Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

ARTICLE TYPE

# From local to global feedback controllers: a symmetry-based approach

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**Abstract**

We introduce a novel procedure for designing global state- or output-feedback controllers for ordinary differential nonlinear systems (ODE) from the knowledge of a local state- or output-feedback controller and a symmetry of the system. Symmetries are one-parameter transformations which leave invariant the structure of the system by mapping solutions into other (parametrized) solutions. We use particular types of symmetry, called contracting symmetry, which uniformly contract solutions into arbitrary small solutions around the origin. This contracting action allows to obtain global feedback controllers from local feedback controller by inverse transformation via the symmetry.

**KEYWORDS:**
ordinary differential nonlinear systems, symmetries, local and global state- and output- feedback controllers

## 1 | INTRODUCTION

Semiglobal and global stabilization via output-feedback controllers is a longstanding problem. A commonly adopted strategy (though not the only one) is to design a state-feedback controller and a state observer separately and then combine them into an output-feedback controller using a *certainty equivalence principle*. The main technical challenges in accomplishing this task are 1) the separate design of semiglobal or global state-feedback controllers and observers 2) the fact that the system may not in general *input-to-state stability* properties, so that, when combining the state-feedback controller and the state observer, even a bounded and converging state estimation error may not guarantee a bounded and convergent state. Locally these issues do not arise and local output-feedback controllers can be readily designed based on the certainty equivalence principle, especially when the system linearization is controllable and observable. In this paper, we study the problem of designing *semiglobal and global* output feedback controllers, with asymptotic, finite or prescribed-time convergence, starting from the availability of a *local* output feedback controller. We address this task by separately considering a *local* state-feedback controller and a *local* state observer, and we show how these can be transformed into semiglobal or global ones via the inverse of a *contracting symmetry*. Symmetries are ubiquitous in physical systems. A contracting symmetry is a one-parameter group of transformations that leaves the system invariant and uniformly contracts its solutions into arbitrary small neighbourhoods of the origin. Although this class of symmetries has already been introduced in the recent works[7] and[9] the results presented in this paper differs substantially from the aforementioned references. The contributions of this paper with respect to[7] and[9] are threefold:

[0]**Abbreviations:** ODE, ordinary differential equations; LGT (GGT), local (global) one-parameter group of $C^\infty$-transformations; LS (GS) local (global) symmetry; SI, state-input contracting symmetry.

- [7] and [9] focus on observer design. Here, we address the control design problem by introducing control inputs and seeking output-feedback controllers. This requires extending the domain over which the symmetry acts; in particular, the control input space must be included.

- we reverse the perspective of [7] and [9] where the system is assumed to be detectable in the first approximation (i.e., a linear observer can be designed for the linearization). Instead, we assume the existence of a local observer for the system, not necessarily designed based on its linearization. A detailed technical comparison is provided in Section 6.1.1.

- The results of this paper go beyond a simple comparison with the existing literature, establishing a novel methodology for transforming a local output-feedback controller (easier to design) into a semiglobal or global output-feedback controller (more difficult to obtain). This kind of results are not even available in the literature.

In Sections 3-4 we recall from [19] and [9] the main notions and results on one-parameter groups, prolongations, and symmetries, including contracting symmetries, i.e., symmetries with a contractive action on both the state and input spaces. We include benchmark examples aimed at illustrating computational and constructive aspects, connections with classical notions (e.g., homogeneity), and the impact on problems related to time-scale transformations (asymptotic or finite-time convergence), as well as state and input space transformations. We further develop several theoretical and computational aspects introduced in [7] and present new ones. This is also instrumental in illustrating the different control design procedures studied in this paper and comparing them with existing approaches.

The main contributions with respect to the existing literature can be summarized as follows:

- In Sections 5.1 and 6.1 we prove several results showing how to transform, via a contracting symmetry, a local controller/observer into semiglobal controllers/observers (Theorems 1 and 3) with asymptotic convergence over an infinite time horizon. In Section 6.1.1 we establish the connections with [9], by discussing the case of system's observability in the first approximation. by discussing the case of observability in the first approximation. On benchmark examples, we compare our semiglobal symmetry-based controllers/observers —derived from local ones —with classical semiglobal high-gain controllers/observers designed directly for the system (see, e.g., Theorem 12.1.1 and Corollary 12.1.2 of [13] or the paper [14] for state-feedback controllers and [1], [14] or [12] for state observers). For the class of triangular systems, we show that a semiglobal feedback controller can always be constructed from a local one, since a contracting symmetry can always be defined for this class (see Section 4.2.3 and the examples following Theorems 1 and 3). This also establishes connections —within the feedback stabilization framework —with more recent approaches for triangular systems based on fuzzy methods (e.g., [2]). However, such approaches typically require state trajectories to remain within a known compact set where the nonlinearities are approximated by basis functions, hence they are not helpful with unbounded state trajectories.

- In Sections 5.2 and 6.2, for linear systems we prove several results showing how to transform, via a (not necessarily contracting) symmetry, a controller/observer with asymptotic convergence into one with *finite- or prescribed- time convergence* (Theorems 2 and 4). On benchmark examples, we compare our symmetry-based controllers/observers with discontinuous sliding-mode controllers/observers ([16], [11]) or smooth time-varying controllers/observers ([21]) designed directly for the system. More generally, for linear systems we show that it is always possible to construct a feedback controller with finite- or prescribed-time convergence starting from one with asymptotic convergence. To the best of our knowledge, no result of comparable generality exists in the literature for linear systems. Theorems 2 and 4 can also be extended to nonlinear triangular systems, at the cost of additional technical complexity (see, e.g., [8] and [21] for prescribed-time and [3] for finite-time with fuzzy-based methods).

- In Section 6.3 we consider systems with unbounded solutions, which represent an even more challenging setting for output-feedback stabilization. We study how to transform a local observer into a global one under an additional condition known as *unboundedness observability* (UO: [22]) , which is a well-known *necessary* condition for the existence of a global observer. In this case, the symmetry group parameter must be time-varying to track the unbounded growth of the state norm. This approach —based on tracking the state norm via a norm estimator —was previously explored in [7] but under stronger assumptions such as output-to-state stability (OSS: [22]). In Example 6.1 we present cases where our global symmetry-based controllers/observers can be successfully designed, while existing approaches based on high-gain observers, sliding-mode observers or fuzzy methods are not applicable due to the unboundedness of the state trajectories.

We consider ODE systems of the form [1]

$$D_t s(t) = F(t, s(t), u(t), d(t)) \tag{1}$$

with states $s(t) \triangleq (y(t), x(t)) \in \mathbb{R}^n$, measured outputs $y(t) \in \mathbb{R}^p$, unmeasured states $x(t) \in \mathbb{R}^{n-p}$ control inputs $u(t) \in \mathbb{R}^m$ and exogenous (in general unknown) inputs $d(t) \in \mathbb{R}^r$. We assume $C^\infty$ functions $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^n$.

**Notational remark**. *We adopt the following notation with regard to* (1) *and its variables:* [2] $t \in \mathbb{R}$ *is the time variable,* $z(t) \triangleq (s(t), u(t), d(t))$ *denote the state, exogenous input, control input and output functions of* $t$ *while* $\mathsf{z} \triangleq (\mathsf{s}, \mathsf{d}, \mathsf{y})$ *represent the coordinates of the state, exogenous input, output space* $\mathcal{Z} \triangleq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ *in such a way that we write* $\mathsf{z} = z(t)$ *with the obvious meaning. Moreover,* $\mathcal{M} \triangleq \mathbb{R} \times \mathcal{Z}$ *is the overall coordinate space for time, states and control and exogenous inputs.*

## 2 | NOTATION I

⊳ (*vector spaces*). Besides the standard notation $\mathbb{R}^n$, we denote by $\mathbb{R}^{n\times s}$ the set of $n \times s$ real matrices. $\mathbb{R}^0_+$ (resp. $\mathbb{R}_+$) denotes the set of non-negative (resp. positive) real numbers. For any vector $\mathsf{v} \in \mathbb{R}^s$ and matrix $A \in \mathbb{R}^{n\times s}$ we denote by $\mathsf{v}_i \in \mathbb{R}$, resp. $(A\mathsf{v})_i \in \mathbb{R}$, the $i$-th component of $\mathsf{v}$, resp. $A\mathsf{v}$. Moreover, $0_{n\times s}$, resp. $I_n$, denote the zero matrix in $\mathbb{R}^{n\times s}$, resp. the identity matrix in $\mathbb{R}^{n\times n}$, and $\mathrm{diag}\{\mathsf{v}_1, \dots, \mathsf{v}_n\}$ or simply $\mathrm{diag}_i\{\mathsf{v}_i\}$ is the diagonal matrix with diagonal entries $\mathsf{v}_1, \dots, \mathsf{v}_n \in \mathbb{R}$. $\mathbb{S}_+(n)$ (resp. $\mathbb{S}_-(n)$) is the set of symmetric positive (resp. negative) definite matrices $S \in \mathbb{R}^{n\times n}$ and $\mathbb{GL}(n)$ is the set of nonsingular matrices $S \in \mathbb{R}^{n\times n}$.
⊳ (*norms*). $|\mathsf{v}|$ denotes the absolute value of $\mathsf{v} \in \mathbb{R}$, $\|\mathsf{v}\| \triangleq \sqrt{\mathsf{v}^\top \mathsf{v}}$ denotes the euclidean (or 2-) norm of $\mathsf{v} \in \mathbb{R}^n$ and the induced norm of $S \in \mathbb{R}^{m\times n}$ is $\|S\| \triangleq \sup_{\mathsf{v}\in\mathbb{R}^n}(\|S\mathsf{v}\|/\|\mathsf{v}\|)$. We identify a vector $\mathsf{v} \in \mathbb{R}^r$ with the $r$-uple $(\mathsf{v}_1, \dots, \mathsf{v}_r)$ and by $\|(\mathsf{v}_1, \dots, \mathsf{v}_r)\|$ we mean the norm of $\mathsf{v} \in \mathbb{R}^r$. For a continuous function $S : \mathbb{R}^0_+ \to \mathbb{R}^{m\times n}$ its sup-norm (or $\infty$-norm) is defined as $\|S\|_\infty = \sup_{t\geq 0} \|S(t)\|$.
⊳ (*monotone functions*). Let $\mathcal{K}_+$ (resp. $\mathcal{K}$, resp. $\mathcal{K}_\infty$) be the set of continuous strictly increasing functions $f : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ such that $f(0) > 0$ (resp. $f(0) = 0$, resp. $f(0) = 0$ and $\lim_{s\to+\infty} f(s) = +\infty$). Let $\mathcal{L}$ be the set of continuous decreasing functions $f : \mathbb{R}^0_+ \to \mathbb{R}_+$ such that $\lim_{s\to+\infty} f(s) = 0$. Finally, let $\mathcal{KL}$ (resp. $\mathcal{K}_+\mathcal{L}$) be the set of continuous functions $f : \mathbb{R}^0_+ \times \mathbb{R}^0_+ \to \mathbb{R}^0_+$ such that $f(\cdot, s) \in \mathcal{K}$ (resp. $f(\cdot, s) \in \mathcal{K}_+$) for each $s \geq 0$ and $f(r, \cdot) \in \mathcal{L}$ for each $r \geq 0$.
⊳ (*saturation functions*). A locally Lipschitz function $(\mathsf{c}, \mathsf{v}) \in \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathrm{sat}_{\mathsf{c}}(\mathsf{v}) \triangleq \left(\mathrm{sat}_{\mathsf{c}}(\mathsf{v}_1) \ \cdots \ \mathrm{sat}_{\mathsf{c}}(\mathsf{v}_n)\right) \in \mathbb{R}^n$ is a *saturation function* if $\mathrm{sat}_{\mathsf{c}}(\mathsf{v}_i) = \mathsf{v}_i$ for all $\mathsf{v}_i \in [-\mathsf{c}, \mathsf{c}]$, $i = 1, \dots, n$, and $|\mathrm{sat}_{\mathsf{c}}(\mathsf{v}_i)| \leq \mathsf{c}$ and $|\mathrm{sat}_{\mathsf{c}}(\mathsf{v}_i) - \mathrm{sat}_{\mathsf{c}}(\mathsf{w}_i)| \leq |\mathsf{v}_i - \mathsf{w}_i|$ for all $\mathsf{v}_i, \mathsf{w}_i \in \mathbb{R}$, $i = 1, \dots, n$.

## 3 | GROUP OF TRANSFORMATIONS AND PROLONGATIONS

The paper considers families of nonlinear transformations $\Psi(\mathfrak{p}, \cdot)$ acting on a coordinate space $\mathcal{M}$ that includes time, states, inputs, and outputs. These transformations are parametrized by $\mathfrak{p} \in \mathbb{R}$ and are endowed with the structure a group, taking the form

$$\Psi(\mathfrak{p}, t, \mathsf{z}) \triangleq (\Psi^t(\mathfrak{p}, t), \Psi^{\mathsf{y}}(\mathfrak{p}, t, \mathsf{y}), \Psi^{\mathsf{x}}(\mathfrak{p}, t, \mathsf{s}), \Psi^{\mathsf{u}}(\mathfrak{p}, t, \mathsf{y}, \mathsf{u}), \Psi^{\mathsf{d}}(\mathfrak{p}, t, \mathsf{s}, \mathsf{d})), \tag{2}$$

where $\mathsf{z} \triangleq (\mathsf{s}, \mathsf{d}, \mathsf{y})$. Here, $\Psi^t$ transforms the time scale, $\Psi^{\mathsf{y}}$ transforms the measured variables, $\Psi^{\mathsf{x}}$ transforms the unmeasured state variables, $\Psi^{\mathsf{u}}$ transforms the control input variables and $\Psi^{\mathsf{d}}$ transforms the exogenous input variables. [3]

### 3.1 | One-parameter group of transformations

To endow the family of transformations (2) with a group structure, we borrow from [19] the definition of one-parameter group of transformations.

[1] $D_t x(t)$ denotes the derivative of the function $x(t)$ w.r.t. $t$ and it is used in place of the more classical notation $\dot{x}(t)$ for underlining the actual time scale with respect to which we are differentiating the function $x(t)$: if $\tau = \mathbb{T}(t)$ is any other continuous time scale (i.e. a strictly increasing continuous function of $t$), $D_\tau \widetilde{x}(\tau)$ denotes the derivative of $\widetilde{x}(\tau) \triangleq x(\mathbb{T}^{-1}(\tau))$ (i.e. the function $x(t)$ in the new time scale $\tau$) w.r.t. the new time scale $\tau$.

[2] This notation is extensively used in standard textbooks on symmetries such as [19].

[3] More general time scales transformations of the form $\Psi^t(\mathfrak{p}, t, \mathsf{x})$ can be considered, including for example Lorentz transformations and, more generally, Poincaré groups of transformations on the time/space domain.

**Definition 1.** (*One-parameter group of transformations*). A *local one-parameter group of $C^\infty$-transformations* (LGT) $\Psi$ acting on $\mathcal{M}$ consists of an open subset $\mathcal{V}$ with $\{0\}\times\mathcal{M}\subset\mathcal{V}\subset\mathbb{R}\times\mathcal{M}$ and a $C^\infty$ mapping $\Psi:\mathcal{V}\to\mathcal{M}$ such that
(i) (*identity*) $\Psi(0,t,\mathsf{z})=(t,\mathsf{z})$ for all $(t,\mathsf{z})\in\mathcal{M}$,
(ii) (*inversion*) if $(\mathfrak{p},t,\mathsf{z})\in\mathcal{V}$ then $(-\mathfrak{p},\Psi(\mathfrak{p},t,\mathsf{z}))\in\mathcal{V}$ and $\Psi(-\mathfrak{p},\Psi(\mathfrak{p},t,\mathsf{z}))=(t,\mathsf{z})$,
(iii) (*composition*) if $(\mathfrak{p}_1,\Psi(\mathfrak{p}_2,t,\mathsf{z}))\in\mathcal{V}$, $(\mathfrak{p}_2,t,\mathsf{z})\in\mathcal{V}$ and also $(\mathfrak{p}_1+\mathfrak{p}_2,t,\mathsf{z})\in\mathcal{V}$ then $\Psi(\mathfrak{p}_1,\Psi(\mathfrak{p}_2,t,\mathsf{z}))=\Psi(\mathfrak{p}_1+\mathfrak{p}_2,t,\mathsf{z})$. $\triangleleft$

The set $\mathcal{V}$, the domain of the mapping $\Psi$, is denoted by $\mathrm{Dom}(\Psi)$, and when $\mathrm{Dom}(\Psi)=\mathbb{R}\times\mathcal{M}$ the group of transformations is *global* (GGT). The underlying group of a LGT or GGT is always $(\mathbb{R},+)$, i.e. the additive group $\mathbb{R}$. [4] A simple example of LGT is the group of *nonlinear* (time) transformations

$$\Psi(\mathfrak{p},t,\mathsf{z})=(\frac{t}{1-t\mathfrak{p}},\mathsf{z}),\tag{3}$$

with $\mathrm{Dom}(\Psi)=\{(\mathfrak{p},t,\mathsf{z})\in\mathbb{R}\times\mathcal{M}:t\mathfrak{p}\neq 1\}$, while an example of GGT is the group of *linear* transformations

$$\Psi(\mathfrak{p},t,\mathsf{z})=(t,e^{\mathfrak{p}A^{\mathsf{y}}}\mathsf{y},e^{\mathfrak{p}A^{\mathsf{x}}}\mathsf{x},e^{\mathfrak{p}A^{\mathsf{u}}}\mathsf{u},e^{\mathfrak{p}A^{\mathsf{d}}}\mathsf{d})\tag{4}$$

for $A^{\mathsf{y}}\in\mathbb{R}^{p\times p}$, $A^{\mathsf{x}}\in\mathbb{R}^{(n-p)\times(n-p)}$, $A^{\mathsf{u}}\in\mathbb{R}^{m\times m}$ and $A^{\mathsf{d}}\in\mathbb{R}^{r\times r}$.

For each $\mathfrak{p}$, $\Psi(\mathfrak{p},\cdot)$ represents a (local) diffeomorphism. Indeed, for each $\mathfrak{p}$ let

$$\mathcal{V}(\mathfrak{p})\triangleq\{(t,\mathsf{z})\in\mathcal{M}:(\mathfrak{p},t,\mathsf{z})\in\mathrm{Dom}(\Psi)\}.$$

The set $\mathcal{V}(\mathfrak{p})$ is open and the mapping $\Psi_{\mathfrak{p}}:\mathcal{V}(\mathfrak{p})\to\mathcal{V}(-\mathfrak{p})$ with $\Psi_{\mathfrak{p}}(t,\mathsf{z})\triangleq\Psi(\mathfrak{p},t,\mathsf{z})$ is a $C^\infty$ transformation with $C^\infty$ inverse $\Psi_{\mathfrak{p}}^{-1}=\Psi_{-\mathfrak{p}}$, i.e. a diffeomorphism. The set $\mathcal{V}(\mathfrak{p})$ (resp. $\mathcal{V}(-\mathfrak{p})$), the domain of the mapping $\Psi_{\mathfrak{p}}$ (resp. $\Psi_{-\mathfrak{p}}$), is denoted by $\mathrm{Dom}(\Psi_{\mathfrak{p}})$ (resp. $\mathrm{Dom}(\Psi_{-\mathfrak{p}})$) for similarities with the notation $\mathrm{Dom}(\Psi)$. For the transformation (3) we have for each $\mathfrak{p}\in\mathbb{R}$

$$\mathrm{Dom}(\Psi_{\mathfrak{p}})=\{(t,\mathsf{z})\in\mathbb{R}\times\mathcal{M}:t\neq\frac{1}{\mathfrak{p}}\},\mathrm{Dom}(\Psi_{-\mathfrak{p}})=\{(t,\mathsf{z})\in\mathbb{R}\times\mathcal{M}:t\neq-\frac{1}{\mathfrak{p}}\}\tag{5}$$

and $\Psi_{\mathfrak{p}}:\mathrm{Dom}(\Psi_{\mathfrak{p}})\to\mathrm{Dom}(\Psi_{-\mathfrak{p}})$ is a diffeomorphism for each $\mathfrak{p}\in\mathbb{R}$.

**Notational remark**. *In what follows, we use interchangeably the notations $\Psi(\mathfrak{p},\cdot)$ and $\Psi_{\mathfrak{p}}(\cdot)$ to emphasize the action of the transformation $\Psi(\mathfrak{p},\cdot)$ on $\mathcal{M}$ for each $\mathfrak{p}$. When no ambiguity arises, we will further simplify the notation by omitting the parameter $\mathfrak{p}$ and writing simply $\Psi(\cdot)$.*

For a vector field $\mathbf{g}$ on $\mathcal{M}$ we denote by $\mathbf{g}|_{(t,\mathsf{z})}$ the tangent vector assigned by $\mathbf{g}$ at $(t,\mathsf{z})\in\mathcal{M}$, i.e. $\mathbf{g}|_{(t,\mathsf{z})}\in T_{(t,\mathsf{z})}\mathcal{M}\cong\mathcal{M}$. The tangent vector $\mathbf{g}|_{(t,\mathsf{z})}$ is represented in the basis $\{\frac{\partial}{\partial t}|_{(t,\mathsf{z})},\frac{\partial}{\partial \mathsf{z}_1}|_{(t,\mathsf{z})},\cdots,\frac{\partial}{\partial \mathsf{z}_{n+m+r}}|_{(t,\mathsf{z})}\}$ of the tangent space $T_{(t,\mathsf{z})}\mathcal{M}$ as [5]

$$\mathbf{g}|_{(t,\mathsf{z})}\triangleq g^t(t,\mathsf{z})\frac{\partial}{\partial t}|_{(t,\mathsf{z})}+\sum_{j=1}^{n+m+r}g^{\mathsf{z}_j}(t,\mathsf{z})\frac{\partial}{\partial \mathsf{z}_j}|_{(t,\mathsf{z})},$$

with associated vector function

$$g(t,\mathsf{z})=(g^t(t,\mathsf{z}),g^{\mathsf{z}_1}(t,\mathsf{z}),\cdots,g^{\mathsf{z}_{n+m+r}}(t,\mathsf{z})).$$

A LGT on $\mathcal{M}$ is the local flow generated by a suitable vector field on $\mathcal{M}$, the *infinitesimal generator* of the group.

**Definition 2.** (*Infinitesimal generator*). The *infinitesimal generator* of a $C^\infty$ LGT $\Psi_{\mathfrak{p}}$ on $\mathcal{M}$ is the $C^\infty$ vector field $\mathbf{g}$ on $\mathcal{M}$ such that $g(t,\mathsf{z})=\frac{\partial}{\partial\mathfrak{p}}\Psi_{\mathfrak{p}}(t,\mathsf{z})|_{\mathfrak{p}=0}$ at each point $(t,\mathsf{z})\in\mathcal{M}$. Moreover, for all $(\mathfrak{p},t,\mathsf{z})\in\mathrm{Dom}(\Psi)$

$$\frac{\partial}{\partial\mathfrak{p}}\Psi_{\mathfrak{p}}(t,\mathsf{z})=g(\Psi_{\mathfrak{p}}(t,\mathsf{z})),\ \Psi_0(t,\mathsf{z})=(t,\mathsf{z}).\triangleleft\tag{6}$$

The infinitesimal generator of (3) and, respectively, (4) are

$$\mathbf{g}|_{(t,\mathsf{z})}=t^2\frac{\partial}{\partial t}\tag{7}$$

$$\mathbf{g}|_{(t,\mathsf{z})}=\sum_{j=1}^{n}(A^{\mathsf{y}}\mathsf{y})_j\frac{\partial}{\partial\mathsf{y}_j}+\sum_{j=1}^{n-p}(A^{\mathsf{x}}\mathsf{x})_j\frac{\partial}{\partial\mathsf{x}_j}+\sum_{j=1}^{m}(A^{\mathsf{u}}\mathsf{u})_j\frac{\partial}{\partial\mathsf{u}_j}+\sum_{j=1}^{n}(A^{\mathsf{d}}\mathsf{d})_j\frac{\partial}{\partial\mathsf{d}_j}.\tag{8}$$

[4] In general, $\mathcal{M}$ can be replaced by any smooth manifold and the additive group $(\mathbb{R},+)$ by any (local) *Lie group* $\mathbb{G}$ endowed with an identity element $\mathfrak{e}\in\mathbb{G}$, a smooth composition law $m:\mathbb{G}\times\mathbb{G}\to\mathbb{G}$ and a smooth inversion law $i:\mathbb{G}\to\mathbb{G}$.

[5] This is a standard notation in differential geometry: see for instance [10] or [19]. Throughout the paper we omit the application point $(t,\mathsf{z})$ in each component of the basis $\{\frac{\partial}{\partial t}|_{(t,\mathsf{z})},\frac{\partial}{\partial\mathsf{z}_1}|_{(t,\mathsf{z})},\cdots,\frac{\partial}{\partial\mathsf{z}_{n+m+p}}|_{(t,\mathsf{z})}\}\in T_{(t,\mathsf{z})}\mathcal{M}$.

The general form of the infinitesimal generator of a one-parameter group (2) is

$$\mathbf{g}|_{(t,z)} \triangleq g^t(t)\frac{\partial}{\partial t} + \sum_{j=1}^{p} g^{\mathsf{y}_j}(t,\mathsf{y})\frac{\partial}{\partial \mathsf{y}_j} + \sum_{j=1}^{n-p} g^{\mathsf{x}_j}(t,\mathsf{s})\frac{\partial}{\partial \mathsf{x}_j} + \sum_{j=1}^{m} g^{\mathsf{u}_j}(t,\mathsf{y},\mathsf{u})\frac{\partial}{\partial \mathsf{u}_j} + \sum_{j=1}^{p} g^{\mathsf{d}_j}(t,\mathsf{s},\mathsf{d})\frac{\partial}{\partial \mathsf{y}_j}.$$

## 3.2 | Time scales transformation

In this section and for later use, we recall from[9] several situations in which $\Psi_{\mathfrak{p}}$ transforms the time variable $t$ or, more generally, a continuous time scale.[6] In order to proceed rigorously, for each $\mathfrak{p} > 0$ and assuming the set $\{a > 0 : [0,a) \subset \mathrm{Dom}(\Psi^t_{\mathfrak{p}})\}$ non-empty, define

$$\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = \sup\{a > 0 : [0,a) \subset \mathrm{Dom}(\Psi^t_{\mathfrak{p}})\}. \tag{9}$$

Clearly $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} > 0$ for all $\mathfrak{p} > 0$ and $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = +\infty$ when $\mathbb{R}^0_+ \subset \mathrm{Dom}(\Psi^t_{\mathfrak{p}})$. We set $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} \triangleq 0$ when the set $\{a > 0 : [0,a) \subset \mathrm{Dom}(\Psi^t_{\mathfrak{p}})\}$ is empty. If we apply the partial derivative $\frac{\partial}{\partial t}$ to both sides of the group generator equation (6)), interchange the order of differentiation $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \mathfrak{p}}$ each other and integrate over $[0,\mathfrak{p}]$, we obtain

$$\frac{\partial \Psi^t_{\mathfrak{p}}}{\partial t}(t) = e^{\int_0^{\mathfrak{p}} \frac{dg^t(\vartheta)}{d\vartheta}|_{\vartheta=\Psi^t_{\mathfrak{s}}(t)} d\mathfrak{s}} > 0$$

for each $\mathfrak{p} > 0$ and $t \in [0, \mathrm{Dom}^+\Psi^t_{\mathfrak{p}})$. Hence, for each $\mathfrak{p} > 0$, $\Psi^t_{\mathfrak{p}}$ is a monotonically increasing function of $t \in [0, \mathrm{Dom}^+\Psi^t_{\mathfrak{p}})$. Equivalently, $\Psi^t_{\mathfrak{p}}$ *transforms non-negative continuous time scales into continuous time scales* for each $\mathfrak{p} > 0$. However, even when $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = +\infty$ for $\mathfrak{p} > 0$ it does not necessarily hold that $\lim_{t\nearrow \mathrm{Dom}^+\Psi^t_{\mathfrak{p}}} \Psi^t_{\mathfrak{p}}(t) = +\infty$. For instance, consider the transformation $\Psi^t_{\mathfrak{p}}(t) = \frac{t}{1+\mathfrak{p}t}$. In this case, although the domain is unbounded, the transformed time remains bounded. In other words, unbounded (or bounded) nonnegative continuous time scales are not necessarily mapped into unbounded continuous time scales.

