Open Communications in Nonlinear Mathematical Physics Special Issue: Hietarinta, 2026
ocnmp:18295 pp 94–103

A Special OCNMP Issue in Honour of Jarmo Hietarinta on the Occasion of his 80th Birthday

Moving Boundary Problems for a Cuspon Equation and Reciprocal Associates: Exact Solution via Painlevé Symmetry Reduction

Colin Rogers

University of New South Wales, Australia

Sandra Carillo

Dipartimento Scienze di Base e Applicate per l'Ingegneria, "Sapienza" Università di Roma, Rome, Italy & Gr. Roma1, IV - Mathematical Methods in NonLinear Physics, National Institute for Nuclear Physics (I.N.F.N.), Rome, Italy

Received May 28, 2026; Accepted June 7, 2026

Citation format for this Article:
Colin Rogers and Sandra Carillo, Moving boundary problems for a Cuspon equation and reciprocal associates: exact solution via Painlevé symmetry reduction, *Open Commun. Nonlinear Math. Phys.*, Special Issue: Hietarinta, ocnmp:18295, 94–103, 2026.

The permanent Digital Object Identifier (DOI) for this Article:
10.46298/ocnmp.18295

Abstract

Here classes of moving boundary problems of Stefan-type for both an established nonlinear evolution equation of cuspon theory and novel reciprocally linked solitonic equations are shown to be solvable via Painlevé II symmetry reduction.

1 Introduction

Moving boundary problems of Stefan-type have had extensive application in continuum mechanics, notably in connection with change of phase in a non-linear heat conduction context and liquid transport through porous media in soil mechanics [34, 12, 15] and [26] (see also references cited therein). In modern soliton theory, classes of moving boundary problems for the canonical Dym equation [37] and reciprocal associates are derived in [17] which admit exact solutions via Painlevé II symmetry reduction. This investigation was originally motivated, in part, by the classical Saffman-Taylor problem with surface tension [36]. The occurrence of the solitonic Dym equation in the Hele-Shaw theory was elucidated in [9]. Moving boundary problems for a range of solitonic equations have been shown to admit exact solution via Painlevé II symmetry reduction [18, 19, 20, 21, 22, 23]. Moving boundary problems of Stefan-type amenable to exact solution have been recently studied in [27, 28, 25]. Reciprocal-type transformations had their origin in the derivation of invariance properties of conservation laws in homentropic gas dynamics in [2] and were subsequently shown in [3] to constitute particular Bäcklund transformations [31, 32]. In modern soliton theory, reciprocal transformations were introduced in [11] and associated with admitted conservation laws. Therein, these were conjugated with the action of the classical Bianchi permutability theorem associated with invariance of the soliton system under a Bäcklund transformation. The important role of Bäcklund Transformations in the construction of new solutions and in revealing new invariance both in the study of soliton equations, both in the commutative and in the non-commutative cases are given in [6](and references therein). Thereby, multi-soliton solutions can be generated iteratively in an algorithmic manner. In [33], reciprocal transformations were applied to link the canonical AKNS and WKI inverse scattering schemes of [1] and [39], respectively. The linkage of certain classes of 1+1-dimensional solitonic hierarchies via reciprocal transformations have been detailed in [30, 29, 5]. Reciprocal transformations in 2+1-dimensions, as originally introduced in [16], have been applied to connect the Kadomtsev-Petviashvili, 2+1-dimensional Dym and modified Kadomtsev-Petviashvili solitonic hierarchies in [13].

In [14], a novel nonlinear evolution equation descriptive of certain cuspon and periodic cuspon phenomena was obtained and notably, in particular, a Lax pair was derived. Here, classes of moving boundary problems of Stefan-type, both for this cuspon equation and integrable extensions linked by reciprocal transformations, are shown to be solvable via Painlevé II symmetry reduction.

