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The effect of loss/gain and Hamiltonian perturbations of the Ablowitz—Ladik lattice on the recurrence of periodic anomalous waves

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Abstract

Using the finite gap method, in this paper we extend the recently developed perturbation theory for anomalous waves (AWs) of the periodic nonlinear Schrödinger (NLS) type equations to lattice equations, using as basic model the Ablowitz–Ladik (AL) lattices, integrable discretizations of the focusing and defocusing NLS equations. We study the effect of physically relevant perturbations of the AL equations, like linear loss, gain, and/or Hamiltonian corrections, on the AW recurrence, in the simplest case of one unstable mode. We show that these small perturbations induce $O(1)$ effects on the periodic AW dynamics, generating three distinguished asymptotic patterns. Since dissipation and higher order Hamiltonian corrections can hardly be avoided in natural phenomena involving AWs, we expect that the asymptotic states described analytically in this paper will play a basic role in the theory of periodic AWs

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in natural phenomena described by discrete systems. The quantitative agreement between the analytic formulas of this paper and numerical experiments is excellent.

Keywords: Ablowitz–Ladik lattice, perturbation theory, integrable lattice, finite-gap theory, anomalous waves, FPUT-recurrence

1. Introduction

The Ablowitz–Ladik (AL) equations [2, 3]:

$$
\begin{aligned}
& i\dot{u}_n + u_{n+1} + u_{n-1} - 2u_n + \eta |u_n|^2 (u_{n-1} + u_{n+1}) = 0, \ \ \eta = \pm 1, \\
& u_n = u(n,t) \in \mathbb{C}, \ \ \dot{u}_n = \frac{\mathrm{d}u_n(t)}{\mathrm{d}t}, \ \ n \in \mathbb{Z}, \ \ t \in \mathbb{R},
\end{aligned} \tag{1}
$$

are distinguished examples of integrable nonlinear differential-difference equations reducing, in the natural continuous limit

$$
u_n(t) = ihv(\xi, \tau), \ \ hn = \xi, \ \ \tau = h^2 t, \ \ h \to 0, \tag{2}
$$

to the celebrated integrable [75] nonlinear Schrödinger (NLS) equations

$$
\begin{aligned}
& iv_\tau + v_{\xi\xi} + 2\eta |v|^2 v = 0, \ \eta = \pm 1, \\
& v(\xi, \tau) \in \mathbb{C}, \ \ v_\tau = \frac{\partial v}{\partial \tau}, \ \ v_{\xi\xi} = \frac{\partial^2 v}{\partial \xi^2}, \ \ \xi, \tau \in \mathbb{R},
\end{aligned} \tag{3}
$$

where h is the lattice spacing. The two cases $\eta = \pm 1$ distinguish between the focusing ($\eta = 1$) and defocusing ($\eta = -1$) NLS regimes.

The AL equations (1) characterize [33] the quantum correlation function of the XY-model of spins [43]. If $\eta = 1$, the AL equation is relevant in the study of anharmonic lattices [65] and plays a role in the description of anomalous waves (AWs) in systems of closely spaced optical fibers. It is gauge equivalent to an integrable discretization of the Heisenberg spin chain [32], and appears in the description of a lossless nonlinear electric lattice ($\eta = 1$) [46]. At last, if $\eta = 1$, the AL hierarchy describes the integrable motions of a discrete curve on the sphere [18].

The well-known Lax pair of equations (1) reads [2, 3]

$$
\begin{aligned}
& \underline{\psi}_{n+1}(t,\lambda) = L_n(t,\lambda)\underline{\psi}_n(t,\lambda), \quad \dot{\underline{\psi}}_n(t,\lambda) = A_n(t,\lambda)\underline{\psi}_n(t,\lambda), \\
& L_n(t,\lambda) = \begin{pmatrix} \lambda & u_n(t) \\ -\eta \overline{u}_n(t) & \frac{1}{\lambda} \end{pmatrix}, \\
& A_n(t,\lambda) = i \begin{pmatrix} \lambda^2 - 1 + \eta u_n \overline{u}_{n-1} & \lambda u_n - \frac{u_{n-1}}{\lambda} \\ \eta \frac{\overline{u}_n}{\lambda} - \eta \lambda \overline{u}_{n-1} & 1 - \frac{1}{\lambda^2} - \eta \overline{u}_n u_{n-1} \end{pmatrix},
\end{aligned} \tag{4}
$$

where $\bar{f}$ is the complex conjugate of f, and the matrices L_n and A_n of the Lax pair (4) possess the two symmetry

$$
\begin{aligned}
L_n(\lambda) &= P_\eta \overline{L_n\left(\frac{1}{\bar{\lambda}}\right)} P_\eta^\dagger = -\sigma_3 L_n(-\lambda)\,\sigma_3, \\
A_n(\lambda) &= P_\eta \overline{A_n\left(\frac{1}{\bar{\lambda}}\right)} P_\eta^\dagger = \sigma_3 A_n(-\lambda)\,\sigma_3,
\end{aligned} \tag{5}
$$

where

$$
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P_\eta = \begin{pmatrix} 0 & -\eta \\ 1 & 0 \end{pmatrix}, \tag{6}
$$

implying that, if $\underline{\psi}_n(\lambda,t) = (\psi_{1n}(\lambda,t), \psi_{2n}(\lambda,t))^T$ is solution of (4), then

$$
\check{\underline{\psi}}_n(\lambda,t) = \begin{pmatrix} -\eta\overline{\psi_{2n}\left(\frac{1}{\bar{\lambda}},t\right)} \\ \overline{\psi_{1n}\left(\frac{1}{\bar{\lambda}},t\right)} \end{pmatrix}, \quad \hat{\underline{\psi}}_n(\lambda,t) = (-1)^n \begin{pmatrix} \psi_{1n}(-\lambda,t) \\ -\psi_{2n}(-\lambda,t) \end{pmatrix} \tag{7}
$$

are also a solution of (4). The inverse scattering transform of the AL equations (1) for localized initial data was developed in [2, 3] (see also [5]), and for non zero boundary conditions and $\eta = 1$ in [1, 10, 55, 57, 59, 60, 69]. The finite gap (FG) method [19, 34, 35, 40–42, 53] for periodic and quasi-periodic solutions was developed in [49].

Unlike the NLS case, for which the homogeneous background solution is unstable under the perturbation of waves with sufficiently small wave numbers κ in the focusing case $\eta = 1$ [8, 9, 73, 74], and always stable in the defocusing case $\eta = -1$, the instability properties of the homogeneous background solution

$$
a\exp\left(2i\eta|a|^2 t\right), \quad a \text{ complex constant parameter}, \tag{8}
$$

of the AL equations (1) are much richer, and can be summarized as follows.

$$
\begin{aligned}
&\text{If } \eta = 1, |\kappa| < \kappa_a := \arccos\left(\frac{1-|a|^2}{1+|a|^2}\right), \ \forall |a| > 0; \\
&\text{if } \eta = -1, |a| > 1, \ \forall \kappa,
\end{aligned} \tag{9}
$$

where κ_a is the smallest positive branch of arccos [6, 15, 56].

The exact AW solutions of the AL equations describing such nonlinear instability in the case of one and two unstable modes, are presented in [15, 51, 55]. In the case of one unstable mode they read

$$
\mathcal{N}(n,t;\kappa,X,T,\rho,\eta) = ae^{2i\eta|a|^2 t + i\rho}\,\frac{\cosh[\sigma(\kappa)(t-T) + 2i\eta\phi] + \eta G\cos[\kappa(n-X)]}{\cosh[\sigma(\kappa)(t-T)] - \eta G\cos[\kappa(n-X)]}, \tag{10}
$$

where

$$
\cos\phi = \sqrt{1+\frac{\eta}{a^2}}\,\sin\left(\frac{\kappa}{2}\right), \quad 0 < \phi < \pi/2, \tag{11}
$$

$$
G = \frac{\sin\phi}{\cos\left(\frac{\kappa}{2}\right)}, \tag{12}
$$

κ is the wave number,

$$
\sigma(\kappa) = 2\sqrt{(1+\eta|a|^2)(1-\cos\kappa)\left[(1+\eta|a|^2)\cos\kappa - (1-\eta|a|^2)\right]} \tag{13}
$$

is the growth rate of the linearized theory in the unstable cases (9), and X, T and ρ are arbitrary real parameters. Since ϕ is defined in terms of (κ, a, η) through (11), the growth rate $\sigma(\kappa)$ and the parameter G can be expressed in terms of (κ, a, η) or in terms of (ϕ, a, η) in the following way:

$$\begin{aligned}\sigma(\kappa) &= 2\sqrt{(1+\eta|a|^2)(1-\cos\kappa)\left[(1+\eta|a|^2)\cos\kappa-(1-\eta|a|^2)\right]}\\ &= 2|a|^2\sin(2\phi),\\ G &= \frac{\sin\theta}{\cos\left(\frac{\kappa}{2}\right)} = \sqrt{1-\frac{\eta}{|a|^2}\frac{1-\cos\kappa}{1+\cos\kappa}} = \frac{\sqrt{|a|^2+\eta}\sin\phi}{\sqrt{|a|^2\sin^2\phi+\eta}}.\end{aligned} \tag{14}$$

If $\eta = 1$, (10) is the Narita solution [51] (see also [6]), discrete analogue of the well-known Akhmediev breather (AB) [7] solution of focusing NLS. If $\eta = -1$, the solution (10) was found in [55], it describes the MI present if $|a| > 1$, $\forall\kappa\in\mathbb{R}$, and blows up at finite time (see also [15]). In this respect we remark that the terminology 'defocusing' NLS equation should not be exported to the AL equation in the regime $\eta = -1$ and $|a| > 1$, since in this case the AL equation is not close to the defocusing NLS equation, reducing in the continuous limit to a focusing NLS type equation with a weak nonlinear dispersion, its background solution is modulationally unstable with respect to monochromatic perturbations of any wave number, and a generic perturbation of it blows up at finite time [15].

The Cauchy problem of the periodic AWs of the focusing NLS equation (3) has been recently solved in [23, 24], to leading order and in terms of elementary functions, for generic periodic initial perturbations of the unstable background, in the case of a finite number N of unstable modes, using a suitable adaptation of the FG method, showing the relevance of the AB solution and of its N-mode generalization [36] in the description of the AW recurrence. See also [25] for an alternative approach to the study of the AW recurrence in the case of a single unstable mode, based on matched asymptotic expansions; see [26] for a finite-gap model describing the numerical instabilities of the AB and [27] for the analytic study of the linear, nonlinear, and orbital instabilities of the AB within the NLS dynamics; see [28] for the analytic study of the phase resonances in the AW recurrence; see [61] and [16] for the analytic study of the AW recurrence in other NLS type $1+1$ dimensional partial differential equations: respectively the PT-symmetric NLS equation [4] and the massive Thirring model [48, 66], showing the universality of the recurrence properties of periodic AWs, but also the specific differences present in different models. A generalization of the above results to multidimensions has been recently developed in [14, 29] on the Davey–Stewartson two equation [17], an integrable generalization of NLS to $2+1$ dimensions. The NLS recurrence of AWs in the periodic setting has been investigated in several numerical and real experiments, see, e.g. [39, 50, 58, 70, 71], and qualitatively studied via a three-wave approximation of NLS [31, 67].