**Definition 3.** (*Time scale transformations*:). A LGT of the form (2)'s maps (continuous) non-negative time scales into (continuous) time scales for each $\mathfrak{p} > 0$.
▹ *Unbounded-to-unbounded (UU) transformations*: If $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = +\infty$ and $\lim_{t\to+\infty} \Psi^t_{\mathfrak{p}}(t) = +\infty$ for each $\mathfrak{p} > 0$ then the LGT transforms bounded non-negative time scales into unbounded time scales. Such transformations are called BU-transforming for $\mathfrak{p} > 0$.
▹ *Bounded-to-unbounded (BU) transformations*: $0 < \mathrm{Dom}^+\Psi^t_{\mathfrak{p}} < +\infty$ and $\lim_{t\nearrow \mathrm{Dom}^+\Psi^t_{\mathfrak{p}}} \Psi^t_{\mathfrak{p}}(t) = +\infty$ for each $\mathfrak{p} > 0$ then the LGT transforms bounded non-negative time scales into unbounded time scales. Such transformations are called BU-transforming for $\mathfrak{p} > 0$.
▹ *Contractively-bounded-to-unbounded (CBU) transformations*: If the transformation is BU and, in addition, the sequence $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} \searrow 0$ as $\mathfrak{p} \to +\infty$ we say that the LGT transforms contractively bounded non-negative time scales into unbounded time scales or it is CBU-transforming for $\mathfrak{p} > 0$. ◃

If $\Psi^t_{\mathfrak{p}}(t) = e^{\mathfrak{p}}t$ (or $\Psi^t_{\mathfrak{p}}(t) = e^{-\mathfrak{p}}t$) then (2) is UU-transforming for $\mathfrak{p} > 0$. On the other hand, if $\Psi^t_{\mathfrak{p}}(t) = \ln \frac{e^{\mathfrak{p}+t}}{1+e^t-e^{\mathfrak{p}+t}}$ (with generator $\mathbf{g}^t|_t = (e^t+1)\frac{\partial}{\partial t}$) then $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = 0$ if $\mathfrak{p} = \ln 2$, $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = \ln \frac{1}{e^{\mathfrak{p}}-1}$ if $0 < \mathfrak{p} < \ln 2$ and $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = +\infty$ if $\mathfrak{p} > \ln 2$. Furthermore, $\lim_{t\to+\infty} \Psi^t_{\mathfrak{p}}(t) = \frac{1}{e^{-\mathfrak{p}}-1} < +\infty$ for $\mathfrak{p} > \ln 2$ and $\lim_{t\nearrow \ln \frac{1}{e^{\mathfrak{p}}-1}} \Psi^t_{\mathfrak{p}}(t) = +\infty$ for $0 < \mathfrak{p} < \ln 2$. Hence, the LGT is not BU-transforming for $\mathfrak{p} > 0$ (actually, it is BU-transforming only for $0 < \mathfrak{p} < \ln 2$). On the other hand, if $\Psi^t_{\mathfrak{p}}(t) = \frac{t}{1-\mathfrak{p}t}$ (with generator $\mathbf{g}^t|_t = t^2\frac{\partial}{\partial t}$) then $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = \frac{1}{\mathfrak{p}}$ if $\mathfrak{p} > 0$. In this case, $\lim_{t\nearrow \frac{1}{\mathfrak{p}}} \Psi^t_{\mathfrak{p}}(t) = +\infty$ for $\mathfrak{p} > 0$, hence the LGT is CBU-transforming for $\mathfrak{p} > 0$. A final example is $\Psi^t_{\mathfrak{p}}(t) = \frac{e^{2\mathfrak{p}}(t-1)+t+1}{t+1-e^{2\mathfrak{p}}(t-1)}$ (with generator $\mathbf{g}^t|_t = (t^2-1)\frac{\partial}{\partial t}$) for which $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = \frac{e^{2\mathfrak{p}}+1}{e^{2\mathfrak{p}}-1}$ for $\mathfrak{p} > 0$, hence the LGT is BU-transforming (but not CBU!) for $\mathfrak{p} > 0$.

We remark that either UU- or BU-transforming LGT's may not guarantee that the transformed time scale $\Psi^t_{\mathfrak{p}}(t)$ is non-negative for all $\mathfrak{p} > 0$ and $t \in [0, \mathrm{Dom}^+\Psi^t_{\mathfrak{p}})$: for instance, consider the case $\Psi^t_{\mathfrak{p}}(t) = t - \mathfrak{p}$ with infinitesimal generator $\mathbf{g}^t|_t = -\frac{\partial}{\partial t}$. For the

[6]A continuous *time scale* $\mathbb{T}$ is a strictly increasing continuous function $\mathbb{T} : \mathcal{I} \to \mathcal{J}$, with $\mathcal{I}, \mathcal{J} \subseteq \mathbb{R}$ intervals of the form $[a,b)$ and $[c,d)$, with $b, d \le +\infty$. The trivial time scale is the identity function $\mathbb{1}$ on $\mathbb{R}$ (i.e. $\mathbb{1}(t) = t$). A time scale $\mathbb{T} : [a,b) \to [c,d)$ is *non-negative* if $c = 0$, *positive* if $c > 0$, it is *bounded* if $\lim_{t\nearrow b} \mathbb{T}(t) < +\infty$ (i.e. $d < +\infty$) and *unbounded* if $\lim_{t\nearrow b} \mathbb{T}(t) = +\infty$ (i.e. $d = +\infty$) .

sake of simplicity, we assume $\Psi^t_{\mathfrak{p}}(0) = 0$ for all $\mathfrak{p} > 0$, i.e. $\Psi_{\mathfrak{p}}$ maps *non-negative time scales into non-negative time scales* for $\mathfrak{p} > 0$ as for instance with $\Psi^t_{\mathfrak{p}}(t) = \frac{t}{1-\mathfrak{p}t}$ and $\Psi^t_{\mathfrak{p}}(t) = e^{a\mathfrak{p}}t$. [7]

## 3.3 | Prolongation of groups and generators

Notice that, with the postulated structure of $\Psi_{\mathfrak{p}}$ and $\mathbf{g}$, the transformation $\Psi^{t,\mathrm{s}}_{\mathfrak{p}} \triangleq (\Psi^t_{\mathfrak{p}}, \Psi^{\mathrm{s}}_{\mathfrak{p}})$ with $\Psi^{\mathrm{s}}_{\mathfrak{p}} \triangleq (\Psi^{\mathrm{y}}_{\mathfrak{p}}, \Psi^{\mathrm{x}}_{\mathfrak{p}})$ is itself a LGT acting on $\mathbb{R} \times \mathbb{R}^n$ with infinitesimal generator $\mathbf{g}^{t,\mathrm{s}}|_{(t,\mathrm{s})} \triangleq g^t(t)\frac{\partial}{\partial t} + \sum_{j=1}^n g^{\mathrm{s}_j}(t,\mathrm{s})\frac{\partial}{\partial \mathrm{s}_j}$. We see how to "prolong" the action of $\Psi^{t,\mathrm{s}}_{\mathfrak{p}}$ on the time derivatives of $s(t)$, in particular $D_t s(t)$. Consider the state and its time derivative functions and define the 1*st-order prolongation* of the function $s(t)$ as

$$\mathrm{pr}^{[1]}s(t) \triangleq (s(t), D_t s(t)).$$

Let $\mathrm{s}^{[1]} \triangleq (\mathrm{s}, \mathrm{s}^{(1)})$ denote the coordinates of the tangent bundle $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ so that we write $\mathrm{s}^{[1]} = \mathrm{pr}^{[1]}s(t)$ with the obvious meaning. There is an induced local action of $\Psi^{t,\mathrm{s}}_{\mathfrak{p}} \triangleq (\Psi^t_{\mathfrak{p}}, \Psi^{\mathrm{s}}_{\mathfrak{p}})$ on $\mathbb{R} \times T\mathbb{R}^n$ called the 1*st-order prolongation* of the group $\Psi^{t,\mathrm{s}}_{\mathfrak{p}}$ and denoted by $\mathrm{pr}^{[1]}\Psi^{t,\mathrm{s}}_{\mathfrak{p}}$. It is defined as the mapping

$$(t, \mathrm{s}^{[1]}) \in \mathbb{R} \times T\mathbb{R}^n \xmapsto{\mathrm{pr}^{[1]}\Psi^{t,\mathrm{s}}_{\mathfrak{p}}} \mathrm{pr}^{[1]}\Psi^{t,\mathrm{s}}_{\mathfrak{p}}(t, \mathrm{s}^{[1]}) \triangleq (\Psi^t_{\mathfrak{p}}(t), \Psi^{\mathrm{s}^{[1]}}_{\mathfrak{p}}(t, \mathrm{s}^{[1]})) \in \mathbb{R} \times T\mathbb{R}^n,$$

where $\Psi^{\mathrm{s}^{[1]}}_{\mathfrak{p}}$ is such that

$$\Psi^{\mathrm{s}^{[1]}}_{\mathfrak{p}}(t, \mathrm{pr}^{[1]}s(t))|_{t=[\Psi^t_{\mathfrak{p}}]^{-1}(\mathrm{t}_{\mathfrak{p}})} = \mathrm{pr}^{[1]}\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}) \triangleq (\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}), D_{\mathrm{t}_{\mathfrak{p}}}\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}))$$

with [8]

$$\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}) \triangleq [\Psi^{\mathrm{s}}_{\mathfrak{p}} \circ (\mathbb{1} \times s)] \circ [\Psi^t_{\mathfrak{p}}]^{-1}(\mathrm{t}_{\mathfrak{p}}).$$

In other words, $\Psi^{\mathrm{s}^{[1]}}_{\mathfrak{p}}(t, \mathrm{pr}^{[1]}s(t))|_{t=[\Psi^t_{\mathfrak{p}}]^{-1}(\mathrm{t}_{\mathfrak{p}})}$ is the 1st-order prolongation of the function $\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})$ which is the transformed function $s(t)$ in the transformed time variable $\mathrm{t}_{\mathfrak{p}} = \Psi^t_{\mathfrak{p}}(t)$. The prolongation of $\Psi^{t,\mathrm{s}}_{\mathfrak{p}}$ is obtained from the following prolongation formula, borrowed from [19].

**Proposition 1.** (*Prolongation formula for* $\Psi^{t,\mathrm{s}}_{\mathfrak{p}}$):

$$\begin{aligned} &\mathrm{pr}^{[1]}\Psi^{t,\mathrm{x}}_{\mathfrak{p}}(t, \mathrm{s}^{[1]}) \triangleq (\Psi^t_{\mathfrak{p}}(t), \Psi^{\mathrm{s}^{[1]}}_{\mathfrak{p}}(t, \mathrm{s}^{[1]})),\ \Psi^{\mathrm{s}^{[1]}}_{\mathfrak{p}}(t, \mathrm{s}^{[1]}) \triangleq (\Psi^{\mathrm{s}}_{\mathfrak{p}}(t, \mathrm{s}), \Psi^{\mathrm{s}^{(1)}}_{\mathfrak{p}}(t, \mathrm{s}^{[1]})), \\ &\Psi^{\mathrm{s}^{(1)}}_{\mathfrak{p}}(t, \mathrm{s}^{[1]}) = (\frac{\partial \Psi^t_{\mathfrak{p}}}{\partial t}(t))^{-1}\left(\frac{\partial \Psi^{\mathrm{s}}_{\mathfrak{p}}}{\partial t}(t, \mathrm{s}) + \frac{\partial \Psi^{\mathrm{s}}_{\mathfrak{p}}}{\partial \mathrm{s}}(t, \mathrm{s})\mathrm{s}^{(1)}\right). \lhd \end{aligned} \tag{10}$$

For example, the 1st-order prolongation of the group (3) is

$$\mathrm{pr}^{[1]}\Psi^{t,\mathrm{s}}_{\mathfrak{p}}(t, \mathrm{s}^{[1]}) = (\frac{t}{1-\mathfrak{p}t}, \mathrm{z}, (1-\mathfrak{p}t)^2\mathrm{s}^{(1)}) \tag{11}$$

where $\Psi^{\mathrm{s}^{(1)}}_{\mathfrak{p}} = (1-\mathfrak{p}t)^2\mathrm{s}^{(1)}$ is the action of $\Psi^{t,\mathrm{s}}_{\mathfrak{p}}$ on the coordinate space of time derivatives $\mathrm{s}^{(1)}$.

Similarly, the 1*st-order prolongation* $\mathrm{pr}^{[1]}\mathbf{g}^{t,\mathrm{s}}$ of the generator $\mathbf{g}^{t,\mathrm{s}}$ of the group $\Psi^{t,\mathrm{s}}_{\mathfrak{p}}$ is defined as the infinitesimal generator of the prolonged group $\mathrm{pr}^{[1]}\Psi^{t,\mathrm{x}}_{\mathfrak{p}}$ according to the following prolongation formula, borrowed from [19].

**Proposition 2.** (*Prolongation formula for* $\mathbf{g}^{t,\mathrm{s}}$):

$$\begin{aligned} &\mathrm{pr}^{[1]}\mathbf{g}^{t,\mathrm{s}}|_{(t,\mathrm{x}^{[1]})} \triangleq g^t(t)\frac{\partial}{\partial t} + \sum_{j=1}^n g^{\mathrm{s}_j}(t,\mathrm{s})\frac{\partial}{\partial \mathrm{s}_j} + \sum_{j=1}^n g^{\mathrm{s}^{(1)}_j}(t, \mathrm{s}^{[1]})\frac{\partial}{\partial \mathrm{s}^{(1)}_j}, \\ &g^{\mathrm{s}^{(1)}_j}(t, \mathrm{s}^{[1]}) = \frac{\partial g^{\mathrm{s}_j}}{\partial \mathrm{s}}(t,\mathrm{s})\mathrm{s}^{(1)} + \frac{\partial g^{\mathrm{s}_j}}{\partial t}(t,\mathrm{s}) - \frac{\partial g^t}{\partial t}(t)\mathrm{s}^{(1)}_j. \lhd \end{aligned} \tag{12}$$

As will be seen in the following sections, we will make extensive use of the prolongation formulas to construct one-parameter groups and study system symmetries. For example, the 1st-order prolongation of (7) is

$$\mathrm{pr}^{[1]}\mathbf{g}^{t,\mathrm{x}}|_{(t,\mathrm{s}^{[1]})} = t^2\frac{\partial}{\partial t} - \sum_{j=1}^n 2\mathrm{s}^{(1)}_j t \frac{\partial}{\partial \mathrm{s}^{(1)}_j}$$

which is also the infinitesimal generator of the prolonged group (11).

[7] Needless to say, our case list is not exhaustive: for instance, if $\Psi^t_{\mathfrak{p}}(t) = \frac{t}{1+\mathfrak{p}t}$ than $\lim_{t\to+\infty} \Psi^t_{\mathfrak{p}}(t) = \frac{1}{\mathfrak{p}} < +\infty$ for each $\mathfrak{p} > 0$ and $\Psi_{\mathfrak{p}}$ transforms *unbounded positive time scales into bounded time scales* for $\mathfrak{p} > 0$.

[8] Here, $\circ$ denotes composition of functions, $\mathbb{1}$ the identity function on $\mathbb{R}$ (i.e. $\mathbb{1}(t) = t$) and $\times$ the Cartesian product of functions

# 4 | SYMMETRIES OF ODE SYSTEMS

One-parameter symmetries of a system are one-parameter groups of transformations that preserve the system's structure. The class of symmetries considered in this paper was introduced in the recent works [7] and [9] for semiglobal and global observer design. For completeness, we begin by recalling from [7] the notion of *system map* and how it is trasformed by the action of a LGT. For each $C^\infty$ function $(t, \mathsf{s}, \mathsf{u}, \mathsf{d}) \overset{F}{\longmapsto} F(t, \mathsf{s}, \mathsf{u}, \mathsf{d}) \in \mathbb{R}^n$ define the *system map* $\Sigma \triangleq \mathfrak{F}(t, \mathsf{s}^{[1]}, \mathsf{u}, \mathsf{d})$ (associated with $F$) as

$$(t, \mathsf{x}^{[1]}, \mathsf{u}, \mathsf{d}) \in \mathbb{R} \times T\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \overset{\mathfrak{F}}{\longmapsto} \mathfrak{F}(t, \mathsf{s}^{[1]}, \mathsf{u}, \mathsf{d}) \triangleq \mathsf{s}^{(1)} - F(t, \mathsf{s}, \mathsf{u}, \mathsf{d}) \in \mathbb{R}^n. \tag{13}$$

## 4.1 | Notation II

In this paragraph, for later use in the main definitions and proofs of the main results, we collect all the notations relative to coordinates, functions, time scales and system maps transformed under the action of a LGT and which will be used in the discussion and proof of the main results. We recall that $\mathsf{s}^{[1]} \triangleq (\mathsf{s}, \mathsf{s}^{(1)})$ and $\mathrm{pr}^{[1]} s(t) \triangleq (s(t), D_t s(t))$.

▷ *transformed coordinates* $(t, \mathsf{s}^{[1]}, \mathsf{u}, \mathsf{d})$ :

$$\begin{aligned}
&\mathsf{t}_{\mathfrak{p}} \triangleq \Psi^t_{\mathfrak{p}}(t),\\
&\mathsf{z}_{\mathfrak{p}} \triangleq (\mathsf{s}_{\mathfrak{p}}, \mathsf{u}_{\mathfrak{p}}, \mathsf{d}_{\mathfrak{p}}) \triangleq (\Psi^{\mathsf{s}}_{\mathfrak{p}}(t, \mathsf{s}), \Psi^{\mathsf{u}}_{\mathfrak{p}}(t, \mathsf{y}, \mathsf{u}), \Psi^{\mathsf{d}}_{\mathfrak{p}}(t, \mathsf{s}, \mathsf{d})),\\
&\mathsf{s}^{(1)}_{\mathfrak{p}} \triangleq \Psi^{\mathsf{s}^{(1)}}_{\mathfrak{p}}(t, \mathsf{s}^{[1]}),\ \mathsf{s}^{[1]}_{\mathfrak{p}} \triangleq (\mathsf{s}_{\mathfrak{p}}, \mathsf{s}^{(1)}_{\mathfrak{p}}),
\end{aligned} \tag{14}$$

▷ *transformed prolongations* $\mathrm{pr}^{[1]} s(t)$ *and functions* $u(t), d(t)$ *in time-scale* $t$ :

$$\begin{aligned}
&z_{\mathfrak{p}}(t) \triangleq (s_{\mathfrak{p}}(t), u_{\mathfrak{p}}(t), d_{\mathfrak{p}}(t)) \triangleq (\Psi^{\mathsf{x}}_{\mathfrak{p}}(t, s(t)), \Psi^{\mathsf{u}}_{\mathfrak{p}}(t, y(t), u(t)), \Psi^{\mathsf{d}}_{\mathfrak{p}}(t, s(t), d(t))),\\
&(D_t s)_{\mathfrak{p}}(t) \triangleq \Psi^{\mathsf{s}^{(1)}}_{\mathfrak{p}}(t, \mathrm{pr}^{[1]} s(t)),\\
&(\mathrm{pr}^{[1]} s)_{\mathfrak{p}}(t) \triangleq (s_{\mathfrak{p}}(t), (D_t s)_{\mathfrak{p}}(t)),
\end{aligned} \tag{15}$$

▷ *transformed prolongations* $\mathrm{pr}^{[1]} s(t)$ *and functions* $u(t), d(t)$ *in the transformed time-scale* $\mathsf{t}_{\mathfrak{p}}$ :

$$\begin{aligned}
&\widetilde{z}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}) \triangleq (\widetilde{s}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}), \widetilde{u}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}), \widetilde{d}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})) \triangleq z_{\mathfrak{p}}(t)|_{[\Psi^t_{\mathfrak{p}}]^{-1}(\mathsf{t}_{\mathfrak{p}})},\\
&\mathrm{pr}^{[1]} \widetilde{s}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}) \triangleq (\mathrm{pr}^{[1]} s)_{\mathfrak{p}}(t)|_{t=[\Psi^t_{\mathfrak{p}}]^{-1}(\mathsf{t}_{\mathfrak{p}})},
\end{aligned} \tag{16}$$

▷ *system map in time-scale* $t$ :

$$\Sigma(t) \triangleq \mathfrak{F}(t, \mathrm{pr}^{[1]} s(t), u(t), d(t)), \tag{17}$$

▷ *transformed system map in time-scale* $t$ :

$$\begin{aligned}
&\Sigma_{\mathfrak{p}} \triangleq \mathfrak{F}(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}^{[1]}_{\mathfrak{p}}, \mathsf{u}_{\mathfrak{p}}, \mathsf{d}_{\mathfrak{p}}),\\
&\Sigma_{\mathfrak{p}}(t) \triangleq \mathfrak{F}(\Psi^t_{\mathfrak{p}}(t), (\mathrm{pr}^{[1]} s)_{\mathfrak{p}}(t), u_{\mathfrak{p}}(t), d_{\mathfrak{p}}(t)),
\end{aligned} \tag{18}$$

▷ *transformed system map in the transformed time-scale* $\mathsf{t}_{\mathfrak{p}}$ :

$$\widetilde{\Sigma}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}) \triangleq \mathfrak{F}(\mathsf{t}_{\mathfrak{p}}, \mathrm{pr}^{[1]} \widetilde{s}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}), \widetilde{u}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}), \widetilde{d}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})) = \Sigma_{\mathfrak{p}}(t)|_{t=[\Psi^t_{\mathfrak{p}}]^{-1}(\mathsf{t}_{\mathfrak{p}})}. \tag{19}$$

With the system map (13) and notation (17), the system (1) can be characterized in the following three equivalent ways

$$(1) \Leftrightarrow \Sigma(t) = 0 \Leftrightarrow \mathfrak{F}(t, \mathrm{pr}^{[1]} x(t), u(t), d(t)) = 0. \tag{20}$$

Moreover, for helping the reader through the forthcoming definitions and examples, we highlight the following equivalences and implications

$$\Sigma = 0 \Leftrightarrow \mathfrak{F}(t, \mathsf{s}^{[1]}, \mathsf{u}, \mathsf{d}) = 0 \Leftrightarrow \mathsf{s}^{(1)} = F(t, \mathsf{s}, \mathsf{u}, \mathsf{d}) \Rightarrow \Sigma_{\mathfrak{p}} = \mathsf{s}^{(1)}_{\mathfrak{p}}|_{\mathsf{s}^{(1)}=F(t,\mathsf{s},\mathsf{u},\mathsf{d})} - F(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}}, \mathsf{u}_{\mathfrak{p}}, \mathsf{d}_{\mathfrak{p}}) = \Sigma_{\mathfrak{p}}|_{\Sigma=0}. \tag{21}$$

We introduce also the vector field $\mathrm{pr}^{[1]}\mathbf{g}$ on $\mathbb{R} \times T\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r$ representing the infinitesimal generator of the prolonged group

$$\mathrm{pr}^{[1]}\mathbf{g}|_{(t,\mathsf{x}^{[1]},\mathsf{u},\mathsf{d})} \triangleq \mathrm{pr}^{[1]}\mathbf{g}^{t,\mathsf{s}}|_{(t,\mathsf{s}^{[1]})} + \sum_{j=1}^{m} g^{\mathsf{u}_j}(t, \mathsf{y}, \mathsf{u}) \frac{\partial}{\partial \mathsf{u}_j} + \sum_{j=1}^{r} g^{\mathsf{d}_j}(t, \mathsf{s}, \mathsf{d}) \frac{\partial}{\partial \mathsf{d}_j} \tag{22}$$

with $\mathrm{pr}^{[1]}\mathbf{g}^{t,\mathsf{x}}$ as in (12) and, finally, by $L_{\mathrm{pr}^{[1]}\mathbf{g}}\Sigma$ we denote the (Lie) derivative of the system map $\mathfrak{F}$ along $\mathrm{pr}^{[1]}\mathbf{g}$ ([10]), i.e.

$$(L_{\mathrm{pr}^{[1]}\mathbf{g}}\Sigma)(q) = \frac{\partial \mathfrak{F}}{\partial t}(q)g^t(t) + \sum_{j=1}^{n} \frac{\partial \mathfrak{F}}{\partial \mathsf{s}_j}(q)g^{\mathsf{s}_j}(t,\mathsf{s}) + \sum_{j=1}^{n} \frac{\partial \mathfrak{F}}{\partial \mathsf{s}_j^{(1)}}(q)g^{\mathsf{s}_j^{(1)}}(t,\mathsf{s}^{[1]}) + \sum_{j=1}^{m} \frac{\partial \mathfrak{F}}{\partial \mathsf{u}_j}(q)g^{\mathsf{u}_j}(t,\mathsf{y},\mathsf{u}) + \sum_{j=1}^{r} \frac{\partial \mathsf{F}}{\partial \mathsf{d}_j}(q)g^{\mathsf{d}_j}(t,\mathsf{s},\mathsf{d})$$

where for brevity we denoted the point $(t, \mathsf{s}^{[1]}, \mathsf{u}, \mathsf{d})$ by $q$.

## 4.2 | Recalls on symmetries and computational issues

A LGT transforms the system map $\Sigma$ into $\Sigma_{\mathfrak{p}}$ (and correspondingly $\Sigma(t)$ into $\Sigma_{\mathfrak{p}}(t)$) and any solution $z(t)$ of the system $\Sigma(t) = 0$ into a transformed function $\widetilde{z}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})$ in transformed time scale $\mathsf{t}_{\mathfrak{p}}$). However, $\widetilde{z}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})$ is not necessarily a solution of the transformed system $\widetilde{\Sigma}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}) = 0$. A LGT is a symmetry of the system $\Sigma(t) = 0$ if $\widetilde{z}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})$ is a solution of $\widetilde{\Sigma}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}) = 0$. For reader's convenience we recall from[7] the definition of symmetry.

**Definition 4.** *(Symmetries).* A LGT (resp. GGT) $\Psi_{\mathfrak{p}}$ on $\mathcal{M}$ with infinitesimal generator $\mathbf{g}$ is a *local (resp. global) $C^\infty$ symmetry (LS, resp. GS)* of the system $\Sigma(t) = 0$ if either of the following equivalent conditions holds:

1. (*Transformation condition*).

$$\forall \mathfrak{p} \in \mathbb{R}, \forall (t,\mathsf{z}) \in \mathrm{Dom}(\Psi_{\mathfrak{p}}) : \Sigma = 0 \Rightarrow \Sigma_{\mathfrak{p}} = 0, \tag{23}$$

2. (*Infinitesimal condition*).

$$\forall (t,\mathsf{z}) \in \mathcal{M} : \Sigma = 0 \Rightarrow L_{\mathrm{pr}^{[1]}\mathbf{g}}\Sigma = 0. \lhd \tag{24}$$

While condition (24) embodies the constructive procedure for finding a symmetry for its generator, condition (23) provides a geometric perspective. By the prolongation formula (10) the vector $\mathsf{s}^{(1)} = F(t,\mathsf{s},\mathsf{u},\mathsf{d})$ is mapped by the group of transformations through its prolonged action into

$$\Psi_{\mathfrak{p}}^{\mathsf{s}^{(1)}}(t,\mathsf{s},F(t,\mathsf{s},\mathsf{u},\mathsf{d})) \triangleq (\frac{\partial \Psi_{\mathfrak{p}}^t}{\partial t}(t))^{-1}\left(\frac{\partial \Psi_{\mathfrak{p}}^{\mathsf{x}}}{\partial t}(t,\mathsf{x}) + \frac{\partial \Psi_{\mathfrak{p}}^{\mathsf{x}}}{\partial \mathsf{x}}F(t,\mathsf{s},\mathsf{u},\mathsf{d})\right).$$

Condition (23) amounts to

$$\Psi_{\mathfrak{p}}^{\mathsf{s}^{(1)}}(t,\mathsf{s},F(t,\mathsf{s},\mathsf{d})) \equiv F(\Psi_{\mathfrak{p}}^t(t), \Psi_{\mathfrak{p}}^{\mathsf{s}}(t,\mathsf{s}), \Psi_{\mathfrak{p}}^{\mathsf{d}}(t,\mathsf{s},\mathsf{d})) \tag{25}$$

Hence, $F$ is mapped into itself or it is left *invariant under the action of* $\Psi_{\mathfrak{p}}$ and system's solutions $s(t)$ are mapped into (parametrized) system's solutions $\widetilde{s}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})$. In particular, the system with time scale $t$

$$\Sigma(t) = 0 \Leftrightarrow D_t s(t) = F(t, s(t), u(t), d(t))$$

is mapped by the symmetry into the system with time scale $\mathsf{t}_{\mathfrak{p}}$

$$\widetilde{\Sigma}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}) = 0 \Leftrightarrow D_{\mathsf{t}_{\mathfrak{p}}}\widetilde{s}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}) = F(\mathsf{t}_{\mathfrak{p}}, \widetilde{s}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}), \widetilde{u}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}), \widetilde{d}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})).$$

This demonstrates how a symmetry transforms the system's solutions while preserving its structural properties. This invariance is a key feature that enables the construction of a (semi)global controller from a locally designed one which is the subject of this paper.

In the following sections we develop in more details some theoretical and computational issues introduced in[7] and we introduce new ones. This is also in some sense necessary to illustrate the different control design procedures we are going to introduce in this paper.

In the following sections, we further elaborate on several theoretical and computational issues previously introduced in[7], while also presenting new developments. This expanded discussion is intended to motivate, clarify and support the various control design procedures that will be introduced throughout the paper.

