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An extended generalized Markov model for the spread risk and its calibration by using filtering techniques in *Solvency II* framework

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Abstract

The *Solvency II* regulatory regime requires the calculation of a capital requirement, the *Solvency Capital Requirement* (SCR), for the insurance and reinsurance companies, that is based on a market-consistent evaluation of the Basic Own Funds probability distribution forecast over a one-year time horizon.

This work proposes an extended generalized Markov model for rating-based pricing of risky securities for spread risk assessment and management within the *Solvency II* framework, under an internal model or partial internal model. This model is based on Jarrow, Lando and Turnbull (1997), Lando (1998) and Gambaro et al. (2018) and models the credit rating transitions and the default process using an extension of a time-homogeneous Markov chain and two subordinator processes. This approach allows simultaneous modeling of credit spreads for different rating classes and credit spreads to fluctuate randomly even when the rating does not change.

The estimation methodologies used in this work are consistent with the scope of the work and the scope of the proposed model, *i.e.*, pricing of defaultable bonds and calculation of SCR for the spread risk sub-module, and with the market-consistency principle required by *Solvency II*. For this purpose, estimation techniques on time series known as filtering techniques are used, which allow the model parameters to be jointly estimated under both the *real-world* probability measure (necessary for risk assessment) and the risk-neutral probability measure (necessary for pricing). Specifically, an appropriate set of time series of credit spread term structures, differentiated by economic sector and rating class, is used.

The proposed model, in its final version, returns excellent results in terms of goodness of fit to historical data, and the projected data are consistent with historical data and the *Solvency II* framework.

The filtering techniques, in the different configurations used in this work (particle filtering with Gauss-Legendre quadrature techniques, particle filtering with Sequential Importance Resampling algorithm, Kalman filter), were found to be an effective and flexible tool for estimating the models proposed, able to handle the high computational complexity of the problem addressed.

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Introduction

With the entry into force of the *Solvency II* regulatory regime on January 1st 2016, *European Insurance and Occupational Pensions Authority*'s (EIOPA) main objective is to adequately protect policy-holders and beneficiaries by providing insurance and reinsurance undertakings with effective solvency requirements and an economic risk-based approach that incentivises them to properly measure and manage their risks. To this end, the new framework Directive [26] provides for the calculation of a new capital requirement, the *Solvency Capital Requirement* (SCR), for insurance and reinsurance companies. This requirement, based on a market-consistent evaluation of the balance sheet, "*should reflect a level of eligible own funds that enables insurance and reinsurance undertakings to absorb significant losses and that gives reasonable assurance of the company solvency to policy-holders and beneficiaries*". In fact, the SCR "*should be determined as the economic capital to be held by insurance and reinsurance undertakings in order to ensure that ruin occurs no more often than once in every 200 cases*" (*i.e.* with a probability of 0,5%). In practice, "*the SCR shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99,5% over a one-year period*".

EIOPA allows insurance and reinsurance undertakings to calculate the SCR in accordance with the *Standard Formula* or using an internal model.
The *Standard Formula* adopts a modular[1] approach and provides standardised model and parameters, calibrated on European insurance market data. If the standardised approach provided by the *Standard Formula* does not adequately reflect the specific risk profile of an undertaking, subject to approval by the supervisory authorities, the undertaking can replace a subset of the standardised parameters with parameters calibrated on its internal data (*Undertaking Specific Parameters*), while maintaining the *Standard Formula* structure.
The internal model, partial or full, is a more sophisticated model developed by the undertaking to evaluate as accurately as possible its risk profile and the consequent requirement. This method is also subject to supervisory authorities approval.

Therefore, in an internal model framework, the SCR calculation requires the *Basic Own Funds*[2] (BOF) probability distribution forecast over a one-year time horizon. Assets and liabilities, and then BOF, shall be valued in accordance with the principle

[1]The individual exposure to each risk category should be assessed in a first step and then aggregated in a second step, a *bottom-up* approach.

[2]According to the Directive [26], BOF are defined as the excess of assets over liabilities.

of market consistency: the risk-neutral[3] probability measure must be used in this evaluation. On the other hand, in order to calculate the SCR, the *real-world*[4] probability distributions of risk factors are also necessary, in order to simulate the evolution of their values over the one-year time horizon required by the Directive. This framework brings out the issue of using estimation techniques based on market time series, which allow to estimate the parameters under both probability measures at the same time.

Within the broader context of market risk, defined as "*the risk of loss or of adverse change in the financial situation resulting, directly or indirectly, from fluctuations in the level and in the volatility of market prices of assets, liabilities and financial instruments*", and credit risk, defined as "*the risk of loss or of adverse change in the financial situation, resulting from fluctuations in the credit standing of issuers of securities, counterparties and any debtors to which insurance and reinsurance undertakings are exposed, in the form of counterparty default risk, or spread risk, or market risk concentrations*", this work focuses only on spread risk, one of the market risks considered by the regulations. In particular, spread risk is, as defined by the Directive: "*the sensitivity of the values of assets, liabilities and financial instruments to changes in the level or in the volatility of credit spreads over the risk-free interest rate term structure.*" [26].

More specifically, this work proposes a methodology for the calculation of the SCR for the spread risk sub-module using a full or partial internal model.

The model proposed to assess the spread risk is a stochastic Markovian model for the rating-based pricing of risky securities: it is an extension of the models proposed by Jarrow, Lando e Turnbull (1997) [36] and by Lando (1998) [47], which aims to represent the credit rating transition and the default process using an extension of a time-homogeneous Markov chain and one or more subordinator processes, as suggested in Gambaro et al. (2018) [28]. Through this approach, term structures of credit spreads for different rating classes can be simultaneously modelled, and the credit spreads are allowed to fluctuate randomly even when the rating does not change.
The model price of a risky zero coupon bond (ZCB) with maturity T and rating i at time t is then defined as:

$$v^i(t,T) = \mathbf{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r(u)\,du} \left(1\,\mathbf{1}^i_{\{\tau > T\}} + \delta\,\mathbf{1}^i_{\{\tau \le T\}} \right) \Big| \mathcal{G}_t \right]$$

where $r(t)$ is the risk-free spot rate process, $\mathbf{1}^i_{\{\tau > T\}}$ and $\mathbf{1}^i_{\{\tau \le T\}}$ are respectively the indicator functions of the default event after and before the maturity T for an i-rated issuer, and δ is the exogenous recovery rate. Under model assumptions, this expectation can be solved in closed form as a function of the eigenvalues and eigenvectors of the generator matrix.

[3]The risk-neutral measure is used primarily for market-consistent valuations, *e.g.* for the pricing of financial and insurance products.

[4]The *real-world* probability measure is used primarily for risk management purposes and SCR assessment.

This model can be used to asses several securities in which credit ratings and default are relevant; specifically, in this work it is used to evaluate risky ZCB for different rating classes and maturities.

According to this framework, Bayesian filtering techniques, specifically particle filtering and Kalman filter, are investigated and implemented for the estimation of the model parameters used in this work. In fact, this estimation technique, widely used in the financial field and discussed in literature from a theoretical and practical point of view (refer to McNeil (2015) [53], Lemke (2006) [49], Doucet et al. (2001) [16] and Durbin and Koopman (2012) [18]), makes it possible to comply with the requirements of the regulations by jointly estimating the *real-world* and risk-neutral parameters, in that the time series make it possible to estimate the parameters of the *real-world* dynamics, while the term structures of the credit spread (which can be interpreted as prices) allow to estimate the parameters of the risk-neutral dynamics. The estimation leverages on the state-space representation, which allows to get around the unobservability of the subordinator processes, and is based on a procedure involving the maximization of the likelihood function computed on the observed data set.
The main difficulty in applying filtering techniques is represented by having to solve multidimensional integrals needed by the algorithm. In this regard, three types of solutions can be identified:

- closed-form solution (the so-called Kalman Filter);
- solution by using quadrature techniques (*e.g.* Gauss-Legendre quadrature);
- solution by using Monte Carlo techniques.

The last two cases represent the class of particle filters. All three of these cases are presented and used in this work.

Markovian models for rating-based term structures are widespread in financial literature, and this work proposes an application of these models in the insurance framework defined by *Solvency II* regulations, offering some solutions to the shortcomings already presented in the literature.
In this work, several configurations of Bayesian filtering techniques are applied for the first time to the estimation of models for rating-based term structures. This estimation procedure allows the model to be estimated under both probability measures relevant for the calculation of the capital requirement in the *Solvency II* framework, *real-world* and risk-neutral[5]. The power of the filtering techniques also allows the estimation procedure to be applied to time series with high depth of credit spreads for all ratings (without modifiers) and for all available maturities jointly. In fact, in the following work, the reference database consists of the time series from 01/01/2007 to 31/12/2021 of credit spreads for all ratings from AAA to CCC (according to *Standard & Poor's Global Ratings* scale) and for different short- and medium-term maturities (from 2 to 15 years), containing more than 120000

[5] The modeling framework is first presented in Lando (1998) [47] for the only purpose of rating-based pricing of risky securities, then only under the risk-neutral probability measure.

observations[6].
In order to obtain a model that is sufficiently adaptable and able to accurately capture different behaviors and temporal trends, even the most extreme ones, of credit spreads considering such a large number of observations, this paper proposes an extension of the Markovian models for rating-based term structures presented by Jarrow, Lando and Turnbull (1997) [36], Lando (1998) [47] and Gambaro et al. (2018) [28]; in particular, the use of two subordinator processes for modelling the process of rating transitions, including default, and a rating-specific liquidity component is proposed[7]. This extension is designed in accordance with the principles of prudence and parsimony in the size of the parameter space required by the regulations.
In addition, this work provides a theoretical and practical overview of the main configurations of Bayesian filtering techniques used in financial model estimation, namely the Kalman filter, particle filtering with Gauss-Legendre quadratures, and particle filtering with Sequential Importance Resampling algorithm.

This work is structured as follows. Chapter 1 introduces credit risk and the different types of credit spread and describes the processing of spread risk in the *Solvency II* framework. In chapter 2 some models for the spread risk are introduced and, specifically, the model object of study in this work and the models from which it derives (Jarrow, Lando and Turnbull (1997) [36], Lando (1998) [47] and Gambaro et al. (2018) [28]) are further investigated. Chapter 2 also presents the theoretical aspects of Markov chains in their different settings. Chapter 3 provides a summary of Bayesian filtering techniques and describes their application to estimation on market price time series; in particular, Kalman filter and the particle filters class are explored. In chapter 4 the estimation problem is formalized and the methodological aspects are detailed. Finally, the results returned by the estimation procedure, both in terms of goodness of fit and in terms of application in *Solvency II* framework, are presented in chapter 5.

[6]Gambaro et al. (2018) [28] estimate the rating transition intensity process by applying the maximum likelihood method proposed in Pearson and Sun (1994) [58]. The reference data for estimation procedure are time series of default probabilities obtained via bootstrap from the `iTraxx Europe CDS Index` spread for a reference rating, from 20/09/2016 to 13/01/2017.

[7]Gambaro et al. (2018) [28] proposes a model with a single subordinator process with intensity modeled with a CIR++ model under the risk-neutral probability measure and with a CIR model under the *real-world* probability measure and sector-specific liquidity component.

Chapter 1

Credit risk and credit spreads

This chapter discusses the theoretical aspects of credit risk and credit spreads.

In section 1.1 the different definitions of credit risk found in the literature are presented and contextualized within the *Solvency II* framework. In addition, the main variables relevant in dealing with credit risk are listed and described, with particular regard to the credit rating, which is of primary importance in this work. Finally, corporate bonds are described, which are among the defaultable claims.

In section 1.2 the main results of the *Standard and Poor's Default, Transition, and Recovery: 2021 Annual Global Corporate Default And Rating Transition Study* are presented.

Section 1.3 focuses on credit spread, in that its different definitions and the several components into which it can be decomposed are analysed. Finally, the various types of data available for measuring spread risk are presented and, specifically, the `iBoxx` indices proposed and calculated by the provider *IHS Markit* are examined in depth.

Section 1.4 presents the main aspects of current regulatory framework *Solvency II*, with a focus on the capital requirement for the spread risk sub-module.

1.1 Credit risk

Credit risk and, in particular, its assessment and management have become increasingly important in recent years. This issue is perfectly in line with the general trend of paying greater attention to risk in all areas, as shown by the recent European regulations in the banking and insurance sectors, *Basel III* and *Solvency II*, in which risk plays a central role in the management and supervision of banking and insurance institutions. The causes of the considerable relevance accorded to credit risk in recent times can be identified above all in:

- the extreme topicality of credit events, such as defaults or downgrades, which have played major roles in the most important recent economic crises, including the global economic crisis of 2008, erupted as a result of the *subprime* mortgage crisis, and the sovereign debt crisis, which led to the *downgrading*[1] of many countries, including Italy, Greece and Portugal;

- the occurred obsolescence of many models for the assessment and management of credit risk developed over the years, due to the continuous evolution and refinement of financial markets and traded securities, especially credit derivative securities.

The definition of credit risk is constantly evolving and updating. Amman (2001) [3] defines credit risk, in its traditional meaning, as the default risk, *i.e.*, the risk that the counterparty of a financial contract fails to meet all or part of its contractual obligations and is therefore insolvent, thereby causing a loss to the creditor.
Over time, credit risk has taken on a more general meaning, less focused on the default event, leading up to Crouhy et al. (2006) [13], that define credit risk as the risk of a loss due to changes in the factors that drive the credit quality of an asset (*e.g.* rating migrations, including default, and recovery rate dynamics), and to *Solvency II* [26], "*the risk of loss or of adverse change in the financial situation, resulting from fluctuations in the credit standing of issuers of securities, counterparties and any debtors to which insurance and reinsurance undertakings are exposed, in the form of counterparty default risk, or spread risk, or market risk concentrations*". In these definitions, the default event is only the extreme case, since the risk that the counterparties may be downgraded by rating agencies, without necessarily leading to default, is also covered.
Two main credit risk paradigms defined by the *Basel Committee* in 1999 can therefore be distinguished: the *default-mode paradigm*, in which the relevant event is exclusively the default of the counterparty, and the *mark-to-market paradigm*, in which the change in value of the credit position as a result of a deterioration in creditworthiness is considered. Depending on the paradigm adopted, two approaches to assessing credit risk can be distinguished: a *default-mode* approach, which envisages only two states for a credit position, default or non-default, and is based solely on the probability of default, and a *multi-state* approach, in which the risk of migration between rating classes is considered through the use of transition probability matrices, and default represents only one of the possible states for a credit position.

Arvanitis et al. (2001) [4] provide a more detailed decomposition of credit risk into the following sub-risks:

- *Default risk*, which can in turn be broken down into:
    - *Counterparty risk*, relevant for derivative securities and loans, that are illiquid positions, for which the risk is evaluated over a long time horizon;
    - *Issuer risk*, relevant for bonds, that are more liquid positions, for which the risk is evaluated over a short time horizon, as traditional market risks.
- *Credit spread risk*, which can in turn be broken down into:

[1] Downgrading of an issuer, company or government, due to the deterioration of its creditworthiness.

  - *Downgrading risk*, the risk of a change in the credit spread due to a change in the rating of a counterparty;
  - *Spread risk*, the risk of a change in the credit spread required by the market, given the same rating;

- *Recovery risk*, the risk that the *recovery rate* for the various counterparties that have defaulted will be lower than estimated;

- *Exposure risk*, the risk that the exposure to a counterparty at the time of default is higher than estimated.

The different components of credit risk generate different types of losses: for example, the default event generates an immediate loss in any approach, while the downgrading event generates an immediate loss only in the *multi-state* approach where the impairment due to the deterioration of creditworthiness is taken into account.

In this work the definition of credit risk that falls under the *mark-to-market paradigm* is adopted, in which in addition to the extreme case of insolvency, the changes in value due to changes in creditworthiness, the so-called credit rating transitions, are also taken into account. Therefore, the *multi-state* approach is followed, in line with the provisions of the *Solvency II* Directive, which explicitly considers insolvency risk and credit spread risk (downgrading and spread fluctuations) for the calculation of the SCR, although distributed over different risk modules.

1.1.1 Main driving factors and variables of credit risk

In order to assess and manage credit risk, in the meaning of the risk that an unexpected change in the creditworthiness of a counterparty generates an unexpected loss, it is necessary to calculate the distribution of the so-called *unexpected loss*. What follows is a brief analysis of the factors driving the credit risk, as defined in [4], and the variables that come into play in this assessment, as defined in Bluhum et al. (2003) [8], with reference to the individual credit position.
The credit risk is driven by:

- *exposure*: it is necessary to calculate the distribution of the future exposures, *i.e.*, their present value discounted at the risk-free rate, at all possible default times;

- *default probabilities*, *i.e.*, the likelihood that a counterparty will be bankrupt or will not honour its obligations at their due times over a given period. There are essentially two approaches to assign a default probability (DP) to every credit counterparty:

  a) calibration based on market data, such as market prices, as in the case of the *expected default frequencies* of the *KMV*-model, and credit spreads of financial products subject to credit risk, such as credit bonds and credit derivatives (*e.g. Credit Default Swaps* (CDS));
  b) rating-based calibration, in which default probabilities are associated with ratings, that are assigned to customers by external rating agencies like

Standard & Poor's Global Ratings (S&P), *Moody's Investors Services* and *Fitch Ratings*, or by internal rating systems. Credit ratings are discussed in more detail in section 1.1.2;

- *credit migration probabilities*: the deterioration of a counterparty's creditworthiness and the resulting rating downgrade lead to a loss; in the context of a marked-to-market valuation, future cash flows are discounted at a higher risky rate, or future expected losses are increased; conversely, creditworthiness may also improve with opposite consequences;
- *recovery rates*, which specify the payment to the contract holder in case of default.

The loss of any obligor is defined by a *loss* variable, $\tilde{L}$, which can be decomposed as follows:

$$\tilde{L} = EAD \times LGD \times L \;, \tag{1.1}$$

where:

- EAD denotes the *exposure at default* subject to be lost in the reference time horizon;
- LGD denotes the *loss given default*, *i.e.*, the expected fraction of loss in case of default. The LGD can be defined through the *recovery rate*, or rather through its ones' complement, and can be interpreted as the expected value of the severity of the loss;
- L denotes the indicator function of the default event D:

$$L = \mathbf{1}_D = \begin{cases} 1 & \text{default,} \\ 0 & \text{non-default.} \end{cases}$$

Considering a generic probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a sample space, $\mathcal{F}$ is a σ-*Algebra* with *measurable* elements, that can be interpreted as the available information[2], and $\mathbb{P}$ is a probability measure, and assuming one of the following conditions:

- EAD and LGD are constant values;
- EAD and LGD are the expectations of the respective random variables and they are independent of each other and of the default event,

the *expected loss* (EL) of any position is defined as the expectation of its corresponding loss variable $\tilde{L}$:

$$EL = \mathbb{E}[\tilde{L}] = EAD \times LGD \times \mathbb{E}[L]. \tag{1.2}$$

Since L is a Bernoulli random variable and its expectation coincides with the probability of the underlying event (the default event), we obtain:

$$EL = EAD \times LGD \times DP. \tag{1.3}$$

[2]The event of default is *measurable* and it is included in $\mathcal{F}$.

Another relevant component, also from a regulatory perspective, in the assessment and management of credit risk is the *unexpected loss*. In fact, both banking and insurance regulations require the coverage of unexpected losses, in addition to expected losses. Bluhum et al. (2003) [8] propose as a measure of the deviation of losses from their expectation (*expected loss*), and therefore as a measure of the *unexpected loss*, the standard deviation of the loss variable $\tilde{L}$:

$$UL = \sqrt{Var[\tilde{L}]} = \sqrt{Var[EAD \times SEV \times L]} \,, \tag{1.4}$$

where SEV denotes the *severity* random variable, whose expectation is the LGD. Under the assumption that the *severity* and the default event are independent, the *unexpected loss* can be written as:

$$UL = EAD \times \sqrt{Var[SEV] \times DP + LGD \times DP(1 - DP)}]. \tag{1.5}$$

For the proof of (1.5) refer to [8].

1.1.2 Credit rating

As defined by the rating agency *Standard & Poor's Global Ratings*, credit ratings are "*forward-looking opinions about the ability and willingness of debt issuers, like corporations or governments, to meet their financial obligations on time and in full, and also the credit quality of an individual debt issue, such as a corporate or government bond, and the relative likelihood that the issue may default*".
Credit ratings provide transparent information and insight to the market participants (*e.g.* investors, portfolio managers, analysts, insurance companies, etc.) through a global language, but they do not indicate investment merit.

Rating agencies usually use a combination of analysts opinions and mathematical models: the latter provide a first assessment of the rating, which is then re-evaluated by rating analysts. In the rating evaluation process, many financial and non-financial factors have to be considered:

- key performance indicators, like future earnings and cashflows,
- debt, short and long-term liabilities, and financial obligations,
- capital structure,
- assets liquidity,
- political and social situation of the country/home country,
- situation of the reference market,
- management quality, company structure, etc.

Each rating agency applies its own methodology and uses a specific rating scale, typically expressed as alphanumeric grades, however comparable with those of other rating agencies. Table 1.1 shows the correspondence between the rating scales of

the three main credit rating agencies, *Standard & Poor's Global Ratings*, *Moody's Investors Services* and *Fitch Ratings*; Table 1.2 and Table 1.3 show in detail the rating scales of *Standard & Poor's Global Ratings* and *Moody's Investors Services*. The two rating scales are almost symmetrical: the first one counts 9 main rating classes, from AAA to D, and 22 rating classes with modifiers, while the second one counts 8 main rating classes, from Aaa to C, and 21 rating classes with modifiers. The modifiers are expressed with +/- in the *Standard & Poor's Global Ratings* scale and with $\{1, 2, 3\}$ in the *Moody's Investor Service* scale, and indicate that the obligor/obligation is, respectively, in the higher end, in the mid-range or in the lower end of that generic rating category.
Rating agencies also have the option of withdrawing an entity's rating for several reasons, including the termination of the rated program(s) and extinguishment of the related debt, the bankruptcy/liquidation of a rated entity, the provision of incorrect, insufficient or otherwise inadequate information, corporate reorganizations or the request of the rated entity itself.

Both rating scales can be relocated within the macro-classification between *Investment Grade* and *Speculative Grade* (or *High Yield*). The first macro-class consists of rating classes that are considered to be more solid, less risky and therefore less speculative: from AAA to BBB- for *Standard & Poor's Global Ratings* and from Aaa to Baa3 for *Moody's Investor Service*; the second macro-class consists of rating classes with significant speculative characteristics, and thus exposed to large uncertainties and adverse conditions.
This differentiation is crucial, since it distinguishes issuers with a good ability to repay capitals and interests at maturity from issuers who have the ability to repay debt but may face many difficulties with adverse economic and financial conditions. Being in the first class provides several benefits, including lower interest rates to be paid to investors, access to a larger number of investors and with it a greater and more stable demand for bonds. As a matter of fact, various regulations, like banking and insurance ones, generally place limits on investments in *Speculative Grade* securities.

In the theoretical approach credit ratings do not necessarily have to be attributed by a commercial rating agency. In fact, many financial institutions have their own credit rating systems, known as *internal ratings*, that can be more adequate for assessing the debt's credit quality and consider changes, improvement or deterioration, in the firm's credit quality more quickly.

Standard & Poor's	**Moody's**	**Fitch**
AAA	Aaa	AAA
AA+	Aa1	AA+
AA	Aa2	AA
AA-	Aa3	AA
A+	A1	A+
A	A2	A
A-	A3	A-
BBB+	Baa1	BBB+
BBB	Baa2	BBB
BBB-	Baa3	BBB-
BB+	Ba1	BB+
BB	Ba2	BB
BB-	Ba3	BB-
B+	B1	B+
B	B2	B
B-	B3	B-
CCC+	Caa1	
CCC	Caa2	
CCC-	Caa3	CCC
CC	Ca	
C		
D	C	DDD
		DD
		D

Table 1.1. Correspondence between *Standard & Poor's Global Ratings*, *Moody's Investor Service* and *Fitch Ratings* rating scales with modifiers.

	Category	Definition
Investment Grade	AAA	Extremely strong capacity to meet its financial commitments. 'AAA' is the highest issuer credit rating assigned by S&P Global Ratings.
	AA+ AA AA-	Very strong capacity to meet its financial commitments.
	A+ A A-	Strong capacity to meet its financial commitments, but somewhat susceptible to economic conditions and changes in circumstances.
	BBB+ BBB BBB-	Adequate capacity to meet its financial commitments, but more susceptible to adverse economic conditions or changing circumstances.
Speculative Grade	BB+ BB BB-	Less vulnerable in the near term but faces major ongoing uncertainties and exposure to adverse business, financial, or economic conditions.
	B+ B B-	More vulnerable to adverse business, financial and economic conditions but currently has the capacity to meet its financial commitments.
	CCC+ CCC CCC-	Currently vulnerable and dependent on favorable business, financial, and economic conditions to meet its financial commitments.
	CC	Highly vulnerable. Default has not yet occurred but is expected to be a virtual certainty, regardless of the anticipated time to default.
	C	Currently highly vulnerable to non-payment, and ultimate recovery is expected to be lower than that of higher rated obligations.
	D	Payment default on a financial commitment or breach of an imputed promise; also used when a bankruptcy petition has been filed.

Table 1.2. *Standard & Poor's Global Ratings* long-term rating scale definitions with modifiers.

	Category	**Definition**
Investment Grade	Aaa	Highest quality, with minimal risk.
	Aa1 Aa2 Aa3	High quality and subject to very low credit risk.
	A1 A2 A3	High quality and subject to low credit risk.
	Baa1 Baa2 Baa3	Medium grade and subject to moderate credit risk.. May possess speculative characteristics.
Speculative Grade	Ba1 Ba2 Ba3	Have speculative elements and subject to substantial credit risk.
	B1 B2 B3	Speculative and subject to high credit risk.
	Caa1 Caa2 Caa3	Of poor standing and subject to very high credit risk.
	Ca	Highly speculative and likely in, or very near, default, with some prospect of recovery in principal and interest.
	C	Lowest-rated class, typically in default, with little prospect for recovery of principal and interest.

Table 1.3. *Moody's Investors Services* long-term rating scale definitions with modifiers.

1.1.3 Corporate bonds

Corporate bonds are debt instruments issued by corporations. The bonds issue involves a commitment by the company to pay specified amounts to the bondholders at specified future dates. However, while a default-free bond (or risk-free bond) pays both coupons and face value to the bondholders with certainty on predetermined dates, the firm issuing corporate bonds can go into default before the bond's maturity and fail to meet its commitments to bondholders, that then will not receive the full promised payments. This is why corporate bonds are also called risky bonds and are included in defaultable claims.
Starting from the corporate bond definition is possible to define the *defaultable term structure* as the term structure of interest rates implied by the yields on the defaultable corporate bonds or on the defaultable sovereign bonds.

Corporate bonds are characterized by:

- recovery rules;
- safety covenants;
- credit ratings;
- default correlations.

Recovery rules generally include clauses on the priority of payment in case of default (*seniority rules*) and on the *recovery payment*, *i.e.*, the timing and the amount of payment in case of default before the bond's maturity (*recovery scheme*). The *recovery payment* generally is defined by a *recovery rate* δ, *i.e.*, the fraction of the bond's face amount paid to the bondholders in case of default.

A defaultable bond with unit face value and maturity T is considered.
If a fixed fraction of the bond's face value is paid to the bondholders at the time of default τ, the recovery scheme is referred to as the *fractional recovery of par value*. The discounted payoff at the time t of the bond is:

$$e^{-\int_t^T r(u)\,du}\,\mathbf{1}_{\{\tau>T\}} + \delta\, e^{-\int_t^\tau r(u)\,du}\,\mathbf{1}_{\{\tau\le T\}}, \tag{1.6}$$

where $e^{-\int_t^T r(u)\,du}$ is the stochastic discount factor at time t for the maturity T, and $\mathbf{1}_{\{E\}}$ is the indicator function of the generic event E. Then the price of the defaultable bond is

$$v^\delta(t,T) = \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}\,\mathbf{1}_{\{\tau>T\}} + \delta\, e^{-\int_t^\tau r(u)\,du}\,\mathbf{1}_{\{\tau\le T\}} \mid \mathcal{G}_t\right], \tag{1.7}$$

where $\mathcal{G}_t$ contains the information about the market variables up to t and about whether and when default has occurred up to time t and $\mathbf{E}^{\mathbb{Q}}$ denotes the risk-neutral expectation in the enlarged probability space supporting τ.

If a fixed fraction of the bond's face value is paid to the bondholders at maturity date T, the recovery scheme is referred to as the *fractional recovery of Treasury value*. Under this rule, the discounted payoff of the bond is

$$\begin{aligned} & e^{-\int_t^T r(u)\,du}\ \mathbf{1}_{\{\tau>T\}} + \delta\ e^{-\int_t^T r(u)\,du}\ \mathbf{1}_{\{\tau\le T\}} \\ &= e^{-\int_t^T r(u)\,du}\ \left(1-\mathbf{1}_{\{\tau\le T\}}\right) + \delta\ e^{-\int_t^T r(u)\,du}\ \mathbf{1}_{\{\tau\le T\}} \end{aligned} \tag{1.8}$$

and the price of the defaultable bond is

$$v^\delta(t,T) = \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}\ \left(1-\mathbf{1}_{\{\tau\le T\}}\right) + \delta\ e^{-\int_t^T r(u)\,du}\ \mathbf{1}_{\{\tau\le T\}}|\mathcal{G}_t\right]. \tag{1.9}$$

If a fraction of the pre-default market value is paid to the bondholders at time of default τ, the recovery scheme is referred to as the *fractional recovery of market value*. Under this rule, the discounted payoff of the bond is

$$e^{-\int_t^T r(u)\,du}\ \mathbf{1}_{\{\tau>T\}} + \delta\ v^\delta(\tau-,T)\ e^{-\int_t^\tau r(u)\,du}\ \mathbf{1}_{\{\tau\le T\}}, \tag{1.10}$$

where $v^\delta(\tau-,T)$ is the price of the defaultable bond just before the default time τ. Then the price of the defaultable bond is

$$v^\delta(t,T) = \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}\ \mathbf{1}_{\{\tau>T\}} + \delta\ v^\delta(\tau-,T)\ e^{-\int_t^\tau r(u)\,du}\ \mathbf{1}_{\{\tau\le T\}}|\mathcal{G}_t\right]. \tag{1.11}$$

It is necessary to stress that the recovery process and/or the recovery value may be specified either exogenously or endogenously based on the current market value of the bond.

Corporate bond is only one of the securities linked to credit risk. For more on defaultable claims and credit derivatives, refer to Bielecki and Rutkowski (2004) [6] and Brigo and Mercurio (2001) [9].

1.2 Overview on 2021 global corporate default and rating transition

Each year *Standard and Poor's* publishes an annual global corporate default and rating transition study, which presents a summary of the overall situation, time trends, and updated rating transition matrices at different levels of granularity (with modifiers, without modifiers, by region, etc.).
This section presents the main results of the *Standard and Poor's Default, Transition, and Recovery: 2021 Annual Global Corporate Default And Rating Transition Study* [63].

The year 2021 was marked by a better-than-expected economic recovery, following a 2020 that was severely affected by the COVID-19 pandemic effects. As in most recovery periods, defaults decreased and for global speculative-grade firms default rates fell below 2% (1.7%) for the eighth time in 41 years (Figure 1.1 and Table 1.4). Downgrades also decreased, outstripped by upgrades of 1.85x.
Although 2020 had seen an improvement in credit quality, the distribution of ratings among companies rated by *Standard and Poor's* remains weak, with 14.5% of companies being rated B- or lower.

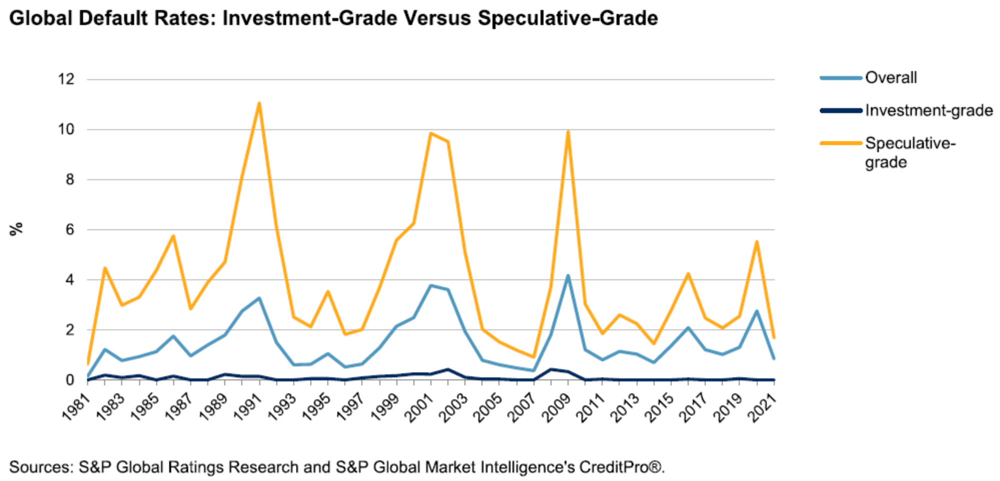


Figure 1.1. Global Default Rates: Investment-Grade Versus Speculative-Grade - Source: *Standard and Poor's*.

Year	Total defaults*	Investment-grade defaults	Speculative-grade defaults	Default rate (%)	Investment-grade default rate (%)	Speculative-grade default rate (%)	Total outstanding debt (bil. $)
2012	83	0	66	1.14	0.00	2.59	86.70
2013	81	0	62	1.03	0.00	2.24	97.29
2014	60	0	45	0.69	0.00	1.44	91.55
2015	113	0	94	1.36	0.00	2.78	110.31
2016	163	1	143	2.09	0.03	4.24	239.79
2017	95	0	83	1.21	0.00	2.47	104.57
2018	82	0	71	1.02	0.00	2.07	131.65
2019	118	2	92	1.30	0.06	2.54	183.21
2020	226	0	198	2.75	0.00	5.52	353.43
2021	72	0	60	0.84	0.00	1.68	66.28

Table 1.4. Global Corporate Default Summary (2012 - 2021). For the complete table (1981-2021), refer to [63] - Source: *Standard and Poor's.*

Consistently with the past six years, the two sectors with the highest number of defaults are consumer services and energy and natural resources, with 29 defaults (40% of the total). Defaults decreased for nearly all sectors except financial institutions, for which they remained constant. Despite the reduction in the number of defaults, the transportation and real estate sectors continued to have annual default rates above their long-term average (Figure 1.2).

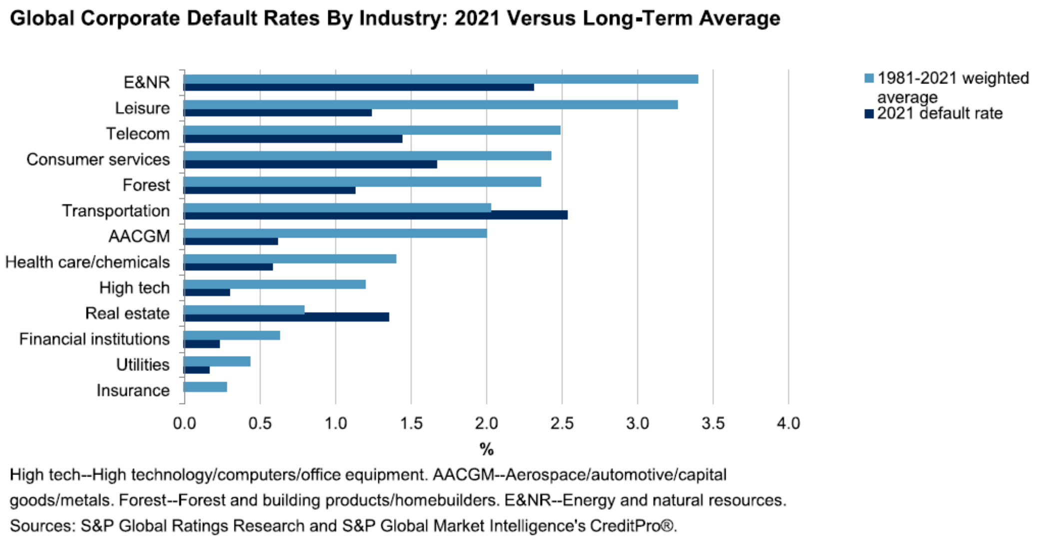


Figure 1.2. Global Corporate Default Rates By Industry: 2021 Versus Long-Term Average - Source: *Standard and Poor's*.

As shown in Figure 1.3, most defaults were concentrated at the lowest ratings: all of the defaulters were rated B or lower, with 83% rated CCC/C. This is a characteristic of the economic recovery years. However, even CCC/C rating presents a default rate for 2021 significantly lower than the long-term (1981-2020) weighted average, *i.e.*, 11% vs. 28.3%.
For the second year in a row, there have been no defaults by investment-grade firms: the default rate for the AAA rating was 0, consistently with its historical trend (Table 1.6).

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC/C**
Minimum	0.00	0.00	0.00	0.00	0.00	0.25	0.00
Maximum	0.00	0.38	0.39	1.02	4.24	13.84	49.46
Weighted long-term average	0.00	0.02	0.05	0.15	0.60	3.18	26.55
Median	0.00	0.00	0.00	0.06	0.58	3.40	25.00
Standard deviation	0.00	0.06	0.10	0.25	0.99	3.25	11.86
2008 default rates	0.00	0.38	0.39	0.49	0.81	4.11	27.27

Table 1.5. Global Corporate Annual Default Rates (%) By Rating Category (2012-2021). For the complete table (1981-2021), refer to [63] - Source: *Standard and Poor's*.

Year	AAA	AA	A	BBB	BB	B	CCC/C
2012	0.00	0.00	0.00	0.00	0.30	1.58	27.52
2013	0.00	0.00	0.00	0.00	0.10	1.52	24.67
2014	0.00	0.00	0.00	0.00	0.00	0.79	17.51
2015	0.00	0.00	0.00	0.00	0.18	2.42	26.67
2016	0.00	0.00	0.00	0.06	0.47	3.76	33.17
2017	0.00	0.00	0.00	0.00	0.08	1.00	26.67
2018	0.00	0.00	0.00	0.00	0.00	0.94	27.18
2019	0.00	0.00	0.00	0.11	0.00	1.49	29.76
2020	0.00	0.00	0.00	0.00	0.94	3.53	47.68
2021	0.00	0.00	0.00	0.00	0.00	0.52	10.99

Table 1.6. Global Corporate Annual Default Rates (%) By Rating Category (2012-2021). For the complete table (1981-2021), refer to [63] - Source: *Standard and Poor's.*

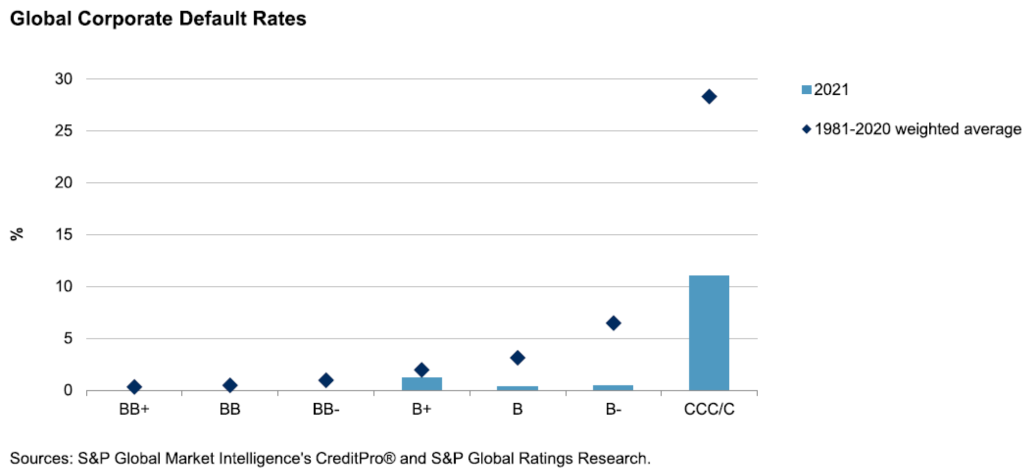


Figure 1.3. Global Corporate Default Rates (%) - Source: *Standard and Poor's.*

Credit ratings also serve as an effective measure of risk over time. In this context, all of *Standard & Poor's Global Ratings Research's default studies* have shown a strong correlation between ratings and defaults: in fact, the better the rating, the lower the frequency of default, and vice versa (Figure 1.4). In addition, studies on rating transitions have shown that better ratings tend to be more stable and that speculative-grade ratings are generally more volatile over a defined time span.

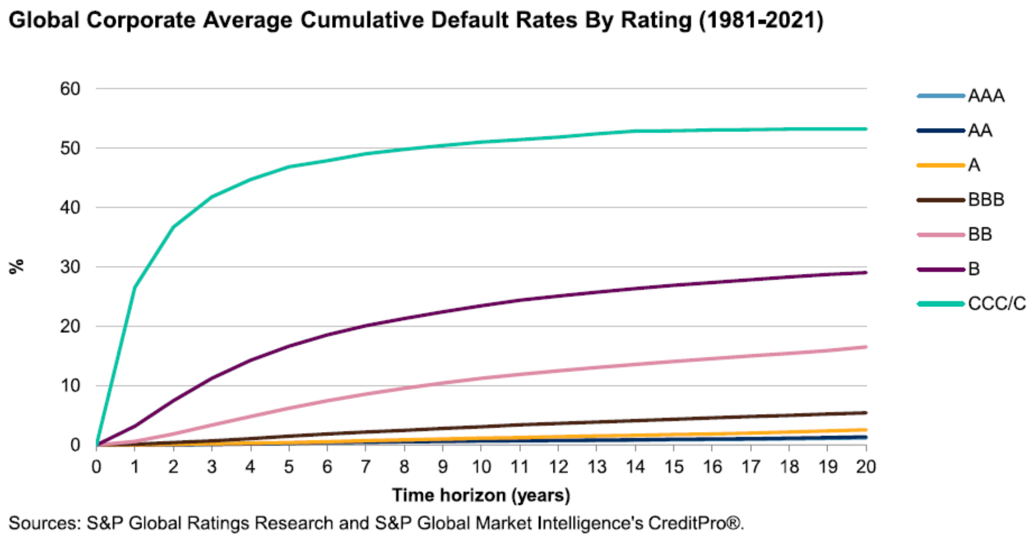


Figure 1.4. Global Corporate Average Cumulative Default Rates (%) By Rating (1981-2021) - Source: *Standard and Poor's.*

In 2021, credit quality has markedly improved. In fact, upgrade and downgrade rates appreciably improved (higher upgrade rates and lower downgrade rates): the downgrade rate fell to its lowest on record, at 5.5%, and the upgrade rate reached its highest since 2013, at 10.2% (Figure 1.5). This produced the lowest downgrade-to-upgrade ratio ever recorded.
Consistent with the decrease of downgrades, the number of large[3] rating changes was limited to one in 2021.

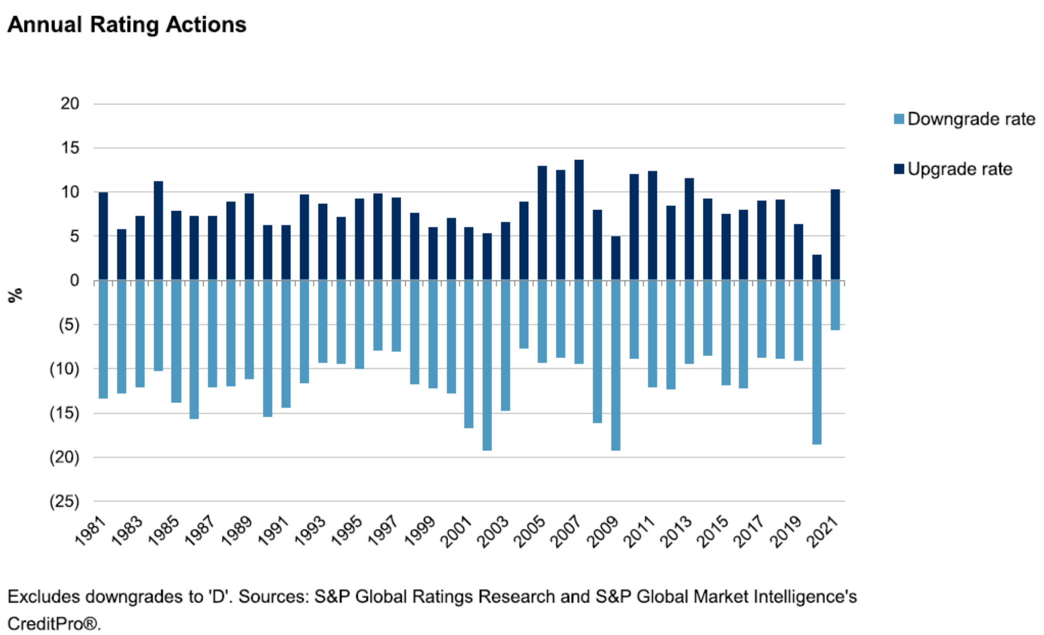


Figure 1.5. Annual Rating Actions - Source: *Standard and Poor's.*

[3]Seven notches or more, including movements to default class.

The 2021 one-year Gini ratio[4] was well above the long-term (1981-2020) one-year weighted average, *i.e.*, 87.7% vs. 82.6%, and the median annual Gini ratio over the 1981-2020 period (85.5%) (Figure 1.6 and Table 1.7).

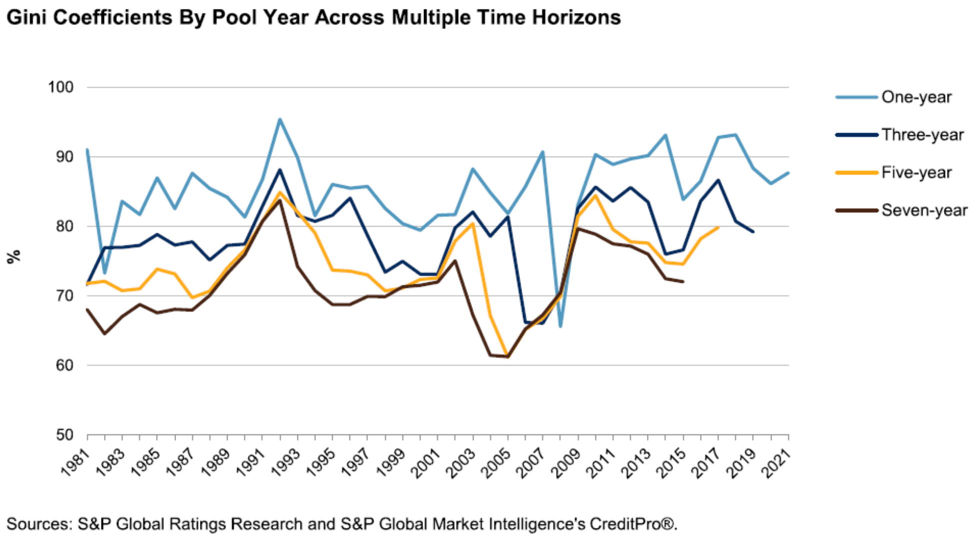


Figure 1.6. Gini Coefficients By Pool Year Across Multiple Time Horizons - Source: *Standard and Poor's.*

	Time horizon			
	One-year	**Three-year**	**Five-year**	**Seven-year**
Weighted average (%)	82.58	75.35	71.59	69.07
Average (%)	85.47	78.62	74.42	71.25
Standard deviation (%)	5.43	5.14	5.34	5.20

Table 1.7. Global Average Gini Coefficients - Sector: Global - Source: Standard and Poor's.

[4]The Gini ratio is a measure of the rank-oredring power of ratings over a given time horizon. It shows the ratio of actual rank-ordering performance to theoretically perfect rank ordering. For details on the Gini methodology, refer to [63] - Appendix II.

Defaults have decreased since 2020 in both number (60 vs. 198) and total amount of debt exposed to debt ($66.3 billion vs. $353.4 billion) and have been recorded mostly in the U.S., which is the nation with the largest number of rated corporate issuers (46%) (Figures 1.7 and 1.8).

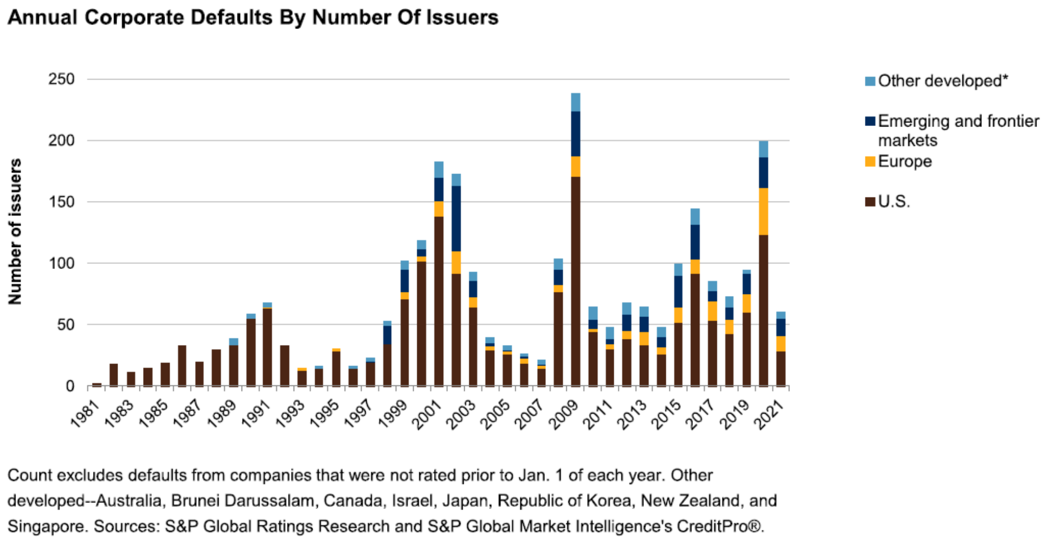


Figure 1.7. Annual Corporate Defaults By Number Of Issuers - Source: *Standard and Poor's.*

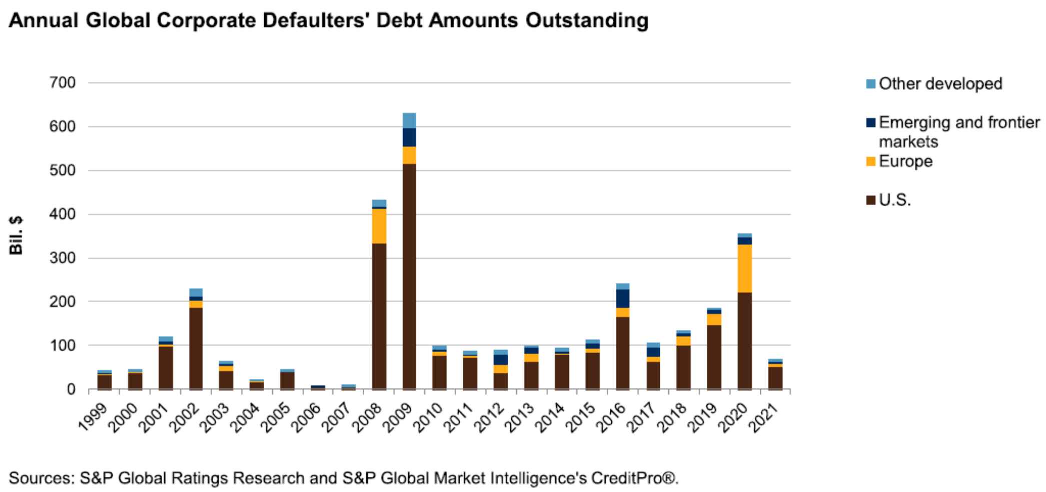


Figure 1.8. Annual Global Corporate Defaulters' Debt Amounts Outstanding - Source: *Standard and Poor's.*

The analysis of 2021 rating transitions shows that the behavior of ratings is consistent with long-term trends: higher ratings are found to be less subject to default and more stable in terms of the frequency of rating transitions. Tables 1.8 and 1.9 show, respectively, the 2021 one-year rating transition matrix and the long-term (1981-2021) average one-year rating transition matrix for the global corporates.

From/To	**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC/C**	**D**	**NR**
AAA	100.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA	0.00	91.72	5.17	0.00	0.00	0.00	0.00	0.00	3.10
A	0.00	0.50	92.74	3.99	0.00	0.00	0.00	0.00	2.78
BBB	0.00	0.00	2.02	91.78	1.74	0.05	0.00	0.00	4.41
BB	0.00	0.00	0.00	3.89	85.30	3.21	0.17	0.00	7.43
B	0.00	0.00	0.00	0.10	4.54	77.20	1.96	0.52	15.68
CCC/C	0.00	0.00	0.00	0.00	0.00	21.76	50.99	10.99	16.26

Table 1.8. 2021 Global One-Year Corporate Transition Rates (%) - Source: S*tandard and Poor's.*

From/To	**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC/C**	**D**	**NR**
AAA	87.09	9.05	0.53	0.05	0.11	0.03	0.05	0.00	3.10
AA	0.48	87.32	7.72	0.46	0.05	0.06	0.02	0.02	3.88
A	0.02	1.56	88.73	4.97	0.25	0.11	0.01	0.05	4.29
BBB	0.00	0.08	3.19	86.72	3.48	0.42	0.09	0.15	5.86
BB	0.01	0.02	0.10	4.52	78.12	6.66	0.53	0.60	9.43
B	0.00	0.02	0.06	0.15	4.54	74.73	4.81	3.18	12.51
CCC/C	0.00	0.00	0.09	0.16	0.49	13.42	43.91	26.55	15.39

Table 1.9. Global Corporate Average One-Year Transition Rates (1981-2021) (%) - Source: *Standard and Poor's.*

1.3 Credit spread: definitions and data

It is essential for the credit risk market to define a measure of the risk premium required by investors to remunerate the assumption of credit risk embedded in a security. Such a measure would make it possible to compare securities that differ by issuer, maturity, coupon or seniority. In practice, there are a number of such measures, which are generally referred to as “credit spreads”, since they attempt to measure the difference in creditworthiness by comparing the yield of the credit risk security with that of some benchmark with higher creditworthiness, usually either a government bond assumed to be risk-free or the LIBOR swap rate at the same maturity.

However, as argued in Amato and Remolona (2003) [2], credit spread should not be considered just an extra return to compensate for the counterparty default risk; rather, it can be determined and defined by more than one component:

- *Expected loss.* Since corporate bonds are issued by companies subject to default risk, some issuers will necessarily go into default; this drives investors to demand a higher yield to compensate for this eventuality. This component, though the most straightforward, is incorrectly considered the main one, as it may account for just a small percentage of the whole spread.

- *Taxation.* With reference to the U.S. market, corporate bonds are taxed at the individual state level, while government bonds are tax-exempt; this leads investors, who reason and operate in terms of net returns, to demand higher returns for corporate bonds to compensate for the effect of taxation. This can represent a significant component of the spread (for Elton et al. (2001) [20] between 28% and 73%, for Driessen (2004) [19] between 34% and 57%).

- *Risk premium.* This component includes premiums for systematic risk, specific risk, and other risks other than default, which are difficult to diversify, driving investors to demand an additional risk premium over that for expected losses and taxation. Driessen (2004) estimates that the weight of this component can vary between 18% and 53% of the spread.

