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Some Remarks to a Theorem of van Geemen

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Abstract. *In [3], Bert van Geemen computed the dimension of the space of the fourth power of the thetanullwerte. In [7], it has been observed that all linear relations are consequences of the quartic Riemann relations. In this note, we want to give a new proof of these results and extend them. to the vector valued case. In a last section we treat the linear dependencies between arbitrary powers of the thetanullwerte. We will show that $k = 4$ is the only case where such dependencies can occur.*

1. Introduction

We consider the Riemann theta series

$$\vartheta[m](\tau, z) = \sum_{p\in\mathbb{Z}^g} \exp\Big(\pi i\tau\Big[p + \frac{a}{2}\Big] + 2\Big(p + \frac{a}{2}\Big)'\Big(z + \frac{b}{2}\Big)\Big).$$

Here $m \in \mathbb{Z}^{2g}$ is a column vector that is divided into two columns $a, b \in \mathbb{Z}^g$,

$$m = \begin{pmatrix} a \\ b \end{pmatrix}.$$

The matrix τ is from the Siegel upper half plane $\mathcal{H}_g$. This means that τ is a symmetric $g \times g$-matrix such that its imaginary part is positive definite. Finally z is a column in $\mathbb{C}^g$. Of particular interest are the thetanullwerte

$$\vartheta[m](\tau) := \vartheta[m](\tau, 0).$$

Up to sign they depend only on m mod two. In this context the columns $m \in \mathbb{Z}^{2g}$ are called characteristics. A characteristic m is called even (odd) if $a'b \equiv 0 \bmod 2$ ($\equiv 1 \bmod 2$). One knows that a thetanullwerte vanishes if and only if m is odd.

Let T_g be the vector space generated by the fourth powers of the thetanullwerte. These are modular forms of weight 2 with respect to the principal congruence subgroup of level two. In [3], Van Geemen showed that the classical Schottky–Jung approach gave a weak solution to the Schottky problem. Trying to give explicit relations for the Schottky locus using Riemann's relations, he obtained, in Corollary 4.10,

$$\dim\,(T_g) = (2^g + 1)(2^{g-1} + 1)/3.$$

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In [7], it has been observed, as consequence of results of Fay [1], that all linear relations between the $\vartheta[m]^4$ are consequences of the quartic Riemann relations.

In this note, we want to give a new proof of these results and extend the method to a vector valued case. Starting from Fay's results about Riemann's relations [1], we prove that the representation of $\mathrm{Sp}(g,\mathbb{F}_2)$, induced by a certain character of the theta group, has two irreducible components that result to be isomorphic to the eigenspaces of Fay's matrix M^+. One component is isomorphic to the space of Riemann's relations, the other is isomorphic to T_g.

Then we prove a similar result for vector valued modular forms that are attached to odd characteristics. Here we have to use Fay's matrix M^-.

In a last section we use this method to treat the linear dependencies between arbitrary powers $\vartheta[m]^k$. As final result we get, cf. Corollary 6.5.

Theorem 1.1. *The thetanullwerte $\vartheta[m]^k$ are linear independent with one exception, $k = 4$.*

It is a pleasure to acknowledge the discussions we had with Bert van Geemen.

2. The symplectic group over the field of two elements

We use the following notations for matrices A. The transposed of A is denoted by A'. Its trace is denoted by $\mathrm{tr}(A)$ and A_0 is the diagonal of A written as column vector. Let A be an $n \times n$-matrix and B an $n \times m$-matrix, then

$$A[B] := B'AB.$$

Let $G = G_g = \mathrm{Sp}(g,\mathbb{F}_2)$ be the symplectic group over the field of two elements. We recall the affine action of G on $\mathbb{F}_2^{2g}$,

$$\sigma\{m\} = \sigma'^{-1}m + \begin{pmatrix} (CD')_0 \\ (AB')_0 \end{pmatrix}.$$

Here

$$\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

denotes the decomposition of σ into 4 $g \times g$-blocks. It is an action in the sense

$$(\sigma\tau)\{m\} = \sigma\{\tau\{m\}\}.$$

In this context the elements of $\mathbb{F}_2^{2g}$ are also called characteristics. A characteristic

$$m = \begin{pmatrix} a \\ b \end{pmatrix}$$

is called even if $a'b = 0$ otherwise odd. We denote the subset of even (odd) characteristics by

$$(\mathbb{F}_2^{2g})_{\text{even}}, \quad (\mathbb{F}_2^{2g})_{\text{odd}}.$$

Their orders are

$$k_g^{\pm} = 2^{g-1}(2^g \pm 1).$$

It is known that the affine action preserves the parity. Moreover, the group G acts double transitively on the set of all even (odd) characteristics and it acts transitively on the set of all pairs (m, n) where m is even and n odd (see [4, Chapt. V, Prop. 2]).