### 4.2.1 | Computing symmetries from the prolongation formulas for generators

A first way of computing a LS of a system $\Sigma(t) = 0$ is by solving the PDE resulting from the infinitesimal condition (24) with *unknown* infinitesimal generator $\mathbf{g}$. Consider the system map $\Sigma$ associated with

$$F(t, \mathsf{s}, \mathsf{u}, \mathsf{d}) = \begin{pmatrix} \mathsf{x} \\ \mathsf{x}^2 + \mathsf{u} + \mathsf{d} \end{pmatrix} \tag{26}$$

with $\mathsf{s} \triangleq (\mathsf{y}, \mathsf{x}) \in \mathbb{R}^2$ and $\mathsf{u}, \mathsf{d} \in \mathbb{R}$. Assuming an infinitesimal generator $\mathbf{g}$ of the form

$$\mathbf{g}|_{(t,\mathsf{z})} \triangleq g^t(t)\frac{\partial}{\partial t} + g^{\mathsf{y}}(t, \mathsf{y})\frac{\partial}{\partial \mathsf{y}} + g^{\mathsf{x}}(t, \mathsf{s})\frac{\partial}{\partial \mathsf{x}} + g^{\mathsf{u}}(t, \mathsf{y}, \mathsf{u})\frac{\partial}{\partial \mathsf{u}} + g^{\mathsf{d}}(t, \mathsf{y}, \mathsf{d})\frac{\partial}{\partial \mathsf{d}}$$

and using the prolongation formula (12) for computing $\mathrm{pr}^{[1]}\mathbf{g}$ in (22), the symmetry condition (24) reads as

$$\begin{aligned} &(\frac{\partial g^{\mathsf{y}}}{\partial \mathsf{y}} - \frac{\partial g^t}{\partial t})\mathsf{x} + \frac{\partial g^{\mathsf{y}}}{\partial t} = g^{\mathsf{x}}, \\ &\frac{\partial g^{\mathsf{x}}}{\partial \mathsf{y}}\mathsf{x} + (\frac{\partial g^{\mathsf{x}}}{\partial \mathsf{x}} - \frac{\partial g^t}{\partial t})(\mathsf{x}^2 + \mathsf{u} + \mathsf{d}) + \frac{\partial g^{\mathsf{x}}}{\partial t} = 2g^{\mathsf{x}}\mathsf{x} + g^{\mathsf{u}} + g^{\mathsf{d}} \end{aligned} \tag{27}$$

A first type of solution for (27) is

$$\mathbf{g}|_{(t,\mathsf{z})} = t\frac{\partial}{\partial t} + (1 - e^{\mathsf{y}})\frac{\partial}{\partial \mathsf{y}} - (1 + e^{\mathsf{y}})\mathsf{x}\frac{\partial}{\partial \mathsf{x}} - (2 + e^{\mathsf{y}})\mathsf{u}\frac{\partial}{\partial \mathsf{u}} + (1 - e^{\mathsf{y}})\frac{\partial}{\partial \mathsf{y}} - (2 + e^{\mathsf{y}})\mathsf{d}\frac{\partial}{\partial \mathsf{d}}. \tag{28}$$

By solving $\frac{\partial \Psi_{\mathfrak{p}}(t,\mathsf{z})}{\partial \mathfrak{p}} = \mathbf{g}|_{\Psi_{\mathfrak{p}}(t,\mathsf{z})}$ with $\Psi_0(t, \mathsf{z}) = (t, \mathsf{z})$ we obtain the LS $\Psi_{\mathfrak{p}}(\mathsf{t}, \mathsf{z}) = (\mathsf{t}_{\mathfrak{p}}, \mathsf{z}_{\mathfrak{p}})$ with

$$\mathsf{t}_{\mathfrak{p}} = e^{\mathfrak{p}}t, \mathsf{s}_{\mathfrak{p}} = \left(\ln \frac{1}{1 - e^{-\mathfrak{p}}(1 - e^{-\mathsf{y}})}, \frac{e^{-2\mathfrak{p}-\mathsf{y}}}{1 - e^{-\mathfrak{p}}(1 - e^{-\mathsf{y}})}\mathsf{x}\right), \mathsf{u}_{\mathfrak{p}} = \frac{e^{-3\mathfrak{p}-\mathsf{y}}}{1 - e^{-\mathfrak{p}}(1 - e^{-\mathsf{y}})}\mathsf{u}, \mathsf{d}_{\mathfrak{p}} = \frac{e^{-3\mathfrak{p}-\mathsf{y}}}{1 - e^{-\mathfrak{p}}(1 - e^{-\mathsf{y}})}\mathsf{d}. \tag{29}$$

Notice that $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \mathcal{M}$ if $\mathfrak{p} > 0$ and $= \{(t, \mathsf{z}) \in \mathcal{M} : \mathsf{y} < -\ln(1 - e^{\mathfrak{p}})\}$ if $\mathfrak{p} < 0$. Moreover, since $\Psi^t_{\mathfrak{p}}(t) = e^{\mathfrak{p}}t$, $\Psi_{\mathfrak{p}}$ is UU-transforming for $\mathfrak{p} > 0$.

A far simpler solution for (27) is

$$\mathbf{g}|_{(t,\mathsf{z})} = t\frac{\partial}{\partial t} - \mathsf{x}\frac{\partial}{\partial \mathsf{x}} - 2\mathsf{u}\frac{\partial}{\partial \mathsf{u}} - 2\mathsf{d}\frac{\partial}{\partial \mathsf{d}}$$

from which we get the GS $\Psi_{\mathfrak{p}}(\mathsf{t}, \mathsf{z}) = (\mathsf{t}_{\mathfrak{p}}, \mathsf{z}_{\mathfrak{p}})$ with

$$\mathsf{t}_{\mathfrak{p}} = e^{\mathfrak{p}}t, \mathsf{s}_{\mathfrak{p}} = (\mathsf{y}, e^{-\mathfrak{p}}\mathsf{x}), \mathsf{u}_{\mathfrak{p}} = e^{-2\mathfrak{p}}\mathsf{u}, \mathsf{d}_{\mathfrak{p}} = e^{-2\mathfrak{p}}\mathsf{d}. \tag{30}$$

Also in this case, $\Psi_{\mathfrak{p}}$ is UU-transforming for all $\mathfrak{p} \in \mathbb{R}$.

### 4.2.2 | Time scale transformation for performances in prescribed time

Consider the system map $\Sigma$ associated with $F(t, \mathsf{s}, \mathsf{w}) = A(t)\mathsf{s} + B(t)\mathsf{w}$, where $\mathsf{w} \triangleq (\mathsf{u}, \mathsf{d})$, and assume a candidate infinitesimal generator $\mathbf{g}$ of the form

$$\mathbf{g}|_{(t,\mathsf{z})} = g^t(t)\frac{\partial}{\partial t} + \sum_{j=1}^{n}(G^{\mathsf{s}}(t)\mathsf{s})_j\frac{\partial}{\partial \mathsf{s}_j} + \sum_{j=1}^{m+r}(G^{\mathsf{w}}(t)\mathsf{w})_j\frac{\partial}{\partial \mathsf{w}_j},$$

where $G^{\mathsf{s}}(t) \in \mathbb{R}^{n\times n}$ and $G^{\mathsf{w}}(t) \in \mathbb{R}^{(m+r)\times(m+r)}$. We seek a function $g^t(t)$ in such a way that the solution $\Psi^t_{\mathfrak{p}}(t)$ of

$$\frac{\partial \Psi^t_{\mathfrak{p}}(t)}{\partial \mathfrak{p}} = g^t(\Psi^t_{\mathfrak{p}}(t)), \Psi^t_0(t) = t, \tag{31}$$

satisfies $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} < +\infty$ and $\lim_{t\nearrow \mathrm{Dom}^+\Psi^t_{\mathfrak{p}}} = +\infty$ for all $\mathfrak{p} > 0$, i.e. $\Psi_{\mathfrak{p}}$ is a BU-transforming LGT for $\mathfrak{p} > 0$. For instance, $g^t(t) = t^2$ with $\Psi^t_{\mathfrak{p}}(t) = \frac{t}{1-\mathfrak{p}t}$ and $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = \frac{1}{\mathfrak{p}}$ or $g^t(t) = t^2 - 3t + 2$ with $\Psi^t_{\mathfrak{p}}(t) = \frac{-e^{\mathfrak{p}}(t-2)+2(t-1)}{t-1-e^{\mathfrak{p}}(t-2)}$ and $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = \frac{2e^{\mathfrak{p}}-1}{e^{\mathfrak{p}}-1}$. When $\Psi^t_{\mathfrak{p}}(t) = \frac{t}{1-\mathfrak{p}t}$, $\Psi_{\mathfrak{p}}$ is also CBU-transforming for $\mathfrak{p} > 0$ (since $\lim_{\mathfrak{p}\to+\infty}\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = \frac{1}{\mathfrak{p}}$). Using the prolongation formula (12) for computing $\mathrm{pr}^{[1]}\mathbf{g}$ in (22), the symmetry condition (24) for $\Sigma(t) = 0$ boils down to

$$\frac{\partial G^{\mathsf{s}}}{\partial t}(t) = A(t)G^{\mathsf{s}}(t) - G^{\mathsf{s}}(t)A(t) + A(t)\frac{\partial g^t}{\partial t}(t) + \frac{\partial A}{\partial t}(t)g^t(t), \tag{32}$$

$$G^{\mathsf{s}}(t)B(t) = B(t)\left(G^{\mathsf{w}}(t) + \frac{\partial g^t}{\partial t}(t)I_{m+s}\right) + \frac{\partial B}{\partial t}(t)t^2. \tag{33}$$

The unique solution of (32), which is a *differential Sylvester equation*, is

$$G^{\mathsf{s}}(t)=\Phi(t,\tau)G^{\mathsf{s}}(\tau)\Phi(\tau,t)+\int_{\tau}^{t}\Phi(t,r)(A(r)\frac{\partial g^{t}}{\partial t}|_{t=r}+\frac{\partial A}{\partial t}|_{t=r}g^{t}(r))\Phi(r,t)dr, \tag{34}$$

$G^{\mathsf{s}}(\tau)\in\mathbb{R}^{n\times n}$ and $\tau\in\mathbb{R}$, where $\Phi(t,\tau)$ is the state transition matrix of the system $D_t s(t)=A(t)s(t)$, unique solution of

$$D_t\Phi(t,\tau)=A(t)\Phi(t,\tau),\Phi(\tau,\tau)=I_n. \tag{35}$$

The remaining (33) is a (algebraic) Sylvester equation. Once we find the matrix $G^{\mathsf{s}}(\tau)$ in (34) satisfying (33) for some $\tau\in\mathbb{R}$ and $G^{\mathsf{w}}(t)\in\mathbb{R}^{(m+s)\times(m+s)}$, we solve

$$\frac{\partial\Psi_{\mathfrak{p}}(t,\mathsf{z})}{\partial\mathfrak{p}}=\mathbf{g}|_{\Psi_{\mathfrak{p}}(t,\mathsf{z})},\ \Psi_0(t,\mathsf{z})=(t,\mathsf{z}),$$

and we get the LS of $\Sigma(t)=0$ in the form

$$\Psi_{\mathfrak{p}}(t,\mathsf{z})=(\Psi^{t}_{\mathfrak{p}}(t),\Phi^{\mathsf{s}}(\mathfrak{p},t)\mathsf{s},\Phi^{\mathsf{w}}(\mathfrak{p},t)\mathsf{w}), \tag{36}$$

where $\Phi^{\mathsf{s}}(\mathfrak{p},t)\in\mathbb{R}^{n\times n}$ and $\Phi^{\mathsf{w}}(\mathfrak{p},t)\in\mathbb{R}^{(m+r)\times(m+r)}$. For instance, if

$$F(t,\mathsf{s},\mathsf{w})=\begin{pmatrix}\mathsf{x}\\ \mathsf{u}\end{pmatrix}, \tag{37}$$

with $g^{t}(t)=t^2$ we get

$$G^{\mathsf{s}}(t)=\begin{pmatrix}t & 0\\ 1 & -t\end{pmatrix},G^{\mathsf{w}}(t)=\begin{pmatrix}-3t & 0\\ 0 & g^{\mathsf{d}}(t)\end{pmatrix}$$

(with arbitrary function $g^{\mathsf{d}}$) as solution of (32)-(33) (by selecting $\tau=0$ and $G^{\mathsf{s}}(0)=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ in (34)). From these matrices, we compute the matrices $\Phi^{\mathsf{s}}(\mathfrak{p},t)$ and $\Phi^{\mathsf{w}}(\mathfrak{p},t)$ in (36) and finally obtain a LS $\Psi_{\mathfrak{p}}$ with

$$(\mathsf{t}_{\mathfrak{p}},\mathsf{s}_{\mathfrak{p}},\mathsf{u}_{\mathfrak{p}},\mathsf{d}_{\mathfrak{p}})=\left(\frac{t}{1-\mathfrak{p}t},\begin{pmatrix}\frac{1}{1-\mathfrak{p}t} & 0\\ \mathfrak{p} & 1-\mathfrak{p}t\end{pmatrix}\mathsf{s},(1-\mathfrak{p}t)^3\mathsf{u},e^{\int_0^{\mathfrak{p}}g^{\mathsf{d}}(\frac{t}{1-\theta t})d\theta}\mathsf{d}\right). \tag{38}$$

Notice that since $\mathrm{Dom}^{+}\Psi^{t}_{\mathfrak{p}}=\frac{1}{\mathfrak{p}}$ and $\lim_{\mathfrak{p}\to+\infty}\mathrm{Dom}^{+}\Psi^{t}_{\mathfrak{p}}=0$ the LS (38) is more specifically CBU-transforming for $\mathfrak{p}>0$.

On the other hand, with $g^{t}(t)=t^2-3t+2$ we get

$$G^{\mathsf{s}}(t)=\begin{pmatrix}-1+t & 0\\ 1 & t+2\end{pmatrix},G^{\mathsf{w}}(t)=\begin{pmatrix}-t+5 & 0\\ 0 & g^{\mathsf{d}}(t)\end{pmatrix}$$

(with arbitrary function $g^{\mathsf{d}}$) (by selecting $\tau=0$ and $G^{\mathsf{s}}(0)=\begin{pmatrix}-1 & -2\\ 1 & 2\end{pmatrix}$ in (34)). From these matrices, we compute the matrices $\Phi^{\mathsf{s}}(\mathfrak{p},t)$ and $\Phi^{\mathsf{w}}(\mathfrak{p},t)$ in (36). Notice that since $\mathrm{Dom}^{+}\Psi^{t}_{\mathfrak{p}}=\frac{2e^{\mathfrak{p}}-1}{e^{\mathfrak{p}}-1}$ and $\lim_{\mathfrak{p}\to+\infty}\mathrm{Dom}^{+}\Psi^{t}_{\mathfrak{p}}=2$ the resulting LS is no more than BU-transforming (i.e. not CBU-transforming) for $\mathfrak{p}>0$.

The above result can be directly generalized to

$$F(t,\mathsf{s},\mathsf{w})=J\mathsf{s}+B\mathsf{u}, \tag{39}$$

with $(J,B)$ in Brunovski form with $J\in\mathbb{R}^{n\times n}$, $n\geq 2$. In this case, with $g^{t}(t)=t^2$, we select $\tau=0$, $G^{\mathsf{s}}(0)$ in (34) and $G^{\mathsf{w}}(t)$ in (33) such that

$$(G^{\mathsf{s}}(0)+Jt^2)e^{-Jt}B=e^{-Jt}B(G^{\mathsf{w}}(t)+2t). \tag{40}$$

The above type of time scale transformation can be used to impose additional control performances on the system such as control in *prescribed time*. Indeed, if the time scale transformation is, for instance, $\Psi^{t}_{\mathfrak{p}}(t)=\frac{t}{1-\mathfrak{p}t}$ then designing the controller in the transformed time scale $t_{\mathfrak{p}}$ automatically defines a controller with prescribed time $\mathrm{Dom}^{+}\Psi^{t}_{\mathfrak{p}}=\frac{1}{\mathfrak{p}}$ in the original time scale $t$ since $\Psi^{t}_{\mathfrak{p}}(t)\to+\infty$ as $t\nearrow\frac{1}{\mathfrak{p}}$.

### 4.2.3 Constructive step-by-step procedures for symmetries of triangular systems

Symmetries can be computed as well by exploiting the system's structure. Consider the *lower triangular* system map $\Sigma$ associated with

$$F(t,\mathsf{s},\mathsf{w}) = \begin{pmatrix} A_1(t,\mathsf{y}) + P_1(t,\mathsf{y})\mathsf{x} \\ A_2(t,\mathsf{s}) + B_2(t,\mathsf{y})\mathsf{u} + P_2(t,\mathsf{s})\mathsf{d} \end{pmatrix} \tag{41}$$

with $\mathsf{s} \triangleq (\mathsf{y},\mathsf{x})$, $\mathsf{y},\mathsf{x},\mathsf{u},\mathsf{d} \in \mathbb{R}^n$, $P_1(t,\mathsf{y})$, $B_2(t,\mathsf{y})$, $P_2(t,\mathsf{x}) \in \mathbb{GL}(n)$. We compute a symmetry of $\Sigma(t)=0$ according to a two-steps procedure. The first step consists of computing, with the help of the prolongation formula (12) and the symmetry condition (24), the infinitesimal generator $\mathbf{g}_1$ of a LS of the system $\Sigma_1(t)=0$, with system map $\Sigma_1$ associated with

$$F_1(t,\mathsf{s}_1,\mathsf{u}_1,\mathsf{d}_1) = A_1(t,\mathsf{y}_1) + P_1(t,\mathsf{y}_1)\mathsf{d}_1.$$

The system $\Sigma_1(t)=0$ is obtained from $\Sigma(t)=0$ by considering the first block of equations $D_t y(t) = A_1(t,y(t)) + P_1(t,y(t))x(t)$ in $\Sigma(t)=0$ and taking $s_1(t)=y(t)$ as state, $d_1(t)=x(t)$ as exogenous input and $y_1(t)=y(t)$ as measured variable. Relying once more on the prolongation formula (12) and the symmetry condition (24), the second step consists of extending the infinitesimal generator $\mathbf{g}_1$ (on the $(t,\mathsf{s}_1,\mathsf{u}_1,\mathsf{d}_1)$-space) to the infinitesimal generator $\mathbf{g}$ (on the $(t,\mathsf{z})$-space) of a LS of $\Sigma(t)=0$ by setting $\mathsf{d}_1 \triangleq \mathsf{x}$ and $\mathsf{y}_1 \triangleq \mathsf{y}$. For instance, the generator $\mathbf{g}_1|_{(t,\mathsf{z}_1)} = g^t(t)\frac{\partial}{\partial t} + g^{\mathsf{y}_1}(t,\mathsf{y}_1)\frac{\partial}{\partial \mathsf{y}_1} + g^{\mathsf{d}_1}(t,\mathsf{s}_1,\mathsf{d}_1)\frac{\partial}{\partial \mathsf{d}_1}$ of a LS of $\Sigma_1(t)=0$ must satisfy by (24)

$$g^{\mathsf{d}_1} = P_1^{-1}\Big[\left(\frac{\partial g^{\mathsf{y}_1}}{\partial \mathsf{y}_1} - \frac{\partial g^t}{\partial t}\right)(A_1 + P_1\mathsf{d}_1) + \frac{\partial g^{\mathsf{y}_1}}{\partial t} - (\frac{\partial A_1}{\partial t} + \frac{\partial P_1}{\partial t}\mathsf{d}_1)g^t - (\frac{\partial A_1}{\partial \mathsf{y}_1} + \frac{\partial P_1}{\partial \mathsf{y}_1}\mathsf{d}_1)g^{\mathsf{y}_1}\Big].$$

The infinitesimal generator $\mathbf{g}$ of a LS of $\Sigma(t)=0$ is obtained by extending the definition of $\mathbf{g}_1$ on the $(t,\mathsf{z})$-space with $\mathsf{d}_1 \triangleq \mathsf{x}$ and $\mathsf{y}_1 \triangleq \mathsf{y}$:

$$\mathbf{g}|_{(t,\mathsf{z})} = g^t(t)\frac{\partial}{\partial t} + g^{\mathsf{y}_1}(t,\mathsf{y})\frac{\partial}{\partial \mathsf{y}} + g^{\mathsf{d}_1}(t,\mathsf{y},\mathsf{x})\frac{\partial}{\partial \mathsf{x}} + g^{\mathsf{u}}(t,\mathsf{y},\mathsf{u})\frac{\partial}{\partial \mathsf{u}} + g^{\mathsf{d}}(t,\mathsf{z})\frac{\partial}{\partial \mathsf{d}} = \mathbf{g}_1|_{\substack{(t,\mathsf{z}_1)\\(\mathsf{d}_1,\mathsf{y}_1)=(\mathsf{x},\mathsf{y})}} + g^{\mathsf{u}}(t,\mathsf{y},\mathsf{u})\frac{\partial}{\partial \mathsf{u}} + g^{\mathsf{d}}(t,\mathsf{s},\mathsf{d})\frac{\partial}{\partial \mathsf{d}}$$

with $g^{\mathsf{u}}$ and $g^{\mathsf{d}}$ satisfying the symmetry condition (24) for $\Sigma(t)=0$

$$g^{\mathsf{u}} = B_2^{-1}\Big[\left(\frac{\partial g^{\mathsf{d}_1}}{\partial \mathsf{x}} - \frac{\partial g^t}{\partial t}\right)B_2 - \frac{\partial B_2}{\partial t}g^t\Big]\mathsf{u}, \tag{42}$$

$$g^{\mathsf{d}} = P_2^{-1}\Big[\frac{\partial g^{\mathsf{d}_1}}{\partial t} + \frac{\partial g^{\mathsf{d}_1}}{\partial \mathsf{y}}(A_1 + P_1\mathsf{x}) + \left(\frac{\partial g^{\mathsf{d}_1}}{\partial \mathsf{x}} - \frac{\partial g^t}{\partial t}\right)(A_2 + P_2\mathsf{d}) - (\frac{\partial A_2}{\partial t} + \frac{\partial P_2}{\partial t}\mathsf{d})g^t - (\frac{\partial A_2}{\partial \mathsf{y}} + \frac{\partial P_2}{\partial \mathsf{y}}\mathsf{d})g^{\mathsf{x}_1} - (\frac{\partial A_2}{\partial \mathsf{x}} + \frac{\partial P_2}{\partial \mathsf{x}}\mathsf{d})g^{\mathsf{d}_1}\Big].$$

For instance, for the system map $\Sigma$ associated with

$$F(t,\mathsf{x},\mathsf{u},\mathsf{d}) = \begin{pmatrix} \mathsf{x} \\ \mathsf{x}\mathsf{y}^k + (1+\mathsf{y}^2)(\mathsf{u}+\mathsf{d}) \end{pmatrix}, \tag{43}$$

$k \geq 1$, we obtain the infinitesimal generator

$$\mathbf{g}|_{(t,\mathsf{z})} = kt\frac{\partial}{\partial t} - \mathsf{y}\frac{\partial}{\partial \mathsf{y}} - (1+k)\mathsf{x}\frac{\partial}{\partial \mathsf{x}} + \left(-(1+2k) + \frac{2\mathsf{y}^2}{1+\mathsf{y}^2}\right)\left(\mathsf{u}\frac{\partial}{\partial \mathsf{u}} + \mathsf{d}\frac{\partial}{\partial \mathsf{d}}\right) \tag{44}$$

and, by integration of $\frac{d\Psi_{\mathfrak{p}}(t,\mathsf{z})}{d\mathfrak{p}} = \mathbf{g}|_{\Psi_{\mathfrak{p}}(t,\mathsf{z})}$, $\Psi_0(t,\mathsf{z}) = (t,\mathsf{z})$, the GS of $\Sigma(t)=0$

$$\Psi_{\mathfrak{p}}(t,\mathsf{z}) = (\mathsf{t}_{\mathfrak{p}},\mathsf{z}_{\mathfrak{p}}) = \left(e^{k\mathfrak{p}}t, \mathrm{diag}\{e^{-\mathfrak{p}}, e^{-(1+k)\mathfrak{p}}\}\mathsf{s}, e^{-(2k-1)\mathfrak{p}}\frac{1+\mathsf{y}^2}{e^{2\mathfrak{p}}+\mathsf{y}^2}\mathsf{w}\right). \tag{45}$$

The GGT $\Psi_{\mathfrak{p}}$ is UU-transforming for $\mathfrak{p} > 0$.

The reader can extract from above a *recursive* symmetry-computing method for higher-dimensional *lower triangular* systems. As well we can outline a *recursive* step-by-step method for *upper triangular* systems. Consider the *upper triangular* system map $\Sigma$ associated with

$$F(t,\mathsf{s},\mathsf{u},\mathsf{d}) = \begin{pmatrix} A_1(t,\mathsf{x}) + B_1(t,\mathsf{u}) + P_1(t,\mathsf{d}) \\ B_2(t,\mathsf{u}) + P_2(t,\mathsf{d}) \end{pmatrix} \tag{46}$$

with $\mathsf{s} \triangleq (\mathsf{y},\mathsf{x})$, $\mathsf{y},\mathsf{x},\mathsf{u},\mathsf{d} \in \mathbb{R}^n$. The first step consists of computing, with the help of the prolongation formula (12) and the symmetry condition (24), the infinitesimal generator $\mathbf{g}_1$ of a LS of the system $\Sigma_1(t)=0$, with system map $\Sigma_1$ associated with

$$F_1(t,\mathsf{s}_1,\mathsf{u}_1,\mathsf{d}_1) = A_1(t,\mathsf{d}_{1,1}) + B_1(t,\mathsf{u}_1) + P_1(t,\mathsf{d}_{1,2}).$$

The system $\Sigma_1(t) = 0$ is obtained from $\Sigma(t) = 0$ by considering the first block of equations $D_t y(t) = A_1(t, x(t)) + B_1(t, u(t)) + P_1(t, d(t))$ in $\Sigma(t) = 0$ and taking $s_1(t) = y(t)$ as state, $d_1(t) = (d_{1,1}(t), d_{1,2}(t)) = (x(t), d(t))$ as exogenous input and $y_1(t) = y(t)$ as measured variable. Relying once more on the prolongation formula (12) and the symmetry condition (24), the second step consists of extending the infinitesimal generator $\mathbf{g}_1$ (on the $(t, \mathsf{s}_1, \mathsf{u}_1, \mathsf{d}_1)$-space) to the infinitesimal generator $\mathbf{g}$ (on the $(t, \mathsf{z})$-space) of a LS of $\Sigma(t) = 0$ by setting $\mathsf{d}_1 = (\mathsf{d}_{1,1}, \mathsf{d}_{1,2}) \triangleq (\mathsf{x}, \mathsf{d})$, $\mathsf{u}_1 \triangleq \mathsf{u}$ and $\mathsf{y}_1 \triangleq \mathsf{y}$. For instance, the generator $\mathbf{g}_1|_{(t,\mathsf{z}_1)} = g^t(t)\frac{\partial}{\partial t} + g^{\mathsf{y}_1}(t, \mathsf{y}_1)\frac{\partial}{\partial \mathsf{y}_1} + g^{\mathsf{u}_1}(t, \mathsf{y}_1, \mathsf{u}_1)\frac{\partial}{\partial \mathsf{u}_1} + g^{\mathsf{d}_{1,1}}(t, \mathsf{y}_1, \mathsf{d}_{1,1})\frac{\partial}{\partial \mathsf{d}_{1,1}} + g^{\mathsf{d}_{1,2}}(t, \mathsf{y}_1, \mathsf{d}_1)\frac{\partial}{\partial \mathsf{d}_{1,2}}$ of a LS of $\Sigma_1(t) = 0$ must satisfy by (24)

$$g^{\mathsf{u}_1} = \left(\frac{\partial B_1}{\partial \mathsf{u}_1}\right)^{-1} \left[ \left(\frac{\partial g^{\mathsf{y}_1}}{\partial \mathsf{y}_1} - \frac{\partial g^t}{\partial t}\right) B_1 - \frac{\partial B_1}{\partial t} g^t \right],$$
$$g^{\mathsf{d}_{1,1}} = \left(\frac{\partial A_1}{\partial \mathsf{d}_{1,1}}\right)^{-1} \left[ \left(\frac{\partial g^{\mathsf{y}_1}}{\partial \mathsf{y}_1} - \frac{\partial g^t}{\partial t}\right) A_1 + \frac{\partial g^{\mathsf{y}_1}}{\partial t} - \frac{\partial A_1}{\partial t} g^t \right], g^{\mathsf{d}_{1,2}} = \left(\frac{\partial P_1}{\partial \mathsf{d}_{1,2}}\right)^{-1} \left[ \left(\frac{\partial g^{\mathsf{y}_1}}{\partial \mathsf{y}_1} - \frac{\partial g^t}{\partial t}\right) P_1 - \frac{\partial P_1}{\partial t} g^t \right] \tag{47}$$

The infinitesimal generator $\mathbf{g}|_{(t,\mathsf{z})}$ of a LS of $\Sigma(t) = 0$ is obtained by extending the definition of $\mathbf{g}_1$ on the $(t, \mathsf{z})$-space with $(\mathsf{d}_{1,1}, \mathsf{d}_{1,2}) \triangleq (\mathsf{x}, \mathsf{d})$, $\mathsf{u}_1 \triangleq \mathsf{u}$ and $\mathsf{y}_1 \triangleq \mathsf{y}$:

$$\mathbf{g}|_{(t,\mathsf{z})} = g^t(t)\frac{\partial}{\partial t} + g^{\mathsf{y}_1}(t, \mathsf{y})\frac{\partial}{\partial \mathsf{y}} + g^{\mathsf{d}_{1,1}}(t, \mathsf{y}, \mathsf{x})\frac{\partial}{\partial \mathsf{x}} + g^{\mathsf{u}}(t, \mathsf{y}, \mathsf{u})\frac{\partial}{\partial \mathsf{u}} + g^{\mathsf{d}_{1,2}}(t, \mathsf{s}, \mathsf{d})\frac{\partial}{\partial \mathsf{d}} = \mathbf{g}_1|^{(t,\mathsf{z}_1)}_{(\mathsf{d}_{1,1},\mathsf{d}_{1,2})=(\mathsf{x},\mathsf{d}),(\mathsf{u}_1,\mathsf{y}_1)=(\mathsf{u},\mathsf{y})}$$

and satisfying the symmetry condition (24) for the overall system $\Sigma(t) = 0$. For instance, for the system map $\Sigma$ associated with

$$F(t, \mathsf{x}, \mathsf{u}, \mathsf{d}) = \begin{pmatrix} \mathsf{x} + \mathsf{u}^2 \\ \mathsf{u} \end{pmatrix}, \tag{48}$$

we obtain the infinitesimal generator

$$\mathbf{g}|_{(t,\mathsf{z})} = -t\frac{\partial}{\partial t} - 3\mathsf{y}\frac{\partial}{\partial \mathsf{y}} - 2\mathsf{x}\frac{\partial}{\partial \mathsf{x}} - \mathsf{u}\frac{\partial}{\partial \mathsf{u}} - g^{\mathsf{d}}\frac{\partial}{\partial \mathsf{d}} \tag{49}$$

with arbitrary $g^{\mathsf{d}}$ and, by integration of $\frac{d\Psi_{\mathfrak{p}}(t,\mathsf{z})}{d\mathfrak{p}} = \mathbf{g}|_{\Psi_{\mathfrak{p}}(t,\mathsf{z})}$, $\Psi_0(t, \mathsf{z}) = (t, \mathsf{z})$, the GS of $\Sigma(t) = 0$

$$\Psi_{\mathfrak{p}}(t, \mathsf{z}) = (\mathsf{t}_{\mathfrak{p}}, \mathsf{z}_{\mathfrak{p}}) = (e^{-\mathfrak{p}}t, \mathrm{diag}\{e^{-3\mathfrak{p}}, e^{-2\mathfrak{p}}\}\mathsf{s}, e^{-\mathfrak{p}}\mathsf{u}, e^{-g^{\mathsf{d}}\mathfrak{p}}\mathsf{d}). \tag{50}$$

The GGT $\Psi_{\mathfrak{p}}$ is UU-transforming for $\mathfrak{p} > 0$.

