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Aggregate Bound Choices about Random and Nonrandom Goods Studied via a Nonlinear Analysis

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Abstract: In this paper, bound choices are made after summarizing a finite number of alternatives. This means that each choice is always the barycenter of masses distributed over a finite set of alternatives. More than two marginal goods at a time are not handled. This is because a quadratic metric is used. In our models, two marginal goods give rise to a joint good, so aggregate bound choices are shown. The variability of choice for two marginal goods that are the components of a multiple good is studied. The weak axiom of revealed preference is checked and mean quadratic differences connected with multiple goods are proposed. In this paper, many differences from vast majority of current research about choices and preferences appear. First of all, conditions of certainty are viewed to be as an extreme simplification. In fact, in almost all circumstances, and at all times, we all find ourselves in a state of uncertainty. Secondly, the two notions, probability and utility, on which the correct criterion of decision-making depends, are treated inside linear spaces over $\mathbb{R}$ having a different dimension in accordance with the pure subjectivistic point of view.

Keywords: consumption matrix; discrete alternatives; aggregate measure; Fréchet class; revealed preference; mean quadratic difference

MSC: 60A05; 60B05; 91B24; 91B16; 91B06; 91B08

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1. Introduction

1.1. Mathematical Preliminaries

Let ${}_1X$ and ${}_2X$ be two marginal random goods. If they are taken into account, then choices under uncertainty and riskiness are based on the budget constraint of the decision maker given by

$$c_1\,\mathbf{P}({}_1X) + c_2\,\mathbf{P}({}_2X) \leq c,$$

where $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ are two barycenters of masses. They are two weighted averages obtained by summarizing two nonparametric marginal distributions of mass. Two marginal distributions of mass always derive from a nonparametric joint distribution of mass. This is because two marginal goods always give rise to a joint good in this paper. We distinguish two stages at all times. At a first stage, the number of admissible weighted averages is infinite. In fact, they are logically undetermined in the only sense that they must not be less than a lower bound, nor greater than an upper bound. The number of points of the budget set of the decision maker is infinite for this reason. Each point of it identifies a weighted average connected with a joint distribution of mass. Since we separate it into its two component parts, this means that each point of the budget set of the decision maker identifies two weighted averages connected with two marginal distributions of mass. At a second stage, only one weighted average is taken into account, so $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ are logically determined. The budget line is expressed by

$$c_1\,\mathbf{P}({}_1X) + c_2\,\mathbf{P}({}_2X) = c.$$

Its nature is endogenous. Given n observed alternatives related to each marginal random good, we pass to consider $n+1$ consumption alternatives referred to each marginal random good. This is because horizontal and vertical intercepts are endogenously established. They are $\frac{c}{c_1}$ and $\frac{c}{c_2}$. The lower bound for $\mathbf{P}({}_1X)$ is zero. Its upper bound is $\frac{c}{c_1}$. Zero and $\frac{c}{c_1}$ are the two distinct end points of a closed line segment. The latter is a part of a horizontal straight line. Similarly, the lower bound for $\mathbf{P}({}_2X)$ is zero. Its upper bound is $\frac{c}{c_2}$. Zero and $\frac{c}{c_2}$ are the two distinct end points of a closed line segment. The latter is a part of a vertical straight line. The number of the possible values for a joint random good is given by $(n+1) \times (n+1) = (n+1)^2$. Since $(n+1)^2$ consumption alternatives are summarized by using $(n+1)^2$ joint masses that are subjectively chosen, the budget set of the decision maker is a convex set. It is a right triangle. Its hypotenuse is the budget line with a negative slope. Each joint mass associated with one of $(n+1)^2$ alternatives can take all values between 0 and 1, end points included, into account. The same is true with regard to $n+1$ marginal masses referred to each marginal random good. At a second stage, whenever the decision maker chooses the prevision bundle given by $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ because it represents the best rational choice for him or her, he or she chooses $(n+1)^2$ joint masses together with $n+1$ marginal masses referred to consumption alternatives connected with each marginal random good. The decision maker needs to make explicit all joint and marginal masses that are subjectively chosen by him or her at a second stage. More data than the observed ones are handled in this way. Only n alternatives related to each marginal random good are directly observed. All other elements are estimated.

Choices under fictitious conditions of certainty are based on the budget constraint of the decision maker written in the form

$$c_1 x_1 + c_2 x_2 \leq c,$$

where the prices of nonrandom good 1 and nonrandom good 2 are given by (c_1, c_2), whereas the amount of money the decision maker has to spend is given by c. The budget line identifying the budget set of the decision maker is of an exogenous nature. It is a hyperplane embedded in $\mathbb{R} \times \mathbb{R}$. Its negative slope given by $-\frac{c_1}{c_2}$ depends on the known prices of the two goods under consideration. The elements identifying the decision maker's budget given by (c_1, c_2, c) are all objective. Nevertheless, c is assumed to be an uncertain or possible element at the time of choice. It can, therefore, be either true or false at a later time unlike the two objective prices that are certainly true. What is chosen for two nonrandom goods is given by (x_1, x_2), where (x_1, x_2) represents the best rational choice for a given decision maker. We write two weighted averages given by

$$x_1 = x_1^1 \, p_1^1 + \ldots + x_1^n \, p_1^n$$

and

$$x_2 = x_2^1 \, p_2^1 + \ldots + x_2^n \, p_2^n,$$

where $\{p_1^i\}$ and $\{p_2^j\}$ are two sets of n masses, with $0 \leq p_i^j \leq 1$, $j = 1, \ldots, n$, $i = 1, 2$, whose sum is always equal to 1 with regard to each of them. Given (x_1, x_2), the estimated quantities of consumption for good 1 belonging to a closed neighborhood of x_1 are expressed by $\{x_1^1, \ldots, x_1^n\}$, whereas the estimated quantities of consumption for good 2 belonging to a closed neighborhood of x_2 are given by $\{x_2^1, \ldots, x_2^n\}$. Note that x_1 is always found between zero and $\frac{c}{c_1}$, whereas x_2 is always found between zero and $\frac{c}{c_2}$. We also deal with the weighted average of n^2 estimated quantities of consumption for good 1 and good 2 that are jointly considered. They derive from the Cartesian product given by $\{x_1^1, \ldots, x_1^n\} \times \{x_2^1, \ldots, x_2^n\}$. The decision maker estimates $\{x_1^1, \ldots, x_1^n\}$ together with n non-negative masses, $\{x_2^1, \ldots, x_2^n\}$ together with n non-negative masses, and $\{x_1^1, \ldots, x_1^n\} \times \{x_2^1, \ldots, x_2^n\}$ together with n^2 non-negative masses such that (x_1, x_2) is actually chosen. In this paper, four Cartesian products are also studied because the Cartesian product of two finite sets of estimated consumption levels identifying different outcomes associated with two marginal nonrandom goods is released from the notion of ordered pair of estimated consumption levels identifying different random events connected with each good under consideration. Aggregate bound choices are shown in this way. An extension of the notion of bundle of goods is caught in this way. More data than the observed ones are treated. Only (x_1, x_2) is directly observed together with (c_1, c_2, c).

If (x_1, x_2) is chosen under ideal conditions of certainty, then n estimated consumption levels belonging to a closed neighborhood of x_1 use n masses such that $n-1$ masses are equal to 0, whereas only one mass of n masses is equal to 1. Similarly, n estimated consumption levels belonging to a closed neighborhood of x_2 use n masses such that $n-1$ masses are equal to 0, whereas only one mass of n masses is equal to 1. It follows that n^2 estimated consumption levels use n^2 masses such that n^2-1 masses are equal to 0, whereas only one mass of n^2 masses is equal to 1. Admissible ordered n-tuples of real numbers referred to each marginal good can be determined by the decision maker. Their number is infinite. One of them is chosen based on the lower and upper bounds that are established. One of them identifies n outcomes such that only one alternative expressed by a real number is true. All others are false. Hence, it is necessary that one of n alternatives related to nonrandom good 1 coincides with x_1. It is also necessary that one of n alternatives related to nonrandom good 2 coincides with x_2.

1.2. Motivations for This Study and Most of Its Novelty Aspects

This paper answers different questions. It is shown that the notion of ordinal utility is a measure obtained as a Euclidean distance. Since the maximizing decision maker chooses the most preferred bundle that can be afforded, it is on the budget line. Each point of the budget set of the decision maker is a measure as well. Each point of it is a barycenter of masses distributed over a finite set of alternatives (see also [1] with regard to probabilistic evaluations). Each point of the space where rational choices take place is a summary of a nonparametric joint distribution of mass (see also [2] with regard to two goods which are chosen). This summary appears as an ordered pair of real numbers. Summarized elements of the Fréchet class are accordingly involved. The best rational choice depends on the decision maker's preferences. Their nature is subjective. Further hypotheses of an empirical nature have to be made to study the best rational choice according to the decision maker's preferences. In this paper, the role played by objective alternatives is essential (see also [3] with regard to issues treated by revealed preference theory). With regard to bound choices being made under conditions of uncertainty and riskiness, objective alternatives are actually observed. With regard to bound choices being made under claimed conditions of certainty, objective alternatives are estimated. The observed consumption levels for two marginal random goods coincide with the contravariant components of two vectors belonging to E^{n+1}. They are assumed to be linearly independent vectors belonging to E^{n+1}. Each observed consumption level is a single event. Different consumption levels are the possible values for a random good. They coincide with a vector belonging to E^{n+1}. With a random event, it is always possible to continue the subdivision. Nevertheless, it is convenient to stop as soon as the subdivision is sufficient for the study under consideration. Otherwise, things become unnecessarily complicated. Any idea which does not consider the subdivision to be of a relative and temporary nature is wrong. In this paper, the subdivision of the notion of possible alternative viewed to be as a random event stops as soon as a joint good arises. For this reason, joint distributions of mass associated with joint goods are handled. In this paper, a quadratic metric is used. More than two marginal goods at a time cannot be studied, so this paper focuses on the two-good assumption for a metric reason (for instance, in statistics, variance, standard deviation, covariance, and correlation are expressed through indices obtained by using a quadratic metric). The two-good assumption is not a restriction from a mathematical point of view. It is not a restriction from a conceptual point of view either (see also [4] with regard to a quadratic function studied by using a two-dimensional diagram). In fact, it is possible to interpret one of the two marginal goods under consideration as representing everything else the decision maker might want to choose (see also [5] with regard to aspects associated with revealed preference theory). Note that the Grassmann coordinates of a two-dimensional linear subspace of E^{n+1} generated by two linearly independent vectors of E^{n+1} coincide with the components of a tensor. Hence, tensors handled in this paper for economic purposes conceptually derive from this geometric matter. They do not derive from a conventional issue. This paper shows that everything can vectorially be studied provided that one takes an appropriate number of dimensions. Not only is it possible to pass from E^{n+1} to $E^{n+1} \otimes E^{n+1}$, but it is also possible to pass from E^{n+1} to a linear space over $\mathbb{R}$ whose dimension is equal to 1. A fundamental theorem has elsewhere been shown by us with respect to this. In effect, there exists a one-to-one correspondence between a one-dimensional linear subspace of E^{n+1} and a one-dimensional straight line on which an origin, a unit of length, and an orientation are chosen. A one-dimensional linear subspace of E^{n+1} contains all collinear vectors (if $\mathbf{x}$ is a vector belonging to E^{n+1}, then all collinear vectors with regard to it are expressed by $\lambda\,\mathbf{x}$, $\forall \lambda \in \mathbb{R}$) with regard to one of the two vectors belonging to E^{n+1} whose contravariant components coincide with the observed consumption levels associated with a marginal random good. Two one-dimensional linear subspaces of E^{n+1} are dealt with. These subspaces identify two one-dimensional straight lines on which an origin, a same unit of length, and an orientation are chosen. They establish the budget set of the decision maker. They accordingly establish an uncountable subset of $\mathbb{R} \times \mathbb{R}$, where the latter is a linear space over $\mathbb{R}$ whose dimension is equal to 2. With regard to choices being made under claimed conditions of certainty, it is possible to pass from E^n to $E^n \otimes E^n$. It is also possible to pass from E^n to a linear space over $\mathbb{R}$. Its dimension is equal to 1. In this paper, what is chosen for each marginal nonrandom good under claimed conditions of certainty is a coherent summary of a nonparametric distribution of mass. Consumption alternatives and their corresponding masses are chosen. They have to be made explicit. This paper shows that to know them is fundamental to study multiple choices.