We introduce the stabilizers

$$H(m) = \{\sigma \in G; \quad \sigma\{m\} = m\}.$$

In the case $m = 0$ this is the so-called theta group. We denote it by $H = H(0)$. We will have to use the following function on $G \times \mathbb{F}_2^{2g}$,

$$\varepsilon_m(\sigma) = (-1)^{\operatorname{tr}(B'C)+(B'D)[a]+(A'C)[b]}.$$

This function arises in the theta transformation formalism as we will see in the next section. At the moment we take it as it is. It has three important properties.

Lemma 2.1. *The following rules hold.*

(a) One has that $\varepsilon_m(\sigma)$ for fixed m is a non-trivial character on $H(m)$. For $h \in H$ one has $\varepsilon_m(h) = \varepsilon_0(h)$.

(b) For $\sigma, \tau \in G$ one has $\varepsilon_m(\tau\sigma) = \varepsilon_m(\sigma)\varepsilon_{\sigma\{m\}}(\tau)$.

(c) For $h \in H$, $\sigma \in G$ one has $\varepsilon_m(h)\varepsilon_m(\sigma) = \varepsilon_m(h\sigma)$.

Proof. It can be proved directly by means of somewhat tedious calculations. Since it is also a direct consequence of the theta transformation formalism which we treat in the next section, we skip the direct proof and leave it to the interested reader as an exercise. □

For a pair of characteristics $m = \begin{pmatrix} a \\ b \end{pmatrix}$ and $n = \begin{pmatrix} c \\ d \end{pmatrix}$, we define the standard symplectic pairing

$$e(m, n) = (-1)^{a'd - b'c}.$$

With these notations we have the following

Lemma 2.2. *The following rule holds: $e(\sigma\{m\}, \sigma\{n\}) = e(m, n)\varepsilon_m(\sigma)e_n(\sigma)$.*

Proof. We use the notations

$$\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \sigma(m) := \sigma'^{-1}m$$

and

$$m = \begin{pmatrix} a \\ b \end{pmatrix}, \quad n = \begin{pmatrix} c \\ d \end{pmatrix}, \quad v = \begin{pmatrix} (CD')_0 \\ (AB')_0 \end{pmatrix}.$$

In the proof we will use

$$\begin{aligned}(D'B)_0 + D'(AB')_0 + B'(CD')_0 = 0,\\ (A'C)_0 + C'(AB')_0 + A'(CD')_0 = 0.\end{aligned}$$

This follows from $\sigma^{-1}\{\sigma\{0\}\} = 0$. Now we get

$$\begin{aligned}e(\sigma\{m\}, \sigma\{n\}) &= e(\sigma(m), \sigma(n))e(\sigma(m), v)e(v, \sigma(n))\\ &= e(m, n)e(\sigma(m + n), v)\\ &= e(m, n)(-1)^{(CD')_0'(B(a+\alpha)+A(b+\beta)-(AB')_0'(D(a+\alpha)+C(b+\beta)}\\ &= e(m, n)(-1)^{(D'B)_0'(a+\alpha)+(A'C)_0'(b+\beta)}\\ &= e(m, n)\varepsilon_m(\sigma)\varepsilon_n(\sigma)\end{aligned}$$

which proves the lemma. □

As special case of Lemma 2.1, we get a character

$$\varepsilon_H := \varepsilon_0 \colon H \to \{\pm 1\}.$$

We induce the character ε_H on H to a G- representation $\mathrm{Ind}_H^G(\varepsilon_H)$. So $\mathrm{Ind}_H^G(\varepsilon_H)$ consists of all functions $f \colon G \to \mathbb{C}$ with the property $f(hg) = \varepsilon_H(h)f(g)$. The group G acts by translation from the right on this space. We mention that this representation has been studied by Frame [2]. He proved that it decomposes into two irreducible non-isomorphic representations and he determined the dimensions of the two components. We will reprove this result with a completely different method.