- *Liquidity risk.* Since the corporate bond market is not always particularly liquid in comparison with the stock or government bond market, it is characterized by higher transaction costs that lead investors to demand additional yield. Driessen (2004) estimates this component to be about 20% of the spread.

Credit spread measures can be classified as follows:

- credit spread measures for fixed rate bonds, including the yield spread, the interpolated spread, the option-adjusted spread (OAS), the asset swap spread (ASW) and the Credit Default Swap spread;

- credit spread measures for Floating Rate Notes (FRNs), including quoted margin, discount margin, and zero discount margin.

The former will be discussed in detail below; for a detailed description of the latter, refer to O’Kane and Sen (2004) [57].

1.3.1 Credit spread measures for fixed rate bonds

This section will define the main credit spread measures for fixed rate bonds, summarized in Figure 1.9.

The yield spread

The yield spread is defined as the difference between the yield-to-maturity of a credit risky bond and the yield-to-maturity[5] of a benchmark treasury bond with similar, but not necessarily coincident, maturity:

$$\text{Yield spread} = y_D - y_B \,, \tag{1.12}$$

where y_D is the yield of the defaultable bond and y_B is the yield of the benchmark bond.

It is the most straightforward and easiest spread measure to use; however, its simplicity has some drawbacks in its use. In fact, being based on yield-to-maturity, it shares its weaknesses in terms of constant reinvestment rate and hold to maturity. In addition, the benchmark can change over time as the bond rolls down the curve, making the yield spread a not consistent measure over time.
For these reasons, the yield spread should be used only as a way to express the price of a bond relative to the benchmark, rather than as a measure of credit risk.

The interpolated Spread

The interpolated Spread (I-spread) is defined as the difference between the yield-to-maturity of the bond and the linearly interpolated yield to the same maturity on an appropriate reference curve:

$$ISpread = y_D - \left[y_{G1} + \left(\frac{y_{G2} - y_{G1}}{T_{G2} - T_{G1}} \right) (T_D - T_{G1}) \right] \,, \tag{1.13}$$

where T_{G1} and T_{G2} are the maturities of the two benchmark bonds, which straddle the maturity of the defaultable bond, and y_{G1} and y_{G2} are the respective yields-to-maturity.

The interpolated spread allows to overcome the problem of maturity mismatch by using a yield at the correct maturity, although interpolated. Anyway, this measure inherits all the drawbacks of the yield-to-maturity, and so it should be used only as a way to express the price of a bond relative to the reference curve.

The option-adjusted spread

The option adjusted spread (OAS) is defined as the parallel shift to the LIBOR zero rate curve that allows to replicate, with the adjusted curve, the price of the bond. The OAS can be referred to as the zero-volatility Spread (ZVS), if the semi-annual

[5] The yield-to-maturity is the constant discounting rate which, when applied to the bond's cash-flows, allows to replicate the price of the bond.

compounding is assumed.

Assuming a discrete compounding for the OAS with the frequency of the bond f, the OAS, denoted by Ω, satisfies the following equation:

$$P^D = \frac{C}{f} \sum_{j=1}^{n} \frac{1}{\left(1 + \frac{(r_{T(j)}+\Omega)}{f}\right)^{f\,T(j)}} + \frac{100}{\left(1 + \frac{(r_{T(n)}+\Omega)}{f}\right)^{f\,T(n)}} \,, \tag{1.14}$$

where P^D is the price of the defaultable bond (including accrued interests), C is the annual coupon of the bond and $r_{T(j)}$ is the LIBOR zero rate at the j-th maturity.

In case of continuous compounding for the OAS, the above equation becomes

$$P^D = \frac{C}{f} \sum_{j=1}^{n} Z_{T(j)} e^{-\Omega\,T(j)} + 100\; Z_{T(n)}\; e^{-\Omega\,T(n)} \,, \tag{1.15}$$

where $Z_{T(j)}$ is the LIBOR discount factor at the j-th maturity.

The option-adjusted spread takes into account the shape of the term structure of interest rates in a robust manner, so it can be used to measure the credit risk embedded in a bond, although it remains essentially a relative value measure.

The asset swap spread

The asset swap spread is defined as the spread over LIBOR paid on the floating leg in a par asset swap package.

The par asset swap package is an instrument for hedging against interest rate risk, in which a fixed-rate component (fixed leg) and a floating-rate component (floating leg) are combined:

- the investor pays the par and receives the bond worth the full price;
- at the same time, the investor enters into an interest rate swap, paying the fixed leg identical in size and timing to the coupon schedule of the bond and receiving a fixed spread over LIBOR, the asset swap spread, with its own frequency, basis and settlement conventions.

The par asset swap spread is given by the formula:

$$A = \frac{P^{LIBOR} - P^D}{PV01} \,, \tag{1.16}$$

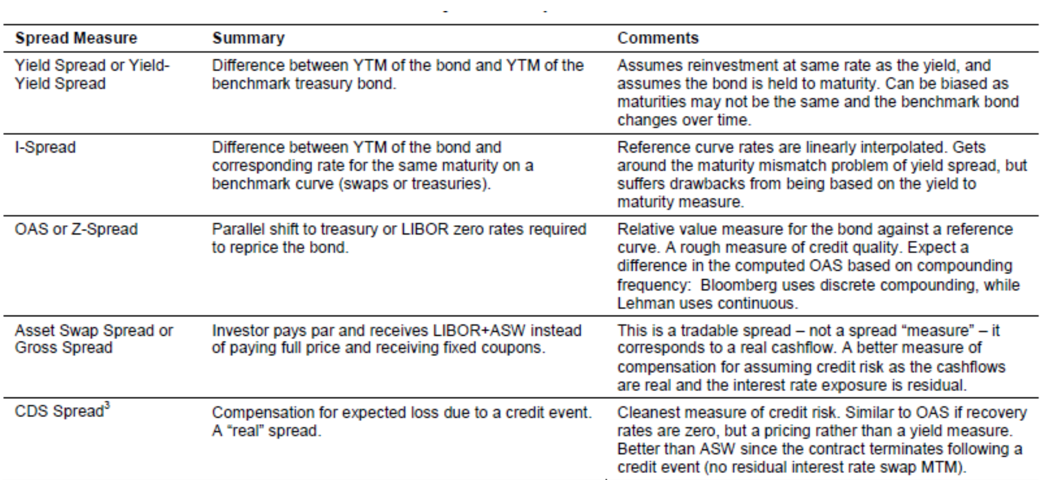
where P^{LIBOR} is the value of the bond's cash-flows discounted at LIBOR, P^D is the market price of the bond, and $PV01$ is the LIBOR discounted present value of a 1 basis point coupon stream, paid according to the frequency, basis and stub conventions of the floating leg of the interest rate swap.

The asset swap spread represents an actually traded spread, and thus can be considered a measure of creditworthiness. In fact, if the bond characteristics are kept constant, changes in price can only result from a change in the perceived creditworthiness of the issuer: the bond price increases, and thus the asset swap spread decreases, only if the implied credit risk decreases and vice versa.

The Credit Default Swap spread

The Credit Default Swap (CDS) spread is defined as the contractual spread which determines the cash-flows paid on the premium leg of a credit default swap, *i.e.*, the contractual premium paid by the protection buyer in a CDS contract.

The CDS spread measures an investor's compensation for assuming the risk of issuer default, that is, the risk of losing the face value of the bond minus the expected recovery rate. So it can be considered the best measure for credit risk. In fact, CDS are contracts affected almost exclusively by credit risk; a term structure of CDS spreads can be observed; and finally, the CDS market is sufficiently liquid, hence the CDS spread can accurately reflect the market price of credit risk.



Spread Measure	Summary	Comments
Yield Spread or Yield-Yield Spread	Difference between YTM of the bond and YTM of the benchmark treasury bond.	Assumes reinvestment at same rate as the yield, and assumes the bond is held to maturity. Can be biased as maturities may not be the same and the benchmark bond changes over time.
I-Spread	Difference between YTM of the bond and corresponding rate for the same maturity on a benchmark curve (swaps or treasuries).	Reference curve rates are linearly interpolated. Gets around the maturity mismatch problem of yield spread, but suffers drawbacks from being based on the yield to maturity measure.
OAS or Z-Spread	Parallel shift to treasury or LIBOR zero rates required to reprice the bond.	Relative value measure for the bond against a reference curve. A rough measure of credit quality. Expect a difference in the computed OAS based on compounding frequency: Bloomberg uses discrete compounding, while Lehman uses continuous.
Asset Swap Spread or Gross Spread	Investor pays par and receives LIBOR+ASW instead of paying full price and receiving fixed coupons.	This is a tradable spread – not a spread "measure" – it corresponds to a real cashflow. A better measure of compensation for assuming credit risk as the cashflows are real and the interest rate exposure is residual.
CDS Spread[3]	Compensation for expected loss due to a credit event. A "real" spread.	Cleanest measure of credit risk. Similar to OAS if recovery rates are zero, but a pricing rather than a yield measure. Better than ASW since the contract terminates following a credit event (no residual interest rate swap MTM).

Figure 1.9. Summary of credit spread measures for fixed rate bonds - Source: O'Kane D., Sen S. *Credit Spreads Explained,* Lehman Brothers Fixed Income Quantitative Credit Research Series, 2004.

1.3.2 Credit spread data

The *Solvency II* Directive gives considerable importance to the data used for expected calculations, particularly for SCR calculation, and to data-quality. In fact, the regulation encourages companies to internally create a data-driven culture and governance through a high level of technology throughout the whole governance structure: "*Member States shall ensure that insurance and reinsurance undertakings have internal processes and procedures in place to ensure the appropriateness,*

completeness and accuracy of the data used in the calculation of their technical provisions." [26].
The data must therefore comply with the principles of:

- completeness, *i.e.*, the data must cover a sufficiently large time period of observations based on the phenomenon being measured, and must contain all information that is fundamental and relevant to the assessments for which they are used;

- appropriateness, *i.e.*, the data must be consistent with the methods and purposes for which they are used;

- accuracy, in terms of correctness and precision, *i.e.*, the data must be free of material errors and must be up-to-date as of the date of assessment.

In the case of full or partial internal models, the data play an even more central role because they are the starting point for the estimation procedures used to estimate model parameters. The correct calculation of SCR requires methodologies that are able to infer:

- both *real-world* and risk-neutral parameters of the models;

- both the parameters of the models used to describe the various elementary sources of risk and the parameters of the dependence structure between the risks, which is also modeled.

To comply with the requirements prescribed by the regulations, the model used to measure spread risk must be able to represent the dynamics of spreads in both *real-world* and risk-neutral measures. Therefore, regardless of the model considered, a data source that allows all model parameters to be estimated under both probability measures must be selected: thus, time series with adequate depth must be available to robustly estimate *real-world* parameters as well. In fact, an estimation on the *cross section* of prices of financial instruments traded on the valuation date can be used for trading and hedging activities but is inadequate for insurance risk management.

There are two types of data available to estimate a model for spread risk under both measures:

- yield spreads, either for individual bonds or for bond indices;

- Credit Default Swap (CDS) index quotes.

In this work, specifically, the *IHS Markit*[6] provider and *IHS Markit* `iBoxx` indices spreads are used, in line with what is described by EIOPA in the *Technical documentation of the methodology to derive EIOPA's risk-free interest rat term structures* (2021) [25] for providing market yield data for corporate bonds aimed at calculating the Volatility Adjustment and Matching Adjustment.

[6] Part of *Standard & Poor's.*

1.3.3 *IHS Markit* `iBoxx` indices

An index is defined as a statistical measure, typically of a price or quantity, calculated from a representative set of underlying data. The main role of an index is to be a benchmark. In fact, an index provides a measure of the performance of a specific segment of a financial market (region, sector, or other asset class) and can be the standard against which the performance of a financial instrument is evaluated.

Bond indices are used to measure the value of various sections of the bond market, being defined by maturity or rating, and are a way to deal with the complexity of the bond market. In fact, bonds, unlike stocks, can be very different by asset class, by type (government, corporate, or sovereign), by tenor, by frequency and coupon rates, and are traded over-the-counter. A further element of complexity is the lower liquidity of the bond market compared to the stock market.

All `iBoxx` indices are basket indices provided by *IHS Markit* and express relative changes in value from the beginning of the relevant period. The composition and weights of the indices are updated at the beginning of each period. The `iBoxx` indeces can be classified by:

- region (global, Europe, North America, Asia Pacific)
- country
- currency (EUR, USD, GBP, ...)
- sector (sovereign, financial, industrial, telecommunications, ...)
- rating (single rating, macro-rating)
- maturity
- type of bond

Table 1.10 shows the existing `iBoxx` index families (updated to September 2022).

In this work, only indices belonging to the `iBoxx` *EUR Indices* and `iBoxx` *EUR High Yield Indices* families are used. The former is designed to reflect the performance of EUR denominated investment grade debt and is split into five major indices: Overall, Eurozone, Collateralized, Corporates and Sub-sovereigns, which in turn are divided into sub-indices based on rating, maturity and sector, as summarized in Figure 1.10. The latter, on the other hand, is designed to reflect the performance of EUR denominated sub-investment grade corporate debt and is split into four different indices: overall, maturity, rating and sector indices, as summarize in Figure 1.11.

Index Family Name	**Region**
iBoxx ABF Indices	APAC
iBoxx ADBI Indices	APAC
iBoxx ALBI Indices	APAC
iBoxx ASIA Indices	APAC
iBoxx AUD Indices	APAC
iBoxx CNH Indices	APAC
iBoxx CNY Indices	APAC
iBoxx JPY Indices	APAC
iBoxx SAR Indices	APAC
iBoxx SGD Indices	APAC
iBoxx USD APAC Indices	APAC
iBoxx USD ASIA Indices	APAC
iBoxx USD Belt & Road Indices	APAC
iBoxx EUR Indices & iBoxx EUR Liquid Investment Grade	Europe
iBoxx EUR FRN Investment Grade Indices	Europe
iBoxx EUR High Yield Indices & iBoxx EUR Liquid High Yield Indices	Europe
iBoxx GBP Indices	Europe
iBoxx GBP High Yield Indices & iBoxx GBP Liquid High Yield Indices	Europe
iBoxx Contingent Convertible Indices	Global
iBoxx USD Emerging Markets Sovereigns Indices	Global
iBoxx USD Emerging Markets Corporates Indices	Global
iBoxx GEMX Indices	Global
iBoxx Global GOV Indices	Global
iBoxx Global HY Indices	Global
iBoxx Global SOV Indices	Global
iBoxx Green Select Indices	Global
iBoxx Green Soc Sust Indices	Global
iBoxx Infrastructure Indices	Global
iBoxx ILB Indices	Global
iBoxx USD Leveraged Loan Indices	North America
iBoxx USD Indices & iBoxx USD Liquid Investment Grade Indices	North America
iBoxx USD FRN Indices	North America
iBoxx USD High Yield Developed Markets Indices & iBoxx USD Liquid High Yield Indices	North America
iBoxx USD MM Indices	North America
iBoxx USD CMBS Indices	North America

Table 1.10. iBoxx index families (updated to September 2022) - Source: *IHS Markit*.

Markit iBoxx EUR Overall			
Sovereigns	**Non-Sovereigns**		
Sovereigns	Sub-Sovereigns	Collateralized	Corporates
• Eurozone Sovereigns > Eurozone country indices	• Other Sovereigns • Agencies • Public Banks • Regions • Supranationals • Other Sub-Sovereigns	• Covered > Country indices • Securitized • Other Collateralized	• Financials > Market sector indices – Market sub-sector indices • Non-Financials > Market sector indices – Market sub-sector indices
Rating and maturity indices			

Figure 1.10. Overview of Markit iBoxx EUR family indices - Source: *IHS Markit*, *Markit* iBoxx *EUR Benchmark Index Guide*, September 2022.

Markit iBoxx EUR High Yield Overall		
Core High Yield	**Special Bond Types**	**Corporate Sectors**
• Fixed coupon bonds • Floating rate notes • Callable bonds • Sinking funds • Rating sensitive bonds • Bonds with poison put option • Bonds with make-whole call or tax changes call provision	• Fixed rate bonds > Step up bonds > Callable bonds > Sinking funds • FRNs • PIK Notes	• Financials > Market sector indices – Market sub-sector indices • Non-Financials > Market sector indices – Market sub-sector indices
Rating and maturity indices (BB, B, CCC) (1-3, 3-5, 5-7, 7-10, 1-5, 5-10, 5+, 1-10, 10+)		

Figure 1.11. Overview of Markit `iBoxx` EUR family indices - Source: *IHS Markit*, *Markit* `iBoxx` *EUR High Yield Index Guide*, September 2022.

The following selection criteria are used to determine the index constituents:

- bond type (*inter alia*, fixed coupon bonds, zero coupon bonds, ...)
- credit rating: all bonds in the *IHS Markit* `iBoxx` *EUR Indices* must have an `iBoxx` rating[7] of investment grade[8], while for `iBoxx` *EUR High Yield Indices* all bonds must have an `iBoxx` rating of sub-investment grade[9].
- time to maturity: all bonds must have a remaining time to maturity of at least one year at rebalancing time.
- issuer eligibility: only EUR denominated debt from corporate issuers is eligible, not considering the country of risk or origin.
- amount outstanding: the minimum required amount outstanding is between 500 million and 1 billion for investment grade bonds and 150 million for high yield bonds.

All bonds are classified based on the principal activities of the issuer and the main sources of the cash flows used to pay coupons and redemptions:

- sovereigns: bonds issued by a central government of a member country of the Eurozone and denominated in Euro or in a pre-Euro currency;
- other sovereigns: bonds issued by a central government that is not a member country of the Eurozone and denominated in Euro or in a pre-Euro currency;
- sub-sovereigns: bonds issued by entities with explicit or implicit government backing due to legal provision, letters of comfort or the public service nature of their business (agencies, supranationals, public banks, regions, ...);

[7] More details on the `iBoxx` Rating Methodology can be found in Appendix A.

[8] BBB- or higher from *Fitch Ratings* and *S&P Global Ratings* and Baa3 or higher from *Moody's Investor Service*.

[9] BB+ or lower from *Fitch Ratings* and *S&P Global Ratings* and Ba1 or lower from *Moody's Investor Service*, but not in default.

- collateralized:
  - covered bonds: bonds which are secured by a general pool of assets in case the issuer becomes insolvent;
  - securitized bonds: bonds secured against specific assets or receivables (ABS), mortgages (MBS) or cash-flows from a whole business segment (Whole Business Securitizations), in each case via a special purpose vehicle;
  - other collateralized bonds;
- corporates: bonds issued by public or private corporations. Corporate bonds are further classified into *Financials* and *Non-Financials* bonds and then into their multiple-level economic sectors, as summarized in Figure 1.12. Corporate bonds are also classified according to the seniority of the debt.

For further details, refer to [31] and [32].

	Economic Sector	Market Sector	Market Sub-Sector
Financials	Core Financials	Banks	Banks
		Insurance	Life Insurance
			Nonlife Insurance
	Financial Services	Financial Services	General Financial
			Equity Investment Instruments
			Nonequity Investment Instruments
		Insurance-wrapped	*
	Real Estate	Real Estate	Real Estate Investment & Services
			Real Estate Investment Trusts
Non-Financials	Basic Materials	Basic Resources	Forestry & Paper
			Industrial Metals
			Mining
		Chemicals	Chemicals
	Consumer Goods	Automobiles & Parts	Automobiles & Parts
		Food & Beverage	Beverages
			Food Producers
		Personal & Household Goods	Household Goods
			Leisure Goods
			Personal Goods
			Tobacco
	Consumer Services	Education	Academic & Educational Services
		Media	Media
		Retail	Food & Drug Retailers
			General Retailers
		Travel & Leisure	Travel & Leisure
	Health Care	Health Care	Health Care Equipment & Services
			Pharmaceuticals & Biotechnology
	Industrials	Construction & Materials	Construction & Materials
		Industrial Goods & Services	Aerospace & Defense
			Electronic & Electrical Equipment
			General Industrials
			Industrial Engineering
			Industrial Transportation
			Support Services
	Oil & Gas	Oil & Gas	Alternative Energy
			Oil Equipment / Services & Distribution
			Oil & Gas Producers
	Technology	Technology	Software & IT Services
			Technology Hardware & Equipment
	Telecommunications	Telecommunications	Integrated Telecommunications
			Wireless Telecommunications
	Utilities	Utilities	Electricity
			Gas / Water & Multiutilities

Figure 1.12. Overview of Markit `iBoxx` Corporates Sectors - Source: IHS Markit, *Markit* `iBoxx` *EUR Benchmark Index Guide*, September 2022.

1.4 The measure of spread risk in *Solvency II*

This chapter discusses the aspects of risk measurement in *Solvency II* affected by spread risk.

Section 1.4.1 recalls the main features of the current regulatory framework for risk measurement and summarizes the structure of the *Standard Formula*. The possibility offered by the regulations to use internal methodologies as an alternative to the *Standard Formula* - so-called internal model - or parts of it - partial internal model - is also recalled.

Section 1.4.2, on the other hand, focuses on the definition and measurement of spread risk within the *Solvency II* framework.

These aspects are particularly relevant in the scope of investigation of this thesis, as the spread risk measurement methodologies proposed in the following chapters are of interest to the insurance sector precisely because of the possibility offered by *Solvency II* to use alternative measurement techniques for individual risk modules, provided one can demonstrate to the national authority in one's country that these are more appropriate and realistic than the *Standard Formula* to represent the risk profile of the insurance company that adopts them.

1.4.1 The current regulatory framework for risk measurement in insurance sector: the Directive 2009/138/EC and the *Standard Formula*

Framework Directive 2009/128/EU [26], known as *Solvency II*, published in Dec. 17$^{\text{th}}$ 2009, marked a profound cultural change in the risk management of the insurance industry. In fact, it did not merely define new solvency requirements, but redefined the management and supervision of an insurance or reinsurance company across the board.
The reasons that led to this veritable revolution in the European insurance system are to be found in the limitations of the previous solvency system, in particular in the fact that it was a poorly risk-sensitive system, providing a capital requirement that was not very sensitive to a company's actual risk profile, and not risk-based, in that it did not take into account all the risks to which an insurance or reinsurance company is exposed, whether on the asset or liability side.

The goals of this legislation are:

- investor protection through preventive control and transparent management of insurance and reinsurance companies;

- to create a uniform European supervisory system through harmonization of the principles and rules on which to base the solvency of any company in the EU;

- improved and increased transparency in risk management. *Solvency II* is based on quantitative aspects of risk measurement that require the use of the insurance company's own internal models and processes.

The system is articulated with a 3-pillar structure, inspired by the similar banking regulation *Basel III*, in which a risk-based supervisory system is promoted to fulfill the need for clear and transparent disclosure to investors and consumers, with the aim to increase the market's ability to assess the solvency of insurance companies. The evaluation of the financial position of the enterprise should be based on an economic balance sheet, in which fair value represents the general valuation principle of assets and liabilities.
Assets shall be valued with a marked-to-market approach, *i.e.*, "*at the amount for which they could be exchanged between knowledgeable willing parties in an arm's length transaction*" [26].
Liabilities shall be valued according to the principle of current exit value, *i.e.*, "*at the amount for which they could be transferred, or settled, between knowledgeable willing parties in an arm's length transaction*" [26]. The fair value of liabilities can be calculated according to two distinct methodologies:

- as a whole, if their value is directly observable in the market;

- as the sum of a best estimate and a risk margin. The best estimate is defined as "*the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure*" [26]. The risk margin is "*such as to ensure that the value of the technical provisions is equivalent to the amount that insurance and reinsurance undertakings would be expected to require in order to take over and meet the insurance and reinsurance obligations*" [26].

The difference between assets and liabilities constitutes the so-called *Basic Own Funds* (BOF), which are the financial resources the company has at its disposal. This variable plays a key role in defining the *Solvency Capital Requirement* (SCR), which companies must hold in order to cope with the risks to which they are subjected.
The SCR is defined as "*the Value-at-Risk of the Basic Own Funds of an insurance or reinsurance undertaking subject to a confidence level of 99,5% over a one-year period*" [26]. That is, if the firm has equity in an amount equal to the SCR, it is probabilistically guaranteed that the ruin event will not occur more than once in 200 cases, *i.e.*, that the firm will still be able, with a probability of at least 99.5%, to meet its obligations to policy-holders and beneficiaries in the following 12 months.

Solvency II regulations provide various methods for calculating the SCR, characterized by increasing levels of complexity and sensitivity to the company's specific risk profile. The *Standard Formula*, which is the basis and benchmark of the Directive, proposes a modular approach, the structure of which is illustrated in Figure 1.13.

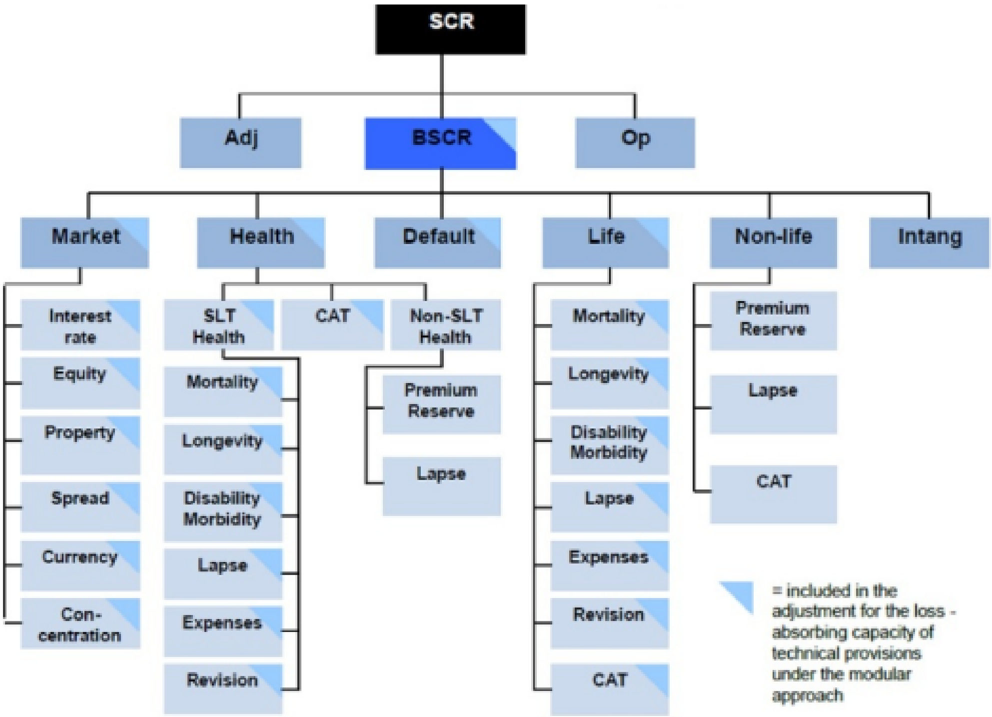


Figure 1.13. Modules and sub-modules of risk of the *EIOPA Standard Formula*.

Marginal SCRs for risk sub-modules are calculated - in factor based[10] or scenario based[11] logic - by means of shocks provided by the regulations, which the company must apply to its assets and liabilities. Once estimated, these are aggregated, by means of correlation matrices provided by EIOPA, firstly at each module level, and then at the level of the *Basic Solvency Capital Requirement* that the insurance company must meet, following a bottom-up approach:

$$BSCR = \sqrt{\sum_{i,j} Cor_{ij}\, SCR_i\, SCR_j} + SCR_{intang}\,. \tag{1.17}$$

Companies are provided by the regulations with the option of using internal methodologies as an alternative to the *Standard Formula* - so-called internal model (IM) - or parts of it - partial internal model (PIM). The model selection is motivated by a number of considerations: some related to the peculiarities of the current situation of financial markets - for example, the observation of negative nominal rates -, others to the principles of *Solvency II* regulations, such as the character of substantiality and adequacy to the company's activities.
The model selection also requires balancing complexity, usability, and readability of results, in search of the right medium, with the guarantee of acting as best as possible (the best effort of the regulations).

[10]Based on the application of one or more factors to a measure of risk exposure.
[11]Based on the study of the impact of adverse scenarios, established by EIOPA.

The selection of models to be used must therefore be defined by:

- ability to represent substantive risks, in a pattern reasonably comparable to that adopted in the *Standard Formula* regulatory approach;
- ability to represent the main features of market risks, in a formally rigorous theoretical framework;
- manageability from the point of view of the computational complexity of the calculation procedures;
- assurance of transparency in the interpretation of the model's findings, for management purposes.

1.4.2 Spread risk in the *Solvency II* framework

The spread risk sub-module, defined as *the sensitivity of the values of assets, liabilities and financial instruments to changes in the level or in the volatility of credit spreads over the risk-free interest rate term structure* [26], can be found within the market risk module.
The *Delegated Regulation* [21] defines the SCR for spread risk sub-module, SCR_{spread}, as the sum of the following capital requirements:

$$SCR_{spread} = SCR_{bonds} + SCR_{securitisation} + SCR_{cd} \ , \tag{1.18}$$

where:

- SCR_{bonds} is the capital requirement for spread risk on bonds and loans;
- $SCR_{securitisation}$ is the capital requirement for spread risk on securitisation positions;
- SCR_{cd} is the capital requirement for spread risk on credit derivatives.

For the purpose of this discussion, it is useful to specifically investigate the calculation of SCR related to loans and bonds, defined as *equal to the loss in the basic own funds that would result from an instantaneous relative decrease in the value of each bond or loan* [21].

Bonds or loans for which a credit assessment by a nominated *External Credit Assesment Institution* (ECAI) is available shall be assigned a risk factor stress, which depends on the credit quality step and the modified duration of the bond or loan, accordingly to Table 1.11.
Spread risk stress factors defined according to creditworthiness class refer to an objective creditworthiness class scale. On the other hand, the creditworthiness assessment provided by the various ECAIs follows internal agency evaluation parameters. In order to standardize the different ratings, EIOPA provided, through *European Commission Implementing Regulation 2016/1800*, the correspondence between ECAIs' credit ratings and the objective scale of creditworthiness classes. Table 1.12 shows this correspondence relative to the ratings provided by *Standard & Poor's Global Ratings*.

Credit quality step		0		1		2		3		4		5 e 6	
Duration (dur_i)	$stress_i$	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i
$dur_i \leq 5$	$b_i \cdot dur_i$		0.9%		1.1%		1.4%		2.5%		4.5%		7.5%
$5 < dur_i \leq 10$	$a_i + b_i \cdot (dur_i - 5)$	4.5%	0.5%	5.5%	0.6%	7.0%	0.7%	12.5%	1.5%	22.5%	2.5%	37.5%	4.2%
$10 < dur_i \leq 15$	$a_i + b_i \cdot (dur_i - 10)$	7.0%%	0.5%	8.4%	0.5%	10.5%	0.5%	20.0%	1%	35.0%	1.8%	58.5%	0.5%
$15 < dur_i \leq 20$	$a_i + b_i \cdot (dur_i - 15)$	9.5%	0.5%	10.9%	0.5%	13.0%	0.5%	25.0%	1.0%	44.0%	0.5%	61.0%	0.5%
$dur_i > 20$	$\min[a_i + b_i \cdot (dur_i - 20); 1]$	12.0%	0.5%	13.4%	0.5%	15.5%	0.5%	30.0%	0.5%	46.6%	0.5%	63.5%	0.5%

Table 1.11. Risk factor stress for bonds or loans for which a credit assessment by a nominated *External Credit Assesment Institution* (ECAI) is available - Source: European Commission, *Delegated Regulation 2015/35*, 2014.

0	1	2	3	4	5	6
AAA	AA	A	BBB	BB	B	CCC, CC, C, D

Table 1.12. Allocation of *Standard & Poor's Global Ratings* credit assessments to the EIOPA objective scale of credit quality steps - Source: European Commission, *Implementing Regulation 2016/1800*, 2016.

In contrast, bonds and loans for which a credit assessment by a nominated ECAI is not available shall be assigned a risk factor stress depending only on the duration of the bond or loan, accordingly to Table 1.13.

Duration (dur_i)	$stress_i$
$dur_i \leq 5$	$3\% \cdot dur_i$
$5 < dur_i \leq 10$	$15\% + 1.7\% \cdot (dur_i - 5)$
$10 < dur_i \leq 20$	$23.5\% + 1.2\% \cdot (dur_i - 10)$
$dur_i > 20$	$\min[35.5\% + 0.5\% \cdot (dur_i - 20); 1]$

Table 1.13. Risk factor stress for bonds or loans for which a credit assessment by a nominated *External Credit Assesment Institution* (ECAI) is not available - Source: European Commission, *Delegated Regulation 2015/35*, 2014.

Chapter 2

Credit risk models

In the scientific literature on credit risk modelling, two main issues are identified: the modelling of the default time, *i.e.*, the random time when the default event occurs, and the modelling of the term structure of credit spreads and, in particular, of the random times of credit migration in an approach that considers intermediate credit events.

The main goal of modelling credit risk is pricing and hedging financial contracts that are sensitive to credit risk, while ensuring internal consistency of the financial model, *i.e.*, compliance with the arbitrage-free principle. In order to model the default/credit migration times and the recovery rate, two main approaches are identified:

- the structural approach, based on the firm's value and some credit-event-triggering threshold (*firm's value models*);
- the reduced-form approach, based on the default process (*intensity models*).

The main difference between structural and reduced-form approaches is the underlying vision. In structural models, the default event is triggered by changes of the market value of the firm's assets; starting from this and their volatility, these models return as output the price of credit-sensitive securities and the default probability. Reduced-form models, on the other hand, are based on reverse-engineering, and model the default process starting from the prices of credit-sensitive securities to return as output the model parameters, such as the risk-neutral default probabilities. This difference is also reflected in the different estimation techniques applied in the two approaches: the parameters of structural models are generally estimated *econometrically* from the time series of share prices, while the parameters of reduced-form models are inferred directly from the prices of securities traded on the market through *cross-sectional* estimation, assuming an arbitrage-free framework.

Another key difference between the two approaches is in the definition of the default event: in structural models it is determined by the deterioration of the firm's assets, while in reduced-form models it coincides with the instant at which a Poisson process makes the first jump.

As for recovery rates, they are generally endogenous in structural models, where the value of the firm's assets at the default time is estimated, and exogenous in reduced-form models, where the value of the firm's assets is not modelled at all.

Another classification for credit risk models is:

- models for the assessment of individual counterparties, based on the structural approach, which include the cornerstone of credit risk models, *i.e.*, the Merton model, the *EDF RiskCalc* model developed by *Moody's KMV* [55] [56] and the *CrediGrades* model by *RiskMetrics*, *Goldman Sachs*, *J.P. Morgan* and *Deutsche Bank*;

- models for the valuation of portfolios, based on both approaches, including the more financial model *CreditMetrics* developed by *J.P. Morgan* [35], the more actuarial model *CreditRisk*$^{+}$ by *Credit Suisse* [14] and the more econometric model *CreditPortfolioView* by *McKinsey* [68].

This chapter is organized as follows. In section 2.1 structural models are presented in detail, with particular focus on the Merton's model. In section 2.2, reduced-form models and Cox processes, on which they are based, are presented. Section 2.3 discusses Markov chains in their discrete-time and continuous-time, conditional and unconditional settings. Section 2.4 focuses on Markovian models for credit migrations, presenting in detail the model of Jarrow, Lando and Turnbull (1997), which is the starting point for the credit spread risk model proposed in this work. Finally, section 2.5 introduces and describes the model for credit spread risk proposed and used in this work.

2.1 Structural models

The key feature of structural models, which owe their name to the fact that they are based on structural variables of the firm, is that the liabilities of a firm, *i.e.*, the bonds it issues, are treated as a contingent claim on the total value of the firm's assets. These models are also called *firm's value models* (or *option-theoretic models*), as they are based on the dynamics of the value of the firm's assets, which is specified through a stochastic differential equation, usually a diffusion or jump-diffusion process, and the evolution of the firm's capital structure. Credit events are therefore triggered by movements of the firm's value relative to some, random or non-random, credit-event-triggering threshold or barrier. In this way, it is possible to link credit events to the firm's economic fundamentals.

The option pricing theory, introduced by Black and Scholes [7], plays a key role in these models, since in this context the liabilities of the firm can be interpreted as a put option on the firm's assets: the fundamental hypothesis is that the underlying process, the firm value, follows a random process similar to the one used to describe generic stocks in equity markets, that is a Geometric Brownian Motion, *i.e.*, the value of company V is assumed to follow a lognormal dynamic[1], and that is possible

[1] This assumption is studied in detail and confirmed by Crouhy et al. in [13].

to observe this value at any time. Hence, the default event can be thoroughly monitored based on default-free market and is less sudden and unexpected.

It is necessary to make a clarification on the definitions of total value of the firm and total value of the firm's assets: the former is equal to the value of the firm's assets, plus tax deductions, less bankruptcy costs, but in some structural models the last two terms are ignored and the two definitions coincide. Then, in these models, the value of the firm V is the sum of the firm equity value S and of the firm debt value D; the firm equity value S can be seen as a kind of option on the value of the firm V, *plain-vanilla* in Merton's model and *barrier-like* in Black and Cox model. Another important difference is that the total value of the firm is complicated to observe, whereas the total value of the firm's assets can be observed at least for traded firms.

Structural models usually refer to only one type of credit event, namely, the firm's default, which is defined as the first time the value of the firm reaches a certain lower threshold, and the recovery rates are usually defined as a function of the firm's value; so both the default event and recovery rates are usually endogenously defined within the model.

In summary, the components that contribute to structural models are the following:

- the dynamics of the total value of the firm's assets,
- the structure of the firm's liabilities,
- the default event (in particular, the default triggering thresholds),
- the recovery rule in case of default,
- other relevant economic quantities, like the short-term interest rate.

The main shortcomings of the structural approach are the assumptions that the value of the firm can be directly observed and that the firm's assets represent a tradeable security, or at least that the firm's value process can be replicated through traded securities. In a framework that provides for periodic and imperfect information about the firm's value (see Duffie and Lando (1997)) is justified in moving to the intensity-based approach.

Some examples of structural models are: Merton (1974), Black and Cox (1976), Brennan and Schwartz (1977, 1978, 1980) and Longstaff and Schwartz (1995).

2.1.1 Default time

In the structural approach, the default time τ is typically defined in terms of the value V and the barrier process v:

$$\tau := \inf\{t > 0 : t \in \mathcal{T}, V_t < v_t\} \tag{2.1}$$

where $\mathcal{T}$ is assumed to be a Borel measurable subset of the time interval $[0, T]$. Furthermore, τ is a $\mathcal{G}$-stopping time and, if the filtration $\mathcal{G}$, *i.e.*, the filtration modeling the information the information flow available to the traders, is generated by a standard Brownian motion, as in most structural models, is a $\mathcal{G}$-predictable stopping time. This means that the random time of default is announced by an increasing sequence of stopping times.

In classical structural models, default may occur only at final time T, if the firm is not able to reimburse all the bond-holders.
First passage time models extend the original Merton framework assuming that the default may occur even before maturity T, so the default time τ is the first time where the firm value hits from above either a deterministic or a stochastic barrier. In these models, the corporate bond is treated as an American put option with *down-and-out* barrier, and the barrier L is below the par value of the bond and represents an absorbing state. Some examples of *first passage time models* are Black and Cox (1976) and Longstaff and Schwartz (1995).

2.1.2 Merton's model (1974)

In Merton's model, which is the starting point for structural models and the study of credit risk in general, it is assumed that there is a corporate bond (the debt) with maturity T and face value L, and the firm default is possible only at the final maturity, if the value of the firm's asset V_T is below the debt L to paid.
The model assumes that the following quantities are listed and therefore observable on the reference market:

- the firm's value at time t, V_t (market value of firm's assets);
- the firm's share value at time t, S_t (market value of firm equity);
- a corporate zero coupon bond with face value L, issued in $t = 0$, with maturity in T and market value at time t $D(t, T)$.

According to Black and Scholes (1973) [7], the issue of the zero coupon bond corresponds to the sale of the firm's assets to the bond-holders and the purchase of a call option to buy back the assets by shareholders, or equivalently to the holding of the firm's assets and the purchase of a put option from the bond-holders. Thus, a corporate bond can be interpreted as a default-free zero coupon bond minus a put option with strike price equivalent to the value of the issue written on the firm's assets. The debt value at time $t < T$ is thus:

$$\begin{aligned} D(t,T) &= \mathbf{E}_t\left[e^{-\int_t^T r(u)\,du}\,\min(V_T, L)\right] \\ &= \mathbf{E}_t\left[e^{-\int_t^T r(u)\,du}\,[V_T - (V_T - L)^+]\right] \\ &= \mathbf{E}_t\left[e^{-\int_t^T r(u)\,du}\,[L - (L - V_T)^+)\right] \\ &= v(t,T)\,L - P(t,T; V_t, L)\ , \end{aligned} \tag{2.2}$$

where $v(t, T)$ is the risk-free discount factor and $P(t, T; V_t, L)$ is the price of an European put option on V_t with strike price L and residual maturity $T - t$. If one assumes deterministic interest rates, then $\mathbf{E}_t\left[e^{-\int_t^T r(u)\,du}\right] = v(t,T)$.

The equity value can be derived as a difference between the value of the firm and the debt, by put-call parity:

$$\begin{aligned} S_t &= V_t - D(t,T) \\ &= V_t - v(t,T)\,L + P(t,T;V_t,L) \\ &= C(t,T;V_t,L)\ , \end{aligned} \tag{2.3}$$

where $C(t,T;V_t,L)$ is the price of an European call option on V_t with strike price L and residual maturity $T-t$. The equity can be interpreted as a call option on the value of the firm.

The valuation of the two options requires the choice of a model for the evolution of the value of the firm. Specifically, Merton assumes that the dynamic of V is described by a diffusion process identical to that used in the Black and Scholes model [7] for the option valuation:

$$dV_t = \mu_V\,V_t\,dt + \sigma\,V_t\,dZ_t. \tag{2.4}$$

And similarly, a constant risk-free spot rate is assumed, $r_t = r$.
The value of the corporate bond at generic instant $t \in [0,T]$ is therefore defined as:

$$D(t,T) = v(t,T)\,L - \left[L\,e^{-rT}N(-d_2) - V_t\,N(-d_1)\right] \tag{2.5}$$

with

$$d_{1/2} = \frac{\ln(V_t/L) + (r \pm \sigma_V^2)\tau}{\sigma_V\sqrt{\tau}}, \qquad \tau = T - t. \tag{2.6}$$

The same result could have been reached by setting [9]:

$$\begin{aligned} D(t,T) &= v(t,T)\,\mathbf{E}^{\mathbb{Q}}\left[L\,\mathbf{1}_{\{V_T \geq L\}} + V_T\mathbf{1}_{\{V_T<L\}}|\mathcal{G}_t\right] \\ &= B(t,T)(1 - \tilde{P}_D\,RR) \end{aligned} \tag{2.7}$$

where:

- $\mathbf{E}^{\mathbb{Q}}$ denotes the expected value under the risk-neutral measure $\mathbb{Q}$;
- $B(t,T) = L\,v(t,T)$ is the risk-free bond value;
- $\tilde{P}_D$ is the risk-neutral probability of default;
- RR is the exogenous recovery rate.

In Merton model the risk-neutral probability of default $\tilde{P}_D$ is:

$$\tilde{P}_D = \mathbf{E}^{\mathbb{Q}}[\mathbf{1}_{\{V_T<L\}}|\mathcal{G}_t] = N(-d_2), \tag{2.8}$$

while the *real-world* probability of default is:

$$P_D = \mathbf{E}^{\mathbb{P}}[\mathbf{1}_{\{V_T<L\}}|\mathcal{G}_t]\ , \tag{2.9}$$

where $\mathbf{E}^{\mathbb{P}}$ denotes the expected value under the *real-world* measure $\mathbb{P}$, and represents the probability that the default occurs.

2.2 Reduced-form models

In the reduced-form approach, the value of the firm's assets and its capital structure are not modelled at all, and the credit events are specified in terms of some exogenously specified jump process[2].
Among the reduced-form models, one can distinguish between *intensity-based models*, that are only focused on the modeling of the default time, and *credit migration models*, that consider migrations between credit rating classes. In the former models, typically, the random default time is defined as the jump time of some one-jump process and a key role is played by the *default intensity process*, which is also known as hazard rate process. In the latter models, which are called also *multiple credit ratings model*, migrations between credit ratings are allowed and are modelled in terms of a (conditional) Markov chain, with a finite state space consisting of the different credit rating classes and the absorbing state of default. The main issue in this approach is modelling the transition intensities matrix for the migration process, both under the risk-neutral and the *real-world* probability measures.

2.2.1 Intensity-based models

Intensity models move from the idea that the default time τ is the first jump time of a Poisson process and it does not depend on market observables and economic fundamentals.
Given the underlying probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, where the probability measure $\mathbb{Q}$ is a *spot martingale measure*, the default time τ is an arbitrary non-negative random variable, and the cumulative distribution function of the default time can be defined as $F(t) = P\left\{\tau \leq t \middle| \mathcal{F}_t\right\}$, with $F_0 = 0$ and $F_t < 1$ for every $t \in \mathbb{R}^+$. The survival process G of the random time τ with respect to the filtration $\mathcal{F}_t$, *i.e.*, the filtration with information about the other relevant default-free variables (interest rates, stock prices, ...), is defined as:

$$G_t := 1 - F_t = P\left\{\tau > t \middle| \mathcal{F}_t\right\}, \quad \forall t \in \mathbb{R}^+. \tag{2.10}$$

The hazard process of the default time τ under $\mathbb{Q}$, given the flow of information represented by $\mathcal{F}_t$, denoted by Γ_t, is defined as:

$$\Gamma_t := -\ln G_t = -\ln(1 - F_t), \quad \forall t \in \mathbb{R}^+, \tag{2.11}$$

with $\Gamma_0 = 0$ and $\lim_{t \to +\infty} \Gamma_t = +\infty$.
In most of the recently developed reduced-form models for credit risk, it is assumed for the hazard process Γ_t of a default time to have absolutely continuous sample paths and to admit the following integral representation:

$$\Gamma_t = \int_0^t \gamma(u)\, du, \quad \forall t \in \mathbb{R}^+ \tag{2.12}$$

for some non-negative, $\mathcal{F}$-progressively measurable stochastic process γ, that is called stochastic intensity of τ.

[2] As a rule, the recovery rates at default are also exogenously given.

For a given default time τ, the associated default process can be introduced as a right-continuous jump process:

$$H_t = \mathbf{1}_{\{\tau \leq t\}} \; . \tag{2.13}$$

The filtration generated by the process H, $\mathcal{H}_t = \sigma\left(H_u : u \leq t\right)$, contains the information about the occurrence of the default event at time t. The default time τ is a stopping time with respect to the filtration $\mathcal{H}_t$.
It is therefore possible to define a third enlarged filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, which contains the filtrations $\mathcal{H}_t$ and $\mathcal{F}_t$ and represents the total information available to agents. The default time τ is not necessarily a stopping time with respect to the filtration $\mathcal{F}_t$, but it is, of course, a stopping time with respect to filtration $\mathcal{G}_t$. It follows that the default time is not predictable with respect to the filtration $\mathcal{G}_t$.

Since the events $\{\tau \leq t\}$ and $\{\tau > t\}$ belong to the filtration $\mathcal{G}_t$, it can be derived that:

$$P\left\{\tau \leq T \middle| \mathcal{G}_t\right\} = \mathbf{1}_{\{\tau \leq t\}} + \mathbf{1}_{\{\tau > t\}} \mathbf{E}^{\mathbb{Q}} \left[1 - e^{-\int_t^T \gamma(u)\, du} \middle| \mathcal{F}_t \right], \tag{2.14}$$

$$P\left\{\tau > T \middle| \mathcal{G}_t\right\} = \mathbf{1}_{\{\tau > t\}} \mathbf{E}^{\mathbb{Q}} \left[e^{-\int_t^T \gamma(u)\, du} \middle| \mathcal{F}_t \right]. \tag{2.15}$$

The presence of two filtrations, and of the third union filtration, serves to make sure that the information contained in the residual variables is not sufficient to predict the default time, thus bypassing the limitation of structural models. While the residual variables do not determine the default event, they do, however, influence the probability of its occurrence.

2.2.2 On Cox processes

Starting from the Poisson processes framework, discussed in detail in Appendix B, let λ_t be a stochastic intensity, besides being time varying, assumed to be at least $\mathcal{F}_t$-adapted[3] and right continuous process. The cumulated intensity or hazard process is the random variable $\Lambda(t) = \int_0^t \lambda(u)\, du$ with $\lambda_t \geq 0$.

Recall that an inhomogeneous Poisson process N with non-negative intensity function λ_t satisfies the following equation:

$$P\left\{N_s - N_t = k\right\} = \frac{\left(\int_t^s \lambda(u)\, du\right)^k}{k!} \, e^{-\int_t^s \lambda(u)\, du}, \qquad k = 0, 1, \ldots \tag{2.16}$$

In particular, assuming $N_0 = 0$, the following relation stands:

$$P\{N_t = 0\} = e^{-\int_0^t \lambda(u)\, du}. \tag{2.17}$$

A Cox process, or a doubly stochastic Poisson process, is a generalization of the Poisson process in which the intensity λ can be stochastic, under the condition that, given a particular realization of the intensity $\lambda_t(\omega)$, the jump process is still a time inhomogeneous Poisson process with intensity $\lambda_t(\omega)$. The doubly stochastic

[3] Given the information $\mathcal{F}_t$ we know λ from 0 to t.

definitions derives from the fact that, besides having stochasticity in the jump component ξ, the process has stochasticity in the probability of jumping, *i.e.*, in the intensity.
For Cox processes, the first jump time can be represented as:

$$\tau := \Lambda^{-1}(\xi) = \inf\left\{t : \int_0^t \lambda(u)\, du \geq \xi\right\}, \tag{2.18}$$

where ξ is a unit exponential random variable, independent of λ. The first jump time can be interpreted as the default time.
With Cox processes, the following relations are valid:

$$P\left\{\tau \in [t, t+dt] \middle| \tau \geq t, \mathcal{F}_t\right\} = \lambda_t\, dt, \tag{2.19}$$

$$\begin{aligned} P\{\tau > t\} &= P\{\Lambda(\tau) \geq \Lambda(t)\} = P\left\{\xi \geq \int_0^t \lambda(u)\, du\right\} \\ &= \mathbf{E}\left[P\left\{\xi \geq \int_0^t \lambda(u)\, du \middle| \mathcal{F}_t\right\}\right] = \mathbf{E}\left[e^{-\int_0^t \lambda(u)\, du}\right], \end{aligned} \tag{2.20}$$

$$P\left\{\tau > t \middle| \mathcal{F}_t\right\} = e^{-\int_0^t \lambda(u)\, du}. \tag{2.21}$$

2.2.3 The evaluation of a zero-coupon bond subject to spread risk.

Some examples of unit corporate ZCB with maturity T, issued by an R-rated issuer, that are subject to various recovery schemes and go on to define their pre-default value at time t, $v^R(t,T)$, are considered in the following.
At any time t, the discounted payoff of a unit defaultable bond is given by:

$$Y_t = \mathbf{1}_{\{\tau \leq T\}}\, h(\tau)\, e^{-\int_t^\tau r(u)\, du} + \mathbf{1}_{\{\tau > T\}}\, e^{-\int_t^T r(u)\, du}, \tag{2.22}$$

where r is the short-term interest rate and $h(\tau)$ is the recovery cash flow.

In case of *zero recovery* scheme, $h(\tau) = 0$ and the pre-default value at time t of such a bond is:

$$v^R(t,T) = \mathbf{1}_{\{\tau > t\}} \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)+\gamma(u)\, du} \middle| \mathcal{F}_t\right] = v(t,T)\, \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T \gamma(u)\, du} \middle| \mathcal{F}_t\right], \tag{2.23}$$

where $v(t,T)$ is the value of a unit risk-free ZCB with maturity T at time t.
Under this recovery scheme, the corporate bond becomes valueless as soon as the default occurs.

In case of *fractional recovery of par value* scheme, $h(\tau) = \delta$, that is a constant coefficient between 0 and 1, and the corporate bond pays at time of default a constant payoff proportional to the bond's face value, in case the bond defaults before or at the bond's maturity T. The pre-default value at time t of such a bond is:

$$v^R(t,T) = \mathbf{1}_{\{\tau > t\}} \mathbf{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(u)+\gamma(u)\, du}\, \delta\, \gamma(s)\, ds + e^{-\int_t^T r(u)+\gamma(u)\, du} \middle| \mathcal{F}_t\right]. \tag{2.24}$$

This is the closest we come to legal practice, in the sense that debt with the same priority is assigned a fractional recovery depending on the outstanding notional amount but not on maturity or coupon, and it is also the measure typically used by rating agencies. In mathematical terms, the formula for pricing a bond is not so pretty, as it requires calculating an integral.

In case of *fractional recovery of market value* scheme, $h(\tau) = \delta\, v^R(\tau^-, T)$, where δ is a constant coefficient between 0 and 1, and $v^R(\tau^-, T)$ is the value of the risky zero coupon bond pre-default, that is, at the instant immediately before default. The corporate bond pays the variable payoff $\delta\, v^R(\tau^-, T)$ at defalut time τ if default occurs before maturity. The pre-default value at time t of such a bond is:

$$\begin{aligned} v^R(t,T) &= \mathbf{1}_{\{\tau>t\}} \mathbf{E}^{\mathbb{Q}} \left[\int_t^T e^{-\int_t^s r(u)+\gamma(u)\,du}\; \delta\, v^R(s-,T)\; \gamma(s)\; ds + e^{-\int_t^T r(u)+\gamma(u)\,du} \Big| \mathcal{F}_t \right] \\ &= \mathbf{1}_{\{\tau>t\}}\; v(t,T)\; \mathbf{E}^{\mathbb{Q}} \left[e^{-\int_t^T (1-\delta)\lambda(u)\,du} \Big| \mathcal{F}_t \right]. \end{aligned} \tag{2.25}$$

Receiving a fraction δ at default time τ of the pre-default value $v^R(\tau^-, T)$ has the same expectation as receiving 0 with probability $1-\delta$ and $v^R(\tau^-, T)$ with probability δ. Since receiving $v^R(\tau^-, T)$ is equivalent to a cancellation of the default event, this formulation of recovery is equal to a zero-recovery assumption in a thinned default-event process. The thinning rate is $1-\delta$ and hence we can use our zero-recovery formulation with the intensity $(1-\delta)\,\lambda(u)$.
This measures the change in market value at the time of default. This has economic meaning since this is the loss in value associated with default. This quantity is extremely convenient to work with for modeling purposes.

In case of *fractional recovery of Treasury value* scheme, $h(\tau) = \delta\; v(t,T)$, and the corporate bond pays the constant payoff δ at maturity T if default occurs before maturity. Under this assumption, the corporate bond in default is replaced with a treasury bond with the same maturity but a reduced payment. The pre-default value at time t of such a bond is:

$$\begin{aligned} v^R(t,T) &= \mathbf{1}_{\{\tau>t\}} \mathbf{E}^{\mathbb{Q}} \left[\int_t^T e^{-\int_t^T r(u)\,du}\; e^{-\int_t^s \gamma(u)\,du}\; \delta\; \gamma(s)\; ds + e^{-\int_t^T r(u)+\gamma(u)\,du} \Big| \mathcal{F}_t \right] \\ &= \mathbf{1}_{\{\tau>t\}}\; v(t,T)\; \mathbf{E}^{\mathbb{Q}} \left[\delta \left(1 - e^{-\int_t^T \gamma(u)\,du} \right) + e^{-\int_t^T \gamma(u)} \Big| \mathcal{F}_t \right]. \end{aligned} \tag{2.26}$$

Under this recovery scheme, the pre-default value of a corporate bond can also be expressed as follows:

$$v^R(t,T) = v(t,T)\; \Big(\delta\; P^{\mathbb{Q}}\{t < \tau \le T\} + P^{\mathbb{Q}}\{\tau > T\} \Big). \tag{2.27}$$

An advantage of the *fractional recovery of Treasury value* scheme is that is permits (at least with an assumption of independence between the short rate r and the default intensity λ) an immediate expression for implied default probabilities.