### 4.2.4 | Symmetries of linear systems with locally linear input nonlinearities

Consider the system map $\Sigma$ associated with

$$F(t, \mathsf{s}, \mathsf{u}, \mathsf{d}) = A\mathsf{s} + B\gamma(\mathsf{u}) + P\mathsf{d},$$

with $\mathsf{u} \in \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ strictly increasing ($C^\infty$) function such that $\gamma(0) = 0$. Consider a LGT $\Psi_{\mathfrak{p}}$ of the form $\Psi_{\mathfrak{p}}(t, \mathsf{z}) = (e^{k\mathfrak{p}}t, e^{G^{\mathsf{s}}\mathfrak{p}}\mathsf{s}, \Psi^{\mathsf{u}}_{\mathfrak{p}}(\mathsf{u}), e^{G^{\mathsf{d}}\mathfrak{p}}\mathsf{d})$, where $k \in \mathbb{R}$, $G^{\mathsf{s}} \in \mathbb{R}^{n\times n}$ and $G^{\mathsf{d}} \in \mathbb{R}^{s\times s}$ are such that

$$AG^{\mathsf{s}} - G^{\mathsf{s}}A + kA = 0,\ \ PG^{\mathsf{d}} - G^{\mathsf{s}}P + kP = 0, \tag{51}$$
$$G^{\mathsf{s}}B = \overline{g}^{\mathsf{u}}B \tag{52}$$

for some $\overline{g}^{\mathsf{u}} \in \mathbb{R}$. Using the prolongation formula (10) for computing $\mathrm{pr}^{[1]}\mathbf{g}$ in (22), the symmetry condition (23) for $\Sigma(t) = 0$ boils down to

$$e^{\mathfrak{p}(G^{\mathsf{s}}-kI_n)}(A\mathsf{s} + B\gamma(\mathsf{u}) + P\mathsf{d}) = Ae^{G^{\mathsf{s}}\mathfrak{p}}\mathsf{s} + B\gamma(\Psi^{\mathsf{u}}_{\mathfrak{p}}(\mathsf{u})) + Pe^{G^{\mathsf{d}}\mathfrak{p}}\mathsf{d} \tag{53}$$

which on account of (51)-(52) is satisfied with $\Psi^{\mathsf{u}}_{\mathfrak{p}}(\mathsf{u}) = \gamma^{-1}(e^{(\overline{g}^{\mathsf{u}}-k)\mathfrak{p}}\gamma(\mathsf{u}))$. Notice that $\Psi_{\mathfrak{p}}$ is a LGT with $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \{(t, \mathsf{z}) \in \mathcal{M} : e^{(\overline{g}^{\mathsf{u}}-k)\mathfrak{p}}\gamma(\mathsf{u}) \in \mathrm{Dom}(\gamma^{-1})\}$. The infinitesimal generator is

$$\mathbf{g}|_{(t,\mathsf{z})} = kt\frac{\partial}{\partial t} + \sum_{j=1}^{n}(G^{\mathsf{s}}\mathsf{s})_j\frac{\partial}{\partial \mathsf{s}_j} + (\overline{g}^{\mathsf{u}} - k)\left(\frac{\partial \gamma}{\partial \mathsf{u}}\right)^{-1}\gamma(\mathsf{u})\frac{\partial}{\partial \mathsf{u}} \tag{54}$$

For instance, if

$$F(t, \mathsf{x}, \mathsf{w}) = \begin{pmatrix} \mathsf{x} \\ \mathrm{arctg}(\mathsf{u}) \end{pmatrix}, \tag{55}$$

we get the LS $\Psi_{\mathfrak{p}}$ with

$$\mathsf{t}_{\mathfrak{p}} = e^{-\mathfrak{p}}t, \mathsf{s}_{\mathfrak{p}} = \mathrm{diag}\{e^{-3\mathfrak{p}}, e^{-2\mathfrak{p}}\}\mathsf{s}, \mathsf{u}_{\mathfrak{p}} = \mathrm{tg}(e^{-\mathfrak{p}}\mathrm{arctg}(\mathsf{u})) \tag{56}$$

and arbitrary $\mathsf{d}_{\mathfrak{p}}$, with $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \mathcal{M}$ for $\mathfrak{p} > 0$ and $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \{(t,\mathsf{z}) \in \mathcal{M} : |\mathrm{arctg}(\mathsf{u})| < e^{\mathfrak{p}}\pi/2\}$ for $\mathfrak{p} < 0$.

### 4.2.5 | Symmetries of systems with locally linear state nonlinearities

Consider the system map $\Sigma$ associated with

$$F(t,\mathsf{s},\mathsf{u},\mathsf{d}) = \begin{pmatrix} \sin(\mathsf{x}) \\ \mathsf{u} \end{pmatrix} \tag{57}$$

with $\mathsf{s} = (\mathsf{y},\mathsf{x}) \in \mathbb{R}^2$. Assuming a candidate infinitesimal generator $\mathbf{g}$ of the form $\mathbf{g}|_{(t,\mathsf{z})} \triangleq t\frac{\partial}{\partial t} + g^{\mathsf{y}}(\mathsf{y})\frac{\partial}{\partial \mathsf{y}} + g^{\mathsf{x}}(\mathsf{s})\frac{\partial}{\partial \mathsf{x}} + g^{\mathsf{u}}(\mathsf{u})\frac{\partial}{\partial \mathsf{u}}$ and using the prolongation formula (12) for computing $\mathrm{pr}^{[1]}\mathbf{g}$ in (22), the symmetry condition (24) reads as

$$\begin{aligned} \left(\frac{\partial g^{\mathsf{y}}}{\partial \mathsf{y}} - \frac{\partial g^{t}}{\partial t}\right)\sin(\mathsf{x}) &= \cos(\mathsf{x})g^{\mathsf{x}}, \\ \frac{\partial g^{\mathsf{x}}}{\partial \mathsf{y}}\sin(\mathsf{x}) + \left(\frac{\partial g^{\mathsf{x}}}{\partial \mathsf{x}} - \frac{\partial g^{t}}{\partial t}\right)\mathsf{u} &= g^{\mathsf{u}}. \end{aligned} \tag{58}$$

We get the simple solution $\mathbf{g}|_{(t,\mathsf{z})} \triangleq t\frac{\partial}{\partial t} - \mathrm{tg}(\mathsf{x})\frac{\partial}{\partial \mathsf{x}} - (1 + \frac{1}{\cos^2(\mathsf{x})})\frac{\partial}{\partial \mathsf{u}}$, $|\mathsf{x}| < \pi/2$, and the LS $\Psi_{\mathfrak{p}}$ with

$$\mathsf{t}_{\mathfrak{p}} = e^{\mathfrak{p}}t, \mathsf{s}_{\mathfrak{p}} = \left(\mathsf{y}, \arcsin(e^{-\mathfrak{p}}\sin(\mathsf{x}))\right), \mathsf{u}_{\mathfrak{p}} = \frac{e^{-\mathfrak{p}}}{\sqrt{e^{2\mathfrak{p}} - \sin^2(\mathsf{x})}}. \tag{59}$$

Moreover, $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \{(t,\mathsf{z}) \in \mathcal{M} : |\mathsf{x}| < \pi/2\}$ for $\mathfrak{p} > 0$ and $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \{(t,\mathsf{z}) \in \mathcal{M} : |\sin(\mathsf{x})| < e^{-|\mathfrak{p}|}, |\mathsf{x}| < \pi/2\}$ for $\mathfrak{p} < 0$.

## 4.3 | Contracting symmetries

As already noted, a LS transforms $\Sigma(t) = 0$ into $\widetilde{\Sigma}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}) = 0$. If the norm of $(s_{\mathfrak{p}}(0), d_{\mathfrak{p}}(t))$ can be made sufficiently small by sufficiently large values of $\mathfrak{p} > 0$ (i.e. the norm of $(s(0), d(t))$ can be made sufficiently small via the action of the symmetry by sufficiently large values of its parameter) then a controller/observer which is designed locally on $\widetilde{\Sigma}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}) = 0$ can be pulled back by the inverse group transformation $[\Psi_{\mathfrak{p}}]^{-1} = \Psi_{-\mathfrak{p}}$ so that to obtain a semiglobal/global controller/observer for $\Sigma(t) = 0$. Obviously, for all of this to happen, the symmetry $\Psi_{\mathfrak{p}}$ must possess certain contraction properties on the $(\mathsf{s},\mathsf{d})$-space. This motivates the following definition, which we borrow from[7] under a simplified version but retaining at the same time the key property of contracting the $(\mathsf{s},\mathsf{d})$-space. We recall that $\mathsf{s}_{\mathfrak{p}}$ and $\mathsf{d}_{\mathfrak{p}}$ denote the functions $\Psi^{\mathsf{s}}_{\mathfrak{p}}(t,\mathsf{s})$ and, respectively, $\Psi^{\mathsf{d}}_{\mathfrak{p}}(t,\mathsf{s},\mathsf{d})$ (see (Def. (14))).

**Definition 5.** (*SI-contraction*). A LGT (resp. GGT) $\Psi_{\mathfrak{p}}$ on $\mathcal{M}$ is *state contracting (SI-contracting)* with contracting maps $(\sigma,\mu)$ if there exist $\sigma \in \mathcal{K}_\infty$ and $\mu \in \mathcal{L}$ such that for all $(\mathfrak{p},t,\mathsf{z}) \in \mathrm{Dom}(\Psi)$ for which $\sigma(\|\mathsf{s}\|) \leq \mathfrak{p}$ we have

$$\|(\mathsf{s}_{\mathfrak{p}},\mathsf{d}_{\mathfrak{p}})\| \leq \mu(\mathfrak{p}).\lhd \tag{60}$$

The above definition of SI-contracting symmetry has been introduced in a slightly weaker form in . Definition 5 actually captures the claimed contracting action of a symmetry $\Psi_{\mathfrak{p}}$ on the $(\mathsf{s},\mathsf{d})$-space. Indeed, for a LGT $\Psi_{\mathfrak{p}}$ with $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \mathcal{M}$ for each $\mathfrak{p} > 0$ and SI-contracting with contracting maps $(\sigma,\mu)$, if we pick any $(t,\mathsf{z}) \in \mathcal{M}$ then there exists $\mathfrak{p}_0 > 0$ such that $\sigma(\|(\mathsf{s},\mathsf{d})\|) \leq \mathfrak{p}$ for $\mathfrak{p} \geq \mathfrak{p}_0$, hence (60) hold for $\mathfrak{p} \geq \mathfrak{p}_0$. Since $\mu \in \mathcal{L}$ we get $\lim_{\mathfrak{p}\to+\infty} \|(\mathsf{s}_{\mathfrak{p}},\mathsf{d}_{\mathfrak{p}})\| = 0$. In conclusion, $\Psi_{\mathfrak{p}}$ acts contractively on the $(\mathsf{s},\mathsf{d})$-space as $\mathfrak{p} \to +\infty$.

Below we provide some examples.

For the GGT (45) we have $\|\mathsf{s}_{\mathfrak{p}}\| \leq e^{-\mathfrak{p}}|\mathsf{y}| + e^{-(1+k)\mathfrak{p}}|\mathsf{x}| \leq e^{-\mathfrak{p}}\|\mathsf{s}\|$ and $|\mathsf{d}_{\mathfrak{p}}| \leq e^{-(2k-1)\mathfrak{p}}|\mathsf{d}|$ for all $\mathfrak{p}, t \geq 0$ and $(\mathsf{s},\mathsf{d}) \in \mathbb{R}^2 \times \mathbb{R}$. Hence, (45) is SI-contracting with contracting maps

$$(\sigma,\mu) = \left(\frac{1}{1-\delta}\ln(r+1), e^{-\delta r}\right) \tag{61}$$

for any $\delta \in (0,1)$.

For the GGT (56) with $\mathsf{d}_{\mathfrak{p}} = e^{-3\mathfrak{p}}\mathsf{d}$ (recall that $\mathsf{d}_{\mathfrak{p}}$ is arbitrary) we have $\|\mathsf{s}_{\mathfrak{p}}\| \leq e^{-2\mathfrak{p}}\|\mathsf{s}\|$ and $|\mathsf{d}_{\mathfrak{p}}| \leq e^{-2\mathfrak{p}}|\mathsf{d}|$ for all $\mathfrak{p}, t \geq 0$ and $(\mathsf{s}, \mathsf{d}) \in \mathbb{R}^2 \times \mathbb{R}$. Hence, (56) is SI-contracting with contracting maps

$$(\sigma, \mu) = (\frac{1}{1-\delta}\ln(r+1), e^{-(1+\delta)r}) \tag{62}$$

for any $\delta \in (0, 1)$.

For the LGT (29) we have $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \mathcal{M}$ for each $\mathfrak{p} \geq 0$

$$\begin{aligned} &\|\mathsf{s}\| \leq \mathfrak{p}\delta \Rightarrow \|\mathsf{s}_{\mathfrak{p}}\| \leq \ln(1 + e^{-\mathfrak{p}(1-\delta)}) + e^{-\mathfrak{p}(1-\delta)}, \\ &|\mathsf{d}| \leq \mathfrak{p}\delta \Rightarrow |\mathsf{d}_{\mathfrak{p}}| \leq e^{-\mathfrak{p}(1-\delta)}, \end{aligned} \tag{63}$$

with any $\delta \in (0, 1)$. Indeed, $|\mathsf{y}| \leq \mathfrak{p}\delta \Rightarrow |\mathsf{y}_{\mathfrak{p}}| = |\ln \frac{1}{1-e^{-\mathfrak{p}}+e^{-\mathfrak{p}-\mathsf{y}}}| \leq \ln(1 + e^{-\mathfrak{p}}(e^{\mathfrak{p}\delta} - 1)) \leq \ln(1 + e^{-\mathfrak{p}(1-\delta)})$. Hence (29) is SI-contracting with contracting maps

$$(\sigma, \mu) = (\frac{r}{\delta}, \ln(1 + e^{-r}(e^{\mathfrak{p}\delta} - 1)) + e^{-r(1-\delta)}) \tag{64}$$

for any $\delta \in (0, 1)$. For the GGT (30) we have for any $\delta \in (0, 1)$

$$\ln(1 + \|\mathsf{s}\|) \leq \mathfrak{p}\delta \Rightarrow \|\mathsf{x}_{\mathfrak{p}}\| \leq e^{-\mathfrak{p}(1-\delta)}, \tag{65}$$

$$\ln(1 + |\mathsf{d}|) \leq \mathfrak{p}\delta \Rightarrow |\mathsf{d}_{\mathfrak{p}}| \leq e^{-\mathfrak{p}(1-\delta)}. \tag{66}$$

However, (30) is not SI-contracting since $|\mathsf{y}_{\mathfrak{p}}| = |\mathsf{y}|$.

## 4.4 | From local controller/observers to semiglobal/global controller/observers

The design of semiglobal or global controllers from local controllers follows the idea outlined below. This approach shares similarities with the frameworks in[7] and[9] where the discussion was restricted to observer design for systems with detectable linearization. However, the methodology pursued here is fundamentally different. In the observer design considered in [7, 9], an observer is constructed for the transformed system, assuming that the original system has a detectable linearization. Since detectability of the linearization is preserved under the symmetry, the observer can be designed based on the linearization of the transformed system. In contrast, in this paper we do not assume detectability of the linearization for the original system. We only require that a local observer is available. Using this local observer, and exploiting the fact that the system structure is preserved under the symmetry, we construct a semiglobal observer for the original system via the inverse group transformation. Finally, the convergence and stability analysis in this framework is significantly more involved than in[9] motivating more technical and careful proofs.

Let us discuss the key points of the idea. Let $d(t)$ be some exogenous input and $\Psi_{\mathfrak{p}}$ a LS of $\Sigma(t) = 0$ with $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \mathcal{M}$ for $\mathfrak{p} > 0$, SI-contracting with contracting maps $(\sigma, \mu)$ and UU-transforming for $\mathfrak{p} > 0$. Similar considerations apply (with proper modifications) for LS that are BU- or CBU-transforming. Pick any $\varrho > 0$. Since $\sigma \in \mathcal{K}_\infty$ and $\mu \in \mathcal{L}$ hence $\mu \circ \sigma \in \mathcal{L}$. Assume that the initial conditions $s(0)$ and exogenous inputs $d(t)$ satisfy for all $t \geq 0$ and for some $\mathfrak{p} > 0$

$$\sigma(\|(s(0), d(t))\| + \omega) \leq \mathfrak{p} \tag{67}$$

where $\omega \geq 0$ is sufficiently large so that

$$(\mu \circ \sigma)(\omega) \leq \varrho \tag{68}$$

On account of the SI-contraction property (60) of $\Psi_{\mathfrak{p}}$ and since $\mu, \mu \circ \sigma \in \mathcal{L}$ it follows that the transformed initial conditions $s_{\mathfrak{p}}(0)$ and exogenous inputs $d_{\mathfrak{p}}(t)$ in time scale $t$ satisfy for all $t \geq 0$

$$\|(s_{\mathfrak{p}}(0), d_{\mathfrak{p}}(t))\| \leq \mu(\mathfrak{p}) \leq (\mu \circ \sigma)(\omega) \leq \varrho, \tag{69}$$

hence in time scale $\mathsf{t}_{\mathfrak{p}}$,

$$\|(\widetilde{s}_{\mathfrak{p}}(0), \widetilde{d}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}))\| \leq \varrho \tag{70}$$

for all $\mathsf{t}_{\mathfrak{p}} \geq 0$. If a controller $\widetilde{u}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})$ is available for the transformed system $\widetilde{\Sigma}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}) = 0$ such that

$$\limsup_{\mathsf{t}_{\mathfrak{p}} \to +\infty} \|\widetilde{s}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})\| \leq \widetilde{\gamma}(\sup_{\mathsf{t}_{\mathfrak{p}} \geq 0} \|\widetilde{d}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})\|)$$

for initial conditions $\widetilde{s}_{\mathfrak{p}}(0)$ and exogenous inputs $\widetilde{d}_{\mathfrak{p}}(t_{\mathfrak{p}})$ satisfying (70), i.e. in a "small" ball of radius $\varrho$, by inverse group transformation $\Psi_{\mathfrak{p}}^{-1}$ we get a controller $u(t)$ for the original system $\Sigma(t) = 0$ such that

$$\limsup_{t\to+\infty} \|s(t)\| \leq \gamma(\sup_{t\geq 0} \|d(t)\|)$$

for all the initial conditions $s(0)$ and exogenous inputs $d(t)$ satisfying (67), i.e. in a "large" ball of radius $\sigma^{-1}(\mathfrak{p})$. This ball can be taken larger and larger as $\mathfrak{p} \to +\infty$. Moreover, asymptotic convergence in both time scales $t$ and $t_{\mathfrak{p}}$ is guaranteed by the symmetry $\Psi_{\mathfrak{p}}$ being $UU$-transforming.

A similar principle can be applied to observer design as long as the solutions $s(t)$ and exogenous inputs $d(t)$ satisfy for all $t \geq 0$ and for some $\mathfrak{p} > 0$

$$\sigma(\|(s(t), d(t))\| + \omega) \leq \mathfrak{p} \tag{71}$$

where $\omega \geq 0$ is as in (68): from a *local* observer for the transformed system $\widetilde{\Sigma}(\mathsf{t}_{\mathfrak{p}}) = 0$ for all its solutions $\widetilde{s}_{\mathfrak{p}}(t_{\mathfrak{p}})$ and exogenous inputs $\widetilde{d}_{\mathfrak{p}}(t_{\mathfrak{p}})$ remaining in a "small" ball of radius $\varrho$ we obtain a *semiglobal* observer for the origin system $\Sigma(t) = 0$ for all its solutions $s(t)$ and exogenous inputs $d(t)$ remaining in a "large" ball of radius $\sigma^{-1}(\mathfrak{p})$. Following a similar philosophy but using a *state-norm* estimator it is possible to extend this type of observer design to the case of *unbounded* solutions $s(t)$, i.e. not remaining in a compact set, with initial conditions $s(0)$ and exogenous inputs $d(t)$ satisfying (71).

# 5 | SYMMETRY-BASED CONTROLLERS

## 5.1 | Semiglobal symmetry-based controllers with asymptotic convergence

In this section we consider the problem of transforming a *local* state-feedback controller, local in the sense of tolerating only sufficiently small initial state conditions and exogenous inputs, into a *semiglobal* controller, semiglobal in the sense of tolerating initial state conditions and exogenous inputs in a *priori* given compact set: $\|(s(0), d(t))\| \leq N$ for some *a priori* given $N \geq 0$ and for all $t \geq 0$. We consider the case of infinite control horizon, i.e. the system time scale is $[0, +\infty)$. To begin with, we assume the existence of a SI-contracting symmetry transforming infinite time scales into infinite time scales.

***Assumption S.1***: $\Psi_{\mathfrak{p}}$ is a LS of $\Sigma(t) = 0$ with $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \mathcal{M}$ for $\mathfrak{p} > 0$, SI-contracting with contracting maps $(\sigma, \mu)$ and UU-transforming for $\mathfrak{p} > 0$. $\lhd$

As announced, we assume to have at hand a *local* state-feedback controller of the form $u(t) = \varphi(t, s(t))$ for $\Sigma(t) = 0$. We formulate this assumption in terms of a *local* dissipational inequality involving energy (or Lyapunov) functions.

***Assumption S.2***: There exist open neighbourhoods $\mathcal{B}^{\mathsf{s}} \subset \mathcal{S}$ and $\mathcal{B}^{\mathsf{d}} \subset \mathbb{R}^s$ of the origin, $\alpha, \underline{\beta}, \overline{\beta}, \gamma \in \mathcal{K}_\infty$ and a $C^1$ function $V : \mathbb{R}^0_+ \times \mathcal{B}^{\mathsf{s}} \to \mathbb{R}^0_+$ and $\varphi : \mathbb{R} \times \mathcal{B}^{\mathsf{s}} \to \mathbb{R}^m$ such that for all $(t, \mathsf{s}, \mathsf{d}) \in \mathbb{R}^0_+ \times \mathcal{B}^{\mathsf{s}} \times \mathcal{B}^{\mathsf{d}}$

$$\underline{\beta}(\|\mathsf{s}\|) \leq V(t, \mathsf{s}) \leq \overline{\beta}(\|\mathsf{s}\|), \tag{72}$$

$$\frac{\partial V(t, \mathsf{s})}{\partial t} + \frac{\partial V(t, \mathsf{s})}{\partial \mathsf{s}} F(t, \mathsf{s}, \varphi(t, \mathsf{s}), \mathsf{d}) \leq -\alpha(V(t, \mathsf{s})) + \gamma(\|\mathsf{d}\|).\lhd \tag{73}$$

Some (mild) extra conditions are required on the state transformation $\Psi^{\mathsf{s}}_{\mathfrak{p}}$ and the inverse input transformation map $\Psi^{\mathsf{u}}_{-\mathfrak{p}}$ for obtaining a converging state $s(t)$ with control input $u(t)$ (in time scale $t$) from a converging transformed state $\widetilde{s}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})$ with transformed control input $\widetilde{u}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})$ (in time scale $\mathsf{t}_{\mathfrak{p}}$). We remind the reader of the notational meaning of the functions $\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}}, \mathsf{d}_{\mathfrak{p}}$ (Def. (14)).

***Assumption S.3***: There exist $\mathfrak{p}_0 > 0$ and $\pi_1 \in \mathcal{K}_+$ such that if $\chi(r) \triangleq (\underline{\beta}^{-1} \circ \overline{\beta} \circ \underline{\beta}^{-1} \circ \alpha^{-1})(2\gamma(s))$ and for all $\mathfrak{p} \geq \mathfrak{p}_0$

$$\begin{aligned} &\|\mathsf{s}_{\mathfrak{p}}\| < 2 \max\{(\underline{\beta}^{-1} \circ \overline{\beta})(\mu(\mathfrak{p})), \chi(\mu(\mathfrak{p}))\} \\ &\Rightarrow \left(\mathsf{s}_{\mathfrak{p}} \in \mathcal{B}^{\mathsf{s}}\right) \wedge \left((\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, \varphi(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}})) \in \mathrm{Dom}(\Psi^{\mathsf{u}}_{-\mathfrak{p}})\right) \wedge \left(\|\Psi^{\mathsf{s}}_{-\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}})\| \leq \frac{\pi_1(\|\mathsf{s}_{\mathfrak{p}}\|)\|\mathsf{s}_{\mathfrak{p}}\|}{\pi_2(\mathfrak{p})}\right).\lhd \end{aligned} \tag{74}$$

The way we formulate Assumption S.3 is motivated by the fact that, even if $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \mathcal{M}$, it may as well happen that $(\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, \varphi(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}})) \notin \mathrm{Dom}(\Psi^{\mathsf{u}}_{-\mathfrak{p}})$ for some $\mathfrak{p} > 0$, while by definition of one-parameter group always $(\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, \mathsf{u}_{\mathfrak{p}}) \in \mathrm{Dom}(\Psi^{\mathsf{u}}_{-\mathfrak{p}})$ for all $\mathfrak{p} > 0$ and $(t, \mathsf{s}, \mathsf{u}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ with $\Psi^{\mathsf{u}}_{-\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, \mathsf{u}_{\mathfrak{p}}) = \mathsf{u}$. Also, by definition of one-parameter group we always have $(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}}) \in \mathrm{Dom}(\Psi^{\mathsf{s}}_{-\mathfrak{p}})$ and $\Psi^{\mathsf{s}}_{-\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}}) = \mathsf{s}$. With Assumption S.3 we ask that for $\mathsf{s}_{\mathfrak{p}}$ in a sufficiently small neighbourhood $\mathcal{B}^{\mathsf{s}}$ we have $(\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, \varphi(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}})) \in \mathrm{Dom}(\Psi^{\mathsf{u}}_{-\mathfrak{p}})$.

***Assumption S.4***: There exist $\pi_3 \in \mathcal{K}$ and $\pi_2 \in \mathcal{L}$ such that

$$\chi(\|\mathsf{d}_{\mathfrak{p}}\|) \leq \pi_2(\mathfrak{p})\pi_3(\|\mathsf{d}\|)$$

for all $\mathfrak{p} > 0$ and $(t, \mathsf{z}) \in \mathcal{M}$. ◁

**Theorem 1.** Under Assumptions S.1-S.4 and for each $N > 0$ consider the solutions $(t, z(t))$ of $\Sigma(t) = 0$ with

$$u(t) \triangleq \Psi^{\mathsf{u}}_{-\mathfrak{p}}(t_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t), \varphi(t_{\mathfrak{p}}(t), s_{\mathfrak{p}}(t))), \tag{75}$$

and $\sup_{t\geq 0} \|(s(0), d(t))\| \leq N$. There exists $\mathfrak{p} > 0$ such that

$$\limsup_{t\to+\infty} \|s(t)\| \leq \pi_1(\sup_{s\in\mathcal{B}^{\mathsf{s}}} \|s\|)\pi_3(\sup_{t\geq 0} \|d(t)\|).◁ \tag{76}$$

In other words, the *local* controller $u(t) = \varphi(t, s(t))$ (here, “local” means specifically that $\varphi(t, \mathsf{s})$ satisfies (73) for $(\mathsf{s}, \mathsf{d})$ in a “small” neighbourhood $\mathcal{B}^{\mathsf{s}} \times \mathcal{B}^{\mathsf{d}}$ of the origin) is transformed by the symmetry $\Psi_{\mathfrak{p}}$ into the *semiglobal* controller (75) (here, “semiglobal” means that (75) satisfies (76) for $(s(0), d(t))$ in a pre-assigned ball of radius $N$).