We will construct a basis $X(m)$ of $\mathrm{Ind}_H^G(\varepsilon_H)$. It is parametrized by even characteristics. Notice that the number of even characteristics equals the index of H in G which is the dimension of $\mathrm{Ind}_H^G(\varepsilon_H)$. So, let m be an even characteristic. We choose $\sigma \in G$ such that $\sigma\{m\} = 0$. Then we define the following function,

$$X(m) \colon G \to \mathbb{C}.$$

It is defined to be 0 outside the coset $H\sigma$, and on this coset it is defined by

$$X(m)(h\sigma) = \varepsilon_H(h)\varepsilon_m(\sigma).$$

It is essential that this is independent on the choice of σ. The proof follows immediately from Lemma 2.1.

We associate to each even m a variable e_m and consider the vector space

$$\sum_{m \text{ even}} \mathbb{C}e_m.$$

We use the basis above to define an isomorphism

$$\mathrm{Ind}_H^G(\varepsilon_H) \cong \sum_{m \text{ even}} \mathbb{C}e_m.$$

Just map $X(m)$ to e_m.

Next we describe an action of G on the right hand side such that this isomorphism is G-equivariant. We have to associate to each $\sigma \in G$ an isomorphism

$$\sigma : \sum_{m \text{ even}} \mathbb{C}e_m \cong \sum_{m \text{ even}} \mathbb{C}e_m.$$

On basis elements it is defined by

$$\sigma(e_m) = \varepsilon_m(\sigma) e_{\sigma\{m\}}.$$

It is easy to check that this defines a representation of G and that the isomorphism

$$\mathrm{Ind}_H^G(\varepsilon_H) \cong \sum_{m \text{ even}} \mathbb{C}e_m$$

is G-equivariant.

3. An operator of Fay

We treat the space

$$\sum_{m \text{ even}} \mathbb{C}e_m$$

together with its G-action. We use the notations

$$\langle m, n\rangle = a'\beta + b'\alpha, \qquad m = \begin{pmatrix} a \\ b \end{pmatrix}, \quad n = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

and

$$e(m,n) = (-1)^{\langle m,n\rangle}.$$

Notice that $\langle m, n\rangle$ is the symplectic pairing whose invariance group is $G = \mathrm{Sp}(g, \mathbb{F}_2)$. Fay [1] defined the following operator

$$M^+ \colon \sum_{m \text{ even}} \mathbb{C}e_m \to \sum_{m \text{ even}} \mathbb{C}e_m, \quad e_m \to \sum_n e(m,n) e_n.$$

(Actually Fay ordered the even characteristics and introduced M^+ as a matrix, see [1, formula (6)].) Thus in Lemma 1.1, he proved

Theorem 3.1 (Fay)**.** *The operator M^+ has two eigenvalues, namely -2^{g-1} and 2^g. Let*

$$V^+ \subset \mathbb{C}^{k_g^+}, \quad W^+ \subset \mathbb{C}^{k_g^+}$$

be the eigenspaces. Then we have

$$\mathbb{C}^{k_g^+} = V^+ \oplus W^+.$$

The dimension of the eigenspaces are

$$\dim V^+ = (2^{2g} - 1)/3, \quad \dim W^+ = (2^g + 1)(2^{g-1} + 1)/3.$$

We need the following result.

Lemma 3.2. *The operator M^+ commutes with the action of G.*

Proof. Use Lemma 2.2. □

As a consequence the two eigenspaces are G-invariant. So we see that $\mathrm{Ind}_H^G(\varepsilon_H)$ admits two invariant subspaces, one isomorphic to V^+ the other to W^+.

We need the decomposition of G into double cosets with respect to H. This can be found also in [2].

Lemma 3.3. *There are two double cosets in $H\backslash G/H$.*

We present a different proof. Let $g_1, g_2 \in G - H$. We have to show that they are in the same double coset. We consider the two characteristics $m_1 = g_1\{0\}$, $m_2 = g_2\{0\}$. They are both different from zero. The group H acts transitively on the set of non-zero characteristics (since G acts double transitively on the set of even characteristics). Hence there exist $h_1 \in H$ such that $h_1\{m_1\} = m_2$. This implies $h_2 := g_2^{-1}h_1g_1 \in H$. It follows $g_2 = h_1g_1h_2^{-1}$.