2.3 On Markov chains

An underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a finite set $\mathcal{K} = \{1, \ldots, K\}$, that represents the state space for all considered Markov chains are set. Since the state space $\mathcal{K}$ is finite, any function $h : \mathcal{K} \to \mathbb{R}$ is bounded and measurable, provided that the state space is endowed with the σ-field of all its subsets.

2.3.1 Discrete-time Markov chains

Let $\{C_t\}_{t=0,1,\ldots}$ be a sequence of random variables on $(\Omega, \mathcal{G}, \mathbb{P})$ with values in $\mathcal{K}$, and let $\mathcal{F}_t^C = \sigma(C_s : s = 0, \ldots, t)$ be the natural filtration generated by the process C. A process C is a discrete-time Markov chain under the original probability measure $\mathbb{P}$ with respect to $\mathcal{G}$ ($\mathcal{G}$-Markov chain) if, for any function $h : \mathcal{K} \to \mathbb{R}$:

$$\mathbf{E}^{\mathbb{P}}\left[h(C_{t+s})|\mathcal{G}_t\right] = \mathbf{E}^{\mathbb{P}}\left[h(C_{t+s})|C_t\right], \quad \forall t, s \in \mathbb{N}^+. \tag{2.28}$$

Therefore a Markov chain is a stochastic process whose future behavior can be determined only by the current state of the process and it is independent of its past. If, in addition,

$$\mathbf{E}^{\mathbb{P}}\left[h(C_{t+s})|C_t\right] = \mathbf{E}^{\mathbb{P}}\left[h(C_{u+s})|C_u\right], \quad \forall t, s, u \in \mathbb{N}^+, \tag{2.29}$$

the Markov chain C is said to be time-homogeneous.
Since the state space $\mathcal{K}$ is finite, the condition (2.28) is equivalent to

$$P\{C_{t+1} = j|\mathcal{G}_t\} = P\{C_{t+1} = j|C_t\}, \quad \forall t \in \mathbb{N}^+,\ \forall j \in \mathcal{K}, \tag{2.30}$$

and the condition (2.29) is equivalent to:

$$P\{C_{t+1} = j|\mathcal{G}_t\} = P\{C_{t+1} = j|C_t\} = P\{C_{s+1} = j|C_s\}, \quad \forall t, s \in \mathbb{N}^+,\ \forall j \in \mathcal{K}. \tag{2.31}$$

The Markov property (2.28) generalizes to the following condition: for any function $\overline{h} : \mathcal{K} \times \ldots \mathcal{K} \to \mathbb{R}$,

$$\mathbf{E}^{\mathbb{P}}\left[\overline{h}(C_{t+s_1}, \ldots, C_{t+s_n})|\mathcal{G}_t\right] = \mathbf{E}^{\mathbb{P}}\left[\overline{h}(C_{t+s_1}, \ldots, C_{t+s_n})|C_t\right], \quad \forall t, s_1, \ldots, s_n \in \mathbb{N}^+. \tag{2.32}$$

From now on, C is assumed to be a time-homogeneous Markov chain under the original probability measure $\mathbb{P}$ with respect to $\mathcal{G}$.

A Markov chain C is governed by the one-step transition probability matrix $\mathbf{P} = [p_{ij}]_{1 \le i,j \le K}$, where the generic element of $\mathbf{P}$, $p_{ij} = P\{C_{t+1} = j|C_t = i\}$, represents the probability of passing from the state i at time t to the state j at time $t+1$. These probabilities are normally assembled in a matrix represented as follows:

$$\mathbf{P} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,K-1} & p_{1,K} \\ \vdots & \ddots & \vdots & \vdots \\ p_{K-1,1} & \cdots & p_{K-1,K-1} & p_{i,K} \\ p_{K,1} & \cdots & p_{K-1,K} & p_{K,K} \end{pmatrix}.$$

The transition probability matrix $\mathbf{P}$ is a stochastic matrix, hence:

- $p_{ij} \geq 0, \;\; \forall i, j \in \mathcal{K};$

- $\sum_{j=1}^{K} p_{ij} = 1, \;$ for any fixed $i \in \mathcal{K}$.

The s-step transition probability matrix as $\mathbf{P}^{(s)} = \left[p_{ij}^{(s)}\right]_{1 \leq i,j \leq K}$ is defined so that, for any $s \in \mathbb{N}^+$,

$$p_{ij}^{(s)} = P\left\{C_{t+s} = j | C_t = i\right\}, \quad \forall i, j \in \mathcal{K}. \tag{2.33}$$

The Chapman-Kolmogorov equations are satisfied, specifically:

$$\mathbf{P}^{(t+s)} = \mathbf{P}^{(t)}\,\mathbf{P}^{(s)} = \mathbf{P}^{(s)}\,\mathbf{P}^{(t)}, \quad \forall t, s \in \mathbb{N}^+. \tag{2.34}$$

More explicitly:

$$p_{ij}^{(t+s)} = \sum_{k=1}^{K} p_{ik}^{(t)}\,p_{kj}^{(s)} = \sum_{k=1}^{K} p_{ik}^{(s)}\,p_{kj}^{(t)}, \quad \forall t, s \in \mathbb{N}^+, \;\; \forall i, j \in \mathcal{K}. \tag{2.35}$$

Then, in case of time-homogeneity:

$$\mathbf{P}^{(s)} = \mathbf{P}^s, \tag{2.36}$$

where $\mathbf{P}^s$ denotes the s-th power of the one-step transition probability matrix $\mathbf{P}$. The following relations can be written:

$$\mathbf{P}^{(t+1)} = \mathbf{P}\,\mathbf{P}^{(t)}, \quad \forall t \in \mathbb{N}^+, \tag{2.37}$$

$$\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)}\,\mathbf{P}, \quad \forall t \in \mathbb{N}^+, \tag{2.38}$$

$$\Delta\mathbf{P}^{(t+1)} = \mathbf{P}^{(t+1)} - \mathbf{P}^{(t)}. \tag{2.39}$$

Starting from equations (2.37) and (2.38), the *backward Kolmogorov* equation:

$$\Delta\mathbf{P}^{(t+1)} = \mathbf{\Lambda}\,\mathbf{P}^{(t)}, \quad \mathbf{P}^{(0)} = \mathbf{I}, \quad \forall t \in \mathbf{N}^+, \tag{2.40}$$

and the *forward Kolmogorov* equation:

$$\Delta\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)}\,\mathbf{\Lambda}, \quad \mathbf{P}^{(0)} = \mathbf{I}, \quad \forall t \in \mathbf{N}^+ \tag{2.41}$$

can be written, where $\mathbf{\Lambda} = \mathbf{P} - \mathbf{I}$ and $\mathbf{I}$ is the K-dimensional identity matrix. The matrix $\mathbf{\Lambda}$ is called the generator matrix associated with the stochastic matrix $\mathbf{P}$ and satisfies the following properties:

- $0 \leq \lambda_{ii}, \;$ for $i \in \mathcal{K}$;

- $\lambda_{ij} \geq 0, \;$ for $i, j \in \mathcal{K}$ with $i \neq j$;

- $\sum_{j=1}^{K} \lambda_{ij} = 0, \;$ for $i \in \mathcal{K}$.

A state $k \in \mathcal{K}$ is defined absorbing for a $\mathcal{G}$-Markov chain C under $\mathbb{P}$ if:

$$P\left\{C_s = k \middle| C_t = k\right\} = 1, \quad \forall t, s \in \mathbb{N}^+, \; t \leq s. \tag{2.42}$$

If the state K is assumed to be the only absorbing state for the $\mathcal{G}$-Markov chain C under $\mathbb{P}$, then the transition probability matrix $\mathbf{P}$ can be expressed as:

$$\mathbf{P} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,K-1} & p_{1,K} \\ \vdots & \ddots & \vdots & \vdots \\ p_{K-1,1} & \cdots & p_{K-1,K-1} & p_{i,K} \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $p_{ii} < 1$ for every $i = 1, \dots, K-1$, and the generator matrix $\mathbf{\Lambda}$ can be expressed as:

$$\mathbf{\Lambda} = \begin{pmatrix} p_{1,1} - 1 & \cdots & p_{1,K-1} & p_{1,K} \\ \vdots & \ddots & \vdots & \vdots \\ p_{K-1,1} & \cdots & p_{K-1,K-1} - 1 & p_{i,K} \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Let the random time τ be the first moment when the process C jumps to the absorbing state K:

$$\tau = \inf\{t \geq 0 : C_t = K\}. \tag{2.43}$$

Its probability distribution under $\mathbb{P}$ is:

$$P\{\tau \leq t | C_0 = i\} = 1 - P\{C_t \neq K | C_0 = i\} = 1 - \sum_{j=1}^{K-1} p_{ij}^{(t)}, \quad \forall t \in \mathbb{N}, \forall i \neq K. \tag{2.44}$$

2.3.1.1 Change of a probability measure

In most financial applications, it is sufficient to study the behaviour of a Markov chain only up to some time $T^* < \infty$. For a fixed T^*, the Radon-Nykodim derivative from the original probability measure $\mathbb{P}$ to an equivalent[4] probability measure $\mathbb{Q}$ is defined as:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_{T^*}} = \psi_{T^*}, \tag{2.45}$$

where ψ_{T^*} is a strictly positive $\mathcal{G}_{T^*}$-measurable random variable with $\mathbf{E}^{\mathbb{P}}\left[\psi_{T^*}\right] = 1$. Then the density process $\psi_t = \mathbf{E}^{\mathbb{P}}\left[\psi_{T^*} | \mathcal{G}_t\right], \; t = 0, \dots, T^*$, follows a strictly positive martingale under $\mathbb{P}$.
If the random variable $\psi_t^{-1}\psi_{t+1}$ is $\sigma(C_t, C_{t+1})$-measurable for any $t = 0, \dots, T^* - 1$, *i.e.*, if $\psi_t^{-1}\psi_{t+1} = g_t(C_t, C_{t+1})$ for some function $g_t : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$, a time-homogeneous $\mathcal{G}$-Markov chain C under $\mathbb{P}$ keeps following a $\mathcal{G}$-Markov chain under $\mathbb{Q}$, and $p_{ij}^*(t) = p_{ij}\, g_t(i, j)$ for arbitrary states $i, j \in \mathcal{K}$ and every $t = 0, \dots, T^* - 1$, where p_{ij}^* denotes the transition probabilities under $\mathbb{Q}$.

[4]Equivalent on $(\Omega, \mathcal{G}_{T^*})$.

Proof. Let $t \in \mathbb{N}^+$ be set. Using the abstract version of the Bayes theorem, for any state $j \in \mathcal{K}$:

$$\begin{aligned} P^*\{C_{t+1} = j|\mathcal{G}_t\} &= \mathbf{E}^{\mathbb{Q}}\left[\psi_t^{-1}\,\psi_{T^*}\mathbf{1}_{\{C_{t+1}=j\}}|\mathcal{G}_t\right] \\ &= \mathbf{E}^{\mathbb{Q}}\left[\mathbf{E}^{\mathbb{Q}}[\psi_{T^*}|\mathcal{G}_{t+1}]\,\psi_t^{-1}\,\mathbf{1}_{\{C_{t+1}=j\}}|\mathcal{G}_t\right] \\ &= \mathbf{E}^{\mathbb{Q}}\left[\psi_t^{-1}\,\psi_{t+1}\mathbf{1}_{\{C_{t+1}=j\}}|\mathcal{G}_t\right] \\ &= \mathbf{E}^{\mathbb{Q}}\left[g_t(C_t, C_{t+1})\mathbf{1}_{\{C_{t+1}=j\}}|\mathcal{G}_t\right] \\ &= \mathbf{E}^{\mathbb{Q}}\left[g_t(C_t, C_{t+1})\mathbf{1}_{\{C_{t+1}=j\}}|C_t\right], \end{aligned}$$

where the last equality follows from applying formula (2.32) to the function $\overline{h}(k,l) = g_t(k,l)\mathbf{1}_{\{j\}}(l)$. Hence, the conditional probability $P^*\{C_{t+1} = j|\mathcal{G}_t\}$ is a $\sigma(C_t)$-measurable random variable, and then:

$$P^*\{C_{t+1} = j|\mathcal{G}_t\} = P^*\{C_{t+1} = j|C_t\},$$

so that C manifestly follows a $\mathcal{G}$-Markov chain under $\mathbb{Q}$. Furthermore:

$$\begin{aligned} p^*_{ij}(t) &= P^*\{C_{t+1} = j|C_t = i\} \\ &= \mathbf{E}^{\mathbb{Q}}\left[\psi_t^{-1}\,\psi_{t+1}\mathbf{1}_{\{C_{t+1}=j\}}|C_t = i\right] \\ &= \mathbf{E}^{\mathbb{Q}}\left[g_t(C_t, C_{t+1})\mathbf{1}_{\{C_{t+1}=j\}}|C_t = i\right] \\ &= p_{ij}\,g_t(i,j). \end{aligned}$$

Typically, a Markov chain C is no longer a time-homogeneous process under an equivalent probability measure $\mathbb{Q}$. For the conditions for preserving the time-homogeneity property under $\mathbb{Q}$, refer to [6].

2.3.1.2 Discrete-time conditionally Markov chain

A probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with some filtrations $\{\mathcal{F}_t\}_{t\in\mathbb{N}^+}$ and $\{\mathcal{G}_t\}_{t\in\mathbb{N}^+}$, such that $\mathcal{F} \subseteq \mathcal{G}$, is considered.
A discrete-time $\mathcal{K}$-valued stochastic process C is a discrete-time conditionally Markov chain under the original probability measure $\mathbb{P}$ with respect to $\mathcal{F}$ if, for any function $h : \mathcal{K} \to \mathbb{R}$,

$$\mathbf{E}^{\mathbb{P}}\left[h(C_s)|\mathcal{G}_t\right] = \mathbf{E}^{\mathbb{P}}\left[h(C_s)|\mathcal{F}_t \vee \sigma(C_t)\right], \quad \forall t, s \in \mathbb{N}^+,\ t \le s. \tag{2.46}$$

The Markov chain C is governed by the one-step $\mathcal{F}$-conditional transition probability matrix process, *i.e.*, an $\mathcal{F}$-adapted, matrix-valued stochastic process:

$$\mathbf{P}(t) = [p_{ij}(t)]_{1\le i,j\le K}, \quad \forall t \in \mathbb{N}^+, \tag{2.47}$$

where for every $t \in \mathbb{N}^+$ and arbitrary $i, j \in \mathcal{K}$, on the set $\{C_t = i\}$:

$$p_{ij}(t) = P\{C_{t+1} = j|\mathcal{F}_t \vee \sigma(C_t)\}. \tag{2.48}$$

A s-step $\mathcal{F}$-conditional transition probability matrix at time t for C can be defined as the $\mathcal{F}_t$-measurable, matrix-valued random variable:

$$\mathbf{P}(t,s) = [p_{ij}(t,s)]_{1\le i,j\le K}\,, \quad \forall t,s \in \mathbb{N}^+, \tag{2.49}$$

where for every $i,j \in \mathcal{K}$, on the set $\{C_t = i\}$:

$$p_{ij}(t,s) = P\{C_{t+s} = j|\mathcal{F}_t \vee \sigma(C_t)\}. \tag{2.50}$$

Similarly to the general case, the $\mathcal{F}$-conditional $\mathcal{G}$-Markov property (2.46) generalizes to the following condition: for any function $\overline{h} : \mathcal{K} \times \ldots \mathcal{K} \to \mathbb{R}$,

$$\mathbf{E}^{\mathbb{P}}\left[\overline{h}(C_{t+s_1},\ldots,C_{t+s_n})\middle|\mathcal{G}_t\right] = \mathbf{E}^{\mathbb{P}}\left[\overline{h}(C_{t+s_1},\ldots,C_{t+s_n})\middle|\mathcal{F}_t \vee \sigma(C_t)\right], \quad \forall t,s_1,\ldots,s_n \in \mathbb{N}^+. \tag{2.51}$$

2.3.2 Continuous-time Markov chains

Let $\{C_t\}_{t\in\mathbb{R}^+}$ be a right-continuous stochastic process on $(\Omega,\mathcal{G},\mathbb{P})$ with values in the finite set $\mathcal{K}$, and let $\mathcal{F}^C \subseteq \mathcal{G}$ be the filtration generated by this process.
A process C is a continuous-time Markov chain with respect to $\mathcal{G}$ under $\mathbb{P}$ if, for an arbitrary function $h : \mathcal{K} \to \mathbb{R}$,

$$\mathbf{E}^{\mathbb{P}}\left[h(C_{t+s})|\mathcal{G}_t\right] = \mathbf{E}^{\mathbb{P}}\left[h(C_{t+s})|C_t\right], \quad \forall t,s \in \mathbb{R}^+. \tag{2.52}$$

If, in addition,

$$\mathbf{E}^{\mathbb{P}}\left[h(C_{t+s})|C_t\right] = \mathbf{E}^{\mathbb{P}}\left[h(C_{u+s})|C_u\right], \quad \forall t,s,u \in \mathbb{R}^+, \tag{2.53}$$

the continuous-time $\mathcal{G}$-Markov chain C is said to be time-homogeneous.

A continuous-time $\mathcal{G}$-Markov chain is governed by a two-parameter family of stochastic transition probability matrices $\mathbf{P}(t,s)$ with $t,s \in \mathbb{R}^+, t \le s$, where the generic entry of $\mathbf{P}(t,s)$, $p_{ij}(t,s) = P\{C_s = j|C_t = i\}$, represents the probability of passing from the state i at time t to the state j at time s.

2.3.2.1 Time-homogeneous Markov chains

A continuous-time time-homogeneous $\mathcal{G}$-Markov chain under $\mathbb{P}$ is governed, instead, by a one-parameter family of stochastic transition probability matrices $\mathbf{P}(s)$, where the generic entry is $p_{ij}(s) = P\{C_{t+s} = j|C_t = i\}$.
The Chapman-Kolmogorov equations are satisfied, specifically:

$$\mathbf{P}(t+s) = \mathbf{P}(t)\,\mathbf{P}(s) = \mathbf{P}(s)\,\mathbf{P}(t), \quad \forall t,s \in \mathbb{R}^+, \tag{2.54}$$

i.e., more explicitly:

$$p_{ij}(t+s) = \sum_{k=1}^{K} p_{ik}(t)\,p_{kj}(s) = \sum_{k=1}^{K} p_{ik}(s)\,p_{kj}(t), \quad \forall t,s \in \mathbb{R}^+, \quad \forall i,j \in \mathcal{K}. \tag{2.55}$$

The family $\mathbf{P}(\cdot)$ is assumed to be right-continuous at time $t = 0$, that is, $\lim_{t\to 0}\mathbf{P}(t) = \mathbf{P}(0)$.
This implies that:

$$\lim_{s\to 0}\mathbf{P}(t+s) = \mathbf{P}(t), \quad \forall t > 0, \tag{2.56}$$

and thus:

$$\lim_{s\to 0} P\{C_{t+s} = j \big| C_t = i\} = \delta_{ij}, \quad \forall i, j \in \mathcal{K}, \ \forall t > 0. \tag{2.57}$$

Since the family $\mathbf{P}(\cdot)$ is right-continuous at time $t = 0$, it is right-hand side differentiable at time $t = 0$, and the following finite limits exists:

$$\lambda_{ij} := \lim_{t\to 0} \frac{p_{ij}(t) - p_{ij}(0)}{t} = \lim_{t\to 0} \frac{p_{ij}(t) - \delta_{ij}}{t}, \quad \forall i, j \in \mathcal{K}, \tag{2.58}$$

with

- $0 \le \lambda_{ii}$, for $i \in \mathcal{K}$;
- $\lambda_{ii} = - \sum_{j=1, i\neq j}^{K} \lambda_{ij} \le 0$;
- $\sum_{j=1}^{K} \lambda_{ij} = 0$, for $i \in \mathcal{K}$.

The matrix $\mathbf{\Lambda} := [\lambda_{ij}]_{1\le i,j\le K}$ is called the infinitesimal generator matrix, or intensity matrix, for a Markov chain governed by $\mathbf{P}(\cdot)$, and each entry λ_{ij} represents the intensity of transition from the state i to the state j. The generator matrix uniquely determines all the relevant probabilistic properties of a time-homogeneous Markov chain.
The issue of the existence and calculation of the generator matrix is discussed in Appendix C.

Starting from the Chapman-Kolmogorov equation (2.58), the *backward Kolmogorov* equation:

$$\frac{\partial \mathbf{P}(t)}{\partial t} = \mathbf{\Lambda}\, \mathbf{P}(t), \quad \mathbf{P}(0) = \mathbf{I}, \tag{2.59}$$

and the *forward Kolmogorov* equation:

$$\frac{\partial \mathbf{P}(t)}{\partial t} = \mathbf{P}(t)\, \mathbf{\Lambda}, \quad \mathbf{P}(0) = \mathbf{I} \tag{2.60}$$

can be derived, and both have the same unique solution:

$$\mathbf{P}(t) = e^{t\mathbf{\Lambda}} := \sum_{n=0}^{\infty} \frac{\mathbf{\Lambda}^n t^n}{n!}, \quad \forall t \in \mathbb{R}^+. \tag{2.61}$$

The state K is assumed to be the only absorbing state for the $\mathcal{G}$-Markov chain C under $\mathbb{P}$; then, denoting λ_{kj} for every $j = 1, \ldots, K$, its infinitesimal generator matrix $\mathbf{\Lambda}$ has the following form:

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,K-1} & \lambda_{1,K} \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_{K-1,1} & \cdots & \lambda_{K-1,K-1} & \lambda_{i,K} \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Moreover, the initial state $C_0 = x \neq K$ is assumed to be set, and τ is assumed to be the random time of absorption at K, *i.e.*, $\tau = \inf\{t > 0 : C_t = K\} < \infty$ $\mathbb{P}$-almost surely, and $H_t^i = \mathbf{1}_{\{C_t=i\}}$ and $H_t = \mathbf{1}_{\{\tau \leq t\}} = \mathbf{1}_{\{C_t=K\}} = H_t^K$. τ is an $\mathcal{F}^c$-stopping time and a $\mathcal{G}$-stopping time.
For any $t, s \in \mathbb{R}^+$, the following equality is valid:

$$P\{\tau > s|\mathcal{G}_t\} = \mathbf{1}_{\{s\leq t\}}\,\mathbf{1}_{\{\tau\geq s\}} + \mathbf{1}_{\{s>t\}} \sum_{i=1}^{K-1} H_t^i\, P\{\tau > s|C_t = i\}, \tag{2.62}$$

and, since:

$$P\{\tau > s|C_t = i\} = 1 - P\{C_s = K|C_t = i\} = 1 - p_{ik}(s-t) \quad \forall 0 \leq t \leq s, \tag{2.63}$$

it can be rewritten it as follows:

$$P\{\tau > s|\mathcal{G}_t\} = \mathbf{1}_{\{s\leq t\}}\,\mathbf{1}_{\{\tau\geq s\}} + \mathbf{1}_{\{s>t\}} \sum_{i=1}^{K-1} H_t^i\, 1 - p_{ik}(s-t). \tag{2.64}$$

2.3.2.2 Time-inhomogeneous Markov chains

In case of a time-inhomogeneous Markov chain, the infinitesimal generator matrix $\mathbf{\Lambda}(t) = [\lambda_{ij}(t)]_{1\leq i,j\leq K}$ and the transition intensities become time-dependent:

$$\lambda_{ij}(t) = \lim_{h\to 0} \frac{p_{ij}(t, t+h) - \delta_{ij}}{h}, \quad \forall i, j \in \mathcal{L}, \tag{2.65}$$

with:

- $\lambda_{ij}(t) \geq 0, \quad \forall i \neq j;$

- $\lambda_{ii}(t) = \lim_{h\to 0} \dfrac{p_{ii}(t, t+h) - 1}{h} = -\lim_{h\to 0} \dfrac{\sum_{j=1, j\neq i}^{K} p_{ij}(t, t+h)}{h} = -\sum_{j=1, j\neq i}^{K} \lambda_{ij}(t),$

where:

$$p_{ij}(t, t+h) = P\{C_{t+h} = j|C_t = i\}, \quad \forall i, j \in \mathcal{K}. \tag{2.66}$$

The two-parameter family of transition probabilities matrices $\mathbf{P}(t, s)$ that governs the time-inhomogeneous Markov chain C satisfies the Chapman-Kolmogorov equation:

$$\mathbf{P}(t, s) = \mathbf{P}(t, u)\,\mathbf{P}(u, s), \quad \forall t \leq u \leq s, \tag{2.67}$$

the *forward Kolmogorov* equation:

$$\frac{\partial \mathbf{P}(t, s)}{\partial s} = \mathbf{P}(t, s)\,\mathbf{\Lambda}(s), \quad \mathbf{P}(t, t) = \mathbf{I}, \tag{2.68}$$

and the *backward Kolmogorov* equation:

$$\frac{\partial \mathbf{P}(t, s)}{\partial t} = -\mathbf{\Lambda}(t)\,\mathbf{P}(t, s), \quad \mathbf{P}(s, s) = \mathbf{I}. \tag{2.69}$$

The Chapman-Kolmogorov equation (2.67) yields:

$$p_{ij}(t, s+h) = \sum_{k=1}^{K} p_{ik}(t, s)\, p_{kj}(s, s+h), \quad \forall i, j \in \mathcal{K},\ 0 \le t \le s \le s+h. \tag{2.70}$$

Maintaining the assumptions of the time-homogeneous case about τ and the initial state C_0, for every $t \in \mathbb{R}^+$ and any $i = 1, \dots, K-1$, the conditional law of the absorption time τ is given by the formula:

$$P\{\tau \le t | C_0 = i\} = 1 - \sum_{j=1}^{K-1} p_{ij}(0, t). \tag{2.71}$$

2.3.2.3 Change of a probability measure

Recall that $H_t^i = \mathbf{1}_{\{C_t = i\}}$ and $H_t = \mathbf{1}_{\{\tau \le t\}} = \mathbf{1}_{\{C_t = K\}} = H_t^K$. Now the number of jumps of the process C from i to j in the interval $(0, t]$, H_t^{ij}, for any fixed $i \ne j$:

$$H_t^{ij} := \sum_{o<u\le t} H_u^i - H_u^j, \quad \forall t \in \mathbb{R}^+, \tag{2.72}$$

and a family $\tilde{k}^{kl}$, $k, l \in \mathcal{K}$, $k \ne l$, of bounded, $\mathcal{F}^{\mathcal{C}}$-predictable, real-valued processes, such that $\tilde{k}_t^{kl} > -1$ and $\tilde{k}^{kk} \equiv 0$ for $k = 1, \dots, K$, can be introduced.
For a fixed $T^* > 0$, the Radon-Nykodim derivate from the reference probability measure $\mathbb{P}$ to an equivalent probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{G}_{T^*})$ can be defined as:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_t} = \psi_t \quad \forall t \in [0, T^*], \tag{2.73}$$

where the process ψ is strictly positive and is defined as:

$$\psi_t = e^{-M_t^C} \prod_{o<u\le t} \left(1 + \sum_{k,l=1}^{K} \tilde{k}_u^{kl} (H_u^{kl} - H_{u-}^{kl}) \right), \tag{2.74}$$

where M_t^C is the path-by-path continuous component of the auxiliary $\mathcal{G}$-martingale process M_t.
To simplify the exposition, only processes $\tilde{k}_t^{kl} = k_{kl}(t)$, where for every $k, l \in \mathcal{K},\ k \ne l$, the function $k_{kl} : \mathbb{R}^+ \to (-1, \infty)$ is Borel measureable and bounded, will be considered.

If the probability measure $\mathbb{Q}$ is defined by (2.73) with the Radon-Nikodym density ψ_{T^*} given by (2.74), then:

i) the process $\{C_t\}_{t\in[0,T^*]}$ is a $\mathcal{G}$-Markov chain under the equivalent probability measure $\mathbb{Q}$,

ii) the infinitesimal generator matrix function $\mathbf{\Lambda}^*(t) = \left[\lambda_{ij}^*(t)\right]_{1\le i,j\le K}$ for C under $\mathbb{Q}$ satisfies, for $i \ne j$,

$$\lambda_{ij}^*(t) = (1 + k_{ij}(t))\, \lambda_{ij}, \quad \forall t \in [0, T^*], \tag{2.75}$$

and

$$\lambda^*_{ij}(t) = -\sum_{j=1, j\neq i}^{K} \lambda^*_{ij}(t), \quad \forall t \in [0, T^*], \tag{2.76}$$

iii) the two-parameter family $\mathbf{P}^*(t,s)$, $0 \leq t \leq s \leq T^*$, of transition probability matrices for C relative to $\mathbb{Q}$ satisfies the *forward Kolmogorov* equation:

$$\frac{\partial \mathbf{P}^*(t,s)}{\partial s} = \mathbf{P}^*(t,s)\,\mathbf{\Lambda}^*(s),\ \mathbf{P}^*(t,t) = \mathbf{I}, \tag{2.77}$$

and the *backward Kolmogorov* equation:

$$\frac{\partial \mathbf{P}^*(t,s)}{\partial t} = -\mathbf{\Lambda}^*(t)\mathbf{P}^*(t,s),\ \mathbf{P}^*(s,s) = \mathbf{I}. \tag{2.78}$$

For a more detailed discussion and the proof of the last proposition refer to [6].

2.3.2.4 Continuous-time conditionally Markov chain

A probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ endowed with some filtrations $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ and $\{\mathcal{G}_t\}_{t\in\mathbb{R}^+}$, such that $\mathcal{F} \subseteq \mathcal{G}$ is considered, and, for the sake of convenience, the following arguments are carried on under the risk-neutral probability measure $\mathbb{Q}$.
A $\mathcal{G}$-adapted $\mathcal{K}$-valued stochastic process C defined on this probability space is a continuous-time conditionally Markov chain relative to $\mathcal{F}$ and under the risk-neutral probability measure $\mathbb{Q}$ if, for any function $h : \mathcal{K} \to \mathbb{R}$:

$$\mathbf{E}^{\mathbb{Q}}[h(C_s)|\mathcal{G}_t] = \mathbf{E}^{\mathbb{Q}}[h(C_s)|\mathcal{F}_t \vee \sigma(C_t)], \quad \forall 0 \leq t \leq s. \tag{2.79}$$

Moreover, since $\sigma(C_t) \subseteq \mathcal{F}_t \vee \sigma(C_t) \subseteq \mathcal{G}_t$, if a process C is a $\mathcal{G}$-Markov chain under $\mathbb{Q}$, then C is also an $\mathcal{F}$-conditional Markov chain under $\mathbb{Q}$, for any choice of a sub-filtration $\mathcal{F}$ of $\mathcal{G}$. However, the opposite is not necessarily true.

Let $\mathbf{\Lambda}^*(t) = [\lambda^*_{ij}(t)]_{1\leq i,j\leq K}$, $t \in \mathbb{R}^+$, denote a $\mathcal{F}$-progressively measurable, bounded, matrix-valued process, and for every $i \in \mathcal{K}$, $t \in \mathbb{R}^+$, and any function $h : \mathcal{K} \to \mathbb{R}$,

$$\mathbf{\Lambda}^*(t)h(i) = \sum_{j=1}^{K} \lambda^*_{ij}(t)h(j). \tag{2.80}$$

$\mathbf{\Lambda}^*$ is an $\mathcal{F}$-conditional infinitesimal generator, or matrix of stochastic intensities, for a $\mathcal{K}$-valued $\mathcal{F}$-conditional $\mathcal{G}$-Markov chain C under $\mathbb{Q}$ if, for any function $h : \mathcal{K} \to \mathbb{R}$ the process M^h, defined as:

$$M^h_t = h(C_t) - h(C_0) - \int_0^t \mathbf{\Lambda} *_u h(C_u)du, \quad \forall t \in \mathbb{R}^+ \tag{2.81}$$

follows a $\mathcal{G}$-martingale under $\mathbb{Q}$. The generic entry $\lambda^*_{ij}(t)$ represent the $\mathcal{F}$-conditional intensity of transition from the state i to the state j.

2.4 Markovian models for credit migrations

Markovian models for credit migrations can be interpreted as an extension of intensity models discussed in section 2.2.1 that allow to consider several possible credit events within the framework of the intensity-based methodology, and not just the default event.
A firm's credit rating is a measure of the firm's creditworthiness and propensity to default. Credit ratings, as seen in section 1.1.2, are typically identified with elements of a certain set. Formally, the credit quality of corporate debt is categorized into a finite number of credit rating classes, and each credit class is represented by an element of a finite set $\mathcal{K} = \{1, \dots, K\}$, where state 1 represent the highest ranking, state $K-1$ represents the lowest ranking and state K corresponds to the default event. It is well known that the credit quality of a given corporate debt changes over time, *i.e.*, the credit quality migrates between various credit classes and credit ratings along with it.
These models aim to dynamically model credit migrations of a corporate bond between different possible credit ratings, that is, changes in their credit quality over time. The most widely used way to model credit migrations is in terms of either discrete- or continuous-time Markov chain (or conditionally Markov chain) C with finite state space, referred to as the credit migration process. The main issue in the Markovian models for the credit migrations is the specification of the transition probabilities matrix (discrete time) or the infinitesimal generator matrix (continuous time) for C, both under the *real-world* and the risk-neutral probability measures.

It is useful to delve into the properties of the probability space in which these models are defined. Typically, the overall filtration involved in a credit risk model that involves credit migrations in terms of a Markov chain C is larger than the natural filtration $\mathcal{F}^C$ of the migration process. Therefore, in general, it is more convenient to deal with either the $\mathcal{G}$-Markov property or the conditional Markov property, rather than the ordinary Markov property of the migration process C. This way, the model involves the presence of several sources of uncertainty (market risk, credit risk, economic factors, ...).
Another important issue in this framework is the preservation of the $\mathcal{G}$-Markov property under an equivalent change of probability measure, as the change from the *real-world* probability to a risk-neutral probability. The conditions imposed in section 2.3.2.3 for the respective Radon-Nikodym densities so that the $\mathcal{G}$-Markov property is preserved are too restrictive for these credit risk models, in which the change from the *real-world* to the risk-neutral probability involves all sources of uncertainty present in the model, since the Radon-Nikodym densities are assumed to be only adapted with respect to the natural filtration of the Markov chain, $\mathcal{F}^C$, rather than adapted to the filtration $\mathcal{G}$. However, if the credit risk model admits structural properties such as some sort of decomposition of the overall risk into market risk and credit risk, mathematically $\mathcal{G} = \mathcal{F} \times \mathcal{F}^C$, where $\mathcal{F}$ and $\mathcal{F}^C$ denote, respectively, the filtrations related to market risk and credit risk. Then it seems natural to expect that the $\mathcal{G}$-Markov property of the migration process will still be preserved, under some conditions for the component of the Radon-Nikodym density corresponding to the filtration $\mathcal{F}^C$. Such a decomposition would correspond to an

underlying product probability space, which may also support the assumption of independence between market risk and default risk (imposed, for instance, in the Jarrow et al. (1997) [36] approach).

2.4.1 Jarrow, Lando and Turnbull model (1997)

Jarrow et al. (1997) [36] propose a Markov model for the term structure of credit risk spreads that incorporates a firm's credit rating as an indicator of the likelihood of default. The model proposed by Jarrow, Lando e Turnbull is based on the Jarrow and Turnbull model (1995).
A frictionless economy with a finite horizon $[0, T^*]$ is considered, where T^* is a fixed horizon date and trading can be discrete or continuous. Moreover, a filtered probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is considered, where the filtration $\mathcal{G}$ represents the total information available to traders and $\mathbb{P}$ is interpreted as the *real-world* probability measure.
The first assumption is that there exists a unique equivalent martingale measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_{T^*})$, such that all the default-free and risky zero coupon bond prices follow a $\mathcal{G}$-martingale, after normalization by the money market account. This assumption is equivalent to the statement that the markets for default-free and risky debt are complete and arbitrage-free.
Let $r(t)$ be the time t default-free spot rate; then, the money market account is denoted in the discrete time case as:

$$B(t) = e^{\sum_{i=0}^{t-1} r(i)}, \tag{2.82}$$

or in the continuous time case as:

$$B(t) = e^{\int_0^t r(u)\, du}. \tag{2.83}$$

The second assumption is that the interest rate risk is modeled by means of an $\mathcal{F}$-adapted stochastic process r of the default-free spot rate, where $\mathcal{F}$ is some sub-filtration of $\mathcal{G}$. Under the first two assumptions, the time t price of a default-free zero coupon bond paying a sure unity of currency at time T, denoted by $v(t,T)$, $0 \leq t \leq T$, can be written as the expected, discounted value of a sure dollar received at time T, that is:

$$v(t,T) = \mathbf{E}_t^{\mathbb{Q}}\left[\frac{B(t)}{B(T)}\right]. \tag{2.84}$$

Let $v^\delta(t,T)$ be the time t price of a risky zero coupon bond promising to pay a unity of currency at time T where $t \leq T \leq \tau$, where τ represent the random time of default.
The third assumption is that the risky zero coupon bond is subject to the *fractional recovery of Treasury value* scheme, as defined in section 2.2.3, and the recovery rate δ is an exogenous constant, that is, the issuer may not pay in full the promised unity of currency, if the firm's default occurs before or at time T. If bankrupt, the firm pays only $\delta < 1$.
Then, $v^\delta(t,T)$ can be written as the expected, discounted value of a "risky" unity of currency received at time T:

$$v^\delta(t,T) = \mathbf{E}_t^{\mathbb{Q}}\left[\frac{B(t)}{B(T)}\left(\delta\, \mathbf{1}_{\{\tau \leq T\}} + \mathbf{1}_{\{\tau > T\}}\right)\Big|\mathcal{G}_t\right]. \tag{2.85}$$

The fourth assumption is that the default time τ is a random variable independent of the stochastic process for default-free spot rates $\{r(t)\}_{0\leq t\leq\tau}$, conditionally upon the filtration $\mathcal{G}$ under the martingale probability measure $\mathbb{Q}$. In other words, for any integrable functional ϕ of the default-free spot rate process r, and any integrable function f of the random time τ:

$$\mathbf{E}^{\mathbb{Q}}\left[\phi(r)f(\tau)|\mathcal{G}_t\right] = \mathbf{E}^{\mathbb{Q}}\left[\phi(r)|\mathcal{G}_t\right]\,\mathbf{E}^{\mathbb{Q}}\left[f(\tau)|\mathcal{G}_t\right], \quad \forall t \in \mathbb{R}^+.$$

This assumption is quite convenient from the perspective of computations. In fact, under the additional structure imposed by the model, the default time is uncorrelated with default-free spot rates under the *real-world* probability measure as well (it is really only needed under the risk-neutral probability measure). This way the model is really only a model of spreads and hence it can be combined with any model of the default-free term structure. Then:

$$\begin{aligned} v^{\delta}(t,T) &= \mathbf{E}_t^{\mathbb{Q}}\left[\frac{B(t)}{B(T)}\Big|\mathcal{G}_t\right]\,\mathbf{E}_t^{\mathbb{Q}}\left[\delta\,\mathbf{1}_{\{\tau\leq T\}} + \mathbf{1}_{\{\tau>T\}}\Big|\mathcal{G}_t\right] \\ &= v(t,T)\,\left(\delta\,P^*\{\tau\leq T|\mathcal{G}_t\} + P^*\{\tau>T|\mathcal{G}_t\}\right) \\ &= v(t,T)\,\left(\delta + (1-\delta)P^*\{\tau>T|\mathcal{G}_t\}\right), \end{aligned} \tag{2.86}$$

where $P^*\{\tau > T|\mathcal{G}_t\}$ is the probability under $\mathbb{Q}$ that default will occur after the maturity T.

If the conditional independence (fourth assumption) is relaxed, one can change numeraire to the treasury bond with maturity T and construct the forward probability measure $\mathbb{Q}_T$:

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}} = \frac{1}{v(0,T)\,B(T)}, \quad \mathbb{Q}\text{-almost surely.} \tag{2.87}$$

Then:

$$v^{\delta}(t,T) = v(t,T)\,\left(\delta + (1-\delta)P^T\{\tau>T|\mathcal{G}_t\}\right), \tag{2.88}$$

where $P^T\{\tau > T|\mathcal{G}_t\}$ is the probability that default will occur after the maturity T under the forward martingale measure for the date $T \leq T^*$ $\mathbb{Q}_T$.

It is interesting to observe that under the fourth assumption, the probability of survival under the risk-neutral probability measure $\mathbb{Q}$ coincides with that under the forward probability measure $\mathbb{Q}_T$:

$$P^*\{\tau > T|\mathcal{G}_t\} = P^T\{\tau > T|\mathcal{G}_t\}.$$

The idea is to make $P^*\{\tau > T|\mathcal{G}_t\}$ depend on the rating of the issuer, i.e. the formula (2.86) become:

$$v^{i}(t,T) = v(t,T)\,\left(\delta + (1-\delta)P_i^*\{\tau>T|\mathcal{G}_t\}\right). \tag{2.89}$$

where i is the initial rating of the issuer. It would be tempting to simply use the empirical transition probabilities as reported by the major rating agencies, but the spreads obtained are totally inconsistent with those observed because of the effects of, among others, different tax treatment of corporate bonds and treasury bonds and different liquidity.

2.4.1.1 The discrete-time case

The dates $t = 0, \dots, T^*$ are considered, where the horizon date T^* is assumed to be a positive integer.
The fifth assumption is that the distribution for the default time and for the credit migration process C is modeled via a discrete-time, time-homogeneous $\mathcal{G}-$Markov chain on a finite state space $\mathcal{K} = \{1, \dots, K\}$, that represents the ordered possible credit classes and the default absorbing state K, under the *real-world* probability measure $\mathbb{P}$. Then, the future probabilistic evolution of credit ratings does not depend on the history of the market and on the past rating, but it is assumed to depend exclusively on the current rating. The Markov chain is specified by the $K \times K$ transition matrix under $\mathbb{P}$:

$$\mathbf{P} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,K-1} & p_{1,K} \\ \vdots & \ddots & \vdots & \vdots \\ p_{K-1,1} & \cdots & p_{K-1,K-1} & p_{i,K} \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where:

- $p_{ij} \geq 0, \quad \forall i, j \in \mathcal{K}$;

- $\sum_{j=1}^{K} p_{ij} = 1$, for any fixed $i \in \mathcal{K}$.

The default time τ is defined as the first time the credit migration process jumps to the absorbing state K:

$$\tau = \inf\{t \in [0, T] : C_t = K\}.$$

The sixth assumption is that the default time and the credit migration process C follow a time-inhomogeneous $\mathcal{G}$-Markov chain under the spot martingale probability measure $\mathbb{Q}$, with the time-dependent transition matrix:

$$\mathbf{P}^*(t, t+1) = \begin{pmatrix} p^*_{1,1}(t, t+1) & \cdots & p^*_{1,K-1}(t, t+1) & p^*_{1,K}(t, t+1) \\ \vdots & \ddots & \vdots & \vdots \\ p^*_{K-1,1}(t, t+1) & \cdots & p^*_{K-1,K-1}(t, t+1) & p^*_{i,K}(t, t+1) \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where:

- $p^*_{ij}(t, t+1) \geq 0, \quad \forall i, j \in \mathcal{K}$;

- $\sum_{j=1}^{K} p^*_{ij}(t, t+1) = 1$, for any fixed $i \in \mathcal{K}$;

- $p^*_{ij}(t, t+1) > 0$ if and only if $p_{ij} > 0$.

The seventh assumption is that the risk premia adjustments are such that credit migration process C under the martingale probability measure $\mathbb{Q}$ satisfy:

$$p^*_{ij}(t,t+1) = \pi_i(t)\, p_{ij} \quad \forall i,j, i \neq j, \tag{2.90}$$

where $\pi_i(t)$ are time-dependent, deterministic coefficients and are interpreted as discrete-time risk premia. These transform the actual probabilities to those used in valuation. Now the t-step transition matrix under $\mathbb{Q}$ can be computed:

$$\mathbf{P}^*(0,t) = \mathbf{P}^*(0,1)\, \mathbf{P}^*(1,2) \ldots \mathbf{P}^*(t-1,t). \tag{2.91}$$

Each one-period transition matrix is a modification of $\mathbf{P}$ using the low-dimensional parameter π. This numerical scheme can be seen as a discrete-time approximation to a row wise adjustment of the generator matrix.

In this kind of a model, the $\mathcal{G}$-Markov property of C on the product space is rather trivial, as it is essentially equivalent to the Markov property of C on the component space. Observe, though, that in such a model the Markov property of the migration process is in fact detached from the market fundamentals (it is sort of superimposed on the market of default able claims).

In view of the seven assumptions just presented, the risk-neutral conditional probability of solvency (*i.e.*, default occurs after T) is:

$$P^*\{\tau > T|\mathcal{G}_t\} = P^*\{\tau > T|\mathcal{C}_t\} = \sum_{j\neq K} p^*_{ij}(t,T) = 1 - p^*_{iK}(t,T), \quad t = 0,\ldots,T, \tag{2.92}$$

where i represents the state the firm is in at time t. Then, the value of a zero coupon bond issued by a firm in credit class i at time t can be rewritten as:

$$v^i(t,T) = v(t,T)\left(\delta + (1-\delta)\sum_{j\neq K} p^*_{ij}(t,T)\right). \tag{2.93}$$

2.4.1.2 The continuous-time case

In the continuous-time case the fifth assumption is replaced by the following: the distribution for the default time and for the credit migration process C are modeled via a continuous-time, time-homogeneous $\mathcal{G}$-Markov chain on a finite state space $\mathcal{K} = \{1,\ldots,K\}$ under the *real-world* probability measure $\mathbb{P}$. The Markov chain is specified by the $K \times K$ infinitesimal generator matrix under $\mathbb{P}$:

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,K-1} & \lambda_{1,K} \\ \vdots & \ddots & \vdots & \vdots \\ \lambda_{K-1,1} & \cdots & \lambda_{K-1,K-1} & \lambda_{i,K} \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

where:

- $0 \leq \lambda_{ii}$, for $i \in \mathcal{K}$;

- $\lambda_{ii} = -\sum_{j=1, i\neq j}^{K} \lambda_{ij} \leq 0.$

- $\sum_{j=1}^{K} \lambda_{ij} = 0, \text{ for } i \in \mathcal{K}.$

The $K \times K$ t-period probability transition matrix is given by

$$\mathbf{P}(t) = e^{t\,\mathbf{\Lambda}} = \sum_{k=0}^{\infty} \frac{(t\mathbf{\Lambda})^k}{k!}. \tag{2.94}$$

As in the discrete-time case, the default time τ is defined as the first time the credit migration process jumps to the absorbing state K:

$$\tau = \inf\{t \in [0, T^*] : C_t = K\}.$$

The sixth assumption is replaced by the following: the default time and the credit migration process C follow a time-inhomogeneous $\mathcal{G}$-Markov chain under the spot martingale probability measure $\mathbb{Q}$, with the time-dependent infinitesimal generator matrix:

$$\mathbf{\Lambda}^*(t) = \begin{pmatrix} \lambda^*_{1,1}(t) & \cdots & \lambda^*_{1,K-1}(t) & \lambda^*_{1,K}(t) \\ \vdots & \ddots & \vdots & \vdots \\ \lambda^*_{K-1,1}(t) & \cdots & \lambda^*_{K-1,K-1}(t) & \lambda^*_{i,K}(t) \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

Finally, the seventh assumption is replaced by the following: the infinitesimal generator matrix of C under the equivalent martingale probability measure $\mathbb{Q}$ is given by:

$$\mathbf{\Lambda}^*(t) = \mathbf{U}(t)\,\mathbf{\Lambda}, \tag{2.95}$$

where $\mathbf{U}(t) = \text{diag}(u_1(t), \ldots, u_{K-1}(t), 1)$ is a $K \times K$ diagonal matrix whose the first $K-1$ entries are strictly positive deterministic functions of t that satisfy:

$$\int_0^T u_i(t)\, dt < +\infty \quad \text{for } i = 1, \ldots, K-1.$$

As in the discrete-time case, the diagonal entries of $\mathbf{U}(t)$ are interpreted as risk premia, that is, the adjustments for risk that transform the actual probabilities into the pseudo-probabilities suitable for valuation purposes.

Similarly as in the discrete-time case:

$$P^*\{\tau > T \mid \mathcal{G}_t\} = P^*\{\tau > T \mid \mathcal{C}_t\} = \sum_{j\neq K} p^*_{ij}(t,T) = 1 - p^*_{iK}(t,T), \quad t \in [0,T], \tag{2.96}$$

hence the same valuation formulae as those derived for the discrete-time case (equation (2.93)) can be applied.

2.5 The model for the spread risk

By using only ratings as the relevant predictor of default, the Jarrow, Lando and Turnbull model does not explain variations in credit spread between different issuers of the same credit quality. Also, the time-inhomogeneous Markov chain assumption implies that bond prices evolve deterministically between ratings changes. The model presented here is a generalization of the Jarrow, Lando and Turnbull model which incorporates state dependence in transition rates and risk premia, thus allowing for stochastic changes in credit spreads between ratings transitions. The process of credit rating transitions and default is modelled using an extension of the classical time-homogenous Markov chain, as presented for the first time in Lando (1998) [47] and later taken up in Gambaro et al. (2018) [28]. This model allows to simultaneously model term structures for different rating classes and spreads to fluctuate stochastically even in periods where the rating of the defaultable issuer does not change, incorporating stochastic transition rates and stochastic default intensities.

Let $\rho(t)$ be the categorical variable "credit rating of a firm at time t". Recalling the Cox process framework defined in Lando (1998) and reported in section 2.2.2, the process of credit rating transitions and default C is assumed to follow a continuous-time, time-inhomogeneous $\mathcal{F}$-conditional $\mathcal{G}$-Markov chain on a finite state space $\mathcal{K} = \{1, \ldots, K\}$, where K is the absorbing state of default, both under the *real-world* probability measure $\mathbb{P}$ and the risk-neutral probability measure $\mathbb{Q}$. The Markov chain is characterized by the $K \times K$ transition matrix for the time interval $(t, s]$:

$$\mathbf{P}(t,s) = \begin{pmatrix} m_{1,1}(t,s) & \cdots & m_{1,K-1}(t,s) & m_{1,K}(t,s) \\ \vdots & \ddots & \vdots & \vdots \\ m_{K-1,1}(t,s) & \cdots & m_{K-1,K-1}(t,s) & m_{i,K}(t,s) \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}(t,s) & \mathbf{q}_D(t,s) \\ & \\ \mathbf{0^T} & 1 \end{pmatrix}, \tag{2.97}$$

where the last row is composed of all zeros and a one because the default state K is an absorbing state.

The transition matrix $\mathbf{P}(t,s)$ is assumed to have the following representation:

$$\mathbf{P}(t,s) = e^{\mathbf{G}(\pi(s)-\pi(t))}, \tag{2.98}$$

where $\mathbf{G}$ is the constant value generator matrix of a time-homogeneous Markov chain, *i.e.*, the generator of the $\mathbf{P}(t,s)$ process, and $\pi(t)$ is a stochastic time. Hence, conditionally to the trajectory of $\pi(t)$, C is a time-inhomogeneous Markov process. Formula (2.98) is an extension of the original Jarrow, Lando e Turnbull model , where instead time evolves deterministically and the generator matrix is time-dependent (depend on t and s):

$$\mathbf{P}(t,s) = e^{\mathbf{G}(t,s)\,(s-t)}. \tag{2.99}$$

In order for the model to be consistent, that is, to have transition probabilities between 0 and 1, the generator matrix $\mathbf{G}$ must have the following properties:

- the elements of the diagonal must be non-positive, $g_{ii} \leq 0$;

- the off-diagonal elements must be non-negative, $g_{ij} \geq 0$, $i \neq j$;
- the sum of the elements on each row must be zero:

$$\sum_{j=1}^{K} g_{ij} = 0, \quad i = 1, \ldots, K;$$

- the elements of the K-th row must all be null, since they correspond to the transition intensities for the state K, which is absorbing.

Moreover, the process π has to be a stochastic time:

- $\pi(t)$ is a real non-negative and increasing right continuous process with left limits (RCLL);
- $\pi(t)$ is a stopping time, for every $t \geq 0$;
- $\pi(t)$ is finite almost surely, for every $t \geq 0$;
- $\pi(0) = 0$;
- $\lim_{t\to\infty} \pi(t) = \infty$.

As demonstrated in Feller (1971) [27], if $\pi(t)$ has stationary non-negative independent increments, then C is unconditionally a Markov chain. A stochastic Lévy process with non-negative values and stationary non-negative independent increments is called a subordinator process; hence, $\pi(t)$ is the subordinator of the subordinated process C.
The financial interpretation of the model is that the transition probabilities of the credit rating process are subjected to a common source of uncertainty, *i.e.*, the subordinator process $\pi(t)$.