In addition, a perturbation theory describing analytically how the Fermi–Pasta–Ulam–Tsingou (FPUT) recurrence [21] of NLS AWs is modified by the presence of a small loss or gain has been recently introduced in [11], showing the these perturbations induce $O(1)$ effects on the periodic AW dynamics, and giving a theoretical explanation of previous interesting real and numerical experiments [39, 64]. The qualitative physical explanation of these analytic results is as follows. In the finite time interval when the AW appears, the non integrable loss/gain perturbation generates a small correction to the unperturbed solution, becoming an $O(1)$ correction when the next AW appears, due to MI. This theory has been applied in [12] to the complex Ginzburg–Landau [52] and Lugiato–Lefever [44] models, treated as perturbations of NLS; see also [13]. The case of a Hamiltonian perturbation was postponed to a subsequent paper, since the leading order term in the main formula is identically zero in this case, and one should go to next order in the asymptotic expansion.

In order to generalize the above results to lattice models, in [15] we investigated the dynamics of AWs arising from the periodic Cauchy problem of the AL equations (1) corresponding to a small initial periodic perturbation of the unstable background (8):

$$\begin{aligned} &u_{n+N}(t) = u_n(t), \quad \forall n \in \mathbb{Z}, \ \forall t \geqslant 0, \\ &u_n(0) = a\left[1 + \epsilon\left(\sum_{j=1}^{p}\left(c_j e^{ik_j n} + c_{-j} e^{-ik_j n}\right) + c_0\right)\right], \quad 0 < \epsilon \ll 1, \end{aligned} \tag{15}$$

where

$$k_j = \frac{2\pi}{N} j, \quad 1 \leqslant j \leqslant p, \quad p := \begin{cases} \frac{N}{2}, & \text{if } N \text{ is even}, \\ \frac{N-1}{2}, & \text{if } N \text{ is odd}, \end{cases} \tag{16}$$

using the matched asymptotics technique introduced in [25]. In particular, if $\eta = 1$, the instability condition $|\kappa| < \kappa_a$ (9) implies that the first $M \leqslant p$ modes $\pm\kappa_j$, $1 \leqslant j \leqslant M$ are unstable, where

$$M := \left\lfloor \frac{N\kappa_a}{2\pi} \right\rfloor \tag{17}$$

and $\lfloor x \rfloor$ is the largest integer less or equal to $x \in \mathbb{R}$, and in the simplest case $M = 1$ ($2\pi/\kappa_a < N < 4\pi/\kappa_a$), the solution of the Cauchy problem (15) is described, to leading order, by the following expression, uniform in space-time, with $t \leqslant t^{(n)} + O(1)$:

$$u(x,t) = \sum_{j=0}^{n} \mathcal{N}\left(x, t; \kappa_1, x^{(j)}, t^{(j)}, \rho_j, 1\right) - a e^{2i|a|^2 t} \frac{1 - e^{4i\phi_1 n}}{1 - e^{4i\phi_1}} + O(\epsilon). \tag{18}$$

where

$$\begin{aligned} &x^{(n)} = x^{(1)} + (n-1)\Delta x, \ x^{(1)} = \frac{\arg(\alpha_1) + \pi/2}{\kappa_1}, \quad \text{mod } N, \\ &t^{(n)} = t^{(1)} + (n-1)\Delta t, \ \ t^{(1)} = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^2}{2|a|^2\epsilon|\alpha_1|\cos(\kappa_1/2)}\right), \\ &\rho_n = 2\phi_1 + (n-1)4\phi_1, \end{aligned} \tag{19}$$

$$\Delta x = \frac{\arg(\alpha_1\beta_1)}{\kappa_1}, \quad \text{mod } N, \qquad \Delta t = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^4}{4|a|^8\epsilon^2|\alpha_1\beta_1|\cos^2(\kappa_1/2)}\right), \tag{20}$$

$$\alpha_1 := \overline{c_1} e^{-i\phi_1} - c_{-1} e^{i\phi_1}, \ \ \beta_1 := \overline{c_{-1}} e^{i\phi_1} - c_1 e^{-i\phi_1}. \tag{21}$$

This is the analytic and quantitative description of the ideal FPUT recurrence of AWs of the AL_+ equation in term of the initial data through elementary functions. $x^{(1)}$ and $t^{(1)}$ are respectively the first appearance time and the position of the maximum of the absolute value of the AW; Δx is the x-shift of the position of the maximum between two consecutive appearances, and Δt is the time interval between two consecutive appearances. The quantitative agreement between formulas (18)–(21) and numerical experiments is excellent, being even better than the expected theoretical estimate of one or more orders of magnitude [15].

As in the NLS case, a very distinguished situation occurs when the initial data in (15) are such that $|c_1| = |c_{-1}| =: |c|$, corresponding to the case $\alpha_1\beta_1 \in \mathbb{R}$:

$$\alpha_1\beta_1 = 2|c|^2 [\cos\gamma - \cos(2\phi_1)] \in \mathbb{R}, \quad \gamma := \arg(c_1) + \arg(c_{-1}). \tag{22}$$

Indeed, if $\alpha_1\beta_1 > 0$, then $\Delta x = 0$ and the FPUT recurrence is periodic with period Δt. If $\alpha_1\beta_1 < 0$, then $\Delta x = N/2$ and the FPUT recurrence is periodic with period $2\Delta t$. Therefore, in terms of the initial data:

$$\begin{aligned} &|c_1| = |c_{-1}|,\ |\gamma| < 2\phi_1 \quad \Leftrightarrow \quad \alpha_1\beta_1 > 0, \\ &|c_1| = |c_{-1}|,\ |\gamma| > 2\phi_1 \quad \Leftrightarrow \quad \alpha_1\beta_1 < 0. \end{aligned} \tag{23}$$

Particularly interesting subcases are $u_n(0) \in \mathbb{R}$ and $u_n(0) \in i\mathbb{R}$ [15].

a) If $u_n(0) \in \mathbb{R}$, then $|c_1| = |c_{-1}|,\ \gamma = 0$, implying $\alpha_1\beta_1 > 0$, $\Delta x = 0$, and a periodic FPUT recurrence with period Δt.
b) If $u_n(0) \in i\mathbb{R}$, then $|c_1| = |c_{-1}|,\ |\gamma| = \pi$, implying $\alpha_1\beta_1 < 0$, $\Delta x = N/2$, and a periodic FPUT recurrence with period $2\Delta t$.

Although the conditions (23) are not generic with respect to the AL dynamics, as we shall see in the following, they describe the generic asymptotic state when the AL dynamics is perturbed by a small loss or gain, as in the NLS case.

In this paper we extend the NLS perturbation theory to the AL lattice, and exemplify this theory on three basic examples: a linear loss, a linear gain, and a quintic Hamiltonian perturbation, showing that these small perturbations induce $O(1)$ effects on the periodic AW dynamics, with the generation of three distinguished asymptotic patterns, and stressing the new features associated with the discrete case. Since dissipation and higher order Hamiltonian corrections can hardly be avoided in natural phenomena involving AWs, and since these perturbations induce $O(1)$ effects on the periodic AW dynamics, we expect that the asymptotic states described analytically in this paper will play a basic role in the theory of periodic AWs in natural phenomena described by discrete systems.

The paper is organized as follows. In section 2 we construct the system of gaps corresponding to the background solution (8) of the AL equations, and to a generic periodic perturbation of the background solution; in section 3 we construct the AW perturbation theory in the simplest case of one unstable mode, and we apply this theory to three relevant examples: a linear loss, a linear gain, and a quintic Hamiltonian perturbation, enabling one to study quantitatively the order one effects of these perturbations on the AL AW dynamics described by formulas (18)–(21). In the appendix we use the Darboux transformations of the AL equations to construct an important ingredient in the derivation of the formulas of this paper: the transition matrix of the background solution. The interesting problem of studying the AW dynamics of the physically relevant discrete NLS equations

$$i\dot{u}_n + u_{n+1} + u_{n-1} - 2u_n + \eta|u_n|^2 u_n = 0, \quad \eta = \pm 1, \tag{24}$$

viewed as a Hamiltonian perturbation of the AL lattices, is postponed to a subsequent paper.

2. The main spectrum of the perturbed background

Let $\Psi_n(t,\lambda)$ be a fundamental matrix solution of (4) for a periodic potential $u_n(t)$ of period N: $u_{n+N}(t) = u_n(t)$. Then it is well-known that the monodromy matrix

$$T(\lambda,t) := \Psi_{1+N}(t,\lambda)\Psi_1^{-1}(t,\lambda) = L_N(t,\lambda)L_{N-1}(t,\lambda)\dots L_1(t,\lambda) \tag{25}$$

satisfies the following properties.

i) $\det T = \prod_{j=1}^{N}\left(1+\eta|u_n(t)|^2\right)$is t- and λ-independent.

ii) $tr\ T$ is a finite Laurent expansion in λ with t-independent coefficients.

To define the main spectrum associated with the periodic AW Cauchy problem, we find it convenient to work in the gauge defined by

$$\underline{w}_n(\lambda,t) = \frac{1}{\prod_{j=0}^{n-1}\sqrt{1+\eta|u_j|^2}}\underline{\psi}_n(\lambda,t), \tag{26}$$

corresponding to the Lax pair (see the appendix of [49])

$$\begin{aligned}
&\underline{w}_{n+1}(\lambda,t) = \tilde{L}_n(t)\underline{w}_n(\lambda,t), \quad \dot{\underline{w}}_n(\lambda,t) = \tilde{A}_n(\lambda,t)\underline{w}_n(\lambda,t),\\
&\quad \tilde{L}_n = \frac{1}{\sqrt{1+\eta|u_n|^2}}L_n, \quad \Rightarrow \quad \det\tilde{L}_n = 1,\\
&\quad \tilde{A}_n = i\begin{pmatrix} \frac{1}{2}\left(\lambda-\frac{1}{\lambda}\right)^2 + \frac{\eta}{2}\left(u_n\overline{u}_{n-1}+\overline{u}_n u_{n-1}\right) & \lambda u_n - \frac{u_{n-1}}{\lambda}\\ -\lambda\eta\overline{u}_{n-1} + \eta\frac{\overline{u}_n}{\lambda} & -\frac{1}{2}\left(\lambda-\frac{1}{\lambda}\right)^2 - \frac{\eta}{2}\left(u_n\overline{u}_{n-1}+\overline{u}_n u_{n-1}\right)\end{pmatrix}.
\end{aligned} \tag{27}$$

The corresponding monodromy matrix $\tilde{T}$ is simply related to T:

$$\tilde{T}(\lambda,t) := \tilde{L}_N(\lambda,t)\tilde{L}_{N-1}(\lambda,t)\dots\tilde{L}_1(\lambda,t) = \frac{1}{\sqrt{\det T}}T(\lambda,t); \tag{28}$$

then

$$\det\tilde{T} = 1, \qquad \mathrm{tr}\tilde{T} = \frac{\mathrm{tr}T}{\sqrt{\det T}} \tag{29}$$

are also t-independent.