Lemma 3.4. *The representation $\mathrm{Ind}_H^G(\varepsilon_H)$ is either irreducible or decomposes into two non-isomorphic irreducible components.*

Proof. The statement is equivalent to

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G(\varepsilon_H), \mathrm{Ind}_H^G(\varepsilon_H)) \le 2.$$

We use Frobenius reciprocity. For a representation of G on a finite dimensional vector space V the evaluation map

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(\varepsilon_H)) \cong \mathrm{Hom}_H(V, \mathbb{C}(\varepsilon_H))$$

is an isomorphism. Here $\mathbb{C}(\varepsilon_H)$ is the one dimensional vector space $\mathbb{C}$ where H acts by means of the character ε_H. So we have to show

$$\dim \mathrm{Hom}_H(\mathrm{Ind}_H^G(\varepsilon_H), \mathbb{C}(\varepsilon_H)) \le 2.$$

Since both representations are real, they are isomorphic to their duals. Hence it is equivalent to prove

$$\dim \mathrm{Hom}_H(\mathbb{C}(\varepsilon_H), \mathrm{Ind}_H^G(\varepsilon_H)) \le 2.$$

Such a homomorphism is defined by the image f of $1 \in \mathbb{C}$ and f must have the property

$$f(hg) = \varepsilon_H(h)f(g) \quad \text{and} \quad f(gh) = \varepsilon_H(h)f(g).$$

Since G has only two double cosets mod H, the dimension of the space of all f is ≤ 2. □

We found two non-zero invariant subspaces of $\mathrm{Ind}_H^G(\varepsilon)$. So Lemma 3.4 shows that they are irreducible and non-isomorphic. So we have proved the following theorem.

Theorem 3.5 (Frame). *The representation* $\mathrm{Ind}_H^G(\varepsilon_H)$ *decomposes into two non-isomorphic representations of dimensions*

$$(2^{2g}-1)/3, \qquad (2^g+1)(2^{g-1}+1)/3.$$

4. Theta series

The symplectic group $\mathrm{Sp}(g,\mathbb{R})$ acts on the Siegel half plane $\mathcal{H}_g$ through

$$\sigma(\tau) = (A\tau+B)(C\tau+D)^{-1}.$$

We recall that the Riemann theta functions are defined by

$$\vartheta[m](\tau,z) = \sum_{p\in\mathbb{Z}^g} \exp\left(\pi i\tau\left[p+\frac{a}{2}\right] + 2\left(p+\frac{a}{2}\right)'\left(z+\frac{b}{2}\right)\right).$$

Here

$$m = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^{2g}$$

is an integral column vector. We denote by

$$\Gamma_g = \mathrm{Sp}(g,\mathbb{Z})$$

the Siegel modular group. We consider the map

$$\Gamma_g \times \mathbb{Z}^{2g} \to \mathbb{Z}^{2g}, \qquad \sigma\{m\} = \sigma'^{-1}m + \begin{pmatrix} (CD')_0 \\ (AB')_0 \end{pmatrix}.$$

It is compatible with the map $\mathrm{Sp}(g,\mathbb{F}_2)\times\mathbb{F}_2^{2g}\to\mathbb{F}_2^{2g}$ that we defined in Sect. 1. Hence we can use the same notation. In [4] at page 85, it is given the following

Theorem 4.1. *We have*

$$\begin{aligned}&\vartheta[\sigma\{m\}](\sigma(\tau),(C\tau+D)'^{-1}z)\\ &\quad = \exp\pi\mathrm{i}\big\{((C\tau+D)^{-1}C)[z]\big\}v(\sigma,m)\sqrt{\det(C\tau+D)}\vartheta[m](\tau,z)\end{aligned}$$

for all $\sigma\in\Gamma_n$. *Here* $v(\sigma,m)$ *denotes a complex number which is independent of* (τ,z). *One has* $v(\sigma,m)^8=1$.