If the generator matrix $\mathbf{G}$ is assumed to have all distinct eigenvectors, *i.e.*, to be diagonalizable, it is possible to perform the so-called spectral decomposition (or diagonalization), a canonical decomposition provided by the spectral theorem:

$$\mathbf{G} = \mathbf{B}\,\mathbf{D}\,\mathbf{B}^{-1}, \tag{2.100}$$

where $\mathbf{B}$ is a $K \times K$ matrix whose columns consist of K eigenvectors of $\mathbf{G}$ and $\mathbf{D}$ is the diagonal matrix whose elements on the diagonal consist of the eigenvalues of $\mathbf{G}$, which are all non-positive:

$$\text{diag}(\mathbf{D}) = (d_1, \ldots, d_{K-1}, d_K = 0) \leq 0.$$

It is relevant to notice that from the properties of $\mathbf{G}$ it follows that the K-th eigenvector has mutually identical components and corresponding null eigenvalues:

$$\mathbf{G}\begin{pmatrix} c \\ \vdots \\ c \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{K} g_{1j}\, c \\ \vdots \\ \sum_{j=1}^{K} g_{Kj}\, c \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \begin{pmatrix} c \\ \vdots \\ c \end{pmatrix}. \tag{2.101}$$

Therefore, the elements of the last column of matrix $\mathbf{B}$ are all equal:

$$b_{1K} = b_{2K} = \cdots = b_{KK}. \tag{2.102}$$

Recall that, by the properties of the exponential of a matrix, in general:

$$e^{\mathbf{G}} = e^{\mathbf{B}\,\mathbf{D}\,\mathbf{B}^{-1}} = \mathbf{B}\,e^{\mathbf{D}}\,\mathbf{B}^{-1} = \mathbf{B}\begin{pmatrix} e^{d_{11}} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{d_{KK}} \end{pmatrix}\mathbf{B}^{-1}. \tag{2.103}$$

Given the spectral decomposition (equation (2.100)) and the property (2.103), the transition matrix can be written as:

$$\begin{aligned}
\mathbf{P}(t,s) &= \mathbf{B}\,e^{\mathbf{D}(\pi(s)-\pi(t))}\,\mathbf{B}^{-1} \\
&= \begin{pmatrix} b_{11} & \dots & b_{1K} \\ \vdots & \ddots & \vdots \\ b_{K1} & \dots & b_{KK} \end{pmatrix}\begin{pmatrix} e^{d_1(\pi(s)-\pi(t))} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{d_K(\pi(s)-\pi(t))} \end{pmatrix}\begin{pmatrix} b^{-1}_{11} & \dots & b^{-1}_{1K} \\ \vdots & \ddots & \vdots \\ b^{-1}_{K1} & \dots & b^{-1}_{KK} \end{pmatrix} \\
&= \begin{pmatrix} b_{11}\,e^{d_1(\pi(s)-\pi(t))} & \dots & b_{1K}\,e^{d_K(\pi(s)-\pi(t))} \\ \vdots & \ddots & \vdots \\ b_{K1}\,e^{d_1(\pi(s)-\pi(t))} & \dots & b_{KK}\,e^{d_K(\pi(s)-\pi(t))} \end{pmatrix}\begin{pmatrix} b^{-1}_{11} & \dots & b^{-1}_{1K} \\ \vdots & \ddots & \vdots \\ b^{-1}_{K1} & \dots & b^{-1}_{KK} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^{K} b_{1j}\,b^{-1}_{j1}\,e^{d_j(\pi(s)-\pi(t))} & \dots & \sum_{j=1}^{K} b_{1j}\,b^{-1}_{jK}\,e^{d_j(\pi(s)-\pi(t))} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{K} b_{Kj}\,b^{-1}_{j1}\,e^{d_j(\pi(s)-\pi(t))} & \dots & \sum_{j=1}^{K} b_{Kj}\,b^{-1}_{jK}\,e^{d_j(\pi(s)-\pi(t))} \end{pmatrix}.
\end{aligned} \tag{2.104}$$

The condition that all the elements in the last row are null except the last one which is equal to 1 (equation (2.97)) implies:

$$\begin{aligned}
d_K &= 0, \\
b_{Ki} &= 0, \quad i = 1, \dots, K-1, \\
b^{-1}_{Ki} &= 0, \quad i = 1, \dots, K-1, \\
b_{KK}\,b^{-1}_{KK} &= 1,
\end{aligned} \tag{2.105}$$

from which it can be derived:

$$\begin{aligned}
&\sum_{j=1}^{K} b_{Kj}\,b^{-1}_{ji}\,e^{d_j(\pi(s)-\pi(t))} = 0, \qquad i = 1, \dots, K_1, \\
&\sum_{j=1}^{K} b_{Kj}\,b^{-1}_{jK}\,e^{d_j(\pi(s)-\pi(t))} = 1.
\end{aligned} \tag{2.106}$$

Finally, the property (2.102) and the last of the four properties (2.105) imply:

$$b_{iK}\,b^{-1}_{KK} = 1, \qquad i = 1, \dots, K. \tag{2.107}$$

The subordinator process $\pi(t)$ is assumed to be defined as an integral of a positive stochastic intensity $\eta(t)$, as defined in [28]:

$$\pi(t) = \int_{t_0}^{t} \eta(u)\, du \; ; \tag{2.108}$$

then, with:

$$\begin{aligned} \mathbf{P}(t,s) &= \mathbf{B}\, e^{\mathbf{D}(\pi(s)-\pi(t))}\, \mathbf{B}^{-1} \\ &= \mathbf{B}\, e^{\mathbf{D}(\int_{t_0}^{s} \eta(u)\, du - \int_{t_0}^{t} \eta(u)\, du)}\, \mathbf{B}^{-1} \\ &= \mathbf{B}\, e^{\mathbf{D} \int_{t}^{s} \eta(u)\, du}\, \mathbf{B}^{-1}, \end{aligned} \tag{2.109}$$

$\mathbf{P}(t,s)$ satisfies the *Kolmogorov backward* equation:

$$\frac{\partial \mathbf{P}(t,s)}{\partial t} = -\mathbf{G}\, \eta(t)\, \mathbf{P}(t,s), \tag{2.110}$$

and is the transition probability matrix of an inhomogeneous Markov chain on $\mathcal{K}$. Hence, the infinitesimal generator matrix of C is $\mathbf{G}\,\eta(t)$, that is, the instantaneous transition probability to jump from the rating i to the rating j, is $g_{ij}\,\eta(t)\,dt$.

Proof. Since:

$$\mathbf{G}\,\mathbf{B} = \mathbf{B}\,\mathbf{D},$$

it can be seen that:

$$\begin{aligned} \frac{\partial \mathbf{P}(t,s)}{\partial t} &= \mathbf{B}\ (-\mathbf{D})\ \eta(t)\, e^{\mathbf{D}\ \int_{t}^{s} \eta(u)\, du}\, \mathbf{B}^{-1} \\ &= -\mathbf{G}\, \eta(t)\, \mathbf{B}\, e^{\mathbf{D}\ \int_{t}^{s} \eta(u)\, du}\, \mathbf{B}^{-1} \\ &= -\mathbf{G}\, \eta(t)\, \mathbf{P}(t,s), \end{aligned}$$

which shows that $\mathbf{P}(t,s)$ is indeed a solution to the *Kolmogorov backward* equation.

With the proposed structure the calculation of the survival probability conditionally on a starting state i is simple:

$$1 - p_{iK}(t,s) = 1 - \sum_{j=1}^{K} -b_{ij} b_{jK}^{-1}\ e^{d_j \int_{t}^{s} \eta(u)\, du} \tag{2.111}$$

This Markov chain describe the evolution of a rating process that is conditional on a particular path of the subordinator process. To get unconditional transition probabilities, it is needed to take the expectation of these transition probabilities over the distribution of the subordinator process.

2.5.1 The price of the zero coupon bond

Consider a unit zero coupon bond maturing at time T issued by a firm whose rating at time t is i, and for tractability assume the *fractional recovery of Treasury value* scheme in case of default, which provides that in case of default the creditor receives in T a fraction $\delta \in (0,1)$ of the face value, *i.e.*, receives in a default-free zero coupon bond with residual maturity $T-\tau$ and face value δ. The price of this bond can be computed as:

$$
\begin{aligned}
v^i(t,T) &= \mathbf{E}_t^{\mathbb{Q}}\left[\frac{B(t)}{B(T)}\left(1\,\mathbf{1}_{\{\tau>T\}}+\delta\,\mathbf{1}_{\{\tau\le T\}}\right)\middle|\mathcal{G}_t\right]\\
&= \mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}\,1\,\mathbf{1}_{\{\tau>T\}}+e^{-\int_t^T r(u)\,du}\,\delta\,\mathbf{1}_{\{\tau\le T\}}\middle|\mathcal{G}_t\right]\\
&= \mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}\,1\,\mathbf{E}_t^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau>T\}}\middle|\mathcal{F}_t\vee\mathcal{G}_t,i\right]+e^{-\int_t^T r(u)\,du}\,\delta\,\mathbf{E}_t^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau\le T\}}\middle|\mathcal{F}_t\vee\mathcal{G}_t,i\right]\middle|\mathcal{G}_t\right]\\
&= \mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}\,1\,(1-q_{iK}(t,T))+e^{-\int_t^T r(u)\,du}\,\delta\,q_{iK}(t,T)\middle|\mathcal{F}_t\right]\\
&= \mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}\,[(1-\delta)\,(1-q_{iK}(t,T))+\delta]\middle|\mathcal{F}_t\right]\\
&= \mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}\left\{(1-\delta)\left[1-\sum_{j=1}^{K}b_{ij}b_{jK}^{-1}\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta(u)\,du}\right]\right]+\delta\right\}\middle|\mathcal{F}_t\right]\\
&= \mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}\left[(1-\delta)\sum_{j=1}^{K-1}(-b_{ij}b_{jK}^{-1})\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta(u)\,du}\right]+\delta\right]\middle|\mathcal{F}_t\right],
\end{aligned}
\tag{2.112}
$$

where $B(t)$ denotes the money market account, τ denotes the default time, $q_{iK}(t,T)$ denotes the probability of default by time T starting from the rating i at time t under the risk-neutral probability measure $\mathbb{Q}$, $\mathcal{G}_t$ and $\mathcal{F}_t$ denote, respectively, the overall filtration and the filtration with the information about the state variables. The term $-|d_j|$ in the risk-neutral expectation coincides with d_j, since the eigenvalues of $\mathbf{G}$ are non-positive ($d_j \le 0$).
The last equality of the (2.112) derives from the relationship:

$$
\begin{aligned}
\sum_{j=1}^{K}b_{ij}b_{jK}^{-1}\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta(u)\,du}\right] &= \sum_{j=1}^{K-1}b_{ij}b_{jK}^{-1}\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta(u)\,du}\right]\\
&\quad+(b_{iK}b_{KK}^{-1})\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_K|\int_t^T \eta(u)\,du}\right],
\end{aligned}
\tag{2.113}
$$

where $(b_{iK}b_{KK}^{-1})=1$ from (2.107) and the expected value is equal to 1 since $d_K=0$.

From here on, information about the reference filtration will be omitted in the expected values to simplify the notation.

The subordinator process $\pi(t)$ and, therefore, the default time τ are assumed to be independent of the default-free spot rate process r, then (2.112) becomes:

$$v^i(t,T) = v(t,T)\left[(1-\delta)\sum_{j=1}^{K-1}(-b_{ij}b_{jK}^{-1})\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta(u)\,du}\right]+\delta\right], \qquad (2.114)$$

where $v(t,T)$ is the price at time t of a unit default-free zero coupon bond with maturity T.
This assumption is really only needed under the risk-neutral probability measure in order to obtain a closed-form valuation formula. Although this assumption is not always empirically confirmed, it is quite common in the literature, see [36], [37], [28], [50] and [15]. Alternatively, this assumption can be relaxed and one can change the numeraire to the treasury bond with maturity T and use the forward probability measure $\mathbb{Q}_T$, as presented in section 2.4.1.

Equation (2.114) shows that the price of the unit zero coupon bond depends on $(K-1)$ intensities, respectively $\eta_j(t) = |d_j|\eta(t)$ with $j = 1,\ldots,K-1$, each proportional to the intensity of the subordinator $\eta(t)$, and therefore perfectly correlated. If affine functions of diffusions with affine drift and volatility for r and η are used, a class of models whose bond prices are expressed as sums of affine models can be obtained.

It is interesting to show the version of the (2.114) in case of zero recovery ($\delta = 0$):

$$v^i(t,T) = v(t,T)\sum_{j=1}^{K-1}(-b_{ij}b_{jK}^{-1})\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta(u)\,du}\right]. \qquad (2.115)$$

The subordinator intensity process $\eta(t)$ is assumed to be modelled as a mean-reverting square root process, for instance as a Cox, Ingersoll and Ross (CIR) model, under both the *real-world* and the risk-neutral probability measures, $\mathbb{P}$ and $\mathbb{Q}$. The *real-world* dynamics is:

$$d\eta(t) = \alpha(\gamma-\eta_t)\,dt + \sigma\sqrt{\eta_t}\,dZ_t^{\mathbb{P}}, \quad \eta(t_0)=\eta_0, \qquad (2.116)$$

with $\alpha,\gamma,\sigma > 0$ such that $2\alpha\gamma > \sigma^2$, $\eta(t) \geq 0$ and where $dZ_t^{\mathbb{P}}$ is a standard Brownian motion under the *real-world* probability measure. Assuming a risk premium defined as:

$$\lambda(t,\eta_t) = \lambda\frac{\sqrt{\eta_t}}{\sigma}, \qquad (2.117)$$

the change of probability measure allows the same functional form for $\eta(t)$ dynamics to be maintained even under the risk-neutral measure:

$$d\eta(t) = \hat{\alpha}(\hat{\gamma}-\eta_t)\,dt + \sigma\sqrt{\eta_t}\,dZ_t^{\mathbb{Q}}, \quad \eta(t_0)=\eta_0, \qquad (2.118)$$

where:

$$\hat{\alpha} = \alpha+\lambda, \quad \hat{\alpha}\hat{\gamma} = \alpha\gamma. \qquad (2.119)$$

Under the CIR model assumption, the distribution for future values, $\eta(s),\ s > t$, conditional on $\mathcal{F}_t$, is a non-central Chi-squared distribution with expected value and variance defined by the following expressions:

$$\begin{aligned}\mathbf{E}_t[\eta(s)] &= \gamma + (\eta_t - \gamma)e^{-\alpha(s-t)},\\ \mathbf{Var}_t[\eta(s)] &= \tfrac{\sigma^2\eta_t}{\alpha}\left[e^{-\alpha(s-t)} - e^{-2\alpha(s-t)}\right] + \tfrac{\sigma^2\gamma}{2\alpha}\left[1 - e^{-\alpha(s-t)}\right]^2,\end{aligned}$$

and the term $\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta(u)\,du}\right]$ has an analytical expression given by [38]:

$$\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta(u)\,du}\right] = A(h)e^{-B(h)|d_j|\,\eta_t},\ \ h = T - t, \tag{2.120}$$

where:

$$\begin{aligned}A(h) &= \left[\frac{\delta_d\, e^{\phi_d\, h}}{\phi_d\,(e^{\delta_d\, h} - 1) + \delta_d}\right]^{\nu},\\ B(h) &= \frac{e^{\delta_d\, h} - 1}{\phi_d\,(e^{\delta_d\, x} - 1) + \delta_d},\end{aligned}$$

where:

$$\delta_d = \sqrt{\hat{\alpha}^2 + 2|d_j|\sigma^2},\quad \phi_d = \frac{\hat{\alpha} + \delta_d}{2},\quad \nu = 2\,\frac{\hat{\alpha}\hat{\gamma}}{\sigma^2}.$$

are the model parameters in Brown and Dybvig parameterization.

Thus, the change of probability measure for the Markov chain transition matrix governing the process of credit rating transitions and default C is defined implicitly, considering the dynamics and the parameters for the intensity $\eta(t)$ of the subordinator process $\pi(t)$, respectively, under the *real-world* and risk-neutral probability measures:

$$\begin{aligned}\mathbf{P}^{\mathbb{P}}(t,s) &= \mathbf{B}\, e^{\mathbf{D}\int_t^s \eta(u)\,du}\,\mathbf{B}^{-1},\\ \mathbf{P}^{\mathbb{Q}}(t,s) &= \mathbf{B}\, e^{\mathbf{D}\int_t^s \hat{\eta}(u)\,du}\,\mathbf{B}^{-1},\end{aligned} \tag{2.121}$$

where $\eta(t)$ and $\hat{\eta}(t)$ denote the intensities of the subordinator process, respectively, under the *real-world* and risk-neutral probability measures, and the $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{B}^{-1}$ matrices of the eigenvalues and eigenvectors of generator matrix $\mathbf{G}$, and consequently the generator matrix $\mathbf{G}$, are assumed invariant between the two probability measures.

2.5.1.1 A special case

It is interesting to consider the limit case in which the only transitions allowed are those to the default state K. In this case the first $(K-1)$ eigenvectors are orthogonal (only one of the components is different from 0). Sorting the first $(K-1)$ eigenvectors by eigenvalue:

$$|d_1| \leq |d_1| \leq \cdots \leq |d_{K-1}|,$$

the intensity $\eta_1(t)$ of the first process is that corresponding to the higher rating class and so on. The $\mathbf{B}$ and $\mathbf{B}^{-1}$ matrices are respectively:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & \dots & 0 & c \\ 0 & 1 & \dots & 0 & c \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & c \\ 0 & 0 & \dots & 0 & c \end{pmatrix}, \quad \mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1/c \end{pmatrix}.$$

It follows that the price of a unit zero coupon bond with maturity T issued by a firm whose rating at time t is i and subject to the *fractional recovery of Treasury value* scheme in case of default becomes:

$$\begin{aligned} v^i(t,T) &= v(t,T)\left[(1-\delta)\sum_{j=1}^{K-1}(-b_{ij}b_{jK}^{-1})\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta(u)\,du}\right]+\delta\right] \\ &= v(t,T)\left[(1-\delta)\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_i|\int_t^T \eta(u)\,du}\right]+\delta\right]. \end{aligned}$$

The terms $(-b_{ij}b_{jK}^{-1})$ have the following property:

$$\sum_{j=1}^{K-1}(-b_{ij}b_{jK}^{-1}) = 1.$$

In fact, since that $(\mathbf{B}\mathbf{B}^{-1})_{ij} = 0$ for $i \neq j$, and then $b_{iK}b_{KK}^{-1} = 1$:

$$\sum_{j=1}^{K}(b_{ij}b_{jK}^{-1}) = \sum_{j=1}^{K-1}(b_{ij}b_{jK}^{-1}) + b_{iK}b_{KK}^{-1} = 0, \quad i \neq K.$$

However, in order to be interpreted as probabilities, the terms $(-b_{ij}b_{jK}^{-1})$ would all have to be non-negative, which does not always occur.

2.5.2 The volatility structure

Let $\hat{y}(t,t+h)$ be the yield to maturity of a unit risky zero coupon bond:

$$\hat{y}(t,t+h) = -\frac{1}{h}\log\hat{v}(t,t+h), \tag{2.122}$$

and let $y(t,t+h)$ be the yield to maturity of a unit default-free zero coupon bond; then, the corresponding spread is defined as:

$$s(t,t+h) = \hat{y}(t,t+h) - y(t,t+h). \tag{2.123}$$

Hence, under the assumption of independence between the subordinator process $\pi(t)$ and the default-free spot rate:

$$s(t,t+h) = -\frac{1}{h}\log\left((1-\delta)\sum_{j=1}^{K-1}(-b_{ij}b_{jK}^{-1})\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^{t+h}\eta(u)\,du}\right]+\delta\right). \tag{2.124}$$

From the assumptions about the intensity $\eta(t)$ of the subordinator process, which follows a CIR model, it can be derived that:

$$\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^{t+h}\eta(u)\,du}\right] = A(h)e^{-B(h)|d_j|\eta(t)},$$

where:

$$\begin{aligned} A(h) &= \left[\frac{\delta_d\, e^{\phi_d\, h}}{\phi_d\,(e^{\delta_d\, h}-1)+\delta_d}\right]^{\nu}, \\ B(h) &= \frac{e^{\delta_d\, h}-1}{\phi_d\,(e^{\delta_d\, h}-1)+\delta_d}, \end{aligned}$$

where δ_d, ϕ_d and ν are defined in section 2.5.1.

In case of zero recovery ($\delta = 0$), if a single contribution is considered in the sum present in (2.124), the model coincides with the model of Duffie and Singleton with default intensity evolving according to a dynamic *à la CIR*:

$$s(t,t+h) = -\frac{\log A(h)}{h} + \frac{B(h)}{h}|d_j|\,\eta(t). \tag{2.125}$$

Therefore, at time t the term structure of credit spreads can be monotone increasing, monotone decreasing or humped. At future time $t+\Delta t$, for each value of h, the distribution of $s(t+\Delta t, t+\Delta t+h)$ is a non-central Chi-squared distribution with variance:

$$\mathbf{Var}_t[s(t{+}\Delta t, t{+}\Delta t{+}h)] = \frac{B^2(h)}{h^2}\left[\frac{\sigma^2\,|d_j|\,\eta(t)}{\alpha}\left(e^{-\alpha\Delta t}-e^{-2\alpha\Delta t}\right)+\frac{\sigma^2\gamma}{2\alpha}\left(1-e^{-\alpha\Delta t}\right)^2\right]. \tag{2.126}$$

Thus, the term structure of spread volatility is governed by the term $B(h)/h$ that zeroes in the limit $h \to \infty$ and tends to 1 in the limit $h \to 0$. Furthermore, the function $B(h)/h$ is monotone decreasing as h increases. In fact, the first derivative with respect to h is equal to:

$$\begin{aligned} \frac{\partial}{\partial h}\left(\frac{B(h)}{h}\right) &= \frac{\delta_d^2 h e^{\delta_d h} - (e^{\delta_d h}-1)\left[\delta_d + \phi_d(e^{\delta_d h}-1)\right]}{h^2\left[\delta_d + \phi_d(e^{\delta_d h}-1)\right]} \\ &= \frac{-\phi_d(e^{\delta_d h}-1)^2 + \delta_d(\delta_d h - 1)(e^{\delta_d h}-1) + \delta_d^2 h}{h^2\left[\delta_d + \phi_d(e^{\delta_d h}-1)\right]}. \end{aligned} \tag{2.127}$$

The denominator of (2.127) is non-negative and zeroes at $h = 0$, where:

$$\lim_{h\to 0}\frac{\partial}{\partial h}\left(\frac{B(h)}{h}\right) = -\frac{(2\phi_d - \delta_d)}{2},$$

which is definitely negative since $(2\phi_d - \delta_d) = \alpha > 0$. For $h > 0$, the requirement that the derivative vanishes has for solution:

$$(e^{\delta_d h}-1) = \frac{\delta_d(\delta_d h - 1) \pm \left[\delta_d^2(\delta_d h-1)^2 + 4\phi_d\delta_d^2 h\right]^{1/2}}{2\phi_d},$$

but of the two solutions, one should be discarded since that $(e^{\delta_d h} - 1)$ is definitely non-negative; the positive solution remains:

$$(e^{\delta_d h} - 1) = \frac{\delta_d}{2\phi_d}\left[(\delta_t - 1) + \sqrt{(\delta_d - 1)^2 + 4\phi_d h}\right],$$

that is derived as the intersection of the two curves:

$$\begin{cases} f(h) = e^{\delta_d h} - 1 \\ g(h) = \frac{\delta_d}{2\phi_d}\left[(\delta_d - 1) + \sqrt{(\delta_d - 1)^2 + 4\phi_d h}\right] \end{cases}, \quad h \geq 0,$$

for which $f(0) = g(0) = 0$. Recalling that $(2\phi_d - \delta_d) = \alpha > 0$, the two curves are both monotone increasing, as can be seen from their respective derivatives:

$$\begin{aligned} f'(h) &= \delta_d\, e^{\delta_d h} \\ g'(h) &= \frac{\delta_d}{2\phi_d}\left[\delta_d + \frac{\delta_d h + (2\phi_d - \delta_d)}{\sqrt{(\delta_d h - 1)^2 + 4\phi_d h}}\right] \end{aligned}$$

and therefore do not intersect at any other point.

Summing up, the first derivative of $B(h)/h$ never vanishes and the function $B(h)/h$ has a maximum in $h = 0$, with derivative definitely negative. Hence, the term that governs the term structure of spread volatility is monotone decreasing.

2.5.3 The extension of the model with double subordinator process

In this section an extension of the model introduced in section 2.5 is presented, involving a double subordinator process and aimed to more accurately capture the behaviours of rating transitions and thus the term structures of credit spreads for all rating classes.

Consider two subordinator processes, $\pi_1(t)$ and $\pi_2(t)$, defined as two integrals of two positive stochastic intensities, $\eta_1(t)$ and $\eta_2(t)$, as in equation (2.108):

$$\pi_1(t) = \int_{t_0}^{t} \eta_1(u)\,du, \quad \pi_2(t) = \int_{t_0}^{t} \eta_2(u)\,du, \tag{2.128}$$

that have the same characteristics and the same properties as the single subordinator process defined in 2.5.

Then, the transition probability matrix $\mathbf{P}(t,s)$ can be expressed through a representation analogous to (2.109):

$$\begin{aligned} \mathbf{P}(t,s) &= e^{\mathbf{G}[(\pi_1(s)+\pi_2(s))-(\pi_1(t)+\pi_2(t))]} \\ &= \mathbf{B}\, e^{\mathbf{D}[(\pi_1(s)+\pi_2(s))-(\pi_1(t)+\pi_2(t))]}\mathbf{B}^{-1} \\ &= \mathbf{B}\, e^{\mathbf{D}\left[\left(\int_{t_0}^{s}\eta_1(u)\,du+\int_{t_0}^{s}\eta_2(u)\,du\right)-\left(\int_{t_0}^{t}\eta_1(u)\,du+\int_{t_0}^{t}\eta_2(u)\,du\right)\right]}\mathbf{B}^{-1} \\ &= \mathbf{B}\, e^{\mathbf{D}\left[\left(\int_{t_0}^{s}\eta_1(u)+\eta_2(u)\,du\right)-\left(\int_{t_0}^{t}\eta_1(u)+\eta_2(u)\,du\right)\right]}\mathbf{B}^{-1} \\ &= \mathbf{B}\, e^{\mathbf{D}\left(\int_{t}^{s}\eta_1(u)+\eta_2(u)\,du\right)}\mathbf{B}^{-1}. \end{aligned} \tag{2.129}$$

The transition probability matrix $\mathbf{P}(t,s)$ thus defined satisfies the *Kolmogorov backward* equation:

$$\frac{\partial \mathbf{P}(t,s)}{\partial t} = -\mathbf{G}(\eta_1(t)+\eta_2(t))\mathbf{P}(t,s), \tag{2.130}$$

and $\mathbf{G}(\eta_1(t)+\eta_2(t))$ is the generator matrix of the inhomogeneous Markov chain that governs C.

The two intensity processes, $\eta_1(t)$ and $\eta_2(t)$, of the subordinator processes and, consequently, the two subordinator processes, $\pi_1(t)$ and $\pi_2(t)$, are assumed to be independent. Thus, the price at time t of the unit risky zero coupon bond with maturity T issued by a firm with rating i at time t, defined in (2.114), becomes:

$$\begin{aligned} v^i(t,T) &= v(t,T)\left[(1-\delta)\sum_{j=1}^{K-1}(-b_{ij}b_{jK}^{-1})\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta_1(u)+\eta_2(u)\,du}\right]+\delta\right] \\ &= v(t,T)\left[(1-\delta)\sum_{j=1}^{K-1}(-b_{ij}b_{jK}^{-1})\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta_1(u)\,du}\right]\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta_2(u)\,du}\right]+\delta\right]. \end{aligned} \tag{2.131}$$

The two intensity processes, $\eta_1(t)$ and $\eta_2(t)$, are assumed to be modelled as a mean-reverting square root process, for instance as a Cox, Ingersoll and Ross (CIR) model, under both the *real-world* and the risk-neutral probability measures, $\mathbb{P}$ and $\mathbb{Q}$. The *real-world* dynamics are:

$$\begin{aligned} d\eta_1(t) &= \alpha_1(\gamma_1-\eta_1(t))\,dt+\sigma_1\sqrt{\eta_1(t)}\,dZ_1^{\mathbb{P}}(t), \quad \eta_1(t_0)=\eta_{1,0}\,, \\ d\eta_2(t) &= \alpha_2(\gamma_2-\eta_2(t))\,dt+\sigma_2\sqrt{\eta_2(t)}\,dZ_2^{\mathbb{P}}(t), \quad \eta_2(t_0)=\eta_{2,0}\,, \end{aligned} \tag{2.132}$$

with $\alpha_j,\gamma_j,\sigma_j>0$ such that $2\alpha_j\gamma_j>\sigma_j^2$, $\eta_j(t)\geq 0$, for $j=1,2$, and where $dZ_j^{\mathbb{P}}(t)$, $j=1,2$, is a standard Brownian motion under the *real-world* probability measure, and the dynamics of $\eta_1(t)$ and $\eta_2(t)$ under the risk-neutral measure are:

$$\begin{aligned} d\eta_1(t) &= \hat{\alpha}_1(\hat{\gamma}_1-\eta_1(t))\,dt+\sigma_1\sqrt{\eta_1(t)}\,dZ_1^{\mathbb{Q}}(t), \quad \eta_1(t_0)=\eta_{1,0}\,, \\ d\eta_2(t) &= \hat{\alpha}_2(\hat{\gamma}_2-\eta_2(t))\,dt+\sigma_2\sqrt{\eta_2(t)}\,dZ_2^{\mathbb{Q}}(t), \quad \eta_2(t_0)=\eta_{2,0}\,. \end{aligned} \tag{2.133}$$

Then the two expectations in (2.131) have an analytical expression given by the CIR model and defined in (2.120).

The model can be extended to more accurately capture market imperfections by adding a rating-specific liquidity spread to the formula of the defaultable zero coupon bond price (2.131):

$$\begin{aligned} v^i(t,T) =& v(t,T)\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T l_i(u)\,du}\right]\times \\ & \left[(1-\delta)\sum_{j=1}^{K-1}(-b_{ij}b_{jK}^{-1})\,\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta_1(u)\,du}\right]\mathbf{E}_t^{\mathbb{Q}}\left[e^{-|d_j|\int_t^T \eta_2(u)\,du}\right]+\delta\right]. \end{aligned} \tag{2.134}$$

The rating-specific liquidity spread intensities $l^i(t)$, for $i = 1, \dots, K-1$, are assumed to be independent of each other of the other risk-factors in the model (the default-free spot rate process and subordinator processes). They are assumed to be modelled as a Vasicek model. Then the rating-specific liquidity spread intensities evolve as an Ornstein-Uhlenbeck process with constant coefficients under both the *real-world* and the risk-neutral probability measures, $\mathbb{P}$ and $\mathbb{Q}$. The *real-world* dynamics is:

$$dl^i(t) = k_i(\theta_i - l^i(t))\, dt + \rho_i\, dZ_i^{\mathbb{P}}(t), \quad l^i(t_0) = l_{i,0}\,, \tag{2.135}$$

with $k_i, \rho_i > 0$, for $i = 1, \dots, K-1$, and $dZ_i^{\mathbb{P}}(t)$, $i = 1, \dots, K-1$, is a standard Brownian motion under the *real-world* probability measure. Assuming a risk premium defined as:

$$\zeta_i(t, l_i(t)) = \zeta_i, \tag{2.136}$$

the change of probability measure allows the same functional form for $l_i(t)$ dynamics to be maintained even under the risk-neutral measure:

$$dl^i(t) = \hat{k}_i(\hat{\theta}_i - l^i(t))\, dt + \rho_i\, dZ_i^{\mathbb{Q}}(t), \quad l^i(t_0) = l_{i,0}\,, \tag{2.137}$$

where:

$$\hat{k}_i = k_i, \quad \hat{\theta}_i = \theta_i - \frac{\zeta_i\, \rho_i}{k_i}. \tag{2.138}$$

Under the Vasicek model assumption, the distribution for future values, $l^i(s)$, $s > t$, $i = 1, \dots, K-1$, conditional on $\mathcal{F}_t$, is a Gaussian distribution with expected value and variance defined by the following expressions:

$$\begin{aligned} \mathbf{E}_t[l_i(s)] &= \theta_i + (l_i(t) - \theta_i)\, e^{-k_i\,(s-t)}, \\ \mathbf{Var}_t[l_i(s)] &= \tfrac{\rho_i^2}{2k_i}\left[1 - e^{-2\,k_i(s-t)}\right], \end{aligned}$$

and the term $\mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T l_i(u)\,du}\right]$ has the following analytical expression:

$$\mathbf{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T l_i(u)\,du}\right] = e^{A(h) - B(h)\, l_i(t)}, \;\; h = T - t, \tag{2.139}$$

where:

$$\begin{aligned} B(h) &= \frac{1 - e^{-\hat{k}_i\, h}}{\hat{k}_i}, \\ A(h) &= \frac{(B(h) - h)\, (\hat{k}_i^2\, \hat{\theta}_i - \rho_i^2/2)}{\hat{k}_i^2} - \frac{\rho_i^2}{4\, \hat{k}_i}\, B^2(h). \end{aligned}$$

Chapter 3

Bayesian filtering techniques

Filtering is an operation that involves extracting information about a variable of interest at a specific time, using the observed/measured data up to and including that time.

Stochastic filtering theory was first established in the early 1940s due to the pioneering work by Norbert Wiener [66] [67] and Andrey N. Kolmogorov [43] [44], and it culminated in 1960 with the publication of classic Kalman filter [39].
The Kalman filter is a recursive method of estimating the state of a dynamic system perturbed by white noise. It is based on the assumption that the observable variables are related to the state of the system by a linear function, and allows estimation of the variables driving such dynamical systems by inferring missing information from the measurement of the observable ones.
The Kalman filter and its numerous variants have been applied in various areas of research: in finance Harvey [30] found in the logic of the Kalman filter the so-called structural time series models, and Babbs and Nowman [5] first applied it in the area of interest rate risk; in the industrial field, the Kalman filter has been used in modern vehicle monitoring systems ([48], [51]), all the way to the space exploration field. In fact, part of the filter's notoriety is due to its use by NASA [52] in the processing guidance system for the spacecraft of the Apollo program. Nowadays, Kalman filters have been applied in various engineering and scientific areas, including communications, machine learning, neuroscience, economics, finance, political science, and many others.
In order to overcome the Kalman filter's very limiting assumptions of linearity and Gaussian distribution of noise process, a variety of extensions have been proposed, including the Extended Kalman Filter and the Unscented Kalman Filter, which allow non-Gaussian and non-linear problems to be dealt with by linearly approximating the observations and equations of state, or by applying approximations to the iterative formulas of the classical Kalman filter.

On this basis, further developing filtering techniques, Gordon [29] first presented the particle filter, later formalized in the work of Kitagawa [42]. Since their introduction in 1993, particle filters have become a very popular class of numerical methods for the solution of optimal estimation problems in non-linear non-Gaussian scenarios, where the Extended Kalman Filter and the Unscented Kalman Filter do not yield reasonable estimates. These methods solve the estimation problems numerically in an online manner, *i.e.*, recursively as observations become available. In comparison with the standard approximation methods, the principal advantage of particle methods is that they do not rely on any local linearization technique or any crude functional approximation. The price that must be paid for this flexibility is computational: these methods are computationally expensive. However, thanks to the availability of ever-increasing computational power, these methods are already used in real-time applications appearing in fields as diverse as chemical engineering, computer vision, financial econometrics, target tracking and robotics.

In the *Solvency II* framework, the application of full or partial internal models requires the use of appropriate estimation techniques for estimating model parameters, allowing:

- to infer the model parameters under both the *real-world* and risk-neutral probability measures;
- to infer both the parameters of the models used to describe individual risk drivers and the parameters of the dependency structure among individual risk drivers.

The estimation technique based on particle filtering is consistent with the requirements of the regulations, as it represents a time series estimation technique that allows the parameters under both probability measures to be estimated and the time series of the latent variable to be reconstructed, through which the dependency structure can be estimated.

This chapter is organized as follows. In section 3.1 Hidden Markov models are presented and the estimation problem is formalized, which justifies the use of filtering techniques. Sections 3.2 and 3.3 present the estimation methodology on time series using filtering techniques, in the particular case of the Kalman filter and the general case, in which the class of particle filters is also placed. This last section then discusses two special cases of particle filters, which utilize, respectively, Gauss-Legendre quadrature and Sequantial Monte Carlo methods with resampling. Section 3.4 shows how the different filtering techniques are applied to the model for spread risk.

3.1 Hidden Markov Models and the estimation problem

Consider a $\mathcal{X}$-valued discrete-time Markov process $\{\boldsymbol{X}_n\}_{n\geq 0}$ such that:

$$\begin{aligned} &\boldsymbol{X}_0 \sim \mu(\boldsymbol{x}_0), \\ &\boldsymbol{X}_n \mid (\boldsymbol{X}_{n-1} = \boldsymbol{x}_{n-1}) \sim f(\boldsymbol{x}_n \mid \boldsymbol{x}_{n-1}), \end{aligned} \tag{3.1}$$

where $\mu(\boldsymbol{x})$ is a probability distribution function and $f(\boldsymbol{x} \mid \boldsymbol{x}')$ denotes the probability density associated with moving from $\boldsymbol{x}'$ to $\boldsymbol{x}$. Only the $\mathcal{Y}$-valued process $\{\boldsymbol{Y}_n\}_{n\geq 1}$ can be observed, and $\{\boldsymbol{X}_n\}_{n\geq 0}$ has to be estimated, under the assumption that, conditional on $\{\boldsymbol{X}_n\}_{n\geq 0}$, $\{\boldsymbol{Y}_n\}_{n\geq 1}$ observations are statistically independent and their marginal densities are:

$$\boldsymbol{Y}_n \mid (\boldsymbol{X}_n = \boldsymbol{x}_n) \sim g(\boldsymbol{y}_n \mid \boldsymbol{x}_n). \tag{3.2}$$

Models defined by equations (3.1) and (3.2) are known as Hidden Markov Models (HMM) or general state-space models (SSM), where:

- $\{\boldsymbol{X}_n\}_{n\geq 0}$ is the (multidimensional) state variable, that is a non-observable Markov process; equation (3.1) is called state equation;
- $\{\boldsymbol{Y}_n\}_{n\geq 1}$ is the (multidimensional) observation variable, whose observations are mutually independent, conditional on the value of the state variable; equation (3.2) is called measure equation or observation equation.

The state equation describes the dynamics of the system, while the measure equation describes the relationship between the state variable and the observation variable. In general, the dynamics is stochastic and the measure equation is affected by a noise due to errors/imperfections in the measure process (*e.g.*, related to the presence of transaction costs).

Example. Finite State-Space Hidden Markov Model
Consider $\mathcal{X} = \{1, \dots, K\}$, so that:

$$\begin{aligned} &P(X_0 = k) = \mu(k), \\ &P(X_n = k \mid X_{n-1} = l) = f(k \mid l). \end{aligned}$$

The observations follow an arbitrary model of the form (3.2). This type of model is extremely general and can be used in areas, such as genetics, in which they can describe imperfectly observed genetic sequences, signal processing, and computer science, in which they can describe, amongst many other things, arbitrary finite-state machines.

In the general representation of such models, the problem of interest is to be able to estimate the expected value of a generic function $h : \mathcal{X} \to \mathbb{R}^n$:

$$I(h) = \mathbf{E}_t\left[h(\boldsymbol{x}_{0:t})\right] = \int_{\mathcal{X}} h(\boldsymbol{x}_{0:t})\, p(\boldsymbol{x}_{0:t} \mid \boldsymbol{y}_{1:t})\, d\boldsymbol{x}_{0:t}, \tag{3.3}$$

where $\boldsymbol{x}_{0:t}$ and $\boldsymbol{y}_{1:t}$ are the time series of the state variable and observations, respectively:

$$\boldsymbol{x}_{0:t} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_t\}, \quad \boldsymbol{y}_{1:t} = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_t\}.$$

The estimation of (3.3) in turn requires knowledge of the probability distribution $p(\boldsymbol{x}_{0:t} \mid \boldsymbol{y}_{1:t})$.

Hence, estimation procedures based on the state-space representation consist in solving the optimal problem related to the following reference quantity:

$$\log(\mathcal{L}) = \log\left(p(\boldsymbol{y}_{1:T})\right) = \log\left(\prod_{t=1}^{T} p(\boldsymbol{y}_t \mid \boldsymbol{y}_{1:t-1})\right), \tag{3.4}$$

where $p(\boldsymbol{y}_{1:T})$ is the (multivariate) probability density function computed for the observed values of the observation variable time series. Then the problem of filtering can be redefined more specifically in the Hidden Markov Model framework as the characterising the distribution and the full path of the state variable at the present time, given the observations up to the present time.
Since at each time in the time series the likelihood function depends on the value of the state variable (which is needed to compute the expected value of the observations), the problem takes the form of a simultaneous estimation problem of the distribution of the values of the state variable and the parameters of its dynamics.

The main challenge in computing (3.4) consists in computing the (multidimensional) integrals that appear in the two recursive equations of the filter. According to the techniques used to solve these integrals, three cases can be distinguished:

- the case where a closed-form solution exists at the iteration on the time pair $(t-1, t)$, because of the (restrictive) assumptions on the state variables and on the observation variables (the so-called Kalman filter);

- the case where integrals are computed using so-called quadrature techniques, *e.g.*, Newton or Gauss quadratures;

- the case where integrals are computed using Monte Carlo methods.

The last two cases represent the class of particle filters: the former are known as deterministic filters and make use of well-established integration schemes, while for the latter there are many alternative algorithms, belonging to the class of Sequential Monte Carlo methods. They will be discussed in detail in the section 3.3. By estimating models using market time series, the use of the techniques of filtering makes it possible to obtain simultaneously the parameters of the dynamics of the risk factors with respect to the two probability measures (*real-world* and risk-neutral), as well as to estimate the values of (unobserved) risk factors and to handle any missing of observed data (*e.g.*, if time series are not equispaced).

3.2 Kalman filter

The Kalman filter provides the solution to the estimation problem in the case where:

- the dynamics is linear (affine) in the state variables;
- the measure equation $g(\boldsymbol{y}_n \mid \boldsymbol{x}_n)$ is linear (affine) in the state variables;
- the process and the observation noises are independent and Gaussian distributed.

The existence of a closed-form solution is due to the property of the Gaussian distribution, whereby linear combinations of Gaussian-distributed random variables are also Gaussian-distributed. For the linearity assumption, the recursive filter equations take the following expression in matrix form:

$$\begin{aligned} \boldsymbol{x}_{t+1} &= \boldsymbol{c}_t + \boldsymbol{T}_t\,\boldsymbol{x}_t + \boldsymbol{R}_t\,\boldsymbol{d}_t, \quad \boldsymbol{d}_t \sim N(\boldsymbol{0}, \boldsymbol{Q}_t) \quad \text{(state equation)},\\ \boldsymbol{y}_t &= \boldsymbol{k}_t + \boldsymbol{Z}_t\,\boldsymbol{x}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t, \sim N(\boldsymbol{0}, \boldsymbol{H}_t) \quad \text{(measure equation)}, \end{aligned} \tag{3.5}$$

with initial condition $\boldsymbol{x}_0 \sim N(\boldsymbol{\alpha}_0, \boldsymbol{P}_0)$. More explicitly:

$$\begin{aligned} \begin{pmatrix} x_1\\ \vdots\\ \vdots\\ x_m \end{pmatrix}_{t+1} &= \underbrace{\begin{pmatrix} c_1\\ c_2\\ \vdots\\ c_m \end{pmatrix}_t}_{m\times 1} + \underbrace{\begin{pmatrix} T_{11} & \dots & T_{1m}\\ \vdots & \ddots & \vdots\\ \vdots & \ddots & \vdots\\ T_{m1} & \dots & T_{mm} \end{pmatrix}_t}_{m\times m} \underbrace{\begin{pmatrix} x_1\\ \vdots\\ \vdots\\ x_n \end{pmatrix}_t}_{n\times 1} + \underbrace{\begin{pmatrix} R_{11} & \dots & R_{1k}\\ \vdots & \ddots & \vdots\\ R_{m1} & \dots & R_{mk} \end{pmatrix}_t}_{m\times k} \underbrace{\begin{pmatrix} d_1\\ \vdots\\ d_k \end{pmatrix}_t}_{k\times 1},\\ \underbrace{\begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_p \end{pmatrix}_t}_{p\times 1} &= \underbrace{\begin{pmatrix} k_1\\ k_2\\ \vdots\\ k_p \end{pmatrix}_t}_{p\times 1} + \underbrace{\begin{pmatrix} z_{11} & \dots & z_{1m}\\ \vdots & \ddots & \vdots\\ z_{p1} & \dots & z_{pm} \end{pmatrix}_t}_{p\times m} \underbrace{\begin{pmatrix} x_1\\ \vdots\\ x_m \end{pmatrix}_t}_{m\times 1} + \underbrace{\begin{pmatrix} \varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_p \end{pmatrix}_t}_{p\times 1}, \end{aligned} \tag{3.6}$$

where:

- $t = 1, 2, \dots, T$, is the time index;
- $\boldsymbol{y}_t$ is the $p \times 1$ vector containing the values of the observed variables;
- $\boldsymbol{x}_t$ is the $m \times 1$ vector containing the values of the state variables;
- $\boldsymbol{\varepsilon}_t \in \mathbb{R}^m$ and $\boldsymbol{d}_t \in \mathbb{R}^k$ are Gaussian-distributed disturbance processes, independent of each other and of $\boldsymbol{x}_t$, serially independent, with covariance matrices, respectively, $\boldsymbol{H}_t$ and $\boldsymbol{Q}_t$;
- $\boldsymbol{c}_t$ and $\boldsymbol{k}_t$ are vectors with – possibly time-varying – elements of size m and p;
- $\boldsymbol{Z}_t$, $\boldsymbol{T}_t$ and $\boldsymbol{R}_t$ are matrices with – possibly time-varying – elements of size $p \times m$, $m \times m$ and $m \times k$;
- T, p, m and k are positive integers.

Assuming that the estimation of $\boldsymbol{x}_t$, conditionally to all the information $\boldsymbol{y}_{1:t}$ available up to that time, $\hat{\boldsymbol{x}}_{t|t} := \boldsymbol{a}_{t|t}$, is available at time t, the expected value and the variance-covariance matrix of $\boldsymbol{x}_{t+1|t}$ can be computed:

$$\begin{aligned} \boldsymbol{a}_{t+1|t} &= \mathbf{E}[\boldsymbol{x}_{t+1|t}] = \boldsymbol{c}_t + \boldsymbol{T}_t\,\boldsymbol{a}_{t|t}, \\ \boldsymbol{P}_{t+1|t} &= \mathbf{Var}[\boldsymbol{x}_{t+1|t}] = \underbrace{\boldsymbol{T}_t\,\boldsymbol{P}_{t|t}\,\boldsymbol{T}'_t + \boldsymbol{R}_t\boldsymbol{Q}_t\boldsymbol{R}'_t}_{\text{sum of variances}}, \end{aligned} \tag{3.7}$$

where $\boldsymbol{P}_{t|t}$ is the variance of $\boldsymbol{x}_t$ conditional on $\boldsymbol{y}_{1:t}$.
Adding the information from the observation of $\boldsymbol{y}_{t+1}$, the variable *innovation* is defined as the difference between the value of the observable variable and its expected value:

$$\boldsymbol{v}_{t+1} = \boldsymbol{y}_{t+1} - \underbrace{\mathbf{E}[\boldsymbol{y}_{t+1}|\boldsymbol{y}_{1:t}]}_{\textit{ex-ante}\text{ expected value}} = \boldsymbol{y}_{t+1} - \boldsymbol{Z}_{t+1}\boldsymbol{a}_{t+1|t}. \tag{3.8}$$

The *innovation* has zero expected value and variance-covariance matrix:

$$\boldsymbol{F}_{t+1} = \mathbf{Var}[\boldsymbol{v}_{t+1}|\boldsymbol{y}_{1:t}] = \boldsymbol{Z}_{t+1}\boldsymbol{P}_{t+1|t}\boldsymbol{Z}'_{t+1} + \boldsymbol{H}_{t+1}. \tag{3.9}$$

The estimate of $\boldsymbol{x}_{t+1}$ updated through the contribution of the new information is:

$$\begin{aligned} \boldsymbol{a}_{t+1|t+1} &= \boldsymbol{a}_{t+1|t} + \underbrace{\boldsymbol{P}_{t+1|t}\boldsymbol{Z}'_{t+1}\boldsymbol{F}^{-1}_{t+1}}_{:=\boldsymbol{G}_t}\boldsymbol{v}_{t+1}, \\ \boldsymbol{P}_{t+1|t+1} &= \boldsymbol{P}_{t+1|t} - \underbrace{\boldsymbol{P}_{t+1|t}\boldsymbol{Z}'_{t+1}\boldsymbol{F}^{-1}_{t+1}}_{\boldsymbol{G}_t}\boldsymbol{Z}_{t+1}\boldsymbol{P}_{t+1|t}. \end{aligned} \tag{3.10}$$

Equations (3.7) and (3.10) define the Kalman filter.

The log-likelihood function then takes this form:

$$\log(\mathcal{L}) = -\frac{T\,p}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\left(\log|\boldsymbol{F}_t| + \boldsymbol{v}'_t\boldsymbol{F}_t^{-1}\boldsymbol{v}_t\right). \tag{3.11}$$

3.3 Particle filter

The Hidden Markov Model defined by (3.1) and (3.2) is a Bayesian model with:

- at time $t = 0$, $\mu_\theta(\boldsymbol{x}_0)$
- for time $t \geq 1$, $f_\theta(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})$ (state equation)
- for time $t \geq 1$, $g_\theta(\boldsymbol{y}_t \mid \boldsymbol{x}_t)$ (measure equation)

where $\mu_\theta(\boldsymbol{x}_0)$ represents the initial probability density of the state variable, $f_\theta(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})$ is the transition probability from time $t-1$ to time t, and $g_\theta(\boldsymbol{y}_t \mid \boldsymbol{x}_t)$ represents the likelihood of the time t observation $\boldsymbol{y}_t$, conditional on the value of the state variable $\boldsymbol{x}_t$ at the same time t. θ is the vector of model parameters (both *real-world* and risk-neutral), and the subscript θ denotes the dependence of the probability distributions on the vector of parameters.
$\mu_\theta(\boldsymbol{x}_0)$ and $f_\theta(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})$ define the prior distribution of the process of interest $\{X_t\}_{t\geq 0}$:

$$p_\theta(\boldsymbol{x}_{0:t}) = \mu_\theta(\boldsymbol{x}_0) \prod_{k=1}^{t} f_\theta(\boldsymbol{x}_k \mid \boldsymbol{x}_{k-1}), \tag{3.12}$$

and $g_\theta(\boldsymbol{y}_t \mid \boldsymbol{x}_t)$ defines the likelihood function:

$$p_\theta(\boldsymbol{y}_{1:t} \mid \boldsymbol{x}_{0:t}) = \prod_{k=1}^{t} g_\theta(\boldsymbol{y}_k \mid \boldsymbol{x}_k). \tag{3.13}$$

In such a Bayesian context, inference about $\boldsymbol{X}_{0:t}$ given a realization of the observations $\boldsymbol{Y}_{1:t} = \boldsymbol{y}_{1:t}$ relies upon the posterior distribution:

$$p_\theta(\boldsymbol{x}_{0:t} \mid \boldsymbol{y}_{1:t}) = \frac{p_\theta(\boldsymbol{x}_{0:t}, \boldsymbol{y}_{1:t})}{p_\theta(\boldsymbol{y}_{1:t})}, \tag{3.14}$$

where:

$$p_\theta(\boldsymbol{x}_{0:t}, \boldsymbol{y}_{1:t}) = p_\theta(\boldsymbol{x}_{0:t})\, p_\theta(\boldsymbol{y}_{1:t} \mid \boldsymbol{x}_{0:t}), \tag{3.15}$$

and:

$$p_\theta(\boldsymbol{y}_{1:t}) = \int_{\Omega_x} p_\theta(\boldsymbol{x}_{0:t}, \boldsymbol{y}_{1:t})\, d\boldsymbol{x}_{0:t}. \tag{3.16}$$

For the finite state-space HMM model discussed in the example of section 3.1, the integrals correspond to finite sums, and all these (discrete) probability distributions can be computed exactly. For the linear Gaussian model, $p_\theta(\boldsymbol{x}_{0:t} \mid \boldsymbol{y}_{1:t})$ is a Gaussian distribution whose mean and covariance can be computed using the Kalman filter, as seen in section 3.2. However, for most non-linear non-Gaussian models, it is not possible to compute these distributions in closed-form and numerical methods need to be implemented. Particle methods are a set of flexible and powerful simulation-based methods which provide samples approximately distributed according to posterior distributions of the form $p_\theta(\boldsymbol{x}_{0:t} \mid \boldsymbol{y}_{1:t})$ and facilitate the approximate calculation of $p_\theta(\boldsymbol{y}_{1:t})$. Such methods are a subset of the class of methods known as Sequential Monte Carlo methods.

The sequential approximation of the marginal distribution $p_\theta(\boldsymbol{x}_t \mid \boldsymbol{y}_{1:t})$ and marginal likelihoods $p_\theta(\boldsymbol{y}_{1:t})$ are of interest.
The recursive procedure of constructing the likelihood by updating the probability distribution of the state variable, expressed in terms of the marginal distribution[1], is:

$$p_\theta(\boldsymbol{x}_t \mid \boldsymbol{y}_{1:t}) = \frac{g_\theta(\boldsymbol{y}_t \mid \boldsymbol{x}_t)\, p_\theta(\boldsymbol{x}_t \mid \boldsymbol{y}_{1:t-1})}{p_\theta(\boldsymbol{y}_t \mid \boldsymbol{y}_{1:t-1})} \quad \text{(updating step)}, \tag{3.17}$$

where:

$$p_\theta(\boldsymbol{x}_t \mid \boldsymbol{y}_{1:t-1}) = \int_{\Omega_x} f_\theta(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})\, p_\theta(x_{t-1} \mid \boldsymbol{y}_{1:t-1})\, d\boldsymbol{x}_{t-1} \quad \text{(prediction step)} \tag{3.18}$$

is the distribution of $\boldsymbol{x}_t$ conditional on the information up to time $t-1$, and:

$$p_\theta(\boldsymbol{y}_t \mid \boldsymbol{y}_{1:t-1}) = \int_{\Omega_x} p_\theta(\boldsymbol{x}_{t-1} \mid \boldsymbol{y}_{1:t-1})\, f_\theta(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})\, g_\theta(\boldsymbol{y}_t \mid \boldsymbol{x}_t)\, d\boldsymbol{x}_{t-1:t}. \tag{3.19}$$

Equations (3.18) and (3.17) allow to calculate $p_\theta(\boldsymbol{x}_t \mid \boldsymbol{y}_{1:t})$ from $p_\theta(\boldsymbol{x}_{t-1} \mid \boldsymbol{y}_{1:t-1})$, and thus use a recursive procedure that starts from $p_\theta(\boldsymbol{x}_0 \mid \boldsymbol{y}_0 = \varnothing) = \mu_\theta(\boldsymbol{x}_0)$. Then, the marginal likelihood $p_\theta(\boldsymbol{y}_{1:t})$ can in turn be recursively evaluated using:

$$\mathcal{L}(\theta) = p_\theta(\boldsymbol{y}_{1:t}) = \prod_{k=1}^{t} p_\theta(\boldsymbol{y}_t \mid \boldsymbol{y}_{1:t-1}). \tag{3.20}$$

This approach is very computationally advantageous because it allows estimates to be updated in real time without having to store the entire sample of observed values.

Particle methods can also be used to address smoothing problems: estimating the distribution of the state at a particular time given all of the observations up to some later time, *i.e.*, attempting to sample from a joint distribution $p(\boldsymbol{x}_{0:T} \mid \boldsymbol{y}_{\boldsymbol{y}_{1:T}})$ and approximating the associated marginals $\{p(\boldsymbol{x}_n \mid \boldsymbol{y}_{1:T})\}$, $n = 1, \ldots, T$.

The operation of the filter can be summarized through the following diagram:

$$\begin{array}{lccccccccc}
t=0 & x_0 & \Rightarrow & x_{1|0} & \rightarrow & x_{2|0} & \rightarrow & x_{3|0} & \rightarrow & x_{4|0} & \rightarrow \\
 & & & \Downarrow & & & & & & & \\
t=1 & & & x_{1|1} & \Rightarrow & x_{2|1} & \rightarrow & x_{3|1} & \rightarrow & x_{4|1} & \rightarrow \\
 & & & & & \Downarrow & & & & & \\
t=2 & & & x_{1|2} & \leftarrow & x_{2|2} & \Rightarrow & x_{3|2} & \rightarrow & x_{4|2} & \rightarrow \\
 & & & & & & & \Downarrow & & & \\
t=3 & & & x_{1|3} & \leftarrow & x_{2|3} & \leftarrow & x_{3|3} & \Rightarrow & x_{4|3} & \rightarrow
\end{array}$$

$\Downarrow$: *filtering*,
$\Rightarrow$: *short-term prediction*,
$\rightarrow$: *long-term prediction*,
$\leftarrow$: *smoothing*.

