{\arctan(2\tilde{h}_{12}/(\tilde{h}_{22} - \tilde{h}_{11}))}{2} \quad (41)$$

Then, in the y -coordinates, the system is described by (38), where z_1, z_2, z_3 are replaced by y_1, y_2, y_3 , respectively. The quadratic function $\mathcal{R}(z)$ can be computed as $\mathcal{R}(z) = \mathcal{R}(\tilde{T}y) = y^T \tilde{T}^T \tilde{H} \tilde{T} y = y^T \tilde{H} y$, with:

$$\tilde{H} = \begin{bmatrix} \tilde{h}_{11} & 0 & \tilde{h}_{13} \\ 0 & \tilde{h}_{22} & \tilde{h}_{23} \\ \tilde{h}_{13} & \tilde{h}_{23} & \tilde{h}_{33} \end{bmatrix} \quad (42)$$

where $\tilde{h}_{22}\tilde{h}_{33} < 0$ due to $\tilde{H}_a\tilde{h}_{33} < 0$. Consequently:

$$\begin{bmatrix} \tilde{h}_{22} & \tilde{h}_{23} \\ \tilde{h}_{23} & \tilde{h}_{33} \end{bmatrix} < 0 \quad (43)$$

which means that given a value $y_2 \neq 0$, there are two values y_3^+ and y_3^- of opposite sign that lead to $\mathcal{R}(\tilde{T}[0, y_2, y_3^+]^T) = \mathcal{R}(\tilde{T}[0, y_2, y_3^-]^T) = 0$. Further calculations show that:

$$\dot{\mathcal{R}}(\tilde{T}[0, y_2, y_3^+]^T) = 2\tilde{h}_{13}\omega y_2 y_3^+ \quad (44)$$

where the fact that $\mathcal{R}(\tilde{T}[0, y_2, y_3^+]^T) = 0$ has been exploited to get rid of the terms that depend on σ . From (44), it is evident that one of the two values y_3^+ and y_3^- corresponds to $\dot{\mathcal{R}}(\tilde{T}[0, y_2, y_3^+]^T) \geq 0$. Hence, we have proven the existence of a state $y^* \in \mathbb{R}^3 \setminus \{0\}$ such that $\mathcal{R}(y^*) = 0$ and $\dot{\mathcal{R}}(y^*) \geq 0$. Consequently, $x^* = T\tilde{T}y^*$ is such that $\mathcal{R}(x^*) = 0$ and $\dot{\mathcal{R}}(x^*) \geq 0$, which completes the proof. $\square$

Lemma 6. Let $H \in \mathbb{S}^4$ be indefinite and the matrix $A \in \mathbb{R}^{4 \times 4}$ have eigenvalues $\lambda_1 = \sigma + j\omega_a$, $\lambda_1^* = \sigma - j\omega_a$, $\lambda_2 = \sigma + j\omega_b$, $\lambda_2^* = \sigma - j\omega_b$, with $\sigma, \omega_a, \omega_b \in \mathbb{R}$. Assume that the eigenvectors v_1 and v_2 associated to λ_1 and λ_2 satisfy (37). Then, the system $\dot{x} = Ax$ is not quadratically repulsive for $\mathcal{R}(x) = x^T H x$.

Proof. Let us apply the change of coordinates $x = Tz$, where $T = [\text{Re}(v_1) \text{ Im}(v_1) \text{ Re}(v_2) \text{ Im}(v_2)]$, thus obtaining:

$$\begin{cases} \dot{z}_1(t) = \sigma z_1(t) + \omega_a z_2(t) \\ \dot{z}_2(t) = -\omega_a z_1(t) + \sigma z_2(t) \\ \dot{z}_3(t) = \sigma z_3(t) + \omega_b z_4(t) \\ \dot{z}_4(t) = -\omega_b z_3(t) + \sigma z_4(t) \end{cases} \quad (45)$$

with $\mathcal{R}(z) = z^T \tilde{H} z$, and $\tilde{H} = T^T H T$. Let us partition the matrix $\tilde{H}$ as follows:

$$\tilde{H} = \begin{bmatrix} \tilde{H}_{aa} & \tilde{H}_{ab} \\ \tilde{H}_{ab}^T & \tilde{H}_{bb} \end{bmatrix} \quad (46)$$