Corollary 4.2. *For the thetanullwerte this means*

$$\vartheta[\sigma\{m\}](\sigma(\tau)) = v(\sigma,m)\sqrt{\det(C\tau+D)}\vartheta[m](\tau).$$

The system of numbers $v(\sigma, m)$, as σ varies, it is very complicated. We will need only the 4th powers. It is known that these depend only on m mod 2 and on the image of σ with respect to the homomorphism $\mathrm{Sp}(g, \mathbb{Z}) \to \mathrm{Sp}(g, \mathbb{F}_2)$. This homomorphism is surjective. So we can define

$$v(\sigma, m)^4 \quad \text{for} \quad \sigma \in \mathrm{Sp}(g, \mathbb{F}_2),\ m \in \mathbb{F}_2^{2g}.$$

This function is known explicitly.

Lemma 4.3. *One has*

$$v(\sigma, m)^4 = \varepsilon_m(\sigma) = (-1)^{\mathrm{tr}(B'C)+(B'D)[a]+(A'C)[b]}.$$

Proof. A proof can be found in [5, Lemma 10]. □

This was the expression we introduced in Sect. 1. It gets now a natural interpretation. Lemma 2.1 is an easy consequences of this interpretation.

For $\sigma \in \Gamma_g$ and a function $f\colon \mathcal{H}_g \to \mathbb{C}$ and an integer r we will use the notation

$$(f|_r\sigma)(\tau) = \det(C\tau + D)^{-r}(f(\sigma\tau).$$

The theta transformation formula shows that

$$f \mapsto f|_2\sigma^{-1}$$

defines a representation of Γ_g on T_g., i.e. the vector space generated by the fourth powers of the thetanullwerte. This representation factors through $G = \mathrm{Sp}(g, \mathbb{F}_2)$. Now we consider the linear map

$$\mathrm{Ind}_H^G(\varepsilon_H) \to T_g, \quad X(m) \to \vartheta[m]^4.$$

The theta transformation formula shows that this map is compatible with the action of G. It is also clear that it is surjective. Its dimension is smaller than that of $\mathrm{Ind}_H^G(\varepsilon_H)$. To show this, one needs that there exists at least one non-trivial relation between the $\vartheta[m]^4$. The quartic Riemann relations give one. (In the next Theorem we will describe a space of such relations.) Therefore one of the two irreducible components must map isomorphically to T_g. We have to work out which one.

For this we formulate some of the Riemann theta relations in a way that has been described by Fay [1]. In Sect. 1 we introduced the operator

$$M^+\colon \sum_{m \text{ even}} \mathbb{C}e_m \to \sum_{m \text{ even}} \mathbb{C}e_m$$

and described the eigenspaces. The original meaning of this operator is also due to Fay [1]. He showed, in formula (22), that the Riemann quartic relations imply the following result.

Theorem 4.4 (Fay)**.** *Let v be an eigenvector of M^+ with eigenvalue -2^{g-1} (i.e. $v \in V^+$). Then the relation*

$$\sum_m v(m)\vartheta[m]^4 = 0$$

holds.

This shows that the irreducible component that maps isomorphically to V^+ goes to zero in T_g. Hence the other component W^+ maps isomorphically to T_g. We obtain the first main result.

Theorem 4.5. *We have*

$$\dim T_g = (2^g + 1)(2^{g-1} + 1)/3.$$

The linear relations between the $\vartheta[m]^4$ are consequences of Riemann's quartic relations.

5. A variant of van Geemen's result

Let $f(\tau, z)$ be a holomorphic function on $\mathcal{H}_g \times \mathbb{C}^g$. Then we can define the holomorphic vector valued function

$$J(f)\colon \mathcal{H}_g \to \mathbb{C}^g, \quad J(f) = \operatorname{grad}_z f(\tau, z)|_{z=0}.$$

We are interested in $J(\vartheta[m])$. This is different from 0 if and only if m is odd.

Theorem 5.1. *For $\sigma \in \Gamma_g$ and odd $m \in \mathbb{Z}^{2g}$ the transformation formula*

$$J(\vartheta[\sigma\{m\}])(\sigma(\tau)) = v(M, \sigma)\sqrt{\det(C\tau + D)}(C\tau + D)(J\vartheta[m])(\tau)$$

holds.

This is a consequence of Theorem 4.2 (see for example [6]).