*Remark 1.* (*Design parameters*). The proof of Theorem 1, reported in the Appendix, points out the procedure for tuning the parameter $\mathfrak{p} > 0$ of the controller (75) which is readily obtained from the local controller $u(t) = \varphi(t, s(t))$. Notice that if $d(t) \equiv 0$ we have exact convergence $x(t) \to 0$ as $t \to +\infty$. Also on account of (67), notice that by increasing $\mathfrak{p} > 0$ we extend automatically the set of initial conditions $s(0)$ and bounded exogenous inputs $d(t)$ for which (76) holds.

*Remark 2.* (*Robustness*). Robustness and sensitivity of the resulting semiglobal controller (75) are inherited from the local controller $u(t) = \varphi(t, s(t))$ and quantitatively evaluated by the input-to-output inequality (76).

### 5.1.1 | The case of stabilizability in the first approximation

The search for functions $V, \alpha, \overline{\beta}, \underline{\beta}, \gamma$ satisfying Assumption S.2 is, say, canonical when the linear approximation of $\Sigma(t) = 0$ is stabilizable. Let

$$A(t) \triangleq \frac{\partial F(t, \mathsf{s}, \mathsf{w})}{\partial \mathsf{s}}\Big|_{(\mathsf{s},\mathsf{w})=0}, B(t) \triangleq \frac{\partial F(t, \mathsf{s}, \mathsf{w})}{\partial \mathsf{u}}\Big|_{(\mathsf{s},\mathsf{w})=0}, G(t) \triangleq \frac{\partial F(t, \mathsf{s}, \mathsf{w})}{\partial \mathsf{d}}\Big|_{(\mathsf{s},\mathsf{w})=0}, \tag{77}$$

$$(\Delta\Sigma_L)(t, \mathsf{s}, \mathsf{w}) \triangleq F(t, \mathsf{s}, \mathsf{w}) - (A(t)\mathsf{s} + B(t)\mathsf{u} + G(t)\mathsf{d}) \tag{78}$$

The following set of “stabilizability in the first approximation” assumptions are considered for $\Sigma(t) = 0$ in place of Assumption S.2.

***Assumption S.2.1***: $\sup_{t\geq 0} \|B(t)\| < +\infty$, $\sup_{t\geq 0} \|G(t)\| < +\infty$ and there exist $F : \mathbb{R}^0_+ \to \mathbb{R}^{m\times n}$, $P : \mathbb{R}^0_+ \to \mathbb{S}_+(n)$ and $\underline{p}, \overline{p} > 0$ such that $\sup_{t\geq 0} \|F(t)\| < +\infty$ and for all $t \geq 0$

$$D_t P(t) + P(t)(A(t) + B(t)F(t)) + (A(t) + B(t)F(t))^{\top}(t)P(t) + 2P(t) \in \mathbb{S}_-(n)$$

with

$$\underline{p}I_n \leq P(t) \leq \overline{p}I_n.◁$$

***Assumption S.2.2***: There exist $\xi \in \mathcal{K}$ such that $\|\Delta\Sigma_L(t, \mathsf{s}, \mathsf{w})\| \leq \xi(\|(\mathsf{s}, \mathsf{w})\|)\|(\mathsf{s}, \mathsf{w})\|$ for all $t \geq 0$ and $(\mathsf{s}, \mathsf{w}) \in \mathbb{R} \times \mathcal{W}$. ◁

We prove that Assumptions S.2.1-S.2.2 imply Assumption S.2.

**Proposition 3.** $(S.2.1) + (S.2.2) \Rightarrow (S.2)$. ◁

The proof is as follows. Define $V(t, \mathsf{s}) \triangleq \mathsf{s}^{\top}P(t)\mathsf{s}$ and $\varphi(t, \mathsf{s}) \triangleq F(t)\mathsf{s}$ and $\mathcal{B}^{\mathsf{s}} \subset \mathbb{R}$ and $\mathcal{B}^{\mathsf{d}} \subset \mathbb{R}^s$ be open neighbourhoods of the origin such that

$$(t, \mathsf{s}, \mathsf{d}) \in \mathbb{R}^0_+ \times \mathcal{B}^{\mathsf{s}} \times \mathcal{B}^{\mathsf{d}} \Rightarrow \xi(\|(\mathsf{s}, \mathsf{w})\|)|_{\mathsf{u}=F(t)\mathsf{s}} \leq \frac{\underline{p}}{2\overline{p}(1 + \sup_{t\geq 0} \|F(t)\|)} \tag{79}$$

Hence,

$$(t, \mathsf{s}, \mathsf{d}) \in \mathbb{R}^0_+ \times \mathcal{B}^{\mathsf{s}} \times \mathcal{B}^{\mathsf{d}} \Rightarrow \|2\mathsf{s}^{\top}P(t)\Delta\Sigma_L(t, \mathsf{s}, \mathsf{w})\|_{\mathsf{u}=F(t)\mathsf{s}} \leq \underline{p}\left(\|\mathsf{s}\|^2 + (\sup_{\mathsf{s}\in\mathcal{B}^{\mathsf{s}}} \|\mathsf{s}\|)\|\|\mathsf{d}\|\right) \leq V(t, \mathsf{x}) + \underline{p}(\sup_{\mathsf{s}\in\mathcal{B}^{\mathsf{s}}} \|\mathsf{s}\|)\|\|\mathsf{d}\|. \tag{80}$$

In conclusion, we have for all $(t, \mathsf{s}, \mathsf{d}) \in \mathbb{R}^0_+ \times \mathcal{B}^{\mathsf{s}} \times \mathcal{B}^{\mathsf{d}}$

$$\underline{p}\|\mathsf{s}\|^2 \leq V(t, \mathsf{s}) \leq \|\mathsf{s}\|^2 \overline{p}, \tag{81}$$

$$\frac{\partial V(t, \mathsf{s})}{\partial t} + \frac{\partial V(t, \mathsf{s})}{\partial \mathsf{s}} F(t, \mathsf{s}, \varphi(t, \mathsf{s}), \mathsf{d}) \leq -V(t, \mathsf{s}) + (\underline{p} + 2\overline{p} \sup_{t \geq 0} \|G(t)\|)(\sup_{\mathsf{s} \in \mathcal{B}^{\mathsf{s}}} \|\mathsf{s}\|)\|\mathsf{d}\| \tag{82}$$

Hence, Assumption S.2 is satisfied with $\underline{\beta}(r) = \underline{p}s$, $\overline{\beta}(r) = \overline{p}s$, $\alpha(r) = r$ and $\gamma(r) = (\underline{p} + 2\overline{p} \sup_{t \geq 0} \|G(t)\|)(\sup_{\mathsf{s} \in \mathcal{B}^{\mathsf{s}}} \|\mathsf{s}\|)r$, which is the claim of our Proposition.

In conclusion, under Assumptions S.2.1-S.2.2 we have at hand a local *linear* controller $u(t) = F(t)s(t)$ for $\Sigma(t) = 0$ (i.e. the controller for the linearization of $\Sigma(t) = 0$) which can be transformed by a symmetry into a semiglobal *nonlinear* controller for $\Sigma(t) = 0$, according to Theorem 1.

***Example*** *4.2.3* (*Triangular systems*). Consider the system map $\Sigma$ associated with (43) and the GS of $\Sigma(t) = 0$ in (45). As already seen, (45) is SI-contracting with contracting maps (62) and UU-transforming (Assumption S.1). Let's see how it is possible to meet the remaining Assumptions S.2-S.4. Since

$$(A(t), B(t), G(t)) = \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \triangleq (A, B, G)$$

with controllable $(A, B)$, Assumption S.2 can be satisfied by showing that Assumptions S.2.1-S.2.2 are met. Using the controllability of $(A, B)$, find $\underline{p}, \overline{p} > 0$, $F \in \mathbb{R}^{1 \times 2}$ and $P \in \mathbb{S}_+(2)$ such that

$$P(A + BF) + (A + BF)^\top P + 2P \in \mathbb{S}_-(2)$$

and

$$\underline{p}I \leq P \leq \overline{p}I.$$

Assumptions S.2.1-S.2.2 are satisfied with $F$ as above, $\mathcal{B}^{\mathsf{s}} \subset \mathbb{R}^2$ and $\mathcal{B}^{\mathsf{d}} \subset \mathbb{R}$ neighbourhoods of the origin, such that

$$(\mathsf{s}, \mathsf{d}) \in \mathcal{B}^{\mathsf{s}} \times \mathcal{B}^{\mathsf{d}} \Rightarrow |\mathsf{x}\mathsf{y}^k + \mathsf{y}^2(F\mathsf{s} + \mathsf{d})| \leq \frac{\underline{p}}{2\overline{p}(1 + \|F\|)}(\|\mathsf{s}\| + |\mathsf{d}|). \tag{83}$$

On account of Proposition 3, Assumption S.2 is satisfied with $\varphi(t, \mathsf{s}) = F\mathsf{s}$, $V(\mathsf{s}) = \mathsf{s}^\top P\mathsf{s}$, $\underline{\beta}(r) = \underline{p}r$, $\overline{\beta}(r) = \overline{p}r$, $\alpha(r) = r$ and $\gamma(r) = (\underline{p} + 2\overline{p})(\sup_{\mathsf{s} \in \mathcal{B}^{\mathsf{s}}} \|\mathsf{s}\|)r$. Furthermore, there exists $\mathfrak{p}_0 > 0$ such that for all $\mathfrak{p} \geq \mathfrak{p}_0$ and any $\delta \in (0, 1)$

$$\|\mathsf{s}_{\mathfrak{p}}\| < \frac{2(\underline{p} + 2\overline{p})\overline{p}}{\underline{p}^2} e^{-\delta\mathfrak{p}} \sup_{\mathsf{s} \in \mathcal{B}^{\mathsf{s}}} \|\mathsf{s}\| \Rightarrow \left(\mathsf{x}_{\mathfrak{p}} \in \mathcal{B}^{\mathsf{s}}\right) \wedge \left((\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, F\mathsf{x}_{\mathfrak{p}}) \in \mathrm{Dom}(\Psi^{\mathsf{u}}_{-\mathfrak{p}})\right) \wedge \left(\|\Psi^{\mathsf{x}}_{-\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}, \mathsf{x}_{\mathfrak{p}})\| \leq e^{(1+k)\mathfrak{p}}\|\mathsf{x}_{\mathfrak{p}}\|\right) \tag{84}$$

i.e. Assumption S.3 with

$$\chi(r) \triangleq \frac{(\underline{p} + 2\overline{p})\overline{p}}{\underline{p}^2}(\sup_{\mathsf{s} \in \mathcal{B}^{\mathsf{s}}} \|\mathsf{s}\|)r, \mu(r) \triangleq e^{-\delta r} \in \mathcal{L}, \pi_1(r) \triangleq 1 + r \in \mathcal{K}_+, \pi_2(r) \triangleq e^{-(1+k)r} \in \mathcal{L}. \tag{85}$$

Finally, since $k \geq 2$ then $|\mathsf{d}_{\mathfrak{p}}| \leq \pi_2(\mathfrak{p})|\mathsf{d}|$ for all $\mathfrak{p} > 0$ and $(t, \mathsf{z}) \in \mathcal{M}$ and Assumption S.4 is met with $\pi_3(r) \triangleq \chi(r)$.

Theorem 1 applies to $\Sigma(t) = 0$ providing the asymptotic bound (76) with the (semiglobal) symmetry-based controller (75) for $\Sigma(t) = 0$

$$u(t) \triangleq e^{2(k-1)\mathfrak{p}} \frac{e^{2\mathfrak{p}} + y^2(t)}{1 + y^2(t)} F\mathrm{diag}\{1, e^{-k\mathfrak{p}}\} s(t). \tag{86}$$

where $\mathfrak{p}$ should be selected as detailed in the proof of Theorem 1. The *local* controller $u(t) = Fs(t)$ (for a set of initial conditions $s(0)$ and disturbances $d(t)$ in the "small" ball $\mathcal{B}_s \times \mathcal{B}_d$) has been transformed by the symmetry of $\Sigma(t) = 0$ into the *semiglobal* controller (86) (for any set of initial conditions $s(0)$ and disturbances $d(t)$ in a pre-assigned ball of radius $N$).

We should compare the semiglobal controller (86), which is obtained by inverse-transforming via symmetry a local controller of $\Sigma(t) = 0$, with a semiglobal controller directly designed on $\Sigma(t) = 0$ of the form

$$u(t) \triangleq Fs(t) \tag{87}$$

where the design of the matrix $F$ takes directly into account the arbitrarily large set of initial conditions $s(0)$ and disturbances $d(t)$ such that $\|(s(0), d(t)\| \leq N$ (see for instance Theorem 12.1.1 and Corollary 12.1.2 of [13] or the paper [14]). ◁

***Example*** 4.2.4 (*Triangular systems with locally linear input nonlinearities*). Consider the system map $\Sigma$ associated with (55) and the GS of $\Sigma(t) = 0$ (56) which is SI-contracting with contracting maps (62). Since

$$(A(t), B(t), G(t)) = \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \triangleq (A, B, G)$$

with controllable $(A, B)$, Assumption S.2 is satisfied also in this case by showing that Assumptions S.2.1-S.2.2 are met. Using the controllability of $(A, B)$ find $F \in \mathbb{R}^{1\times 2}$, $\underline{p}, \overline{p} > 0$ and $P \in \mathbb{S}_+(2)$ such that

$$P(A + BF) + (A + BF)^\top P + 2I_n \in \mathbb{S}_-(2)$$

and

$$\underline{p}I \le P \le \overline{p}I.$$

Find $\mathcal{B}^{\mathsf{s}} \subset \mathbb{R}^2$, neighbourhood of the origin, such that

$$\mathsf{s} \in \mathcal{B}^{\mathsf{s}} \Rightarrow |\mathrm{arctg}(F\mathsf{s}) - F\mathsf{s}| \le \frac{\underline{p}}{2\overline{p}(1 + \|F\|)}\|\mathsf{s}\| \tag{88}$$

and define $\varphi(t, \mathsf{s}) = F\mathsf{s}$, $\underline{\beta}(r) = \underline{p}r$, $\overline{\beta}(r) = \overline{p}r$, $\alpha(r) = r$ and $\gamma(r) = 2\overline{p}(\sup_{\mathsf{s}\in\mathcal{B}^{\mathsf{s}}} \|\mathsf{s}\|)r$. These choices secure Assumptions S.2.1-S.2.2, hence Assumption S.2.

Furthermore, since $\mathrm{Dom}(\Psi^{\mathsf{u}}_{-\mathfrak{p}}) = \{(\mathsf{t}, \mathsf{y}, \mathsf{u}) : |e^{\mathfrak{p}}\mathrm{arctg}(\mathsf{u})| < \frac{\pi}{2})\}$, there exists $\mathfrak{p}_0 > 0$ such that for all $\mathfrak{p} \ge \mathfrak{p}_0$ and any $\delta \in (0, 1)$

$$\|\mathsf{s}_{\mathfrak{p}}\| < \frac{4\overline{p}^2}{\underline{p}^2} e^{-(1+\delta)\mathfrak{p}} \sup_{\mathsf{s}\in\mathcal{B}^{\mathsf{s}}} \|\mathsf{s}\| \Rightarrow \left(\mathsf{s}_{\mathfrak{p}} \in \mathcal{B}^{\mathsf{s}}\right) \wedge \left((\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, F\mathsf{s}_{\mathfrak{p}}) \in \mathrm{Dom}(\Psi^{\mathsf{u}}_{-\mathfrak{p}})\right) \wedge \left(\|\Psi^{\mathsf{s}}_{-\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}})\| \le e^{3\mathfrak{p}}\|\mathsf{s}_{\mathfrak{p}}\|\right) \tag{89}$$

i.e. Assumption S.3 with

$$\chi(r) \triangleq \frac{2\overline{p}^2}{\underline{p}^2}(\sup_{\mathsf{s}\in\mathcal{B}^{\mathsf{s}}} \|\mathsf{s}\|)r, \mu(r) \triangleq e^{-(1+\delta)r} \in \mathcal{L}, \pi_1(r) \triangleq 1 + r \in \mathcal{K}_+, \pi_2(r) \triangleq e^{-3r} \in \mathcal{L}. \tag{90}$$

Finally, $|\mathsf{d}_{\mathfrak{p}}| \le \pi_2(\mathfrak{p})|\mathsf{d}|$ for all $\mathfrak{p} \ge 0$ and $(t, \mathsf{z}) \in \mathcal{M}$ and Assumption S.4 is met with $\pi_3(r) \triangleq \chi(r)$.

Theorem 1 applies to $\Sigma(t) = 0$ providing the asymptotic bound (76) with the symmetry-based controller (75)

$$u(t) \triangleq \mathrm{tg}(e^{\mathfrak{p}}\mathrm{arctg}(e^{-2\mathfrak{p}}F\mathrm{diag}\{1, e^{-\mathfrak{p}}\}s(t))), \tag{91}$$

where $\mathfrak{p}$ should be selected as detailed in the proof of Theorem 1. The *local* controller $u(t) = Fs(t)$ has been transformed by the symmetry of $\Sigma(t) = 0$ into the *semiglobal* controller (86).

We should compare the semiglobal controller (91), which is obtained by inverse-transforming via symmetry a local controller of $\Sigma(t) = 0$, with a semiglobal controller directly designed on $\Sigma(t) = 0$ of the form

$$u(t) \triangleq Fs(t) \tag{92}$$

where the design of the matrix $F$ takes directly into account the arbitrarily large set of initial conditions and disturbances $\|(s(0), d(t)\| \le N$ (see for instance Proposition 14.1.5 of[13]). ⊲

## 5.2 | Symmetry-based controllers with finite- or prescribed-time convergence

In this section we consider the problem of transforming a state-feedback controller with asymptotic-time convergence (i.e. infinite time scale) into a state-feedback controller with finite- or prescribed-time convergence (i.e. finite time scale). To this aim, we focus our attention on linear systems $\Sigma(t) = 0$, i.e.

$$F(t, \mathsf{s}, \mathsf{w}) = A(t)\mathsf{s} + B(t)\mathsf{u} + G(t)\mathsf{d} \tag{93}$$

and we assume the existence of a (not necessarily SI-contracting) symmetry which is non-expanding on time scales, i.e. a BU-transforming symmetry.

***Assumption PT.S.1***: $\Psi_{\mathfrak{p}}$ is a LS of $\Sigma(t) = 0$ with $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \{(t, \mathsf{z} \in \mathcal{M} : t \in \mathrm{Dom}(\Psi^t_{\mathfrak{p}})\}$ and BU-transforming for $\mathfrak{p} > 0$. ⊲

The next assumption asserts, through a dissipation inequality, the existence of a controller $u(t) = F(t)s(t)$ for $\Sigma(t) = 0$ with asymptotic time convergence.

***Assumption PT.S.2***: There exist $\alpha, \underline{\beta}, \overline{\beta}, \gamma > 0$ and a $C^1$ function $P : \mathbb{R}^0_+ \to \mathbb{S}_+(n)$ and $F : \mathbb{R}^0_+ \to \mathbb{R}^{m\times n}$ such that for all $(t, \mathsf{s}, \mathsf{d}) \in \mathbb{R}^0_+ \times \mathbb{R}^n \times \mathbb{R}^s$ and with $V(t, \mathsf{s}) \triangleq \mathsf{s}^\top P(t)\mathsf{s}$

$$\underline{\beta}\|\mathsf{s}\|^2 \leq V(t, \mathsf{s}) \leq \overline{\beta}\|\mathsf{s}\|^2, \tag{94}$$

$$\frac{\partial V(t, \mathsf{s})}{\partial t} + \frac{\partial V(t, \mathsf{s})}{\partial \mathsf{s}}((A(t) + B(t)F(t))\mathsf{s} + G(t)\mathsf{d}) \leq -\alpha V(t, \mathsf{s}) + \gamma\|\mathsf{d}\|^2. \triangleleft \tag{95}$$

***Assumption PT.S.3***: There exist $\psi \in \mathcal{L}$, $\pi_1 > 0$ and $\pi_2 : \mathbb{R}^0_+ \times \mathbb{R}^0_+ \to \mathbb{R}_+$ such that for all $\mathfrak{p} > 0$ and $(t, \mathsf{z}) \in \mathcal{M}$ for which $t \in [\Psi^t_{-\mathfrak{p}}(\tau), \mathrm{Dom}^+\Psi^t_{\mathfrak{p}})$, with $\tau \in [0, \Psi^t_{\mathfrak{p}}(\mathrm{Dom}^+\Psi^t_{\mathfrak{p}}))$, there holds that

$$\|\Psi^{\mathsf{x}}_{-\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}})\| \leq \frac{\pi_1\|\mathsf{s}_{\mathfrak{p}}\|}{\pi_2(\mathsf{t}_{\mathfrak{p}}, \mathfrak{p})}, \tag{96}$$

$$\left|\frac{\partial \ln \pi_2(r, \mathfrak{p})}{\partial r}\right|_{r=\mathsf{t}_{\mathfrak{p}}} \leq \psi(\mathsf{t}_{\mathfrak{p}}), \tag{97}$$

$$\|\mathsf{d}_{\mathfrak{p}}\| \leq \pi_2(\mathsf{t}_{\mathfrak{p}}, \mathfrak{p})\|\mathsf{d}\|. \triangleleft \tag{98}$$

Notice that since $\Sigma(t) = 0$ is linear we are not requiring the symmetry to be SI-contracting (see Assumption PT.S.1 and compare with Assumption S.1).

***Assumption PT.S.4***: $(\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, F(\mathsf{t}_{\mathfrak{p}})\mathsf{s}_{\mathfrak{p}}) \in \mathrm{Dom}(\Psi^{\mathsf{u}}_{-\mathfrak{p}})$ for all $\mathfrak{p} > 0$ and $(t, \mathsf{z}) \in \mathcal{M}$ such that $t \in [0, \mathrm{Dom}^+\Psi^t_{\mathfrak{p}})$. $\triangleleft$

**Theorem 2.** Under Assumptions PT.S.1-PT.S.4 and for all the solutions $(t, z(t))$ of $\Sigma(t) = 0$, with

$$u(t) \triangleq \Psi^{\mathsf{u}}_{-\mathfrak{p}}(t_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t), F(t_{\mathfrak{p}}(t))s_{\mathfrak{p}}(t)), \tag{99}$$

and $\sup_{t\geq 0}\|d(t)\| \leq N$, for any $\mathfrak{p} > 0$ we have

$$\limsup_{t \nearrow \mathrm{Dom}^+\Psi^t_{\mathfrak{p}}} \|x(t)\| \leq \pi_1\sqrt{\frac{2\gamma}{\underline{\beta}\alpha}} \sup_{t\geq 0}\|d(t)\|. \tag{100}$$

If in addition $\Psi_{\mathfrak{p}}$ is CBU-transforming, for each $T^* > 0$ there exists $\mathfrak{p} > 0$ such that (100) holds with $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} < T^*$. $\triangleleft$

Hence, if the symmetry is BU-transforming the global controller $u(t) = Fs(t)$ with asymptotic time convergence is transformed by the symmetry into the global controller (99) with finite-time convergence, while if in addition the symmetry is CBU-transforming then the global controller $u(t) = Fs(t)$ with asymptotic time convergence is transformed by the symmetry into the global controller (99) with prescribed-time convergence $< T^*$. The characterization of Theorem 2 is, at this level of generality and as far as we know, not existing in the literature. Moreover, the resulting controllers are not of the sliding-mode type ([11]), rather smooth time-varying ([21]).

***Example** 4.2.4* (*Triangular linear systems*). Consider the system map $\Sigma$ associated with (37) and the CBU-transforming LS $\Psi_{\mathfrak{p}}$ of $\Sigma(t) = 0$ in (38) with $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = \frac{1}{\mathfrak{p}}$. Since

$$(A(t), B(t), G(t)) = \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right) \triangleq (A, B, G)$$

with controllable $(A, B)$, Assumption PT.S.2 is satisfied with $\underline{p}, \overline{p} > 0$ and $P \in \mathbb{S}_+(2)$ such that $P(A+BF)+(A+BF)^\top P+2I_n \in \mathbb{S}_-(2)$ for some $F \in \mathbb{R}^{1\times 2}$. Furthermore, for all $\mathfrak{p} > 0$, $\mathsf{s} \in \mathbb{R}^2$ and $t \in [\Psi^t_{-\mathfrak{p}}(\tau), \mathrm{Dom}^+\Psi^t_{\mathfrak{p}}) \triangleq [\frac{1}{1+\mathfrak{p}}, \frac{1}{\mathfrak{p}})$ (pick $\tau \triangleq 1$), there holds that $(\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, F\mathsf{s}_{\mathfrak{p}}) \in \mathrm{Dom}(\Psi^{\mathsf{u}}_{-\mathfrak{p}})$ and

$$\|\Psi^{\mathsf{s}}_{-\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}, \mathsf{s}_{\mathfrak{p}})\| = \left\|\begin{pmatrix}\frac{1}{1+\mathsf{t}_{\mathfrak{p}}\mathfrak{p}} & 0\\ \mathfrak{p} & 1 + \mathsf{t}_{\mathfrak{p}}\mathfrak{p}\end{pmatrix}\mathsf{s}_{\mathfrak{p}}\right\| \leq \frac{\|\mathsf{s}_{\mathfrak{p}}\|}{\pi_2(\mathsf{t}_{\mathfrak{p}}, \mathfrak{p})} \tag{101}$$

with $\pi_2(r, \mathfrak{p}) \triangleq \frac{1}{(1+r\mathfrak{p})^3}$ satisfying

$$\left|\frac{\partial \ln \pi_2(r, \mathfrak{p})}{\partial r}\right|_{r=\mathsf{t}_{\mathfrak{p}}} = 3\left|\frac{\mathfrak{p}}{1 + \mathsf{t}_{\mathfrak{p}}\mathfrak{p}}\right| \leq \psi(\mathsf{t}_{\mathfrak{p}}) \triangleq \frac{6}{1 + \mathsf{t}_{\mathfrak{p}}}. \tag{102}$$

By setting $g^d(t) = -3t$ (recall that $g^d$ is arbitrary) we also get $|\mathsf{d}_{\mathfrak{p}}| \leq \pi_2(\mathsf{t}_{\mathfrak{p}}, \mathfrak{p})|\mathsf{d}|$ for all $\mathfrak{p} > 0$, $t \in [0, \mathrm{Dom}^+\Psi^t_{\mathfrak{p}})$ and $\mathsf{d} \in \mathbb{R}$, hence Assumption PT.S.3 is satisfied. Also Assumption PT.S.4 is satisfied for $\mathfrak{p} > 0$ and $t \in [0, \frac{1}{\mathfrak{p}})$. Theorem 2 applies to

$\Sigma(t) = 0$ providing the asymptotic bound (100) with the symmetry-based controller (99)

$$u(t) \triangleq \frac{1}{(1-\mathfrak{p}t)^3} F \begin{pmatrix} \frac{1}{1-\mathfrak{p}t} y(t) \\ \mathfrak{p}y(t) + (1-\mathfrak{p}t)x(t) \end{pmatrix} \tag{103}$$

and prescribed-time convergence $\mathrm{Dom}^+\Psi^t_{\mathfrak{p}} = \frac{1}{\mathfrak{p}}$ less than any a priori given positive number $< T^*$ (simply pick $\mathfrak{p} > \frac{1}{T^*}$). The global controller $u(t) = Fs(t)$ with asymptotic time convergence is transformed by the symmetry $\Psi_{\mathfrak{p}}$ into the global controller (103) with prescribed-time convergence in any a priori given prescribed time $< T^*$.

We should compare the controller (103) with finite-time convergence, which is obtained by inverse-transforming via a symmetry a controller of $\Sigma(t) = 0$ with asymptotic time convergence, with a sliding-mode controller directly designed on $\Sigma(t) = 0$ taking directly into account initial conditions and disturbances such that $\sup_{t\geq 0} \|(s(0), d(t))\| \leq N$ (see for instance [16], [11]). Moreover, a sliding mode controller is a *time-invariant discontinuous* function of the state $s(t)$, while our controller is a *time-varying continuous* (actually smooth) function of the state $s(t)$. Our controllers are more resemblant to the ones used in [21]. As also noticed in [21], the time-varying gain-based finite-time control is built upon regular state feedback, not on fractional-power or discontinuous state feedback, thus resulting in smooth control action everywhere during the entire operation of the system. Moreover, the finite-time control is characterized with uniformly pre-specifiable convergence time that is independent of initial condition and any other design parameters and can be preassigned as needed within the physically allowable range. $\triangleleft$

# 6 | SYMMETRY-BASED OBSERVERS

## 6.1 | Semiglobal symmetry-based observers with asymptotic convergence

In this section we consider the problem of transforming a *local* state-observer, local in the sense of tolerating only sufficiently small state and exogenous inputs, into a *semiglobal* controller, semiglobal in the sense of tolerating state and exogenous inputs in a *priori* given compact set: $\|(s(t), d(t))\| \leq N$ for some *a priori* given $N \geq 0$ and for all $t \geq 0$. We consider the case of infinite control horizon, i.e. the system time scale is $[0, +\infty)$. To begin with, we assume the existence of a SI-contracting symmetry for the system.