with $\tilde{H}_{aa}, \tilde{H}_{bb} \in \mathbb{S}^2$ and $\tilde{H}_{ab} \in \mathbb{R}^{2 \times 2}$. If any among $\tilde{H}_{aa}$ or $\tilde{H}_{bb}$ is indefinite, then by virtue of Lemma 4 we can conclude that $\dot{x} = Ax$ is not quadratically repulsive for $\mathcal{R}(x)$. Hence, we will discuss in detail the non-trivial case where $\tilde{H}_{aa}$ and $\tilde{H}_{bb}$ are both definite, in which case (37) implies that they are of opposite definiteness. Without loss of generality, let us assume that $\tilde{H}_{aa} > 0$ and $\tilde{H}_{bb} < 0$, and let us consider the change of basis $z = \tilde{T}y$ where $\tilde{T}$ has the following structure:

$$\tilde{T} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\psi) & \sin(\psi) \\ 0 & 0 & -\sin(\psi) & \cos(\psi) \end{bmatrix} \quad (47)$$

with θ chosen as (41) and ψ chosen as follows:

$$\psi = \frac{\arctan(2\tilde{h}_{34}/(\tilde{h}_{44} - \tilde{h}_{33}))}{2} \quad (48)$$

Then, in the y -coordinates, the system is described by (45) where z_1, z_2, z_3, z_4 are replaced by y_1, y_2, y_3, y_4 , respectively. The quadratic function $\mathcal{R}(z)$ can be computed as $\mathcal{R}(z) = \mathcal{R}(\tilde{T}y) = y^T \tilde{T}^T \tilde{H} \tilde{T} y = y^T \tilde{H} y$, with:

$$\tilde{H} = \begin{bmatrix} \tilde{h}_{11} & 0 & \tilde{h}_{13} & \tilde{h}_{14} \\ 0 & \tilde{h}_{22} & \tilde{h}_{23} & \tilde{h}_{24} \\ \tilde{h}_{13} & \tilde{h}_{23} & \tilde{h}_{33} & 0 \\ \tilde{h}_{14} & \tilde{h}_{24} & 0 & \tilde{h}_{44} \end{bmatrix} \quad (49)$$

where $\tilde{h}_{22} > 0$ and $\tilde{h}_{33} < 0$ due to $\tilde{H}_{aa} > 0$ and $\tilde{H}_{bb} < 0$, respectively. Consequently:

$$\begin{bmatrix} \tilde{h}_{22} & \tilde{h}_{23} \\ \tilde{h}_{23} & \tilde{h}_{33} \end{bmatrix} < 0 \quad (50)$$

which means that given a value $y_2 \neq 0$, there are two values y_3^+ and y_3^- of opposite sign that lead to $\mathcal{R}(\tilde{T}[0, y_2, y_3^+, 0]^T) = \mathcal{R}(\tilde{T}[0, y_2, y_3^-, 0]^T) = 0$. Further calculations show that:

$$\dot{\mathcal{R}}(\tilde{T}[0, y_2, y_3^+, 0]^T) = 2(\tilde{h}_{13}\omega_a - \tilde{h}_{24}\omega_b) y_2 y_3^+ \quad (51)$$

where the fact that $\mathcal{R}(\tilde{T}[0, y_2, y_3^+, 0]^T) = 0$ has been exploited to get rid of the terms that depend on σ . From (51), it is evident that one of the two values y_3^+ and y_3^- corresponds to $\dot{\mathcal{R}}(\tilde{T}[0, y_2, y_3^+, 0]^T) \geq 0$. Hence, we have proven the existence of a state $y^* \in \mathbb{R}^4 \setminus \{0\}$ such that $\mathcal{R}(y^*) = 0$ and $\dot{\mathcal{R}}(y^*) \geq 0$. Consequently, $x^* = T\tilde{T}y^*$ is such that $\mathcal{R}(x^*) = 0$ and $\dot{\mathcal{R}}(x^*) \geq 0$, which completes the proof. $\square$

We can now complete the proof of Theorem 2:

- (i) by considering the subset of states $x \in \mathbb{R}^n$ such that $x = [v_1 \ v_2] z$ with $z \in \mathbb{R}^2$, one gets that $\mathcal{R}(x) = \mathcal{R}([v_1 \ v_2] z) = z^T \tilde{H} z$ with $\tilde{H} = [v_1 \ v_2]^T H [v_1 \ v_2]$, where the dynamics of z is described by (31). If v_1 and v_2 were such that $\mathcal{R}(v_1)\mathcal{R}(v_2) < 0$, then $\tilde{H}$ would be indefinite and, according to Lemma 3, there exist states $z^* \in \mathbb{R}^2 \setminus \{0\}$, hence states $x^* = [v_1 \ v_2] z^* \in \mathbb{R}^n \setminus \{0\}$, such that $\mathcal{R}(x^*) = 0$ and $\dot{\mathcal{R}}(x^*) \geq 0$, which would make the system (1) not repulsive for $\mathcal{R}(x)$;
- (ii) by considering the subset of states $x \in \mathbb{R}^n$ such that $x = [\text{Re}(v) \ \text{Im}(v)] z$ with $z \in \mathbb{R}^2$, one gets that $\mathcal{R}(x) = \mathcal{R}([\text{Re}(v) \ \text{Im}(v)] z) = z^T \tilde{H} z$ with $\tilde{H} = [\text{Re}(v) \ \text{Im}(v)]^T H [\text{Re}(v) \ \text{Im}(v)]$, where the dynamics of z is described by (32). If v were such that $\mathcal{R}(v)\mathcal{R}(v^*) < 0$, then $\tilde{H}$ would be indefinite and, according to Lemma 4, there exist states $z^* \in \mathbb{R}^2 \setminus \{0\}$, hence states $x^* = [\text{Re}(v) \ \text{Im}(v)] z^* \in \mathbb{R}^n \setminus \{0\}$, such that $\mathcal{R}(x^*) = 0$ and $\dot{\mathcal{R}}(x^*) \geq 0$, which would make the system (1) not repulsive for $\mathcal{R}(x)$;
- (iii) by considering the subset of states $x \in \mathbb{R}^n$ such that $x = [\text{Re}(v_1) \ \text{Im}(v_1) \ v_2] z$ with $z \in \mathbb{R}^3$, one gets that $\mathcal{R}(x) = z^T \tilde{H} z$, where the dynamics of z is described by (38). If v_1 and v_2 were such that $\mathcal{R}(v_1)\mathcal{R}(v_2) < 0$, then it follows from Lemma 5 that there exist states $z^* \in \mathbb{R}^3 \setminus \{0\}$, hence states $x^* = [\text{Re}(v_1) \ \text{Im}(v_1) \ v_2] z^* \in \mathbb{R}^n \setminus \{0\}$, such that $\mathcal{R}(x^*) = 0$ and $\dot{\mathcal{R}}(x^*) \geq 0$, which would make the system (1) not repulsive for $\mathcal{R}(x)$;
- (iv) by considering the subset of states $x \in \mathbb{R}^n$ such that $x = [\text{Re}(v_1) \ \text{Im}(v_1) \ \text{Re}(v_2) \ \text{Im}(v_2)] z$ with $z \in \mathbb{R}^4$, one gets that $\mathcal{R}(x) = z^T \tilde{H} z$, where the dynamics of z is described by (45). If v_1 and v_2 were such that $\mathcal{R}(v_1)\mathcal{R}(v_2) < 0$, then it follows from Lemma 6 that there exist states $z^* \in \mathbb{R}^4 \setminus \{0\}$, hence states $x^* = [\text{Re}(v_1) \ \text{Im}(v_1) \ \text{Re}(v_2) \ \text{Im}(v_2)] z^* \in \mathbb{R}^n \setminus \{0\}$, such that $\mathcal{R}(x^*) = 0$ and $\dot{\mathcal{R}}(x^*) \geq 0$, which would make the system (1) not repulsive for $\mathcal{R}(x)$.

Appendix B

Proof of Theorem 3. (continued) The following lemmas are instrumental in completing the proof of Theorem 3. They establish that

when the matrix H is reduced by removing certain rows and columns, the eigenvalues of the resulting matrix $\tilde{H}$ lie within specific intervals determined by the eigenvalues of the original matrix H (Lemma 7). Furthermore, they show that quadratic repulsiveness of the system $\dot{x} = Ax$ for an indefinite quadratic form is incompatible with the following scenarios:

- the leftmost eigenvector of A lies in the region of the state-space where $\mathcal{R}(x) < 0$ (Lemma 8);
- the rightmost eigenvector of A lies in the region of the state-space where $\mathcal{R}(x) > 0$ (Lemma 9).