We consider the fourth symmetric tensor product $\operatorname{Sym}^4(\mathbb{C}^g)$ and define the functions

$$S(\vartheta[m])\colon \mathcal{H}_g \to \operatorname{Sym}^4(\mathbb{C}^g), \quad \tau \to \operatorname{Sym}^4 J(\vartheta[m])(\tau).$$

The space generated by these k_g^- tensors is called by S_g. A special case of Theorem 5.1 states

$$S(\vartheta[\sigma\{m\}])(\sigma(\tau)) = \varepsilon_m(\sigma)\det(C\tau + D)^2 \operatorname{Sym}^4(C\tau + D)S(\vartheta[m])(\tau).$$

This shows that the $S(\vartheta[m])$ are vector valued modular forms on the principal congruence subgroup of level two with the character ε_0 and with respect to the $\mathrm{GL}(n, \mathbb{C})$ representation

$$\det(CZ + D)^2 \operatorname{Sym}^4(CZ + D).$$

This representation is irreducible and has highest weight $(6, 2, \dots, 2)$.

Up to the weight this is the same transformation formula as for the $\vartheta[m]^4$ in the even case.

The tensors $S(\vartheta[m])$ satisfy similar linear relations as the $\vartheta[m]^4$. Instead of the theta group H we now have to use the stabilizer $K = H(n)$ of a fixed odd characteristic n. To be concrete, we take

$$n' = (1, 0, \dots, 0, 1, 0, \dots, 0).$$

Now we have to consider the non trivial character $\varepsilon_K = \varepsilon_n$. The induced representation $\mathrm{Ind}_K^G(\varepsilon_K)$ comes into the game. Its dimension equals the number k_g^- of odd characteristics. We will construct a basis $Y(m)$ of $\mathrm{Ind}_K^G(\varepsilon_K)$. It is parametrized by odd characteristics. So, let m be an odd characteristic. We choose $\sigma \in G$ such that $\sigma\{m\} = n$. Then we define the following function, $Y(m)\colon G \to \mathbb{C}$ analoguos to $X(m)$. It is defined to be 0 outside the coset $K\sigma$, and on this coset it is defined by

$$X(m)(k\sigma) = \varepsilon_K(k))\varepsilon_m(\sigma).$$

It is also independent of the choice of σ. Again this follows from Lemma 2.1.

We associate to each odd m a variable e_m and consider the vector space

$$\sum_{m \text{ odd}} \mathbb{C}e_m.$$

We use the basis above to define an isomorphism

$$\mathrm{Ind}_K^G(\varepsilon_K) \cong \sum_{m \text{ odd}} \mathbb{C}e_m.$$

Just map $Y(m)$ to e_m.

Besides M^+, Fay [1] defined the following operator

$$M^-\colon \sum_{m \text{ odd}} \mathbb{C}e_m \to \sum_{m \text{ odd}} \mathbb{C}e_m, \quad e_m \to \sum_n e(m,n)e_n.$$

Theorem 5.2. *The operator M^- has two eigenvalues, namely 2^{g-1} and -2^g. Let*

$$V^- \subset \mathbb{C}^{k_g^+}, \quad W^- \subset \mathbb{C}^{k_g^+}$$

be the eigenspaces. Then we have

$$\mathbb{C}^{k_g^-} = V^- \oplus W^-.$$

The dimension of the eigenspaces are

$$\dim V^- = (2^{2g} - 1)/3, \quad \dim W^- = (2^g - 1)(2^{g-1} - 1)/3.$$

The same proof as in Lemma 3.3 shows that M^- commutes with G. Hence the two spaces V^- and W^- are G-invariant subspaces.

With the same method, using Fay's matrix M^- instead of M^+, one can get now a new proof of the following result of Frame.