What we say in this paper is more general than one might think at first. In fact, the possible values for a random and nonrandom good are of an objective nature in the same way as a sample of n observations on a given variable. The possible values for a joint good are of an objective nature in the same way as a sample of n pairs of observations on two given variables. In particular, this means that the two-variable linear model based on the least-squares criterion can be studied inside a subset of a linear space over $\mathbb{R}$. Its dimension is equal to 2. It is accordingly possible to extend the least-squares model by studying multilinear relationships between variables inside a subset of a linear space over $\mathbb{R}$ of dimension 2. Such relationships are studied whenever two or more than two marginal variables are the components of a variable of order 2 or greater than 2. A multilinear regression model based on this approach is made by us, and the connected paper is now under review by an international journal. It is possible to show that mean quadratic differences, the correlation coefficient, Jensen's inequality, and principal component analysis can be based on intrinsic conditions of uncertainty characterized by objective and subjective elements that are studied inside subsets of linear spaces over $\mathbb{R}$ provided with a specific dimension.

2. Goods Demanded by the Decision Maker under Different Conditions

2.1. Random Goods Demanded under Conditions of Uncertainty and Riskiness

We establish the following:

Definition 1. *A random good is a random quantity (we do not use the term random variable, but we use the term random quantity because to say random variable might suggest that we are thinking of the statistical interpretation, where many trials in which the random quantity can vary are involved. The random quantity could assume different values from trial to trial according to the statistical interpretation, but this interpretation is contrary to our way of understanding the problem. If the random quantity assumes different values from trial to trial, then the term random variable can be used. Since we do not use the word event in a generic sense, we do not say trials of the* *same event to mean single and similar events. In this paper, an event is always a single event whose sense is specific. A random quantity is characterized by a nonparametric distribution of probability. This distribution can vary from individual to individual. It can also vary with the information) viewed to be as a specification of what will be chosen in each different outcome of a random process. The different outcomes of a random process are different random events. A random good is intrinsically characterized by a probability distribution consisting of a list of different outcomes and the probability associated with each outcome. The decision maker chooses a probability distribution of getting different random events.*

For example, random goods are random gains, where the term gain has to be meant in an algebraic sense. A loss is therefore a negative gain. Random goods are risky assets, whose return is not known at the moment. It is possible to observe n alternatives associated with a risky asset to make a prevision about its return. Given a family of n possible and observed alternatives, a marginal random good denoted by ${}_1X$ is written in the form expressed by

$${}_1X = {}_{(1)}x^1 \, |{}_{(1)}E_1| + {}_{(1)}x^2 \, |{}_{(1)}E_2| + \ldots + {}_{(1)}x^n \, |{}_{(1)}E_n|, \tag{1}$$

where $|{}_{(1)}E_i|$, $i = 1, \ldots, n$, coincides with 0 or 1 whenever uncertainty ceases. In this paper, n possible alternatives identify a finite partition of n mutually exclusive events or states of the world of a contingent consumption plan (see also [6] with regard to the more general notion of random quantity). Such a plan is contingent because the decision maker can choose a weighted average of n possible values for ${}_1X$ given by ${}_{(1)}x^1, {}_{(1)}x^2, \ldots, {}_{(1)}x^n$. This paper firstly focuses on all the weighted averages of n possible values for ${}_1X$ given by ${}_{(1)}x^1, {}_{(1)}x^2, \ldots, {}_{(1)}x^n$. This is because the budget set of the decision maker consists of them. Secondly, he or she chooses one of them. This happens after putting a specific hypothesis of an empirical nature about his or her subjective preferences. Note that we mathematically use the contravariant components of an n-dimensional vector with regard to an orthonormal basis of a linear space over $\mathbb{R}$ (whenever an orthonormal basis of a linear space over $\mathbb{R}$ denoted by E^n and having a Euclidean structure is considered, the contravariant and covariant components of a same vector coincide. They are the same real numbers. If $\mathcal{B}_n^{\perp} = \{\mathbf{e}_i\} = \{\mathbf{e}^i\}$ is an orthonormal basis of E^n, where E^n is the space of possible alternatives, then we write

$${}_{(1)}\mathbf{x} = {}_{(1)}x^1 \, \mathbf{e}_1 + {}_{(1)}x^2 \, \mathbf{e}_2 + \ldots + {}_{(1)}x^n \, \mathbf{e}_n$$

or

$${}_{(1)}\mathbf{x} = {}_{(1)}x_1 \, \mathbf{e}^1 + {}_{(1)}x_2 \, \mathbf{e}^2 + \ldots + {}_{(1)}x_n \, \mathbf{e}^n$$

without ambiguity. Hence, we observe ${}_{(1)}x^1 = {}_{(1)}x_1, \ldots, {}_{(1)}x^n = {}_{(1)}x_n$. A located vector at the origin of E^n is entirely established by its end point. In view of this, an ordered n-tuple of real numbers can be called either a point of an affine space denoted by $\mathcal{E}^n$ or a vector of a linear space denoted by E^n. Such spaces are therefore isomorphic. In this paper, contravariant components are used to denote possible alternatives. They are always summarized. Every vector of E^n admits one and only one set of contravariant components with regard to an arbitrary basis of E^n denoted by $\mathcal{B}_n^{\perp}$. Every vector of E^n is uniquely individuated by one and only one set of contravariant components with regard to an arbitrary basis of E^n denoted by $\mathcal{B}_n^{\perp}$). Note that it is also possible to write

$${}_1X = {}_{(1)}x^1 \, |{}_{(1)}E_1| \, \mathbf{e}_1 + {}_{(1)}x^2 \, |{}_{(1)}E_2| \, \mathbf{e}_2 + \ldots + {}_{(1)}x^n \, |{}_{(1)}E_n| \, \mathbf{e}_n. \tag{2}$$

In this paper, a possible value for ${}_1X$ is a single event. In general, a single event is intrinsically a well-determined proposition. Since we focus on random goods, every proposition is always expressible through a real number (see also [7] with regard to what is objectively possible). (Let E be a single event. The objects to which judgments of probability apply are called propositions if one is thinking in terms of the expressions in which they are formulated. Accordingly, E is a proposition. If $E = \emptyset$, then we write $\mathbf{P}(E) = 0$, where $\mathbf{P}$ stands for probability. Conversely, if $E = \Omega$, then we write $\mathbf{P}(E) = 1$. However, according to the subjectivistic point of view, it is possible to observe $E \neq \emptyset$ such that $\mathbf{P}(E) = 0$ as well as $E \neq \Omega$ such that $\mathbf{P}(E) = 1$). Uncertainty about a possible alternative stands for ignorance by the decision maker (see also [8] with regard to a complete explanation of the subjectivistic point of view). In this paper, uncertainty consists of two different aspects. Possibility and probability are the two aspects of it (see also [9] for realizing how operational rules associated with the exploration of probabilistic evaluations work). They are studied inside linear spaces over $\mathbb{R}$. Its objective aspect is given by what is possible, whereas its subjective element is given by what is probable (see also [10] with regard to the constitutive elements of the notion of probability). With regard to

$$\mathbf{P}({}_1X) = {}_{(1)}x^1\,{}_{(1)}p_1 + {}_{(1)}x^2\,{}_{(1)}p_2 + \ldots + {}_{(1)}x^n\,{}_{(1)}p_n, \tag{3}$$

possibility and probability are expressed by two vectors of E^n used to obtain $\mathbf{P}({}_1X)$, where $\mathbf{P}({}_1X)$ is mathematically obtained via a scalar or inner product. We write

$${}_{(1)}\mathbf{x} = ({}_{(1)}x^1, {}_{(1)}x^2, \ldots, {}_{(1)}x^n)$$

to denote what is objectively possible. The different outcomes of a random process are different random events. They are objectively possible. We write

$${}_{(1)}\mathbf{p} = ({}_{(1)}p_1, {}_{(1)}p_2, \ldots, {}_{(1)}p_n)$$

to denote what is subjectively probable. The probability of a possible alternative is not a first principle within this context, but it is a practical notion of a relative and subjective nature (see also [11] with regard to the study connected with its subjective elements associated with what a given individual feels). We write

$${}_{(1)}p_1 + {}_{(1)}p_2 + \ldots + {}_{(1)}p_n = 1, \tag{4}$$

with $0 \leq {}_{(1)}p_i \leq 1$, $i = 1, \ldots, n$. All those evaluations such that (4) holds are coherent. Their number is equal to ∞^{n-1}. Given ${}_{(1)}x^1, {}_{(1)}x^2, \ldots, {}_{(1)}x^n$, a contingent consumption plan consists of ∞^{n-1} admissible choices of masses at a first stage such that a weighted average of n values given by ${}_{(1)}x^1, {}_{(1)}x^2, \ldots, {}_{(1)}x^n$ takes place. Whenever the decision maker chooses $\mathbf{P}({}_1X)$ at a second stage because it represents the best rational choice for him or her, those n masses used to obtain the prevision or the mathematical expectation of ${}_1X$ have to be made explicit. In particular, if ${}_{(1)}x^1 = 0$ and ${}_{(1)}p_1 = 0$, since we observe ${}_{(1)}p_2 + \ldots + {}_{(1)}p_n = 1$, then nothing changes. We consider concrete probability distributions measuring uncertainty (see also [12] with regard to the study about uncertainty). They are discrete distributions. The same is true with regard to ${}_2X$. Let ${}_1X$ and ${}_2X$ be two random goods studied inside the budget set of the decision maker, where each of them has n (with $n > 2$ that is an integer) possible values denoted by $I({}_1X) = \{{}_{(1)}x^1, \ldots, {}_{(1)}x^n\}$ and $I({}_2X) = \{{}_{(2)}x^1, \ldots, {}_{(2)}x^n\}$. Since we have ${}_{(1)}x^1 < \ldots < {}_{(1)}x^n$ as well as ${}_{(2)}x^1 < \ldots < {}_{(2)}x^n$ without loss of generality, we write $\inf I({}_1X) = {}_{(1)}x^1$ and $\sup I({}_1X) = {}_{(1)}x^n$, as well as $\inf I({}_2X) = {}_{(2)}x^1$ and $\sup I({}_2X) = {}_{(2)}x^n$. Both ${}_1X$ and ${}_2X$ are bounded from above and below. Observe that $I({}_1X)$ contains the contravariant components of ${}_{(1)}\mathbf{x} \in E^n$ associated with ${}_1X$ before transferring them on a one-dimensional straight line, whereas $I({}_2X)$ contains the contravariant components of ${}_{(2)}\mathbf{x} \in E^n$ associated with ${}_2X$ before transferring them on another one-dimensional straight line (note that a theorem about this is proved by us in another paper currently under review by an international journal). We have

$$\mathbf{P}({}_1X) = {}_{(1)}x^1\,{}_{(1)}p_1 + \ldots + {}_{(1)}x^n\,{}_{(1)}p_n$$

and

$$\mathbf{P}({}_2X) = {}_{(2)}x^1\ {}_{(2)}p_1 + \ldots + {}_{(2)}x^n\ {}_{(2)}p_n,$$

where we write

$${}_{(1)}p_1 + \ldots + {}_{(1)}p_n = 1,$$

with $0 \leq {}_{(1)}p_i \leq 1$, $i = 1, \ldots, n$, as well as

$${}_{(2)}p_1 + \ldots + {}_{(2)}p_n = 1,$$

with $0 \leq {}_{(2)}p_j \leq 1$, $j = 1, \ldots, n$.