The eigenvalues of $\tilde{T}$:

$$\chi_\pm = \frac{\mathrm{tr}\tilde{T}(\lambda)}{2} \pm \sqrt{\left(\frac{\mathrm{tr}\tilde{T}(\lambda)}{2}\right)^2 - 1} \tag{30}$$

and its eigenvectors live on a two-sheeted covering Γ of the λ plane, due to the square root in (30), and play a crucial role in the construction of the Bloch eigenvectors, defined by the equations

$$\begin{aligned}
&\vec{w}_{n+1}^{B,\pm}(\gamma) = \tilde{L}_n(t)\vec{w}_n^{B,\pm}(\gamma),\\
&\vec{w}_{n+N}^{B,\pm}(\gamma) = \chi_\pm(\gamma)\vec{w}_n^{B,\pm}(\gamma), \quad \gamma\in\Gamma,
\end{aligned} \tag{31}$$

where the Floquet multipliers $\chi_\pm(\gamma)$ are just the eigenvalues of the monodromy matrix $\tilde{T}$. The main spectrum is defined by the condition

$$|\chi(\gamma)| = 1 \quad \Leftrightarrow \quad -2 \leqslant \mathrm{tr}\tilde{T}(\lambda) \leqslant 2, \tag{32}$$

and, generically, consists of disconnected curves (the bands) in the complex λ plane. The end points of the main spectrum, corresponding to periodic and anti-periodic Bloch eigenvectors and characterized by the conditions

$$\chi(\gamma) = \pm 1 \quad \Leftrightarrow \quad \mathrm{tr}\tilde{T} = \pm 2, \tag{33}$$

are the branch points and the multiple points (arising from the coalescence of two or more branch points). These end points are gauge independent and, in the original gauge (4), they are characterized instead by the condition

$$\mathrm{tr}T = \pm 2\sqrt{\det T}. \tag{34}$$

We remark that the symmetries (5) imply the relations

$$\mathrm{tr}\tilde{T}(\lambda) = (-\eta)^N \mathrm{tr}\tilde{T}\left(1/\overline{\lambda}\right) = -\mathrm{tr}\tilde{T}(-\lambda). \tag{35}$$

Consequently the main spectrum and its end points are invariant under the transformations $\lambda \to 1/\overline{\lambda}$ and $\lambda \to -\lambda$ (if λ belongs to the main spectrum, also $1/\overline{\lambda}, -\lambda, -1/\overline{\lambda}$ belong to the main spectrum). In particular, if λ is a branch point or a multiple point, then also $1/\overline{\lambda}, -\lambda, -1/\overline{\lambda}$ are respectively branch points or multiple points.

2.1. The main spectrum of the background

For the background solution $u_n^{[0]}(t) = ae^{2i\eta|a|^2 t}$ the above quantities are explicit. The n-periodic matrix fundamental solution of the Lax pair (27) reads:

$$W_n^{[0]}(t) = e^{i\left(\eta|a|^2 t + \frac{\arg a}{2}\right)\sigma_3} \begin{pmatrix} e^{s(\lambda)\exp(i\eta\mu)t} e^{i\eta\mu n} & e^{s(\lambda)\exp(-i\eta\mu)t} e^{-i\eta\mu n} \\ g_+(\lambda)\, e^{s(\lambda)\exp(i\eta\mu)t} e^{i\eta\mu n} & g_-(\lambda)\, e^{s(\lambda)\exp(-i\eta\mu)t} e^{-i\eta\mu n} \end{pmatrix}, \tag{36}$$

where λ and μ satisfy the equation

$$\cos\mu = \frac{1}{2\sqrt{1+\eta|a|^2}}\left(\lambda + \lambda^{-1}\right), \tag{37}$$

and

$$g_\pm(\lambda) = \frac{1}{|a|}\left(-\frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) \pm \sqrt{\frac{1}{4}\left(\lambda - \frac{1}{\lambda}\right)^2 - \eta|a|^2}\right),$$
$$s(\lambda) = i\sqrt{1+\eta|a|^2}\left(\lambda - \frac{1}{\lambda}\right). \tag{38}$$

From equations (37) and (11) it follows that

$$\sin\phi = \frac{1}{2|a|\sqrt{\eta}}\left(\lambda - \frac{1}{\lambda}\right), \quad \cos\phi = \frac{1}{|a|}\sqrt{|a|^2 - \eta\left(\frac{\lambda-\lambda^{-1}}{2}\right)^2},$$
$$\sin\mu = \frac{\sqrt{\eta|a|^2 - \left(\frac{\lambda-\lambda^{-1}}{2}\right)^2}}{\sqrt{1+\eta|a|^2}}, \tag{39}$$

implying

$$g_{\pm}(\lambda) = \pm\frac{i}{\sqrt{\eta}}e^{\pm i\phi}, \tag{40}$$

and

$$W_n^{[0]}(t) = e^{i\left(\eta|a|^2 t + \frac{\arg a}{2}\right)\sigma_3}\begin{pmatrix} e^{i\eta\mu n} & e^{-i\eta\mu n} \\ \frac{i}{\sqrt{\eta}}e^{i\eta(\mu n+\phi)} & -\frac{i}{\sqrt{\eta}}e^{-i\eta(\mu n+\phi)}\end{pmatrix}e^{-\frac{\sigma}{2}t\sigma_3 + i\nu t},$$
$$\nu = 2|a|\sqrt{\eta+|a|^2}\sin\phi\cos\mu t. \tag{41}$$

Then the monodromy matrix reads

$$\tilde{T}^{[0]}(\lambda,t) = W_{N+1}^{[0]}(t)\left(W_1^{[0]}(t)\right)^{-1} = W_N^{[0]}(t)\left(W_0^{[0]}(t)\right)^{-1}$$
$$= \frac{1}{\cos\phi}\begin{pmatrix} \cos(\mu N - \phi) & \frac{1}{\sqrt{\eta}}\sin(\mu N)e^{i\left(2\eta|a|^2 t + \arg a\right)} \\ -\sqrt{\eta}\sin(\mu N)e^{-i\left(2\eta|a|^2 t + \arg a\right)} & \cos(\mu N + \phi)\end{pmatrix} \tag{42}$$

and

$$\mathrm{tr}\tilde{T}^{[0]}(\lambda) = 2\cos(\mu N). \tag{43}$$

The definition (32) of the main spectrum implies that $\mu \in \mathbb{R}$ and, from (37):

$$\begin{aligned}
&\{\lambda \in \mathbb{R},\ 1/\lambda_0 \leqslant |\lambda| \leqslant \lambda_0\} \cup \{\lambda \in \mathbb{C},\ |\lambda| = 1\}, && \text{if } \eta = 1,\\
&\left\{\lambda = e^{i\theta},\ \theta \in \mathbb{R},\ \pi - \theta_m \leqslant |\theta| \leqslant \theta_m\right\}, && \text{if } \eta = -1,\ |a| < 1,\\
&\{\lambda = ip,\ p \in \mathbb{R},\ 1/p_0 \leqslant p \leqslant p_0\}, && \text{if } \eta = -1,\ |a| > 1,
\end{aligned} \tag{44}$$

where

$$\lambda_0 = \sqrt{1+|a|^2} + |a|, \ \ \theta_m = \arccos\left(\sqrt{1-|a|^2}\right), \ \ p_0 = \sqrt{|a|^2 - 1} + |a|. \tag{45}$$

The end points of the main spectrum, corresponding to $\mathrm{tr}\tilde{T}^{[0]} = \pm 2$, are characterized by

$$\mu_n = \frac{\pi}{N}n, \quad 0 \leqslant n \leqslant N. \tag{46}$$

From (37), for a given μ_n, $0 \leqslant n \leqslant N$, we have two values $\lambda_n^{\pm}$ of λ, corresponding to the $2N+2$ end points

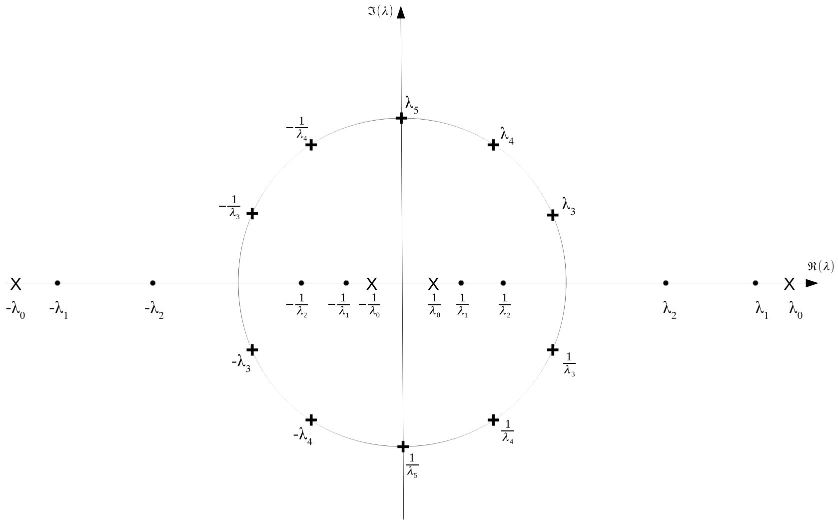

Figure 1. Spectrum in the complex λ plane associated with the background solution of the AL equation, with periodicity $N = 10$ and $a = 1$. The double points of the spectrum associated with the unstable modes are the black points on the real axis, those associated with the stable modes, indicated by the symbol '+', are located on the unit circle. The four branch points are real and indicated by the symbol 'X'.

$$\lambda_n^{\pm} = \sqrt{1+\eta|a|^2}\cos\mu_n \pm \sqrt{(1+\eta|a|^2)\cos^2\mu_n - 1}, \ \ 0 \leqslant n \leqslant N, \tag{47}$$

with the following symmetries:

$$\lambda_n^- = 1/\lambda_n^+ = -\lambda_{N-n}^+, \ \ 0 \leqslant n \leqslant N. \tag{48}$$

Since

$$\begin{aligned} \partial_\lambda \mathrm{tr}\tilde{T}_0(\lambda) &= \frac{N}{\sqrt{1+\eta|a|^2}} \frac{\left(1-\lambda^{-2}\right)\sin(\mu N)}{\sin\mu}, \\ \partial_\lambda^2 \mathrm{tr}\tilde{T}_0(\lambda) &= \frac{N}{\sqrt{1+\eta|a|^2}} \left\{ 2\frac{\sin(\mu N)}{\lambda^3 \sin\mu} + \left(1-\lambda^{-2}\right)\left[N\frac{\cos(\mu N)}{\sin\mu} - \frac{\sin(\mu N)\cos\mu}{\sin^2(\mu)}\right]\right\}, \end{aligned} \tag{49}$$

it follows that

$$\partial_\lambda \mathrm{tr}\tilde{T}_0(\lambda)\Big|_{\lambda_0^{\pm}} \neq 0, \ \ \partial_\lambda \mathrm{tr}\tilde{T}_0(\lambda)\Big|_{\lambda_N^{\pm}} \neq 0, \tag{50}$$

implying that the four points $\lambda_0^{\pm}$, $\lambda_N^{\pm}$, are branch points (indicated by the symbol 'X' in figure 1); it also follows that

$$\begin{aligned} &\partial_\lambda T_0(\lambda)\Big|_{\lambda_n^{\pm}} = 0, \\ &\partial_\lambda^2 T_0(\lambda)\Big|_{\lambda_n^{\pm}} = \frac{N^2}{\sqrt{1+\eta|a|^2}} \frac{(-1)^n}{\sin\mu_n}\left(1-\left(\lambda_n^{\pm}\right)^2\right) \neq 0, \quad n \neq 0, N, \end{aligned} \tag{51}$$

implying that the remaining end points are double points (the black points and those indicated by the symbol '+' in figure 1).

We have the following picture depending on η and $|a|$.

i) $\eta = 1$. The symmetries (48) suggest the following numeration rule

$$\lambda_n := \lambda_n^+, \quad 0 \leqslant n \leqslant \left\lfloor \frac{N}{2} \right\rfloor, \tag{52}$$

where

$$\begin{aligned} &\lambda_n > 1, \quad \lambda_n > \lambda_{n+1}, \quad 0 \leqslant n \leqslant M, \\ &\lambda_n = e^{i \arg \lambda_n}, \quad \arg \lambda_n < \arg \lambda_{n+1}, \quad M+1 \leqslant n \leqslant \left\lfloor \frac{N}{2} \right\rfloor, \end{aligned} \tag{53}$$

where M is the number of unstable modes, as in (17).