[1]This recursion can also be expressed in terms of posterior distribution $p_\theta(\boldsymbol{x}_{0:t} \mid \boldsymbol{y}_{1:t})$.

3.3.1 Iterative Gauss-Legendre quadrature

Iterative quadrature is an important numerical approximation method, which was widely used in computer graphics and physics in the early days. One of the most popular quadrature methods is Gaussian quadrature: a quadrature rule for obtaining an approximation of a finite integral of a function, defined as a weighted sum of n function values at specified points, called nodes, within the domain of integration:

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} w_k\, f(x_k), \tag{3.21}$$

where:

- n is the number of sample points used,
- w_k are quadrature weights,
- x_k are quadrature points or nodes.

In order to integrate the function $f(x)$ over $[-1, 1]$ with Gauss–Legendre quadrature, the k-th quadrature node is the k-th root of the n-th order Legendre polynomial $P_n(x)$ normalized so that $P_n(1) = 1$ and the weights are given by the formula [1]:

$$w_k = \frac{2}{(1 - x_k^2)[P_n^i(x_k)]^2}. \tag{3.22}$$

In case of an integration interval $[a, b]$ other than $[-1, 1]$, it is necessary to change the integral to the interval $[-1, 1]$ before applying the Gaussian quadrature rule as follows:

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{b-a}{2}\,\xi + \frac{a+b}{2}\right) \frac{dx}{d\xi}\,d\xi\,, \tag{3.23}$$

with $\frac{dx}{d\xi} = \frac{b-a}{2}$. Then the quadrature nodes and weights become:

$$\begin{aligned} x_k' &= \frac{b-a}{2}\,x_k + \frac{a+b}{2}, \\ w_k' &= \frac{b-a}{2}\,w_k, \end{aligned} \tag{3.24}$$

where x_k and w_k are the quadrature nodes and weights computed with the integration interval $[-1, 1]$.

3.3.2 Sequential Monte Carlo methods for filtering

Sequential Monte Carlo methods are a general class of Monte Carlo methods that sample sequentially from a sequence of target probability densities $\{\pi_n(\boldsymbol{x}_{1:n})\}$ of increasing dimension, where each distribution $\pi_n(\boldsymbol{x}_{1:n})$ is defined on the product space $\mathcal{X}^n$. The distributions are defined as:

$$\pi_n(\boldsymbol{x}_{1:n}) = \frac{\gamma(\boldsymbol{x}_{1:n})}{Z_n}, \tag{3.25}$$

and $\gamma_n : \mathcal{X}^n \rightarrow \mathbb{R}^+$ is required to be known pointwise, while the normalising constant:

$$Z_n = \int_{\Omega_x} \gamma_n(\boldsymbol{x}_{1:n})\, d\boldsymbol{x}_{1:n} \tag{3.26}$$

might be unknown. Sequential Monte Carlo provides an approximation of $\pi_1(\boldsymbol{x}_1)$ and an estimate of Z_1 at time 1, an approximation of $\pi_2(\boldsymbol{x}_2)$ and an estimate of Z_2 at time 2, and so on.

Monte Carlo methods approximate $\pi_n(\boldsymbol{x}_{1:n})$ using the empirical measure:

$$\hat{\pi}_n(\boldsymbol{x}_{1:n}) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^i_{1:n}}(\boldsymbol{x}_{1:n}), \tag{3.27}$$

where $\delta_{x_0}(x)$ denotes the Dirac delta mass located at x_0 and $X^i_{1:n}$, for $i = 1, \ldots, N$, are N independent random variables sampled from $\pi_n(\boldsymbol{x}_{1:n})$.

This basic Monte Carlo approach is affected by at least two problems:

1. if $\pi_n(\boldsymbol{x}_{1:n})$ is a complex high-dimensional probability distribution, sampling is impossible;
2. an algorithm sampling exactly from $\pi_n(\boldsymbol{x}_{1:n})$, sequentially for each value of n, would have a computational complexity increasing at least linearly with n.

3.3.2.1 Importance Sampling

The first problem can be solved through the importance sampling method, that relies on an importance density $q_n(\boldsymbol{x}_{1:n})$ such that:

$$\pi_n(\boldsymbol{x}_{1:n}) > 0 \Rightarrow q_n(\boldsymbol{x}_{1:n}) > 0.$$

Then:

$$\pi_n(\boldsymbol{x}_{1:n}) = \frac{w_n(\boldsymbol{x}_{1:n})\, q_n(\boldsymbol{x}_{1:n})}{Z_n}, \tag{3.28}$$

$$Z_n = \int_{\Omega_x} w_n(\boldsymbol{x}_{1:n})\, q_n(\boldsymbol{x}_{1:n})\, d\boldsymbol{x}_{1:n}, \tag{3.29}$$

where $w_n(\boldsymbol{x}_{1:n})$ is the unnormalised weight function:

$$w_n(\boldsymbol{x}_{1:n}) = \frac{\gamma_n(\boldsymbol{x}_{1:n})}{q_n(\boldsymbol{x}_{1:n})}. \tag{3.30}$$

An importance density $q_n(\boldsymbol{x}_{1:n})$ from which it easy to draw samples (*e.g.*, a multivariate Gaussian) has been selected, and N independent samples $X^i_{1:n} \sim q_(\boldsymbol{x}_{1:n})$ have been drawn. Then the empirical measure of the samples $X^i_{1:n}$ is the Monte Carlo approximation of $q_n(\boldsymbol{x}_{1:n})$, and:

$$\hat{\pi}_n(\boldsymbol{x}_{1:n}) = \sum_{i=1}^{N} W^i_n\, \delta_{X^i_{1:n}}(\boldsymbol{x}_{1:n}), \tag{3.31}$$

$$\hat{Z}_n = \frac{1}{N}\sum_{i=1}^{N} w_n(X_{1:n}^i), \tag{3.32}$$

where:

$$W_n^i = \frac{w_n(X_{1:n}^i)}{\sum_{j=1}^{N} w_n(X_{1:n}^j)}. \tag{3.33}$$

3.3.2.2 Sequential Importance Sampling

The second problem can be solved through an algorithm that admits a fixed computational complexity at each time step. This solution is based on selecting an importance distribution of the form:

$$\begin{aligned} q_n(\boldsymbol{x}_{1:n}) &= q_{n-1}(\boldsymbol{x}_{1:n-1})\, q_n(\boldsymbol{x}_n \mid \boldsymbol{x}_{1:n-1}) \\ &= q_1(\boldsymbol{x}_1) \prod_{k=2}^{n} q_k(\boldsymbol{x}_k \mid \boldsymbol{x}_{1:k-1}). \end{aligned} \tag{3.34}$$

Practically, $X_1^i \sim q_1(\boldsymbol{x}_1)$ at time 1, and then $X_k^i \sim q_k(\boldsymbol{x}_k \mid X_{1:k-1}^i)$ at time $k = 2, \ldots, n$ are sampled. The associated unnormalised weights can be computed recursively using the decomposition:

$$\begin{aligned} w_n(\boldsymbol{x}_{1:n}) &= w_{n-1}(\boldsymbol{x}_{1:n-1})\, \alpha_n(\boldsymbol{x}_{1:n}) \\ &= w_1(\boldsymbol{x}_1) \prod_{k=2}^{n} \alpha_k(\boldsymbol{x}_{1_k}), \end{aligned} \tag{3.35}$$

where the incremental importance weight function $\alpha_n(\boldsymbol{x}_{1:n})$ is defined as

$$\alpha_n(\boldsymbol{x}_{1:n}) = \frac{\gamma_n(\boldsymbol{x}_{1:n})}{\gamma_{n-1}(\boldsymbol{x}_{1:n-1})\, q_n(\boldsymbol{x}_n \mid \boldsymbol{x}_{1:n-1})}. \tag{3.36}$$

At any time n, the estimates $\hat{\pi}_n(\boldsymbol{x}_{1:n})$ and $\hat{Z}_n$ defined by (3.31) and (3.32) can be obtained.

With appropriate selection of the importance distribution $q_n(\boldsymbol{x}_n \mid \boldsymbol{x}_{1:n-1})$, the time required to sample from $q_n(\boldsymbol{x}_n \mid \boldsymbol{x}_{1:n-1})$ and to compute $\alpha_n(\boldsymbol{x}_{1:n})$ is independent of n.

3.3.2.3 Resampling

The Importance Sampling and the Sequential Importance Sampling methods provide estimates whose variances increase, typically exponentially, with n. To mitigate this problem, the resampling technique is used.

Consider an Importance Sampling approximation $\hat{\pi}_n(\boldsymbol{x}_{1:n})$ of the target distribution $\pi_n(\boldsymbol{x}_{1:n})$. This approximation is based on weighted samples from $q_n(\boldsymbol{x}_{1_n})$, and does not provide samples approximately distributed according to $\pi_n(\boldsymbol{x}_{1:n})$. In order to obtain approximate samples from $\pi_n(\boldsymbol{x}_{1:n})$, it is possible to sample from its Importance Sampling approximation $\hat{\pi}_n(\boldsymbol{x}_{1:n})$, *i.e.*, to select $X_{1_n}^i$ with probability

W_n^i. This technique is called resampling. In order to obtain N samples from $\hat{\pi}_n(\boldsymbol{x}_{1:n})$, one can simply resample N times from $\hat{\pi}_n(\boldsymbol{x}_{1:n})$. This is equivalent to associating a number of offspring N_n^i with each particle $X_{1:n}^i$ such that $N_n^{1:N} = (N_n^1, \ldots, N_n^N)$ follow a multinomial distribution with parameter vector $(N, W_n^{1:N})$ and associating a weight of $1/N$ with each offspring. Then, an approximation of $\hat{\pi}_n(\boldsymbol{x}_{1:n})$, defined as the resampled empirical measure, can be obtained:

$$\overline{\pi}_n(\boldsymbol{x}_{1:n}) = \sum_{i=1}^{N} \frac{N_n^i}{N}\, \delta_{X_{1:n}^i}(\boldsymbol{x}_{1:n}), \tag{3.37}$$

with $\mathbf{E}\left[N_n^i \mid W_n^{1;N}\right] = N\, W_n^i$.

Several resampling schemes have been proposed in the literature; the three main ones are:

- Systematic Resampling: sample $U_1 \sim \text{Unif}\left[0, \frac{1}{N}\right]$ and define $U_i = U_1 + \frac{i-1}{N}$ for $i = 2, \ldots, N$, then set:

$$N_n^i = \left|\{U_j : \sum_{k=1}^{i-1} W_n^k \leq U_j \leq \sum_{k=1}^{i} W_n^k\}\right|,$$

  with the convention $\sum_{k=1}^{0} := 0$;

- Residual Resampling: set $\tilde{N}_n^i = \lfloor N\, W_n^i \rfloor$, sample $\overline{N}_n^{1:N}$ from a multinomial of parameters $\left(N, \overline{W}_n^{1:N}\right)$, where $\overline{W}_n^i \propto W_n^i - \tilde{N}_n^i/N$, then set $N_n^i = \tilde{N}_n^i + \overline{N}_n^i$;

- Multinomial Resampling: sample $N_n^{1:N}$ from a multinomial of parameters $(N, W_n^{1:N})$.

An important advantage of resampling techniques is that they allow to remove particles with low weights with a high probability, which is extremely useful in the sequential framework, as the computational effor ts should be focused on regions of high probability mass.

3.3.2.4 A Sequential Importance Resampling algorithm for filtering

In the filtering context, the aim is to compute a numerical approximation of the distribution $\{p(\boldsymbol{x}_{0:t} \mid \boldsymbol{y}_{1:t})\}_{t\geq 0}$ sequentially in time.

If $\gamma_t(\boldsymbol{x}_{0:t}) = p(\boldsymbol{x}_{0:t}, \boldsymbol{y}_{1:t})$ is selected, then:

$$\pi_t(\boldsymbol{x}_{0:t}) = p(\boldsymbol{x}_{0:t} \mid \boldsymbol{y}_{1:t}), \tag{3.38}$$

$$Z_t = p(\boldsymbol{y}_{1:t}). \tag{3.39}$$

In order to minimize the variance of the importance weights at time t, the following importance distribution has been selected:

$$\begin{aligned} q_t(\boldsymbol{x}_t \mid \boldsymbol{x}_{0:t-1}) &= \pi_t(\boldsymbol{x}_t \mid \boldsymbol{x}_{0:t-1}) \\ &= p(\boldsymbol{x}_t \mid \boldsymbol{y}_t, \boldsymbol{x}_{t-1}) \\ &= \frac{g(\boldsymbol{y}_t \mid \boldsymbol{x}_t)\, f(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})}{p(\boldsymbol{y}_t \mid \boldsymbol{x}_{t-1})}, \end{aligned} \tag{3.40}$$

and the associated incremental importance weight is:

$$\alpha_t(\boldsymbol{x}_{0:n}) = q(\boldsymbol{x}_t \mid \boldsymbol{y}_t, \boldsymbol{x}_{t-1}). \tag{3.41}$$

Assuming an importance distribution of the form $q_t(\boldsymbol{x}_t \mid \boldsymbol{x}_{0:t-1}) = q(\boldsymbol{x}_t \mid \boldsymbol{y}_t, \boldsymbol{x}_{t-1})$, the incremental weight becomes:

$$\alpha_t(\boldsymbol{x}_{0:t}) = \alpha_t(\boldsymbol{x}_{t-1:t}) = \frac{g(\boldsymbol{y}_t \mid \boldsymbol{x}_t)\, f(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})}{q(\boldsymbol{x}_t \mid \boldsymbol{y}_t, \boldsymbol{x}_{t-1})}. \tag{3.42}$$

Hence, at time t:

$$\hat{p}(\boldsymbol{x}_{0:t} \mid \boldsymbol{y}_{1:t}) = \sum_{i=1}^{N} W_t^i \delta_{X_{1:t}^i}(\boldsymbol{x}_{0:t}), \tag{3.43}$$

$$\hat{p}(\boldsymbol{y}_t \mid \boldsymbol{y}_{1:t-1}) = \sum_{i=1}^{N} W_{t-1}^i\, \alpha_t\left(X_{t-1:t}^i\right). \tag{3.44}$$

The algorithm can be summarised as follows [42]:

1. for $i = 1, \ldots, N$ draw samples from the importance distribution:

$$X_k^i \sim q_t(\boldsymbol{x}_t \mid \boldsymbol{x}_{0:t-1}) = p(\boldsymbol{x}_t \mid \boldsymbol{y}_t, \boldsymbol{x}_{t-1});$$

2. for $i = 1, \ldots, N$ update the importance weights up to a normalizing constant:

$$\alpha_k\left(X_{k-1:k}^i\right) = \frac{g(\boldsymbol{y}_k \mid X_k^i)\, f(X_k^i \mid X_{k-1}^i)}{q(X_k^i \mid \boldsymbol{y}_k, X_{k-1}^i)},$$

$$\hat{W}_k^i = W_{k-1}^i\, \alpha_k\left(X_{k-1:k}^i\right),$$

   Note that if the transition prior probability distribution $f(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})$ is used as the importance function, $q_t(\boldsymbol{x}_t \mid \boldsymbol{x}_{0:t-1}) = f(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1})$, the last equation simplifies to the following:

$$\hat{W}_k^i = W_{k-1}^i\, g\left(\boldsymbol{y}_k \mid X_k^i\right):$$

3. for $i = 1, \ldots, N$ compute the normalized importance weights:

$$W_k^i = \frac{\hat{W}_k^i}{\sum_{j=1}^{N} \hat{W}_k^j};$$

4. for $i = 1, \ldots, N$ resample $\{W_k^i, X_{1:k}^i\}$ to obtain N new equally-weighted particles $\{1/N, \overline{X}_{1:k}^i\}$: draw N particles from the current particle set with probabilities proportional to their weights, then set $W_k^i = 1/N$.

3.4 Filtering techniques applied to the model for spread risk

In this section it is described how the different filtering techniques presented above are applied to the model for spread risk presented in section 2.5 and its extensions. Specifically, the estimation of the model with one subordinator process (2.114) exploits particle filtering with Gauss-Legendre quadrature techniques; the estimation of the model with two subordinator processes (2.131) drops the quadrature techniques for solving integrals, because of the high computational complexity due to the presence of multidimensional integrals, and instead exploits particle filtering with the Sequential Importance Resampling algorithm; finally, the estimation of the liquidity component within the model for spread and liquidity risks (2.134), due to modeling assumptions (Vasicek model and thus Gaussian distributions), can exploit the Kalman filter.

3.4.1 The model for spread risk with one subordinator process

In the model for spread risk summarized by (2.114), the reference variable is the intensity $\eta(t)$ of the subordinator process, which is a latent variable on which the observed values of the credit spreads of the different rating classes depend. It can thus be configured as a Hidden Markov Model, where:

- the intensity $\eta(t)$ of the subordinator process is the state variable;
- the term structures of credit spreads for a given sector and the different rating classes considered are the observation variables.

The measure equation $g\left(\boldsymbol{\sigma}_t(s) \mid \eta_t\right)$ is assumed to have a Gaussian distribution with expected values equal to model values and diagonal variance-covariance matrix Δ with identical elements for "macro-rating": ratings AAA, AA, and A have variance δ_A, ratings BBB, BB, and B have variance δ_B, and rating CCC has variance δ_C:

$$\Delta = \operatorname{diag}(\delta_A, \delta_A, \delta_A, \delta_B, \delta_B, \delta_B, \delta_C). \tag{3.45}$$

The assumption of using different variances by "macro-rating" is justified by the significantly different statistics observed for the time series of credit spreads for different ratings, which will be presented later. The measure equation is:

$$g(\boldsymbol{\sigma}_t(\boldsymbol{s}) \mid \eta_t) \sim N\left(\boldsymbol{\sigma}_t(\boldsymbol{s}); \boldsymbol{\sigma}_t^M(\boldsymbol{s}), \Delta\right), \tag{3.46}$$

where $\boldsymbol{\sigma}_t(\boldsymbol{s})$ are the term structures of credit spread for the different rating classes considered at time t for the vector of residual maturity $\boldsymbol{s}$, and $\boldsymbol{\sigma}_t^M(\boldsymbol{s})$ are the corresponding model values.
Instead, the state equation $f_\theta(\eta_t \mid \eta_{t-1})$ is implicitly defined by the model assumption for the intensity of the subordinator process (CIR model), *i.e.*, it is a non-central Chi-squared distribution.

Since the distributions provided by the model are not Gaussian and the relationship between the state variable and the observed variables is non-linear, the closed-form solutions provided by the Kalman filter cannot be used, but the numerical approximation technique of Gauss-Legendre quadratures with $n = 1024$ quadrature nodes over the interval $[0, 25]$ must be implemented. Therefore, the recursive filter equations (3.17) and (3.18) become:

$$\widehat{p}_\theta\left(\eta_t^i \mid \boldsymbol{\sigma}_{1:t-1}(\boldsymbol{s})\right) = \sum_{k=1}^{n} f_\theta(\eta_t^i \mid \eta_{t-1}^k)\, \tilde{p}_\theta(\eta_{t-1}^k \mid \boldsymbol{\sigma}_{1:t-1}(\boldsymbol{s}))\, w^k, \tag{3.47}$$

$$\tilde{p}_\theta\left(\eta_t^i \mid \boldsymbol{\sigma}_{1:t}(\boldsymbol{s})\right) = \frac{g_\theta\left(\boldsymbol{\sigma}_t(\boldsymbol{s}) \mid \eta_t^i\right)\, \hat{p}_\theta\left(\eta_t^i \mid \boldsymbol{\sigma}_{1:t-1}(\boldsymbol{s})\right)}{\underbrace{\sum_{k=1}^{n} w^k g_\theta\left(\boldsymbol{\sigma}_t(\boldsymbol{s}) \,\middle|\, \eta_t^k\right) \widehat{p}_\theta\left(\eta_{t-1}^k \,\middle|\, \boldsymbol{\sigma}_{1:t-1}(\boldsymbol{s})\right)}_{p_\theta\left(\boldsymbol{\sigma}_t(\boldsymbol{s}) \mid \boldsymbol{\sigma}_{1:t-1}(\boldsymbol{s})\right)}}, \tag{3.48}$$

where θ is the vector of model parameters (both *real-world* and risk-neutral).

Then the optimization problem (3.4) becomes:

$$\max_{\theta \in \Omega_\theta} \mathcal{L}(\theta) = \max_{\theta \in \Omega_\theta} \prod_{t=1}^{T} p_\theta\left(\boldsymbol{\sigma}_t(\boldsymbol{s}) \mid \boldsymbol{\sigma}_{t-1}(\boldsymbol{s})\right). \tag{3.49}$$

3.4.2 The model for spread risk with two subordinator processes

In the extension of the model with two subordinator processes (2.131), the reference variables are the intensities, $\eta_1(t)$ and $\eta_2(t)$, of the two subordinator processes. These two variables thus represent the latent variables of the Hidden Markov Model, while the observation variables are still the term structures of the credit spreads for the different rating classes.

The assumption for the measure equation $g\left(\boldsymbol{\sigma}_t(\boldsymbol{s}) \mid \eta_1(t), \eta_2(t)\right)$ remains the same as that made for the model with only one subordinator process: a Gaussian distribution with expected values equal to model values and diagonal variance-covariance matrix with identical elements for “macro-rating”.
With two state variables, the state equation becomes:

$$f_\theta\left(\eta_{1,t}, \eta_{2,t} \mid \eta_{1,t-1}, \eta_{2,t-1}\right) = f_{\theta_1}\left(\eta_{1,t} \mid \eta_{1,t-1}\right)\, f_{\theta_2}\left(\eta_{2,t} \mid \eta_{2,t-1}\right),$$

where θ is the overall vector of model parameters, θ_1 is the vector of parameters related to the intensity $\eta_1(t)$ of the first subordinator process, and θ_2 is that related to the intensity $\eta_2(t)$ of the second subordinator process. The two transition prior probability distributions, $f_{\theta_1}\left(\eta_{1,t} \mid \eta_{1,t-1}\right)$ and $f_{\theta_2}\left(\eta_{2,t} \mid \eta_{2,t-1}\right)$, are implicitly defined by the model assumption for the intensities of the subordinator processes (CIR model), i.e., they are non-central Chi-squared distributions.

Since, as in the single-subordinator process case, the closed-form solutions provided by the Kalman filter cannot be used and the level of computational complexity increases compared to the single-subordinator process case, the Sequential Importance Resampling algorithm for particle filtering is used. The importance distribution $q_t\left(\eta_{1,t}, \eta_{2,t} \mid \eta_{1,0:t-1}, \eta_{2,0:t-1}\right)$ is assumed to coincide with the transition prior probability distribution $f_\theta\left(\eta_{1,t}, \eta_{2,t} \mid \eta_{1,t-1}, \eta_{2,t-1}\right)$, then the Sequential Importance Resampling algorithm shown in section 3.3.2.4 with $N = 1024$ independent samples is used.

3.4.3 The liquidity component in the model for spread and liquidity risks

In the model for spread and liquidity risk (2.134), for the liquidity component the reference variable is the rating-specific liquidity spread intensity $l^i(t)$, for $i = 1, \ldots, K-1$. These variables represent the latent variables of the Hidden Markov Model, while the observation variables are the term structures of the credit spreads for the rating class i.

The liquidity component is estimated separately from the spread risk component, and the estimation is performed individually on each rating $i = 1, \ldots, K-1$. Therefore, since the rating-specific liquidity spread intensity $l^i(t)$ has been assumed to follow a Vasicek model, and thus follows a Gaussian distribution and respects the property of linearity, it is possible to exploit the closed-form solutions provided by the Kalman filter. Based on the properties of Vasicek model, particularly (2.139), the measure equation becomes:

$$\begin{aligned}\boldsymbol{\sigma}_t^i(\boldsymbol{s}) &= \tilde{\boldsymbol{A}}_t^i(\boldsymbol{s}) + \boldsymbol{B}^i(\boldsymbol{s})\, l_t^i + \boldsymbol{\varepsilon}_t^i \\ &= \underbrace{\begin{pmatrix} \frac{-\log v^i_{Spr}(t,s_1) - A^i(s_1)}{s_1} \\ \frac{-\log v^i_{Spr}(t,s_2) - A^i(s_2)}{s_2} \\ \vdots \\ \frac{-\log v^i_{Spr}(t,s_n) - A^i(s_n)}{s_n} \end{pmatrix}}_{\boldsymbol{k}_t = \tilde{\boldsymbol{A}}_t^i(\boldsymbol{s})} + \underbrace{\begin{pmatrix} \frac{B^i(s_1)}{s_1} \\ \frac{B^i(s_2)}{s_2} \\ \vdots \\ \frac{B^i(s_n)}{s_n} \end{pmatrix}}_{\boldsymbol{Z}_t = \boldsymbol{B}^i(\boldsymbol{s})} l_t^i + \begin{pmatrix} \varepsilon^i_{s_1} \\ \varepsilon^i_{s_2} \\ \vdots \\ \varepsilon^i_{s_n} \end{pmatrix}_t ,\end{aligned} \tag{3.50}$$

where $v^i_{Spr}(t, s_k)$ denotes the price at time t of the unit risky zero coupon bond with maturity s_k issued by a firm with rating i at time t, defined in (2.131) with the parameters obtained from the estimation of the spread risk component, $\boldsymbol{A}^i(\boldsymbol{s})$ and $\boldsymbol{B}^i(\boldsymbol{s})$ are defined by the Vasicek model (2.139), and $\boldsymbol{\varepsilon}_t^i \sim N(\boldsymbol{0}, \boldsymbol{H}^i)$, where $\boldsymbol{H}^i$ is a diagonal matrix whose elements on the diagonal are equal to h_i^2, for $i = 1, \ldots, K-1$, since the measurement errors are assumed to be independent from each other.
Instead, the state equation is implicitly defined by the model assumption for the rating-specific liquidity spread intensity $l^i(t)$ (Vasicek model) and becomes:

$$l_{t+1}^i = \underbrace{\theta_i \left(1 - e^{-k_i \Delta t}\right)}_{\boldsymbol{c}_t^i} + \underbrace{e^{-k_i \Delta t}}_{\boldsymbol{T}_t^i}\, l_t^i + \boldsymbol{d}_t^i, \tag{3.51}$$

where $\boldsymbol{d}_t^i \sim N(\boldsymbol{0}, \boldsymbol{Q}^i)$, with $\boldsymbol{Q}^i = \frac{\rho_i^2}{2\,k_i}\left[1 - e^{-2\,k_i(s-t)}\right]$.

Chapter 4

Estimation problem

The purpose of the following chapter is to present:

- the data underlying the estimation problem (section 4.1);
- the methodological aspects of the estimation procedure (section 4.2), with detail on the optimization problem and on the synthetic measures used in determining the goodness of fit of the models to the data.

Finally, in section 4.3 the estimation problem is framed within the *Solvency II* framework through a practical application.

4.1 Reference database

This section presents the input data to the estimation procedure, and specifically the time series of term structures of credit spreads for different rating classes (section 4.1.1) and the rating transition matrix (section 4.1.2).

4.1.1 Credit spread time series

For time series of term structures of credit spreads, `iBoxx` indices provided by *IHS Markit* are used, consistent with the EIOPA guidelines in [25] for calculating fundamental spreads for matching adjustment. Specifically, *IHS Markit* `iBoxx` indices for the *Financials* sector, EUR currency, and credit ratings ranging from AAA to CCC, according to *Standard & Poor's Global Ratings* classification, are used. Table 4.1 shows the identifiers of the *IHS Markit* `iBoxx` indices that have been considered.

Index name/Maturity bucket	1-3yr	3-5yr	5-7yr	y-10yr	10+yr
EUR Financials AAA	DE000A0JZBM9	DE000A0JZBP2	DE000A0JZBR8	DE000A0JZBT4	DE000A0JZBK3
EUR Financials AA	DE000A0JZBB2	DE000A0JZBD8	DE000A0JZBF3	DE000A0JZBH9	DE000A0JZA95
EUR Financials A	DE000A0JZA12	DE000A0JZA38	DE000A0JZA53	DE000A0JZA79	DE000A0JZAZ3
EUR Financials BBB	DE000A0JZBX6	DE000A0JZBZ1	DE000A0JZB11	DE000A0JZB37	DE000A0JZBV0
EUR High Yield Financials BB	GB00BDDN5909	GB00BYMX3P02	GB00BYX97S82	GB00BDDN5C33	GB00BL09VP55
EUR High Yield Financials B	GB00BDDN5H87	GB00BYMX3M70	GB00BYX97Q68	GB00BDDN5K17	GB00BL09VP55
EUR High Yield Financials CCC	GB00BDDN5P61	GB00BYMX3R26	GB00BYX97V12	GB00BDDN5R85	

Table 4.1. ISIN codes of *IHS Markit* `iBoxx` indices for the *Financials* sector, EUR currency and AAA to CCC ratings - Source: *IHS Markit*.

Among the various types of spreads available for *IHS Markit* `iBoxx` indices, the *Annual Benchmark Spread* is considered, and the *Average Time to Maturity/Average Expected Remaining Life* is used as a proxy for maturity.

The *Annual Benchmark Spread* is defined as a premium above the yield on a default-free bond necessary to compensate for the additional risk associated with holding the bond. It is calculated as the difference between the yield of the bond and the benchmark bond. The selection criteria for a benchmark bond used by the provider *IHS Markit* are:

- a government bond is selected as an approximation of a default-free bond;
- the difference between maturities of a bond and the benchmark bond is the smallest in comparison to other alternatives.

The *Annual Benchmark Spread* of a bond i at time t is:

$$BMS^a_{i,t} = \begin{cases} 0 & \text{Benchmark bonds} \\ Y^a_{i,t} - Y^a_{BM(i),t} & \text{Other bonds,} \end{cases} \tag{4.1}$$

and the *Annual Benchmark Spread* of a index at time t is:

$$BMS^a_t = \sum_{i=1}^{n} BMS^a_{i,t}\, W^D_{i,t} \tag{4.2}$$

where n is the number of bonds/indices that make up the index, $BMS^a_{i,t}$ is the *Annual Benchmark Spread* of a bond/index i at time t and $W^D_{i,t}$ is the duration-adjusted market value weight of a bond or index, *i.e.*, the adjusted share of each bonds market value in the aggregate adjusted market value of the index, or the current weight of the sub-index in the overall index as of the last rebalancing.

The *Average Time to Maturity* is defined as a weighting of the *Average Expected Remaining Life* of the bonds/sub-indices that make up the index, in accordance with the amount outstanding:

$$LFU_t = \begin{cases} \sum_{i=1}^{n} LF_{i,t}\, W^N_{i,t} & \text{Index of bonds} \\ \sum_{i=1}^{n} LFU_{i,t}\, W^N_{i,t} & \text{Index of indices} \end{cases} \tag{4.3}$$

where $LF_{i,t}$ is the *Average Expected Remaining Life* of the bond i at time t, $LFU_{i,t}$ is the *Average Time to Maturity* of the sub-index i at time t and $W^N_{i,t}$ is the nominal value weighting, *i.e.*, the share of the notional of bond i in the aggregate notional of the index, or the fixed weight of the sub-index i in the overall index as of the last rebalancing.

The valuation date is 31/12/2021 and the depth of the time series is 15 years, from 01/01/2007 to 31/12/2021. The periodicity of the time series is daily. Table 4.2 shows the overall availability of credit spread data for the *IHS Markit* `iBoxx` indices considered, while table 4.3 shows the year-by-year detail.

IHS Markit iBoxx index	Maturity	Market value	Observations
EUR Financials AAA - 1-3	2	5 516 476 643	2327
EUR Financials AAA - 3-5	4	4 927 871 934	2399
EUR Financials AAA - 5-7	6	3 916 205 602	2358
EUR Financials AAA - 7-10	8	5 363 510 971	1629
EUR Financials AAA - 10+	14	4 084 164 057	1244
EUR Financials AA - 1-3	2	52 650 850 957	3939
EUR Financials AA - 3-5	4	45 591 680 392	3939
EUR Financials AA - 5-7	6	25 086 644 930	3939
EUR Financials AA - 7-10	8	23 015 840 644	3939
EUR Financials AA - 10+	15	5 657 217 749	3889
EUR Financials A - 1-3	2	95 846 586 646	3889
EUR Financials A - 3-5	4	101 038 447 078	3890
EUR Financials A - 5-7	6	67 401 880 719	3890
EUR Financials A - 7-10	8	61 345 065 649	3890
EUR Financials A - 10+	13	10 751 186 522	3890
EUR Financials BBB - 1-3	2	49 707 321 550	3890
EUR Financials BBB - 3-5	4	57 304 285 173	3890
EUR Financials BBB - 5-7	6	42 814 002 345	3890
EUR Financials BBB - 7-10	8	35 452 841 869	3890
EUR Financials BBB - 10+	12	8 217 208 981	3204
EUR High Yield Financials BB - 1-3	2	16 312 748 668	3899
EUR High Yield Financials BB - 3-5	4	16 775 285 602	3898
EUR High Yield Financials BB - 5-7	6	8 195 328 585	3832
EUR High Yield Financials BB - 7-10	8	4 317 310 019	3898
EUR High Yield Financials BB - 10+	21	1 046 066 465	1742
EUR High Yield Financials B - 1-3	2	2 179 708 553	3579
EUR High Yield Financials B - 3-5	4	3 275 042 164	3514
EUR High Yield Financials B - 5-7	6	983 786 900	3469
EUR High Yield Financials B - 7-10	8	697 699 434	1470
EUR High Yield Financials B - 10+	39	124 824 727	651
EUR High Yield Financials CCC - 1-3	2	1 055 441 671	2734
EUR High Yield Financials CCC - 3-5	4	937 284 795	2906
EUR High Yield Financials CCC - 5-7	6	308 918 134	644
EUR High Yield Financials CCC - 7-10	8	410 998 647	1061

Table 4.2. Overall availability of credit spread data for the *IHS Markit* iBoxx indices.

IHS Markit iBoxx index	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
EUR Financials AAA - 1-3	256	258	259	261	260	0	0	237	260	21	130	258	127	0	0
EUR Financials AAA - 3-5	256	258	259	261	260	236	259	22	132	261	128	0	0	0	67
EUR Financials AAA - 5-7	256	258	259	261	260	22	132	259	128	0	0	0	0	238	264
EUR Financials AAA - 7-10	256	258	259	261	260	258	127	0	0	0	0	0	0	0	0
EUR Financials AAA - 10+	256	258	259	261	260	0	0	0	0	0	0	0	0	0	0
EUR Financials AA - 1-3	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials AA - 3-5	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials AA - 5-7	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials AA - 7-10	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials AA - 10+	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials A - 1-3	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials A - 3-5	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials A - 5-7	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials A - 7-10	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials A - 10+	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials BBB - 1-3	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials BBB - 3-5	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials BBB - 5-7	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials BBB - 7-10	256	258	259	261	260	258	259	259	260	261	258	258	258	260	264
EUR Financials BBB - 10+	0	0	87	261	260	258	259	259	260	261	258	258	258	260	264
EUR High Yield Financials BB - 1-3	256	258	258	261	260	258	259	259	261	262	261	260	262	262	264
EUR High Yield Financials BB - 3-5	256	258	258	261	260	258	259	259	261	262	261	260	262	262	264
EUR High Yield Financials BB - 5-7	190	258	258	261	260	258	259	259	261	262	261	260	262	262	264
EUR High Yield Financials BB - 7-10	256	258	258	261	260	258	259	259	261	262	261	260	262	262	264
EUR High Yield Financials BB - 10+	0	174	259	261	41	129	259	150	260	196	0	0	0	0	23
EUR High Yield Financials B - 1-3	107	87	258	261	260	258	259	259	261	262	261	260	262	262	264
EUR High Yield Financials B - 3-5	149	44	194	261	260	258	259	259	261	262	261	260	262	262	264
EUR High Yield Financials B - 5-7	0	258	258	261	260	258	259	259	261	262	261	260	242	108	264
EUR High Yield Financials B - 7-10	0	0	132	173	64	218	259	259	173	154	0	0	22	0	67
EUR High Yield Financials B - 10+	0	0	153	54	0	0	105	109	260	21	0	0	0	0	0
EUR High Yield Financials CCC - 1-3	0	108	258	151	66	258	195	259	261	262	152	0	240	262	264
EUR High Yield Financials CCC - 3-5	0	128	153	106	22	194	213	259	261	262	261	260	262	262	264
EUR High Yield Financials CCC - 5-7	0	41	72	0	106	151	127	88	110	0	0	0	0	0	0
EUR High Yield Financials CCC - 7-10	0	0	11	21	116	151	127	0	0	0	0	0	110	262	264

Table 4.3. Year-by-year detail of credit spread data for the *IHS Markit* iBoxx indices.

In order to obtain estimates that are as representative as possible, having in mind the availability, the reliability and the robustness of data, in terms of the number of observations, the market value of indices and the number of constituents, the following decisions have been adopted for pragmatic reasons:

- *EUR Financials AAA* credit spread indices have not been available for several maturity buckets and several observation years; moreover, the number of index constituents and market value are very low. In order to solve the current lack of data, in accordance with the assumption taken by EIOPA as part of the Volatility Adjustment and Matching Adjustment calculations, the market *Financials* AAA credit spread will be 85% of market *Financials* AA credit spread. The 0.85 reduction factor is based on historical experience: the average ratio between the average values of AAA and AA *Financials* credit spread among different maturity buckets is 0.83;

- *EUR High Yield Financials BB* credit spread index for 10+ years-maturity bucket have not been available for several observation years; moreover, the number of index constituents and market value are very low. In order to avoid lack of representativeness due to the lack of data, for the BB rating, only the maturity buckets 1-3 years, 3-5 years, 5-7 years and 7-10 years are considered;

- *EUR High Yield Financials B* credit spread indices have the same problems as *EUR Financials AAA* credit spread indices. Therefore, in order to maintain maturity information, a flat index is not used, in contrast to the EIOPA guidelines, but an adjustment is applied consistently with that applied to *EUR Financials AAA* credit spread indices: the market *Financials* B credit spread will be 180% of market *Financials* BB credit spread. The 1.80 increase factor is based on historical experience: the average ratio between the average values of B and BB Financials credit spread among different maturity buckets is 1.80[1];

- *EUR High Yield Financials CCC* credit spread indices have the same problems as *EUR Financials AAA* and *EUR High Yield Financials B* credit spread indices. Therefore, in order to maintain maturity information, a flat index is not used, in contrast to the EIOPA guidelines, but an adjustment is applied consistently with that applied to *EUR High Yield Financials B* credit spread indices: the market *Financials* CCC credit spread will be 365% of market Financials BB credit spread. The 3.65 increase factor is based on historical experience: the average ratio between the average values of CCC and BB *Financials* credit spread among different maturity buckets is 3.65.

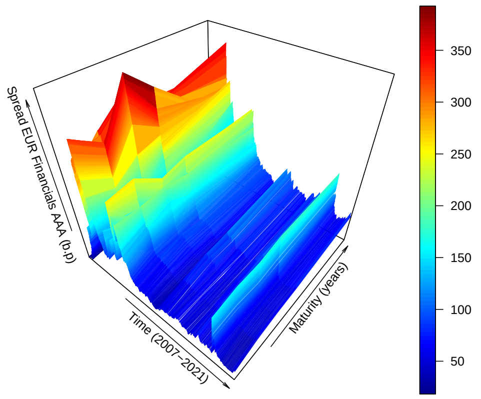
Therefore, the number of input observations is 19445 for *EUR Finacials AAA* indices, 19445 for *EUR Financials AA* indices, 19445 for *EUR Financials A* indices, 18759 for *EUR Financials BBB* indices, 15538 for *EUR High Yield Financials BB* indices, 15538 for *EUR High Yield Financials B* indices and 15538 for *EUR High Yield Financials CCC* indices, for a total of 123708 observations processed in the

[1]For the calculation of this average ratio, the 10+ year maturity bucket was excluded, as the reference maturity for *Financials* B is almost twice that of Financial BB (39 years vs. 21 years).

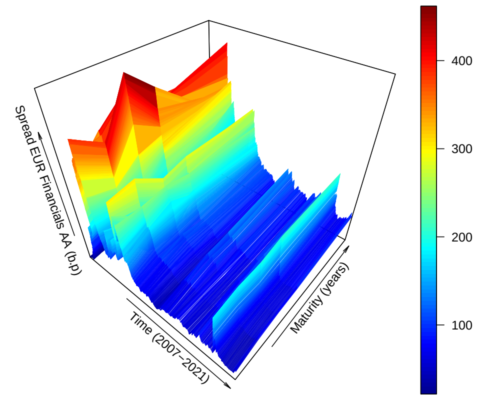
estimation procedure. This amount is evidence of the numerical complexity of the estimation problem considered.

Figures 4.1 and 4.2 show the surfaces of the credit spreads observed in the market[2] for the Investment Grade (AAA to BBB) and High Yield (BB to CCC) rating classes, respectively. Figure 4.3 compares the time series of credit spreads of different rating classes, from AAA to CCC, for common maturities (2, 4, 6 and 8 years). It shows that credit spreads trended similarly for all rating classes and maturities considered, with peaks corresponding to the 2008-2009 (subprime mortgages), 2012 (European sovereign debt), and 2020 (COVID-19 pandemic) crises. In addition, spread risk estimation raises the issue of controlling dominance between risks corresponding to different rating classes. Figure 4.3 shows a general dominance among the different rating classes, that is, as the rating worsens, the credit spread increases. This behavior is also confirmed by table 4.4, which shows the inversion frequencies between credit spreads of adjacent rating classes, computed both at the level of the central curve and at the level of the "Up" and "Down" curves, obtained respectively by summing and subtracting a standard deviation[3] from the central values. The table shows that at the level of the central curves inversions are almost totally absent, and shows that the difference between the central curves is small compared to their volatility.
Tables 4.5 and 4.6-4.12 show the means, volatilities, and 99.5% quantiles of credit spreads for all rating classes and maturities considered, computed over the entire observation period (2007-2021) and over subperiods of annual span, respectively. With respect to the overall statistics, it can be noticed that as the rating deteriorates, the means and volatilities increase, while there is no uniform trend as the maturity increases, with the volatilities of the latest maturities always being lower than those of the earliest maturities. With respect to the annual statistics, on the other hand, it can be seen that for all rating classes and all maturities considered, the variability of credit spreads decreases significantly from 2013 onward. Figure 4.4 compares historical volatilities, over 1-year rolling windows, of credit spreads for all rating classes considered and for common maturities (2, 4, 6 and 8 years).

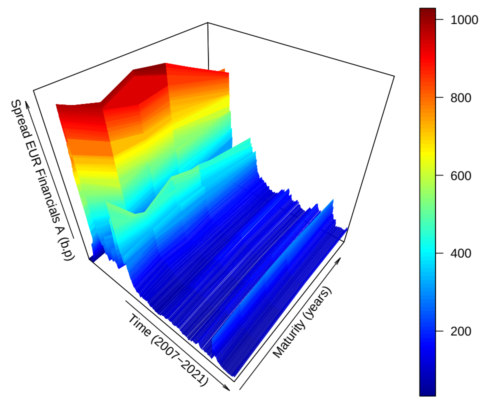
[2] As a result of the decisions described above.
[3] Computed over the whole time series.



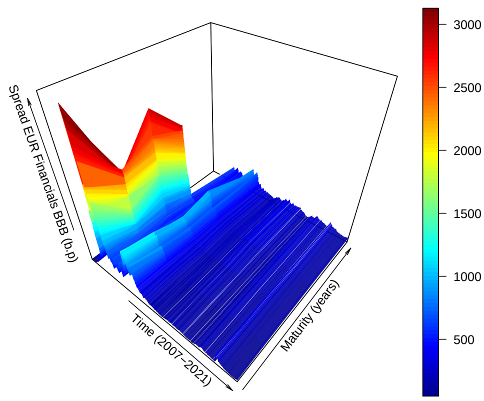
(a) *EUR Financials AAA*



(b) *EUR Financials AA*

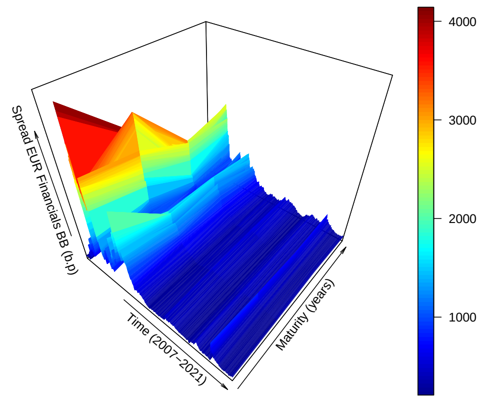


(c) *EUR Financials A*

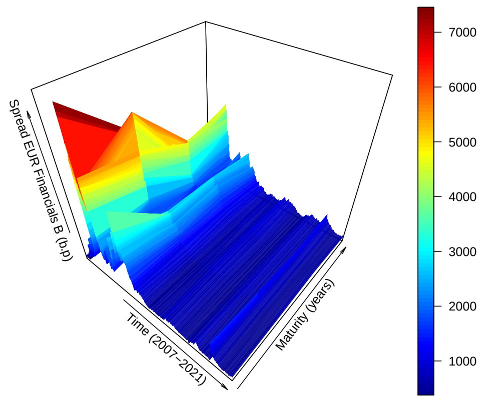


(d) *EUR Financials BBB*

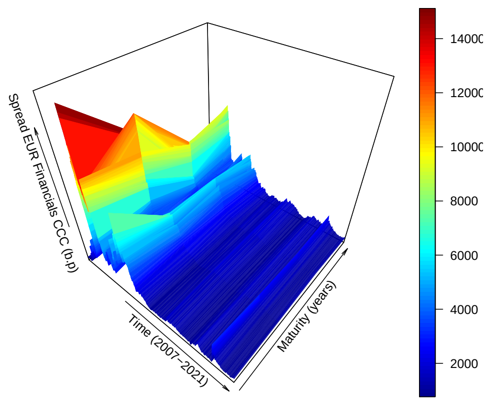
Figure 4.1. Credit spread term structures time series - from 01/01/2007 to 31/12/2021 - Investment Grade.



(a) *EUR High Yield Financials BB*



(b) *EUR High Yield Financials B*



(c) *EUR High Yield Financials CCC*

Figure 4.2. Credit spread term structures time series - from 01/01/2007 to 31/12/2021 - High Yield.

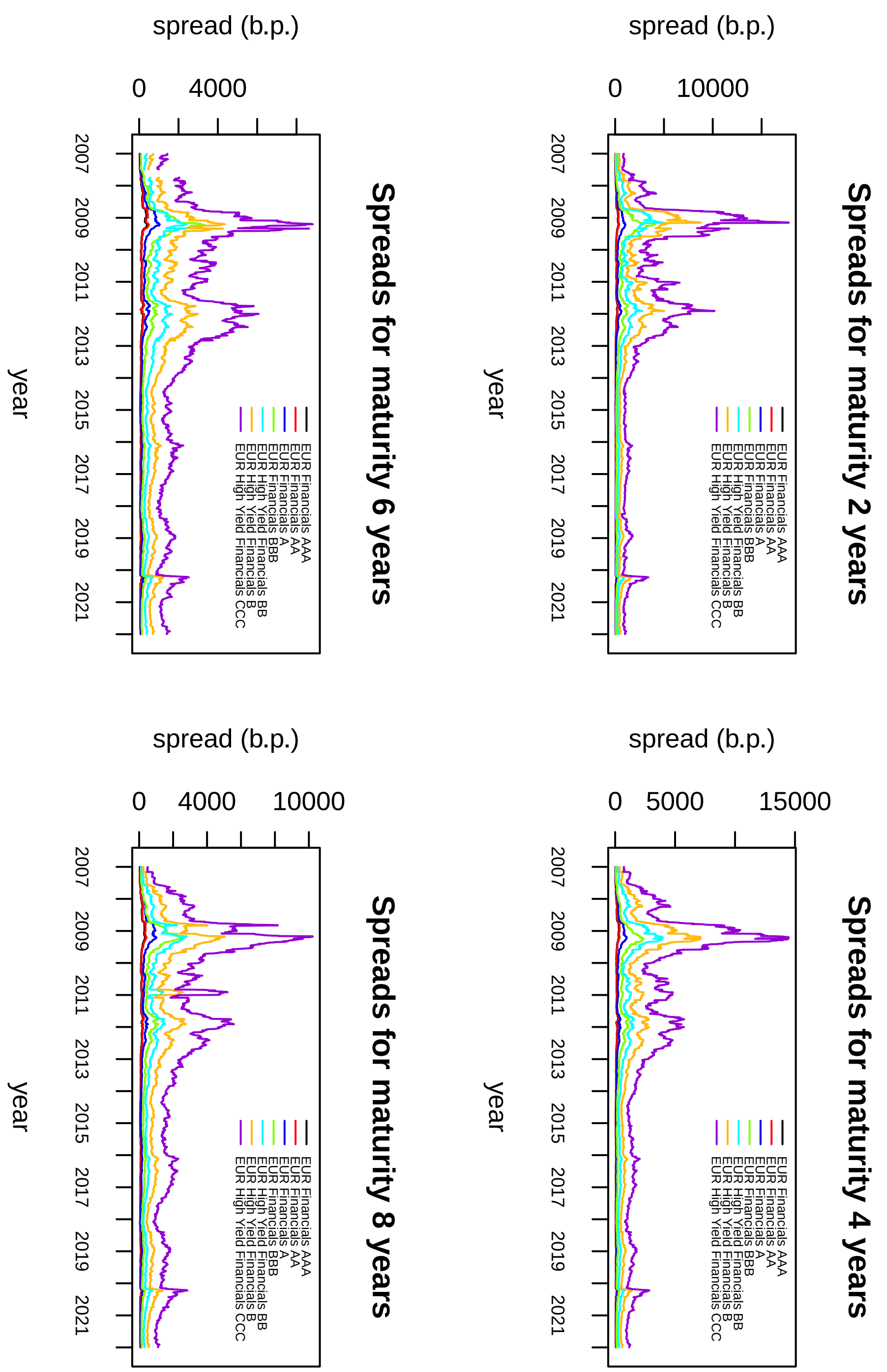


Figure 4.3. Comparison of credit spread time series for AAA to CCC rating classes for maturities 2, 4, 6 and 8 years.

Event	**Central vs Central**	**Down vs Up**	**Down vs Central**	**Down vs Down**
EUR Financials AA $<$ *EUR Financials AAA*	0.00	100.00	99.98	29.67
EUR Financials A $<$ *EUR Financials AA*	0.00	93.42	87.68	73.91
EUR Financials BBB $<$ *EUR Financials A*	0.02	93.87	90.22	77.01
EUR High Yield Financials BB $<$ *EUR Financials BBB*	1.98	98.13	89.16	22.67
EUR High Yield Financials B $<$ *EUR High Yield Financials BB*	0.00	94.74	86.23	48.58
EUR High Yield Financials CCC $<$ *EUR High Yield Financials B*	0.00	91.79	81.34	48.58

Table 4.4. Frequencies (%) of inversion between credit spreads of adjacent rating classes.

Bucket	Maturity	Mean	Std. Dev.	Min	Max	Quant. 99.5%
			EUR Financials AAA			
1-3yr	2	79.53	60.48	13.14	351.68	322.69
3-5yr	4	89.57	57.64	21.45	331.18	315.58
5-7yr	6	97.52	64.44	27.22	419.19	392.68
7-10yr	8	102.34	60.17	31.77	368.62	340.63
10+yr	15	118.22	56.13	41.54	384.13	332.90
			EUR Financials AA			
1-3yr	2	93.56	71.15	15.46	413.74	379.64
3-5yr	4	105.38	67.81	25.23	389.62	371.27
5-7yr	6	114.73	75.81	32.02	493.16	461.98
7-10yr	8	120.39	70.79	37.38	433.67	400.74
10+yr	15	139.08	66.04	48.87	451.92	391.65
			EUR Financials A			
1-3yr	2	171.27	181.13	24.37	1039.05	996.76
3-5yr	4	178.83	154.58	40.00	968.92	911.70
5-7yr	6	201.87	168.81	45.60	1044.73	973.28
7-10yr	8	216.16	167.30	73.61	1013.47	952.37
10+yr	13	206.23	124.95	93.76	819.50	775.01
			EUR Financials BBB			
1-3yr	2	393.16	535.93	39.70	4125.30	3842.83
3-5yr	4	365.41	370.03	54.02	2381.88	2237.60
5-7yr	6	391.46	422.00	82.80	3324.18	2883.55
7-10yr	8	401.85	363.94	101.06	2571.68	2419.64
10+yr	12	316.11	125.38	145.89	789.21	747.08
			EUR High Yield Financials BB			
1-3yr	2	706.10	707.67	184.95	4874.89	4007.63
3-5yr	4	696.13	605.64	188.17	3969.93	3808.96
5-7yr	6	627.85	379.08	234.30	2419.64	2198.16
7-10yr	8	638.70	419.05	129.88	2804.49	2579.93
			EUR High Yield Financials B			
1-3yr	2	1270.98	1273.80	332.91	8774.80	7213.73
3-5yr	4	1253.03	1090.16	338.71	7145.87	6856.13
5-7yr	6	1130.14	682.35	421.74	4355.35	3956.69
7-10yr	8	1149.66	754.29	233.78	5048.08	4643.87
			EUR High Yield Financials CCC			
1-3yr	2	2577.27	2582.99	675.07	17793.35	14627.85
3-5yr	4	2540.86	2210.60	686.82	14490.24	13902.70
5-7yr	6	2291.67	1383.65	855.19	8831.69	8023.28
7-10yr	8	2331.25	1529.54	474.06	10236.39	9416.74

Table 4.5. Means, standard deviations, minimum, maximum and 99.5% quantiles of market data (b.p.).

		EUR Financial AAA														
		2Y			4Y			6Y			8Y			15Y		
Start Date	End Date	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%
01-01-2007	31-12-2007	29.92	16.54	65.64	41.53	22.50	88.54	54.45	26.34	108.98	56.97	24.60	108.86	59.69	15.24	89.78
01-01-2008	31-12-2008	156.97	76.66	309.54	180.96	79.58	330.14	216.74	88.01	379.15	206.19	70.24	340.63	204.68	70.51	377.78
01-01-2009	31-12-2009	184.44	88.01	351.11	177.99	78.31	311.64	204.89	102.16	417.83	208.53	85.57	368.31	219.27	35.20	294.01
01-01-2010	31-12-2010	112.87	24.45	162.60	119.50	16.01	160.17	109.45	17.58	147.36	127.68	22.56	174.79	161.05	10.44	183.70
01-01-2011	31-12-2011	156.69	48.73	252.14	147.47	45.54	263.54	126.78	35.72	222.70	150.29	34.23	238.19	183.89	36.47	249.24
01-01-2012	31-12-2012	104.99	36.61	200.58	128.61	35.91	203.95	139.35	40.90	213.32	137.60	32.70	204.97	148.13	25.48	205.68
01-01-2013	31-12-2013	55.12	3.52	64.30	74.62	5.38	86.07	88.91	5.52	102.43	87.48	3.76	97.21	100.21	5.92	113.36
01-01-2014	31-12-2014	43.31	7.91	56.05	55.16	10.06	72.90	62.49	11.93	81.51	64.99	11.45	84.41	76.87	13.54	95.96
01-01-2015	31-12-2015	43.85	5.70	52.11	58.62	8.50	72.92	70.82	12.61	94.29	76.13	14.78	102.65	94.10	21.48	134.85
01-01-2016	31-12-2016	51.97	3.80	63.20	62.55	5.39	76.03	72.53	9.68	93.60	79.36	9.37	96.13	100.36	14.80	131.88
01-01-2017	31-12-2017	54.47	5.40	63.38	57.67	6.09	66.59	61.18	5.64	70.24	65.05	5.87	76.88	83.78	6.25	91.37
01-01-2018	31-12-2018	50.31	6.68	66.04	57.74	10.31	79.75	65.19	12.33	83.11	69.15	13.48	94.55	87.10	13.10	106.58
01-01-2019	31-12-2019	51.90	5.69	70.87	62.20	7.11	86.64	64.32	8.60	91.43	73.44	9.66	102.40	87.73	9.67	113.07
01-01-2020	31-12-2020	61.37	29.01	154.03	70.08	28.33	164.73	71.80	26.66	165.93	75.03	29.35	178.37	89.88	29.27	184.28
01-01-2021	31-12-2021	34.99	4.62	50.93	49.25	4.29	63.98	54.62	4.69	69.93	57.65	4.42	72.60	76.54	7.68	99.41

Table 4.6. Year by year statistics: means, standard deviations and 99.5% quantiles of *EUR Financial AAA* indices (b.p.).