2.2. Random Goods Being Chosen under Uncertainty and Riskiness: the Decision Maker's Demand Functions

The possible values for ${}_1X$ and ${}_2X$ are random events. It is possible to show that they are always found on two half-lines, where each half-line extends indefinitely on the right of zero before being restricted (see further in this subsection). This is a very important issue. This is because the budget set of the decision maker is a right triangle belonging to the first quadrant of a two-dimensional Cartesian coordinate system with regard to random goods as well. The vertex of the right angle of this triangle coincides with the point given by $(0,0)$. Its hypotenuse is the budget line, whose slope is negative. Since it is possible to pursue the subdivision of possible alternatives viewed to be as random events, two marginal random goods denoted by ${}_1X$ and ${}_2X$ always give rise to a joint random good denoted by ${}_1X\,{}_2X$. We accordingly pass from n alternatives associated with each marginal random good to $n \times n = n^2$ outcomes associated with ${}_1X\,{}_2X$. A coherent prevision of ${}_1X\,{}_2X$ is denoted by $\mathbf{P}({}_1X\,{}_2X)$ (see also [13] with regard to a coherent assessment of masses). Let $\mathcal{P}$ be the set of all coherent previsions denoted by $\mathbf{P}$ connected with ${}_1X\,{}_2X$. The set denoted by $\mathcal{P}$ is a two-dimensional convex set. It is the budget set of the decision maker. Possible pairs of real numbers denoted by $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ are the Cartesian coordinates of possible points belonging to $\mathcal{P}$. Note that $\mathbf{P}({}_1X\,{}_2X)$ is a bilinear measure coinciding with a two-dimensional point (see also [14] with regard to probabilistic evaluations). We always project the point denoted by $\mathbf{P}({}_1X\,{}_2X) = (\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ onto the two mutually orthogonal axes of a two-dimensional Cartesian coordinate system whose intersection is given by the point $(0,0)$. All coherent previsions of each marginal random good identify two one-dimensional convex sets. Each mass associated with one of n alternatives can take all values between 0 and 1, end points included, into account. The same is true with respect to each mass associated with one of n^2 alternatives.

In general, given ${}_1X'$ and ${}_1X''$, whose possible values are on the same horizontal axis of a two-dimensional Cartesian coordinate system, if $\mathbf{P}$ is additive, then we write

$$\mathbf{P}({}_1X' + {}_1X'') = \mathbf{P}({}_1X') + \mathbf{P}({}_1X''). \tag{5}$$

Accordingly, ${}_1X'$ and ${}_1X''$ have the same number of possible values. Such a number is equal to n within this context. Hence, the sum of two n-dimensional vectors is taken into account. Moreover, n masses associated with n possible alternatives for ${}_1X'$ have to be the same as the ones associated with n possible alternatives for ${}_1X''$. Otherwise, (5) does not work. It follows that a more extensive one-dimensional convex set is individuated on one of the two half-lines under consideration. In fact, given $I({}_1X) = \{{}_{(1)}x^1, \ldots, {}_{(1)}x^n\}$, where we have ${}_{(1)}x^1 < \ldots < {}_{(1)}x^n$, if $\mathbf{P}$ is convex, then we write

$${}_{(1)}x^1 \leq \mathbf{P}({}_1X) \leq {}_{(1)}x^n. \tag{6}$$

For every real number denoted by a, it follows that if we write

$$\mathbf{P}((a)\,{}_1X) = a\,\mathbf{P}({}_1X), \tag{7}$$

then $\mathbf{P}$ is linear ($\mathbf{P}$ is also bilinear. Whenever $\mathbf{P}$ is bilinear, we decompose it into two linear measures coinciding with two one-dimensional points. Each of them is still denoted by $\mathbf{P}$. Moreover, ${}_1X$ is always non-negative. In fact, if a is a real number, then it is always between a' and a''. We consequently observe that $a\,{}_1X$ is found between $a'\,{}_1X$ and $a''\,{}_1X$. Since we can write ${}_1X = {}_1X' - {}_1X''$ every time, where ${}_1X' = {}_1X$ (with ${}_1X \geq 0$) and ${}_1X'' = -{}_1X$ (with ${}_1X \leq 0$), ${}_1X'$ and ${}_1X''$ are non-negative quantities at all times. It follows that ${}_1X$ is non-negative. The same is true with regard to ${}_2X$. This means that only the first quadrant of a two-dimensional Cartesian coordinate system is necessary to study ${}_1X$ and ${}_2X$. On the other hand, the same is true with regard to nonrandom goods handled inside the budget set of the decision maker). Additivity and convexity of $\mathbf{P}$ represent all that is necessary for the foundation of the whole theory of probability.

From

$$c_1\,({}_1X) + c_2\,({}_2X) \leq c, \tag{8}$$

it follows that the budget constraint of the decision maker is given by

$$c_1\,\mathbf{P}({}_1X) + c_2\,\mathbf{P}({}_2X) \leq c. \tag{9}$$

The budget line is given by

$$c_1\,\mathbf{P}({}_1X) + c_2\,\mathbf{P}({}_2X) = c. \tag{10}$$

Its negative slope is given by $-\frac{c_1}{c_2}$. Note that c_1 and c_2 are the two objective prices of the two marginal random goods under consideration, whereas c is the amount of money the decision maker has to spend. Formally, c_1 and c_2 are the two real coefficients identifying the negative slope of a hyperplane embedded in $\mathbb{R} \times \mathbb{R}$. By definition, this hyperplane never separates $\mathbf{P}$ from the set of possible values for ${}_1X$, ${}_2X$, and ${}_1X\,{}_2X$. Since $\mathbf{P}({}_1X)$, $\mathbf{P}({}_2X)$, and $\mathbf{P}({}_1X\,{}_2X)$ belong to uncountable sets, the quantities demanded by the decision maker and denoted by $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ depend on the three objective elements identifying a two-dimensional convex set, where $\mathbf{P}({}_1X\,{}_2X)$ is decomposed into $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ respectively. It follows that an axiomatic approach to the theory of decision-making is not alone sufficient to explain bound choices being made by the decision maker. We have to consider subjective elements as well. In fact, $\mathbf{P}$ is intrinsically of a subjective nature (see also [15] with regard to studies about subjective probability). The budget line can always be drawn. Its nature is endogenous. This is because it is possible to establish its horizontal and vertical intercepts every time. This means that we pass from n to $n+1$ possible alternatives for each marginal random good. Structures open to the adjunction of new entities as new circumstances arise are considered in this way. In this paper, they are linear spaces over $\mathbb{R}$ having a different dimension. Structures open are considered because the notion of a possible alternative viewed to be as a random event is intrinsically subdivisible. The prices of the two random goods under consideration are endogenously determined whenever the budget line is drawn. Three convex sets are established. They are two one-dimensional convex sets and one two-dimensional convex set. The first one-dimensional convex set is found between $(0,0)$ and the horizontal intercept of the budget line given by $\frac{c}{c_1}$. Weighted averages of $n+1$ values are firstly handled. Their number is infinite. The second one is found between $(0,0)$ and the vertical intercept of it given by $\frac{c}{c_2}$. Weighted averages of $n+1$ values are firstly treated. Their number is infinite. The third two-dimensional convex set is given by all the points that are found inside the plane region bounded by the right triangle into account. Weighted averages of $(n+1) \times (n+1)$ values are firstly studied. Their number is infinite. Boundary points that are found on each restricted half-line identify degenerate averages. Note that (10) always passes through the point whose coordinates are given by

$$\big(\sup I({}_1X),\, \sup I({}_2X)\big).$$

Note that it is now possible to pass from E^{n+1} to a linear space over $\mathbb{R}$ whose dimension is equal to 1. There exists a one-to-one correspondence between a one-dimensional linear subspace of E^{n+1} and a one-dimensional straight line, on which an origin, a unit of length, and an orientation are chosen. We study two marginal random goods, so two one-dimensional linear subspaces of E^{n+1} are dealt with. These subspaces identify two one-dimensional straight lines, on which an origin, a same unit of length, and an orientation are chosen. They establish the budget set of the decision maker.

We establish the following:

Definition 2. *After decomposing $\mathbf{P}({}_1X\,{}_2X)$ inside an uncountable subset of $\mathbb{R}\times\mathbb{R}$, the decision maker's demand functions giving what is chosen with regard to the two random goods under consideration are expressed by*

$$\mathbf{P}({}_1X) = \{\mathbf{P}({}_1X)[(c_1, c_2, c)]\} \tag{11}$$

and

$$\mathbf{P}({}_2X) = \{\mathbf{P}({}_2X)[(c_1, c_2, c)]\}, \tag{12}$$

where $\mathbf{P}$ is additive and convex as a consequence of its coherence.

A remarkable point of this paper is the following. The decision maker makes explicit both marginal masses associated with possible alternatives related to ${}_1X$ and ${}_2X$ and the joint ones associated with possible alternatives related to ${}_1X\,{}_2X$ whenever the best rational choice is made by him or her. He or she is subjected to $2(n+1)-1$ constraints in order to estimate all joint masses of ${}_1X\,{}_2X$. Such constraints coincide with $2(n+1)-1$ marginal masses. Marginal masses associated with possible alternatives related to ${}_1X$ and ${}_2X$ give rise to $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$. Given the best rational choice expressed by $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$, a bilinear and disaggregate measure coinciding with $\mathbf{P}({}_1X\,{}_2X)$ is a summarized element of the Fréchet class. It is known that the set of all joint distributions of mass, with the same given marginal masses, constitutes the Fréchet class. Given the prevision bundle expressed by $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$, he or she also makes explicit a summarized element of the Fréchet class such that $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ never change. He or she can choose a coherent summary of a joint distribution of mass identifying a summarized element of the Fréchet class such that there is no linear correlation between random good 1 and random good 2, so they are stochastically independent. In other words, given the same marginal masses, ${}_1X$ and ${}_2X$ can be stochastically independent if each joint mass in a joint distribution is the product of its corresponding marginal masses. In particular, if ${}_1X$ and ${}_2X$ are two risky assets, then the decision maker is risk neutral. He or she could also choose a coherent summary of a joint distribution of mass such that there is an inverse or direct linear relationship between ${}_1X$ and ${}_2X$. This means that the decision maker is, respectively, risk averse or risk loving. In fact, given the same marginal masses, an aggregation of joint masses such that ${}_1X$ tends to increase when ${}_2X$ increases shows a direct linear relationship between ${}_1X$ and ${}_2X$. Conversely, given the same marginal masses, an aggregation of joint masses such that ${}_1X$ tends to decrease when ${}_2X$ increases shows an inverse linear relationship between ${}_1X$ and ${}_2X$. It follows that our model supports the notion of risk to be intrinsically of a subjective nature.

2.3. Random Goods Being Chosen under Uncertainty and Riskiness: Prevision and Utility Are the Two Sides of the Same Coin

In this subsection, we show that prevision and utility are formally the two sides of the same coin. Accordingly, we prove the following:

Theorem 1. *Let ${}_1X$ and ${}_2X$ be two logically independent random goods. They are jointly considered inside the budget set of the decision maker. Their possible values are expressed by $I({}_1X)\cup\{\frac{c}{c_1}\}$ and $I({}_2X)\cup\{\frac{c}{c_2}\}$. If each prevision of ${}_1X\,{}_2X$ denoted by $\mathbf{P}({}_1X\,{}_2X)$ is decomposed into two linear previsions, then its properties coincide with the ones of well-behaved preferences.*

Proof. If ${}_1X$ and ${}_2X$, where each of them has $n+1$ possible values, are two logically independent random goods, then the number of the possible values for ${}_1X\,{}_2X$, where ${}_1X\,{}_2X$ is a joint random good, is equal to $(n+1) \times (n+1) = (n+1)^2$. The decision maker ranks all the prevision possibilities. He or she ranks bundles of two goods. He or she uses the notion of Euclidean distance between two points measured along the 45-degree line. One of them is given by $(0,0)$. Farther points from $(0,0)$ are preferred because their ordinal utility is greater. Usual assumptions of completeness, reflexivity, and transitivity about preferences are valid. The additivity and convexity of $\mathbf{P}$ with respect to two logically independent random goods correspond to the monotonicity and convexity of well-behaved preferences considered inside the budget set of the decision maker. Well-behaved preferences are monotonic because more of both goods is better. They are also convex because averages are weakly preferred to extremes. We are talking about goods, not bads. We imagine indifference curves which are parallel lines restricted to the first quadrant of a two-dimensional Cartesian coordinate system. All indifference curves we graphically imagine are contained in the budget set of the decision maker. They have the same slope as (10). We think of indifference curves representing perfect substitutes, so the weighted average of two indifferent and extreme prevision bundles is not preferred to the two extreme prevision bundles, but it is as good as the two extreme prevision bundles. Every prevision bundle obtains a utility level and those prevision bundles on higher indifference curves obtain larger utility levels. The direction of increasing preference is up and to the right. It is towards the direction of increased random good 1 average consumption and increased random good 2 average consumption. With regard to (10), we write

$$\frac{\Delta\mathbf{P}({}_2X)}{\Delta\mathbf{P}({}_1X)} = -\frac{c_1}{c_2} \tag{13}$$

because if the decision maker increases $\mathbf{P}({}_1X)$, then he or she must decrease $\mathbf{P}({}_2X)$ and vice versa in order to move along it. If he or she chooses $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ inside the budget set, then we write

$$MU_1 = \frac{\Delta U}{\Delta\mathbf{P}({}_1X)} = \frac{u(\mathbf{P}({}_1X) + \Delta\mathbf{P}({}_1X), \mathbf{P}({}_2X)) - u(\mathbf{P}({}_1X), \mathbf{P}({}_2X))}{\Delta\mathbf{P}({}_1X)}, \tag{14}$$

where MU_1 measures the rate of change in utility, denoted by ΔU, associated with a small change in the amount of random good 1 expressed by $\Delta\mathbf{P}({}_1X)$. MU_1 is the marginal utility with respect to random good 1. The amount of random good 2 is held fixed. We can multiply the change in average consumption of random good 1 by the marginal utility with respect to random good 1. This allows to calculate the change in utility associated with a small change in average consumption of random good 1. We therefore write

$$\Delta U = MU_1\,\Delta\mathbf{P}({}_1X). \tag{15}$$

On the other hand, the marginal utility with respect to random good 2 is

$$MU_2 = \frac{\Delta U}{\Delta\mathbf{P}({}_2X)} = \frac{u(\mathbf{P}({}_1X), \mathbf{P}({}_2X) + \Delta\mathbf{P}({}_2X)) - u(\mathbf{P}({}_1X), \mathbf{P}({}_2X))}{\Delta\mathbf{P}({}_2X)}. \tag{16}$$