Then the remaining end points are the sets

$$\left\{ 1/\lambda_n,\ 0 \leqslant n \leqslant \left\lfloor \frac{N}{2} \right\rfloor \right\}, \left\{ -\lambda_n,\ 0 \leqslant n \leqslant \left\lfloor \frac{N}{2} \right\rfloor \right\}, \left\{ -1/\lambda_n,\ 0 \leqslant n \leqslant \left\lfloor \frac{N}{2} \right\rfloor \right\}. \tag{54}$$

If N is even, $\lambda_{\lfloor \frac{N}{2} \rfloor} = \lambda_{\frac{N}{2}} = i$, and in this case the four points reduce to two, since $-\lambda_{\frac{N}{2}} = 1/\lambda_{\frac{N}{2}}$.

We remark that the reality condition $(1+|a|^2)\cos^2\mu_n > 1$ for the end points in (47) is equivalent to the instability condition (9), with $\kappa_n = 2\mu_n$, Therefore the real double points

$$\{\lambda_n, 1/\lambda_n, -\lambda_n, -1/\lambda_n,\ 1 \leqslant n \leqslant n_c\}, \tag{55}$$

correspond to the unstable modes (the black points in figure 1), while the remaining double points, located on the unit circle (the points indicated by '+'), correspond to the stable modes.

2.2. *The main spectrum of a periodically perturbed background*

Here we concentrate on the construction of the end points of the main spectrum associated with a periodic perturbation of the background. As we have seen, apart from the branch points $\lambda_0, 1/\lambda_0, -\lambda_0, -1/\lambda_0$, the remaining end points of the main spectrum of the background are double points. Therefore a generic periodic perturbation of the background will resolve such a degeneration in agreement with the following basic principles of perturbation theory for doubly degenerate eigenvalues.

Consider the eigenvalue equation for the unperturbed operator $\mathcal{L}_0$:

$$\mathcal{L}_0 |\psi >= \lambda |\psi >, \tag{56}$$

where λ is a doubly degenerate eigenvalue corresponding to the two independent eigenfunctions $|f^{\pm} >$. Then the perturbed operator $\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_1$ resolves the degeneration in the following way:

$$\mathcal{L}|\psi, \pm >= \left(\lambda + \epsilon \Delta^{\pm}\right)|\psi, \pm >, \tag{57}$$

where $\Delta^{\pm}$ are the eigenvalues of the 2×2 matrix

$$V=\begin{pmatrix} <f^{+}|\mathcal{L}_1|f^{+}> & <f^{+}|\mathcal{L}_1|f^{-}> \\ <f^{-}|\mathcal{L}_1|f^{+}> & <f^{-}|\mathcal{L}_1|f^{-}> \end{pmatrix}, \tag{58}$$

made of the matrix elements of the perturbation $\mathcal{L}_1$ with respect to the invariant subspace $\{|f^{-}>,|f^{+}>\}$ associated with the eigenvalue λ, $<f^{\pm}|$ are the corresponding elements of the dual basis such that

$$<f^{-}|f^{-}>=<f^{+}|f^{+}>=1,\ \ <f^{\mp}|f^{\pm}>=<f^{\mp}|f>=0, \tag{59}$$

and $|f>$ is any other element of the basis of eigenfunctions of $\mathcal{L}_0$.

To apply this result to our case, we first observe that the first equation in (27) (the spectral problem) can be written as the eigenvalue problem [49]

$$\mathcal{L}_n\vec{w}_n=\lambda\vec{w}_n,\quad \mathcal{L}_n=\begin{pmatrix}\sqrt{1+\eta|u_n|^2}\mathbf{E} & -u_n\\ -\eta\bar{u}_{n-1} & \sqrt{1+\eta|u_{n-1}|^2}\mathbf{E}^{-1}\end{pmatrix}, \tag{60}$$

where $\mathbf{E}$ is the forward lattice shift: $\mathbf{E}f_n=f_{n+1}$. Let us specialize the previous formulas to the case $\eta=1$ and compute the effect of the initial monochromatic perturbation

$$u_n=a\left(1+\epsilon\left(c_je^{ik_jn}+c_{-j}e^{-i\kappa_jn}\right)\right) \tag{61}$$

on the spectrum discussed in the previous section.

Then $\mathcal{L}=\mathcal{L}_0+\epsilon\mathcal{L}_1+O(\epsilon^2)$, where

$$\mathcal{L}_0=\begin{pmatrix}\sqrt{1+|a|^2}\mathbf{E} & -a\\ -\bar{a} & \sqrt{1+|a|^2}\mathbf{E}^{-1}\end{pmatrix}, \tag{62}$$

$$\mathcal{L}_1=\begin{pmatrix}\frac{|a|^2}{2\sqrt{1+|a|^2}}\left[\left(c_j+\overline{c_j}\right)e^{2i\mu_jn}+\left(c_{-j}+\overline{c_j}\right)e^{-2i\mu_jn}\right]\mathbf{E} & -a\left(c_je^{2i\mu_jn}+c_{-j}e^{-2i\mu_jn}\right)\\ -\bar{a}\left(\overline{c_j}e^{-2i\mu_j(n-1)}+\overline{c_{-j}}e^{2i\mu_j(n-1)}\right) & \frac{|a|^2}{2\sqrt{1+|a|^2}}\left[\left(c_j+\overline{c_{-j}}\right)e^{2i\mu_j(n-1)}+\left(c_{-j}+\overline{c_j}\right)e^{-2i\mu_j(n-1)}\right]\mathbf{E}^{-1}\end{pmatrix}.$$

We use the following notation for the Bloch eigenfunctions (41) at $t=0$, in the generic double points $\lambda_j,\ 1/\lambda_j$:

$$|\lambda_j,\pm>=e^{i\frac{\arg a}{2}\sigma_3}\begin{pmatrix}1\\ \pm i\,e^{\pm i\phi_j}\end{pmatrix}e^{\pm i\mu_jn},\qquad |\lambda_j^{-1},\pm>=e^{i\frac{\arg a}{2}\sigma_3}\begin{pmatrix}1\\ \pm i\,e^{\mp i\phi_j}\end{pmatrix}e^{\pm i\mu_jn},$$

where

$$\cos\phi_j=\frac{\sqrt{1+|a|^2}}{|a|}\sin\mu_j,\quad \kappa_j=2\mu_j. \tag{63}$$

Then

$$\begin{aligned}V(\lambda_j)&=\begin{pmatrix}<+,\lambda_j|\,\mathcal{L}_1\,|\lambda_j,+> & <+,\lambda_j|\,\mathcal{L}_1\,|\lambda_j,-> \\ <-,\lambda_j|\,\mathcal{L}_1\,|\lambda_j,+> & <-,\lambda_j|\,\mathcal{L}_1\,|\lambda_j,->\end{pmatrix}\\ &=\frac{|a|\lambda_j}{2\sqrt{1+|a|^2}\sin\phi_j}\begin{pmatrix}0 & \beta_je^{-i\mu_j-i\phi_j}\\ \alpha_je^{i\mu_j+i\phi_j} & 0\end{pmatrix},\end{aligned}$$

its eigenvalues are $\Delta_j^{\pm} = \pm \frac{|a|}{2\sqrt{1+|a|^2}} \frac{\lambda_j \sqrt{\alpha_j \beta_j}}{\sin \phi_j}$, and the double point λ_j splits into the two branch points

$$E^{\pm}(\lambda_j) = \lambda_j \left(1 \pm \epsilon \frac{|a|}{2\sqrt{1+|a|^2}} \frac{\sqrt{\alpha_j \beta_j}}{\sin \phi_j} + O\left(\epsilon^2\right) \right), \tag{64}$$

generating the gap

$$E^{+}(\lambda_j) - E^{-}(\lambda_j) = \frac{\epsilon |a| \lambda_j \sqrt{\alpha_j \beta_j}}{\sqrt{1+|a|^2} \sin \phi_j}. \tag{65}$$

Proceeding in a similar way, one constructs the matrices $V(\lambda_j^{-1})$, $V(-\lambda_j)$,$V(-\lambda_j^{-1})$; for instance

$$V\left(\lambda_j^{-1}\right) = -\frac{|a|}{2\sqrt{1+|a|^2} \sin \phi_j \lambda_j} \begin{pmatrix} 0 & e^{i\phi_j - i\mu_j} \tilde{\beta}_j \\ e^{-i\phi_j + i\mu_j} \tilde{\alpha}_j & 0 \end{pmatrix},$$

where

$$\tilde{\alpha}_j = \overline{c_j} e^{i\phi_j} - c_{-j} e^{-i\phi_j}; \qquad \tilde{\beta}_j = \overline{c_{-j}} e^{-i\phi_j} - c_j e^{i\phi_j}.$$

Therefore the double point λ_j^{-1} splits into the two branch points

$$E^{\pm}\left(\lambda_j^{-1}\right) = \lambda_j^{-1} \left(1 \mp \frac{\epsilon |a|}{2\sqrt{1+|a|^2}} \frac{\sqrt{\tilde{\alpha}_j \tilde{\beta}_j}}{\sin \phi_j} + O\left(\epsilon^2\right) \right). \tag{66}$$

If $\lambda_j \in \mathbb{R}$ (the corresponding mode is unstable), the splitting is generic: $\alpha_j \beta_j$ is an arbitrary complex parameter and the corresponding gap $E^+ - E^-$ has an arbitrary inclination and length. In addition, since $\tilde{\alpha}_j \tilde{\beta}_j = \overline{\alpha_j \beta_j}$, the branch points associated with λ_j and λ_j^{-1} satisfy the symmetry relation

$$E^{\pm}\left(\lambda_j^{-1}\right) = \frac{1}{\overline{E^{\pm}(\lambda_j)}}. \tag{67}$$

If λ_j is on the unit circle: $\lambda_j = e^{i \arg \lambda_j}$ (the corresponding mode is stable), then $\phi_j \in i\mathbb{R}$ and

$$\cos \mu_j = \frac{\cos(\arg \lambda_j)}{\sqrt{1+|a|^2}}, \quad \sin \phi_j = i \frac{\sin(\arg \lambda_j)}{\sqrt{1+|a|^2}}, \quad \beta_j = -\overline{\alpha_j}; \tag{68}$$

therefore

$$E^{\pm}(\lambda_j) = \lambda_j \left(1 \pm \frac{\epsilon |a|}{2\sqrt{1+|a|^2}} \frac{|\alpha_j|}{\sin(\phi_j)} + O\left(\epsilon^2\right) \right). \tag{69}$$

It follows that the corresponding branch points are on the line from the origin to λ_j (the splitting is radial), and satisfy the relation (see figure 2)

$$E^{-}(\lambda_j) = \frac{1}{\overline{E^{+}(\lambda_j)}}. \tag{70}$$

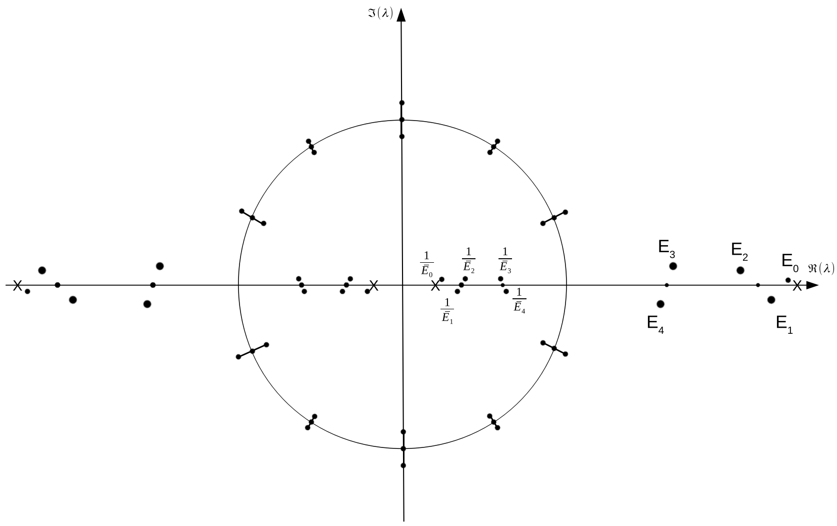

Figure 2. The effect of an initial generic periodic perturbation on the spectrum of figure 1. The degenerate double points are now open gaps. The ones corresponding to the double points on the unit circle are radial; the ones corresponding to double points on the real axis have arbitrary inclination.