Lemma 7. Let $H \in \mathbb{S}^n$, and let $I \subseteq \{1, \dots, n\}$ with $\text{card}(I) = m$, $1 \leq m \leq n$. Define $\tilde{H}$ as the principal submatrix obtained from H by maintaining the rows and columns indexed by I . Then, the eigenvalues of H and $\tilde{H}$ satisfy:

$$\lambda_i(H) \geq \lambda_i(\tilde{H}) \geq \lambda_{i+m-m}(H), \quad i = 1, \dots, m \quad (52)$$

Proof. It follows from applying Poincaré separation theorem (Horn & Johnson, 2012) to H using vectors in the standard basis. $\square$

Lemma 8. Let $\tilde{H} \in \mathbb{S}^n$ be indefinite with $e_n^T \tilde{H} e_n < 0$ and no null eigenvalues, and let $\Lambda \in \mathbb{C}^{n \times n}$ be diagonal. Then, for the system $\dot{z} = Az$, there exists $z^* \in \mathbb{R}^n \setminus \{0\}$ so that $\mathcal{R}(z^*) = 0$ and $\dot{\mathcal{R}}(z^*) \geq 0$.

Proof. We can distinguish two cases: (i) $\lambda_n \in \mathbb{R}$; and (ii) $\lambda_n \in \mathbb{C}$. If $\lambda_n \in \mathbb{R}$, let us note that without any loss of generality we can assume that $e_n^T \tilde{H} e_n = -1$ (this is straightforwardly obtained by a proper scaling of the x_n -axis), which leads to $e_n^T \tilde{H} \Lambda e_n = -\lambda_n$. Let $v \in \mathbb{C}^n$ denote a unitary eigenvector of $\tilde{H}$ associated to one of its positive eigenvalues $\rho > 0$, i.e., $\tilde{H}v = \rho v$ with $v^T v = 1$, and let us consider the vector $y = \alpha e_n + v$, with $\alpha \in \mathbb{R}$ chosen so that $\mathcal{R}(y) = 0$. Since:

$$\begin{aligned} \mathcal{R}(y) &= \mathcal{R}(\alpha e_n + v) = (\alpha e_n + v)^T \tilde{H} (\alpha e_n + v) \\ &= \alpha^2 e_n^T \tilde{H} e_n + 2\alpha e_n^T \tilde{H} v + v^T \tilde{H} v = -\alpha^2 + 2\alpha e_n^T \tilde{H} v + \rho \end{aligned} \quad (53)$$

by virtue of Descartes' rule of sign, there exist two real solutions of opposite sign $\alpha_+ > 0$ and $\alpha_- < 0$. Moreover, it is clear from (53) that:

$$-\alpha^2 = -\rho - 2\alpha e_n^T \tilde{H} v \quad (54)$$

Then, one obtains:

$$\begin{aligned} \dot{\mathcal{R}}(y) &= \text{He} \{ y^T \tilde{H} \Lambda y \} = \text{He} \{ (\alpha e_n + v)^T \tilde{H} \Lambda (\alpha e_n + v) \} \\ &= \text{He} \{ \alpha^2 e_n^T \tilde{H} \Lambda e_n + \alpha e_n^T \tilde{H} \Lambda v + \alpha v^T \tilde{H} \Lambda e_n + v^T \tilde{H} \Lambda v \} \\ &= \text{He} \{ -\alpha^2 \lambda_n + \alpha(e_n^T \tilde{H} \Lambda v + \rho v^T \Lambda e_n) + \rho v^T \Lambda v \} \end{aligned} \quad (55)$$

Replacing (54) into (55) and recalling that $v^T v = 1$ leads to:

$$\dot{\mathcal{R}}(y) = \rho v^T \text{He} \{ \Lambda - \lambda_n I \} v + \alpha \text{He} \{ e_n^T (\tilde{H} \Lambda + \rho \Lambda^H - 2\lambda_n \tilde{H}) v \} \quad (56)$$

Note that either $\alpha = \alpha_+$ or $\alpha = \alpha_-$ will make the term $\alpha \text{He} \{ e_n^T (\tilde{H} \Lambda + \rho \Lambda^H - 2\lambda_n \tilde{H}) v \}$ non-negative. On the other hand $\text{He} \{ \Lambda - \lambda_n I \} \geq 0$, so that one can choose z^* as the vector y that makes (56) non-negative.