If we calculate the marginal utility with respect to random good 2, then we keep the amount of random good 1 constant. We can evidently write

$$\Delta U = MU_2\,\Delta\mathbf{P}({}_2X). \tag{17}$$

Marginal utility is used to calculate the marginal rate of substitution (abbreviated to MRS). It is the rate at which a given individual is willing to substitute a small amount of random good 2 for random good 1. We focus on that indifference curve whose utility level is larger. It coincides with the budget line. We consider a change in the average consumption of each random good such that it keeps utility constant. It is denoted by $(\Delta\mathbf{P}({}_1X), \Delta\mathbf{P}({}_2X))$. This change moves the decision maker along that indifference curve whose utility level is larger. We write

$$MU_1\,\Delta\mathbf{P}({}_1X) + MU_2\,\Delta\mathbf{P}({}_2X) = \Delta U = 0. \tag{18}$$

If we solve for the slope of the indifference curve, then we obtain

$$\text{MRS} = \frac{\Delta\mathbf{P}({}_2X)}{\Delta\mathbf{P}({}_1X)} = -\frac{MU_1}{MU_2}. \tag{19}$$

Since the MRS measures the slope of the indifference curve under consideration, if we consider its algebraic sign, then we write

$$\text{MRS} = -\frac{\Delta\mathbf{P}({}_2X)}{\Delta\mathbf{P}({}_1X)}$$

because it is negative. Nevertheless, (19) tells us that we consider the absolute value of the MRS by means of the ratio of marginal utilities. The ratio of marginal utilities is independent of the particular way being chosen by the decision maker to represent his or her preferences. Let $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ be a point belonging to the indifference curve, whose utility level is larger. After projecting $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ onto the two mutually orthogonal axes of a two-dimensional Cartesian coordinate system, the additivity and convexity of $\mathbf{P}$ with respect to marginal previsions of ${}_1X$ and ${}_2X$ correspond to the monotonicity and convexity of well-behaved preferences referred to each axis of a two-dimensional Cartesian coordinate system. Hence, whenever we say that more is better, we mean that a line segment is increasingly large on the horizontal axis, and a line segment is increasingly large on the vertical one. Additionally, any line segment on the horizontal axis is a one-dimensional convex set in the same way as any line segment on the vertical one. Preferences for perfect substitutes are expressed by a utility function whose form is additive, so

$$u(\mathbf{P}({}_1X), \mathbf{P}({}_2X)) = \mathbf{P}({}_1X) + \mathbf{P}({}_2X) \tag{20}$$

is constant along all indifference curves we graphically imagine. In particular, it is constant along the budget line. □

Since indifference curves cannot cross, given any two prevision bundles belonging to two different indifference curves, Theorem 1 tells us that the decision maker can rank them as to their distance from $(0,0)$ measured along the 45-degree line. One of the prevision bundles is strictly better than the other if and only if its distance from $(0,0)$ measured along the 45-degree line is greater than the other. A numerical example about this can easily be shown by using the Pythagorean theorem. In fact, it is possible to write

$${}^2d(O,\mathbf{P}) = \sqrt{\sum_{i=1}^{2}\mathbf{P}({}_iX)^2} \tag{21}$$

to denote the distance of $\mathbf{P}$ from O, where $\mathbf{P}$ stands for $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$, namely,

$$\mathbf{P} = \begin{pmatrix}\mathbf{P}({}_1X)\\ \mathbf{P}({}_2X)\end{pmatrix}.$$

The bundles for which the decision maker is indifferent to $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ form the indifference curve, whose slope is negative. It is imagined by identifying preferences for perfect substitutes without loss of generality. It intersects the 45-degree line in a point only. All other indifference curves intersect the 45-degree line. Each of them intersects the 45-degree line in a point only.

We note the following:

Remark 1. *The budget line contains those prevision bundles, whose utility level is larger. An optimal choice for the decision maker depends on his or her subjective preferences. For instance, such a choice can be where the indifference curve is tangent to the budget line. This indifference curve is therefore rounded. Its slope is negative. Whenever we are not interested in proving that prevision (probability) and utility are formally the two sides of the same coin, we do not think of indifference curves representing perfect substitutes. We can think of rounded indifference curves. Nevertheless, nothing changes. This is because further hypotheses of an empirical nature have always to be made at a second stage to study the best rational choice for a given decision maker.*

2.4. Nonrandom Goods Demanded under Claimed Conditions of Certainty

A nonrandom good is not characterized by a random process. There is no specification of what will be chosen in each different state of the world. This is because there are not states of the world. For example, normal and ordinary goods are nonrandom goods. The demand for a normal good increases when income increases. The demand for an ordinary good increases when its price decreases. Nonetheless, this paper shows that nonrandom goods are chosen under claimed conditions of certainty. They are fictitious conditions of certainty. Bound choices being made by a given decision maker under claimed conditions of certainty are intrinsically characterized by the incompleteness of the state of information and knowledge associated with him or her. The conditions of certainty are not real, but they are ideal. They are an extreme simplification obtained avoiding uncertain factors that are always present. In fact, in almost all circumstances, and at all times, we all find ourselves in a state of ignorance (see also [16] with regard to the incompleteness of the state of information and knowledge associated with a given decision maker). Given two nonrandom goods having downward-sloping demand curves, $(x_1,\, x_2)$ represents what is actually chosen for each of them by the decision maker inside the budget set. We establish the following:

Definition 3. *The quantity of consumption for two nonrandom goods actually demanded by the decision maker under claimed conditions of certainty is an average quantity. We write*

$$x_1 = x_1^1\, p_1^1 + \ldots + x_1^n\, p_1^n \tag{22}$$

and

$$x_2 = x_2^1\, p_2^1 + \ldots + x_2^n\, p_2^n, \tag{23}$$

where $\{p_1^i\}$ *and* $\{p_2^j\}$ *are two sets of n masses, with* $0 \leq p_i^j \leq 1$, $j = 1, \ldots, n$, $i = 1, 2$, *whose sum is always equal to* 1 *with regard to each of them. Given the best rational choice expressed by* $(x_1,\, x_2)$, *the estimated quantities of consumption for good* 1 *are expressed by* $\{x_1^1, \ldots, x_1^n\}$, *whereas the estimated quantities of consumption for good* 2 *are given by* $\{x_2^1, \ldots, x_2^n\}$.

We are found inside the budget set of the decision maker. We are found inside a subset of $\mathbb{R} \times \mathbb{R}$. Note that we also deal with the weighted average of n^2 estimated quantities of consumption for good 1 and good 2 that are jointly considered (see also [17] with regard to what is demanded for goods). They derive from the Cartesian product given by $\{x_1^1, \ldots, x_1^n\} \times \{x_2^1, \ldots, x_2^n\}$, where n^2 non-negative masses are associated with each pair of this product. Given $(x_1,\, x_2)$, the weighted average of n^2 estimated quantities of consumption for good 1 and good 2 is a summarized element of the Fréchet class. We establish the following:

Definition 4. *The set of all weighted averages of* n^2 *estimated quantities of consumption for good* 1 *and good* 2 *that are jointly considered, with the same given marginal weighted averages of n estimated quantities of consumption for good* 1 *and n estimated quantities of consumption for good* 2, *constitutes the Fréchet class.*

It is clear that x_1 is the given marginal weighted average of n quantities of consumption for good 1, whereas x_2 is the given marginal weighted average of n quantities of consumption for good 2 (given the best rational choice expressed by (x_1, x_2), we firstly handle a closed neighborhood of x_1 denoted by $[x_1 - \epsilon\,;\, x_1 + \epsilon']$ on the horizontal axis, and a closed neighborhood of x_2 denoted by $[x_2 - \epsilon\,;\, x_2 + \epsilon']$ on the vertical one, where both ϵ and ϵ' are two small positive quantities. Since the state of information and knowledge associated with a given decision maker is assumed to be incomplete at the time of choice, n estimated quantities of consumption for good 1 belong to $[x_1 - \epsilon\,;\, x_1 + \epsilon']$ and n estimated quantities of consumption for good 2 belong to $[x_2 - \epsilon\,;\, x_2 + \epsilon']$. These quantities belong to two one-dimensional convex sets. It is not necessary that one of n alternatives coincides with x_1. The same is true with regard to x_2. It follows that n^2 estimated quantities of consumption for good 1 and good 2 jointly considered are handled. After determining $\{x_1^1, \ldots, x_1^n\}$, $\{x_2^1, \ldots, x_2^n\}$, and $\{x_1^1, \ldots, x_1^n\} \times \{x_2^1, \ldots, x_2^n\}$, two nonparametric marginal distributions of mass together with a nonparametric joint distribution of mass take place in such a way that (x_1, x_2) is their chosen summary). With regard to the two goods that are separately considered, it is evident that every weighted average of n quantities of consumption for each of them is always found between the lowest quantity of consumption and the highest one for each good under consideration. The same is true with regard to n^2 quantities of consumption for the two goods into account that are jointly considered, where each pair of n^2 pairs is handled by taking the arithmetic product of its corresponding elements into account with regard to a one-dimensional straight line. Note that (x_1, x_2) is a bilinear and disaggregate measure belonging to a subset of $\mathbb{R} \times \mathbb{R}$, where $\mathbb{R} \times \mathbb{R}$ is the direct product of $\mathbb{R}$ and $\mathbb{R}$. It is decomposed into two linear measures, where each of them belongs to a subset of $\mathbb{R}$. Accordingly, we observe reductions of dimension by passing from n^2 to 2 (being equal to 2 the dimension of the plane), and from n to 1 (being equal to 1 the dimension of the straight line). All coherent weighted averages of n^2 quantities of consumption for the two goods into account identify a two-dimensional convex set coinciding with a subset of $\mathbb{R} \times \mathbb{R}$. They are obtained by taking all values lying between 0 and 1, end points included, into account with regard to each mass of n^2 masses (weights). The number of these admissible values is infinite. Moreover, all these averages also identify two one-dimensional convex sets coinciding with two closed line segments belonging to the two mutually orthogonal axes of a two-dimensional Cartesian coordinate system. They are obtained by taking all values lying between 0 and 1, end points included, into account, with regard to each mass of n masses (weights). The number of these admissible values is infinite. Strictly speaking, we refer ourselves to two half-lines, where each of them extends indefinitely in a positive direction from zero before being restricted. Note that this framework is the same as the one characterizing bound choices being made by the decision maker under uncertainty and riskiness. Boundary points that are found on each restricted half-line identify degenerate averages. The budget constraint of the decision maker is written in the form

$$c_1\, x_1 + c_2\, x_2 \leq c,$$

where the prices of good 1 and good 2 are given by (c_1, c_2), whereas the amount of money the decision maker has to spend is given by c. The budget line identifying the budget set of the decision maker is of an exogenous nature. It is a hyperplane embedded in $\mathbb{R} \times \mathbb{R}$. Its negative slope given by $-\frac{c_1}{c_2}$ depends on the known prices of the two goods under consideration. Note that the elements identifying the decision maker's budget given by (c_1, c_2, c) are all objective. Nevertheless, c is assumed to be an uncertain or possible element at the time of choice. It can therefore be either true or false at a later time unlike the two objective prices that are certainly true (bound choice being made by the decision maker is always relative to a given state of information and knowledge associated with him or her. It is assumed to be incomplete. If there is no ignorance anymore because further information is later acquired, then it is possible to observe a parallel shift outward or inward of the budget line. Its slope is accordingly unchanged. Moreover, it is also possible that the budget line does not shift). The budget set of the decision maker is a right triangle belonging to the first quadrant of a two-dimensional Cartesian coordinate system. The vertex of the right angle of this triangle coincides with the point given by $(0,0)$. We note the following:

Remark 2. *The prices of good* 1 *and good* 2 *are the two real coefficients identifying the negative slope of a hyperplane embedded in* $\mathbb{R} \times \mathbb{R}$. *Since* $(x_1,\ x_2)$ *is a point belonging to a two-dimensional convex set, the budget line does not separate* $(x_1,\ x_2)$ *from the set of points denoted by* $\{x_1^1, \ldots, x_1^n\} \times \{x_2^1, \ldots, x_2^n\}$. *It does not separate* x_1 *from* $\{x_1^1, \ldots, x_1^n\}$, *nor* x_2 *from* $\{x_2^1, \ldots, x_2^n\}$. *All the elements of* $\{x_1^1, \ldots, x_1^n\}$ *are found between zero and* $\frac{c}{c_1}$. *All the elements of* $\{x_2^1, \ldots, x_2^n\}$ *are found between zero and* $\frac{c}{c_2}$. *By definition, the budget line is a hyperplane, so possible alternatives rightly come into play. Possible consumption levels are not directly observed, but they are estimated together with their corresponding masses. Possible consumption levels have to be made explicit by the decision maker together with their corresponding masses.*

The objects of the decision maker choice studied inside the budget set of the decision maker have to maximize his or her utility (see also [18] with regard to the study about the subjective notion of utility). Accordingly, these objects of decision maker choice can identify his or her optimal choices, whose nature is always relative to a given set of information and knowledge associated with the decision maker under consideration. In this paper, such objects of decision maker choice deal with average quantities of consumption (see also [3] with regard to issues treated by revealed preference theory).

3. Revealed Preference Applied to Choices Being Made under Conditions of Uncertainty and Riskiness

We denote by E^2 a two-dimensional linear space over $\mathbb{R}$ having a Euclidean structure. (The space of alternatives is firstly denoted by E^{n+1}. Since we study two marginal random goods at a time, two one-dimensional linear subspaces of E^{n+1} are considered. Each of them is transferred on a one-dimensional straight line, on which an origin, a same unit of length, and an orientation are chosen. We do not handle two one-dimensional straight lines, but we deal with two half-lines. We pass from two half-lines to two closed-line segments. This is because $\mathbf{P}$ is involved together with its coherence properties. We accordingly use $n+1$ non-negative and finitely additive masses with regard to $n+1$ possible alternatives referred to each marginal random good, so E^2 coincides with $\mathbb{R} \times \mathbb{R}$. On the other hand, we write $\dim E^2 = \dim(\mathbb{R} \times \mathbb{R}) = 2$. To use $n+1$ masses with regard to each marginal random good such that ∞^n admissible choices of them can firstly be made implies that two one-dimensional convex sets are handled). E^2 consists of ordered pairs of real numbers. The set of all $\mathbf{x} \in E^2$, with ${}_1x = \mathbf{P}({}_1X) \geq 0$ and ${}_2x = \mathbf{P}({}_2X) \geq 0$, is denoted by E^2_+. The set of all $\mathbf{x} \in E^2$, with ${}_1x = \mathbf{P}({}_1X) > 0$ and ${}_2x = \mathbf{P}({}_2X) > 0$, is denoted by E^2_{++}. A different budget set can be observed whenever the budget line changes its negative slope. Each choice being made by the decision maker is associated with a budget set characterized by a budget line. All decision maker's previsions concerning joint random goods that are chosen whenever the budget line changes its negative slope can identify a finite sequence belonging to E^2 and denoted by

$$\{\mathbf{x}^k \,|\, k = 1, \ldots, K\}. \tag{24}$$

For each k, it is possible to consider a pair of real numbers written in the form $(x_1^k,\ x_2^k)$. If we suppose, to fix ideas, that it turns out to be $K = 2$, then we are faced with a balanced sequence of pairs given by

$$(x_1^1,\ x_2^1),\ (x_1^2,\ x_2^2). \tag{25}$$

With regard to such a sequence, it suffices to observe how upper and lower indices appear. The space where the decision maker chooses is denoted by E^2_+. It coincides with the first quadrant of a two-dimensional Cartesian coordinate system. We consider a collection denoted by $\mathcal{U}$ of utility functions written in the form

$$U\colon E^2_+ \to \mathbb{R}, \tag{26}$$

where we have $U \in \mathcal{U}$. Each decision maker's prevision concerning a joint random good consists of a two-dimensional vector $\mathbf{x} \in E^2_+$ obtained from the budget set denoted by

$$B(\mathbf{c}, c) = \{\mathbf{x} \in E^2_+ \,|\, \mathbf{c} \cdot \mathbf{x} \leq c\} \tag{27}$$

within this context, where $\mathbf{c} = (c_1, c_2)$ is a price vector, whereas c is the amount of money the decision maker has to spend. We wrote the same budget constraint treated before. Note that $\mathbf{c} \cdot \mathbf{x}$ is a scalar or inner product characterizing E^2 from a metric point of view. A generic pair denoted by

$$(\mathbf{x}, \mathbf{c}) \in E^2_+ \times E^2_{++} \tag{28}$$

represents all we need to know about a coherent prevision of a random good being made by the decision maker and about the budget. It follows that a finite collection of pairs written in the form

$$\{(\mathbf{x}^1, \mathbf{c}^1), \ldots, (\mathbf{x}^K, \mathbf{c}^K)\} \tag{29}$$

expresses a dataset. We deal with random goods, whose possible values can explicitly be considered of a monetary nature. Since in this paper each point of the budget set of the decision maker is a summarized element of the Fréchet class, concave or convex utility functions whose nature is ordinal can be considered. These utility functions are used to satisfy a preference ordering. A linear utility function representing the identity of monetary value and utility can also be considered (see also [19] with regard to a specific optimization problem). For instance, the decision maker can estimate all the joint masses under consideration in such a way that he or she is risk averse. Given a collection $\mathcal{U}$ of strictly increasing and concave utility functions, a dataset expressed by (29) is $\mathcal{U}$-rational if there exists $U \in \mathcal{U}$ such that we write, for each k,

$$\mathbf{x}^k \in \operatorname{argmax} \{U(\mathbf{x}) \,|\, \mathbf{x} \in B(\mathbf{c}^k, \mathbf{c}^k \cdot \mathbf{x}^k)\}. \tag{30}$$

On the other hand, the decision maker can also estimate all the joint masses under consideration in such a way that he or she is risk loving. Given a collection $\mathcal{U}'$ of strictly increasing and convex utility functions, a dataset expressed by (29) is $\mathcal{U}'$-rational if there exists $U' \in \mathcal{U}'$ such that we write, for each k,

$$\mathbf{x}^k \in \operatorname{argmax} \{U'(\mathbf{x}) \,|\, \mathbf{x} \in B(\mathbf{c}^k, \mathbf{c}^k \cdot \mathbf{x}^k)\}. \tag{31}$$

If the decision maker estimates all the joint masses under consideration in such a way that he or she is risk neutral, then his or her linear utility function coincides with the 45-degree line. A dataset expressed by (29) is $\mathcal{U}''$-rational if there exists one and one only $U'' \in \mathcal{U}''$ such that we write, for each k,

$$\mathbf{x}^k \in \operatorname{argmax} \{U''(\mathbf{x}) \,|\, \mathbf{x} \in B(\mathbf{c}^k, \mathbf{c}^k \cdot \mathbf{x}^k)\}. \tag{32}$$

The decision maker maximizes his or her subjective utility associated with each bundle of two random goods belonging to (29) when and only when his or her choices are found on the corresponding budget lines, where the 45-degree line exactly crosses them. Note that in order for $U(\mathbf{x})$ and $U'(\mathbf{x})$ to appear linearly in the representation, it is necessary to introduce a new axis. The same is not true for $U''(\mathbf{x})$. This is because it is possible to use the 45-degree line inside the budget set of the decision maker where there are only two axes. (Though the optimal-choice problem studied in this paper is a topic concerning a constrained optimization problem, it is not solved by using an auxiliary function known as the Lagrangian. In fact, the preference-maximization problem is solved through the notion of distance. With regard to a quadratic metric, we use measures compatible with concave, convex, and linear utility functions. We do not use tools compatible with a concave utility function only. We exercise great care in not going far beyond the consideration of cases immediately at hand and directly interesting. In this paper, they are objective alternatives whose number is finite. We do not substitute the abstraction of schematized models for the changing and temporary reality. Accordingly, in saying something about the particular case of interest, we do not prefer to race on ahead playing around with illusory problems contemplating infinite cases. They are all possible cases. After contemplating infinite cases, it is conceptually possible to choose one of them. If one of them is chosen, then a prediction appears. This is because to make a prediction means to venture to try to guess, among the possible alternatives, the one that will happen. On the other hand, it is mathematically possible to obtain a result based on infinite cases without any risk. If it is obtained, then a sure prediction takes place. This is because a powerful mathematical method is used. A sure prediction is a uniquely determined answer to a problem based on infinite cases. They are all possible cases. In this paper, we are conversely interested in the notion of prevision, so barycenters of masses distributed over a finite set of possible alternatives are chosen. Prevision is not prediction).

4. An Extension of the Notion of Bundle of Nonrandom Goods: A Consumption Matrix

Given (x_1, x_2), (22) and (23) are obtained by decomposing inside the budget set of the decision maker the bilinear measure expressed by

$$x_1\, x_2 = x_1^1\, x_2^1\, p_{11} + \ldots + x_1^n\, x_2^n\, p_{nn}, \tag{33}$$

where we write

$$p_{11} + p_{12} + \ldots + p_{1n} + \ldots + p_{nn} = 1, \tag{34}$$

with $0 \leq p_{ij} \leq 1$, $i, j = 1, \ldots, n$. We deal with n^2 joint masses characterizing (33). We can think of putting them into a two-way table having n rows and n columns. Such masses are mathematically the covariant components of an affine tensor of order 2 studied outside the budget set of the decision maker (see also [20] with regard to the study connected with nonparametric distributions of mass). Nevertheless, together with $\{x_1^1, \ldots, x_1^n\} \times \{x_2^1, \ldots, x_2^n\}$, they also identify a point belonging to a subset of $\mathbb{R} \times \mathbb{R}$. This subset coincides with the budget set of the decision maker. Given (x_1, x_2), all the n^2 joint masses are subjectively chosen in such a way that the marginal masses identifying the sets $\{p_1^i\}$ and $\{p_2^j\}$ remain unchanged. It being understood that the marginal masses always remain unchanged whenever (x_1, x_2) is chosen, we also consider

$$x_1\, x_1 = x_1^1\, x_1^1\, p_{11} + \ldots + x_1^n\, x_1^n\, p_{nn}, \tag{35}$$

where all off-diagonal masses are necessarily equal to 0,

$$x_2\, x_2 = x_2^1\, x_2^1\, p_{11} + \ldots + x_2^n\, x_2^n\, p_{nn}, \tag{36}$$

where all off-diagonal masses are necessarily equal to 0, and

$$x_2\, x_1 = x_2^1\, x_1^1\, p_{11} + \ldots + x_2^n\, x_1^n\, p_{nn}, \tag{37}$$

where $x_2\, x_1$ has the same joint masses as $x_1\, x_2$. It follows that we write a symmetric matrix of order 2 denoted by

$$C = \begin{pmatrix} x_1\, x_1 & x_1\, x_2 \\ x_2\, x_1 & x_2\, x_2 \end{pmatrix}. \tag{38}$$

We call it a consumption matrix. Aggregate bound choices are studied in this way. A nonlinear analysis is used. Whenever we deal with $x_1\, x_1$ and $x_2\, x_2$, the slope of the corresponding budget line is equal to -1. This is because the two catheti of the right triangle under consideration are equal.