We finally remark that, since the product $\alpha_1\beta_1$ plays a basic role in the FPUT recurrence of AL AWs, appearing in the definitions (20) of Δx and Δt, and since such a product appears also in the definition (65) of the gap $E^+(\lambda_1) - E^-(\lambda_1)$, it is possible to express Δx and Δt in terms of the gap, obtaining the spectral representation of the FPUT recurrence:

$$\Delta x = \frac{2}{\kappa_1} \arg\left(E^+(\lambda_1) - E^-(\lambda_1)\right),$$
$$\Delta t = \frac{1}{\sigma_1} \log\left(\frac{\sigma_1^4 \lambda_1^2}{4|a|^6 (1+|a|^2) \sin^2(\phi_1) \cos^2(\kappa_1/2) \left|E^+(\lambda_1) - E^-(\lambda_1)\right|^2}\right). \quad (71)$$

3. FPUT recurrence of AWs for the perturbed AL_+ equation

The main spectrum is a constant of motion with respect to the AL_+ time evolution. If one perturbs such evolution:

$$\dot{u}_n = i\left(u_{n+1} + u_{n-1} - 2u_n + |u_n|^2 (u_{n-1} + u_{n+1})\right) + f[u_n], \quad |f[u_n]| \ll 1, \quad (72)$$

or, in matrix form

$$\dot{L}_n = A_{n+1} L_n - L_n A_n + F[u_n], \quad F[u_n] = \begin{pmatrix} 0 & f[u_n] \\ -\overline{f[u_n]} & 0 \end{pmatrix}, \quad (73)$$

the main spectrum evolves in time generically in a non integrable fashion, and the theory of perturbations of integrable nonlinear evolution equations is the proper tool to have an analytic description of the effect of such a small perturbation on the dynamics under scrutiny.

From the variation of the monodromy matrix T in (25):

$$\delta T(\lambda,t) = \sum_{n=1}^{N} L_N \dots L_{n+1} (\delta L_n) L_{n-1} \dots L_1,$$
$$\delta L_n = \begin{pmatrix} 0 & \delta u_n \\ -\overline{\delta u_n} & 0 \end{pmatrix}, \tag{74}$$

it follows that the trace of such a variation can be expressed in terms of the so-called transition matrix

$$\hat{T}(n,m,\lambda,t) := \Psi_n(t,\lambda)\Psi_m^{-1}(t,\lambda) = L_{n-1}L_{n-2}\dots L_m \tag{75}$$

as follows

$$\begin{aligned}\mathrm{tr}(\delta T(\lambda,t)) &= \sum_{n=1}^{N} \mathrm{tr}(L_{n-1}\dots L_1 L_N \dots L_{n+1}(\delta L_n)) \\ &= \sum_{n=1}^{N} \mathrm{tr}(L_{N+n-1}\dots L_{n+1}(\delta L_n)) = \sum_{n=1}^{N} \mathrm{tr}\left(\hat{T}(N+n,n+1,\lambda,t)\,\delta L_n\right),\end{aligned} \tag{76}$$

where we have used first the permutation properties of the trace and then the periodicity of L_n.

Equation (76) is valid for any variation; in our case $\delta u_n(t) = \dot{u}_n \mathrm{d}t$, then $\delta T = \dot{T}\mathrm{d}t$, $\delta L_n = \dot{L}_n \mathrm{d}t$, and equation (76) becomes

$$\begin{aligned}\mathrm{tr}\left(\dot{T}(\lambda,t)\right) &= \sum_{n=1}^{N} \mathrm{tr}\left(\hat{T}(n+N,n+1,\lambda,t)\,\dot{L}_n\right) \\ &= \sum_{n=1}^{N} \mathrm{tr}\left(\hat{T}(n+N,n+1,\lambda,t)\,[A_{n+1}L_n - L_nA_n + F[u_n]]\right) \\ &= \sum_{n=1}^{N} \mathrm{tr}\left(\hat{T}(n+N,n+1,\lambda,t)\,F[u_n]\right) = \sum_{n=1}^{N} \mathrm{tr}\left(\hat{T}(n+N,n,\lambda,t)\,L_n^{-1}F[u_n]\right),\end{aligned} \tag{77}$$

where we have used first the fact that the AL vector field $A_{n+1}L_n - L_nA_n$ is not responsible for the time evolution of tr$T$, and second the definitions (74) and (75) of δL_n and of the transition matrix.

Recalling the relation (29) between the trace of T and that of $\tilde{T}$, we obtain the time derivative of tr$\tilde{T}$ in terms of the perturbation:

$$\left(\mathrm{tr}\tilde{T}\right)_t = \left(\frac{\mathrm{tr}(T(\lambda,t))}{\sqrt{\det T(\lambda,t)}}\right)_t = \frac{\mathrm{tr}\left(\sum_{n=1}^{N}\left[2\hat{T}(n+N,n,\lambda,t) - \mathrm{tr}T(\lambda,t)\right]L_n^{-1}F[u_n]\right)}{2\sqrt{\det T(t)}}. \tag{78}$$

Equation (78) describes in a rather implicit and nonlinear way the time evolution of the main spectrum, due to the nonintegrable perturbation $F[u_n]$. A crucial simplification arises from the observation that, in the AW recurrence, during the linear stages of MI, characterized by the background solution, tr$\tilde{T}$ is essentially constant (see figures 3 and 6 below). Therefore the variation takes place in the finite intervals in which the AWs appear, described to leading order by the AW solution, and since the Narita solution is exponentially localized in time, one can replace the integral over the time interval of appearance by an integral over the real time axis.

In conclusion the variation of $\mathrm{tr}\tilde{T}$ at each appearance of the AW is described by the following basic formula of the perturbation theory

$$\Delta\left(\mathrm{tr}\tilde{T}(\lambda)\right) = \int_{-\infty}^{\infty} \frac{\mathrm{tr}\left(\sum_{n=1}^{N}\left[2\hat{T}(n+N,n,\lambda,t) - \mathrm{tr}T(\lambda,t)\right]L_n^{-1}F\left[u_n(t)\right]\right)}{2\sqrt{\det T(\lambda,t)}}\mathrm{d}t, \tag{79}$$

where all the quantities inside the integral (the transition and monodromy matrices, L_n and u_n) correspond to the AW solution (10). For instance (see the appendix for details):

$$\hat{T}(n+N,n;\lambda_1)$$
$$= \left(1+|a|^2\right)^{\frac{N}{2}}\left[-1+\frac{2N|a|\sqrt{1+|a|^2}\sin^2\phi\cos\frac{k}{2}\,g_n(t)}{\cos^2\phi\,H_n(t)}\begin{pmatrix}q_1(n)\overline{q_2(n)} & -\overline{q_2(n)q_2(n)}\\ \overline{q_1(n)q_1(n)} & -\overline{q_1(n)}q_2(n)\end{pmatrix}\right], \tag{80}$$

where

$$q_{1_n} = 2\cos\left(\frac{\kappa(n-x_1)+i\eta\sigma(t-t_1)-\phi}{2}\right)e^{i\eta|a|^2t+i\frac{\arg a}{2}+i\nu t},$$
$$q_{2_n} = -2\sqrt{\eta}\sin\left(\frac{\kappa(n-x_1)+i\eta\sigma(t-t_1)+\phi}{2}\right)e^{-i\eta|a|^2t-i\frac{\arg a}{2}+i\nu t}, \tag{81}$$

and

$$g_n(t) = \overline{q_1(n)q_1(n)}\,e^{-2i|a|^2t} + q_2(n)q_2(n)\,e^{2i|a|^2t} + 2q_1(n)q_2(n)\sin\phi = 4\cos^2\phi\,, \tag{82}$$

$$H_n(t) = \left(\frac{|q_1(n)|^2}{\lambda_1}+\lambda_1|q_2(n)|^2\right)\left(\lambda_1|q_1(n)|^2+\frac{|q_2(n)|^2}{\lambda_1}\right)$$
$$= 16\left(1+|a|^2\right)\left[\cos\frac{k}{2}\cosh\left(\sigma(t-t_1)\right)-\cos\left(k(n-x_1)\right)\sin\phi\right]$$
$$\times\left[\cos\frac{k}{2}\cosh\left(\sigma(t-t_1)\right)-\cos\left(k(n-x_1-1)\right)\sin\phi\right]. \tag{83}$$

On the other hand, reasoning as in [11], for $\lambda\sim\lambda_1$:

$$\mathrm{tr}\tilde{T}(\lambda)\sim\frac{\mathrm{tr}\,T(\lambda_1)}{\sqrt{\det T}}+\left(\frac{N\sin\phi_1}{\lambda_1\cos\phi_1}\right)^2(\lambda-\lambda_1)^2. \tag{84}$$

Evaluating this formula at $\lambda=E^+(\lambda_1)$ and recalling that $E^+(\lambda_1)-E^-(\lambda_1)=2(E^+(\lambda_1)-\lambda_1)$ and $\mathrm{tr}\tilde{T}(E^+(\lambda_1))=-2$, we have

$$\mathrm{tr}\tilde{T}(\lambda_1)=\frac{Tr\,T(\lambda_1)}{\sqrt{\det T}}\sim -2-\left(\frac{N\sin\phi_1}{2\lambda_1\cos\phi_1}\right)^2\left(E^+(\lambda_1)-E^-(\lambda_1)\right)^2, \tag{85}$$

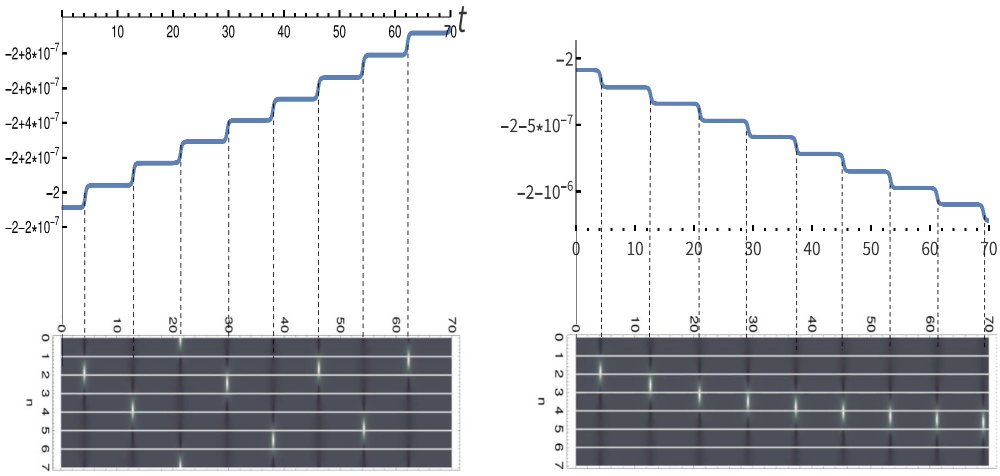

Figure 3. Dynamics of the real part $\Re(\frac{Tr(T)}{\sqrt{\det T}})$ of $Tr(\tilde{T})$, in the presence of the small loss $\nu = 10^{-8}$ or gain $\nu = -10^{-8}$. The variation corresponds to the AW appearance. The initial condition is: $u_n(0) = a(1+\epsilon(c_+ e^{ik_1 n} + c_- e^{-ik_1 n}))$, with $a = 1.1$, $\epsilon = 10^{-4}$, $k_1 = \frac{2\pi}{N}$, $N = 7$, $c_+ = 0.53 - i\,0.86$ and $c_- = -0.26 + i\,0.22$.

implying that

$$\Delta\left(\mathrm{tr}\tilde{T}(\lambda_1)\right) \sim -\left(\frac{N\sin\phi_1}{2\lambda_1\cos\phi_1}\right)^2 \Delta\left(E^+(\lambda_1) - E^-(\lambda_1)\right)^2 . \tag{86}$$

Equations (79) and (86) complete the perturbation theory. In the following we apply it to three physically relevant perturbations, a linear loss, a linear gain, and a quintic Hamiltonian perturbation.