If $\lambda_n \in \mathbb{C}$, let us note that $\lambda_{n-1} = \lambda_n^*$ and, without any loss of generality, we can assume that

$$(e_{n-1} + e_n)^T \tilde{H} [e_{n-1} \quad e_n] = [-1/2 \quad -1/2]$$

(this is obtained by scaling and rotating the x_{n-1} - and x_n -axes of appropriate amounts). By similar arguments as previously stated, it follows that $(e_{n-1} + e_n)^T \tilde{H} \Lambda (e_{n-1} + e_n) = -\text{Re}(\lambda_n)$. By considering the vector $y = \alpha(e_{n-1} + e_n) + v$, with α chosen so that $\mathcal{R}(y) = 0$, one eventually finds that:

$$\dot{\mathcal{R}}(y) = \rho v^T \text{He} \{ \Lambda - \text{Re} \{ \lambda_n \} I \} v$$

$$+ \alpha \text{He} \{ (e_{n-1} + e_n)^T (\tilde{H} \Lambda + \rho \Lambda^H - 2\text{Re} \{ \lambda_n \} \tilde{H}) v \} \quad (57)$$

Following similar arguments as in the case $\lambda_n \in \mathbb{R}$, the fact that $\text{He} \{ \Lambda - \text{Re} \{ \lambda_n I \} \} \geq 0$ completes the proof. $\square$

Lemma 9. Let $\tilde{H} \in \mathbb{S}^n$ be indefinite with $e_1^T \tilde{H} e_1 > 0$ and no null eigenvalues, and let $\Lambda \in \mathbb{C}^{n \times n}$ be diagonal. Then, for the system $\dot{z} = Az$, there exists $z^* \in \mathbb{R}^n \setminus \{0\}$ so that $\mathcal{R}(z^*) = 0$ and $\dot{\mathcal{R}}(z^*) \geq 0$.

Proof. The proof mirrors that of Lemma 8, albeit some minor changes that are explicitly detailed hereafter for the case where $\lambda_1 \in \mathbb{R}$. The case $\lambda_1 \in \mathbb{C}$ is omitted. Without any loss of generality, we can assume that $e_1^T \tilde{H} e_1 = 1$ so that $e_1^T \tilde{H} \Lambda e_1 = \lambda_1$. By considering the vector $y = \alpha e_1 + v$, with α chosen such that $\mathcal{R}(y) = 0$, where $v \in \mathbb{C}^n$ denotes a unitary eigenvector of $\tilde{H}$ associated to one of its negative eigenvalues $\rho < 0$, one finds that:

$$\dot{\mathcal{R}}(y) = \rho v^T \text{He} \{ \Lambda - \lambda_1 I \} v + \alpha \text{He} \{ e_1^T (\tilde{H} \Lambda + \rho \Lambda^H - 2\lambda_1 \tilde{H}) v \} \quad (58)$$

where $\text{He} \{ \Lambda - \lambda_1 I \} \leq 0$. $\square$

We can now complete the proof of Theorem 3:

- (i) Let us assume that A has $\nu + 1$ eigenvectors such that $\mathcal{R}(v_i) = v_i^T H v_i < 0$, hereafter referred to, with a slight abuse of language, as the *negative* eigenvectors of A . By applying the change of variables $x = Ty$, we obtain the diagonalized form $\dot{y} = \Lambda y$. Moreover, $\mathcal{R}(x) = \mathcal{R}(Ty) = y^T H_T y$ where $H_T = T^T H T$ has the same inertia as H . Let $I_n \subseteq \{1, \dots, n\}$ with $\text{card}(I_n) = \nu + 1$ denote the indexes of vectors e_i in the transformed coordinates corresponding to the negative eigenvectors of A , and let $\tilde{\Lambda}_n$ and $\tilde{H}_n$ denote the principal submatrices obtained from Λ and H_T by maintaining the rows and columns indexed by I_n . According to Lemma 7:

$$\lambda_1(\tilde{H}_n) \geq \lambda_\pi(H_T) > 0 \quad (59)$$

so the matrix $\tilde{H}_n$ is indefinite. Moreover, $\tilde{H}_n$ has negative values on the diagonal, which means that all eigenvectors of $\tilde{\Lambda}_n$, among which the leftmost one, lie in the region where $\tilde{\mathcal{R}}_n(z_n) = z_n^T \tilde{H}_n z_n < 0$. Hence, the subsystem $\dot{z}_n = \tilde{\Lambda}_n z_n$ is not quadratically repulsive for $\tilde{\mathcal{R}}_n(z_n)$. In fact, according to Lemma 8, there exists $z_n^* \in \mathbb{R}^{\nu+1} \setminus \{0\}$ so that $\tilde{\mathcal{R}}_n(z_n^*) = 0$ and $\text{He} \{ z_n^{*T} \tilde{H}_n z_n^* \} \geq 0$. Let us consider the vector $y_n^* \in \mathbb{R}^n$ such that the $I_n(j)$ -th element is equal to the j th element of z_n for $j = 1, \dots, \nu + 1$ and the remaining elements of y_n^* are equal to zero. This vector induces another vector $x_n^* = T y_n^*$ for which, by construction, $\mathcal{R}(x_n^*) = 0$ and $\dot{\mathcal{R}}(x_n^*) \geq 0$, which contradicts the assumption of $\dot{x} = Ax$ being quadratically repulsive for $\mathcal{R}(x)$.

- (ii) Let us assume now that A has $\pi + 1$ positive eigenvectors v_i such that $\mathcal{R}(v_i) = v_i^T H v_i > 0$. By following a similar reason as described above, but with $I_p \subseteq \{1, \dots, n\}$, $\text{card}(I_p) = \pi + 1$, denoting the indexes of vectors e_i in the transformed coordinates corresponding to the positive eigenvectors of A , one obtains matrices $\tilde{\Lambda}_p$ and $\tilde{H}_p$ by maintaining the rows and columns indexed by I_p in Λ and H_T , respectively. According to Lemma 7:

$$0 > \lambda_{\pi+1}(H_T) \geq \lambda_{\pi+1}(\tilde{H}_p) \quad (60)$$

so the matrix $\tilde{H}_p$ is indefinite. Moreover, $\tilde{H}_p$ has positive values on the diagonal, which means that all eigenvectors of $\tilde{\Lambda}_p$, among which the rightmost one, lie in the region where $\tilde{\mathcal{R}}_p(z_p) = z_p^T \tilde{H}_p z_p > 0$. Hence, the subsystem $\dot{z}_p = \tilde{\Lambda}_p z_p$ is not quadratically repulsive for $\tilde{\mathcal{R}}_p(z_p)$. In fact, according to Lemma 9, there exists $z_p^* \in \mathbb{R}^{\pi+1} \setminus \{0\}$ so that $z_p^{*T} \tilde{H}_p z_p^* = 0$ and $\text{He} \{ z_p^{*T} \tilde{H}_p z_p^* \} \geq 0$. Let us consider the vector $y_p^* \in \mathbb{R}^n$ such that the $I_p(j)$ -th element is equal to the j th element of z_p for $j = 1, \dots, \pi + 1$ and the remaining elements of y_p^* are equal to zero. This vector induces another vector $x_p^* = T y_p^*$ for which, by construction, $\mathcal{R}(x_p^*) = 0$

and $\dot{\mathcal{R}}(x_p^*) \geq 0$, which contradicts the assumption of $\dot{x} = Ax$ being quadratically repulsive for $\mathcal{R}(x)$.

- (iii) If there were an eigenvector of A such that $\mathcal{R}(v_i) = 0$, then one could select $x(0) = v_i$ and obtain that the corresponding state trajectory $\tilde{x}(t)$ is such that $\mathcal{R}(\tilde{x}(t)) = 0$ for all $t \in \mathbb{R}_+$, which contradicts the repulsiveness assumption.

(i)-(iii) imply that A must have exactly ν negative eigenvectors and π positive eigenvectors of A . Let us now prove that the eigenvalues corresponding to the former eigenvectors cannot have smaller real parts than those corresponding to the latter eigenvectors. By considering the diagonalized form $\dot{y} = Ay$, let i_n and i_p denote the indexes of two vectors e_{i_n} and e_{i_p} corresponding to a negative and a positive eigenvector of A , respectively. The corresponding principal submatrix obtained from H_T by maintaining the rows and columns i_n and i_p is necessarily indefinite. If $\text{Re}(\lambda_{i_p}) \geq \text{Re}(\lambda_{i_n})$, then the conditions for the application of Lemmata 8–9 are (simultaneously) true, which contradicts the assumption of quadratic repulsiveness for $\dot{x} = Ax$. Hence, $\text{Re}(\lambda_{i_n}) > \text{Re}(\lambda_{i_p})$ and this inequality holds for each pair of eigenvalues $\lambda_{i_n}, \lambda_{i_p}$ associated to a negative and a positive eigenvector of A , which completes the proof.

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