***Assumption O.1***: $\Psi_{\mathfrak{p}}$ is a LS of $\Sigma(t) = 0$ with $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \mathcal{M}$ for $\mathfrak{p} > 0$, SI-contracting with contracting maps $(\sigma, \mu)$ and UU-transforming for $\mathfrak{p} > 0$. $\triangleleft$

As already mentioned, we assume the existence of a *local* observer for $\Sigma(t) = 0$ of the form

$$D_t\zeta(t) = F(t, \zeta(t), u(t), 0) + \psi(t, y(t), y(t) - \zeta^{\mathsf{y}}(t))$$

with $\zeta = (\zeta^{\mathsf{y}}, \zeta^{\mathsf{x}})$.

***Assumption O.2***: There exist open neighbourhoods $C^{\mathsf{s}} \triangleq C^{\mathsf{y}} \times C^{\mathsf{x}} \subset \mathcal{S}$, $C^{\mathsf{e}} \triangleq C^{\mathsf{e}^{\mathsf{y}}} \times C^{\mathsf{e}^{\mathsf{x}}} \supset C^{\mathsf{s}}$, $C^{\mathsf{d}} \subset \mathbb{R}^r$ of the origin, $\delta, \underline{\theta}, \overline{\theta}, \eta \in \mathcal{K}_\infty$ and a $C^1$ function $W : \mathbb{R}^0_+ \times C^{\mathsf{e}} \to \mathbb{R}^0_+$ and $\psi : \mathbb{R} \times C^{\mathsf{y}} \times C^{\mathsf{e}^{\mathsf{y}}} \to \mathbb{R}^m$ such that

$$\underline{\theta}(\|\mathsf{e}\|) \leq W(t, \mathsf{e}) \leq \overline{\theta}(\|\mathsf{e}\|), \tag{104}$$

$$\frac{\partial W(t,\mathsf{e})}{\partial t} + \frac{\partial W(t,\mathsf{e})}{\partial \mathsf{e}}\Big(F(t,\mathsf{s},\mathsf{u},\mathsf{d}) - F(t,\mathsf{s}-\mathsf{e},\mathsf{u},0) - \psi(t,\mathsf{y},\mathsf{e}^{\mathsf{y}})\Big) \leq -\delta(W(t,\mathsf{e})) + \eta(\|\mathsf{d}\|) \tag{105}$$

for all $(t, \mathsf{s}, \mathsf{e}, \mathsf{u}, \mathsf{d}) \in \mathbb{R}^0_+ \times C^{\mathsf{s}} \times C^{\mathsf{e}} \times \mathbb{R}^m \times C^{\mathsf{d}}$ with $\mathsf{e} \triangleq (\mathsf{e}^{\mathsf{y}}, \mathsf{e}^{\mathsf{x}})$. $\triangleleft$

As in the context of state-feedback controllers, some (mild) conditions are required on $\Psi^{\mathsf{s}}_{-\mathfrak{p}}$ for obtaining a converging estimate $\hat{s}(t)$ of $s(t)$ (in time scale $t$) from a converging estimate $\tilde{\zeta}(\mathsf{t}_{\mathfrak{p}})$ of the transformed state $\tilde{s}_{\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}})$ (in time scale $\mathsf{t}_{\mathfrak{p}}$).

***Assumption O.3***: There exist $\mathfrak{p}_0 > 0$ and $\pi_1 \in \mathcal{K}_+$ such that if $\chi(r) \triangleq (\underline{\theta}^{-1} \circ \overline{\theta} \circ \underline{\theta}^{-1} \circ \delta^{-1})(2\eta(s))$ and for all $\mathfrak{p} \geq \mathfrak{p}_0$

$$\begin{aligned} &\|\zeta\| < 2\max\{(\underline{\theta}^{-1}\circ\overline{\theta})(\mu(\mathfrak{p})), \chi(\mu(\mathfrak{p}))\} + \mu(\mathfrak{p}) \\ &\Rightarrow \Big((\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, \zeta^{\mathsf{x}}) \in \mathrm{Dom}(\Psi^{\mathsf{x}}_{-\mathfrak{p}})\Big) \wedge \left(\|\frac{\partial \Psi^{\mathsf{s}}_{-\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, \zeta^{\mathsf{x}})}{\partial \zeta^{\mathsf{x}}}\| \leq \frac{\pi_1(\|(\mathsf{y}_{\mathfrak{p}}, \zeta^{\mathsf{x}})\|)}{\pi_2(\mathfrak{p})}\right) . \triangleleft \end{aligned} \tag{106}$$

***Assumption O.4***: There exist $\pi_3 \in \mathcal{K}$ and $\pi_2 \in \mathcal{L}$ such that

$$\chi(\|\mathsf{d}_{\mathfrak{p}}\|) \leq \pi_2(\mathfrak{p})\pi_3(\|\mathsf{d}\|)$$

for all $\mathfrak{p} > 0$ and $(t, \mathsf{z}) \in \mathcal{M}$. $\triangleleft$

**Theorem 3.** Under Assumptions O.1-O.4 and for each $N > 0$ consider the solutions $(t, z(t))$ of $\Sigma(t) = 0$, with observer

$$\begin{aligned}
\hat{s}(t) &\triangleq \begin{pmatrix} y(t) \\ \Psi^{\mathsf{x}}_{-\mathfrak{p}}(t_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t), \zeta^{\mathsf{x}}(t)) \end{pmatrix}, \zeta(t) \triangleq (\zeta^{\mathsf{y}}(t), \zeta^{\mathsf{x}}(t)), \\
D_t\zeta(t) &= \frac{\partial t_{\mathfrak{p}}(t)}{\partial t}\Big(F(t_{\mathfrak{p}}(t), \zeta(t), u_{\mathfrak{p}}(t), 0) + \psi(t_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t) - \zeta^{\mathsf{y}}(t))\Big), \zeta(0) = 0,
\end{aligned} \tag{107}$$

and $\sup_{t\geq 0} \|(s(t), d(t))\| \leq N$. There exists $\mathfrak{p} > 0$ such that

$$\limsup_{t\to+\infty} \|s(t) - \hat{s}(t)\| \leq \pi_1(2 \sup_{\mathsf{s}\in C^{\mathsf{s}}} \|\mathsf{s}\| + \sup_{\mathsf{e}\in C^{\mathsf{e}}} \|\mathsf{e}\|)\pi_3(\sup_{t\geq 0} \|d(t)\|).\triangleleft \tag{108}$$

*Remark 3.* The proof of Theorem 3, reported in the Appendix, points out the procedure for tuning the parameter $\mathfrak{p} > 0$ of the observer (107) which is obtained via symmetry from the local observer $D_t\zeta(t) = F(t, \zeta(t), u(t), 0)+\psi(t, y(t), y(t)-\zeta^{\mathsf{y}}(t))$. Notice that if $d(t) \equiv 0$ we have exact asymptotic convergence $s(t) - \hat{s}(t) \to 0$ as $t \to +\infty$. Also in this case by increasing $\mathfrak{p}$ we enlarge the set of bounded solutions $s(t)$ and exogenous inputs $d(t)$ for which (108) holds. Robustness and sensitivity properties of the observer (107) are ineherited from the local observer.

*Remark 4.* (*Robustness*). Robustness and sensitivity of the resulting semiglobal observer (75) are inherited from the local observer and quantitatively evaluated by the input-to-output inequality (108).

### 6.1.1 | The case of detectability in the first approximation

The search for functions $\delta, \underline{\theta}, \overline{\theta}, \eta$ and $W$ satisfying Assumption O.2 is, say, canonical when the linear approximation of $\Sigma(t) = 0$ is detectable. This is exactly the context in which the semiglobal results of[7] must be considered and compared with the one given in Theorem 3. Let

$$\begin{aligned}
A^{\mathsf{x}}(t) &\triangleq \frac{\partial F(t, \mathsf{s}, \mathsf{w})}{\partial \mathsf{x}}\Big|_{(\mathsf{s},\mathsf{w})=0}, G(t) \triangleq \frac{\partial F(t, \mathsf{s}, \mathsf{w})}{\partial \mathsf{d}}\Big|_{(\mathsf{s},\mathsf{w})=0}, \\
(\Delta\Sigma_L)(t, \mathsf{s}, \mathsf{w}) &\triangleq F(t, \mathsf{s}, \mathsf{w}) - (A^{\mathsf{x}}(t)\mathsf{x} + G(t)\mathsf{d}),
\end{aligned} \tag{109}$$

$A(t) \triangleq \begin{pmatrix} 0_{n\times p} & A^{\mathsf{x}}(t)\end{pmatrix}$ and $C(t) \triangleq \begin{pmatrix} I_p & 0_{p\times n}\end{pmatrix}$. We introduce two simpler assumptions which are dual of Assumptions S.2.1. and S.2.2 and imply Assumption O.2.

***Assumption O.2.1***: $\sup_{t\geq 0} \|G(t)\| < +\infty$ and there exist $K : \mathbb{R}^0_+ \to \mathbb{R}^{n\times p}$, $Q : \mathbb{R}^0_+ \to \mathbb{S}_+(n)$ and $\underline{q}, \overline{q} > 0$ such that $\sup_{t\geq 0} \|K(t)\| < +\infty$ and

$$D_tQ(t) + Q(t)(A(t) - K(t)C(t)) + (A(t) - K(t)C(t))^{\top}(t)Q(t) + 2Q(t) \in \mathbb{S}_-(n)$$

with $\underline{q}I_n \leq Q(t) \leq \overline{q}I_n$ for all $t \geq 0$. $\triangleleft$

***Assumption O.2.2***: There exist $\xi \in \mathcal{K}$ such that

$$\|\frac{\partial \Delta\Sigma_L(t, \mathsf{s}, \mathsf{w})}{\partial(\mathsf{x}, \mathsf{d})}\| \leq \xi(\|(\mathsf{s}, \mathsf{d})\|)$$

for all $t \geq 0$ and $(\mathsf{s}, \mathsf{w}) \in \mathcal{S} \times \mathcal{W}$. $\triangleleft$

We prove below that Assumptions O.2.1 and O.2.2 imply Assumption O.2.

**Proposition 4.** (O.2.1)+(O.2.2) $\Rightarrow$(O.2). $\triangleleft$

The proof is as follows. Define $W(t, \mathsf{e}) \triangleq \mathsf{e}^{\top}Q(t)\mathsf{e}$, $\mathsf{e} \triangleq (\mathsf{e}^{\mathsf{y}}, \mathsf{e}^{\mathsf{x}})$ and $\psi(t, \mathsf{y}, \mathsf{e}^{\mathsf{y}}) \triangleq K(t)\mathsf{e}^{\mathsf{y}} = K(t)C(t)\mathsf{e}$ and $C^{\mathsf{s}} \subset \mathcal{S}$, $C^{\mathsf{e}} \supset C^{\mathsf{s}}$ and $C^{\mathsf{d}} \subset \mathbb{R}^r$ be open neighbourhoods of the origin such that

$$\mathsf{s} \in C^{\mathsf{s}}, \mathsf{e} \in C^{\mathsf{e}}, \mathsf{d} \in C^{\mathsf{d}} \Rightarrow \xi(\|(\mathsf{s}, \mathsf{e}, \mathsf{d})\|) \leq \frac{\underline{q}}{2\overline{q}} \tag{110}$$

Hence,

$$\begin{aligned}
&(t, \mathsf{s}, \mathsf{e}, \mathsf{u}, \mathsf{d}) \in \mathbb{R}^0_+ \times C^{\mathsf{s}} \times C^{\mathsf{e}} \times \mathbb{R}^m \times C^{\mathsf{d}} \\
&\Rightarrow \|2\mathsf{e}^{\top}Q(t)(\Delta\Sigma_L(t, \mathsf{y}, \mathsf{x}, \mathsf{u}, \mathsf{d}) - \Delta\Sigma_L(t, \mathsf{y}, \mathsf{x} - \mathsf{e}^{\mathsf{x}}, \mathsf{u}, 0))\| \leq \underline{q}\left(\|\mathsf{e}\|^2 + (\sup_{\mathsf{e}\in C^{\mathsf{e}}} \|\mathsf{e}\|)\|\|\mathsf{d}\|\right) \leq W(t, \mathsf{x}) + \underline{q}(\sup_{\mathsf{e}\in C^{\mathsf{e}}} \|\mathsf{e}\|)\|\|\mathsf{d}\|.
\end{aligned}$$

In conclusion, we have for all $(t, \mathsf{s}, \mathsf{e}, \mathsf{u}, \mathsf{d}) \in \mathbb{R}^0_+ \times \mathcal{C}^{\mathsf{s}} \times \mathcal{C}^{\mathsf{e}} \times \mathbb{R}^m \times \mathcal{C}^{\mathsf{d}}$

$$\underline{q}\|\mathsf{x}\|^2 \leq W(t,\mathsf{x}) \leq \|\mathsf{x}\|^2\overline{q}, \tag{111}$$

$$\frac{\partial W(t,\mathsf{x})}{\partial t} + \frac{\partial W(t,\mathsf{x})}{\partial \mathsf{x}}(F(t,\mathsf{s},\mathsf{u},\mathsf{d}) - F(t,\mathsf{s}-\mathsf{e},\mathsf{u},0) - \psi(t,\mathsf{y},\mathsf{e}^{\mathsf{y}})) \leq -W(t,\mathsf{x}) + (\underline{q} + 2\overline{q}\sup_{t\geq 0}\|G(t)\|)(\sup_{\mathsf{e}\in\mathcal{C}^{\mathsf{e}}}\|\mathsf{e}\|)\|\mathsf{d}\| \tag{112}$$

and Assumption O.2 is satisfied with $\underline{\theta}(r) = \underline{q}s$, $\overline{\theta}(r) = \overline{q}r$, $\delta(r) = r$ and $\eta(r) = (\underline{q} + 2\overline{q}\sup_{t\geq 0}\|G(t)\|)(\sup_{\mathsf{e}\in\mathcal{C}^{\mathsf{e}}}\|\mathsf{e}\|)r$. Hence, the claim of the Proposition.

Under Assumptions O.2.1-O.2.2 we have at hand a local observer $D_t\zeta(t) = F(t,\zeta(t),u(t),0) + K(t)(y(t) - \zeta^{\mathsf{y}}(t))$ for $\Sigma(t) = 0$ which can be transformed by a symmetry into a semiglobal observer for $\Sigma(t) = 0$.

***Example** 4.2.3* (*Triangular systems*). Consider the system map $\Sigma$ associated with (43) and the GS of $\Sigma(t) = 0$ in (45). As already seen, (45) is SI-contracting with contracting maps (62) and UU-transforming (Assumption O.1). Let's see how it is possible to meet the remaining Assumptions O.2-O.4. Since

$$(A^{\mathsf{x}}(t), G(t)) = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) \triangleq (A^{\mathsf{x}}, G), A = \begin{pmatrix}0 & A^{\mathsf{x}}\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, C \triangleq \begin{pmatrix}1 & 0\end{pmatrix} \tag{113}$$

with observable $(C, A)$, Assumption O.2 is satisfied once we show that Assumptions O.2.1 and O.2.2 are met. Using the observability of $(C, A)$, find $\underline{q}, \overline{q} > 0$ and $Q \in \mathbb{S}_+(2)$ such that $Q(A - KC) + (A - KC)^\top Q + 2Q \in \mathbb{S}_-(2)$ for some $K \in \mathbb{R}^{2\times 1}$. Assumptions O.2.1 and O.2.2 are satisfied with $K$ as above and $\mathcal{C}^{\mathsf{s}}$, $\mathcal{C}^{\mathsf{e}}$ and $\mathcal{C}^{\mathsf{d}}$ such that (see inequality (110))

$$\mathsf{s} \in \mathcal{C}^{\mathsf{s}}, \mathsf{e} \in \mathcal{C}^{\mathsf{e}}, \mathsf{d} \in \mathcal{C}^{\mathsf{d}} \Rightarrow \|(\mathsf{s},\mathsf{e},\mathsf{d})\|^k + \|(\mathsf{s},\mathsf{e},\mathsf{d})\|^2 \leq \frac{\underline{q}}{2\overline{q}}. \tag{114}$$

Correspondingly, Assumption O.2 is satisfied with $\psi(t,\mathsf{y},\mathsf{e}^{\mathsf{y}}) = K\mathsf{e}^{\mathsf{y}}$, $W(\mathsf{e}) = \mathsf{e}^\top Q\mathsf{e}$, $\underline{\theta}(r) = \underline{q}r$, $\overline{\theta}(r) = \overline{q}r$, $\delta(r) = r$ and $\eta(r) = (\underline{q} + 2\overline{q}(\sup_{\mathsf{e}\in\mathcal{C}^{\mathsf{e}}}\|\mathsf{e}\|)r$.

Furthermore, there exists $\mathfrak{p}_0 > 0$ such that for all $\mathfrak{p} \geq \mathfrak{p}_0$ and any $\delta \in (0,1)$

$$\|\zeta\| < \left(\frac{2(\underline{q}+2\overline{q})\overline{q}}{\underline{q}^2}(\sup_{\mathsf{e}\in\mathcal{C}^{\mathsf{e}}}\|\mathsf{e}\|) + 1\right)e^{-\delta\mathfrak{p}} \Rightarrow \left((\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, \zeta^{\mathsf{x}}) \in \mathrm{Dom}(\Psi^{\mathsf{x}}_{-\mathfrak{p}})\right) \wedge \left(\|\frac{\partial\Psi^{\mathsf{x}}_{-\mathfrak{p}}(\mathsf{t}_{\mathfrak{p}}, \mathsf{y}_{\mathfrak{p}}, \zeta^{\mathsf{x}})}{\partial\zeta^{\mathsf{x}}}\| \leq e^{(k+1)\mathfrak{p}}\right) \tag{115}$$

i.e. Assumption O.3 with $\mu$, $\pi_1$ and $\pi_2$ as in (90) and

$$\chi(r) = \frac{(\underline{q}+2\overline{q})\overline{q}}{\underline{q}^2}(\sup_{\mathsf{e}\in\mathcal{C}^{\mathsf{e}}}\|\mathsf{e}\|)r.$$

Finally, since $k \geq 2$, $|\mathsf{d}_{\mathfrak{p}}| \leq \pi_2(\mathfrak{p})|\mathsf{d}|$ for all $\mathfrak{p} > 0$ and $(t,\mathsf{z}) \in \mathcal{M}$, i.e. Assumption O.4 with $\pi_3(r) \triangleq \chi(r)$. Theorem 3 applies to $\Sigma(t) = 0$ providing the asymptotic bound (108) with the symmetry-based observer (75)

$$\begin{aligned}\widehat{s}(t) &\triangleq \begin{pmatrix} y(t)\\ e^{(1+k)\mathfrak{p}}\zeta^{\mathsf{x}}(t)\end{pmatrix}\\ D_t\zeta(t) &= e^{k\mathfrak{p}}\begin{pmatrix}\zeta^{\mathsf{x}}(t)\\ \zeta^{\mathsf{x}}(t)(\zeta^{\mathsf{y}}(t))^k + e^{-(2k+1)\mathfrak{p}}(1+y^2(t))u(t)\end{pmatrix} + Ke^{k\mathfrak{p}}(e^{-\mathfrak{p}}\mathsf{y}(t) - \zeta^{\mathsf{y}}(t)),\end{aligned} \tag{116}$$

where $\zeta = (\zeta^{\mathsf{y}}, \zeta^{\mathsf{x}})$ and $\mathfrak{p}$ should be selected as detailed in the proof of Theorem 3. We should compare the semiglobal observer (116), which is obtained by inverse-transforming via symmetry a local observer of $\Sigma(t) = 0$, with a semiglobal observer directly designed on $\Sigma(t) = 0$ by taking into account that $\sup_{t\geq 0}\|s(t), d(t)\| \leq N$ (see for instance [1], [14] and [12]). In general, our symmetry-based semiglobal observers are much less sensitive to the peaking phenomenon: as it can be seen from (116), while in classical high-gain observers the observer gain peaks as $(e^{\mathfrak{p}}k_1, e^{\mathfrak{p}(1+2k)}k_2)^\top$, in our observers the observer gain peaks as $(e^{\mathfrak{p}k}k_1, e^{\mathfrak{p}k}k_2)^\top$ .

## 6.2 | Semiglobal symmetry-based observers with prescribed- or finite-time convergence

In this section we consider the problem of transforming an observer with asymptotic-time convergence (i.e. infinite time scale) into an observer with finite- or prescribed-time convergence (i.e. finite time scale). We still focus our attention on linear systems (93) and we assume the existence of a (not necessarily SI-contracting) symmetry which is non-expanding on time scales, i.e. a BU-transforming symmetry.

***Assumption PT.O.1***: $\Psi_{\mathfrak{p}}$ is a LS of $\Sigma(t) = 0$ with $\mathrm{Dom}(\Psi_{\mathfrak{p}}) = \{(t,\mathsf{z}) \in \mathcal{M} : t \in \mathrm{Dom}(\Psi^t_{\mathfrak{p}})\}$ and BU-transforming for $\mathfrak{p} > 0$.$\triangleleft$

Let $C(t) \triangleq \left(I_{p\times p}\ \ 0_{p\times n}\right)$. The next assumption asserts the existence of an observer of the form

$$D_t\zeta(t) = A(t)\zeta(t) + B(t)u(t) + K(t)(y(t) - C(t)\zeta(t)) \tag{117}$$

for (93) with asymptotic error convergence.

***Assumption PT.O.2***: There exist $\delta, \underline{\theta}, \overline{\theta}, \eta > 0$, $P : \mathbb{R}^0_+ \to \mathbb{S}_+(n)$ and $K : \mathbb{R} \to \mathbb{R}^{n\times p}$ such that for all $t \geq 0$, $\mathsf{e} \in \mathbb{R}^n$ and with $W(t,\mathsf{s}) = \mathsf{e}^\top P(t)\mathsf{e}$

$$\underline{\theta}\|\mathsf{e}^2\| \leq W(t,\mathsf{e}) \leq \overline{\theta}\|\mathsf{e}\|^2, \tag{118}$$

$$\frac{\partial W(t,\mathsf{e})}{\partial t} + \frac{\partial W(t,\mathsf{e})}{\partial \mathsf{e}}\Big((A(t) - K(t)C(t))\mathsf{e} + G(t)\mathsf{d}\Big) \leq -\delta W(t,\mathsf{e}) + \eta\|\mathsf{d}\|^2.\lhd \tag{119}$$

***Assumption PT.O.3***: $(\mathsf{t}_\mathfrak{p}, \mathsf{y}_\mathfrak{p}, \zeta^\mathsf{x}) \in \mathrm{Dom}(\Psi^\mathsf{x}_{-\mathfrak{p}})$ for all $\mathfrak{p} > 0$, $\zeta^\mathsf{x} \in \mathbb{R}^n$ and $(t,\mathsf{z}) \in \mathcal{M}$ such that $t \in [0, \mathrm{Dom}^+\Psi^t_\mathfrak{p})$. $\lhd$

***Assumption PT.O.4***: There exist $\psi \in \mathcal{L}$, $\pi_1 > 0$ and $\pi_2 : \mathbb{R}^0_+ \times \mathbb{R}^0_+ \to \mathbb{R}_+$ such that for all $\mathfrak{p} > 0$ and $(t,\mathsf{z}) \in \mathcal{M}$ with $t \in [\Psi^t_{-\mathfrak{p}}(\tau), \mathrm{Dom}^+\Psi^t_\mathfrak{p})$ and $\tau \in [0, \Psi^t_\mathfrak{p}(\mathrm{Dom}^+\Psi^t_\mathfrak{p}))$, there holds that

$$\left\|\frac{\partial \Psi^\mathsf{x}_{-\mathfrak{p}}(\mathsf{t}_\mathfrak{p}, \mathsf{y}_\mathfrak{p}, \zeta^\mathsf{x})}{\partial \zeta^\mathsf{x}}\right\| \leq \frac{\pi_1}{\pi_2(\mathsf{t}_\mathfrak{p}, \mathfrak{p})}, \tag{120}$$

$$\left|\frac{\partial \ln \pi_2(r,\mathfrak{p})}{\partial r}\right|_{r=\mathsf{t}_\mathfrak{p}} \leq \psi(\mathsf{t}_\mathfrak{p}), \tag{121}$$

$$\|\mathsf{d}_\mathfrak{p}\| \leq \pi_2(\mathsf{t}_\mathfrak{p}, \mathfrak{p})\|\mathsf{d}\|.\lhd \tag{122}$$

**Theorem 4.** Under Assumptions PT.O.1-PT.O.4 and for all the solutions $(t, z(t))$ of $\Sigma(t) = 0$ for which $\sup_{t\geq 0}\|d(t)\| < +\infty$, there exists $\mathfrak{p} > 0$ such that

$$\limsup_{t\nearrow T(\mathfrak{p})} \|s(t) - \hat{s}(t)\| \leq \pi_1\sqrt{\frac{2\eta}{\underline{\beta}\delta}}\sup_{t\geq 0}\|d(t)\| \tag{123}$$

where

$$\hat{s}(t) = \begin{pmatrix} y(t) \\ \Psi^\mathsf{x}_{-\mathfrak{p}}(t_\mathfrak{p}(t), y_\mathfrak{p}(t), \zeta^\mathsf{x}(t)) \end{pmatrix},$$

$$D_t\zeta(t) = \frac{\partial t_\mathfrak{p}(t)}{\partial t}\Big((A(t_\mathfrak{p}(t)) - K(t_\mathfrak{p}(t))C(t_\mathfrak{p}(t))\zeta(t) + K(t_\mathfrak{p}(t))y_\mathfrak{p}(t)\Big), \zeta(0) = 0. \tag{124}$$

If in addition $\Psi_\mathfrak{p}$ is CBU-transforming for each $T^* > 0$ there exists $\mathfrak{p} > 0$ such that (123) holds with $\mathrm{Dom}^+\Psi^t_\mathfrak{p} < T^*$. $\lhd$

Finite-time sliding mode-based observers for linear systems have been preliminarily studied in [23] (only to cite one). Sliding mode-based observers are discontinuous in the output innovation, while ours are continuous (actually smooth) with time-varying gains. This type of observers is more resemblant to the ones discussed in [?], in which there is also a quite detailed study of the different performances of the two types of observers.

***Example** 4.2.2 **(cont'ed)***. Consider the system map $\Sigma$ associated with (55). The LGT (56) is a LS of $\Sigma(t) = 0$, with $\mathrm{Dom}(\Psi_\mathfrak{p}) = \{(t,\mathsf{z}) \in \mathcal{M} : t \neq \frac{1}{\mathfrak{p}}\}$ and CBU-transforming for $\mathfrak{p} > 0$ (Assumption PT.O.1). Assumptions PT.O.2-PT.O.3 are met with $\tau \triangleq 1$, $\psi(t) \triangleq \frac{6}{1+t}$, $\pi_2(t,\mathfrak{p}) \triangleq \frac{1}{(1+\mathfrak{p}t)^3}$ and $\pi_1 \triangleq 1$. By Theorem 4 an observer for $\Sigma(t) = 0$ with prescribed finite-time convergence $< T^*$ is provided by (124) with $\mathfrak{p} > \frac{1}{T^*}$

$$\hat{s}(t) = \begin{pmatrix} y(t) \\ -\mathfrak{p}\zeta^\mathsf{y}(t) + \frac{1}{1-\mathfrak{p}t}\zeta^\mathsf{x}(t) \end{pmatrix},$$

$$D_t\chi(t) = \frac{1}{(1-\mathfrak{p}t)^2}\left(\begin{pmatrix} \zeta^\mathsf{x}(t) \\ 0 \end{pmatrix} + K\Big(\frac{y(t)}{1-\mathfrak{p}t} - \zeta^\mathsf{y}(t)\Big)\right),$$

with $\zeta = (\zeta^\mathsf{y}, \zeta^\mathsf{x})$ and $K \in \mathbb{R}^2$ such that $A - KC = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - K\begin{pmatrix} 1 & 0 \end{pmatrix}$ is Hurwitz.

## 6.3 | Symmetry-based observers with asymptotic convergence and unbounded state solutions

Theorem (3) gives a procedure for transforming a local observer for $\Sigma(t) = 0$ into a semiglobal observer, where "semiglobal" means that the observer tolerates only the state solutions and disturbances which remain for all times in a prescribed compact set: $\sup_{t\geq 0} \|(s(t), d(t))\| \leq N$. Hence, unbounded state solutions are not tolerated by the observer and hinder or undermine its convergence. Moreover, Theorem 3 is a bit out of the paradigm provided by Theorem 1 in which only $s(0)$ and the exogenous inputs $d(t)$ are bound to a prescribed compact set: $\sup_{t\geq 0} \|(s(0), d(t))\| \leq N$. With this motivation, we want to study how to transform a local observer for $\Sigma(t) = 0$ into a semiglobal observer when only $s(0)$ and the exogenous inputs $d(t)$ are bound to a prescribed compact set (in this sense, our observer may be considered *global*). To this aim and since a *known and constant* bound $N$ on the state solutions is not available in this context, we introduce an assumption which is also *necessary* for the existence of an observer and provides a *time-varying bound* $N(t)$ for the state solutions of $\Sigma(t) = 0$.