3.1. Linear loss or gain perturbations

If the perturbation is a linear loss or gain:

$$\begin{aligned} &f[u_n] = \nu u_n, \quad |\nu| \ll 1, \quad \text{loss}: \ \nu < 0; \quad \text{gain}: \ \nu > 0, \\ &\quad \Rightarrow \quad F[u_n] = \nu U_n(t), \quad U_n := \begin{pmatrix} 0 & u_n \\ -\bar{u}_n & 0 \end{pmatrix}, \end{aligned} \tag{87}$$

we can express the variation of $\mathrm{tr}\tilde{T}$ after the jth appearance of the AW, through the analytic formula:

$$\begin{aligned} \Delta_j\left(\mathrm{tr}\tilde{T}\right) &= \Delta_j\left(\frac{Tr(T)}{\sqrt{\det T}}\right) = \frac{2N\sqrt{1+|a|^2}\sin^2\phi\cos\frac{k}{2}}{\cos^2\phi}\int_{-\infty}^{\infty}\sum_{n=1}^{N}\frac{g_n(t)\,h_n(t)}{H_n(t)\,(1+|u_n(t)|^2)}\mathrm{d}t \\ &= -\nu\frac{2N|a|^2\sin^2\phi\cos\left(\frac{k}{2}\right)}{(1+|a|^2)}I(\phi, x_j), \end{aligned} \tag{88}$$

where $g_n(t), H_n(t)$ are defined in (82),

$$\begin{aligned} h_n(t) &= \lambda_1 \overline{u}_n \overline{q_2}^2 + \frac{u_n}{\lambda_1} \overline{q_1}^2 \\ &= -4|a|\sqrt{1+|a|^2} \left(\cos\frac{k}{2}\cos(2\phi) + \cos(k(n-x_j) - i\sigma t)\sin\phi\right) \\ &\quad \times \frac{\left(\cos\frac{k}{2}\cosh(\sigma t) - \cos(k(n-x_j-1))\sin\phi\right)}{\left(\cos\left(\frac{k}{2}\right)\cosh(\sigma(t-t_1)) - \sin\phi\cos(k(n-x_j))\right)}, \end{aligned}$$

and

$$I(\phi, x_j) = \int_{-\infty}^{\infty} \left[\sum_{n=1}^{N} \frac{\cos\frac{\kappa}{2}\cos(2\phi) + \sin\phi\cos(\kappa_1(n-x_j))\cosh(\sigma_1 t)}{\left[\cos\frac{k}{2}\cosh(\sigma t) - \cos(k(n-x_j+1))\sin\phi\right]\left[\cos\frac{k}{2}\cosh(\sigma t) - \cos(k(n-x_j-1))\sin\phi\right]}\right] \mathrm{d}t\,.$$

Note that $1+|u_n(t)|^2$ can be written as:

$$\begin{aligned} 1+|u_n|^2 &= \left(1+|a|^2\right)\left[\cos\left(\frac{k}{2}\right)\cosh(\sigma(t-t_1)) - \sin\phi\cos(k(n-x_1+1))\right] \\ &\quad \times \frac{\left[\cos\left(\frac{k}{2}\right)\cosh(\sigma(t-t_1)) - \sin\phi\cos(k(n-x_1-1))\right]}{\left[\cos\left(\frac{k}{2}\right)\cosh(\sigma(t-t_1)) - \sin\phi\cos(k(n-x_1))\right]^2}. \end{aligned}$$

Comparing equations (88) and (86) we infer that the effect of the jth AW appearance is to modify the gap according to the formula:

$$\Delta_j\left(E^+(\lambda_1) - E^-(\lambda_1)\right)^2 = \nu\frac{8|a|^2\lambda_1^2\cos^2\phi\cos\frac{\kappa_1}{2}}{N(1+|a|^2)} I\left(\phi_1, x^{(j)}\right),$$

that, combined with

$$\left(E^+(\lambda_1) - E^-(\lambda_1)\right)^2\Big|_{t=0} = \frac{\epsilon^2|a|^2\lambda_1^2\alpha_1\beta_1}{(1+|a|^2)\sin^2\phi_1} \tag{89}$$

provides the analytic formula for the position of the gap after the nth appearance in terms of the initial data:

$$\left(E^+(\lambda_1) - E^-(\lambda_1)\right)_n^2 = \frac{\epsilon^2|a|^2\lambda_1^2\alpha_1\beta_1}{(1+|a|^2)\sin^2\phi_1} + \nu\frac{8|a|^2\lambda_1^2\cos^2\phi\cos\frac{\kappa_1}{2}}{N(1+|a|^2)}\sum_{j=1}^{n} I\left(\phi, x^{(j)}\right). \tag{90}$$

We can also define the useful sequence of complex numbers $\{Q_n\}$

$$Q_n = \frac{\left(1+|a|^2\right)\sin^2\phi_1}{\epsilon^2|a|^2\lambda_1^2}\left(E^+(\lambda_1) - E^-(\lambda_1)\right)_n^2, \qquad Q_0 = \alpha_1\beta_1, \tag{91}$$

where the subscript indicates the index of the AW appearence, obtaining

$$Q_n = \alpha_1\beta_1 + \frac{\nu}{\epsilon^2}\frac{8\cos^2(\phi)\sin^2(\phi)\cos\frac{\kappa_1}{2}}{N}\sum_{j=1}^{n} I\left(\phi_1, x^{(j)}\right), \quad n \geqslant 0. \tag{92}$$

From (71) and (91) we can conveniently express the x-shifts and recurrence times $\Delta_n t$ in terms of the sequence $\{Q_n\}$ of complex numbers:

$$\Delta_n t = t^{(n+1)} - t^{(n)} = \frac{1}{\sigma_1} \log\left(\frac{\sigma_1^4}{4\epsilon^2 |a|^8 |Q_n| \cos^2 \frac{\kappa_1}{2}}\right),$$
$$\Delta_n x = x^{(n+1)} - x^{(n)} = \frac{\arg(Q_n)}{\kappa_1}. \tag{93}$$

Summarizing the results of this section, the FPUT recurrence in the presence of a small loss or gain is described as follows.

Main result. *Consider the periodic Cauchy problem* (15) *with period $N \in \mathbb{N}^+$ for the AL equation perturbed by a small linear loss or gain*

$$i\dot{u}_n - 2u_n + \left(1 + \eta |u_n|^2\right)(u_{n-1} + u_{n+1}) = i\nu u_n, \quad \nu \in \mathbb{R},\ |\nu| \ll 1, \tag{94}$$

in the finite time interval $[0, T]$, in the simplest case of one unstable mode only ($\frac{2\pi}{\kappa_a} < N < \frac{4\pi}{\kappa_a}$). Then the solution is given, to leading order and up to $O(\epsilon)$ corrections, by the same analytic expression describing the AL FPUT recurrence (18):

$$u(x,t) = \sum_{j=0}^{n} \mathcal{N}\left(x, t; \kappa_1, x^{(j)}, t^{(j)}, \rho_j, 1\right) - a e^{2i|a|^2 t} \frac{1 - e^{4i\phi_1 n}}{1 - e^{4i\phi_1}} + O(\epsilon), \tag{95}$$

where the position and time of the first appearance $(x^{(1)}, t^{(1)})$ of the AW, described by the Narita solution (10)*, are essentially the same as in the unperturbed case and described by equations* (19) *and* (21)*, while the position and time of the subsequent appearances of the AWs, again described by the Narita solution* (10)*, are given by formulas* (93) (see figure 4).

We first remark that, as for the NLS case, to obtain these results, we have implicitly assumed that

$$|\nu| T,\ |a|^2 |\nu| T^2 \ll 1, \tag{96}$$

and the meaning of these conditions can be explained observing that the background solution

$$\tilde{u}_0(t, \nu) = a \exp(-\nu t) \exp\left(i \frac{|a|^2}{\nu} (1 - \exp(-2\nu t))\right) \tag{97}$$

of (94) behaves as follows

$$\tilde{u}_0(t, \nu) = a \exp(\nu t) \exp\left(2i|a|^2 t \left(1 + \nu t + O(\nu t)^2\right)\right), \quad |\nu| \ll 1. \tag{98}$$

Therefore the amplitude and the oscillation frequency of the background slowly decrease if $\nu < 0$ (loss), and slowly increase if $\nu > 0$ (gain). The condition $|\nu| T \ll 1$ means that we can neglect the slow decay/growth of the amplitudes of the background and of the AWs; the condition $|a|^2 |\nu| T^2 \ll 1$ means that we can neglect the slow decay/growth of the oscillation frequency and its effects. In particular, a can be treated as a constant parameter under the above assumptions.

We also remark that, *unlike the unperturbed AL case, and unlike the perturbed NLS case, the x-shifts $\Delta_n x$ and the recurrence times $\Delta_n t$ after the nth appearance depend, through Q_n, on the positions $x^{(j)}$, $j = 1, \ldots, n$ of the first n AW appearances.*

In addition, since the functions $I(\phi_1, x^{(j)})$ are real and positive, the second term in the expression (92) of Q_n consists of a sum of positive terms in the case of a small gain ($\nu > 0$), or of negative terms in the case of a small loss ($\nu < 0$), and this second term becomes dominant in the sum:

$$Q_n \sim \frac{\nu}{\epsilon^2} \frac{8\cos^2(\phi)\sin^2(\phi)\cos\frac{\kappa_1}{2}}{N} \sum_{j=1}^{n} I\left(\phi_1, x^{(j)}\right), \quad n \gg 1 \tag{99}$$

as n increases. Therefore the AW recurrence tends to a lower dimensional asymptotic state (an attractor) characterized by the condition

$$\begin{aligned} &\Delta x = 0, \quad \text{if} \quad \nu > 0, \text{ small gain}, \\ &\Delta x = \frac{N}{2}, \quad \text{if} \quad \nu < 0, \text{ small loss}. \end{aligned} \tag{100}$$

From (92) we distinguish three different situations.

1) If $|\nu| \gg \epsilon^2$, after the first appearance, essentially the same as in the unperturbed case, the dynamics enters immediately one of the two attractors, depending on the sign of ν (see figures 4(a) and (e)).
2) If $|\nu| = O(\epsilon^2)$, after the first appearance, essentially the same as in the unperturbed case, the dynamics reaches the above attractors after a suitable transient (see figures 4(b) and (d)).
3) If $|\nu| \ll \epsilon^2$, the dynamics is essentially the same as in the unperturbed case.

To have an idea on how good this analytic theory is in describing the FPUT recurrence of AWs in the presence of loss or gain, in the following table we compare the numerical output of the experiment described in figure 4(b) ($\epsilon = 10^{-4}, \nu = -10^{-8}$) with the above theoretical predictions:

	Numeric	Theor
$(\Delta X_1, \Delta T_1)$	(2.010 914, 8.671 074)	(2.010 913, 8.671 072)
$(\Delta X_2, \Delta T_2)$	(2.631 1495, 8.529 947)	(2.631 1491, 8.529 944)
$(\Delta X_3, \Delta T_3)$	(2.919 0295, 8.380 007)	(2.919 0291, 8.380 003)
$(\Delta X_4, \Delta T_4)$	(3.071 6356, 8.256 583)	(3.071 6351, 8.256 578)
$(\Delta X_5, \Delta T_5)$	(3.162 6266, 8.155 82)	(3.162 6261, 8.156 81)

The worst difference between numerics and the theory is in the 5th decimal digit, while the expected error is $O(\epsilon) = O(10^{-4})$. Therefore this perturbation theory does even better than expected from theoretical arguments.