Theorem 5.3. *Let $K = H(m)$ be the stabilizer of an odd characteristic m. Denote by ε_K the distinguished character of K. Then the representation $\mathrm{Ind}_K^G(\varepsilon_K)$ decomposes into two non-isomorphic irreducibles representation of dimensions*

$$(2^{2g}-1)/3, \qquad (2^g-1)(2^{g-1}-1)/3.$$

As we have seen, the group $\mathrm{Sp}(g,\mathbb{F}_2)$ acts one the space S_g. The linear map

$$\mathrm{Ind}_K^G(\varepsilon_K) \to S_g, \qquad Y(m) \to S(\vartheta[m]),$$

is G-linear. It is surjective. Hence it is an isomorphism of one of the two irreducible complements maps isomorphically. This can be obtained also as consequence of Theorem 2.3 in [2]. We recall that this paper is written in the orthogonal language. We will show that the second case holds. For this we have to make use of Fay's theta relations. A special case of [1, (23)] says for $z_1 - z_2 = z_3 - z_4$ and $w \in W^-$

$$\sum_{m \text{ odd}} w_m \vartheta[m](\tau, z_1)\vartheta[m](\tau, z_2)\vartheta[m](\tau, z_3)\vartheta[m](\tau, z_4) = 0$$

or

$$\sum_{m \text{ odd}} w_m \vartheta[m](\tau, z_1)\vartheta[m](\tau, z_2)\vartheta[m](\tau, z_3)\vartheta[m](\tau, -z_1 + z_2 + z_3) = 0$$

with three independent variables z_1, z_2, z_3. We take the gradient with respect to the variable z_3 and take then $z_3 = 0$. Since $\vartheta[m](\tau, 0) = 0$ for odd m, we obtain

$$\sum_{m \text{ odd}} w_m \vartheta[m](\tau, z_1)\vartheta[m](\tau, z_2)\vartheta[m](\tau, -z_1 + z_2) J(\vartheta[m])(\tau) = 0.$$

Taking the z_2-gradient gives

$$\sum_{m \text{ odd}} w_m \vartheta[m](\tau, z_1)(\vartheta[m](\tau, -z_1) J(\vartheta[m](\tau)) \otimes J(\vartheta[m])(\tau) = 0.$$

The theta functions $\vartheta[m](\tau, z)$ are invariant under $z \mapsto -z$ up to a sign. Now differentiating twice by z_1 and then setting $z_1 = 0$ gives

$$\sum_{m \text{ odd}} w_m S(\vartheta[m]) = 0.$$

So we get non trivial linear relations between the $S(\vartheta[m])$.

Theorem 5.4. *The dimension of the space spanned by the tensors $S(\vartheta[m])$ has dimension $(2^{2g}-1)/3$.*

6. Arbitrary powers of the thetanullwerte

The question arises whether one can get similar results for the space $T_g(k)$ generated by kth powers of thetanullwerte where k is a natural number (so $T_g = T_g(4)$).

We start with the case $k \equiv 4 \mod 8$. In this case we have a natural surjective G-linear map

$$\mathrm{Ind}_H^G(\varepsilon_H) \to T_g(k)$$

that generalizes that described in Sect. 3.

Theorem 6.1. *Assume $k \equiv 4 \mod 8$, $k \neq 4$. The map*

$$\mathrm{Ind}_H^G(\varepsilon_H) \to T_g(k)$$

is an isomorphism. Hence $T_g(k)$ is the direct sum of the two irreducible representations of dimensions

$$(2^{2g} - 1)/3, \qquad (2^g + 1)(2^{g-1} + 1)/3.$$

Proof. We have to show that none of the two irreducible components goes to 0. Actually we find in each of the two irreducible components one element that does not go to 0. We use the description of V^+, W^+ as in Theorem 3.1. Clearly

$$(2^g - 1)X(0)^k - \sum_{m \neq 0} X(m)^k \in V^+,$$

$$(2^g + 1)X(0)^k + \sum_{m \neq 0} X(m)^k \in W^+.$$

We want to show that their images are not 0. This means

$$(2^g - 1)\vartheta[0]^k - \sum_{m \neq 0} \vartheta[m]^k \neq 0,$$

$$(2^g + 1)\vartheta[0]^k + \sum_{m \neq 0} \vartheta[m]^k \neq 0.$$

We apply to the two functions the Siegel Φ-operator $g - 1$ times, cf. [4, page 203]. In the first case one gets

$$2^{g-1}(\vartheta \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^k - \vartheta \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^k - \vartheta \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^k)$$

This is different from zero for $k \neq 4$ since

$$\vartheta \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]^4 - \vartheta \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^4 - \vartheta \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^4 = 0$$

is the defining relation between the three thetanullwerte in genus 1. The second case is similar. □