We establish the following:

Definition 5. *Given two marginal goods, whose chosen quantities are expressed through pure numbers, a consumption matrix is a square matrix of order* 2 *containing four bilinear measures,* *where each of them is decomposed into two linear measures inside the budget set of the decision maker.*

We firstly handle all the points belonging to the budget set of the decision maker. All the points belonging to the budget set of the decision maker are barycenters of masses distributed over finite sets of alternatives. We secondly focus on $x_1\,x_2$. Hence, we decompose $x_1\,x_2$ into two linear measures. Since we want to release the notion of bundle of goods from the one of ordered pair of quantities of consumption being chosen by a given decision maker, we obtain four metric measures coinciding with all elements of the square matrix of order 2 denoted by C. An aggregate measure of a bilinear nature is accordingly given by

$$x_{12} = \begin{vmatrix} x_1\,x_1 & x_1\,x_2 \\ x_2\,x_1 & x_2\,x_2 \end{vmatrix} = x_1\,x_1\,x_2\,x_2 - x_1\,x_2\,x_2\,x_1. \tag{39}$$

Possible alternatives are firstly estimated together with their corresponding masses. They are secondly aggregated. The measure given by (39) derives from four α-products. They are $x_1\,x_1$, $x_1\,x_2$, $x_2\,x_1$, and $x_2\,x_2$. Each of them is obtained as a scalar or inner product by using joint masses. The measure given by (39) represents the average quantity of consumption for a double and stand-alone nonrandom good consisting of good 1 and good 2. A double good is nothing but a multiple good of order 2. Its components are good 1 and good 2. Their multilinear relationships are studied. The measure given by (39) is the α-norm of a particular tensor of order 2. Each α-product contained in (39) is calculated outside the budget set of the decision maker. In fact, it uses n^2 joint masses that are the components of an affine tensor of order 2 belonging to $E^n \otimes E^n$, where we have

$$\dim(E^n \otimes E^n) = n^2.$$

Even though each α-product contained in (39) is a real number, it does not appear as such inside the budget set of the decision maker. This real number appears in a disaggregate fashion. This is because only a two-dimensional point expressed by an ordered pair of real numbers appears.

Another Consumption Matrix: Changes of Origin

Given x_1, it is possible to consider a change of origin expressed by

$$d_1 = (x_1^1 - x_1)\,p_1^1 + \ldots + (x_1^n - x_1)\,p_1^n, \tag{40}$$

where we write $(x_1^i - x_1) = d_1^i$, $i = 1, \ldots, n$. All deviations from x_1 of the estimated alternatives associated with good 1 are considered in this way. Given x_2, it is similarly possible to consider another change of origin given by

$$d_2 = (x_2^1 - x_2)\,p_2^1 + \ldots + (x_2^n - x_2)\,p_2^n, \tag{41}$$

where we write $(x_2^j - x_2) = d_2^j$, $j = 1, \ldots, n$. We obtain

$$d_1\,d_2 = d_1^1\,d_2^1\,p_{11} + \ldots + d_1^n\,d_2^n\,p_{nn}, \tag{42}$$

where we have

$$p_{11} + p_{12} + \ldots + p_{1n} + \ldots + p_{nn} = 1,$$

with $0 \leq p_{ij} \leq 1$, $i, j = 1, \ldots, n$. We also consider

$$d_1\,d_1 = d_1^1\,d_1^1\,p_{11} + \ldots + d_1^n\,d_1^n\,p_{nn}, \tag{43}$$

$$d_2\,d_2 = d_2^1\,d_2^1\,p_{11} + \ldots + d_2^n\,d_2^n\,p_{nn}, \tag{44}$$

and

$$d_2\, d_1 = d_2^1\, d_1^1\, p_{11} + \ldots + d_2^n\, d_1^n\, p_{nn}. \tag{45}$$

All marginal and joint masses do not change (see also [21] with regard to the study about changes of origin). They are the same masses that are established by the decision maker with regard to the estimated alternatives connected with x_1, x_2, and $x_1\, x_2$. We write another symmetric matrix of order 2 denoted by

$$C' = \begin{pmatrix} d_1\, d_1 & d_1\, d_2 \\ d_2\, d_1 & d_2\, d_2 \end{pmatrix}. \tag{46}$$

It contains four bilinear measures identifying four α-products. They are all considered outside the budget set of the decision maker. It follows that another aggregate measure of a bilinear nature expressing the variability of consumption for a multiple good of order 2 is given by

$$d_{12} = \begin{vmatrix} d_1\, d_1 & d_1\, d_2 \\ d_2\, d_1 & d_2\, d_2 \end{vmatrix} = d_1\, d_1\, d_2\, d_2 - d_1\, d_2\, d_2\, d_1. \tag{47}$$

It is based on changes of origin.

5. How to Check the Weak Axiom of Revealed Preference by Using Aggregate Measures

5.1. The Bravais–Pearson Correlation Coefficient Associated with Each Bundle of Two Nonrandom Goods Being Chosen by the Decision Maker Inside His or Her Budget Set

Given (x_1, x_2), we consider two aggregate measures based on changes of origin. The former is expressed by (47), whereas the latter coincides with

$$\hat{d}_{12} = \begin{vmatrix} d_1\, d_1 & 0 \\ 0 & d_2\, d_2 \end{vmatrix}. \tag{48}$$

After some mathematical steps, we write

$$-1 \le \left(1 - \frac{\begin{vmatrix} d_1\, d_1 & d_1\, d_2 \\ d_2\, d_1 & d_2\, d_2 \end{vmatrix}}{\begin{vmatrix} d_1\, d_1 & 0 \\ 0 & d_2\, d_2 \end{vmatrix}} \right)^{1/2} \le 1, \tag{49}$$

where it is possible to realize that the expression within the parentheses coincides with the Bravais–Pearson correlation coefficient referred to a double and stand-alone nonrandom good consisting of good 1 and good 2. We write it in the following form given by

$$r_{12} = \frac{d_1\, d_2}{\sqrt{d_1\, d_1}\,\sqrt{d_2\, d_2}}. \tag{50}$$

It is a measure of a linear relationship between two sets of estimated quantities of consumption for good 1 and good 2. It geometrically measures the angle between two vectors of E^n, where the contravariant components of each of them represent all deviations from a mean value. Since metric measures can be considered inside linear spaces over $\mathbb{R}$ having a different dimension, this paper shows that everything can vectorially be studied, provided one takes an appropriate number of dimensions.

We note the following:

Remark 3. *Whenever we consider a bilinear measure that is decomposed into two linear measures, we refer ourselves to a continuous subset of $\mathbb{R} \times \mathbb{R}$. Such a bilinear measure is decomposed into two linear measures inside a convex set coinciding with a subset of $\mathbb{R} \times \mathbb{R}$. The budget set of the decision maker is generated by two one-dimensional straight lines, on which an origin, a same unit of length, and an orientation are established. They identify two mutually orthogonal axes of a two-dimensional*

Cartesian coordinate system. With regard to two one-dimensional straight lines, we consider two "reductions of dimension". We firstly pass from 2 *to* 1*. In fact, a point of a two-dimensional convex set is always decomposed into two points of two one-dimensional convex sets. We secondly pass from n to* 1*. In fact, since the decision maker summarizes n estimated consumption alternatives corresponding to n one-dimensional points by using n non-negative masses subjectively chosen, he or she obtains a real number with respect to each closed line segment belonging to each half-line.*

Remark 4. *Whenever we consider an aggregate measure referred to a multiple good of order* 2*, we take four bilinear measures into account. Each of them is obtained by considering the covariant components of an affine tensor of order* 2*. Such components coincide with the n^2 joint masses that are estimated by the decision maker. After establishing the marginal masses, the decision maker has to take them into account to estimate n^2 joint masses. The aggregate measure under consideration coincides with the determinant of a square matrix of order* 2*. It is a real number obtained by considering a bilinear function. The two columns of the matrix under consideration are two column vectors, where each of them has two components expressed by two real numbers. Whenever we consider an aggregate measure, we go away from the budget set of the decision maker.*

Remark 5. *The Bravais–Pearson correlation coefficient is intrinsically based on the bilinear object being chosen by the decision maker inside the budget set. Such an object is a bundle of two goods. Nevertheless, whenever we use the Bravais–Pearson correlation coefficient, we go away from the budget set of the decision maker.*

5.2. A Violation of the Weak Axiom of Revealed Preference

Suppose that (x_1, x_2) is chosen by the decision maker at prices (b_1, b_2) (see also [22] with regard to new developments in revealed preference theory). We denote by r_{12} the Bravais–Pearson correlation coefficient associated with (x_1, x_2). Such a coefficient is expressed by (50). Remind that the quantity of consumption that is chosen for good 1 is found on the horizontal axis, whereas the quantity of consumption that is chosen for good 2 is found on the vertical one. If good 1 becomes more expensive and good 2 becomes less expensive, then the budget line changes its negative slope. This is because the state of information and knowledge associated with a given decision maker changes (see also [23] with regard to probabilistic aspects). The budget line becomes steeper. Let (y_1, y_2) be the bundle of goods being chosen at prices (q_1, q_2), where it is evident that we have $q_1 > b_1$ and $q_2 < b_2$. The Bravais–Pearson correlation coefficient associated with (y_1, y_2), where we write $(y_1, y_2) \neq (x_1, x_2)$, is denoted by

$$r'_{12} = \frac{d'_1\, d'_2}{\sqrt{d'_1\, d'_1}\,\sqrt{d'_2\, d'_2}}. \tag{51}$$

We have

$$d'_1 = (y_1^1 - y_1)\, {p'}_1^1 + \ldots + (y_1^n - y_1)\, {p'}_1^n \tag{52}$$

as well as

$$d'_2 = (y_2^1 - y_2)\, {p'}_2^1 + \ldots + (y_2^n - y_2)\, {p'}_2^n, \tag{53}$$

where the estimated quantities of consumption for good 1 at price q_1 and connected with y_1 are expressed by $\{y_1^1, \ldots, y_1^n\}$, whereas the estimated quantities of consumption for good 2 at price q_2 and connected with y_2 are given by $\{y_2^1, \ldots, y_2^n\}$. Note that $\{{p'}_1^i\}$ and $\{{p'}_2^j\}$ are two sets of n non-negative masses such that each mass of them is found between 0 and 1, end points included. The sum of these masses is always equal to 1 with regard to each set of them.

If a violation of the weak axiom of revealed preference is observed, then the quantity of consumption that is chosen for good 1 and denoted by y_1 does not decrease, but it increases (see also [24] with regard to the study referred to demand functions). Moreover, the quantity of consumption that is demanded for good 2 and denoted by y_2 does not increase, but it decreases. A violation of the principle according to which the demand curve for each of the two goods under consideration slopes downwards is consequently observed. This implies that if a negative number is used to show r_{12} because there exists an inverse relationship between the quantity of consumption associated with a good and its price, then a positive number has to be used to show r'_{12}. This is because we have to consider a change of sign. On the other hand, this paper shows that each point of the budget set of the decision maker is a summarized element of the Fréchet class. Accordingly, the decision maker can estimate all the joint masses expressing the variability of consumption for the two marginal goods under consideration based on the state of information and knowledge associated with him or her (see also [25] with regard to multilinear measures).

5.3. Decision Maker Choices That Satisfy the Weak Axiom of Revealed Preference

Suppose (x_1, x_2) to be chosen by the decision maker at prices (b_1, b_2). We denote by r_{12} the Bravais–Pearson correlation coefficient associated with it. Such a coefficient is expressed by (50). If good 1 becomes less expensive and good 2 becomes more expensive, then the budget line changes its negative slope. It becomes flatter. Let (y_1, y_2) be the bundle of goods being chosen at prices (q_1, q_2), where it is clear that we have $q_1 < b_1$ and $q_2 > b_2$. The Bravais–Pearson correlation coefficient associated with (y_1, y_2), where we write $(y_1, y_2) \neq (x_1, x_2)$, is denoted by (51).

If the weak axiom of the revealed preference is satisfied, then the quantity of consumption that is chosen for good 1 and denoted by y_1 does not decrease, but it increases. Moreover, the quantity of consumption that is chosen for good 2 and denoted by y_2 does not increase, but it decreases. Hence, we do not observe a violation of the principle according to which the demand curve for each of the two goods under consideration slopes downwards. This implies that if a negative number is used to show r_{12}, then a negative number has to be used to show r'_{12}. This is because we do not need to consider a change of sign.