From (90) it follows that

$$\left(E^+(\lambda_1) - E^-(\lambda_1)\right)_n^2 \to \nu \frac{8|a|^2\lambda_1^2\cos^2\phi\cos\frac{\kappa_1}{2}}{N(1+|a|^2)} \sum_{j=1}^{n} I\left(\phi, x^{(j)}\right) \tag{101}$$

as n increases. Therefore the gap, whose initial inclination is arbitrary, tends to become horizontal in $\nu > 0$ (gain), and vertical if $\nu < 0$ (loss) (see figure 5).

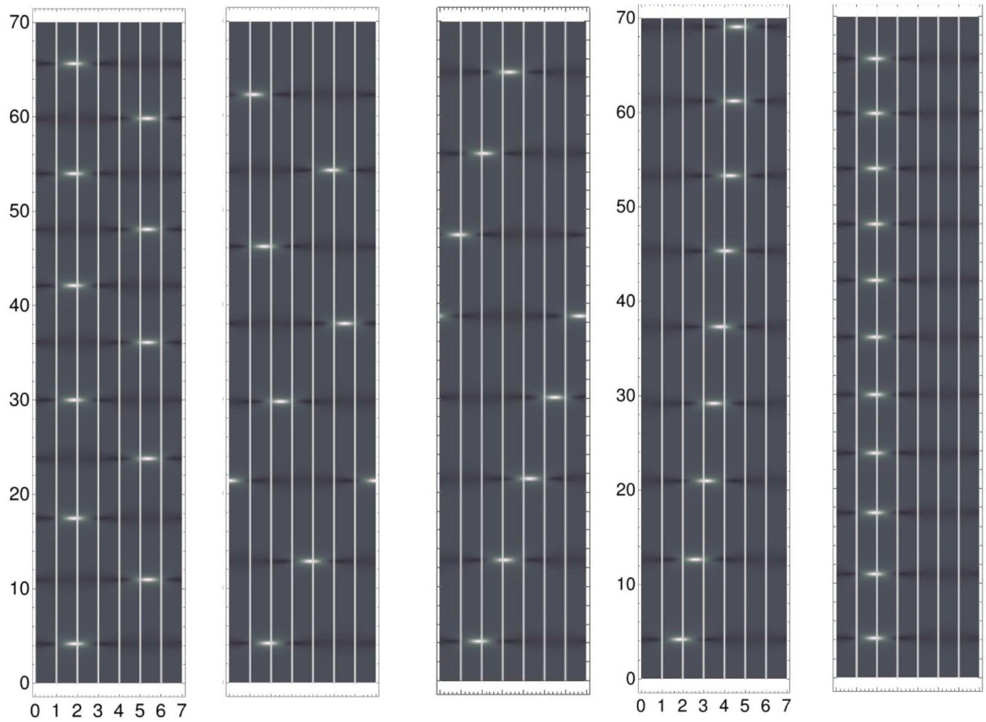

Figure 4. Density plot of $|u_n(t)|$ with a small loss/gain. Initial condition $u_n(0) = a(1+\epsilon(c_+e^{ikn}+c_-e^{-ikn}))$, where $N=7$, $a=1.1$, $\epsilon=10^{-4}$, $c_+=0.53-i0.86$ and $c_-=-0.26+i0.22$. The integration is performed using the 6th order Runge-Kutta [62]. a) $\nu=-10^{-6}$; the system enters immediately the attractor of the loss dynamics characterized by $\Delta x = N/2$. b) $\nu=-10^{-8}$; the system enters the loss attractor after a transient of few appearences. c) $\nu=0$; pure AL dynamics. d) $\nu=10^{-8}$; the system enters the attractor of the gain dynamics, characterized by $\Delta x=0$, after a transient of few appearences. e) $\nu=10^{-6}$; the system enters immediately the attractor of the gain dynamics.

3.2. Hamiltonian perturbation

In many physical contexts, a quintic term is introduced in NLS type models to describe higher order Hamiltonian effects. The matrix perturbation of the Lax pair is chosen as follows:

$$F[u_n] = -i\gamma \begin{pmatrix} 0 & u_n \\ \overline{u_n} & 0 \end{pmatrix} |u_n|^4, \quad |\gamma| \ll 1, \tag{102}$$

and the variation of $\mathrm{tr}\tilde{T}$ after the jth appearance of the AW is described by the analytic formula

$$\Delta_j\left(\frac{Tr(T)}{\sqrt{\det T}}\right) = -i\gamma\left(\frac{2N|a|^6\cos\phi_1\sin^2\phi_1\cos\frac{k_1}{2}}{1+|a|^2}\right)\tilde{I}\left(\phi_1, x^{(j)}\right), \tag{103}$$

$$\tilde{I}\left(\phi_1, x^{(j)}\right) = \int_{-\infty}^{\infty}\sum_{n=1}^{N}\frac{\mathcal{N}\left(n,t;x^{(j)}\right)}{\mathcal{D}\left(n,t;x^{(j)}\right)}\mathrm{d}t, \tag{104}$$

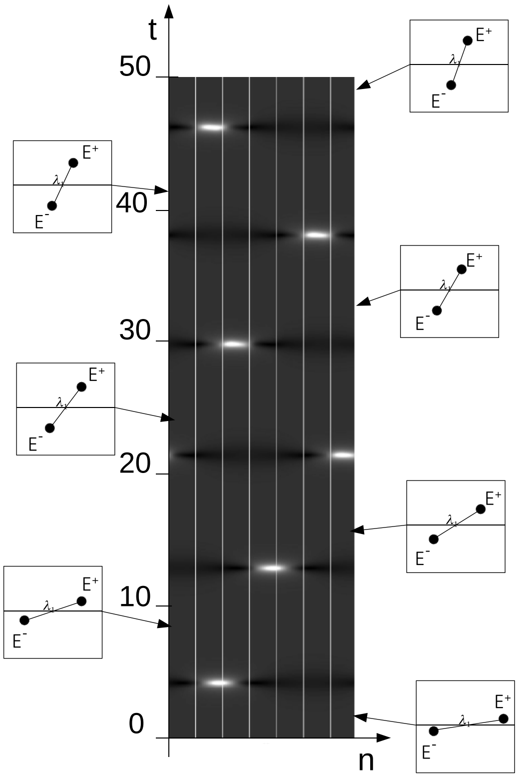

Figure 5. Density plot of $|u_n(t)|$ and the (discrete) evolution of the gap $E^+ - E^-$, due to the AW appearence, in the case of a small loss. The numerical experiment is the same as in figure 4(b).

where

$$\mathcal{N}(n,t;x^{(j)}) = \left|\cos(\frac{\kappa_1}{2})\cosh(\sigma_1(t) + 2i\phi_1) + \sin\phi_1\cos(k_1(n - x^{(j)}))\right|^4$$
$$\times \left\{ \sin(k_1(n - x^{(j)}))\left[2\cos\frac{k_1}{2}\sin^2\phi_1 - \cos\frac{k_1}{2}\cosh(\sigma_1 t)^2\right.\right.$$
$$\left. - \sin\phi_1\cos(k_1(n - x^{(j)}))\cosh(\sigma_1 t\right] + i\sinh(\sigma_1 t)\left[\cos\frac{k_1}{2}\cos(k_1(n - x^{(j)}))\right.$$
$$\left.\left.\times\cosh(\sigma_1 t - \sin\phi_1\left(\cos k_1 + \sin^2(k_1(n - x^{(j)}))\right)\right]\right\}, \quad (105)$$

and

$$\mathcal{D}(n,t;x^{(j)}) = [\cos\frac{k_1}{2}\cosh(\sigma_1 t) - \cos(k_1(n - x^{(j)}))\sin\phi_1]^5$$
$$\times[\cos\frac{k_1}{2}\cosh(\sigma_1 t) - \cos(k_1(n - x^{(j)} + 1))\sin\phi_1]$$
$$\times[\cos\frac{k_1}{2}\cosh(\sigma_1 t) - \cos(k_1(n - x^{(j)} - 1))\sin\phi_1]. \quad (106)$$

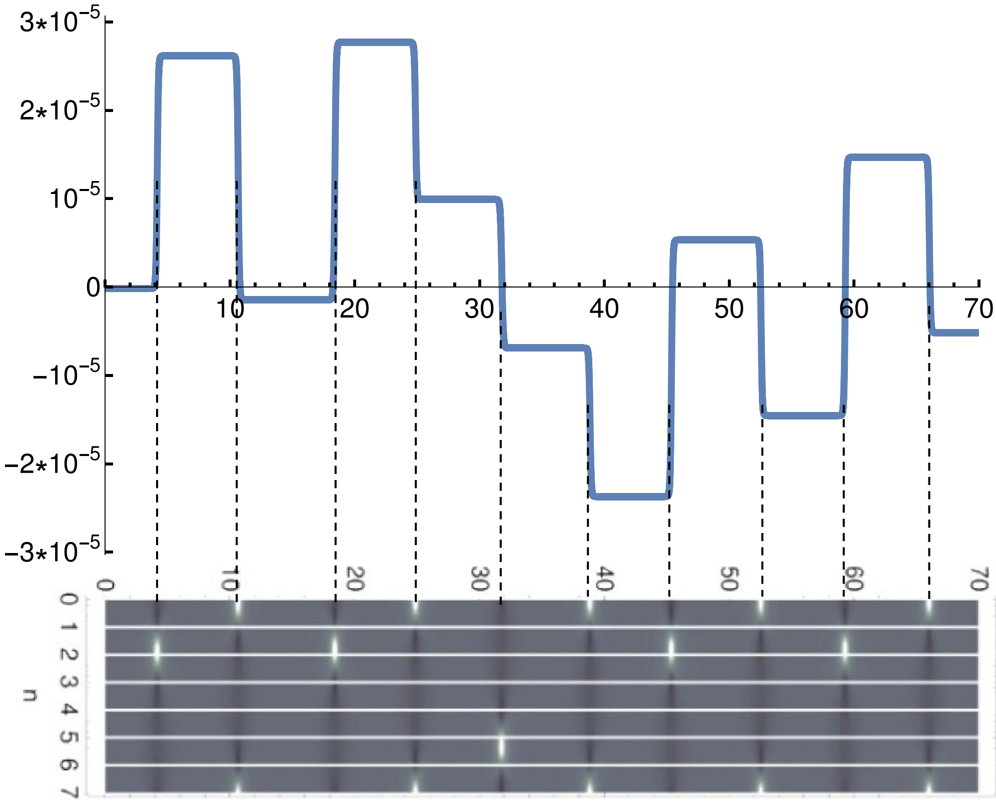

Figure 6. Dynamics of the imaginary part of trace $\Im(\frac{Tr(T)}{\sqrt{\det T}})$, in the presence of the Hamiltonian perturbation $\gamma = 10^{-8}$. The variations correspond to the AW appearances. The initial condition is the same as before: $u_n(0) = a(1+\epsilon(c_+e^{ik_1n}+c_-e^{-ik_1n}))$, with $a=1.1$, $\epsilon = 10^{-4}$, $k_1 = \frac{2\pi}{N}$, $N=7$, $c_+ = 0.53 - i*0.86$ and $c_- = -0.26 + i*0.22$.