2 A Class of Reciprocal Moving Boundary Problems for a Solitonic Cuspon Equation: Painlevé II Symmetry Reduction

The reciprocal transformation

$$dx^* = v\,dx + (-v_{xx} + 3v^2)\,dt, \quad t^* = t \tag{2.1}$$

with compatibility condition for the canonical solitonic Korteweg-de Vries equation

$$v_t - 6vv_x + v_{xxx} = 0 \tag{2.2}$$

with $v = -1/m^*$ yields

$$dx = m^*\, dx^* + \left(\frac{1}{2} \frac{\partial^2}{\partial x^{*2}} \left(\frac{1}{m^{*2}} \right) - \frac{3}{m^*} \right) dt^*. \tag{2.3}$$

The latter has compatibility condition

$$m^*_t = \frac{1}{2} \left(\frac{1}{m^*} \right)_{x^* x^* x^*} - 3 \left(\frac{1}{m^*} \right)_{x^*}, \tag{2.4}$$

namely, the solitonic cuspon equation in [14]. In [22], exact solutions of a class of Korteweg-de Vries moving boundary problems has been solved via application of the Miura transformation

$$v = u_x + u^2 \tag{2.5}$$

which connects the KdV equation (2.2) to the mKdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0. \tag{2.6}$$

This important link may be derived in the context of a class of classical Bäcklund transformations due to to Clairin [31]. In [22], application was made of Painlevé II symmetry reduction to solve a class of KdV moving boundary problems of Stefan-type problems governed by the system

$$\begin{gathered} v_t - 6vv_x + v_{xxx} = 0 \ , \quad 0 < x < S(t) = \gamma(t+a)^{1/3} \ , t > 0 \\ \left. \begin{array}{l} v_{xx} - 3v^2 = L_m S^i \dot{S} \\ v = P_m S^j \end{array} \right\} \text{ on } x = S(t), \quad t > 0 \\ (v_{xx} - 3v^2)|_{x=0} = H_0 (t+a)^k, t > 0, \\ S(0) = S_0 \ . \end{gathered} \tag{2.7}$$

The mKdV equation (2.6) may be be shown to admit a Painlevé II symmetry reduction with

$$u = (t+a)^p \Psi(\xi), \quad \xi = \frac{x}{(t+a)^q}. \tag{2.8}$$

Thus, on substitution of the latter representation into (2.6) there results

$$p\Psi - q\xi\Psi' - 6(t+a)^{2p-q+1}\Psi^2\Psi' + (t+a)^{-3q+1}\Psi''' = 0 \tag{2.9}$$

whence $p = -1/3, q = 1/3$ and $\Psi(\xi)$ is governed by

$$\Psi''' - 6\Psi^2\Psi' - \frac{1}{3}(\xi\Psi)' = 0 \tag{2.10}$$

so that, on integration,

$$\Psi'' - 2\Psi^3 - \frac{1}{3}\xi\Psi = \kappa^* \ , \ \kappa^* \in \mathbf{R}. \tag{2.11}$$

On introduction of the scalings $\Psi = \delta w$, $\xi = \epsilon z$, into the latter there results the classical Painlevé II equation

$$w_{zz} = zw^3 + zw + \alpha \ , \tag{2.12}$$

with parameter $\alpha = \kappa^* \epsilon^2/\delta$. Under the Miura transformation (2.5), the class of solutions

$$v = (t+a)^{-2/3}(\Psi'(\xi) + \Psi^2(\xi)) := (t+a)^{-2/3}\Lambda(\xi) \tag{2.13}$$

of the KdV equation (2.2) is obtained wherein $\Psi(\xi)$ is given by the scaled version (2.11) of the Painlevé II equation. The following KdV moving boundary conditions are considered. In detail :

I

$$v_{xx} - 3v^2 = L_m S^i \dot{S} \text{ on } x = S(t) = \gamma(t+a)^{1/3} \ , \quad t > 0 \ .$$

Insertion of the relation (2.13) yields

$$(t+a)^{-4/3}(\Lambda''(\gamma) - 3\Lambda^2(\gamma)) = \frac{1}{3} L_m \gamma^{4i/3}(t+a)^{(i-2)/3}. \tag{2.14}$$

where $i = -2$ on alignment of the powers in $t + \alpha$ $i = -2$ and

$$L_m = 3\gamma^{-4i/3}[\Lambda''(\gamma) - 3\Lambda^2(\gamma)]. \tag{2.15}$$

II

$$v = P_m S^i \dot{S} \quad \text{on} \quad x = S(t) = \gamma(t+a)^{1/3} \ , \quad t > 0 \ .$$