***Assumption OU.1***: (Unboundedness observability[5]). There exist $\chi_1, \chi_2, \chi_3, \sigma_1, \sigma_2 \in \mathcal{K}_\infty$ such that the solutions $(t, z(t))$ of $\Sigma(t) = 0$ satisfy for all $t \geq 0$

$$\|s(t)\| \leq \chi_1(t) + \chi_2(\|s(0)\|) + \chi_3\left(\int_0^t \sigma_1(\|(u(r), d(r))\rangle dr + \int_0^t \sigma_2(\|y(r)\|)dr)\|\right). \lhd \tag{125}$$

Equivalent conditions for Assumption OU.1 can be stated in term of Lyapunov functions ([5]). The systems (26), (43) or (46) have the above unboundedness observability property. In particular, for a system $\Sigma(t) = 0$ of the form

$$D_t s(t) = f(y(t))s(t) + g_1(y(t))u + g_2(y(t))d(t)$$

we always have the existence of $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ such that the solutions $(t, z(t))$ of $\Sigma(t) = 0$ satisfy for all $t \geq 0$

$$\|s(t)\| \leq t + \|s(0)\| + e^{\int_0^t \sigma_1(\|(u(r),d(r))\|)dr + \int_0^t \sigma_2(\|y(r)\|)dr}. \tag{126}$$

i.e. Assumption OU.1 with any $\chi_1(r) = r$, $\chi_2(r) = r$, $\chi_3(r) = e^r$.

If the state solutions are unbounded, the parameter $\mathfrak{p}$ of the symmetry must be *time-varying* to keep track of $\|s(t)\|$ for all $t \geq 0$, taking advantage of the upper bound (125) postulated by Assumption OU.1. This type of approach consisting of tracking the state norm $\|s(t)\|$ with a *norm estimator* has been pursued for global observer design in the recent paper[7], however with more stringent assumptions than Assumption OU.1 such as output-to-state-stability ([22]).

One technical problem which arises when $\mathfrak{p}$ is time-varying is that $\Psi^t_\mathfrak{p}(t)$ is no more guaranteed to be increasing as a function of time (as when $\mathfrak{p}$ is constant) even if $\frac{\partial \Psi^t_\mathfrak{p}(t)}{\partial t} > 0$ for each $\mathfrak{p}$ and $t \in \mathrm{Dom}(\Psi^t_\mathfrak{p})$ which is the case of a UU-transforming $\Psi_\mathfrak{p}$ for each $\mathfrak{p} > 0$. Even worse, $\lim_{t\to+\infty} \Psi^t_\mathfrak{p}(t) < +\infty$ and the UU-transforming property of $\Psi_\mathfrak{p}$ is lost. The following example is clarifying: $\Psi^t_\mathfrak{p}(t) = e^{-\mathfrak{p}}t$ and $\mathfrak{p} = \ln(1 + t^2)$. Our next assumption OU.2 takes care exactly of this aspect. Let

$$L_\omega[u, y](t) \triangleq \omega + \chi_1(t) + \chi_2(\omega) + \chi_3\left(\int_0^t \sigma_1(\|u(r)\| + \omega)dr + \int_0^t \sigma_2(\|y(r)\|\right)dr$$

with $\omega > 0$ . Notice that $L_\omega[u, y](t) > 0$ for all $t \geq 0$.

***Assumption OU.2***: For each $N > 0$, the following inequality holds for all the solutions $(t, z(t))$ of $\Sigma(t) = 0$ such that $\|(y(0), u(0))\| \leq N$, $\omega > 0$ and $t \geq 0$ such that $g^t|_{\Psi^t_\mathfrak{p}(t)|_{\mathfrak{p}=\sigma(L_\omega[u,y](t))}} < 1$:

$$D_t L_\omega[u, y](t) \leq \frac{\frac{\partial \Psi^t_\mathfrak{p}(t)}{\partial t}}{1 - g^t|_{\Psi^t_\mathfrak{p}(t)}} \left(\frac{d\sigma(r)}{dr}\right)^{-1}\Bigg|_{\substack{\mathfrak{p}=\sigma(L_\omega[u,y](t))\\ r=L_\omega[u,y](t)}}, \tag{127}$$

$$-\frac{\frac{\partial \Psi^t_\mathfrak{p}(t)}{\partial t}}{D_t L_\omega[u, y](t)} < g^t|_{\Psi^t_\mathfrak{p}(t)} \frac{d\sigma(r)}{dr}\Bigg|_{\substack{\mathfrak{p}=\sigma(L_\omega[u,y](t))\\ r=L_\omega[u,y](t)}}. \lhd \tag{128}$$

*Remark 5.* Assumption OU.2 is a bit tricky one but eventually much easier to be satisfied than one might think. If for instance $\Psi^t_\mathfrak{p}(t) = e^{k\mathfrak{p}}t$ with generator $\mathbf{g}^t|_t = kt\frac{\partial}{\partial t}$ and SI-contracting map $\sigma(r) = \frac{1}{1-\delta}\ln(1 + r)$ with any $\delta \in (0, 1)$ (see for example (43)), Assumption OU.2 boils down to the single condition

$$D_t L_\omega[u, y](t) \leq \frac{(1 + L_\omega[u, y](t))^{\frac{k}{1-\delta}+1}}{1 + kt(1 + L_\omega[u, y](t))^{\frac{k}{1-\delta}}} \tag{129}$$

for all $t \geq 0$ such that $kt(1 + L_\omega[u, y](t))^{\frac{k}{1-\delta}} < 1$ and with $\|(y(0), u(0))\| \leq N$ for some given $N > 0$. Since $L_\omega[u, y](t) \geq \omega$ for all $t \geq 0$, the above condition holds true if more simply

$$D_t L_\omega[u, y](t) \leq \frac{1}{2}(1 + \omega)^{\frac{k}{1-\delta}+1} \tag{130}$$

for all non-negative $t < \frac{1}{k}(1 + \omega)^{-\frac{k}{1-\delta}}$ and with $\|(y(0), u(0))\| \leq N$. Since

$$D_t L_\omega[u, y](t) \triangleq D_t \chi_1(t) + \frac{d\chi_3(r)}{dr}\Big|_{r=\chi_3^{-1}(L_\omega[u,y](t)-\omega-\chi_1(t)-\chi_2(\omega))}\Big(\sigma_1(\|u(t)\| + \omega) + \sigma_2(\|y(t)\|\Big) \tag{131}$$

there clearly exists $\delta^* \in (0, 1)$ such that (130) holds true for all $\delta > \delta^*$ and non-negative $t < \frac{1}{k}(1+\omega)^{-\frac{k}{1-\delta}}$ with $\|(y(0), u(0))\| \leq N$. $\triangleleft$

Next, we introduce our assumptions for the existence of a local observer. More specifically, for a *modified* system $\Sigma_M(t) = 0$ obtained from $\Sigma(t) = 0$ and defined as

$$D_t s(t) = (1 - \Delta(t)\mathbf{g}^t|_t)F(t, s(t), u(t), d(t)) + \Delta(t)\mathbf{g}^{\mathsf{s}}|_{(t,s(t))}$$

with $\Delta(\cdot)$ a known function such that $|\Delta(t)| \leq 1$ for all $t \geq 0$, we assume the existence of a *local* observer of the form

$$D_t\zeta(t) = (1 - \Delta(t)\mathbf{g}^t|_t)\Big(F(t, \zeta(t), u(t), 0) + \psi(t, y(t), y(t) - \zeta^{\mathsf{y}}(t))\Big) + \Delta(t)\mathbf{g}^{\mathsf{s}}|_{(t,\zeta(t))} \tag{132}$$

with $\zeta = (\zeta^{\mathsf{y}}, \zeta^{\mathsf{x}})$.

***Assumption OU.3***: There exist open neighbourhoods $C^{\mathsf{s}} \triangleq C^{\mathsf{y}} \times C^{\mathsf{x}} \subset \mathcal{S}$, $C^{\mathsf{e}} \triangleq C^{\mathsf{e^y}} \times C^{\mathsf{e^x}} \supset C^{\mathsf{s}}$, $C^{\mathsf{d}} \subset \mathbb{R}^r$ of the origin, $\delta, \underline{\theta}, \overline{\theta}, \eta \in \mathcal{K}_\infty$ and a $C^1$ function $W : \mathbb{R}^0_+ \times C^{\mathsf{e}} \to \mathbb{R}^0_+$ and $\psi : \mathbb{R} \times C^{\mathsf{y}} \times C^{\mathsf{e^y}} \to \mathbb{R}^m$ such that

$$\underline{\theta}(\|\mathsf{e}\|) \leq W(t, \mathsf{e}) \leq \overline{\theta}(\|\mathsf{e}\|), \tag{133}$$

$$\begin{aligned}&\frac{\partial W(t, \mathsf{e})}{\partial \mathsf{e}}\Big((1 - \Delta(t)\mathbf{g}^t|_t)(F(t, \mathsf{s}, \mathsf{u}, \mathsf{d}) - F(t, \mathsf{s} - \mathsf{e}, \mathsf{u}, 0) - \psi(t, \mathsf{y}, \mathsf{e^y})) + \Delta(t)(\mathbf{g}^{\mathsf{s}}|_{(t,\mathsf{s})} - \mathbf{g}^{\mathsf{s}}|_{(t,\mathsf{s}-\mathsf{e})})\Big) + \frac{\partial W(t, \mathsf{e})}{\partial t}\\ &\leq -\delta(W(t, \mathsf{e})) + \eta(\|\mathsf{d}\|)\end{aligned} \tag{134}$$

for $(t, \mathsf{x}, \mathsf{e}, \mathsf{u}, \mathsf{d}) \in \mathbb{R}^0_+ \times C^{\mathsf{s}} \times C^{\mathsf{e}} \times \mathbb{R}^m \times C^{\mathsf{d}}$ with $\mathsf{e} \triangleq (\mathsf{e^y}, \mathsf{e^x})$ and $\Delta(t) \in [-1, 1]$. $\triangleleft$

The following result is proved as Theorem 3, with the only difference that now the parameter $\mathfrak{p}$ is time-varying.

**Theorem 5.** Under Assumptions O.1, O.3, O.4 and OU.1-OU.3 and for each $N > 0$ consider the solutions $(t, z(t))$ of $\Sigma(t) = 0$, with $\sup_{t\geq 0} \|(s(0), u(0), d(t))\| \leq N$ and observer

$$\hat{s}(t) \triangleq \begin{pmatrix} y(t) \\ \Psi^{\mathsf{x}}_{-\mathfrak{p}}(t_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t), \zeta^{\mathsf{x}}(t)) \end{pmatrix}, \zeta(t) \triangleq (\zeta^{\mathsf{y}}(t), \zeta^{\mathsf{x}}(t)),$$

$$D_t\zeta(t) = \frac{\partial \Psi^t_{\mathfrak{p}}(t)}{\partial t}\Big(F(t_{\mathfrak{p}}(t), \zeta(t), u_{\mathfrak{p}}(t), 0) + \psi(t_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t) - \zeta^{\mathsf{y}}(t))\Big) + \mathbf{g}^{\mathsf{s}}|_{(t_{\mathfrak{p}}(t),\zeta(t))} D_t\mathfrak{p}(t),\ \zeta(0) = 0, \tag{135}$$

$$D_t\mathfrak{p}(t) = \frac{\sigma(r)}{dr}\Big|_{r=\sigma^{-1}(\mathfrak{p}(t))}\Big(D_t\chi_1(t) + \frac{d\chi_3(r)}{dr}\Big|_{r=\chi_3^{-1}(\sigma^{-1}(\mathfrak{p}(t))-\omega-\chi_1(t)-\chi_2(\omega))}(\sigma_1(\|u(t)\| + \omega) + \sigma_2(\|y(t)\|)\Big),\ \mathfrak{p}(0) = \sigma(\omega + \chi_2(\omega)).$$

There exists $\omega > 0$ such that

$$\limsup_{t\to+\infty} \|s(t) - \hat{s}(t)\| \leq \pi_1(\sup_{\mathsf{s}\in C^{\mathsf{s}}} \|\mathsf{s}\| + \sup_{\mathsf{e}\in C^{\mathsf{e}}} \|\mathsf{e}\|)\pi_3(\sup_{t\geq 0} \|d(t)\|).\triangleleft \tag{136}$$

*Remark 6.* The burdensome expression for $D_t\mathfrak{p}(t)$ in (135) is nothing but a real-time implementation of the time-varying variable $\mathfrak{p}(t) = \sigma(L_\omega[u, y](t))$ which depends on the input and output segments over $[0, t]$ (see (125)). Hence, $\mathfrak{p}(t)$ is defined for all $t \geq 0$.

**Example 6.1.** *(Global symmetry-based observers versus others).* Consider the system map $\Sigma$ associated with

$$F(t, \mathsf{s}, \mathsf{u} = \left(\begin{pmatrix} A_1(\mathsf{y}) + \mathsf{x} \\ A_2(\mathsf{s}) + \mathsf{d} \end{pmatrix}, \mathsf{x}_1\right), \tag{137}$$

$\mathsf{y1}, \mathsf{x}, \mathsf{d} \in \mathbb{R}$, and

$$A_1(\mathsf{y}) = a_1\mathsf{y}^{\frac{\gamma+r_1}{r_1}}, A_2(\mathsf{s}) = a_{2,1}\mathsf{y}^{\frac{\gamma+r_2}{r_1}} + a_{2,2}\mathsf{x}^{\frac{\gamma+r_2}{r_2}} \tag{138}$$

for some $a_1, a_{2,1}, a_{2,2} \in \mathbb{R}$, $r_1 > 0$, $\gamma \geq 0$ with $r_2 \triangleq r_1 + \gamma$. Hence, $F$ and $H$ are homogeneous with weights $(r_1, r_2)$ and degrees $\gamma$ and $r_1$, respectively. It is possible to design a global homogeneous observer for $\Sigma(t) = 0$ (see for instance[4] for chains of integrators) if

$$|A_1(\mathsf{x}_1^a) - A_1(\mathsf{y}^b)| \leq c_1 |\mathsf{y}^a - \mathsf{y}^b|^{\frac{\gamma+r_1}{r_1}},$$
$$|A_2(\mathsf{s}^a) - A_2(\mathsf{s}^b)| \leq c_2 (|\mathsf{y}^a - \mathsf{y}^b|^{\frac{\gamma+r_2}{r_1}} + |\mathsf{x}^a - \mathsf{x}^b|^{\frac{\gamma+r_2}{r_2}}) \tag{139}$$

for all $\mathsf{x}^a, \mathsf{x}^b \in \mathbb{R}^2$ and for some $c_1, c_2 > 0$, which holds only if $\gamma = 0$ (the case $\gamma < 0$ may be considered but this would imply $F \notin C^1$). It is not even possible to design a sliding-mode observer for $\Sigma(t) = 0$ if $\sup_{t \geq 0} \|(s(t), d(t)\| = +\infty$. On the other hand, as already discussed in the case of (41) in Example 4.2.3, we find that $\Psi_{\mathfrak{p}}(t, \mathsf{z}) = (e^{\mathfrak{p}\gamma} t, e^{-\mathfrak{p}r_1}\mathsf{x}_1, e^{-\mathfrak{p}r_2}\mathsf{x}_2, e^{-\mathfrak{p}(\gamma+r_2)}\mathsf{u}, e^{-\mathfrak{p}r_1}\mathsf{y})$ is a GS of $\Sigma(t) = 0$, SI-contracting with contracting maps $(\frac{1}{\delta}\ln(1+s), e^{-s(1-\delta)})$ for any $\delta \in (0, 1)$. Furthermore, from[4] we borrow a $C^1$ function $V : \mathbb{R}^n \to \mathbb{R}^0_+$ of the form $V(\mathsf{x}) = \int_{\mathsf{x}_2^{\frac{r_1}{r_2}}}^{d_1 \mathsf{x}_1} (s^{\frac{d_2 - r_1}{r_1}} - \mathsf{x}_2^{\frac{d_2 - r_1}{r_2}}) ds + |\mathsf{x}_2|^{\frac{d_2}{r_2}}$ such that $D_t V(x(t))|_{\Sigma(t)=0} \leq -\ell_1 V^{\frac{d_2+\gamma}{r_2}}(x(t)) + \ell_2 |y(t)|^{\frac{d_2+\gamma}{r_1}} + \ell_3 (\sup_{t\geq 0} \|d(t)\|)^{\frac{d_2+\gamma}{r_2+\gamma}}$ for $t \geq 0$ and for some $\ell_1, \ell_2, \ell_3 > 0$ and sufficiently large $d_1 > 0$ and $d_2 > \max\{r_1, r_2\}$. Moreover, $\|x(t)\| \leq b_1 V^{\frac{r_2}{d_2}} + b_0$ for all $t \geq 0$ and for some $b_1, b_0 > 0$. Hence, $\Sigma(t) = 0$ has the unboundedness observability property (Assumption OU.1) and we can easily find $\delta^* \in (0, 1)$ such that Assumption OU.2 is satisfied for all $\delta \in [\delta^*, 1)$ (see the short discussion in (130) -(131)). The remaining assumptions of Theorem 5 can be satisfied with a symmetry designed as in Example 4.2.3 and a simple *local* observer for $\Sigma(t) = 0$. In conclusion, while for a system $\Sigma(t) = 0$ with $F$ in (137)-(138) we can design a global symmetry-based observer, we cannot use any other existing observer unless $\Sigma(t) = 0$ satisfies either (139) with $\gamma = 0$ or $\sup_{t\geq 0} \|(s(t), d(t)\| < +\infty$.

Similar considerations can be brought in for the system map $\Sigma$ associated to (26) and the GS of $\Sigma(t) = 0$ in (29). Notice that (26) does not fall in the class (137)-(138) for any choice of $r_1 > 0$ and $\gamma \geq 0$. As already seen, (29) is SI-contracting. Moreover, it is a well-known fact that $\Sigma(t) = 0$ has the unboundedness observability property (Assumption OU.1). The remaining assumptions of Theorem 5 can be satisfied with the symmetry (29) and a simple *local* observer for $\Sigma(t) = 0$. To the best of our knowledge, this is the first type of global observer proposed for a system $\Sigma(t) = 0$ associated with (26). $\lhd$

## 7 | OUTPUT FEEDBACK CONTROLLERS

The results and methodologies of Sections 5 and 6 can be put together to give a symmetry-based procedure for transforming local output feedback controller with infinite-time convergence into semiglobal/global output feedback controller with infinite-time/prescribed-time convergence. No additional technicality or adjustment is needed.

## 8 | CONCLUSIONS

One-parameter groups and (contracting) symmetries of ODE systems have been recalled from[9]. Adopting a symmetry-based approach, the problem of transforming a local output-feedback controller with infinite-time convergence into a semiglobal or global output-feedback controller with either infinite-time or prescribed-time convergence has been thoroughly investigated. In particular, the state-feedback control problem and the observer design problem have been addressed separately. While some nonlinear generalizations for prescribed-time control have been proposed in[8] and with delays in[7], future research will focus on extending these methodologies to more general classes of systems like PDE (partial differential equation) systems.

$\square$

## APPENDIX

## A PROOFS OF MAIN RESULTS

**Lemma 1.** Consider the system

$$D_t s(t) = F(t, s(t), d(t)) \tag{A1}$$

and assume that for some open neighbourhoods $\mathcal{B}^{\mathsf{s}} \subset \mathbb{R}^n$ and $\mathcal{B}^{\mathsf{d}} \subset \mathbb{R}^s$ of the origin

$$s(t) \in \mathcal{B}^{\mathsf{s}}, d(t) \in \mathcal{B}^{\mathsf{d}} \tag{A2}$$

for all $t \in [0, t^*)$ with some $t^* > 0$ and

$$\sup_{t\in[0,t^*)} \|d(t)\| = N. \tag{A3}$$

Moreover,

$$\underline{\beta}(\|\mathsf{s}\|) \le V(t,\mathsf{s}) \le \overline{\beta}(\|\mathsf{s}\|), \tag{A4}$$

$$\frac{\partial V(t,\mathsf{s})}{\partial t} + \frac{\partial V(t,\mathsf{s})}{\partial \mathsf{x}} F(t,\mathsf{s},\mathsf{d}) \le -\alpha(V(t,\mathsf{s})) + \gamma(\|\mathsf{d}\|) \tag{A5}$$

for all $t \in [0,t^*)$, $\mathsf{s} \in \mathcal{B}^{\mathsf{s}}$ and $\mathsf{d} \in \mathcal{B}^{\mathsf{d}}$ and for some $\alpha, \underline{\beta}, \overline{\beta}, \gamma \in \mathcal{K}_\infty$ and $C^1$ function $V : \mathbb{R}^0_+ \times \mathcal{B}^{\mathsf{s}} \to \mathbb{R}^0_+$. For all $t \in [0,t^*)$

$$\|s(t)\| \le \underline{\beta}^{-1}\left(\max\left\{\overline{\beta}(\|s(0)\|) - \gamma(N)t, \underline{\beta}(\chi(\sup_{t\in[0,t^*)}\|d(t)\|))\right\}\right) \tag{A6}$$

where $\chi(r) \triangleq (\underline{\beta}^{-1}\circ\overline{\beta}\circ\underline{\beta}^{-1}\circ\alpha^{-1})(2\gamma(s)) \in \mathcal{K}_\infty$.

*Proof.* Let $c \triangleq (\underline{\beta}\circ\chi)(N)$. We have using (A3), (A4) and (A5)

$$\begin{aligned} &V(t,s(t)) = c, t \in [0,t^*) \Rightarrow \|s(t)\| \ge \overline{\beta}^{-1}(c) = \underline{\beta}^{-1}(\alpha^{-1}(2\gamma(N))) \Rightarrow \alpha(V(t,s(t))) \ge 2\gamma(\|d(t)\|) \\ &\Rightarrow D_t V(t,s(t)) \le 0 \Rightarrow V(\tau, s(\tau)) \le c, \forall \tau \in [t,t^*). \end{aligned} \tag{A7}$$

Hence, for any $\bar{t} \in (0,t^*)$

$$V(\bar{t}, s(\bar{t})) \le c \Rightarrow V(t,s(t)) \le c, \forall t \in [\bar{t}, t^*) \Rightarrow \|s(t)\| \le \underline{\beta}^{-1}(c) = \chi(N) \le \chi(\sup_{t\in[0,t^*)}\|d(t)\|), \forall t \in [\bar{t}, t^*). \tag{A8}$$

On the other hand, if $V(0,s(0)) > c$ then there exists $\bar{t} \in (0,t^*]$ such that either $V(\bar{t}, s(\bar{t})) = c$ with $\bar{t} < t^*$ or $V(t,s(t)) > c$ for all $t \in [0,\bar{t})$ with $\bar{t} = t^*$. In either $\bar{t} < t^*$ or $\bar{t} = t^*$ cases, using (A3), (A4) and (A5) one more time

$$\begin{aligned} &V(t,x(t)) > c, \forall t \in [0,\bar{t}) \Rightarrow \|x(t)\| > \overline{\beta}^{-1}(c) = \underline{\beta}^{-1}(\alpha^{-1}(2\gamma(N))) \Rightarrow \alpha(V(t,x(t))) > 2\gamma(N), \forall t \in [0,\bar{t}) \\ &\Rightarrow D_t V(t,s(t)) < -\gamma(N), \forall t \in [0,\bar{t}). \end{aligned} \tag{A9}$$

Hence,

$$\|x(t)\| \le \underline{\beta}^{-1}\left(\max\left\{\overline{\beta}(\|s(0)\|) - \gamma(N)t, \underline{\beta}(\chi(N))\right\}\right) \tag{A10}$$

for all $t \in [0,\bar{t})$. If $\bar{t} < t^*$ then $V(\bar{t}, s(\bar{t})) = c$ and on account of (A8)

$$\|x(t)\| \le \chi(\sup_{t\in[0,t^*)}\|d(t)\|) = \chi(N) \tag{A11}$$

for all $t \in [\bar{t}, t^*)$, otherwise $\bar{t} = t^*$ and (A10) holds for all $t \in [0,t^*)$. From these inequalities it readily follows (A6). □

*Theorem 1.*

*Proof.* Pick $N > 0$ and consider initial conditions $s(0)$ and exogenous inputs $d(t)$ for which

$$\sup_{t\ge 0} \|(s(0), d(t))\| \le N. \tag{A12}$$

Define

$$\mathfrak{p} \triangleq \sigma(\omega) \text{ where } \omega > \max\{\sigma^{-1}(\mathfrak{p}_0), N\} \tag{A13}$$

and $\mathfrak{p}_0$ introduced in Assumption S.3. By Assumption S.1 $\Psi_{\mathfrak{p}}$ is SI-contracting with contracting maps $(\sigma, \mu)$. Since $\sigma \in \mathcal{K}_\infty$ and from (A12) and (A13) it follows

$$\sup_{t\ge 0} \sigma(\|(s(0), d(t))\|) \le \sigma(N) \le \mathfrak{p}.$$

Hence, by the SI-contracting property of $\Psi_{\mathfrak{p}}$

$$\|s_{\mathfrak{p}}(0)\|, \sup_{t\ge 0}\|d_{\mathfrak{p}}(t)\| \le \mu(\mathfrak{p}) \tag{A14}$$

Using this and the fact that all the functions $\mu, \underline{\beta}^{-1}\circ\overline{\beta}\circ\mu$ and $\chi\circ\mu$ are in $\mathcal{L}$, pick $\omega_1 > \max\{\sigma^{-1}(\mathfrak{p}_0), N\}$ be such that for all $\omega \geq \omega_1$ (recall that $\mathfrak{p} = \sigma(\omega)$)

$$s_{\mathfrak{p}}(0) \in \mathcal{B}^{\mathrm{s}}, d_{\mathfrak{p}}(t) \in \mathcal{B}^{\mathrm{d}},\ \forall t \geq 0, \tag{A15}$$

and

$$\mathcal{N}(\mathfrak{p}) \triangleq \{\mathrm{s} \in \mathbb{R}^n : \|\mathrm{s}\| < 2\max\{(\underline{\beta}^{-1}\circ\overline{\beta})(\mu(\mathfrak{p})), \chi(\mu(\mathfrak{p}))\}\} \subset \mathcal{B}^{\mathrm{s}} \tag{A16}$$

with $\underline{\beta}, \overline{\beta}, \chi$ introduced in Assumption S.2 and S.3. Clearly,

$$s_{\mathfrak{p}}(0) \in \mathcal{N}(\mathfrak{p})$$

by (A14) and since $(\underline{\beta}^{-1}\circ\overline{\beta})(s) \geq s$ for all $s \geq 0$. Let $t^* \leq +\infty$ be the first time for which $s_{\mathfrak{p}}(t^*) \in \partial\mathcal{N}(\mathfrak{p})$, the boundary of $\mathcal{N}(\mathfrak{p})$. We claim that $t^* = +\infty$ whatever is $\omega \geq \omega_1$. Indeed, assume that for at least one value of $\omega \geq \omega_1$ we have $t^* < +\infty$. Recalling (A15), this means that for some finite $t^*$

$$s_{\mathfrak{p}}(t) \in \mathcal{N}(\mathfrak{p}) \subset \mathcal{B}^{\mathrm{s}}, d_{\mathfrak{p}}(t) \in \mathcal{B}^{\mathrm{d}},\ t \in [0, t^*), \tag{A17}$$

then

$$(t_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t), \varphi(t_{\mathfrak{p}}(t), s_{\mathfrak{p}}(t))) \in \mathrm{Dom}(\Psi^{\mathrm{u}}_{-\mathfrak{p}}),\ t \in [0, t^*),$$

by Assumption S.3 and the control input function $u(t)$ in (75) is well-defined. Moreover, by the group properties of $\Psi_{\mathfrak{p}}$

$$u_{\mathfrak{p}}(t) = \Psi^{\mathrm{u}}_{\mathfrak{p}}(t, y(t), u(t)) = \Psi^{\mathrm{u}}_{\mathfrak{p}}(\Psi^{t}_{-\mathfrak{p}}(t_{\mathfrak{p}}(t)), \Psi^{\mathrm{y}}_{-\mathfrak{p}}(t_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t)), \Psi^{\mathrm{u}}_{-\mathfrak{p}}(t_{\mathfrak{p}}(t), y_{\mathfrak{p}}(t), \varphi(t_{\mathfrak{p}}(t), s_{\mathfrak{p}}(t))))) = \varphi(t_{\mathfrak{p}}(t), s_{\mathfrak{p}}(t)). \tag{A18}$$