		EUR Financial AA														
		2Y			4Y			6Y			8Y			15Y		
Start Date	End Date	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%
01-01-2007	31-12-2007	35.20	19.45	77.22	48.86	26.47	104.17	64.06	30.99	128.21	67.02	28.94	128.07	70.22	17.93	105.62
01-01-2008	31-12-2008	184.67	90.19	364.17	212.90	93.63	388.40	254.99	103.54	446.06	242.57	82.64	400.74	240.80	82.95	444.45
01-01-2009	31-12-2009	216.99	103.54	413.07	209.41	92.13	366.63	241.05	120.18	491.56	245.33	100.67	433.31	257.97	41.41	345.90
01-01-2010	31-12-2010	132.79	28.76	191.29	140.59	18.84	188.43	128.76	20.68	173.37	150.21	26.54	205.63	189.48	12.28	216.12
01-01-2011	31-12-2011	184.35	57.34	296.64	173.49	53.58	310.05	149.15	42.03	262.00	176.81	40.27	280.22	216.34	42.90	293.22
01-01-2012	31-12-2012	123.51	43.07	235.98	151.31	42.25	239.94	163.94	48.12	250.97	161.88	38.48	241.14	174.27	29.98	241.98
01-01-2013	31-12-2013	64.85	4.14	75.65	87.79	6.33	101.26	104.60	6.50	120.51	102.92	4.43	114.36	117.90	6.96	133.36
01-01-2014	31-12-2014	50.95	9.31	65.94	64.89	11.83	85.76	73.52	14.03	95.89	76.46	13.48	99.30	90.44	15.93	112.89
01-01-2015	31-12-2015	51.59	6.70	61.31	68.96	10.00	85.79	83.31	14.84	110.93	89.56	17.39	120.77	110.70	25.27	158.65
01-01-2016	31-12-2016	61.14	4.47	74.35	73.59	6.34	89.45	85.33	11.38	110.12	93.36	11.02	113.09	118.07	17.41	155.15
01-01-2017	31-12-2017	64.08	6.35	74.56	67.84	7.17	78.34	71.98	6.64	82.64	76.53	6.90	90.45	98.57	7.35	107.49
01-01-2018	31-12-2018	59.19	7.86	77.69	67.93	12.12	93.82	76.70	14.51	97.78	81.35	15.86	111.23	102.47	15.42	125.39
01-01-2019	31-12-2019	61.06	6.69	83.38	73.17	8.37	101.93	75.67	10.11	107.56	86.40	11.36	120.47	103.21	11.37	133.02
01-01-2020	31-12-2020	72.20	34.13	181.21	82.45	33.33	193.80	84.46	31.36	195.21	88.27	34.52	209.85	105.74	34.43	216.80
01-01-2021	31-12-2021	41.16	5.44	59.92	57.94	5.05	75.27	64.26	5.51	82.27	67.82	5.19	85.41	90.05	9.04	116.95

Table 4.7. Year by year statistics: means, standard deviations and 99.5% quantiles of *EUR Financial AA* indices (b.p.).

		EUR Financial A														
		2Y			4Y			6Y			8Y			13Y		
Start Date	End Date	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%
01-01-2007	31-12-2007	50.28	27.49	111.80	76.23	43.44	174.27	93.04	46.32	189.45	129.04	57.72	257.10	137.31	42.30	238.52
01-01-2008	31-12-2008	337.29	217.87	761.18	392.42	185.44	783.61	442.12	216.24	876.63	491.74	224.72	957.01	386.41	145.44	818.99
01-01-2009	31-12-2009	598.91	259.38	1034.97	518.76	240.14	968.63	556.36	251.51	1044.66	553.35	243.33	1012.58	488.33	163.08	800.39
01-01-2010	31-12-2010	266.09	42.44	371.90	233.63	26.57	294.14	276.96	33.25	355.03	283.98	40.36	370.02	224.03	23.23	267.11
01-01-2011	31-12-2011	336.50	145.47	649.38	261.67	88.65	451.83	330.34	106.48	548.11	312.78	93.63	519.55	284.57	91.00	442.24
01-01-2012	31-12-2012	249.45	99.98	505.44	279.20	81.68	459.21	283.26	90.16	474.27	299.67	82.00	463.67	286.86	56.31	383.79
01-01-2013	31-12-2013	99.47	8.07	115.06	141.70	12.36	166.39	141.06	11.38	173.75	154.71	12.87	187.41	166.57	11.50	193.42
01-01-2014	31-12-2014	78.88	13.67	125.37	95.99	12.71	120.07	107.99	9.67	130.29	113.88	10.32	134.81	128.66	9.10	151.41
01-01-2015	31-12-2015	73.88	9.35	90.76	93.41	12.34	117.12	114.66	16.28	146.99	133.52	23.09	178.74	148.38	18.96	187.64
01-01-2016	31-12-2016	87.93	7.50	107.49	108.39	10.50	135.85	126.49	18.39	168.89	149.42	18.41	196.54	152.31	23.32	207.58
01-01-2017	31-12-2017	76.75	8.82	91.38	90.59	12.06	111.54	104.35	14.67	128.37	117.19	15.08	144.86	123.16	17.13	156.59
01-01-2018	31-12-2018	75.75	12.51	105.68	93.77	18.88	137.26	112.73	23.33	160.97	125.71	22.34	170.09	140.02	28.99	201.80
01-01-2019	31-12-2019	77.68	9.71	112.27	101.03	15.05	149.20	116.37	18.94	175.66	128.06	18.59	183.43	146.76	23.83	205.27
01-01-2020	31-12-2020	101.86	45.08	238.06	117.85	44.68	264.91	132.15	46.82	289.57	144.16	45.16	299.17	165.96	45.92	286.72
01-01-2021	31-12-2021	58.91	5.72	80.78	79.15	5.21	101.31	91.35	6.88	115.64	106.85	8.17	135.25	115.68	7.09	140.69

Table 4.8. Year by year statistics: means, standard deviations and 99.5% quantiles of *EUR Financial A* indices (b.p.).

		EUR Financial BBB														
		2Y			4Y			6Y			8Y			12Y		
Start Date	End Date	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%
01-01-2007	31-12-2007	86.20	43.94	185.48	116.87	63.60	261.01	157.84	69.42	277.73	149.73	51.75	261.50			
01-01-2008	31-12-2008	647.91	478.03	1845.05	603.25	303.38	1410.89	700.81	405.63	1720.85	653.85	409.67	1620.57			
01-01-2009	31-12-2009	1762.94	1000.45	4098.93	1277.70	586.22	2341.91	1532.28	747.73	3279.29	1296.23	665.91	2568.46	447.91	44.60	541.20
01-01-2010	31-12-2010	584.62	82.11	787.81	553.67	66.62	689.77	497.90	73.66	638.89	486.72	57.02	620.65	432.77	57.40	584.55
01-01-2011	31-12-2011	803.40	269.01	1297.13	679.77	239.23	1151.82	576.49	186.07	951.65	638.45	256.07	1132.13	486.88	133.70	784.52
01-01-2012	31-12-2012	732.29	224.77	1239.67	660.02	182.40	1097.29	591.72	144.89	909.88	593.32	145.32	949.88	531.59	120.95	750.28
01-01-2013	31-12-2013	269.20	54.01	360.32	296.65	39.46	373.87	287.68	41.47	387.49	346.99	32.49	420.88	284.16	25.92	348.57
01-01-2014	31-12-2014	146.88	26.84	219.56	173.35	17.25	217.35	195.69	11.41	227.13	251.82	13.85	288.01	236.39	31.76	309.49
01-01-2015	31-12-2015	120.85	16.79	159.41	157.08	19.99	196.56	191.38	25.85	250.41	267.42	35.54	338.39	297.03	36.11	383.35
01-01-2016	31-12-2016	134.84	12.37	172.51	194.05	16.67	243.62	236.29	22.77	305.77	323.49	31.15	420.42	325.26	26.38	416.34
01-01-2017	31-12-2017	103.18	15.41	136.91	142.01	28.54	193.44	169.52	26.59	213.58	221.09	41.36	293.68	259.39	52.17	347.11
01-01-2018	31-12-2018	124.36	31.80	193.26	158.10	41.46	249.86	189.89	48.40	294.55	222.35	44.56	303.29	235.39	48.38	341.22
01-01-2019	31-12-2019	128.99	24.36	199.67	164.35	29.82	249.36	207.03	40.26	306.10	218.92	34.63	304.23	251.18	35.42	334.02
01-01-2020	31-12-2020	170.86	75.90	356.04	188.43	66.22	363.41	209.97	61.85	373.97	213.04	54.84	367.42	248.16	49.70	372.82
01-01-2021	31-12-2021	83.24	8.63	107.67	117.56	7.77	144.97	130.48	8.23	158.22	146.21	7.68	173.11	162.57	11.00	189.40

Table 4.9. Year by year statistics: means, standard deviations and 99.5% quantiles of *EUR Financial BBB* indices (b.p.).

		EUR High Yield Financial BB											
		2Y			4Y			6Y			8Y		
Start Date	End Date	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%
01-01-2007	31-12-2007	373.22	188.54	855.32	455.83	224.82	922.75	404.27	131.68	644.67	348.46	182.47	749.34
01-01-2008	31-12-2008	1324.13	981.02	3600.11	1309.92	647.31	2844.26	794.98	290.80	1531.21	993.49	372.10	2212.20
01-01-2009	31-12-2009	2145.30	1090.43	4800.39	2161.85	988.96	3958.23	1402.32	451.57	2401.93	1655.99	556.44	2794.80
01-01-2010	31-12-2010	883.44	182.99	1257.34	924.72	210.30	1312.42	875.14	95.56	1070.27	869.49	201.71	1403.67
01-01-2011	31-12-2011	1564.35	429.04	2761.67	1098.59	263.54	1570.21	964.45	305.19	1584.87	959.55	296.49	1519.21
01-01-2012	31-12-2012	1312.47	316.18	2031.46	1096.35	190.40	1533.90	1189.40	225.14	1633.94	958.85	144.08	1305.36
01-01-2013	31-12-2013	536.01	62.24	659.29	541.86	69.46	712.14	656.90	74.82	769.29	571.00	65.61	706.23
01-01-2014	31-12-2014	281.44	27.32	362.53	329.19	44.66	446.47	402.58	42.03	487.74	423.46	35.03	494.35
01-01-2015	31-12-2015	273.93	16.35	309.36	362.65	26.54	421.90	390.34	37.43	455.79	405.61	23.58	451.58
01-01-2016	31-12-2016	378.32	30.64	476.25	440.90	35.79	562.26	474.74	43.04	608.15	540.70	34.43	626.75
01-01-2017	31-12-2017	286.16	23.37	349.32	334.07	49.12	428.49	328.78	44.58	423.36	355.11	76.98	484.28
01-01-2018	31-12-2018	308.95	79.73	494.44	328.55	70.66	490.18	359.71	70.16	502.63	355.67	78.04	502.56
01-01-2019	31-12-2019	290.78	53.37	459.59	374.25	45.80	493.48	378.44	57.56	503.15	407.35	36.86	500.48
01-01-2020	31-12-2020	408.96	183.11	920.92	429.04	124.34	781.43	424.08	117.58	686.74	464.59	105.79	763.69
01-01-2021	31-12-2021	254.67	18.93	305.26	282.13	22.64	344.73	332.62	34.97	423.37	288.98	23.65	348.70

Table 4.10. Year by year statistics: means, standard deviations and 99.5% quantiles of *EUR High Yield Financial BB* indices (b.p.).

		EUR High Yield Financial B											
		2Y			4Y			6Y			8Y		
Start Date	End Date	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%
01-01-2007	31-12-2007	671.79	339.38	1539.58	820.49	404.68	1660.95	727.68	237.02	1160.41	627.22	328.44	1348.81
01-01-2008	31-12-2008	2383.44	1765.84	6480.20	2357.86	1165.15	5119.67	1430.96	523.44	2756.18	1788.28	669.77	3981.96
01-01-2009	31-12-2009	3861.54	1962.78	8640.70	3891.34	1780.14	7124.81	2524.18	812.82	4323.47	2980.77	1001.60	5030.64
01-01-2010	31-12-2010	1590.20	329.37	2263.21	1664.50	378.54	2362.36	1575.25	172.01	1926.49	1565.09	363.07	2526.61
01-01-2011	31-12-2011	2815.82	772.26	4971.01	1977.46	474.38	2826.38	1736.01	549.34	2852.77	1727.18	533.69	2734.58
01-01-2012	31-12-2012	2362.44	569.12	3656.63	1973.43	342.72	2761.02	2140.92	405.25	2941.09	1725.92	259.34	2349.65
01-01-2013	31-12-2013	964.83	112.02	1186.72	975.35	125.03	1281.85	1182.43	134.68	1384.72	1027.79	118.11	1271.21
01-01-2014	31-12-2014	506.59	49.17	652.55	592.54	80.38	803.65	724.64	75.65	877.93	762.24	63.05	889.83
01-01-2015	31-12-2015	493.07	29.44	556.85	652.78	47.77	759.42	702.61	67.37	820.42	730.09	42.44	812.84
01-01-2016	31-12-2016	680.97	55.16	857.25	793.63	64.42	1012.07	854.53	77.46	1094.67	973.27	61.97	1128.15
01-01-2017	31-12-2017	515.09	42.07	628.78	601.33	88.41	771.28	591.80	80.25	762.05	639.20	138.57	871.70
01-01-2018	31-12-2018	556.10	143.52	889.99	591.38	127.19	882.32	647.47	126.29	904.73	640.20	140.48	904.61
01-01-2019	31-12-2019	523.41	96.07	827.26	673.65	82.43	888.26	681.18	103.61	905.67	733.23	66.35	900.86
01-01-2020	31-12-2020	736.12	329.60	1657.66	772.27	223.82	1406.57	763.34	211.65	1236.13	836.26	190.43	1374.64
01-01-2021	31-12-2021	458.41	34.07	549.47	507.83	40.74	620.51	598.72	62.95	762.07	520.17	42.57	627.66

Table 4.11. Year by year statistics: means, standard deviations and 99.5% quantiles of *EUR High Yield Financial B* indices (b.p.).

		EUR High Yield Financial CCC											
		2Y			4Y			6Y			8Y		
Start Date	End Date	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%	Mean	Std. Dev.	Quant. 99.5%
01-01-2007	31-12-2007	1362.24	688.18	3121.92	1663.78	820.59	3368.04	1475.57	480.62	2353.05	1271.87	666.00	2735.09
01-01-2008	31-12-2008	4833.08	3580.74	13140.40	4781.21	2362.67	10381.55	2901.68	1061.42	5588.92	3626.24	1358.15	8074.53
01-01-2009	31-12-2009	7830.35	3980.07	17521.42	7890.76	3609.72	14447.54	5118.48	1648.21	8767.04	6044.35	2031.02	10201.02
01-01-2010	31-12-2010	3224.57	667.90	4589.29	3375.24	767.60	4790.33	3194.25	348.79	3906.49	3173.65	736.22	5123.40
01-01-2011	31-12-2011	5709.87	1565.98	10080.10	4009.84	961.94	5731.27	3520.24	1113.93	5784.78	3502.35	1082.20	5545.12
01-01-2012	31-12-2012	4790.51	1154.05	7414.83	4001.68	694.96	5598.74	4341.31	821.76	5963.88	3499.79	525.88	4764.56
01-01-2013	31-12-2013	1956.45	227.16	2406.41	1977.80	253.52	2599.31	2397.70	273.09	2807.91	2084.14	239.49	2577.74
01-01-2014	31-12-2014	1027.26	99.70	1323.23	1201.54	162.99	1629.62	1469.42	153.41	1780.25	1545.65	127.84	1804.38
01-01-2015	31-12-2015	999.84	59.69	1129.16	1323.68	96.87	1539.93	1424.73	136.61	1663.63	1480.47	86.05	1648.27
01-01-2016	31-12-2016	1380.86	111.84	1738.31	1609.30	130.63	2052.25	1732.80	157.08	2219.75	1973.57	125.66	2287.64
01-01-2017	31-12-2017	1044.48	85.32	1275.02	1219.37	179.28	1563.99	1200.03	162.73	1545.26	1296.15	280.99	1767.62
01-01-2018	31-12-2018	1127.66	291.02	1804.71	1199.20	257.92	1789.16	1312.93	256.09	1834.60	1298.18	284.86	1834.34
01-01-2019	31-12-2019	1061.35	194.82	1677.50	1366.01	167.16	1801.20	1381.29	210.11	1836.50	1486.82	134.55	1826.75
01-01-2020	31-12-2020	1492.69	668.35	3361.36	1565.98	453.85	2852.22	1547.88	429.18	2506.60	1695.76	386.14	2787.47
01-01-2021	31-12-2021	929.54	69.08	1114.20	1029.78	82.62	1258.26	1214.08	127.65	1545.30	1054.79	86.33	1272.75

Table 4.12. Year by year statistics: means, standard deviations and 99.5% quantiles of *EUR High Yield Financial CCC* indices (b.p.).

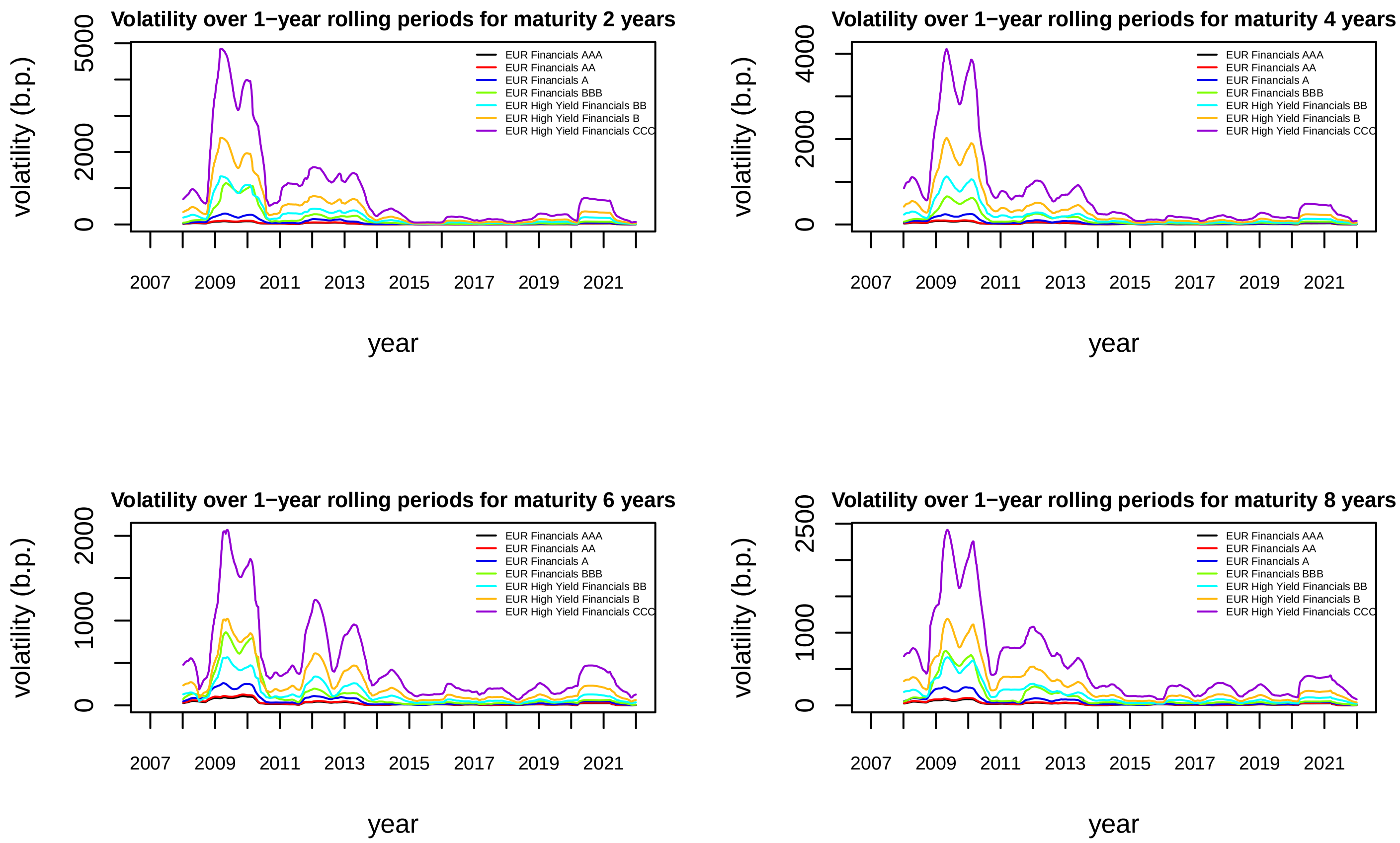


Figure 4.4. Volatilities over 1-year rolling periods for AAA to CCC rating classes for maturities 2, 4, 6 and 8 years.

4.1.2 The transition matrix

The infinitesimal generator matrix **G** is initialised using the historical transition matrices published by rating agencies. Specifically, the generator matrix is obtained from the *Global Corporate Average One-Year Transition Rates* (1981-2921) provided by *Standard & Poor's Global Ratings* (table 4.13). The choice of this input transition matrix is consistent with the EIOPA guidelines [25] for calculating the matching adjustment. Therefore, the transition probabilities refer to the 1-year average calculated from 1981 to 2021.
The use of an average transition matrix calculated over a long time span (1981-2021) instead of the one-year transition probability matrix fits with the EIOPA guidelines[4] and use in a time series estimation procedure. Moreover, it allows the number of elements equal to zero to be reduced, as sparse matrices are not consistent from a theoretical point of view and complicate estimation on market data.
Having in mind the limited number of exposures per geographical area, a global transition matrix, referring to all countries, is used.
The credit ratings considered are those ranging from AAA to CCC, according to *Standard & Poor's Global Ratings* notation, without taking into account rating modifiers. Furthermore, having in mind the definition of the market source for ratings below CCC, those categories are included as defaults.

From/To	AAA	AA	A	BBB	BB	B	CCC	D	NR
AAA	87.09	9.05	0.53	0.05	0.11	0.03	0.05	0.00	3.10
AA	0.48	87.32	7.72	0.46	0.05	0.06	0.02	0.02	3.88
A	0.02	1.56	88.73	4.97	0.25	0.11	0.01	0.05	4.29
BBB	0.00	0.08	3.19	86.72	3.48	0.42	0.09	0.15	5.86
BB	0.01	0.02	0.10	4.52	78.12	6.66	0.53	0.60	9.43
B	0.00	0.02	0.06	0.15	4.54	74.73	4.81	3.18	12.51
CCC	0.00	0.00	0.09	0.16	0.49	13.42	43.91	26.55	15.39

Table 4.13. Global Corporate Average One-Year Transition Rates (1981-2021) (%) - Source: Standard and Poor's.

Then the *withdrawn rating/not rated* class is excluded and its associated probability mass is allocated proportionally to the other rating classes. This adjustment is performed by dividing the elements of the original matrix by the difference between one and the probability of transition to the *not rated* class:

$$\hat{p}_{ij} = \frac{p_{ij}}{\sum\limits_{j \neq NR} p_{ij}}, \tag{4.4}$$

where p_{ij} is the i, j-th element of the original matrix and $\hat{p}_{ij}$ is the corresponding element of the matrix without *not rated* class. Therefore the adjusted transition matrix has seven rating classes (*i.e.*, eight rows and columns, including the situation of being defaulted, which is considered to be an absorbing state – no return to rated categories) and is reported in table 4.14.

[4]EIOPA uses an average transition matrix calculated over the 1986-2016 time span.

From/To	AAA	AA	A	BBB	BB	B	CCC	D
AAA	89.87	9.34	0.55	0.05	0.11	0.03	0.05	0.00
AA	0.50	90.84	8.03	0.48	0.05	0.06	0.02	0.02
A	0.02	1.63	92.72	5.19	0.26	0.11	0.01	0.05
BBB	0.00	0.08	3.39	92.13	3.70	0.45	0.10	0.16
BB	0.01	0.02	0.11	4.99	86.26	7.35	0.59	0.66
B	0.00	0.02	0.07	0.17	5.19	85.42	5.50	3.63
CCC	0.00	0.00	0.11	0.19	0.58	15.86	51.89	31.38

Table 4.14. Adjusted Global Corporate Average One-Year Transition Rates (1981-2021) (%).

Finally, the generator matrix used as input (table 4.15) is obtained as a solution of the *Quasi-Optimization* algorithm (refer to Appendix C) using the `R` package `ctmcd`, and satisfies the properties of a generator matrix: non-positivity of the elements of the diagonal, non-negativity of off-diagonal elements, sums of each row null and elements of the last row null.

From/To	AAA	AA	A	BBB	BB	B	CCC	D
AAA	-0,10715	0,10336	0,00145	0,00024	0,00123	0,00017	0,0007	0
AA	0,00554	-0,09717	0,08754	0,00277	0,00042	0,00063	0,00028	0
A	0,00019	0,01776	-0,07741	0,05618	0,00172	0,00109	0	0,00047
BBB	0	0,0006	0,03667	-0,08416	0,04139	0,00316	0,00107	0,00128
BB	0	0,00023	0,00017	0,05605	-0,15152	0,08538	0,00488	0,0048
B	0	0,00025	0,00068	0,00015	0,06061	-0,16853	0,08214	0,0247
CCC/C	0	0	0,00137	0,00246	0,00077	0,23737	-0,66761	0,42564
D	0	0	0	0	0	0	0	0

Table 4.15. Input infinitesimal generator matrix.

Tables 4.16 and 4.17 show the eigenvectors and eigenvalues, respectively, of the input infinitesimal generator matrix. The eigenvectors are all distinct, so it is possible to perform the spectral decomposition of the generator matrix and the K-th eigenvector has mutually identical components and corresponding null eigenvalues.

AAA	AA	A	BBB	BB	B	CCC	D
-0.00108	0.05011	0.91594	0.99078	-0.98211	0.87173	-0.57316	0.35355
-0.00035	-0.06388	-0.35934	-0.10595	-0.15886	0.41356	-0.51986	0.35355
0.00041	0.09719	0.15063	-0.03864	0.04921	0.13329	-0.45882	0.35355
-0.00195	-0.24690	-0.07787	0.05937	0.04454	-0.08062	-0.36306	0.35355
0.01512	0.78798	-0.01355	-0.00956	-0.04300	-0.16385	-0.21372	0.35355
-0.15317	-0.48492	0.04982	-0.04133	-0.05832	-0.12472	-0.10735	0.35355
0.98808	-0.25872	0.02275	-0.01770	-0.02374	-0.04881	-0.04141	0.35355
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.35355

Table 4.16. Eigenvectors of input infinitesimal generator matrix.

AAA	AA	A	BBB	BB	B	CCC	D
-0.70440	-0.22323	-0.14747	-0.11827	-0.09043	-0.05821	-0.01155	0.00000

Table 4.17. Eigenvalues of input infinitesimal generator matrix.

4.2 On the methodology of estimation procedures

The model parameters for the intensities of the subordinating processes (η in the case of the model with one subordinator process, η_1 and η_2 in the case of the model with two subordinator processes) are estimated using the time series of credit spreads for all rating classes and all maturities considered jointly. Particle filtering is used in both cases, using the Gauss-Legendre quadrature technique and the Sequential Importance Resampling algorithm, respectively, as described in section 3.4.
In the estimation stage, a zero recovery rate has been assumed.

The estimation of the rating-specific liquidity components is a subsequent step to estimation of the spread risk component, as described in 3.4. The model parameters for the intensity of the liquidity component are estimated considering only the time series of credit spreads referred to each rating, using the Kalman filter technique.

4.2.1 On the optimization problem

The `R` package `nlopt`, and specifically the algorithm `cobyla` (Constrained Optimization By Linear Approximations), a numerical optimization method for constrained problems where the derivative of the objective function is not known, is used to solve the optimization problem in the estimation stage. This method, presented by Powell in [59] and [60], has as its main feature robustness, although at a high computational cost. This choice is motivated by the fact that the function (3.20) to be optimized is very irregular, and gradient optimization algorithms do not allow for stable results. The maximum number of iterations has been set equal to 1000 for each estimation step.

4.2.2 Quality measures

The estimation results have been evaluated according to several quality measures. To measure the goodness of fit to the input data, a graphical comparison of the market and model time series for all rating classes and all maturities considered have been performed, accompanied by calculation of means and standard deviations and analysis of normality of residuals, defined as the difference between market and model values. Furthermore, three summary indicators have been considered:

- the root mean square error (RMSE), since it represents a measure of absolute error expressed in the same unit of measurement as the estimated data;

- the overall coefficient of determination (R^2), as it is a measure of relative error;

- the *single-rating* coefficients of determination (R_i^2, for i = AAA, AA, ..., CCC);

- δ_A, δ_B and δ_C parameters, which, under the assumptions of randomness in market prices, serve to capture the (slight) imperfections of the market, and therefore can be considered representative of effects such as the differential bid-ask spread or liquidity effects.

4.3 Application in *Solvency II* framework

Finally, the forecasting ability of the model for spread risk has been tested by comparing the SCR for spread risk sub-module for unsubordinated risky zero coupon bonds with maturities from 1 to 30 years calculated in internal model framework and the corresponding SCR provided by the *Standard Formula*. The SCR of the internal model is calculated as the value at risk of the risky zero coupon bond price distribution over a 1-year forecast time horizon.

In order to forecast the value of the risky zero coupon bond over a fixed time horizon $[t, t + \Delta t]$, the following steps are required:

- eliciting the *real-world* probability distribution of the risk factors, *i.e.*, the intensities of the subordinator processes $\eta_1(t + \Delta t)$ and $\eta_2(t + \Delta t)$, conditional on the information available at time t;
- obtaining the probability distribution of the bond price, through the model bond price formula (2.134), involving $\eta_1(t+\Delta t)$, $\eta_2(t+\Delta t)$, the eigenvectors and eigenvalues of the generator matrix, and the model's risk-neutral parameters.

In the model presented in this work, the spread component at time $t + \Delta t$ with residual life τ for the rating class i is defined by the following expression:

$$
\begin{aligned}
Q^i(t+\Delta t, t+\Delta t+\tau) =& \delta + (1-\delta) \sum_{j=1}^{K-1} (-b_{ij} b_{jK}^{-1})\, \mathbf{E}^{\mathbb{Q}}_{t+\Delta t} \left[e^{-|d_j| \int_{t+\Delta t}^{t+\Delta t+\tau} \eta_1(u)\, du} \right] \times \\
& \times \mathbf{E}^{\mathbb{Q}}_{t+\Delta t} \left[e^{-|d_j| \int_{t+\Delta t}^{t+\Delta t+\tau} \eta_2(u)\, du} \right] \\
=& \delta + (1-\delta) \sum_{j=1}^{K-1} (-b_{ij} b_{jK}^{-1})\, A(\tau; \hat{\theta}_1) e^{-B(\tau;\hat{\theta}_1)\, |d_j|\, \eta_1(t+\Delta t)} \times \\
& \times A(\tau; \hat{\theta}_2) e^{-B(\tau;\hat{\theta}_2)\, |d_j|\, \eta_2(t+\Delta t)},
\end{aligned}
\tag{4.5}
$$

where $\hat{\theta}_1$ and $\hat{\theta}_2$ are the vectors of the risk-neutral parameters for the processes of the two subordinator process intensities, and $A(\tau; \hat{\theta})$ and $B(\tau; \hat{\theta})$ are functions of τ and $\hat{\theta}$ defined by the modeling assumptions, as described in section 2.5.1.

Once the probability distribution of $Q^i(t + \Delta t, t + \Delta t + \tau)$ has been defined, the SCR value for spread risk in internal model framework can be represented as:

$$
\widehat{SCR}^{\,i}_{Spread} = \frac{\mathbf{E}^{\mathbb{P}} \left[Q^i(t+\Delta t, t+\Delta t+\tau) \mid \mathcal{G}_t \right] - \overline{Q}^i(t+\Delta t, t+\Delta t+\tau)}{Q^i(t, t+\tau)}, \tag{4.6}
$$

where $\overline{Q}^i(t+\Delta t, t+\Delta t+\tau)$ is the 0.5% quantile of the distribution of $Q^i(t+\Delta t, t+\Delta t+\tau)$ and $Q^i(t, t+\tau)$ is the value at time t of the spread risk component.

SCR is also calculated in an integrated modeling approach, whereby spread risk, migration risk, and default risk are considered jointly. In this approach, the probability distribution of the price of the risky zero coupon bond also depends on the rating transition (including default) that occurs in the time horizon $[t, t+\Delta t]$, which is simulated with the transition matrix implied in the model. In case of transition from rating i to rating j in the time horizon $[t, t+\Delta t]$, the spread component at time $t+\Delta t$ with residual life τ for the rating class i, is defined as:

$$Q^i_{S+M}(t+\Delta t, t+\Delta t+\tau) = Q^j(t+\Delta t, t+\Delta t+\tau) \tag{4.7}$$

where $Q^j(t+\Delta t, t+\Delta t+\tau)$ is defined by (4.5).
In case of default, a *fractional recovery of market value scheme* is assumed, *i.e.*, the price of the defaulted bond is assumed to be 55% of the price at the instant immediately prior to default:

$$Q^i_{S+M}(t+\Delta t, t+\Delta t+\tau) = 0.55\, Q^i(t+\Delta t, t+\Delta t+\tau). \tag{4.8}$$

55% represents the average recovery rate (1987-2022) for unsubordinated U.S.[5] corporate bonds in terms of nominal recovery, as shown in [62].
Then the SCR value for spread risk in internal model framework in an integrated modeling approach can be represented as:

$$\widehat{SCR}^i_{S+M} = \frac{\mathbf{E}^{\mathbb{P}}\left[Q^i_{S+M}(t+\Delta t, t+\Delta t+\tau) \mid \mathcal{G}_t\right] - \overline{Q}^i_{S+M}(t+\Delta t, t+\Delta t+\tau)}{Q^i(t, t+\tau)}. \tag{4.9}$$

[5] *Standard & Poor's* has published the recovery study only for U.S. corporate debt.

Chapter 5

Numerical results

This chapter presents the results returned by the estimation procedure, both in terms of the goodness of fit of the model to historical data and in terms of its application within the *Solvency II* framework. Specifically, for the first two model configurations, spread risk model with one subordinator process (2.114) and spread risk model with two subordinator processes (2.131), only the results in terms of goodness of fit are presented; for the full model (spread and liquidity risks) (2.134) the results in terms of *Solvency Capital Requirement* are also presented.

5.1 The model for spread risk with one subordinator process

In this section, results on the goodness of fit of the model with a single subordinator process are presented.

5.1.1 The goodness of fit of the model to the historical data

Table 5.1 shows the starting values, upper bounds, lower bounds and optimal parameters returned by the estimation procedure. Unlike other models for spread risk, such as the Duffie-Singleton model, the financial interpretation of the model parameters is less straightforward because the parameters relate to the intensity of the subordinator process underlying the transition matrix of the Markov chain that models the rating process.
Table 5.2 presents the quality measures that can be used to assess the goodness of fit of the model to historical data. Overall, the estimation of the model is found to be fair. In fact, the overall coefficient of determination R^2 amounts to 0.72 and the overall *RMSE* is 601 b.p.. At the individual rating class level, there is a better goodness of fit for the central ratings (A to B), with R^2 ranging from 0.60 to 0.88, while for the extreme ratings (AAA, AA and CCC) there is a significantly worse goodness of fit, with R^2 ranging from 0.21 to 0.35. The *RMSE* shows similar behavior, with proportionally smaller values at the central rating classes.
The δ parameters for the three rating macroclasses, A, B and C, are found to be in line with the corresponding *RSME*.
The consistent use of the model requires that all rating classes considered have at

least sufficient goodness of fit. To this end, the extension of the model with two subordinator processes is proposed.

	Plow	Pupp	Par0	Parfit
α	0.001000	7.000000	1.500000	2.744029
γ	0.000000	5.000000	0.050000	0.365893
σ	0.001000	5.000000	2.500000	1.417053
$\widehat{\alpha}$	0.001000	7.000000	0.200000	0.261695
$\widehat{\gamma}$	0.000000	5.000000	2.500000	3.836619
η_0	0.000000	25.000000	5.000000	0.822875
δ_A	0.000100	0.100000	0.005000	0.006986
δ_B	0.000100	1.000000	0.100000	0.030607
δ_C	0.000100	2.000000	0.500000	0.163798

Table 5.1. Starting values (***Par0***), lower bounds (***Plow***), upper bounds (***Pupp***) e estimated values (***Parfit***) of model parameters for the model with one subordinator process.

Quality Measure	Overall	AAA	AA	A	BBB	BB	B	CCC
RMSE (b.p.)	601.04	54.52	63.71	80.53	249.87	189.39	397.07	1609.50
$\mathbf{R^2}$	0.72	0.21	0.22	0.75	0.60	0.88	0.84	0.35
$\delta_\mathbf{A}$ **(b.p.)**	69.86							
$\delta_\mathbf{B}$ **(b.p.)**	306.07							
$\delta_\mathbf{C}$ **(b.p.)**	1637.98							

Table 5.2. Quality measures for the goodness of fit of the model with one subordinator process.

5.2 The model for spread risk with two subordinator processes

In this section, results on the goodness of fit of the model with two subordinator processes are presented.

5.2.1 The goodness of fit of the model to the historical data

Table 5.3 shows the starting values, upper bounds, lower bounds and optimal parameters returned by the estimation procedure.
Table 5.4 presents the quality measures that can be used to assess the goodness of fit of the model to historical data. The overall goodness of fit improves, with the R^2 increasing to 0.77 (+7%) and the *RMSE* decreasing to 542 b.p. (-10%) compared with the model with one subordinator process. At the individual rating class level, there is a partial leveling of goodness of fit. For the extreme rating classes (AAA, AA, CCC), poorly replicated by the model with one subordinator process, there is a marked improvement, with R^2 now above 0.45. For the BBB, BB, and B rating classes, already fairly well replicated by the model with one subordinator process, there is minimal worsening. The A rating class, on the other hand, improves reaching an excellent R^2 of 0.87. The *RMSE* shows similar behavior, with proportionally

smaller values at the central rating classes.
The δ parameters for the three rating macroclasses, A, B and C, are found to be in line with the corresponding *RMSE*.
Although the goodness of fit has reached at least sufficient levels, in order to improve the consistency of the model, the full model with two subordinator processes and a rating-specific liquidity component is needed.

	Plow	**Pupp**	**Par0**	**Parfit**
α_1	0.001000	7.000000	2.744029	0.413639
γ_1	0.000000	5.000000	0.365893	4.997962
σ_1	0.001000	5.000000	1.417053	3.608986
$\widehat{\alpha}_1$	0.001000	7.000000	0.261695	0.313215
$\widehat{\gamma}_1$	0.000000	5.000000	3.836619	3.880962
α_2	0.001000	7.000000	1.500000	4.469894
γ_2	0.000000	5.000000	0.050000	0.000425
σ_2	0.001000	5.000000	2.500000	0.061724
$\widehat{\alpha}_2$	0.001000	7.000000	0.200000	6.810526
$\widehat{\gamma}_2$	0.000000	5.000000	2.500000	0.000279
δ_A	0.000100	0.100000	0.005000	0.004738
δ_B	0.000100	1.000000	0.100000	0.029691
δ_C	0.000100	2.000000	0.500000	0.202653

Table 5.3. Starting values (***Par0***), lower bounds (***Plow***), upper bounds (***Pupp***) e estimated values (***Parfit***) of model parameters for the model with two subordinator processes.

Quality Measure	**Overall**	**AAA**	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
RMSE (b.p.)	542.97	44.64	51.54	57.67	253.37	234.12	460.78	1411.65
$\mathbf{R^2}$	0.77	0.47	0.49	0.87	0.59	0.82	0.78	0.50
$\delta_\mathbf{A}$ **(b.p.)**	47.38							
$\delta_\mathbf{B}$ **(b.p.)**	296.92							
$\delta_\mathbf{C}$ **(b.p.)**	2026.54							

Table 5.4. Quality measures for the goodness of fit of the model with two subordinator processes.

5.3 The model for spread risk with two subordinator processes and a rating-specific liquidity component

This section presents results on the goodness of fit and the application within the *Solvency II* framework of the full model with two subordinator processes and a rating-specific liquidity component.

5.3.1 The goodness of fit of the model to the historical data

Table 5.5 shows the starting values, upper bounds, lower bounds and optimal parameters for the rating-specific liquidity model, returned by the estimation procedure. The parameters for the spread component remain those estimated for the model with two subordinator processes, shown in table 5.3.
Table 5.6 presents the quality measures that can be used to assess the goodness of fit of the model to historical data. The overall goodness of fit markedly improves with the R^2 increasing to an excellent 0.92 (+19%) and the *RMSE* decreasing to 318 b.p. (-42%) compared with the model with two subordinator processes. The *RMSE* value is sufficiently small in proportion to the order of magnitude of the data. The excellent goodness of fit is also confirmed at the individual rating class level, with *single-rating* R^2 reaching a minimum level of 0.84 (CCC rating) to a maximum level of 0.97 (BBB rating) and single-rating *RMSE* decreasing significantly, more than halving compared to the model with two subordinator processes. It is also noticeable that the central rating classes are those best replicated by the model; this finding is consistent with the choice to estimate the model on the credit spreads data for all rating classes and all maturities considered jointly.
Figures 5.1 to 5.7 show the comparison of model and market values for all rating classes and all maturities considered. The graphical analysis also confirms the excellent goodness of fit of the model to the observed data, and shows how the model succeeds in capturing the trend of credit spreads over time. Table 5.7 and figures 5.8 to 5.14 provide a summary of the mean and volatility (standard deviation) values of the market and model time series for all the rating classes and maturities that have been considered. The analysis of model volatilities confirms the model characteristic of decreasing trend as maturity increases, as described in section 2.5.2, which does not always match what is observed on market data.
Figures 5.15 to 5.21 show normality analyses for the residuals referred to all rating classes and all maturities considered. The analysis consists of comparing empirical and theoretical densities, quantiles and CDFs. It is clear that the assumption of normality of the residuals is not met, as the distributions have very heavy tails.
Table 5.8 contains the transition matrix implied in the model, *i.e.*, the expected transition matrix, calculated empirically. In comparison with the transition matrix provided by *Standard & Poor's Global Ratings* (table 4.14) from which the generator matrix is obtained, the probabilities of remaining in the initial state are slightly lower, while the probabilities of default are slightly higher. In order to use this transition matrix within the *Solvency II* framework, it is useful to point out that high yield ratings (BB, B and CCC) have a probability of default of more than 5‰.

	Plow	Pupp	Par0	Parfit						
				AAA	AA	A	BBB	BB	B	CCC
k	0.00100	5.00000	1.00000	0.89903	0.88936	2.27326	0.82893	0.74335	1.33046	0.62264
θ	-5.00000	5.00000	0.01000	0.00009	0.00078	0.00125	0.04311	0.03671	0.00009	0.02809
ρ	0.00100	2.00000	0.15000	0.13304	0.13426	0.12760	0.03818	0.02868	0.02255	0.08647
$\widehat{k}$	0.00100	5.00000	0.50000	0.56224	0.57002	0.51399	0.11167	0.12829	0.25330	0.85245
$\widehat{\theta}$	-5.00000	5.00000	0.05000	0.02252	0.02240	0.02970	0.03775	0.03215	0.05278	0.06040
l_0	-5.00000	5.00000	-0.05000	-0.04188	-0.08839	2.62786	2.35371	-0.06981	-0.19110	2.51472
h	0.00010	2.000000	0.50000	0.50168	0.49925	0.52611	0.40589	0.40948	0.50222	0.87847

Table 5.5. Starting values (***Par0***), lower bounds (***Plow***), upper bounds (***Pupp***) e estimated values (***Parfit***) of model parameters for the rating-specific liquidity components of the model.

Quality Measure	Overall	AAA	AA	A	BBB	BB	B	CCC
RMSE (b.p.)	318.47	23.68	25.69	43.90	75.26	147.28	312.80	822.97
$\mathbf{R^2}$	0.92	0.85	0.87	0.93	0.97	0.93	0.90	0.84

Table 5.6. Quality measures for the goodness of fit of the full model with two subordinator processes and a rating-specific liquidity component.

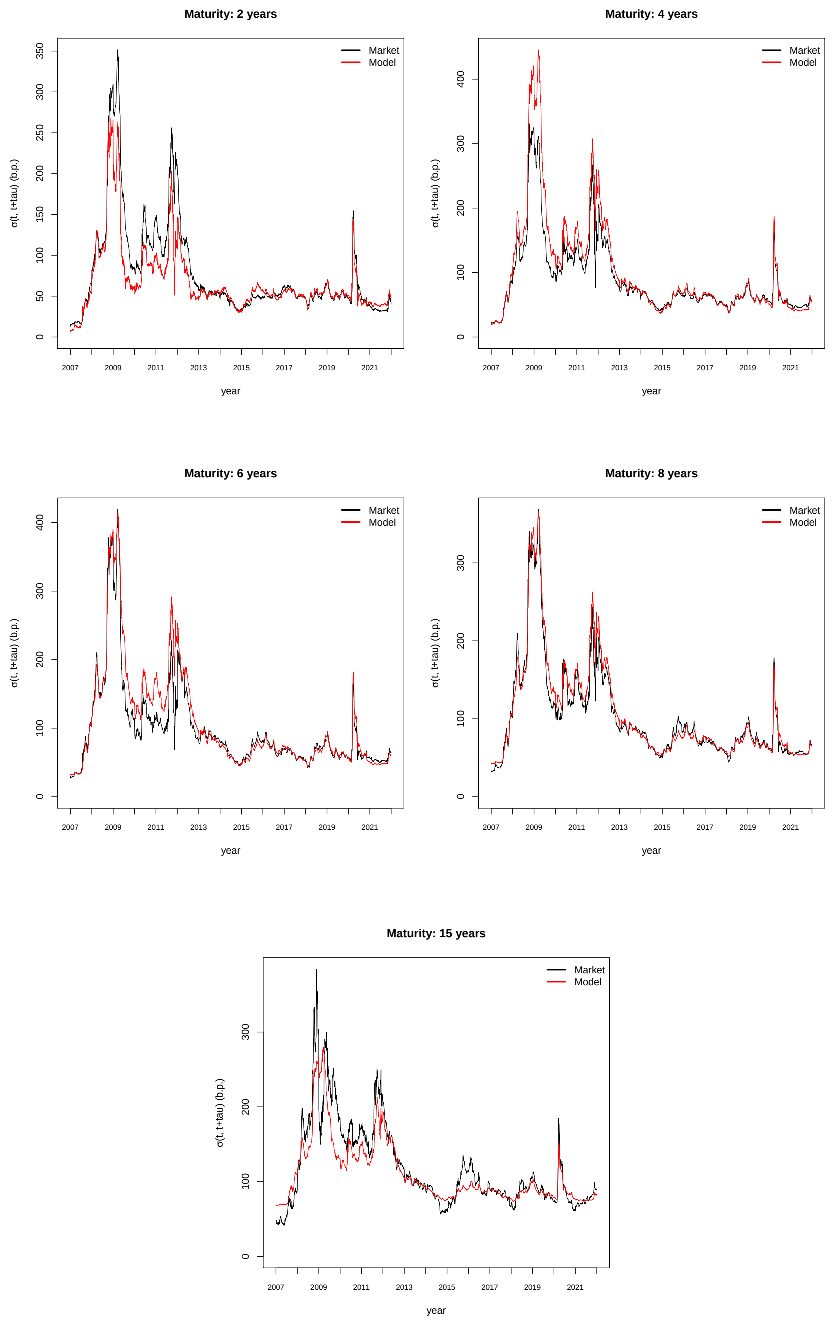


Figure 5.1. Comparison of model (red) and market (black) credit spread time series for maturities: 2, 4, 6, 8 and 15 years - *EUR Financials AAA* index.

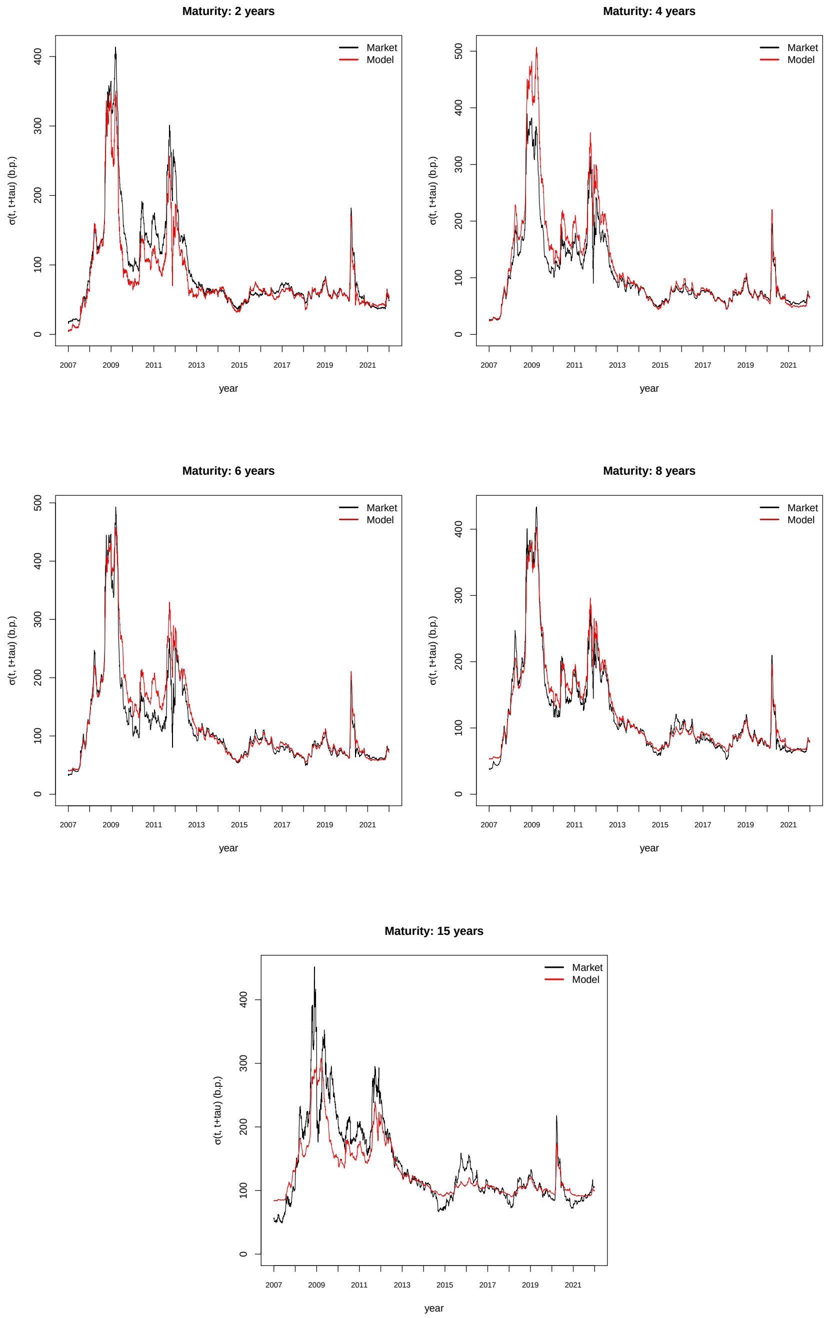


Figure 5.2. Comparison of model (red) and market (black) credit spread time series for maturities: 2, 4, 6, 8 and 15 years - *EUR Financials AA* index.

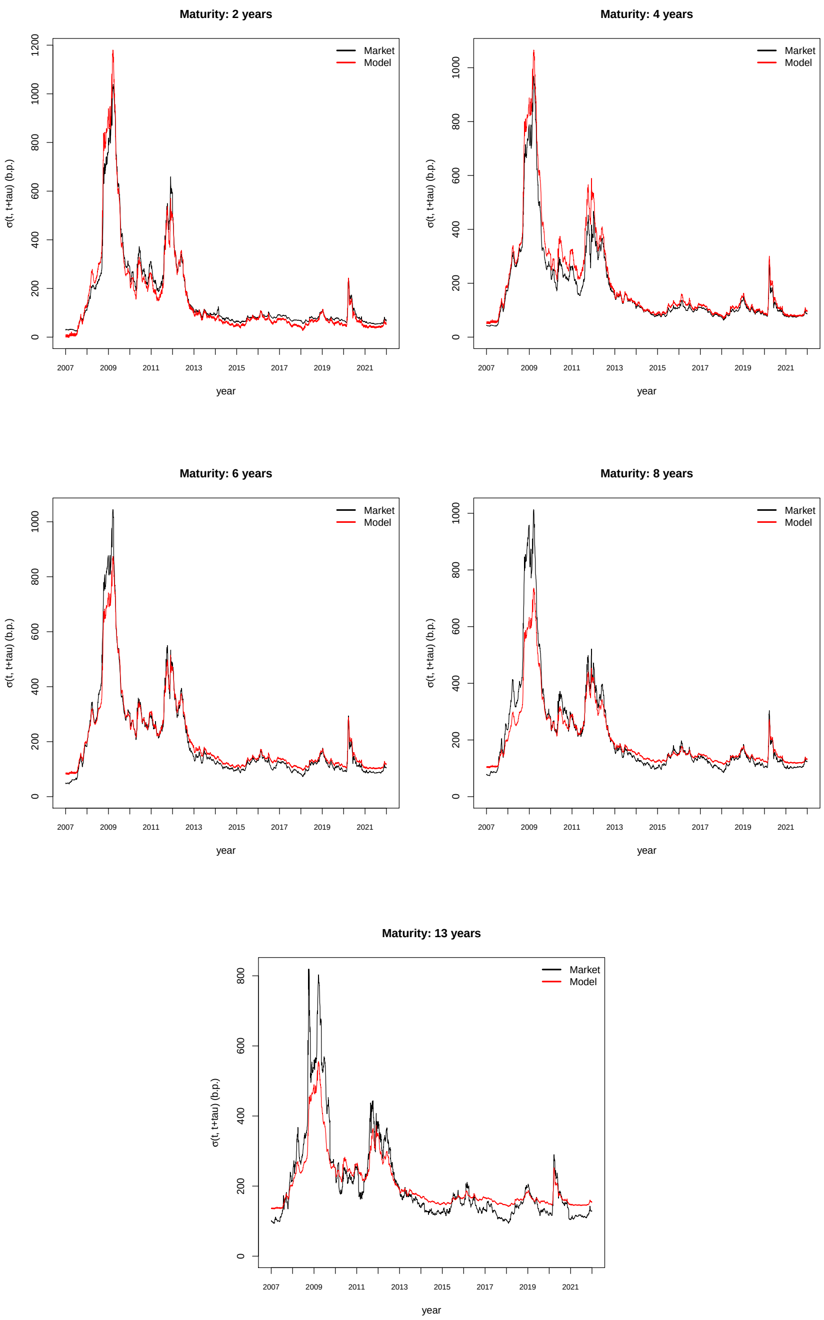


Figure 5.3. Comparison of model (red) and market (black) credit spread time series for maturities: 2, 4, 6, 8 and 13 years - *EUR Financials A* index.