This requires $j = -2$ and

$$P_m = \gamma^2 \Lambda(\gamma) = \gamma^2[\Psi'(\gamma) + \Psi^2(\gamma)]. \tag{2.16}$$

III

$$(v_{xx} - 3v^2)|_{x=0} = H_0(t+a)^k \ , \quad t > 0 \ .$$

This boundary condition yelds $k = -4/3$ together with

$$H_0 = \Lambda''(0) - 3\Lambda^2(0) \ . \tag{2.17}$$

The moving boundary problems for the cuspon equation (2.4) under the reciprocal transformation (2.1) with $v = 1/m^*$ become:

$$m^*_t = \frac{1}{2}\left(\frac{1}{m^{*2}}\right)_{x^*x^*x^*} - 3\left(\frac{1}{m^*}\right)_{x^*} , \qquad x^*|_{x=0} < x^* < x^*|_{x=S(t)} := S^*(t^*), \quad t^* > 0.$$

$$\left.\begin{array}{l} \frac{1}{2}\left(\frac{1}{m^{*2}}\right)_{x^*x^*} - 3\left(\frac{1}{m^*}\right) = m^* L_m S^i \dot{S} \\ \frac{1}{m^*} = P_m S^j \end{array}\right\} \quad \text{on } x^* = S^*(t^*), \quad t^* > 0 \tag{2.18}$$

$$\left[\frac{1}{2}\left(\frac{1}{m^{*2}}\right)_{x^*x^*} - 3\left(\frac{1}{m^*}\right)\right]\Bigg|_{x^*|_{x=0}} = m^*\big|_{x^*|_{x=0}} H_0(t+a)^k, \qquad t^* > 0$$

$$S^*(0) = S_0^*.$$

In the preceding, $x^* = S^*(t^*)$ is the reciprocal moving boundary obtained by application of (2.1) so $x = S(t) = \gamma(t+a)^{1/3}$. Thus,

$$\begin{aligned} dx^*|_{x=S(t)} &= \left[v dx - (v_{xx} - 3v^2)dt\right]|_{x=S(t)} \\ &= (P_m S^j \dot{S} + L_m S^i \dot{S})dt \end{aligned} \tag{2.19}$$

wherein $i = j = -2$. Accordingly,

$$dx^*|_{x=S(t)} = (P_m + L_m)\,(1/3\gamma)(t^* + a)^{-4/3} dt \tag{2.20}$$

so that

$$S^*(t^*) = \gamma^*(t^*+a)^{-1/3} + \delta^* \quad , \quad \gamma^*, \delta^* \in \mathbf{R} \tag{2.21}$$

with $\gamma^* = -\gamma^{-1}(P_m + L_m)$. The associated reciprocal initial boundary condition becomes:

$$S^*(t^*) = \gamma^* a^{-1/3} + \delta^* \quad . \tag{2.22}$$

In addition,

$$dx^*|_{x=0} = (-v_{xx} + 3v^2)dt|_{x=0} = H_0(t^*+a)^k dt \tag{2.23}$$

with $k = -1/3$, whence

$$x^*|_{x=0} = -3H_0(t^*+a)^{-1/3} + \epsilon^* \quad , \quad \epsilon^* \in \mathbf{R}. \tag{2.24}$$

Thus, the region

$$x^*|_{x=0} < x^* < x^*|_{x=S(t)} := S^*(t^*) \ ,$$

reciprocally associated with $0 < x < S(t)$, is given by

$$-3H_0(t^*+a)^{-1/3} + \epsilon^* < x^* < \gamma^*(t^*+a)^{-1/3} + \delta^* \ . \tag{2.25}$$

It is remarked that moving boundary problems constrained by a pair of time-dependent boundaries occur, in particular, both in the context of resonant nonlinear Schrödinger boundary analysis [8] and certain Stefan-type problems in the context of nonlinear heat conduction incorporating a source term [4].