5.4. A Summary of Consumption Data Based on Subjective Elements as Well

In general, the chain of direct comparisons can be of any finite length. Assumptions about how the decision maker's preferences work tell us that any two bundles of goods can directly be compared. Hence, to say that any two bundles of goods can directly be compared means that the decision maker can choose between any two given bundles of goods. Any two bundles of goods belonging to two different indifference curves can directly be compared for a reason of a metric nature. This is because the preferred bundle is always more distant from $(0, 0)$ than the other one (see Theorem 1). (In this paper, a bilinear measure coincides with a two-dimensional point, whereas a linear measure coincides with a one-dimensional point. The bilinear objects being chosen by the decision maker inside his or her budget set coincide with two-dimensional points. They are bundles of two goods. They are always measured within this context).

Let (x_1, x_2) be the consumption bundle being chosen when prices are (c_1, c_2). Let (y_1, y_2) be another consumption bundle such that we write

$$c_1 x_1 + c_2 x_2 \geq c_1 y_1 + c_2 y_2. \tag{54}$$

If the decision maker is choosing the most preferred bundle he or she can afford, then (x_1, x_2) is strictly preferred to (y_1, y_2). It is strictly preferred to (y_1, y_2) for a reason of a metric nature. This is because distances of (x_1, x_2) and (y_1, y_2) from the point given by $(0, 0)$ can be measured, so the meaning of (54) is of a metric nature.

After observing several choices of bundles of goods at different prices, we obtain different measures to check the weak axiom of revealed preference (see also [26] with regard to empirical aspects associated with choice). We are indirectly interested in knowing how much it costs the decision maker to purchase each bundle of goods at each corresponding set of prices. This is because we want to consider subjective elements as well. They are always connected with choices of bundles of goods being made at different prices (see also [27] with regard to the study about individual behavior). Thus, we are directly interested in knowing the sign of each correlation coefficient associated with each observation characterized by a given set of prices and quantities of consumption being chosen by the decision maker. Each observation identifies a choice relative to a given set of information and knowledge associated with a given decision maker (see also [28] with regard to a primordial and fundamental study connected with revealed preference). It is assumed to be incomplete at the time of choice. In this paper, we focus on possible and objective alternatives that are estimated after recognizing that the budget line is formally a hyperplane embedded in $\mathbb{R} \times \mathbb{R}$ whose slope depends on the prices. In particular, we observe the same sign expressed by the minus symbol referred to each correlation coefficient whenever the decision maker chooses the best things he or she can afford. We conversely observe different signs referred to all correlation coefficients that have been calculated in this paper whenever the decision maker does not choose the best things he or she can afford. It follows that all bound choices being made by him or her are not coherent with revealed preference theory (see also [29] with regard to the notion of utility function).

6. Other Variability Measures Characterizing a Bayesian Approach to the Theory of Decision-Making: Mean Quadratic Differences Referred to Random Goods

The notion of α-product is of a metric nature. It is a scalar or inner product whose mathematical character is bilinear. Contravariant and covariant components of vectors belonging to E^{n+1} are used to obtain it. Contravariant components of vectors belonging to E^{n+1} are mathematically used outside the budget set of the decision maker. For example, from the following Table 1,

Table 1. Contravariant and covariant components of vectors

Random Good 1 \ Random Good 2	**0**	**4**	**5**	**Sum**
0	0	0	0	0
2	0	0.1	0.2	0.3
3	0	0.5	0.2	0.7
Sum	0	0.6	0.4	1

it follows that we write $\mathbf{P}({}_1X\,{}_2X) = 11.8$. Given the contravariant components of ${}_{(2)}\mathbf{x}$ identifying the following column vector

$$\begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix},$$

its covariant components are expressed by

$$0 \cdot 0 + 4 \cdot 0 + 5 \cdot 0 = 0,$$

$$0 \cdot 0 + 4 \cdot 0.1 + 5 \cdot 0.2 = 1.4,$$

and

$$0 \cdot 0 + 4 \cdot 0.5 + 5 \cdot 0.2 = 3,$$

so it is possible to write the following result

$$\left\langle \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.4 \\ 3 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_\alpha = \mathbf{P}({}_1X\,{}_2X) = 11.8.$$

On the other hand, after calculating the covariant components of ${}_{(1)}\mathbf{x}$ in a similar way, we write

$$\left\langle \begin{pmatrix} 0 \\ 1.7 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_\alpha = \mathbf{P}({}_1X\,{}_2X) = 11.8.$$

A marginal distribution of ${}_1X$ can be interpreted as a joint distribution of ${}_1X$ and ${}_2X = \boldsymbol{\phi}$. For instance, from the following Table 2,

Table 2. A marginal distribution viewed to be as a joint distribution

${}_2X = \boldsymbol{\phi}$ / ${}_1X$	**1**	**1**	**1**	**Sum**
0	0	0	0	0
2	0	0.3	0	0.3
3	0	0	0.7	0.7
Sum	0	0.3	0.7	1

it follows that it is possible to write $\mathbf{P}({}_1X) = \mathbf{P}({}_1X\,{}_2X) = \mathbf{P}({}_2X\,{}_1X) = 2.7$. Since we have $\mathbf{P}({}_1X\,{}_1X) = 7.5$ and $\mathbf{P}({}_2X\,{}_2X) = 1$, the variability of consumption for ${}_1X$ can be expressed by

$$\sigma^2_{{}_1X} = \begin{vmatrix} \mathbf{P}({}_1X\,{}_1X) = 7.5 & \mathbf{P}({}_1X\,{}_2X) = 2.7 \\ \mathbf{P}({}_2X\,{}_1X) = 2.7 & \mathbf{P}({}_2X\,{}_2X) = 1 \end{vmatrix} = 0.21. \quad (55)$$

It is clear that (55) expresses a known index in a more general fashion. The variability of consumption for ${}_1X$ is processed outside the budget set of the decision maker, whereas $\mathbf{P}({}_1X)$ is actually chosen inside the budget set of the decision maker. If $\mathbf{P}({}_1X)$ is chosen by the decision maker inside the budget set, then ${}_1X$ is associated with another random good, whose possible values are all different. This is because they identify a finite partition of states of the world of a contingent consumption plan. ${}_1X$ can also be associated with ${}_1X$ itself. The variability of consumption for ${}_1X$ is processed on the basis of what is actually chosen. A nonparametric distribution of mass characterizing a marginal random good denoted by ${}_1X$ is summarized by using the notion of α-norm of a tensor of order 2 denoted by ${}_{(1)}f$. A measure of variability connected with consumption for ${}_1X$ is obtained by calculating the α-norm of ${}_{(1)}f$ denoted by $\|{}_{(1)}f\|^2_\alpha$ (see also [30] with regard to multilinear measures applied to the notion of utility). The relation between the mean quadratic difference of ${}_1X$, given by

$${}^2\Delta^2({}_1X) = \|{}_{(1)}f\|^2_\alpha = \frac{1}{2}\begin{vmatrix} 2\,\|{}_{(1)}\mathbf{x}\|^2_\alpha & 2\,{}_{(1)}\bar{\mathbf{x}} \\ 2\,{}_{(1)}\bar{\mathbf{x}} & 2 \end{vmatrix}, \quad (56)$$

and its standard deviation has been established by Corrado Gini (see also [31] with regard to indices obtained through an approach put forward by Corrado Gini). Note that ${}_{(1)}\bar{\mathbf{x}}$ has its contravariant components that are all equal because it vectorially denotes the expected value of ${}_1X$. The variability of a distribution of mass depends on how the decision maker estimates all the masses under consideration. These masses are estimated by him or her according to a given knowledge hypothesis. It is made explicit by him or her from time to time. We write

$${}^2\Delta^2({}_1X) = 2\,\sigma^2_{{}_1X}. \quad (57)$$

The linear mean quadratic difference of X_{12} given by

$${}^2_L\Delta^2(X_{12}) = \|{}_{(1)}\mathbf{d} - {}_{(2)}\mathbf{d}\|^2_\alpha = \|{}_{(1)}\mathbf{d}\|^2_\alpha + \|{}_{(2)}\mathbf{d}\|^2_\alpha - 2\langle {}_{(1)}\mathbf{d}, {}_{(2)}\mathbf{d} \rangle_\alpha \quad (58)$$

is obtained by using a linear and quadratic metric (see also [32] with regard to measures where the variability is not standardized). It is clear that X_{12} is a multiple random good of order 2 (think of a two-risky asset portfolio). The nonlinear (multilinear) mean quadratic difference of X_{12} expressed by

$$ {}^{2}_{NL}\Delta^2(X_{12}) = \frac{3^2}{3!}\left[\|_{(1)}\mathbf{d}\|_\alpha^2\,\|_{(2)}\mathbf{d}\|_\alpha^2 - \left(\langle_{(1)}\mathbf{d},\,_{(2)}\mathbf{d}\rangle_\alpha\right)^2\right] \tag{59} $$

is obtained by using a nonlinear and quadratic metric (see also [33] with regard to a different approach to variability). All possible values identifying $_{(1)}\mathbf{d}$ represent deviations from a mean value. All possible values identifying $_{(2)}\mathbf{d}$ represent deviations from a mean value. After observing that

$$ r_{12} = \frac{\langle_{(1)}\mathbf{d},\,_{(2)}\mathbf{d}\rangle_\alpha}{\|_{(1)}\mathbf{d}\|_\alpha\,\|_{(2)}\mathbf{d}\|_\alpha} \tag{60} $$

is the measure of correlation with respect to X_{12} whose possible values for its components denoted by $_1X$ and $_2X$ are subjected to changes of origin, we observe

$$ {}^{2}_{NL}\Delta^2(X_{12}) = \frac{3}{2}\|_{(1)}\mathbf{d}\|_\alpha^2\,\|_{(2)}\mathbf{d}\|_\alpha^2\left(1 - r_{12}^2\right). \tag{61} $$

We already obtained the measure of correlation with respect to a multiple good of order 2 that is chosen under claimed conditions of certainty. Note that ${}^{2}_{L}\Delta^2(X_{12})$ and ${}^{2}_{NL}\Delta^2(X_{12})$ measure the riskiness of X_{12} whenever X_{12} is viewed to be as a two-risky asset portfolio. On the other hand, portfolio choices are bound choices as well.

7. Multiple Physical Goods of Order 2: A Numerical Example

For convenience, we consider a simplified dataset, whose real observations are only two. The state of information and knowledge associated with a given decision maker is assumed to be incomplete at the time of choice.

In Table 3, the prices of the two single physical goods under consideration are denoted by c_1 and c_2, whereas the quantities actually chosen by the decision maker inside the budget set are denoted by x_1 and x_2. For example, these quantities are associated with two different kinds of cheese. These two different kinds of cheese are, respectively, good 1 and good 2. With regard to the first observation, a joint distribution of mass is estimated together with two marginal distributions of mass. We estimate a joint distribution by taking into account that our goal is also to check the weak axiom of revealed preference. In this section, we estimate a joint distribution of mass together with two marginal distributions of mass by taking two logical criteria into account. They obey the rules of the logic of prevision and the ones of ordinary logic. Ordinary logic characterizes choices being made under conditions of ideal certainty. The logic of prevision is involved whenever the state of information and knowledge associated with a given decision maker is incomplete. Given a finite number of alternatives that can be observed or estimated, such a logic admits an infinite number of admissible values connected with each non-negative mass. Conversely, ordinary logic admits only two values associated with each non-negative mass, either true $= 1$ or false $= 0$, whenever the state of information and knowledge associated with a given decision maker is assumed to be complete. If ignorance ceases, then a consumption level is not uncertain or possible anymore, but it is either true or false. (Two types of logical reasoning can separately be handled. Given n observed consumption levels belonging to a set whose nature is objective, if we want to obtain their coherent summary, then a deductive reasoning takes place. On the other hand, given an observed quantity contained in the dataset under consideration (see Table 3), if we pass from n estimated consumption levels (belonging to a closed neighborhood of the observed quantity into account) to their coherent summary coinciding with this observed quantity, then a deductive reasoning still takes place. In general, a deductive reasoning always uses n non-negative and finitely additive masses, where each of them can take infinite admissible values between 0 and 1, end points included, into account. It is not necessary that one of the n alternatives coincides with their summary. Real situations where the state of information and knowledge associated with a given decision maker is incomplete at the time of choice are therefore dealt with by the logic of prevision. Conversely, if we pass from an observed value contained in the dataset under consideration to n estimated consumption levels (belonging to a closed neighborhood of the observed quantity into account), then an inductive reasoning takes place. It uses n masses such that $n-1$ masses are coherently equal to 0, whereas only one mass of n masses is coherently equal to 1. Ordinary logic is consequently involved, so degenerate distributions of mass to be summarized appear. Since linear spaces and subspaces over $\mathbb{R}$ are taken into account to study bound choices in this paper, admissible ordered n-tuples of real numbers (belonging to a closed neighborhood of the observed quantity under consideration) can be estimated by the decision maker before transferring them on a one-dimensional straight line, on which an origin, a unit of length, and an orientation are chosen. Their number is infinite. The decision maker chooses one of them based on lower and upper bounds that are established. One of them identifies n outcomes such that only one alternative expressed by a real number is true. All others are false. It is true because it is observed under ideal conditions of certainty. This number is therefore that value appearing in the dataset under consideration. Hence, it is absolutely necessary that one of n alternatives coincides with the observed value contained in the dataset into account. In this section, if the incompleteness of the state of information and knowledge associated with a given decision maker ceases, then the bound choice does not change. This means that a new piece of information later acquired in such a way that there is no ignorance anymore is unimportant with regard to the bound choice). From the following Table 4,