But $\Psi^{t}_{\mathfrak{p}}(t)$ is a monotonically increasing function of $t \in [0, t^*)$ with $\Psi^{t}_{\mathfrak{p}}(0) = 0$ (in general $\Psi^{t}_{\mathfrak{p}}(0) \neq 0$ but we assume $\Psi^{t}_{\mathfrak{p}}(0) = 0$ for simplicity) and $\lim_{t\to+\infty} \Psi^{t}_{\mathfrak{p}}(t) = +\infty$, since $\Psi_{\mathfrak{p}}$ is UU-transforming by Assumption S.1. Set $\mathrm{t}^*_{\mathfrak{p}} = \Psi^{t}_{\mathfrak{p}}(t^*)$ and consider the variables $(t, z(t))$ in the new unbounded positive time scale $\mathrm{t}_{\mathfrak{p}} = \Psi^{t}_{\mathfrak{p}}(t)$, i.e. $(\mathrm{t}_{\mathfrak{p}}, \widetilde{z}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})) = (\Psi^{t}_{\mathfrak{p}}(t), z_{\mathfrak{p}}\circ[\Psi^{t}_{\mathfrak{p}}]^{-1}(\mathrm{t}_{\mathfrak{p}}))$, with $\mathrm{t}_{\mathfrak{p}} \in [0, \mathrm{t}^*_{\mathfrak{p}})$. By Assumption S.1 $\Psi_{\mathfrak{p}}$ is a LS of $\Sigma(t) = 0$, hence $\Psi_{\mathfrak{p}}$ maps $\Sigma(t) = 0$ into $\widetilde{\Sigma}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}) = 0$ which on account of (A18) reads as

$$D_{\mathrm{t}_{\mathfrak{p}}}\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}) = F(\mathrm{t}_{\mathfrak{p}}, \widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}), \varphi(\mathrm{t}_{\mathfrak{p}}, \widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})), \widetilde{d}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})) \tag{A19}$$

On account of (A17)

$$\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}) \in \mathcal{N}(\mathfrak{p}) \subset \mathcal{B}^{\mathrm{s}}, \widetilde{d}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}) \in \mathcal{B}^{\mathrm{d}},\ \mathrm{t}_{\mathfrak{p}} \in [0, \mathrm{t}^*_{\mathfrak{p}}), \tag{A20}$$

and on account of (A14)

$$\|\widetilde{s}_{\mathfrak{p}}(0)\|, \sup_{\mathrm{t}_{\mathfrak{p}}\geq 0} \|\widetilde{d}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})\| \leq \mu(\mathfrak{p}). \tag{A21}$$

By Assumption S.2 and (A20)

$$\underline{\beta}(\|\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})\|) \leq V(\mathrm{t}_{\mathfrak{p}}, \widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})) \leq \overline{\beta}(\|\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})\|), \tag{A22}$$

$$D_{\mathrm{t}_{\mathfrak{p}}} V(\mathrm{t}_{\mathfrak{p}}, \widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})) \leq -\alpha(V(\mathrm{t}_{\mathfrak{p}}, \widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}))) + \gamma(\|\widetilde{d}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})\|),\ \mathrm{t}_{\mathfrak{p}} \in [0, \mathrm{t}^*_{\mathfrak{p}}), \tag{A23}$$

for some $\alpha, \underline{\beta}, \overline{\beta}, \gamma \in \mathcal{K}_\infty$ and a $C^1$ function $V : \mathbb{R}^0_+ \times \mathcal{B}^{\mathrm{s}} \to \mathbb{R}^0_+$ and $\varphi : \mathbb{R} \times \mathcal{B}^{\mathrm{s}} \to \mathbb{R}^m$. From lemma 1 it follows that

$$\|\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})\| \leq \underline{\beta}^{-1}\left(\max\left\{\overline{\beta}(\|\widetilde{s}_{\mathfrak{p}}(0)\|) - \gamma(N)\mathrm{t}_{\mathfrak{p}}, \underline{\beta}(\chi(\sup_{\mathrm{t}_{\mathfrak{p}}\geq 0}\|\widetilde{d}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})\|))\right\}\right) \tag{A24}$$

with $\chi(s) \triangleq (\underline{\beta}^{-1}\circ\overline{\beta}\circ\underline{\beta}^{-1}\circ\alpha^{-1})(2\gamma(s)) \in \mathcal{K}_\infty$. Using (A21)

$$\|\widetilde{s}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})\| \leq \underline{\beta}^{-1}\left(\max\left\{\overline{\beta}(\|\widetilde{s}_{\mathfrak{p}}(0)\|) - \gamma(N)\mathrm{t}_{\mathfrak{p}}, \underline{\beta}(\chi(\sup_{\mathrm{t}_{\mathfrak{p}}\geq 0}\|\widetilde{d}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}})\|))\right\}\right) \leq \max\left\{\left(\underline{\beta}^{-1}\circ\overline{\beta}\right)(\mu(\mathfrak{p})), \chi(\mu(\mathfrak{p}))\right\} \tag{A25}$$

from which it follows that $\widetilde{s}_{\mathfrak{p}}(\mathrm{t}^*_{\mathfrak{p}}) \in \mathcal{N}(\mathfrak{p})$, i.e. a contradiction on account of the definition of $\mathrm{t}^*_{\mathfrak{p}}$. Hence, $t^* = +\infty$ (and $\mathrm{t}^*_{\mathfrak{p}} = +\infty$) whatever is $\omega \geq \omega_1$. Pick any $\omega \geq \omega_1$ then, using lemma 1 one more time,

$$\widetilde{x}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}) \in \mathcal{N}(\mathfrak{p}) \subset \mathcal{B}^{\mathrm{s}}, \widetilde{d}_{\mathfrak{p}}(\mathrm{t}_{\mathfrak{p}}) \in \mathcal{B}^{\mathrm{d}}, \forall \mathrm{t}_{\mathfrak{p}} \geq 0, \tag{A26}$$

and

$$\limsup_{\mathfrak{t}_\mathfrak{p}\to+\infty} \|\widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\| \le \chi(\sup_{\mathfrak{t}_\mathfrak{p}\ge 0} \|\widetilde{d}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\|). \tag{A27}$$

Furthermore, by Assumption S.4

$$\chi(\sup_{\mathfrak{t}_\mathfrak{p}\ge 0} \|\widetilde{d}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\|) \le \pi_2(\mathfrak{p})\pi_3(\sup_{t\ge 0}\|d(t)\|). \tag{A28}$$

By the group properties of $\Psi_\mathfrak{p}$

$$(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) \in \mathrm{Dom}(\Psi^{\mathsf{s}}_{-\mathfrak{p}})$$

for all $\mathfrak{t}_\mathfrak{p} \ge 0$, hence $\Psi^{\mathsf{s}}_{-\mathfrak{p}}(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p}))$ is well-defined for $\mathfrak{t}_\mathfrak{p} \ge 0$. Moreover, since $\mathfrak{p} \ge \mathfrak{p}_0$, on account of Assumption S.3 and (A26) we have

$$\|\Psi^{\mathsf{s}}_{-\mathfrak{p}}(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p}))\| \le \frac{\pi_1(\|\widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\|)\|\widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\|}{\pi_2(\mathfrak{p})} \le \frac{\pi_1(\sup_{s\in\mathcal{B}^s}\|s\|)\|\widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\|}{\pi_2(\mathfrak{p})}. \tag{A29}$$

On account of (A27), (A28) and (A29) and since $\Psi_\mathfrak{p}$ is UU-transforming by Assumption S.1

$$\limsup_{t\to+\infty}\|s(t)\| = \limsup_{t\to+\infty}\|\Psi^{\mathsf{s}}_{-\mathfrak{p}}(\Psi^t_\mathfrak{p}(t), \Psi^{\mathsf{s}}_\mathfrak{p}(t, s(t)))\| = \limsup_{\mathfrak{t}_\mathfrak{p}\to+\infty}\|\Psi^{\mathsf{s}}_{-\mathfrak{p}}(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p}))\| \le \pi_1(\sup_{s\in\mathcal{B}^s}\|s\|)\pi_3(\sup_{t\ge0}\|d(t)\|).$$

□

*Theorem 2.*

*Proof.* Let $\mathfrak{p} > 0$. By Assumption PT.S.2 $(t_\mathfrak{p}(t), y_\mathfrak{p}(t), F(t_\mathfrak{p}(t))s_\mathfrak{p}(t)) \in \mathrm{Dom}(\Psi^{\mathsf{u}}_{-\mathfrak{p}})$ for all $t \in [0, \mathrm{Dom}^+\Psi^t_\mathfrak{p})$ and $u(t)$ in (99) is well-defined. Moreover, by the properties of $\Psi_\mathfrak{p}$

$$u_\mathfrak{p}(t) = \Psi^{\mathsf{u}}_\mathfrak{p}(t, y(t), u(t)) = \Psi^{\mathsf{u}}_\mathfrak{p}(\Psi^t_{-\mathfrak{p}}(t_\mathfrak{p}(t)), \Psi^{\mathsf{y}}_{-\mathfrak{p}}(t_\mathfrak{p}(t), y_\mathfrak{p}(t)), \Psi^{\mathsf{u}}_{-\mathfrak{p}}(t_\mathfrak{p}(t), y_\mathfrak{p}(t), F(t_\mathfrak{p}(t))s_\mathfrak{p}(t)) = F(t_\mathfrak{p}(t))s_\mathfrak{p}(t). \tag{A30}$$

But $\Psi^t_\mathfrak{p}(t)$ is a monotonically increasing function of $t \in [0, \mathrm{Dom}^+\Psi^t_\mathfrak{p})$ with $\Psi^t_\mathfrak{p}(0) = 0$ and $\mathrm{Dom}^+\Psi^t_\mathfrak{p} < +\infty$ ($\Psi_\mathfrak{p}$ is BU-transforming by Assumption PT.S.1). Consider the variables $(t, z(t))$ in the new unbounded positive time scale $\mathfrak{t}_\mathfrak{p} = \Psi^t_\mathfrak{p}(t)$, i.e. $(\mathfrak{t}_\mathfrak{p}, \widetilde{z}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) = (\Psi^t_\mathfrak{p}(t), z_\mathfrak{p}\circ[\Psi^t_\mathfrak{p}]^{-1}(\mathfrak{t}_\mathfrak{p}))$, with $\mathfrak{t}_\mathfrak{p} \ge 0$. By Assumption PT.S.1 $\Psi_\mathfrak{p}$ is a LS of $\Sigma(t) = 0$, hence $\Psi_\mathfrak{p}$ maps $\Sigma(t) = 0$ into $\widetilde{\Sigma}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p}) = 0$ which reads as on account of (A30)

$$D_{\mathfrak{t}_\mathfrak{p}}\widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p}) = F(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p}), F(\mathfrak{t}_\mathfrak{p})\widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p}), \widetilde{d}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) \tag{A31}$$

By Assumption PT.S.2 with $V(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}) = \widetilde{s}_\mathfrak{p}^\top P(\mathfrak{t}_\mathfrak{p})\widetilde{s}_\mathfrak{p}$ and for all $\mathfrak{t}_\mathfrak{p} \ge 0$

$$\underline{\beta}\|\widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\|^2 \le V(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) \le \overline{\beta}\|\widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\|^2, \tag{A32}$$

$$D_{\mathfrak{t}_\mathfrak{p}}V(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) \le -\alpha V(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) + \gamma\|\widetilde{d}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\|^2 \tag{A33}$$

for some $\alpha, \underline{\beta}, \overline{\beta}, \gamma > 0$. On the other hand, with $W(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}) = \frac{V(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p})}{\pi_2^2(\mathfrak{t}_\mathfrak{p}, \mathfrak{p})}$

$$D_{\mathfrak{t}_\mathfrak{p}}W(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) \le -\left(\alpha - 2\left|\frac{\partial \ln \pi_2(r, \mathfrak{p})}{\partial r}\right|_{r=\mathfrak{t}_\mathfrak{p}}\right)W(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) + \gamma\frac{\|\widetilde{d}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\|^2}{\pi_2^2(\mathfrak{t}_\mathfrak{p}, \mathfrak{p})} \tag{A34}$$

Using (97) and (98) of Assumption PT.S.3, for $\mathfrak{t}_\mathfrak{p} \ge \mathfrak{t}^\tau_\mathfrak{p} \triangleq \Psi^t_\mathfrak{p}(\tau)$

$$D_{\mathfrak{t}_\mathfrak{p}}W(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) \le -(\alpha - 2\psi(\mathfrak{t}_\mathfrak{p}))W(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) + \gamma\sup_{t\ge0}\|d(t)\|^2 \tag{A35}$$

where $\psi \in \mathcal{L}$. By integrating (A36) and using Gronwall's inequality, for $\mathfrak{t}_\mathfrak{p} \ge \mathfrak{t}^\alpha_\mathfrak{p} \triangleq \max\{\psi^{-1}(\min\{\frac{\alpha}{2}, \psi(0)\}), \mathfrak{t}^\tau_\mathfrak{p}\}$ we find

$$W(\mathfrak{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})) \le e^{-\frac{\alpha}{2}(\mathfrak{t}_\mathfrak{p}-\bar{\mathfrak{t}}_\mathfrak{p})}W(\bar{\mathfrak{t}}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\bar{\mathfrak{t}}_\mathfrak{p})) + \frac{2\gamma}{\alpha}\sup_{t\ge0}\|d(t)\|^2. \tag{A36}$$

Hence, for $\mathfrak{t}_\mathfrak{p} \ge \mathfrak{t}^\alpha_\mathfrak{p}$ and on account of (94) in Assumption PT.S.2

$$\|\widetilde{s}_\mathfrak{p}(\mathfrak{t}_\mathfrak{p})\| \le \left(\sqrt{\frac{\overline{\beta}}{\underline{\beta}}}e^{-\frac{\alpha}{4}(\mathfrak{t}_\mathfrak{p}-\mathfrak{t}^\alpha_\mathfrak{p})}\frac{\|\widetilde{s}_\mathfrak{p}(\mathfrak{t}^\alpha_\mathfrak{p})\|}{\pi_2(\mathfrak{t}^\alpha_\mathfrak{p}, \mathfrak{p})} + \sqrt{\frac{2\gamma}{\underline{\beta}\alpha}}\sup_{t\ge0}\|d(t)\|\right)\pi_2(\mathfrak{t}_\mathfrak{p}, \mathfrak{p}) \tag{A37}$$

By the properties of $\Psi_\mathfrak{p}$, $(\mathrm{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathrm{t}_\mathfrak{p})) \in \mathrm{Dom}(\Psi^{\mathrm{s}}_{-\mathfrak{p}})$ for all $\mathrm{t}_\mathfrak{p} \geq 0$, hence $\Psi^{\mathrm{s}}_{-\mathfrak{p}}(\mathrm{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}))$ is well-defined for $\mathrm{t}_\mathfrak{p} \geq 0$. Moreover, on account of (96) of Assumption PT.S.3 we have for $\mathrm{t}_\mathfrak{p} \geq \mathrm{t}^\alpha_\mathfrak{p}$

$$\left\|\Psi^{\mathrm{s}}_{-\mathfrak{p}}(\mathrm{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}))\right\| \leq \frac{\pi_1 \|\widetilde{s}_\mathfrak{p}(\mathrm{t}_\mathfrak{p})\|}{\pi_2(\mathrm{t}_\mathfrak{p}, \mathfrak{p})}. \tag{A38}$$

On account of (A37) and (A38) and since $\Psi_\mathfrak{p}$ is BU-transforming (Assumption PT.S.1)

$$\limsup_{t \nearrow \mathrm{Dom}^+\Psi^t_\mathfrak{p}} \|s(t)\| = \limsup_{t \nearrow \mathrm{Dom}^+\Psi^t_\mathfrak{p}} \|\Psi^{\mathrm{s}}_{-\mathfrak{p}}(\Psi^t_\mathfrak{p}(t), \Psi^{\mathrm{s}}_\mathfrak{p}(t, s(t)))\| = \limsup_{\mathrm{t}_\mathfrak{p} \to +\infty} \|\Psi^{\mathrm{s}}_{-\mathfrak{p}}(\mathrm{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}))\| \leq \pi_1 \sqrt{\frac{2\gamma}{\underline{\beta}\alpha}} \sup_{t \geq 0} \|d(t)\|).$$

If $\Psi_\mathfrak{p}$ is in addition CBU-transforming, for each $T^* > 0$ we can select $\mathfrak{p} > 0$ in such a way that $\mathrm{Dom}^+\Psi^t_\mathfrak{p} < T^*$. □

*Theorem 3.*

*Proof.* Pick $N > 0$ and consider solutions $(t, z(t))$ such that

$$\sup_{t \geq 0} \|(s(t), d(t)\| \leq N \tag{A39}$$

Define $\mathfrak{p} \triangleq \sigma(\omega)$ with $\omega > \max\{\sigma^{-1}(\mathfrak{p}_0), N\}$ and $\mathfrak{p}_0$ introduced in Assumption O.3. By Assumption O.1 $\Psi_\mathfrak{p}$ is SI-contracting with contracting maps $(\sigma, \mu)$. Since $\sigma \in \mathcal{K}_\infty$ it follows $\sup_{t \geq 0} \sigma(\|(s(t), d(t))\|) \leq \sigma(N) \leq \mathfrak{p}$. Hence, by the SI-contracting property of $\Psi_\mathfrak{p}$

$$\sup_{t \geq 0} \|(s(t), d(t)\| \leq \mu(\mathfrak{p}). \tag{A40}$$

Using this and the fact that $\mu \in \mathcal{L}$, pick $\omega_1 > \max\{\sigma^{-1}(\mathfrak{p}_0), N\}$ such that for all $\omega \geq \omega_1$

$$\sup_{t \geq 0} \|(s_\mathfrak{p}(t), d_\mathfrak{p}(t)\| \leq \mu(\mathfrak{p}) \Rightarrow s_\mathfrak{p}(t) \in \mathcal{C}^{\mathrm{s}}, d_\mathfrak{p}(t) \in \mathcal{C}^{\mathrm{d}}, \forall t \geq 0 \tag{A41}$$

and

$$\mathcal{O}(\mathfrak{p}) \triangleq \{\mathrm{e} \in \mathbb{R}^n \,:\, \|\mathrm{e}\| < 2\max\{(\underline{\theta}^{-1} \circ \overline{\theta})(\mu(\mathfrak{p})), \chi(\mu(\mathfrak{p}))\}\} \subset \mathcal{C}^{\mathrm{e}} \tag{A42}$$

with $\underline{\theta}, \overline{\theta}, \chi$ introduced in Assumptions O.2 and O.3. By (A40) and since $\zeta(0) = 0$ and $(\underline{\theta}^{-1} \circ \overline{\theta})(s) \geq s$ for all $s \geq 0$, $e(0) \in \mathcal{O}(\mathfrak{p})$ for all $t \geq 0$ with estimation error

$$e(t) \triangleq (e^{\mathrm{y}}(t), e^{\mathrm{x}}(t)) := (y_\mathfrak{p}(t) - \zeta^{\mathrm{y}}(t), x_\mathfrak{p}(t) - \zeta^{\mathrm{x}}(t)).$$

Let $t^* \leq +\infty$ be the first time for which $e(t^*) \in \partial\mathcal{O}(\mathfrak{p})$ (the boundary of $\mathcal{O}(\mathfrak{p})$). We claim that $t^* = +\infty$ whatever is $\omega \geq \omega_1$. Indeed, assume that for at least one value of $\omega \geq \omega_1$ we have $t^* < +\infty$. In what follows we consider $t \in [0, t^*)$. But $\Psi^t_\mathfrak{p}(t)$ is a monotonically increasing function of $t \in [0, t^*)$ with $\lim_{t \to +\infty} \Psi^t_\mathfrak{p}(t) = +\infty$ (by Assumption O.1 $\Psi_\mathfrak{p}$ is UU-transforming). Set $\mathrm{t}^*_\mathfrak{p} = \Psi^t_\mathfrak{p}(t^*)$ and consider the variables $(t, z(t))$ in the new unbounded positive time scale $\mathrm{t}_\mathfrak{p} = \Psi^t_\mathfrak{p}(t)$, i.e. $(\mathrm{t}_\mathfrak{p}, \widetilde{z}_\mathfrak{p}(\mathrm{t}_\mathfrak{p})) = (\Psi^t_\mathfrak{p}(t), z_\mathfrak{p} \circ [\Psi^t_\mathfrak{p}]^{-1}(\mathrm{t}_\mathfrak{p}))$, with $\mathrm{t}_\mathfrak{p} \in [0, \mathrm{t}^*_\mathfrak{p})$. By Assumption O.1 $\Psi_\mathfrak{p}$ is a LS of $\Sigma(t) = 0$, hence $\Psi_\mathfrak{p}$ maps $\Sigma(t) = 0$ into $\widetilde{\Sigma}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}) = 0$ which reads as

$$D_{\mathrm{t}_\mathfrak{p}} \widetilde{s}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}) = F(\mathrm{t}_\mathfrak{p}, \widetilde{s}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}), \widetilde{w}_\mathfrak{p}(\mathrm{t}_\mathfrak{p})) \tag{A43}$$

with

$$\widetilde{s}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}) \in \mathcal{C}^{\mathrm{s}}, \widetilde{d}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}) \in \mathcal{C}^{\mathrm{d}} \tag{A44}$$

on account of (A41) and

$$\sup_{\mathrm{t}_\mathfrak{p} \geq 0} \|(\widetilde{s}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}), \widetilde{d}_\mathfrak{p}(\mathrm{t}_\mathfrak{p})\| \leq \mu(\mathfrak{p}) \tag{A45}$$

on account of (A40). Furthermore, by considering $\zeta(t)$ and $e(t)$ in the new time scale $\mathrm{t}_\mathfrak{p}$, i.e. $\widetilde{\zeta}(\mathrm{t}_\mathfrak{p}) = \zeta \circ [\Psi^t_\mathfrak{p}]^{-1}(\mathrm{t}_\mathfrak{p}))$ and $\widetilde{e}(\mathrm{t}_\mathfrak{p}) = e \circ [\Psi^t_\mathfrak{p}]^{-1}(\mathrm{t}_\mathfrak{p}) = \widetilde{s}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}) - \widetilde{\zeta}(\mathrm{t}_\mathfrak{p})$ with $\widetilde{e}(\mathrm{t}_\mathfrak{p}) = (\widetilde{e}^{\mathrm{y}}(\mathrm{t}_\mathfrak{p}), \widetilde{e}^{\mathrm{x}}(\mathrm{t}_\mathfrak{p}))$, the observer's equations result to be

$$D_{\mathrm{t}_\mathfrak{p}} \widetilde{\zeta}(\mathrm{t}_\mathfrak{p}) = F(\mathrm{t}_\mathfrak{p}, \widetilde{\zeta}(\mathrm{t}_\mathfrak{p}), \widetilde{u}_\mathfrak{p}(\mathrm{t}_\mathfrak{p}), 0) + \psi(\mathrm{t}_\mathfrak{p}, \mathrm{y}_\mathfrak{p}, \widetilde{e}^{\mathrm{y}}(\mathrm{t}_\mathfrak{p})) \tag{A46}$$

with

$$\|\widetilde{e}_\mathfrak{p}(0)\| \leq \mu(\mathfrak{p}) \tag{A47}$$

on account of (A40) and since $\zeta(0) = 0$. By Assumption O.2 and (A44)

$$\underline{\theta}(\|\widetilde{e}(\mathrm{t_p})\|) \le W(\mathrm{t_p}, \widetilde{e}(\mathrm{t_p})) \le \overline{\theta}(\|\widetilde{e}(\mathrm{t_p})\|), \tag{A48}$$

$$D_{\mathrm{t_p}} W(\mathrm{t_p}, \widetilde{e}_{\mathrm{p}}(\mathrm{t_p})) \le -\delta(W(\mathrm{t_p}, \widetilde{e}_{\mathrm{p}}(\mathrm{t_p}))) + \eta(\|\widetilde{d}_{\mathrm{p}}(\mathrm{t_p})\|) \tag{A49}$$

for some $\delta, \underline{\theta}, \overline{\theta}, \eta \in \mathcal{K}_\infty$ and a $C^1$ function $W : \mathbb{R}^0_+ \times \mathcal{C}^{\mathrm{s}} \times \mathcal{C}^{\mathrm{e}} \to \mathbb{R}^0_+$ and $\psi : \mathbb{R}^0_+ \times \mathcal{C}^{\mathrm{y}} \times \mathcal{C}^{\mathrm{e^y}} \to \mathbb{R}^m$. From lemma 1 it follows that

$$\|\widetilde{e}(\mathrm{t_p})\| \le \underline{\theta}^{-1}\Big( \max\{\overline{\theta}(\|\widetilde{e}_{\mathrm{p}}(0)\|) - \eta(N)\mathrm{t_p}, \overline{\theta}\circ\chi(\sup_{t\in[0,\mathrm{t_p^*})} \|\widetilde{d}_{\mathrm{p}}(\mathrm{t_p})\|)\} \Big) \tag{A50}$$

with $\chi \triangleq \underline{\theta}^{-1}\circ\overline{\theta}\circ\underline{\theta}^{-1}\circ\delta^{-1}\circ(2\eta) \in \mathcal{K}_\infty$. Using this and (A45)

$$\|\widetilde{e}(\mathrm{t_p})\| \le \max\{(\underline{\theta}^{-1}\circ\overline{\theta})(\mu(\mathfrak{p})), \chi(\mu(\mathfrak{p}))\} \tag{A51}$$

from which it follows that $\widetilde{e}(\mathrm{t_p^*}) \in \mathcal{O}(\mathfrak{p})$, i.e. a contradiction on account of the definition of $\mathrm{t_p^*}$. Hence, if we choose any $\omega \ge \omega_1$ then, using Lemma 1 once more,

$$\widetilde{e}(\mathrm{t_p}) \in \mathcal{O}(\mathfrak{p}) \subset \mathcal{C}^{\mathrm{e}}, \widetilde{d}_{\mathrm{p}}(\mathrm{t_p}) \in \mathcal{C}^{\mathrm{d}}, \forall \mathrm{t_p} \ge 0, \tag{A52}$$

$$\limsup_{\mathrm{t_p}\to+\infty} \|\widetilde{e}(\mathrm{t_p})\| \le \chi(\sup_{\mathrm{t_p}\ge 0} \|\widetilde{d}_{\mathrm{p}}(\mathrm{t_p})\|). \tag{A53}$$

In particular, from (A45) and (A52)

$$\|\widetilde{\zeta}(\mathrm{t_p})\| \le 2\max\{(\underline{\theta}^{-1}\circ\overline{\theta})(\mu(\mathfrak{p})) + \mu(\mathfrak{p}), \forall \mathrm{t_p} \ge 0. \tag{A54}$$

Furthermore, by Assumption O.4

$$\chi(\sup_{\mathrm{t_p}\ge 0} \|\widetilde{d}_{\mathrm{p}}(\mathrm{t_p})\|) \le \pi_2(\mathfrak{p})\pi_3(\sup_{t\ge 0}\|d(t)\|) \tag{A55}$$

and by Assumption O.3 and (A54) we have $(\mathrm{t_p}, \widetilde{y}_{\mathrm{p}}(\mathrm{t_p}), \widetilde{\zeta}^{\mathrm{x}}(\mathrm{t_p})) \in \mathrm{Dom}(\Psi^{\mathrm{x}}_{-\mathrm{p}})$ with $\widetilde{\zeta}(\mathrm{t_p}) \triangleq (\widetilde{\zeta}^{\mathrm{y}}(\mathrm{t_p}), \widetilde{\zeta}^{\mathrm{x}}(\mathrm{t_p}))$, hence $\Psi^{\mathrm{x}}_{-\mathrm{p}}(\mathrm{t_p}, \widetilde{y}_{\mathrm{p}}(\mathrm{t_p}), \widetilde{\zeta}^{\mathrm{x}}(\mathrm{t_p})))$ is well-defined. On account of (A53) and (A55), Assumption O.3 and since $\Psi_{\mathrm{p}}$ is UU-transforming

$$\limsup_{t\to+\infty} \|s(t) - \widehat{s}(t)\| = \limsup_{\mathrm{t_p}\to+\infty} \|\Psi^{\mathrm{x}}_{-\mathrm{p}}(\mathrm{t_p}, \widetilde{y}_{\mathrm{p}}(\mathrm{t_p}), \widetilde{x}_{\mathrm{p}}(\mathrm{t_p})) - \Psi^{\mathrm{x}}_{-\mathrm{p}}(\mathrm{t_p}, \widetilde{y}_{\mathrm{p}}(\mathrm{t_p}), \widetilde{\zeta}^{\mathrm{x}}(\mathrm{t_p}))\|$$

$$\le \left( \limsup_{\mathrm{t_p}\to+\infty} \int_0^1 \| \frac{\partial \Psi^{\mathrm{x}}_{-\mathrm{p}}(\mathrm{t_p}, \widetilde{\mathrm{y}}_{\mathrm{p}}(\mathrm{t_p}), \mathrm{x})}{\partial \mathrm{x}} \|_{\mathrm{x}=r\widetilde{e}^{\mathrm{x}}_{\mathrm{p}}(\mathrm{t_p})+\widetilde{\zeta}^{\mathrm{x}}(\mathrm{t_p})} dr \right) \limsup_{\mathrm{t_p}\to+\infty} \|\widetilde{e}(\mathrm{t_p})\| \le \pi_1(\sup_{\mathrm{s}\in\mathcal{C}^{\mathrm{s}}} \|\mathrm{s}\| + \sup_{\mathrm{e}\in\mathcal{C}^{\mathrm{e}}} \|\mathrm{e}\|)\pi_3(\sup_{t\ge 0}\|d(t)\|). \lhd$$

$\square$

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