3 An Extended Solitonic Cuspon Equation: Reciprocal Moving Boundary Problems

In [24], a novel solitonic extension of the cuspon equation (2.4) was introduced, namely:

$$m^*_t = \frac{1}{2}(m^{*-2})_{x^*x^*x^*} + \delta^*(m^{*-1})_{x^*} + \epsilon^*(m^{*-2})_{x^*} \quad , \quad \delta^*, \epsilon^* \in \mathbf{R} \tag{3.1}$$

which is linked via a reciprocal transformation to the canonical solitonic Gardner equation

$$v_\tau + 6v(1-v)v_y + v_{yyy} = 0. \tag{3.2}$$

The latter has diverse physical applications, notably in plasma physics [35], optical lattice theory [38], nonlinear wave, propagation phenomena in both hydrodynamics [10] and elastodynamics [7]. On application to (3.2) of the reciprocal transformation

$$dx^* = v\,dy - [v_{yy} + 3v^2 - 2v^3]d\tau, \quad t^* = \tau \tag{3.3}$$

with $v = 1/m^*$, there results

$$dy = m^* dx^* + \left[\frac{1}{2}\left(m^{*-2}\right)_{x^*x^*} + 3m^{*-1} - 2m^{*-2}\right] dt^*, \tag{3.4}$$

with compatibility condition the extended cuspon equation (3.1) with parameters $\delta^* = 3,\ \beta = -2$.

A Class of Moving Boundary Problems

It was recently established in [24] that a class of nonlinear moving boundary problems for the Gardner equation (3.2) admits exact solution via a Painlevé II symmetry reduction on application of a mKdV connection. This class was determined by the nonlinear system

$$v_\tau + 6v(1-v)v_y + v_{yyy} = 0, \qquad \frac{3\tau}{2} < y < \gamma(\tau+a)^{1/3} + \frac{3\tau}{2}, \quad \tau > 0,$$

$$\left.\begin{aligned} v_{yy} - 2\left(v - \frac{1}{2}\right)^3 &= L_m S^j \dot{S}, \\ v - \frac{1}{2} &= P_m S^j \end{aligned}\right\} \quad \text{on} \quad y = \gamma(\tau+a)^{1/3} + \frac{3\tau}{2},\ \tau > 0, \tag{3.5}$$

and

$$\left[v_{yy} - 2\left(v - \frac{1}{2}\right)^3\right]_{y=3\tau/2} = H_0(\tau+a)^k, \qquad \tau > 0,$$

$$S(0) = S_0,$$

wherein $S(\tau) = \gamma(\tau+a)^{1/3}$. The preceding system (3.5) was obtained in [24] by setting

$$x = -\frac{3}{2}\tau + y, \qquad t = \tau, \qquad v = \frac{1}{2} + u, \tag{3.6}$$

in the class of mKdV moving boundary problems

$$
\begin{aligned}
& u_t - 6u^2 u_x + u_{xxx} = 0, \qquad 0 < x < S(t),\ t > 0, \\
& \left.\begin{aligned} u_{xx} - 2u^3 = L_m S^i, \dot{S}, \\ u = P_m S^i, \end{aligned}\right\} \quad \text{on} \quad x = S(t),\ t > 0, \\
& \left[u_{xx} - 2u^3\right]_{x=0} = H_0 (t+a)^k, \qquad t > 0, \\
& S(0) = S_0 \ \ \text{with} \ \ S(t) = \gamma (t+a)^{1/3} \ .
\end{aligned}
\tag{3.7}
$$

This nonlinear system has been shown to admit exact solution via Painlevé II symmetry reduction [21]. In particular it admits the exact solution

$$
u = -\delta (t+a)^{-1/3} \phi' \left(\frac{x}{\epsilon (t+a)^{1/3}} \right) \left[\phi \left(\frac{x}{\epsilon (t+a)^{1/3}} \right) \right]^{-1},
$$

where ϕ is governed by an Airy equation. A novel class of exactly solvable moving boundary problems for the extended solitonic cuspon equation results via the action of the reciprocal transformation (3.3) on the Gardner moving boundary system (3.5). The reciprocal extended cuspon moving boundary system inherits the property of admittance of exact solution via Painlevé II symmetry reduction.

Acknowledgements

S.C. acknowledges the Italian National Institute of Nuclear Physics (CSN4- MMNLP - Mathematical methods of nonlinear Physics- RM1), the Italian National Group of Mathematical Physics (GNFM–INdAM), SBAI Dept. of Sapienza University of Rome, Italy and the project CTE of Regione Lazio, for supporting the present research activity.

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