Next we consider the case $k \equiv 0 \mod 8$. Here we have an obvious surjective G-invariant map

$$\mathrm{Ind}_H^G(1) \to T_g(k)$$

where 1 denotes the trivial character of H. The same argument as in the case of the nontrivial character ε shows that this representation has at most two irreducible components. One is the trivial one dimensional representation consisting of all constant functions $f\colon H \to \mathbb{C}$. So we have two irreducible components, one of dimension 1, the other of dimension $k_g^+ - 1$. The representation of dimension $k_g^+ - 1$ cannot map to zero, since the dimension of $T_g(k)$ is greater than one. The function $\sum_m \vartheta[m]^k$ is non-zero and is invariant under G. Hence we obtain the following result.

Theorem 6.2. *Assume $k \equiv 0 \mod 8$. Then the map*

$$\mathrm{Ind}_H^G(1) \to T_g(k)$$

is an isomorphism. As a consequence $T_g(k)$ decomposes under the action of G into two irreducible components of dimension

$$1, \quad 2^{g-1}(2^g+1)-1.$$

Finally we treat the case that k is not divisible by 4. Since we want to include also odd k we have to replace Γ_g by its two fold metaplectic cover $\tilde{\Gamma}_g$. Recall that this is the set of all pairs $(\sigma, \sqrt{\det(C\tau+D)})$, $\sigma \in \Gamma_g$, where $\sqrt{\det(C\tau+D)}$ is one of the holomorphic square roots of $\det(C\tau+D)$. We set for a function $f\colon \mathcal{H}_g \to \mathbb{C}$

$$f \mid (\sigma, \sqrt{\det(C\tau+D)}) = \sqrt{\det(C\tau+D)}^{\,-k} f(\sigma\tau).$$

Then $f \mapsto f \mid \sigma^{-1}$ gives a representation of $\tilde{\Gamma}_g$ on $T_g(k)$. If k is even, it factors through Γ_g and through G if k is divisible by 4. We denote by $\tilde{\Gamma}_g[2]$ the inverse image of $\Gamma_g[2]$ in $\tilde{\Gamma}_g[2]$. The group $\tilde{\Gamma}_g$ and hence $\tilde{\Gamma}_g[2]$ act on the space $T_g(k)$.

Lemma 6.3. *Assume that 4 does not divide k. The space $T_g(k)$ decomposes under $\tilde{\Gamma}_g[2]$ into the one dimensional spaces $\mathbb{C}\vartheta[m]^k$ which are pairwise non-isomorphic (their characters are pairwise different).*

Proof. We know that the $\vartheta[m]^k$ are modular forms on $\tilde{\Gamma}_g[2]$ with respect to some character v_m. We have to show that the characters v_m, v_n for $m \neq n$ are different. Since Γ_g acts doubly transitive, we can assume $m = 0$ and $n = \binom{a}{0}$, $a \neq 0$. One checks that v_m and v_n disagree on translations matrices $\left(\begin{smallmatrix} E & 2S \\ 0 & E \end{smallmatrix}\right)$. □

Theorem 6.4. *Assume that 4 does not divide k. Then the space $T_g(k)$ is irreducible under the action of $\tilde{\Gamma}_g$. Its dimension is*

$$k_g^+ = 2^{g-1}(2^g+1).$$

Proof. Let $V \subset T_g(k)$ be a non-zero invariant subspace. Under the action of $\tilde{\Gamma}_g[2]$ it decomposes into certain sums of the one-dimensional spaces $\mathbb{C}\vartheta[m]$. The group $\tilde{\Gamma}$ acts transitively on the even characteristics m. Hence $V = T_g(k)$. □

An immediate consequence of Theorems 6.1, 6.2, 6.4 is the following result.

Corollary 6.5. *The thetanullwerte $\vartheta[m]^k$ are linear independent with one exception, $k = 4$.*

Proof. For all $k \neq 4$, the statement is an immediate consequence of the fact that the even thetanullwerte are k_g^+. When $k = 4$ we recall that we have relations of the form

$$\sum_{m \in (\mathbb{F}_2^{2g})_{\text{even}}} v(m)\vartheta[m]^4 = 0.$$

Here v is an eigenvector of M^+ with eigenvalue -2^{g-1}. $\square$

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