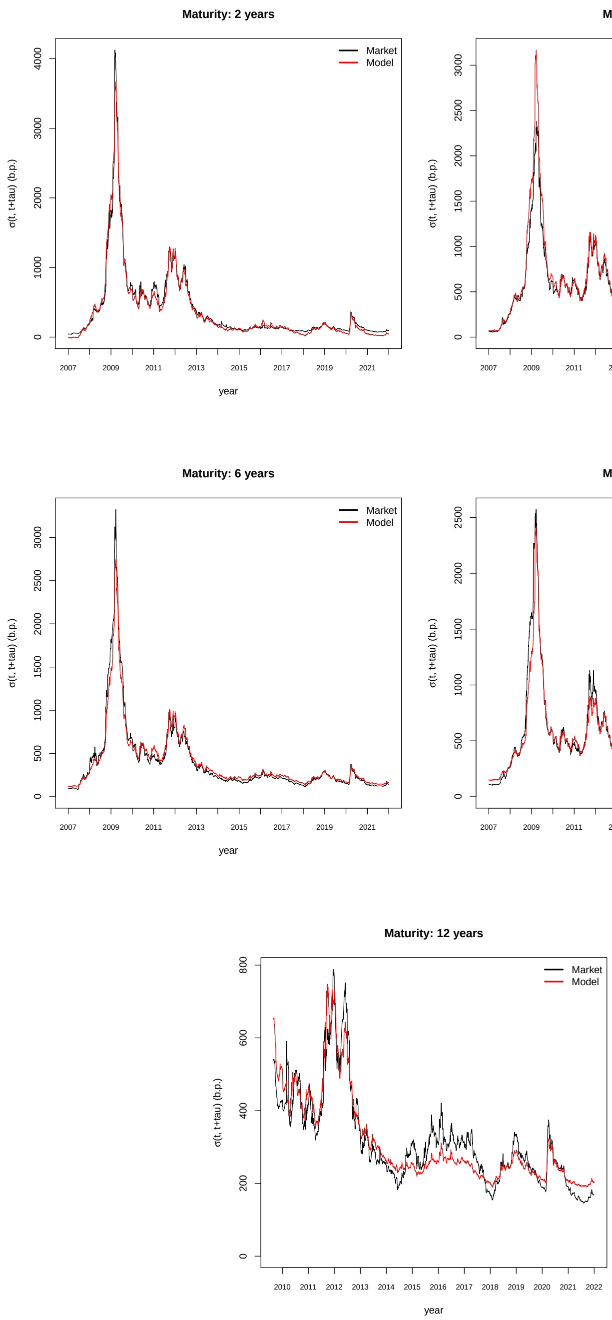


Figure 5.4. Comparison of model (red) and market (black) credit spread time series for maturities: 2, 4, 6, 8 and 12 years - *EUR Financials BBB* index.

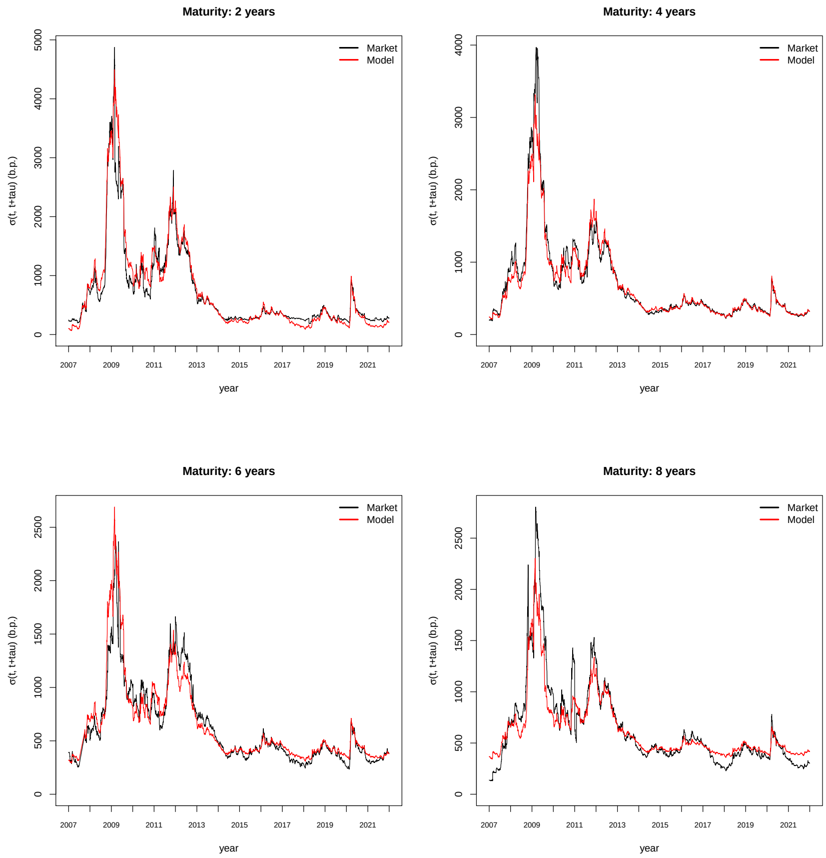


Figure 5.5. Comparison of model (red) and market (black) credit spread time series for maturities: 2, 4, 6 and 8 years - *EUR High Yield Financials BB* index.

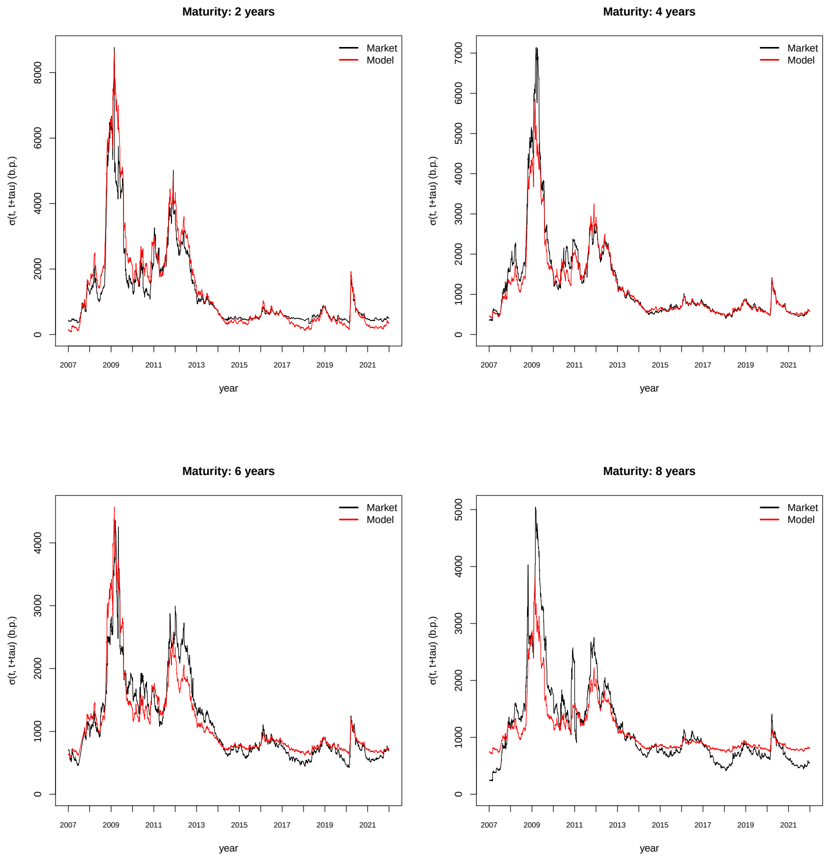


Figure 5.6. Comparison of model (red) and market (black) credit spread time series for maturities: 2, 4, 6 and 8 years - *EUR High Yield Financials B* index.

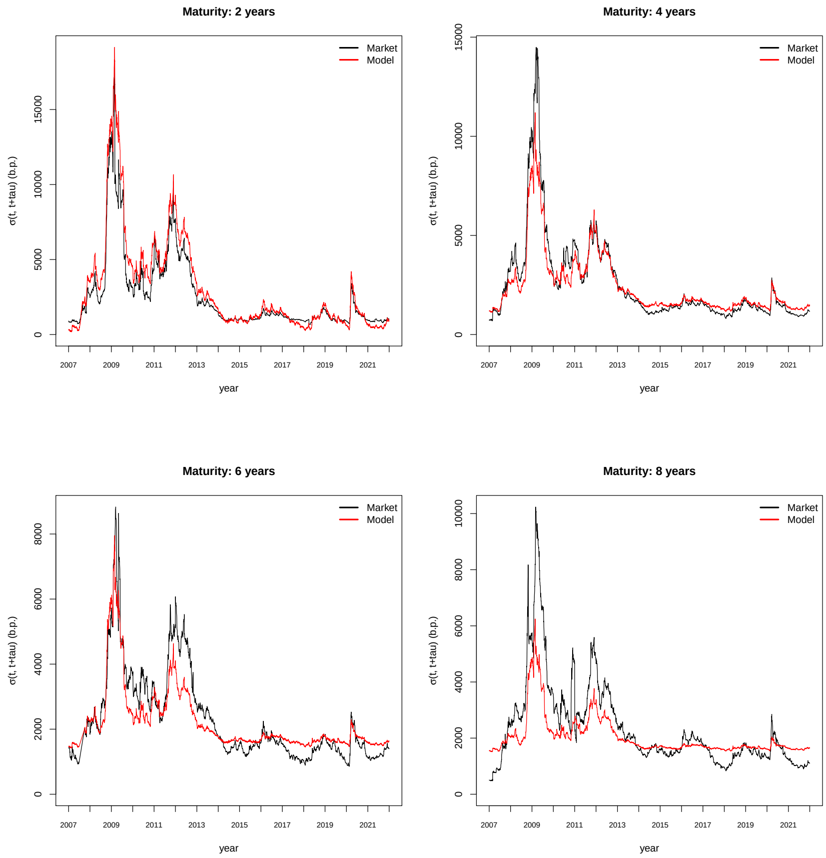


Figure 5.7. Comparison of model (red) and market (black) credit spread time series for maturities: 2, 4, 6 and 8 years - *EUR High Yield Financials CCC* index.

	Mean		Volatility	
τ	MKT	MDL	MKT	MDL
EUR Financials AAA				
2	79.53	67.65	60.48	42.29
4	89.57	102.51	57.64	79.81
6	97.52	106.81	64.44	74.60
8	102.34	105.92	60.17	64.17
15	118.22	110.16	56.13	42.84
EUR Financials AA				
2	93.56	80.53	71.15	57.06
4	105.38	120.38	67.81	91.28
6	114.73	124.75	75.81	82.00
8	120.39	124.16	70.79	69.63
15	139.08	129.67	66.04	45.61
EUR Financials A				
2	171.27	161.21	181.13	195.98
4	178.83	204.44	154.58	180.97
6	201.87	206.59	168.81	144.32
8	216.16	204.24	167.30	116.69
13	206.23	204.27	124.95	78.72
EUR Financials BBB				
2	393.16	372.70	535.93	530.56
4	365.41	402.36	370.03	444.45
6	391.46	397.01	422.00	369.13
8	401.85	385.41	363.94	315.44
12	316.11	313.25	125.38	124.41
EUR High Yield Financials BB				
2	706.10	726.01	707.67	786.66
4	696.13	673.24	605.64	528.64
6	627.85	644.99	379.08	396.27
8	638.70	624.63	419.05	313.35
EUR High Yield Financials B				
2	1270.98	1367.47	1273.80	1514.49
4	1253.03	1172.44	1090.16	901.15
6	1130.14	1127.98	682.35	635.71
8	1149.66	1110.18	754.29	479.01
EUR High Yield Financials CCC				
2	2577.27	3036.56	2582.99	3217.47
4	2540.86	2426.24	2210.60	1597.70
6	2291.67	2200.31	1383.65	1006.70
8	2331.25	2034.43	1529.54	706.75

Table 5.7. Means and volatilities of historical (MKT) and model-reconstructed (MDL) credit spreads (b.p.).

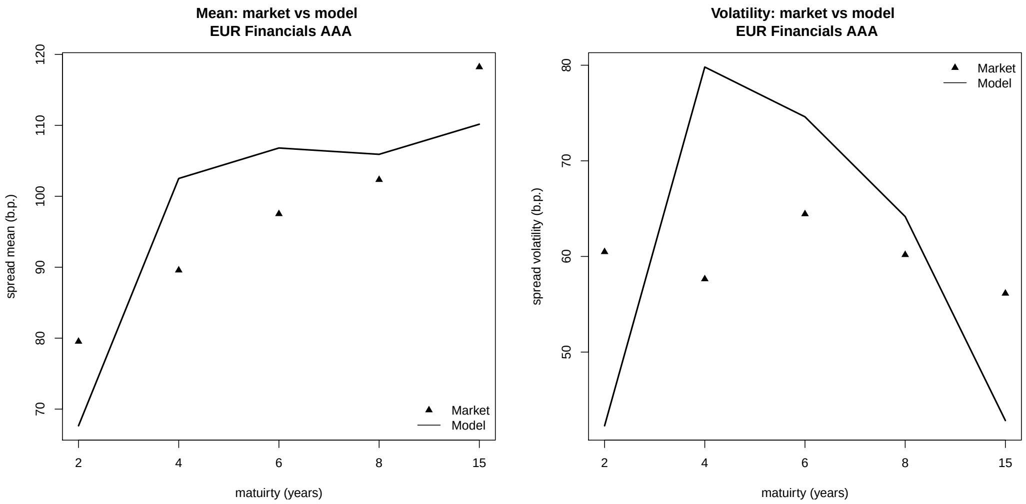


Figure 5.8. Comparison of sample means and volatilities, calculated on the credit spread time series reconstructed with the model (solid line) and market values (triangles) - *EUR Financials AAA* index.

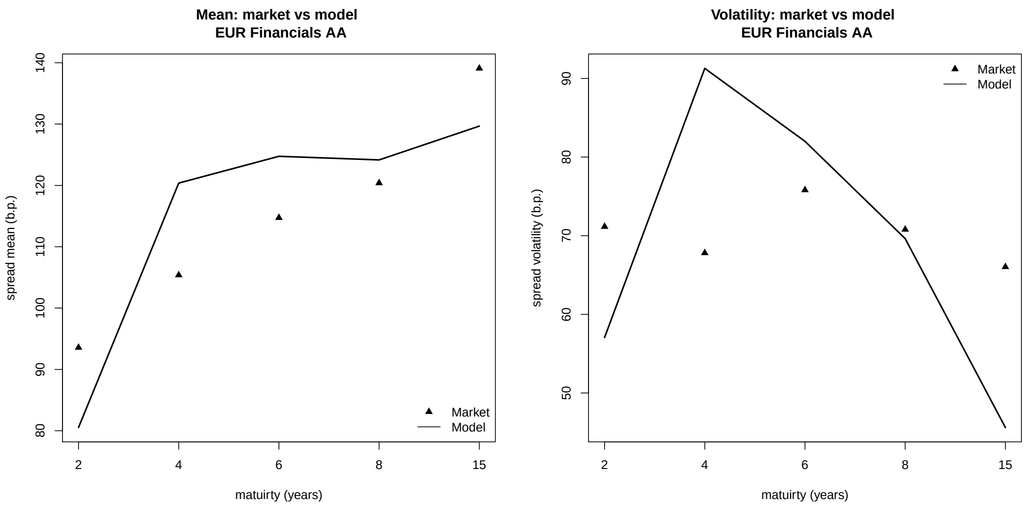


Figure 5.9. Comparison of sample means and volatilities, calculated on the credit spread time series reconstructed with the model (solid line) and market values (triangles) - *EUR Financials AA* index.

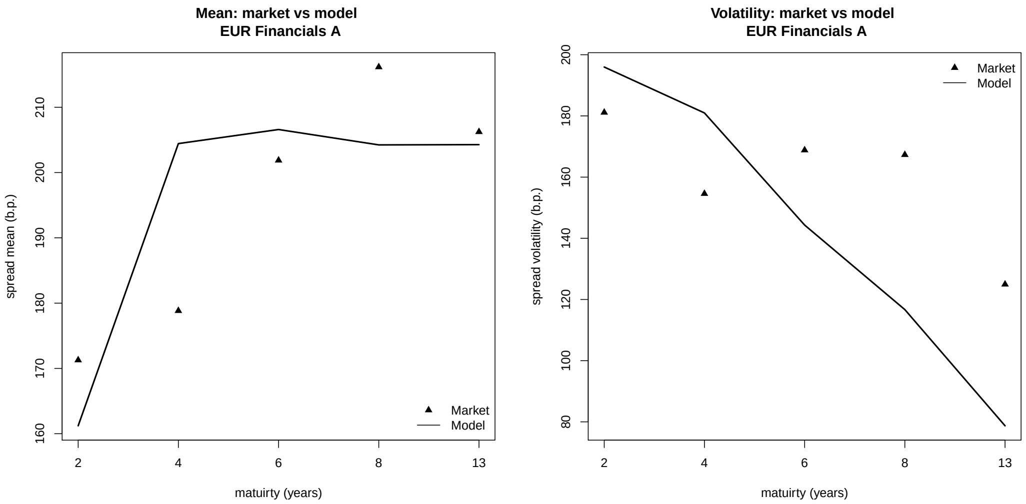


Figure 5.10. Comparison of sample means and volatilities, calculated on the credit spread time series reconstructed with the model (solid line) and market values (triangles) - *EUR Financials A* index.

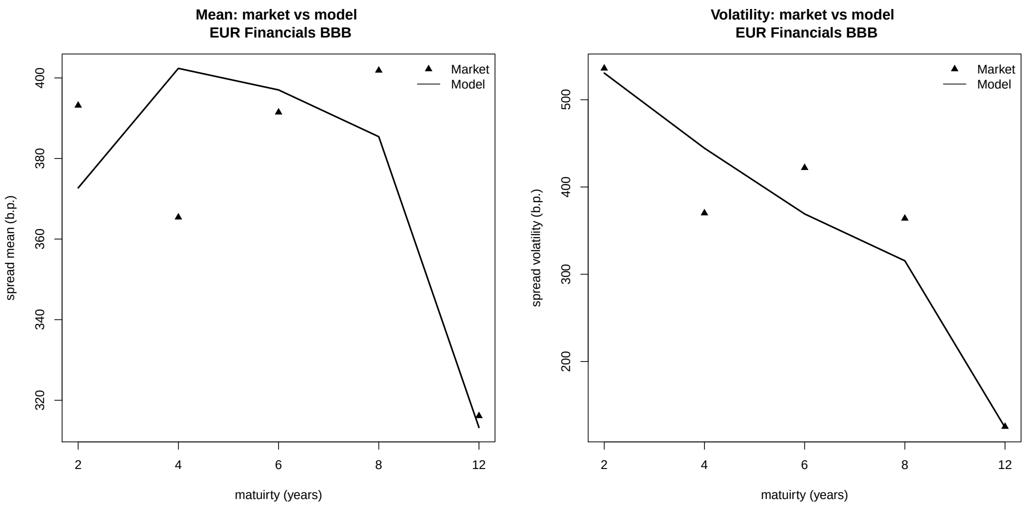


Figure 5.11. Comparison of sample means and volatilities, calculated on the credit spread time series reconstructed with the model (solid line) and market values (triangles) - *EUR Financials BBB* index.

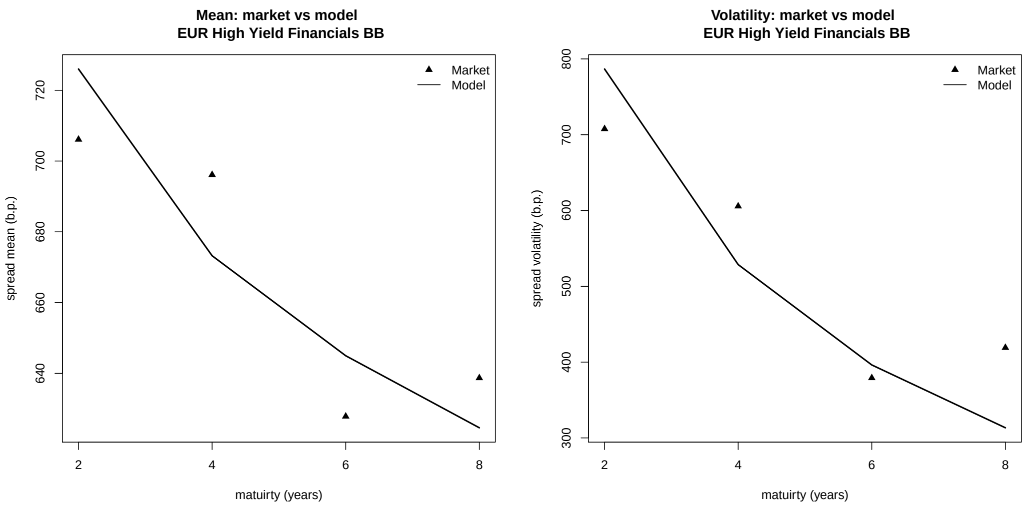


Figure 5.12. Comparison of sample means and volatilities, calculated on the credit spread time series reconstructed with the model (solid line) and market values (triangles) - *EUR High Yield Financials BB* index.

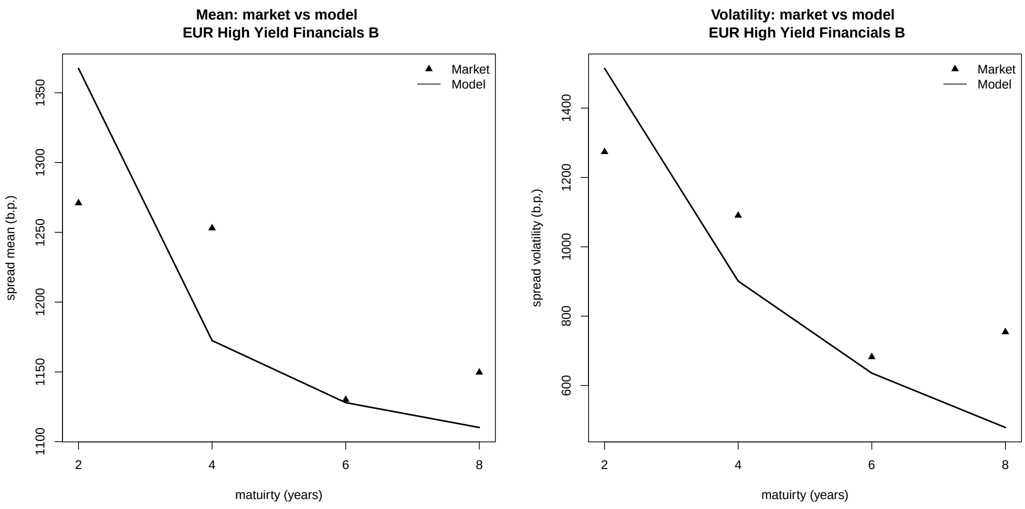


Figure 5.13. Comparison of sample means and volatilities, calculated on the credit spread time series reconstructed with the model (solid line) and market values (triangles) - *EUR High Yield Financials B* index.

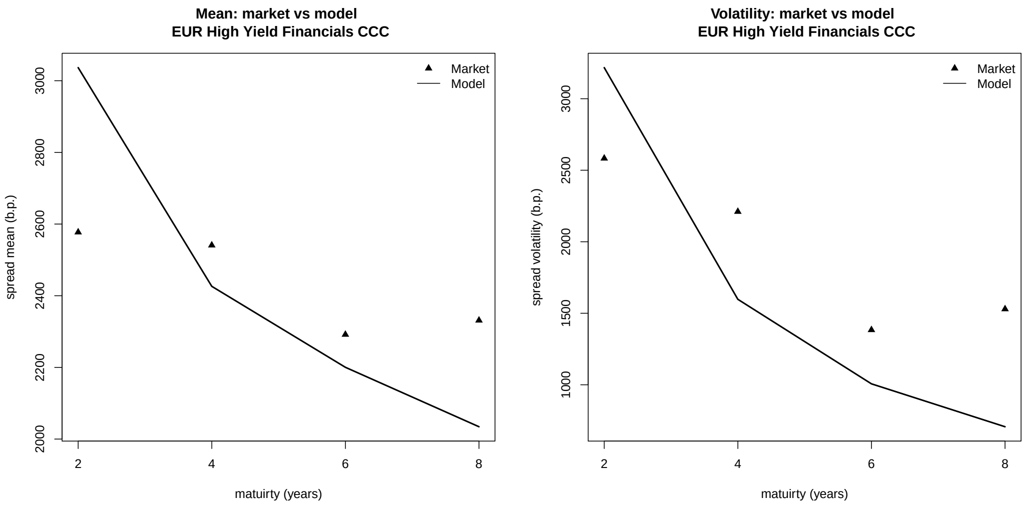
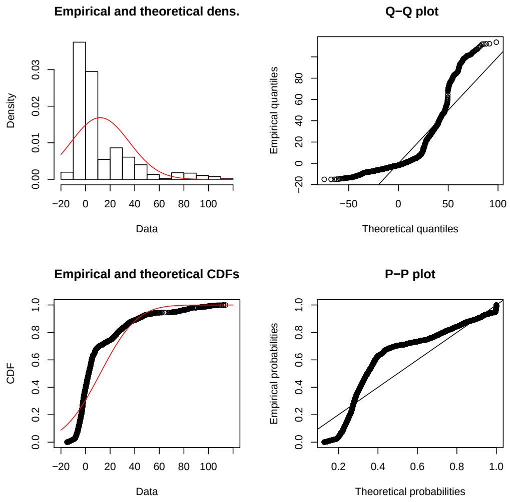
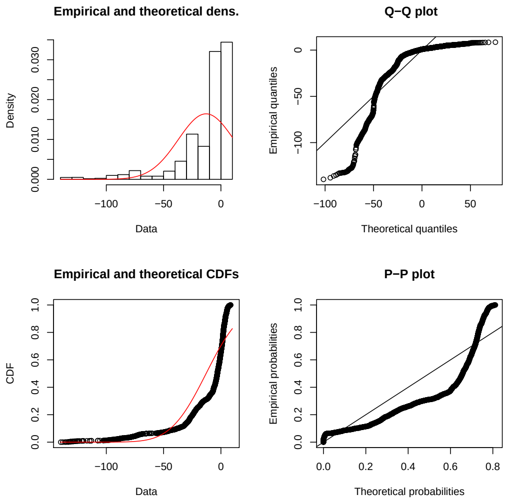


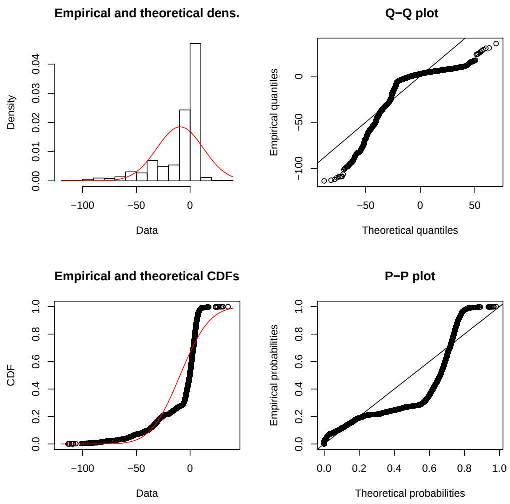
Figure 5.14. Comparison of sample means and volatilities, calculated on the credit spread time series reconstructed with the model (solid line) and market values (triangles) - *EUR High Yield Financials CCC* index.



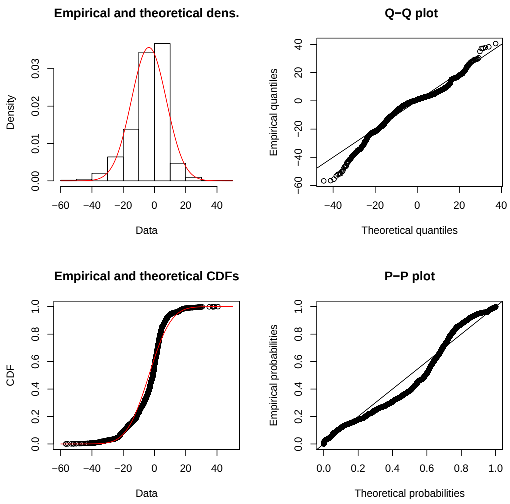
(a) Maturity: 2 years



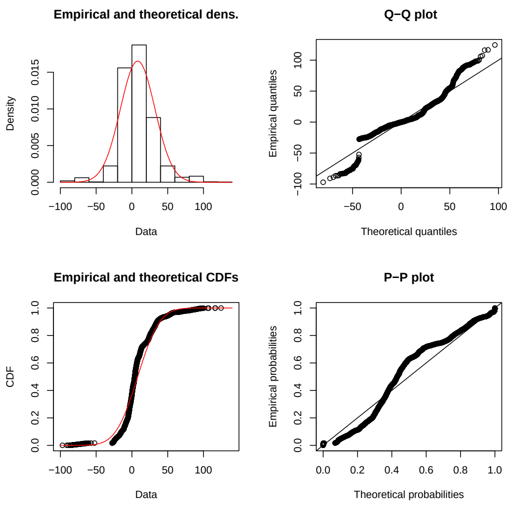
(b) Maturity: 4 years



(c) Maturity: 6 years

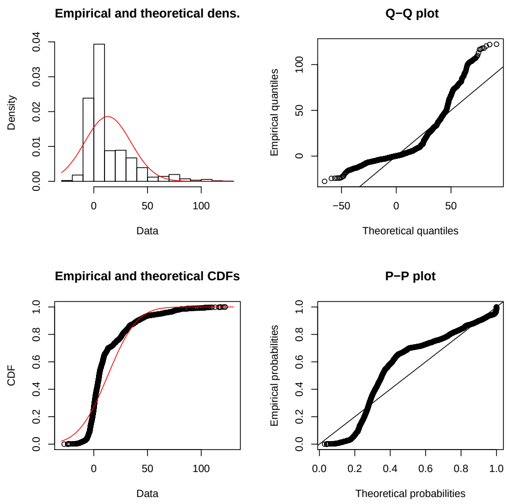


(d) Maturity: 8 years

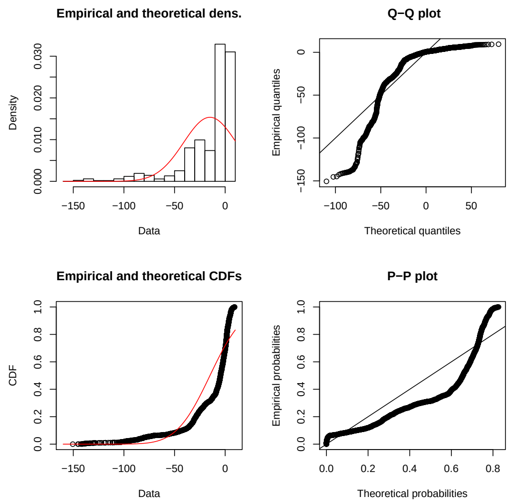


(e) Maturity: 15 years

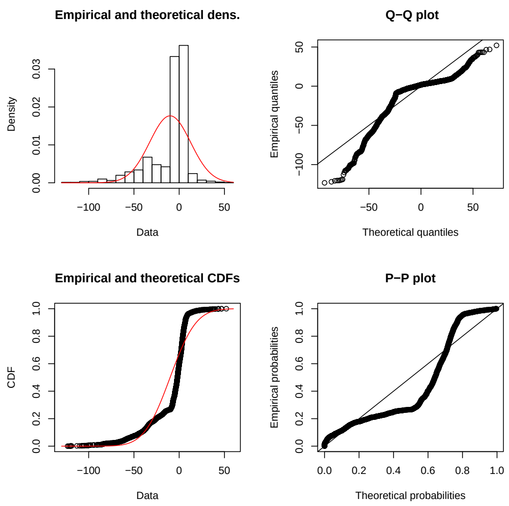
Figure 5.15. Normality analysis of residuals: comparison of empirical (histogram) and theoretical (red line) densities, Q-Q plot, comparison of empirical (black line) and theoretical (red line) CDFs and P-P plot - *EUR Financials AAA* index.



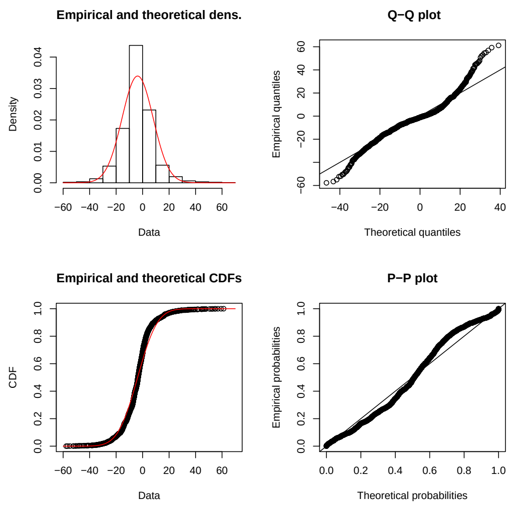
(a) Maturity: 2 years



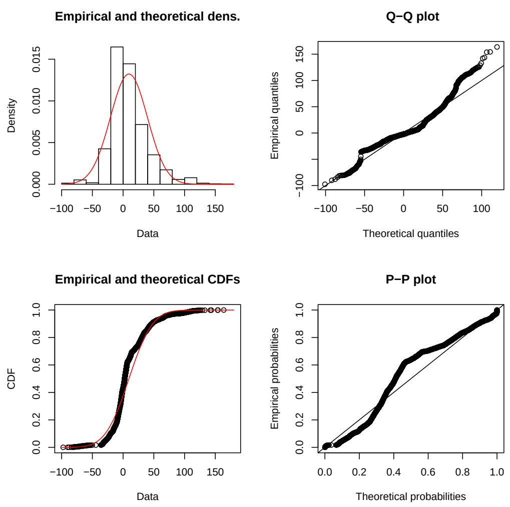
(b) Maturity: 4 years



(c) Maturity: 6 years

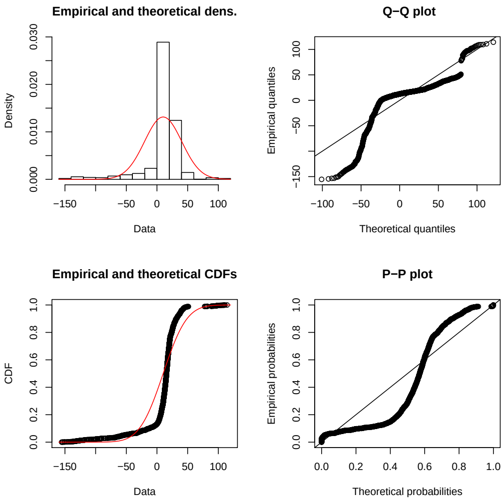


(d) Maturity: 8 years

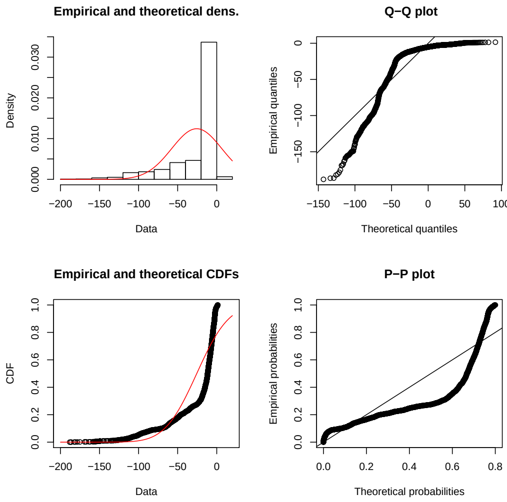


(e) Maturity: 15 years

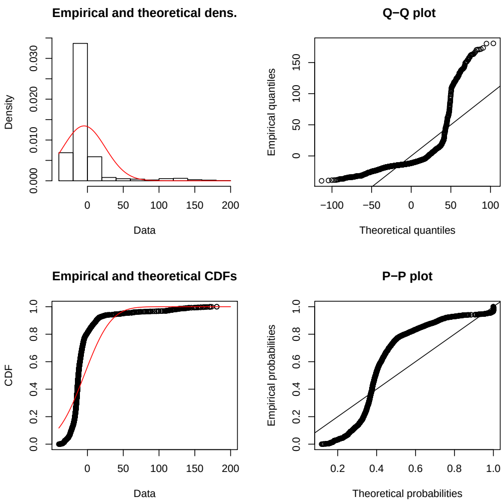
Figure 5.16. Normality analysis of residuals: comparison of empirical (histogram) and theoretical (red line) densities, Q-Q plot, comparison of empirical (black line) and theoretical (red line) CDFs and P-P plot - *EUR Financials AA* index.



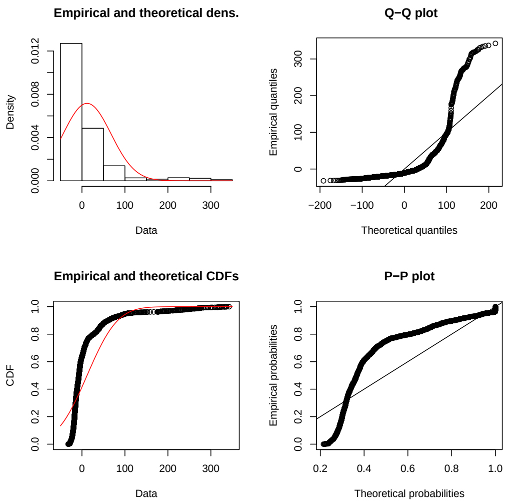
(a) Maturity: 2 years



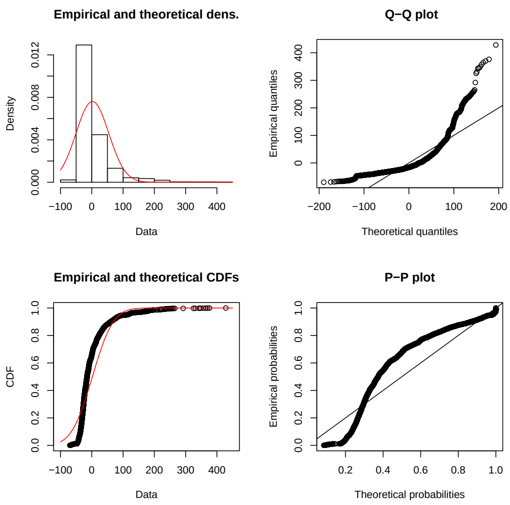
(b) Maturity: 4 years



(c) Maturity: 6 years

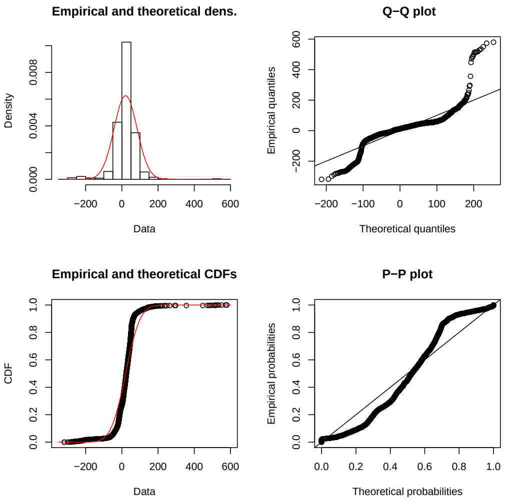


(d) Maturity: 8 years

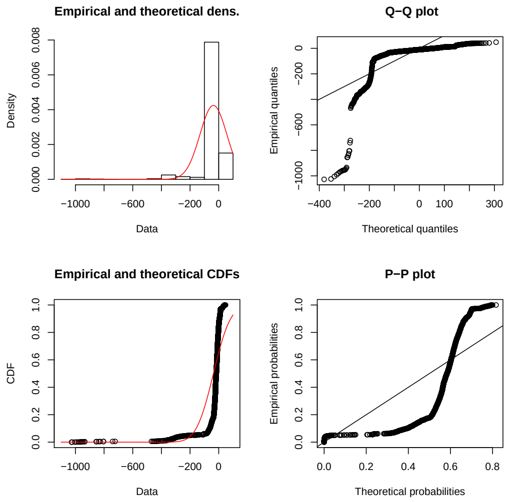


(e) Maturity: 13 years

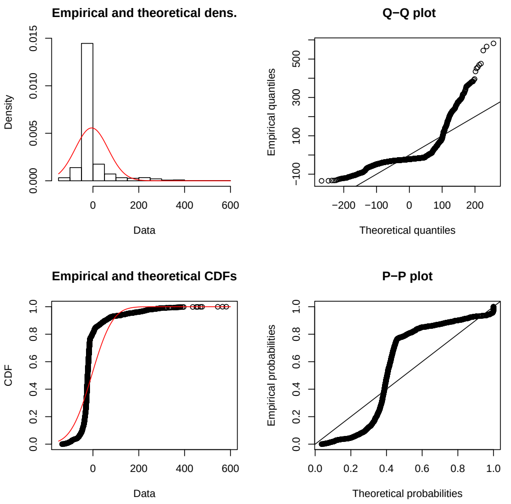
Figure 5.17. Normality analysis of residuals: comparison of empirical (histogram) and theoretical (red line) densities, Q-Q plot, comparison of empirical (black line) and theoretical (red line) CDFs and P-P plot - *EUR Financials A* index.



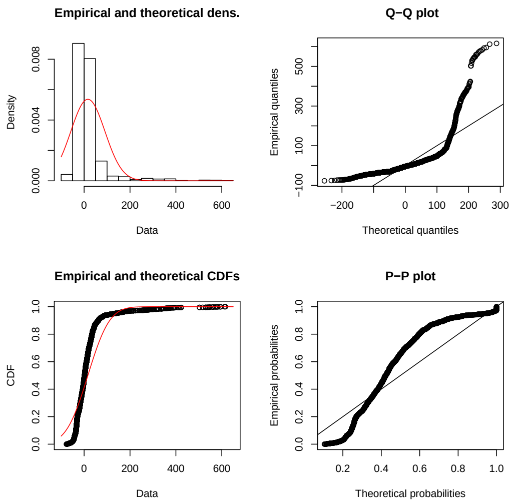
(a) Maturity: 2 years



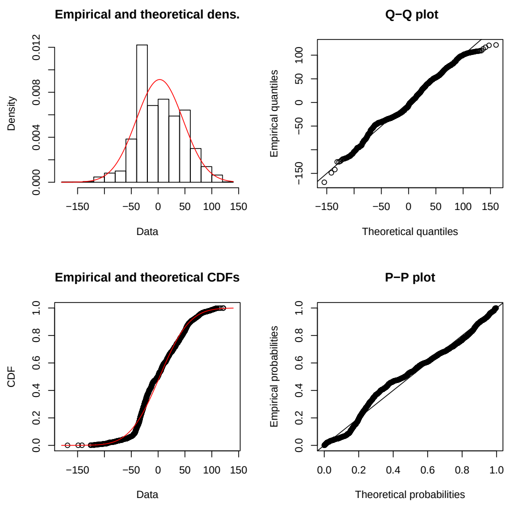
(b) Maturity: 4 years



(c) Maturity: 6 years

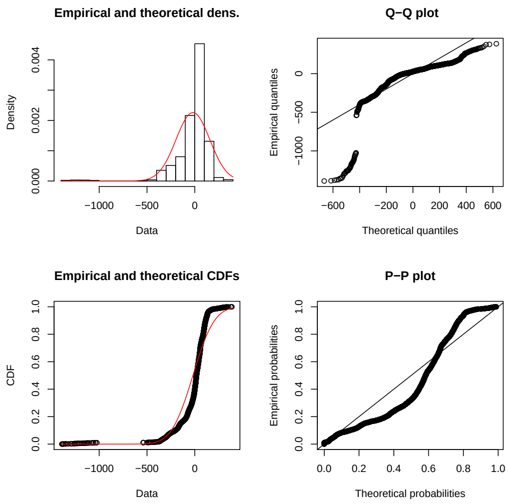


(d) Maturity: 8 years

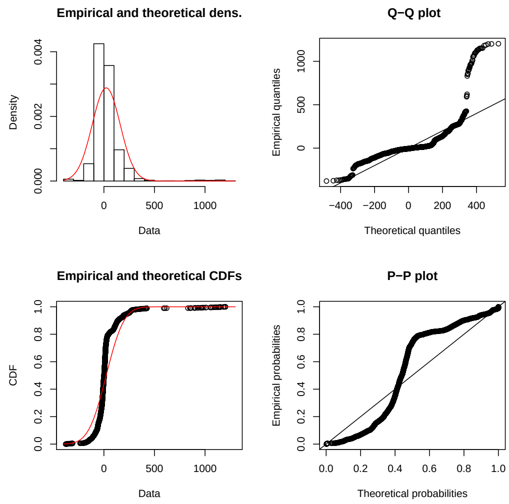


(e) Maturity: 12 years

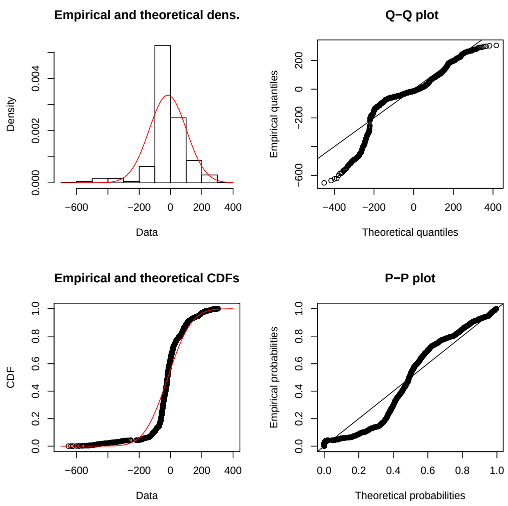
Figure 5.18. Normality analysis of residuals: comparison of empirical (histogram) and theoretical (red line) densities, Q-Q plot, comparison of empirical (black line) and theoretical (red line) CDFs and P-P plot - *EUR Financials BBB* index.



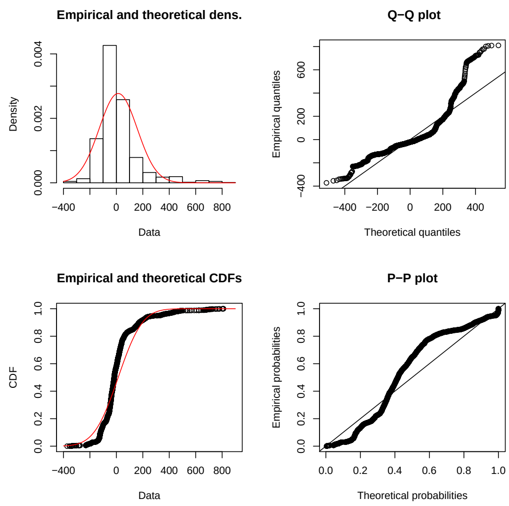
(a) Maturity: 2 years



(b) Maturity: 4 years

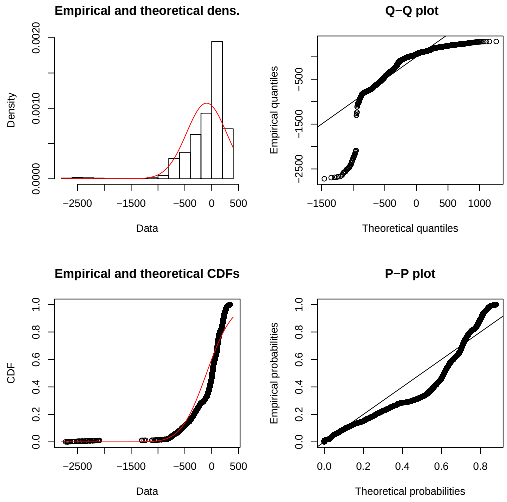


(c) Maturity: 6 years

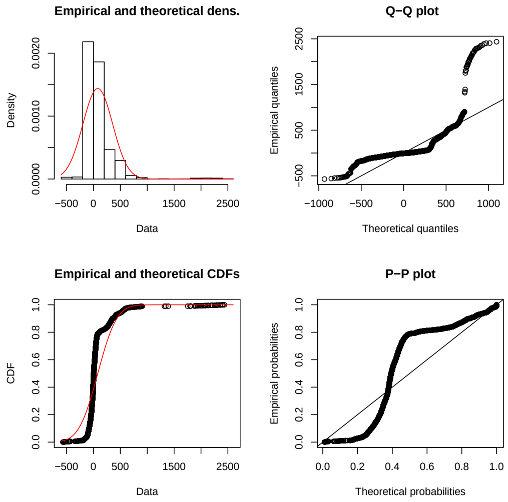


(d) Maturity: 8 years

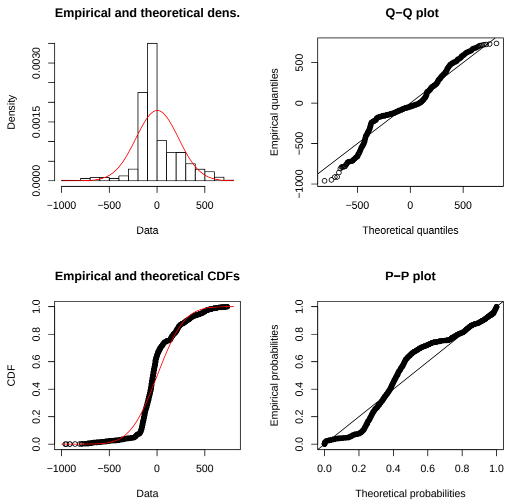
Figure 5.19. Normality analysis of residuals: comparison of empirical (histogram) and theoretical (red line) densities, Q-Q plot, comparison of empirical (black line) and theoretical (red line) CDFs and P-P plot - *EUR High Yield Financials BB* index.



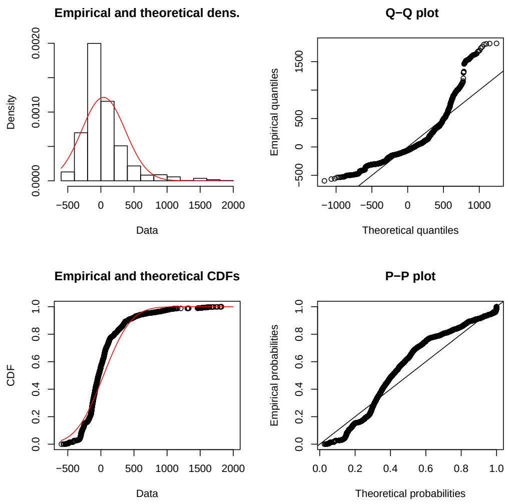
(a) Maturity: 2 years



(b) Maturity: 4 years

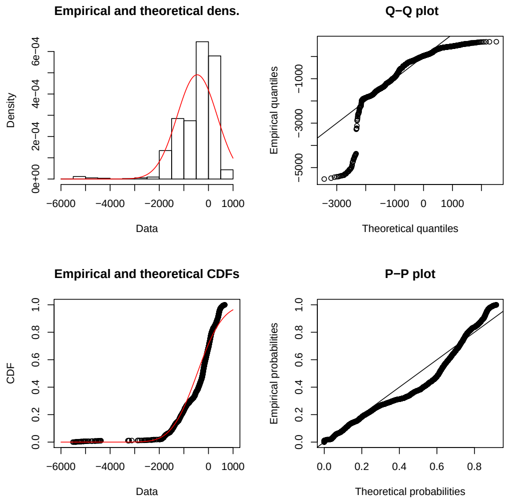


(c) Maturity: 6 years

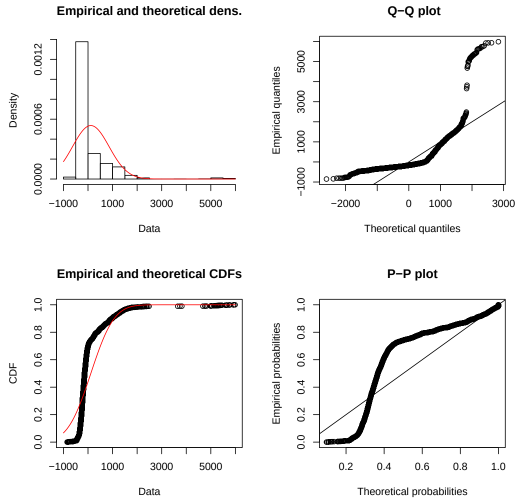


(d) Maturity: 8 years

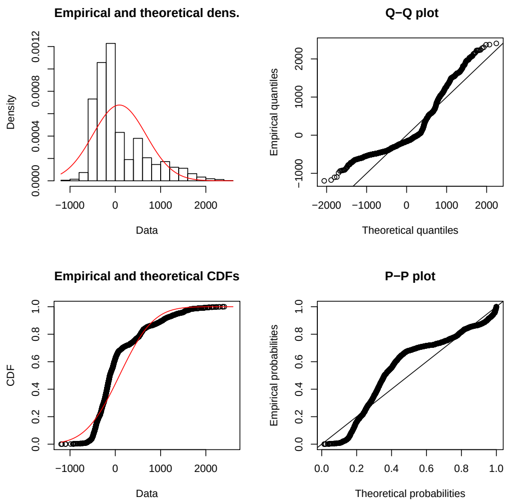
Figure 5.20. Normality analysis of residuals: comparison of empirical (histogram) and theoretical (red line) densities, Q-Q plot, comparison of empirical (black line) and theoretical (red line) CDFs and P-P plot - *EUR High Yield Financials B* index.



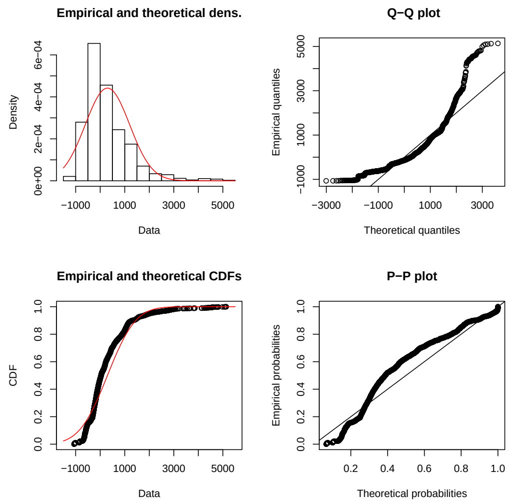
(a) Maturity: 2 years



(b) Maturity: 4 years



(c) Maturity: 6 years



(d) Maturity: 8 years

Figure 5.21. Normality analysis of residuals: comparison of empirical (histogram) and theoretical (red line) densities, Q-Q plot, comparison of empirical (black line) and theoretical (red line) CDFs and P-P plot - *EUR High Yield Financials CCC* index.