Proceeding as in the previous section, one obtains the same **main result** as before, but now

$$Q_n = \alpha_1\beta_1 - i\frac{\gamma}{\epsilon^2}\frac{8a^4\cos^3(\phi)\sin^2(\phi)\cos\frac{\kappa_1}{2}}{N}\sum_{j=1}^{n}\tilde{I}\left(\phi_1, x^{(j)}\right), \quad n \geqslant 0. \tag{107}$$

As in the loss/gain case, the x-shifts $\Delta_n x$ and the recurrence times $\Delta_n t$ after the nth appearance depend, through Q_n in (107), on the positions $x^{(j)}$, $j = 1,\ldots,n$ of the first n AW appearances. But now this contributions are purely imaginary and their imaginary parts can be either positive or negative. Therefore one cannot have, in general, asymptotic attractors. On the other hand, if $\gamma\,\epsilon^{-2} = O(1)$ as in figure 6, then these purely imaginary terms prevail over the term $|\alpha_1\beta_1| = O(1)$ in (107), due to the quintic perturbation. Therefore, at each appearance

$$\Delta_j x \sim \pm\frac{N}{4} \tag{108}$$

(see figure 6).

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix

To evaluate explicitly the right hand side of (79), the simplest way is to make use of the Darboux transformations (DTs) of the AL equations (see [22]), concentrating on the AL_+ equation.

Proposition ([22]). *Let* $\left(u_n^{[0]}, \underline{\psi}_n^{[0]}(\lambda)\right)$ *be a solution of the Lax pair* (4) *for* $\eta = 1$*, then* $(u_n^{[1]}, \underline{\psi}_n^{[1]}(\lambda))$ *is also a solution of* (4)*, where*

$$\underline{\psi}_n^{[1]}(\lambda) = D_n(\lambda)\,\underline{\psi}_n^{[0]}(\lambda), \quad u_n^{[1]} = a_{n+1}u_n^{[0]} + b_{n+1}, \tag{109}$$

$$D_n(\lambda) = \frac{1}{\lambda_1|q_2|^2 + \frac{|q_1|^2}{\lambda_1}} \times \begin{pmatrix} \left(\frac{\lambda}{\lambda_1} - \frac{\lambda_1}{\lambda}\right)|q_{1n}|^2 + \left(\lambda\lambda_1 - \frac{1}{\lambda\lambda_1}\right)|q_{2n}|^2 & -q_{1n}\bar{q}_{2n}\left(\lambda_1^2 - \frac{1}{\lambda_1^2}\right) \\ q_{2n}\bar{q}_{1n}\left(\lambda_1^2 - \frac{1}{\lambda_1^2}\right) & -\left(\lambda\lambda_1 - \frac{1}{\lambda\lambda_1}\right)|q_{1n}|^2 - \left(\frac{\lambda}{\lambda_1} - \frac{\lambda_1}{\lambda}\right)|q_{2n}|^2 \end{pmatrix}, \tag{110}$$

$$a_n = \frac{\lambda_1|q_{1n}|^2 + |q_{1n}|^2\lambda_1}{\lambda_1|q_{2n}|^2 - q_{1n}|^2/\lambda_1}, \quad b_n = \frac{\lambda_1^2 - 1/\lambda_1^2}{\lambda_1|q_{2n}|^2 - |q_{1n}|^2/\lambda_1}, \tag{111}$$

and $\vec{q}_n = (q_{1n}, q_{2n})^T$ *is a linear combination of two independent solutions of* (4) *for* $u_n = u_n^{[0]}$*, evaluated at* $\lambda = \lambda_1$*, where* λ_1 *is a real parameter in the interval* $(1, \lambda_0)$ *(an unstable double point) (see figure* 1*).*

Now we specialize this construction choosing $u_n^{[0]}(t) = a\,e^{2i\eta|a|^2t}$ (the background solution); then a matrix fundamental solution of the Lax pair (4) reads

$$\Psi_n^{[0]} = \left(1 + \eta|a|^2\right)^{\frac{n}{2}} e^{i\left(|a|^2t + \frac{\arg a}{2}\right)\sigma_3} \begin{pmatrix} e^{i\frac{\eta}{2}(kn-\phi) - \frac{\sigma t}{2}} & -e^{-i\frac{\eta}{2}(kn-\phi) + \frac{\sigma t}{2}} \\ \frac{i}{\sqrt{\eta}}e^{i\frac{\eta}{2}(kn+\phi) - \frac{\sigma t}{2}} & \frac{i}{\sqrt{\eta}}e^{-i\frac{\eta}{2}(kn+\phi) + \frac{\sigma t}{2}} \end{pmatrix} e^{i\nu t}, \tag{112}$$

where $\kappa = 2\mu = \frac{2\pi}{N}$, and a linear combination of its columns leads to

$$\begin{aligned} q_{1n} &= 2\cos\left(\frac{\kappa(n - x_1) + i\eta\sigma(t - t_1) - \phi}{2}\right) e^{i\eta|a|^2t + i\frac{\arg a}{2} + i\nu t}, \\ q_{2n} &= -2\sqrt{\eta}\sin\left(\frac{\kappa(n - x_1) + i\eta\sigma(t - t_1) + \phi}{2}\right) e^{-i\eta|a|^2t - i\frac{\arg a}{2} + i\nu t}, \end{aligned} \tag{113}$$

where x_1, t_1 are arbitrary real parameters.

Therefore (109) leads to the AW solution (10):

$$u_n^{[1]}(t) = -a\, e^{2i\eta|a|^2 t}\left[\frac{\cosh(\sigma(t-T)+2i\eta\phi)+\frac{\sin\phi}{\cos\left(\frac{\kappa}{2}\right)}\cos(\kappa(n-X))}{\cosh(\sigma(t-T))-\frac{\sin\phi}{\cos\left(\frac{\kappa}{2}\right)}\cos(\kappa(n-X))}\right], \tag{114}$$

where $X = x_1 - \frac{\eta}{2} - \frac{\pi}{2\kappa}$ and $T = t_1$. Correspondingly, the Darboux (dressing) matrix reads:

$$D_n(\lambda) = \frac{1}{\lambda_1|q_2|^2 + \frac{|q_1|^2}{\lambda_1}} \times \begin{pmatrix} \left(\frac{\lambda}{\lambda_1}-\frac{\lambda_1}{\lambda}\right)|q_1|^2 + \left(\lambda\lambda_1 - \frac{1}{\lambda\lambda_1}\right)|q_2|^2 & -q_1\bar{q}_2\left(\lambda_1^2-\frac{1}{\lambda_1^2}\right) \\ q_2\bar{q}_1\left(\lambda_1^2-\frac{1}{\lambda_1^2}\right) & -\left(\lambda\lambda_1-\frac{1}{\lambda\lambda_1}\right)|q_1|^2 - \left(\frac{\lambda}{\lambda_1}-\frac{\lambda_1}{\lambda}\right)|q_2|^2 \end{pmatrix}. \tag{115}$$

To calculate the remaining ingredients appearing in the basic formula (79), we observe that

$$\begin{aligned} T^{[1]}(n+N,n;\lambda_1) &= \Psi_{n+N}(\lambda_1)\,\Psi_{n+N}^{-1}(\lambda_1) \\ &= \lim_{\lambda\to\lambda_1} D_{n+N}(\lambda)\,\Psi_{n+N}^{[0]}(\lambda_1)\left(\Psi_n^{[0]}(\lambda_1)\right)^{-1} D_n^{-1}(\lambda) \\ &= \lim_{\lambda\to\lambda_1} D_{n+N}(\lambda)\, T^{[0]}(n+N,n,\lambda)\, D_n^{-1}(\lambda) \\ &= \lim_{\lambda\to\lambda_1}\left[\left(D_{n+N}(\lambda_1) + (\lambda-\lambda_1)\,\partial_\lambda D_{n+N}(\lambda)\big|_{\lambda_1}\right)\right. \\ &\qquad \left.\times\left(T^{[0]}(n+N,n,\lambda_1) + (\lambda-\lambda_1)\,\partial_\lambda T^{[0]}(n+N,n,\lambda)\big|_{\lambda_1}\right) D_n^{-1}(\lambda)\right]. \end{aligned} \tag{116}$$

Therefore we need to evaluate the following quantities associated with the Darboux matrix (115):

$$D_n(\lambda_1) = \frac{\left(\lambda_1^2 - \frac{1}{\lambda_1^2}\right)}{\lambda_1|q_{2_n}|^2 + \frac{|q_{1_n}|^2}{\lambda_1}}\begin{pmatrix}\overline{q_{2_n}} \\ \overline{q_{1_n}}\end{pmatrix}(q_{2_n}, -q_{1_n}), \tag{117}$$

$$\partial_\lambda D_n(\lambda)\Big|_{\lambda=\lambda_1} = \frac{\mathrm{diag}\left(2|q_{1_n}|^2 + |q_{2_n}|^2\left(\lambda_1^2+\frac{1}{\lambda_1^2}\right), -2|q_{2_n}|^2 - |q_{1_n}|^2\left(\lambda_1^2+\frac{1}{\lambda_1^2}\right)\right)}{\lambda_1\left(\lambda_1|q_{2_n}|^2 + \frac{|q_{1_n}|^2}{\lambda_1}\right)}, \tag{118}$$

$$D_n^{-1}(\lambda\sim\lambda_1) = \frac{\tilde{D}}{\lambda-\lambda_1} + O(1),\quad \tilde{D} := \frac{\lambda_1}{2\left(\frac{|q_{2_n}|^2}{\lambda_1}+\lambda_1|q_{1_n}|^2\right)}\begin{pmatrix}q_{1_n}\\ q_{2_n}\end{pmatrix}(\overline{q_{1_n}}, -\overline{q_{2_n}}), \tag{119}$$

and those associated with the transition matrix of the background solution:

$$T^{[0]}(n,m,\lambda,t) = \Psi_n^{[0]}(\lambda,t)\left(\Psi_m^{[0]}(\lambda,t)\right)^{-1} = \left(1+\eta a^2\right)^{\frac{n-m}{2}} \times \begin{pmatrix} \cos(\mu(n-m)) + i\sin(\mu(n-m))\frac{g_-+g_+}{g_--g_+} & -2i\frac{\sin(\mu(n-m))}{g_--g_+}e^{2i|a|^2t+i\arg a} \\ 2i\eta\frac{\sin(\mu(n-m))}{g_--g_+}e^{-2i|a|^2t-i\arg a} & \cos(\mu(n-m)) - i\sin(\mu(n-m))\frac{g_-+g_+}{g_--g_+} \end{pmatrix}, \tag{120}$$

$$\partial_\lambda T^{[0]}(n,m;t) = \left[\frac{(n-m)\sin\phi}{\lambda\cos\phi} \begin{pmatrix} \sin(\mu(n-m)) - \cos(\mu(n-m))\frac{\sin\phi}{\cos\phi} & -\frac{\cos(\mu(n-m))}{\cos\phi}e^{2i|a|^2 t} \\ \frac{\cos(\mu(n-m))}{\cos\phi}e^{-2i|a|^2 t} & \sin(\mu(n-m)) + \cos(\mu(n-m))\frac{\sin\phi}{\cos\phi} \end{pmatrix} + \frac{\sin(\mu(n-m))\cos\left(\frac{k}{2}\right)}{\lambda\sin\left(\frac{k}{2}\right)\cos^2\phi} \begin{pmatrix} 1 - \sin\phi\, e^{2i|a|^2 t} + \\ \sin\phi\, e^{-2i|a|^2 t} - 1 \end{pmatrix} \right] \left(1+\eta a^2\right)^{\frac{n-m}{2}}. \quad (121)$$

At last, substituting these formulas in (116), we obtain $T^{[1]}(n+N,n;\lambda_1)$ as in (80).

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