Table 3. A simplified dataset

Observation	c_1	c_2	x_1	x_2
1	4	5	2	3
2	6	3	1.5	4

Table 4. An estimated joint distribution associated with two marginal choices

Good 1 \ Good 2	0	2	3	4	Sum
0	0	0	0	0	0
1	0	0	0	1/3	1/3
2	0	0	1/3	0	1/3
3	0	1/3	0	0	1/3
Sum	0	1/3	1/3	1/3	1

we observe $x_1\,x_2 = x_2\,x_1 = 5.33$, $x_1 = 2$, $x_2 = 3$. From the following Table 5,

Table 5. How to estimate a joint distribution when the two marginal goods are the same

Good 1 \ Good 1	**0**	**1**	**2**	**3**	**Sum**
0	0	0	0	0	0
1	0	1/3	0	0	1/3
2	0	0	1/3	0	1/3
3	0	0	0	1/3	1/3
Sum	0	1/3	1/3	1/3	1

it is possible to write $x_1\, x_1 = 4.67$. All off-diagonal elements have to be equal to 0. The slope of the corresponding budget line is equal to -1. With regard to the budget set of the decision maker containing points whose number is infinite, each joint probability can take all values from 0 to 1, end points included, into account. Conversely, how to estimate $x_1\, x_1$ is constrained. From the following Table 6,

Table 6. A possible estimate of a joint distribution when the two marginal goods are the same

Good 2 \ Good 2	**0**	**2**	**3**	**4**	**Sum**
0	0	0	0	0	0
2	0	1/3	0	0	1/3
3	0	0	1/3	0	1/3
4	0	0	0	1/3	1/3
Sum	0	1/3	1/3	1/3	1

it is possible to obtain $x_2\, x_2 = 9.67$, so we write

$$x_{12} = \begin{vmatrix} 4.67 & 5.33 \\ 5.33 & 9.67 \end{vmatrix} = 16.75.$$

Our simplified dataset contains pure numbers, so 16.75 represents the average quantity of consumption referred to a multiple physical good of order 2. The two single physical goods, which are the components of this multiple physical good of order 2, are not fused together from a physical point of view. If this happens, then we obtain another single physical good. Conversely, we want to obtain a multiple physical good of order 2, whose components are two single physical goods. A multiple physical good of order 2 is a portfolio containing two single physical goods. They give rise to an aggregate good. It is viewed to be as a stand-alone good from a conceptual point of view. It does not live from a material point of view. Conversely, its components live from a material point of view.

With regard to the second observation, a joint distribution of mass is estimated together with two marginal distributions of mass. In this section, we estimate them by taking two logical criteria into account. We estimate a joint distribution by taking into account that our goal is also to check the weak axiom of revealed preference.

From Table 7, we have $x_1\, x_2 = x_2\, x_1 = 5.33$, $x_1 = 1.5$, $x_2 = 4$.

Table 7. An estimated joint distribution associated with other two marginal choices

Good 2 / Good 1	**0**	**3**	**4**	**5**	**Sum**
0	0	0	0	0	0
0.5	0	0	0	1/3	1/3
1.5	0	0	1/3	0	1/3
2.5	0	1/3	0	0	1/3
Sum	0	1/3	1/3	1/3	1

From Table 8, it is possible to write $x_1\, x_1 = 2.917$. All off-diagonal elements have to be equal to 0.

Table 8. An estimate of a joint distribution when the two marginal goods are the same

Good 1 / Good 1	**0**	**0.5**	**1.5**	**2.5**	**Sum**
0	0	0	0	0	0
0.5	0	1/3	0	0	1/3
1.5	0	0	1/3	0	1/3
2.5	0	0	0	1/3	1/3
Sum	0	1/3	1/3	1/3	1

From Table 9, it is possible to obtain $x_2\, x_2 = 16.67$, so we write

$$x_{12} = \begin{vmatrix} 2.917 & 5.33 \\ 5.33 & 16.67 \end{vmatrix} = 20.21749.$$

Table 9. An estimate of a joint distribution when the two marginal goods are the same

Good 2 / Good 2	**0**	**3**	**4**	**5**	**Sum**
0	0	0	0	0	0
3	0	1/3	0	0	1/3
4	0	0	1/3	0	1/3
5	0	0	0	1/3	1/3
Sum	0	1/3	1/3	1/3	1

It is clear that 20.21749 represents the average quantity of consumption referred to a multiple physical good of order 2. This quantity was obtained by considering the quantities actually chosen by the decision maker inside the budget set characterized by a specific pair of prices. If we consider changes of origin, then it is possible to study the variability of consumption referred to a multiple physical good of order 2. It is also possible to check the weak axiom of revealed preference.

From Table 10, it follows that it is possible to consider aggregate measures as well. We calculate some aggregate measures. They are x_{12}, d_{12}, and r_{12}. The weak axiom of revealed preference is satisfied. Choices connected with two single physical goods are rational. Accordingly, multiple choices connected with a multiple physical good of order 2 are rational as well. The best rational choice depends on the decision maker's preferences. Their nature is subjective. (Optimal choices are always referred to the variable state of information and knowledge associated with a given decision maker. Actual situations are intrinsically uncertain, so a less arbitrary origin has to be considered. By studying average quantities, the possibility that bound choices are relative to a specific state of information and knowledge associated with a given decision maker is handled. Thus, variations in the total amount of money the decision maker has to spend could happen. Risks of external origin determining variations in income could occur as well. Since a point of the budget set of the decision maker is chosen whenever the best rational choice for a given decision maker takes place, a specific weighted average is not logically undetermined anymore, but it is logically determined. It consists in distributing among all the alternatives into account subjective expectations and sensations identified with non-negative and finitely additive masses. In this paper, barycenters of masses distributed over finite sets of alternatives are specifically treated). The best rational choice depends on further hypotheses of an empirical nature. In this paper, it is not possible to show computational simulations and algorithms underlying measures that identify bound choices for space constraints only (see also [34] with regard to an interesting issue in biomedicine).

Table 10. Some aggregate measures connected with bound choices

Observation	c_1	c_2	x_1	x_2	x_{12}	d_{12}	r_{12}
1	4	5	2	3	16.75	0	−1
2	6	3	1.5	4	20.21749	0	−1

8. Discussion, Conclusions, and Future Perspectives

This study focuses on metric measures. The variability of choice for two marginal nonrandom goods that are the components of a multiple good is expressed by using the Bravais–Pearson correlation coefficient. This coefficient is intrinsically associated with aggregate measures. We use it because the variability of a joint distribution of mass is expressed by its numerator. This variability always depends on how the decision maker estimates all the joint masses under consideration. The Bravais–Pearson correlation coefficient associated with each bundle of two nonrandom goods is used in order to check the weak axiom of revealed preference. In this paper, mean quadratic differences connected with multiple random goods are also proposed. This is because the origin of the variability of a nonparametric distribution of mass is not standardized within this context. It is not standardized because the decision maker makes explicit, from time to time, the knowledge hypothesis underlying it. This origin is not connected with the theory of measurement errors, where such errors are of a random nature. Different measures based on this origin can be used. In general, aggregate measures can be established to study bilinear or multilinear relationships between variables. In particular, it is possible to consider such measures together with parametric probability distributions, such as normal distributions to solve specific inference problems treated by statistics and econometrics. In this paper, we firstly deal with four Cartesian products, where each of them is the product of two finite sets identifying the estimated consumption levels for two marginal nonrandom goods being chosen under claimed conditions of certainty. Non-negative and finitely additive masses connected with each Cartesian product are considered, so a square matrix of order 2 is secondly obtained. The average quantity of consumption associated with a multiple good of order 2 and its variability are obtained by taking four real numbers into account, where each of them derives from an α-product. Four Cartesian products are studied because the Cartesian product of two finite sets of estimated consumption levels associated with two marginal nonrandom goods is released from the notion of ordered pair of estimated consumption levels connected with each good under consideration. Four ordered pairs of marginal nonrandom goods are handled. An extension of the notion of bundle of goods is caught in this way. A nonlinear analysis is used.

In this paper, we apply to random goods the principles of revealed preference. This is because the budget line can always be drawn. It is possible to establish its horizontal and vertical intercepts every time. The prices of the two random goods under consideration are always determined in this way. Three convex sets are established. It is possible to check how the budget line changes its negative slope from an observation to another one. $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ are chosen by the decision maker. He or she has to be a maximizing decision maker. (It is possible to study real data given by time series connected with annual returns referred to marginal risky assets. It is possible to make a coherent prevision about the return associated with each marginal risky asset based on observed data in different stock markets. Each time series is associated with a stock market. Real data given by time series are possible alternatives. They can be summarized. Their nature is objective. From the slope of the budget line, which can endogenously be drawn, it is possible to observe the prices of the two risky assets viewed to be as two marginal random goods. It is possible to wonder if the decision maker under consideration maximizes, or does not maximize, his or her utility). If two marginal random goods are studied inside the budget set of the decision maker, then we can also establish which is the price that the decision maker is willing to pay to purchase the right to participate in a bet that places him or her in the uncertain situation identified with X_{12}, where X_{12} is a multiple random good of order 2. If we write $X_{12} = \{{}_1X, {}_2X\}$, then ${}_1X$ and ${}_2X$ are the two components of X_{12}. We can determine its mathematical expectation denoted by $\mathbf{P}(X_{12})$. In particular, if X_{12} is a two-risky asset portfolio, then $\mathbf{P}(X_{12})$ represents its expected return. It is possible to consider the cardinal utility function associated with X_{12}. It depends on a subjective attitude towards risk. Anyway, we have to go away from the budget set after observing what is chosen inside it with respect to the two components of X_{12}. If we go away from the budget set, then the criteria of rational decision making consist of the choice of any coherent evaluation of the probabilities (probability evaluations can be based on symmetric probabilities implying a judgment of equal probability being made by the decision maker who deals with all the single alternatives into account. They can also be based on frequencies. Nevertheless, a judgment of equal probability is subjective itself, whereas relating probability back to frequency has meaning if and only if the meaning and conditions are subjectively specified) being made by the decision maker and any utility function referred to X_{12} having the necessary mathematical properties. Such properties have to comply with a subjective attitude towards risk. The decision maker also fixes as his or her goal the maximization of the prevision or mathematical expectation of his or her cardinal utility associated with X_{12}. Accordingly, it is possible to extend to multiple random goods the notion of moral expectation, which was put forward by Daniel Bernoulli and developed by John von Neumann and Oskar Morgenstern at a later time. On the other hand, it is possible to behave rationally with respect to decisions being made by the decision maker based on subjective preferences without knowing anything about probability and utility. It is not correct to think that every subjective attitude is a fact on which it is not possible to express an objective judgment based on rational conditions made clear by economists, mathematicians, and statisticians. It is true that everyone has a subjective attitude towards risk caught by a specific cardinal utility function associated with a single or multiple good. It is true that everyone can evaluate all probabilities under consideration according to one's own subjective judgment. It is false that everyone can choose which rational rules have to be complied with. Such rules cannot arbitrarily be chosen by anyone. They are logical rules. In this paper, we extend them by studying multiple choices connected with multiple goods of order 2. On the other hand, multiple goods of order greater than 2 can similarly be studied.

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