From/To	AAA	AA	A	BBB	BB	B	CCC	D
AAA	84.48	12.93	1.95	0.28	0.18	0.07	0.06	0.05
AA	0.70	86.03	11.54	1.37	0.17	0.13	0.03	0.04
A	0.04	2.35	88.94	7.60	0.67	0.25	0.03	0.13
BBB	0.00	0.21	4.95	88.15	5.09	1.01	0.16	0.42
BB	0.00	0.05	0.43	6.83	80.44	9.51	0.93	1.81
B	0.00	0.04	0.14	0.61	6.66	79.84	5.40	7.30
CCC	0.00	0.01	0.14	0.31	1.29	15.55	46.73	35.98

Table 5.8. Transition matrix implied in the model (%).

5.3.2 Application within the *Solvency II* framework

Figures 5.22 to 5.28 show the comparison between the SCR curves provided by the *Standard Formula* and those calculated within the internal model framework, in the two configurations of the modular approach (not including rating transitions and default) and the integrated approach (including rating transitions and default). The SCR calculated within the internal model framework with integrated approach always turn out to be significantly higher than those provided by the *Standard Formula*. On the other hand, the SCR calculated within the internal model framework with a modular approach turn out to be slightly higher than those under the *Standard Formula* for investment grade rating classes (AAA, AA, A, and BBB), while medium- to long-term maturities of high-yield rating classes (BB, B, CCC) show slightly lower SCR than those provided by the *Standard Formula*.
The greater prudence of the defined partial internal model compared with the *Standard Formula* may be justified by the different calibration procedure and the different data used in it. In fact, as reported in the *Solvency II Calibration Paper* [11], the *Standard Formula* SCR for the spread risk sub-module is calibrated using the time series of Corporate Bond Indices provided by *Merril Lynch* indices with a depth of 9 years (02/1999 - 02/2010), which is significantly less than the one used in this work (01/01/2007 - 31/12/2021). By stopping at 2010, these time series do not consider a crucial period for the trend of credit spreads and their volatilities, such as that of the European sovereign debt crisis in 2012. Moreover, the credit spreads of the *Merril Lynch* indices have significantly lower levels and volatilities than the *IHS Markit* `iBoxx` indices used in this work. Furthermore, in the *Standard Formula* calibration procedure each portfolio spread series (rating and maturity) was first transformed into a 3 month moving average function in order to smooth out short-term spikes. All these aspects result in the variability and thus the riskiness estimated to the *IHS Markit* `iBoxx` indices being higher than those of the *Standard Formula*, thus leading to higher SCR in internal model framework.

If the two different approaches, modular and integrated, are compared, SCR are always higher if computed using the integrated approach versus using the modular approach, as the former covers all facets of credit risk, including migration and default risks. This result is consistent with what is observed in the *Results from the 2020 MCRCS* [24], where it is reported that credit risk charges are generally higher for firms adopting an integrated approach than for those adopting a modular approach. It is also observed that in both cases SCR are higher for the lowest rating classes.

Table 5.9 shows the statistics of simulated 1-year rating transitions. It can be seen that the average "transition" consists of remaining in the initial rating class, and that for initial high yield ratings the 99.5% quantile transition coincides with default state.
Figures 5.29 to 5.35 show the 1-year probability distribution forecast of term structures of credit spreads for all rating classes and maturities considered. The effect of rating transitions and default in the simulation is evident: the distributions simulated with the modular approach turn out to be smoothed, while those simulated with the integrated approach turn out to be multi-modal.

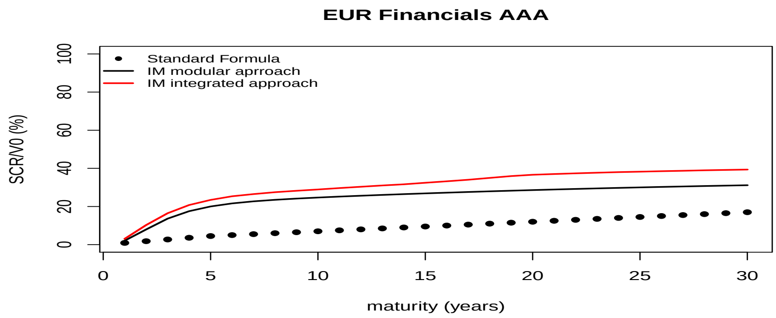


Figure 5.22. SCR provided by *Standard Formula* (black dots), in internal model logic with modular approach (black line) and in internal model logic with integrated approach (red line) for an unsubordinated risky ZCB as maturity changes - *EUR Financials AAA* index.

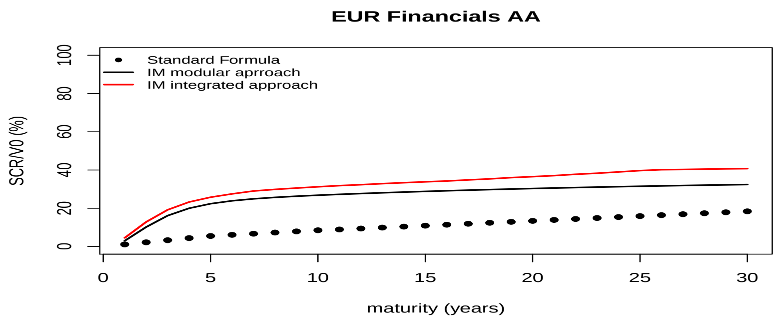


Figure 5.23. SCR provided by *Standard Formula* (black dots), in internal model logic with modular approach (black line) and in internal model logic with integrated approach (red line) for an unsubordinated risky ZCB as maturity changes - *EUR Financials AA* index.

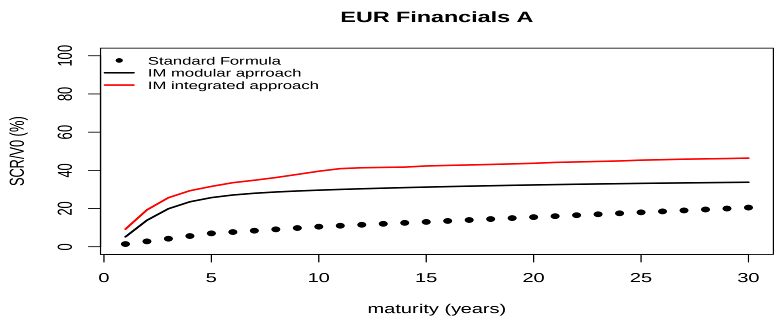


Figure 5.24. SCR provided by *Standard Formula* (black dots), in internal model logic with modular approach (black line) and in internal model logic with integrated approach (red line) for an unsubordinated risky ZCB as maturity changes - *EUR Financials A* index.

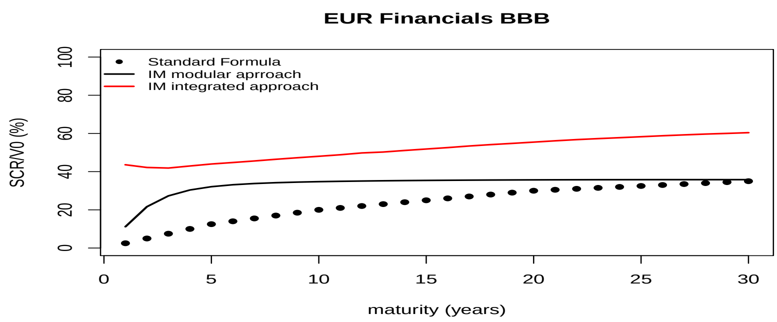


Figure 5.25. SCR provided by *Standard Formula* (black dots), in internal model logic with modular approach (black line) and in internal model logic with integrated approach (red line) for an unsubordinated risky ZCB as maturity changes - *EUR Financials BBB* index.

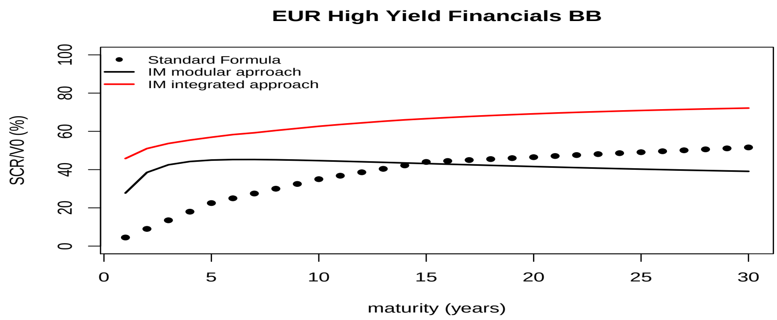


Figure 5.26. SCR provided by *Standard Formula* (black dots), in internal model logic with modular approach (black line) and in internal model logic with integrated approach (red line) for an unsubordinated risky ZCB as maturity changes - *EUR High Yield Financials BB* index.

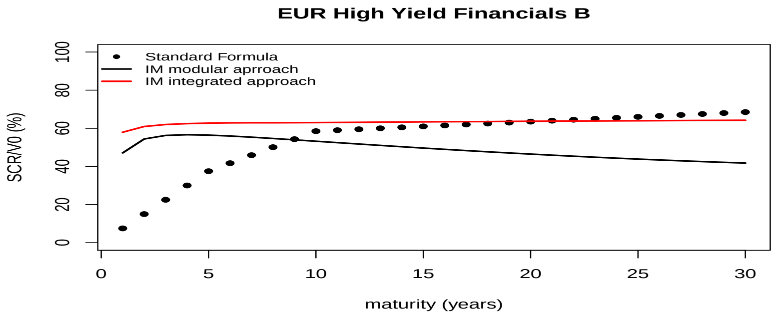


Figure 5.27. SCR provided by *Standard Formula* (black dots), in internal model logic with modular approach (black line) and in internal model logic with integrated approach (red line) for an unsubordinated risky ZCB as maturity changes - *EUR High Yield Financials B* index.

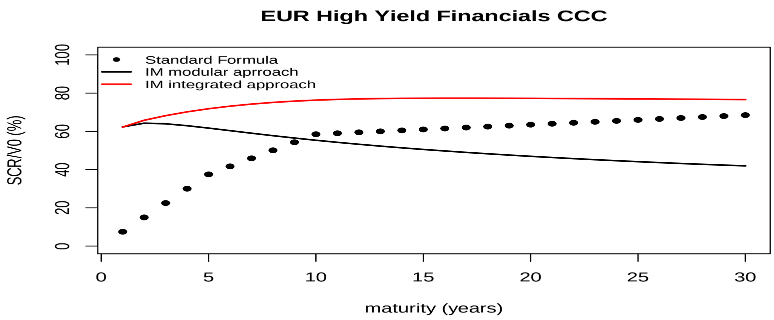


Figure 5.28. SCR provided by *Standard Formula* (black dots), in internal model logic with modular approach (black line) and in internal model logic with integrated approach (red line) for an unsubordinated risky ZCB as maturity changes - *EUR High Yield Financials CCC* index.

Stat/Starting rating	AAA	AA	A	BBB	BB	B	CCC
Mean	AAA	AA	A	BBB	BB	B	CCC
Quantile 99.5%	BBB	BBB	BB	CCC	D	D	D

Table 5.9. Means and 99.5 % quantiles of simulated 1-year rating transitions.

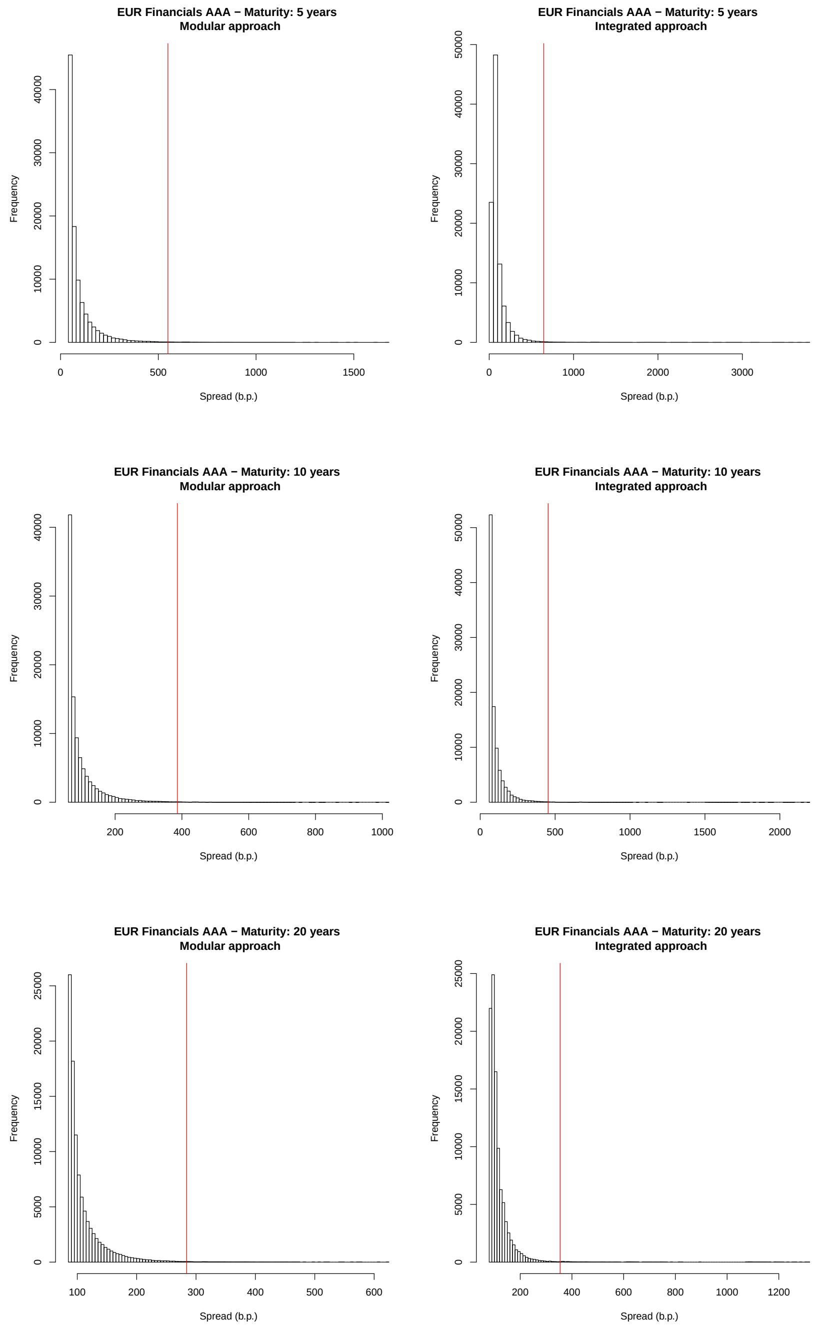


Figure 5.29. Probability distribution forecast of credit spreads $\sigma_h(T, T+\tau)$ with $T = 1$ and $\tau = 5, 10, 20$ years with modular (left) and integrated (right) approaches. The red line denotes the quantile at 99.5% - *EUR Financials AAA*.

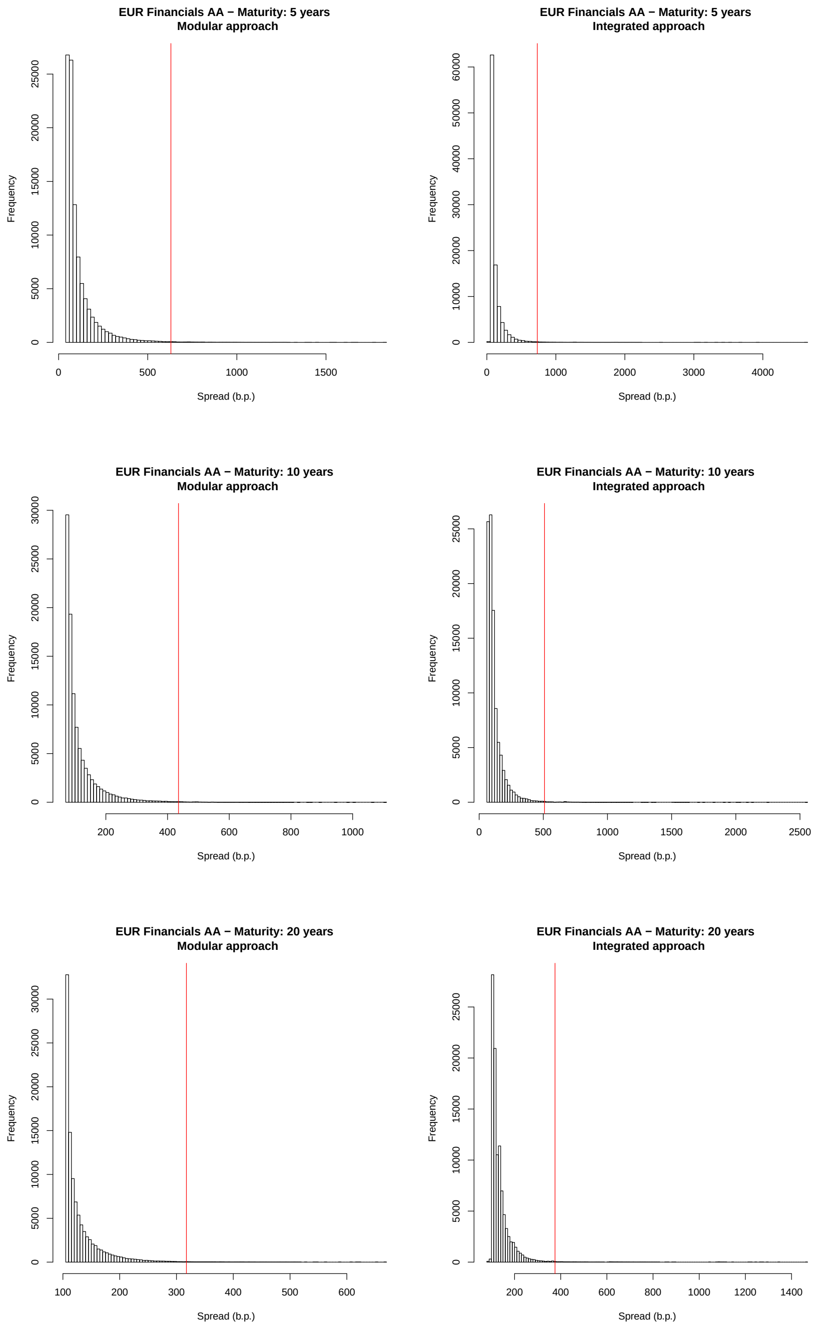


Figure 5.30. Probability distribution forecast of credit spreads $\sigma_h(T, T+\tau)$ with $T = 1$ and $\tau = 5, 10, 20$ years with modular (left) and integrated (right) approaches. The red line denotes the quantile at 99.5% - *EUR Financials AA*.

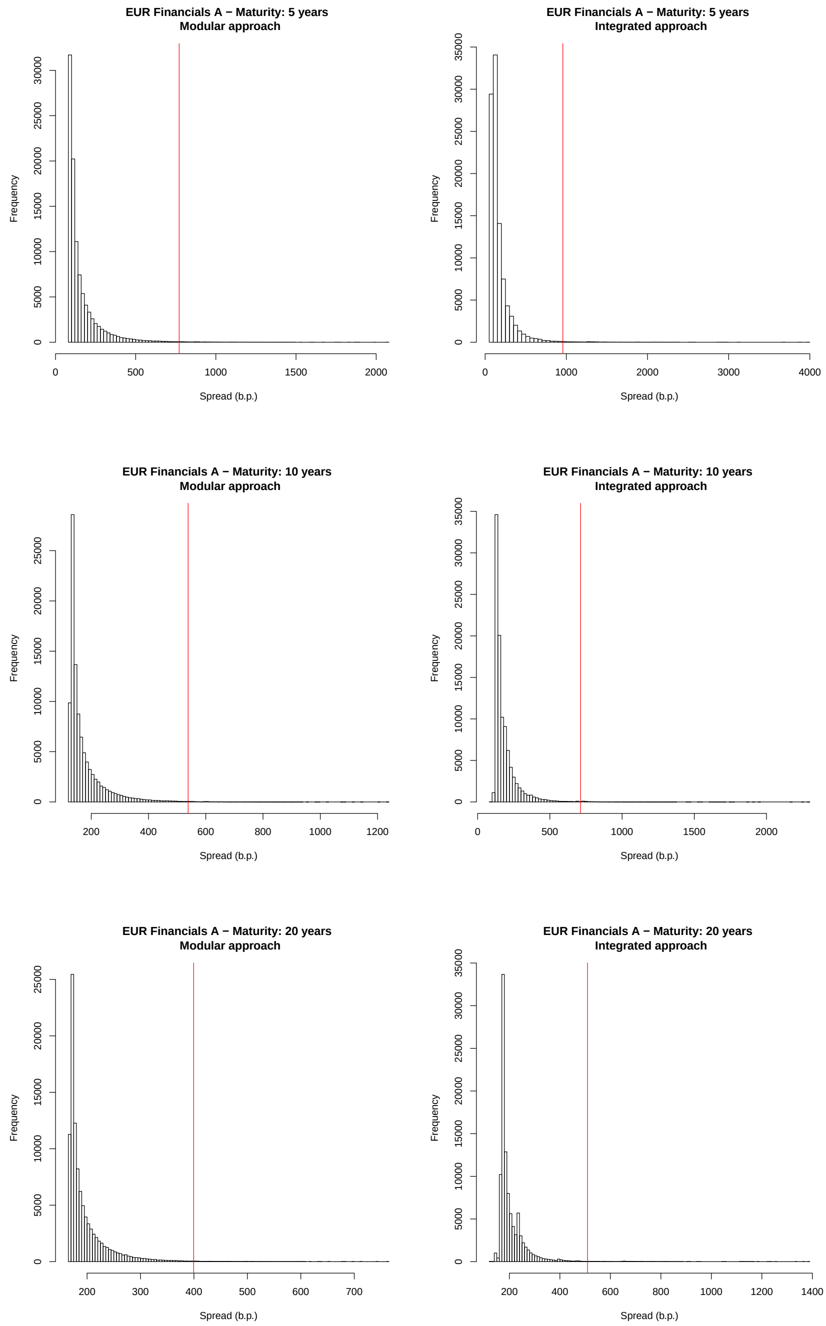


Figure 5.31. Probability distribution forecast of credit spreads $\sigma_h(T, T+\tau)$ with $T = 1$ and $\tau = 5, 10, 20$ years with modular (left) and integrated (right) approaches. The red line denotes the quantile at 99.5% - *EUR Financials A*.

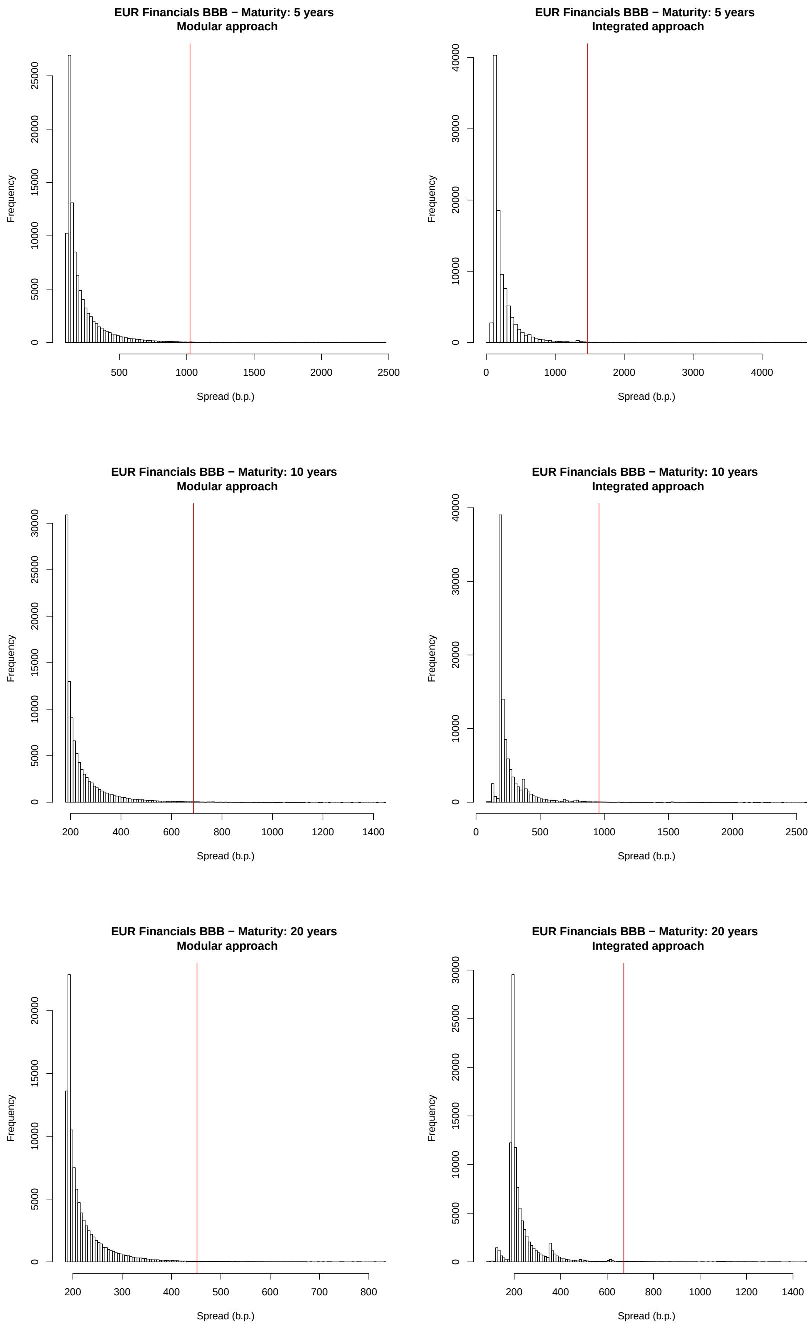


Figure 5.32. Probability distribution forecast of credit spreads $\sigma_h(T, T+\tau)$ with $T = 1$ and $\tau = 5, 10, 20$ years with modular (left) and integrated (right) approaches. The red line denotes the quantile at 99.5% - *EUR Financials BBB*.

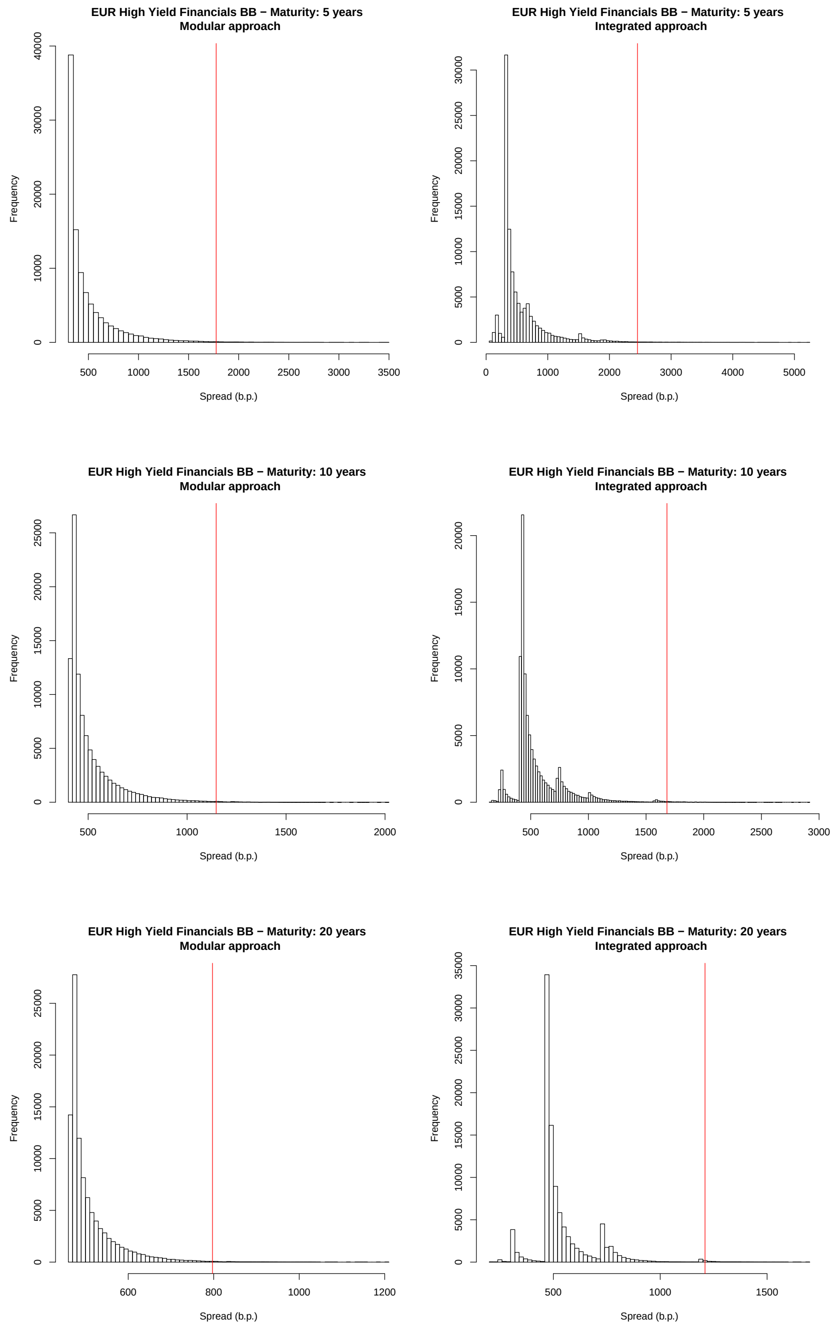


Figure 5.33. Probability distribution forecast of credit spreads $\sigma_h(T, T+\tau)$ with $T = 1$ and $\tau = 5, 10, 20$ years with modular (left) and integrated (right) approaches. The red line denotes the quantile at 99.5% - *EUR High Yield Financials BB*.

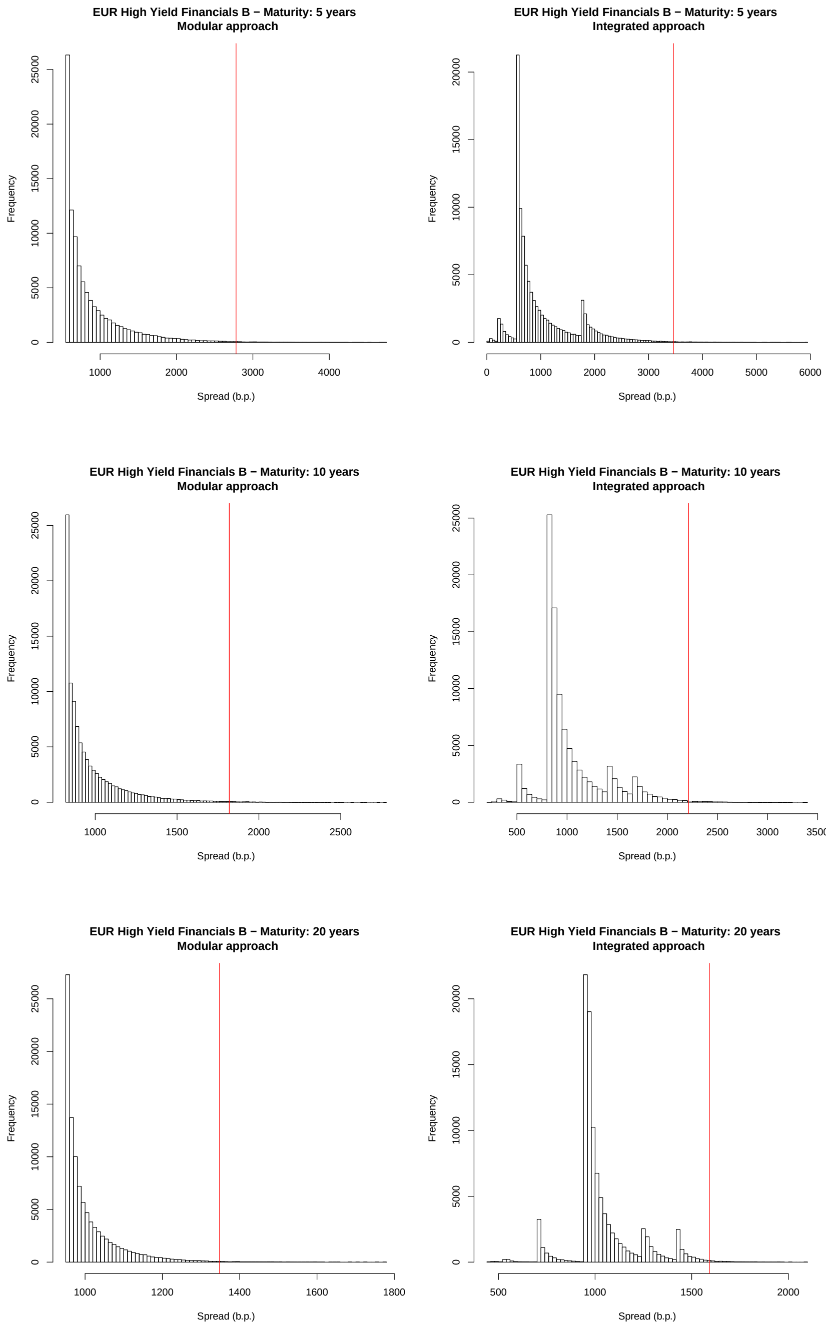


Figure 5.34. Probability distribution forecast of credit spreads $\sigma_h(T, T+\tau)$ with $T = 1$ and $\tau = 5, 10, 20$ years with modular (left) and integrated (right) approaches. The red line denotes the quantile at 99.5% - *EUR High Yield Financials B*.

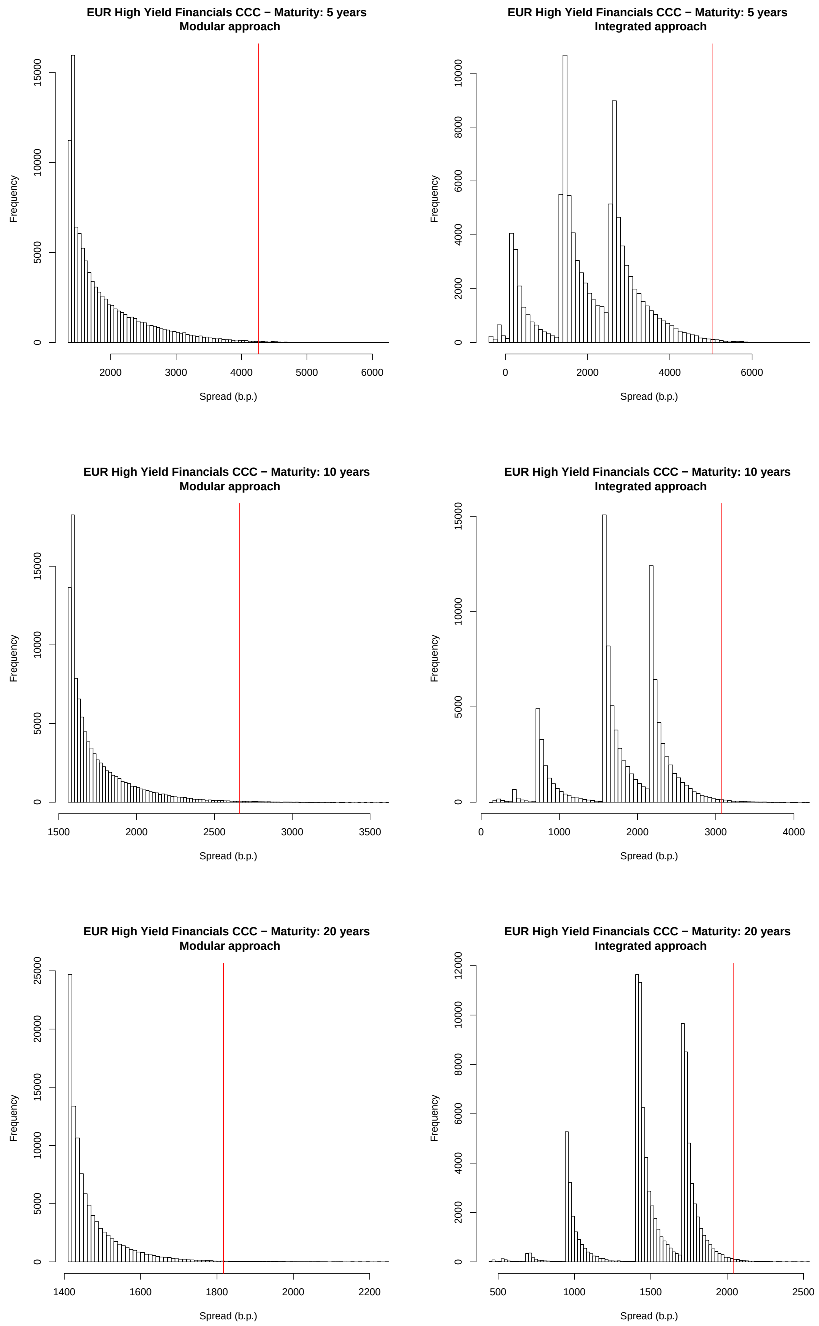


Figure 5.35. Probability distribution forecast of credit spreads $\sigma_h(T, T+\tau)$ with $T = 1$ and $\tau = 5, 10, 20$ years with modular (left) and integrated (right) approaches. The red line denotes the quantile at 99.5% - *EUR High Yield Financials CCC.*

Chapter 6

Conclusions

The model proposed in this work, in its different configurations, allows the issue of rating transitions to be explicitly addressed within a model for spread risk, as defined in *Solvency II*. Corporate credit rating may change over time according to the improvement or deterioration of credit quality and is considered the fundamental variable that drives the default process.
Regarding the comparison of the quality of the estimates obtained from the three different versions of the model for the spread risk treated (model with one subordinator process, model with two subordinator processes, and full model with two subordinator processes and a rating-specific liquidity component), it is evident that the model with a subordinator process fails to satisfactorily fit the market data, and that the extension to a second subordinator process has a positive effect in terms of goodness of fit. However, the model should not be evaluated solely in terms of goodness of fit, but should be considered within the scope of the analysis, in this case a partial internal model for calculating the *Solvency Capital Requirement* for the spread risk sub-module. Therefore, it is required to define a model that can capture credit spread trends, especially the most extreme ones, for all rating classes considered. Indeed, in a model with an integrated approach, in which the different components of credit risk, including migration and default risks, are considered, the credit spreads for each risk class considered contribute to the determination of the SCR for a given rating class. To this end, the extended version of the model with two subordinator processes including a rating-specific liquidity component has been proposed; it returned an excellent goodness of fit, and makes it possible to capture the particular trends in credit spreads of different rating classes, particularly the different crises observed during the period considered.
This extension has been proposed in accordance with the principles of prudence and parsimony in the size of the parameter space required by the regulations.
The proposed model has the drawback of being more difficult to understand than classic reduced-form models for credit risk, because of the use of one or more subordinator processes in the definition of the rating transition matrix, which also makes the financial interpretation of model parameters complicated. However, it is not contrary to the regulations that require governability and awareness on the part of management in the use of an internal model.

In terms of *Solvency Capital Requirement*, the proposed model is more prudential than the *Standard Formula* and consistent with the findings of the *Market and Credit Risk Comparative Studies* proposed by EIOPA. The greater prudentiality of the proposed model, as described in the previous chapters, can be attributed to the estimation procedure used, which does not include smoothing techniques, and to the *IHS Markit* `iBoxx` indices used in the estimation, which provide credit spreads with particularly high averages and volatilities, when compared with those used for the *Standard Formula* calibration.

The application framework made it necessary to estimate the model under both probability measures involved in calculating the SCR, the *real-world* measure for the evolution of risk factors over a 1-year time horizon and the risk-neutral measure for pricing securities from the projected risk factors. This framework required particular estimation techniques on time series, and in particular filtering techniques were used. The filtering techniques, in the different configurations used in this work (particle filtering with Gauss-Legendre quadrature techniques, particle filtering with Sequential Importance Reampling algorithm, Kalman filter), were found to be an effective and flexible tool for estimating the models considered and returned substantially good estimates. Indeed, the full model excellently replicates the data used for the estimation and projects consistently with the *Standard Formula* and observed market dynamics.
One of the advantages of using filtering techniques is the ability to process a very large amount of input data, as seen in this work. In fact, the structure filtering techniques allows exceptional savings in memory space: since they only operate on date pairs, once the estimate is updated, the memory location containing the value can be cleaned out and overwritten. This makes it possible to apply the estimation procedure to time series with high depth. However, filtering techniques also have some drawbacks. The high computational complexity of the problem addressed through the use of filtering techniques and the very large amount of data processed result in a significant increase in estimation time compared with cross-sectional estimation procedures, which consider a single date. Moreover, the estimation results may be affected by the choice of the depth of the time series.
In the case of particle filtering, the high computational complexity is also due to the numerical calculation of the integrals present in the filter formulas. In this work, two different techniques are used for the numerical approximation of the integrals: the Gauss-Legendre quadrature approximation and the Sequential Importance Resampling algorithm, based on Monte Carlo simulation techniques.

Further detailed analyses concerning the use of a stochastic recovery rate and different configurations of the simulation of rating transitions for a portfolio of risky securities are deferred to future research work. An additional relevant topic deferred to future work is the development of a procedure for backtesting the obtained estimations, which compares the observed values with the distribution estimated by the model. This issue is of primary importance for supervisory approval of the internal model, either partial or full.

Appendix A

iBoxx Rating Methodology

IHS Markit uses information provided by the three major rating agencies - *Standard & Poor's Global Ratings*, *Moody's Investors Services*, and *Fitch Ratings* - to calculate the rating of an `iBoxx` index.

If a bond is rated by several agencies, then the average rating is attached to the bond. If a tranche is not rated, the rating of its parent is applied. The rating is consolidated to the nearest rating grade.

Investment grade is defined as BBB- or higher from *Fitch Ratings* and *Standard & Poor's Global Ratings*, and Baa3 or higher from *Moody's Investor Services*. If at least one of the above credit rating agencies provides "D" ("default") or "SD" ("selective default") rating, all available ratings from the agencies are consolidated into the "D" `iBoxx` rating.

The `iBoxx` average rating is determined as the average of the ratings of the three credit rating agencies where available: the available credit ratings are converted into scores according to Table A.1, the numerical average of all scores is calculated and rounded to the nearest integer, and finally the rounded average score is converted back into the `iBoxx` index rating according to Table A.2.

Standrd & Poor's	**Moody's**	**Fitch**	**Score**
AAA	Aaa	AAA	1
AA+	Aa1	AA+	2
AA	Aa2	AA	3
AA-	Aa3	AA	4
A+	A1	A+	5
A	A2	A	6
A-	A3	A-	7
BBB+	Baa1	BBB+	8
BBB	Baa2	BBB	9
BBB-	Baa3	BBB-	10
BB+	Ba1	BB+	11
BB	Ba2	BB	12
BB-	Ba3	BB-	13
B+	B1	B+	14
B	B2	B	15
B-	B3	B-	16
CCC+	Caa1	CCC+	17
CCC	Caa2	CCC	18
CCC-	Caa3	CCC-	19
CC	Ca	CC	20
C	C	C	21
D/RD		D	22

Table A.1. Correspondence between the ratings of the three rating agencies - *Standard & Poor's Global Rating*, *Moody's Investors Services*, and *Fitch Ratings* - and the numerical scores - Source: *IHS Markit* - iBoxx Rating Methodology.

Score	**iBoxx rating**
1	AAA
2 3 4	AA
5 6 7	A
8 9 10	BBB
11 12 13	BB
14 15 16	B
17 18 19	CCC
20	CC
21	C
22	D

Table A.2. Correspondence between the numerical scores and the iBoxx ratings - Source: *IHS Markit* - iBoxx Rating Methodology.

Appendix B

Poisson Processes

Poisson processes are the purely jump analogous of the Brownian motion and play a key role in modeling default. Like the Brownian motion, Poisson processes belong to the family of Levy processes, *i.e.*, particular cases of processes with stationary independent increments and with right continuous and left limit paths.

B.1 Time homogeneous Poisson Processes

A time homogeneous Poisson process is a unit-jump increasing, right continuous process M_t with the following properties:

- independent increments, for any $0 < t < s < u$:

$$M_u(\omega) - M_s(\omega) \text{ independent of } M_s(\omega) - M_t(\omega)\,; \tag{B.1}$$

- stationary increments, for any $0 < t < s$ and any $h > 0$:

$$M_{s+h}(\omega) - M_{t+h}(\omega) \sim M_s(\omega) - M_t(\omega)\,; \tag{B.2}$$

- $M_0 = 0$,

where ω is the experiment result.
Further properties of time homogeneous Poisson processes are as follows:

- there exists a positive number $\overline{\gamma} \in \mathbb{R}$ such that $P\{M_t = 0\} = e^{-\overline{\gamma}t}$ for all t, *i.e.*, the probability of having no jumps up to some given time t is an exponential function of minus that time;
- $\lim_{t\to 0} P\{M_t \geq 2\}/t = 0$, *i.e.*, the probability of having more than one jump in an arbitrary small time going to zero goes to zero faster then the time itself;
- $\lim_{t\to 0} P\{M_t = 1\}/t = \overline{\gamma}$, *i.e.*, the probability of having exactly one jump in an arbitrary small time, re-scaled by the time itself, is the constant $\overline{\gamma}$;
- $P\{M_s - M_t = k\} = e^{-\overline{\gamma}(s-t)}(\overline{\gamma}(s-t))^k/k!$, *i.e.*, the number of jumps of a Poisson process follows the Poisson law;
- $P\{\tau \in [t, t+dt] | \tau \geq t\} = \overline{\gamma}dt$, *i.e.*, $\overline{\gamma}$ can be interpreted as the probability of having a new jump at time t given there have not been any before t.

B.2 Time inhomogeneous Poisson Processes

Let M_t be a standard Poisson process and Γ_t the cumulated intensity or cumulated hazard rate:

$$\Gamma_t = \int_0^t \gamma(u)\, du\,, \tag{B.3}$$

with a deterministic time-varying intensity $\gamma(t)$; then a time inhomogeneous Poisson process N_t with intensity $\gamma(t)$ is defined as

$$N_t = M_{\Gamma_t}. \tag{B.4}$$

Essentially, a time inhomogeneous Poisson process is just a time-changed standard Poisson process.
The time inhomogeneous Poisson process still has independent but no longer stationary increments, due to the time distortion introduced by Γ.
By construction, the process N jumps the first time at τ if and only if the process M jumps the first time at Γ_τ, then, since M is a standard Poisson process for which the first jump time is exponentially distributed:

$$\Gamma_\tau := \xi \sim \text{ exponential}(1). \tag{B.5}$$

By inverting equation B.5 it can be obtained:

$$\tau = \Gamma^{-1}(\xi)\,; \tag{B.6}$$

hence, the probability of first jump between t and s is:

$$\begin{aligned} P\left\{t < \tau < s\right\} &= P\left\{\Gamma_t < \Gamma_\tau < \Gamma_s\right\} = P\left\{\Gamma_t < \xi < \Gamma_s\right\} \\ &= P\left\{\xi > \Gamma_t\right\} - P\left\{\xi > \Gamma_s\right\} = e^{-\Gamma_t} - e^{-\Gamma_s} \\ &= e^{-\int_0^t \gamma(u)\, du} - e^{-\int_0^s \gamma(u)\, du} \\ &= e^{-\int_0^t \gamma(u)\, du}\left(1 - e^{-\int_t^s \gamma(u)\, du}\right). \end{aligned} \tag{B.7}$$

Similarly to what was seen for the time homogeneous case, the probability of having a new jump at time t given that there have not been any before t is:

$$P\left\{\tau \in [t, t+dt] | \tau \geq t\right\} = \gamma(t)dt. \tag{B.8}$$

Appendix C

The embedding problem and the *Quasi-Optimization* of the generator matrix

In risk management practices and reduced-form pricing models based on Markov chains, the knowledge of the annual transition matrix, which is provided by several specialized agencies, may be insufficient. But computing a fractional root of an annual transition matrix is a badly placed problem: in fact, it may return invalid (with negative values) or non-unique transition matrices.

One approach of obtaining transition matrices for periods of arbitrary length involves embedding a discrete-time Markov chain into a continuous-time Markov chain. For a continuous-time Markov chain, in fact, any transition matrix can be expressed as the exponential of the generator (like in Equation 2.61). Therefore, solving the embedding problem essentially allows one to find a generator consistent with the annual transition matrix of the discrete-time Markov chain. However, computing the generator of an existing transition matrix by taking its logarithm still raises the problem of existence and uniqueness. In fact, observed transition matrices typically have characteristics that preclude the existence of the generator (Israel et al. (2001) [34]). Alternatively, more than one generator matrix could be associated with the same transition matrix (Kingman (1962), Carette (1995)). Moreover, the problem of non-uniqueness of the generator also compromises the no-arbitrage property assumed by the models; in fact, this relies on the existence and uniqueness of an equivalent martingale measure, which in Markov credit migration models cannot be independent of the existence and uniqueness of the transition matrix generator.

However, there are regularization algorithms that make it possible to compute the generator that returns the best approximation of the original transition matrix, without running into the problems of existence and uniqueness.

C.1 The embedding problem

Suppose that $\{C_t\}_{t\in\mathbb{R}^+}$ is a time-homogeneous, continuous-time Markov chain with finite state space $\mathcal{K} = \{1, \dots K\}$ under some probability measure $\mathbb{P}$. Suppose that the transition probability matrix for C corresponding to time $t = 1$, $\mathbf{P}(1) = [p_{ij}(1)]_{1\le i,j\le K}$ is given, where, for every $i, j = 1, \dots K$ and every $t \in \mathbb{R}^+$:

$$p_{ij}(1) = P\{C_1 = j \mid C_0 = i\} = P\{C_{t+1} = j \mid C_t = i\}. \tag{C.1}$$

The classic embedding problem for a Markov chain C relative to $\mathbb{P}$ can be summarized as follows: find a $K \times K$ matrix $\hat{\mathbf{\Lambda}} = [\hat{\lambda}_{ij}]_{1\le i,j\le K}$ with:

- non-negative off-diagonal entries: $\hat{\lambda}_{ij} \ge 0$ for every $i, j \in \mathcal{K}$ with $i \ne j$;
- all rows summing to 0: $\hat{\lambda}_{ii} = -\sum_{j\ne i} \hat{\lambda}_{ij}, \quad \forall i \in \mathcal{K}$,

such that $e^{\hat{\mathbf{\Lambda}}} = \mathbf{P}(1)$.

In case of a time-homogeneous, continuous-time Markov chain C, if the transition probability matrix function $\mathbf{P}(t)$, $t \in \mathbb{R}^+$ satisfies some mild regularity conditions, then there exists a unique infinitesimal generator matrix $\mathbf{\Lambda}$ such that $\mathbf{P}(t) = e^{t\mathbf{\Lambda}}$. Consequently, one of the solutions of the embedding problem for C relative to $\mathbb{P}$ in this case is $\hat{\mathbf{\Lambda}} = \mathbf{\Lambda}$, so that the infinitesimal generator matrix is a solution of the embedding problems. Unfortunately, it is known that the embeddability problem is nearly unavoidable in credit risk modeling. Israel et al. (2001) [34] provide several necessary conditions for the non-existence of an exact valid generator. One of this is: if there are states i and j such that j is accessible from i, but $p_{ij} = 0$, an exact generator matrix does not exist for the transition probability matrix $\mathbf{P}$.
This condition is likely to hold for the majority of empirical rating transition matrices. For example, high investment grades tend to exhibit zero default probability in the empirical transition probability matrix, even if the true probability is not zero. However, default state is accessible from the same high investment grades if successive downgrades are considered. Hence, the above condition is almost unavoidable and a simple matrix logarithm of an empirical transition matrix is very likely to contain negative off-diagonal elements.

C.2 The *Quasi-Optimization* of the generator matrix

There are several ways to cope with the embeddability problem. One way is to adjust the matrix logarithm of $\mathbf{P}$ such that the adjusted $\mathbf{\Lambda}$ represents a valid generator matrix. Specifically, the approach first sets the negative off-diagonals to zero and then adds the extra value to the other elements to compensate (see Inamura (2006) [33]). There are also a variety of numerical procedures, from ad-hoc adjustments to optimization-based adjustment, as shown in [34] and in [45].
Kreinin and Sidelnikova (2001) [45] present an algorithm of *Quasi-Optimization* of the generator matrix. The regularization problem can be described as follows: find a generator matrix $\mathcal{X}$ that is a valid generator and, when raised to the power

t, most closely matches the annual transition probability matrix $\mathbf{P}$. The set of generator matrices, $\mathbf{G}(K)$, consisting of all matrices of dimension $K \times K$ that are valid generator matrices, is defined. The problem of *Quasi-Optimization* of the generator matrix is: find $\hat{\mathbf{\Lambda}} \in \mathbf{G}(K)$ such that:

$$\|\hat{\mathbf{\Lambda}} - \ln \mathbf{P}\| = \min_{\mathcal{X} \in \mathbf{G}(K)} \{\|\mathcal{X} - \ln(\mathbf{P})\|\}. \tag{C.2}$$

The space of the generator matrices, $\mathbf{G}(K)$, is a Cartesian product of K-dimensional cones. Each row of a generator has the property that its elements sum to zero and non-diagonal elements are non-negative. By permuting the row elements, one can always represent them as a point in a standard cone, $\gamma(K)$, defined as:

$$\gamma(K) = \left\{(x_1, \ldots, x_K) \in \mathbb{R}^K, \sum_{i=1}^{K} x_i = 0,\ x_1 \leq 0,\ x_i \geq 0,\ \text{for } i \geq 2\right\}. \tag{C.3}$$

Note that $\gamma(K)$ is contained in the hyperplane $H(K)$:

$$H(K) = \left\{(x_1, \ldots, x_K) \in \mathbb{R}^K,\ \sum_{i=1}^{K} x_i = 0\right\}. \tag{C.4}$$

The problem of *Quasi-Optimization* of the generator matrix can be solved on a row by row basis by projecting a point $a \in \mathbb{R}^K$, *i.e.*, a row of the matrix $\ln(\mathbf{P})$, onto the cone defined in C.3. Thus, the problem can be reduced to K independent instances of the following distance minimization problem: for a given point $a \in \mathbb{R}^K$, $a = (a_1, \ldots, a_K)$, find $g^* \in \gamma(K)$ such that:

$$\text{dist}(a, g^*) = \min_{g \in \gamma(K)} \{\text{dist}(a, g)\}. \tag{C.5}$$

The optimal solution to this problem can be obatained as follows:

1. let b be the projection of a on $H(K)$: set $b_i = a_i - \lambda$, where $\lambda = \frac{1}{K}\left(\sum_{i=1}^{K} a_i\right)$;

2. let $a = \pi(b)$, where π is a permutation that orders the coordinates of b in descending sequence;

3. find l^*, that is the smallest integer $2 \leq l \leq K-1$ that satisfies:

$$(K - l + 1)a_{l+1} \geq \sum_{i=0}^{K-(l+1)} a_K - i\,;$$

4. define $\mathcal{B} = \{i : 2 \leq i \leq l^*\}$. Construct the vector $g \in \gamma(K)$ as follows:

$$g_i = \begin{cases} 0 & i \in \mathcal{B}, \\ a_i - \frac{1}{(K-l^*+1)} \sum_{j \notin \mathcal{B}} a_j & \text{otherwise;} \end{cases}$$

5. apply the inverse permutation π^{-1} to g. $\pi^{-1}(g)$ is the solution to problem.

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