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# Shear viscosity in QFTs dual to AdS spherical black holes

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**Abstract.** In the framework of the AdS/CFT correspondence, we define and compute the spherical analogue of the shear viscosity for QFTs dual to five-dimensional charged AdS black holes in general relativity (GR) and Gauss-Bonnet (GB) gravity. We show that the ratio between this quantity and the entropy density,  $\tilde{\eta}/s$ , exhibits a temperature-dependent hysteresis.

## 1. Introduction

Nowadays, holography plays a central role in theoretical gravitational physics and has produced several successful applications as, for example, the AdS/CFT correspondence. Moreover it gives insights on quantum gravity and it is a powerful tool for the description of phase transitions and the computation of transport coefficients in strongly coupled QFTs [1–5] in the hydrodynamic limit. In particular, it has been proposed that the shear viscosity to entropy density ratio  $\eta/s$  satisfies a fundamental bound  $\eta/s \geq 1/4\pi$ , known as the KSS bound [1], which found support both from string theory [2] and quark-gluon plasma experimental data [6]. By now, it is well-known that the KSS bound is violated by higher curvature terms in the Einstein-Hilbert action [7] or by breaking of translational or rotational symmetry of the black brane background [8–12]. Typically, when the KSS bound is violated,  $\eta/s$  exhibits a non-trivial dependence on the temperature [13].

Until now, these investigations have been restricted to planar topologies in the bulk (black branes) and have not concerned spherical topologies (black holes). The main obstruction to this generalisation is the absence of the usual hydrodynamic limit for QFTs dual to spherical black holes (BHs). Indeed, differently from the black brane case, the spherical geometry of the horizon breaks the translational symmetry in the dual QFT preventing the existence of conserved charges. However, it is still possible to define a relativistic hydrodynamics in curved spacetimes without translational symmetry as an expansion in the derivatives of the hydrodynamic fields of the stress-energy tensor [14] and a related Kubo formula for the shear viscosity.

In this work we discuss whether it is possible to use transport coefficients of the dual QFT to learn about the complicated thermodynamical phase portrait of BHs, in particular five-dimensional AdS BHs coupled to an electromagnetic field in GR [15] and GB gravity [16].

## 2. Hydrodynamics in curved spacetimes

The hydrodynamic limit of a QFT living on a curved spacetime can be defined in the same way as for a QFT in the plane. We just consider the system at large relaxation times (small



frequencies) and large scales compared to the microscopic scale of the system. When the latter is unknown, we can still give a thermal description of the system and associate this microscopic scale with the inverse of the temperature  $T$ . Thus, the hydrodynamic limit corresponds to consider excitations of the system with wavelength  $\lambda \gg 1/T$ . In this limit, the macroscopic behaviour of the QFT living in a curved background is described by a stress-energy tensor, which can be written as [14, 17]

$$T^{ab} = (\epsilon + P) u^a u^b + P g^{ab} + \Pi^{ab}, \quad (1)$$

where  $\epsilon$  and  $P$  are the energy density and the thermodynamical pressure and  $u^a$  is the fluid velocity, usually considered in the frame in which the fluid is at rest. The tensor  $\Pi^{ab}$  contains all the dissipative contributions to the stress-energy tensor. At first order in the velocity expansion, it depends on three transport coefficients  $\kappa$ ,  $\tau_\Pi$  and  $\eta$ , the latter known as the shear viscosity.

The previous considerations hold for a QFT in a generic curved space. Working in the AdS/CFT framework, we can apply Eq. (1) to a four dimensional CFT dual to a five-dimensional AdS bulk spherical BH [18–20]. To derive a Kubo formula for CFTs living on the boundary of AdS<sub>5</sub>, whose spatial section is the three-sphere, we consider small perturbations around the boundary background metric, i.e.  $g_{ab} = \bar{g}_{ab} + h_{ab}$ . In general we can consider three different types of perturbations: shear, sound and transverse (scalar) modes. The behaviour of these modes will be encoded in three different correlators  $G_{1,2,3}(\omega, k)$ , where  $k$  is the momentum. In the translationally invariant case (and also when translation invariance is broken by external matter fields) at  $k = 0$  these three functions are equal, owing to rotational symmetry [7]. By contrast, in the spherical case under consideration, the momentum cannot be taken to zero by construction (see later for details) and the correlators will be different. Thus, in general, any definition of the shear viscosity in a spherical background based on linear response to small disturbances will be channel-dependent [21]. In this work we will focus on the transverse (and traceless) perturbations. The computations for the sound and shear channel is left for future investigations. Under these assumptions and in linear approximation, Eq. (1) becomes

$$T^{ij} = -P h_{ij} - \eta \dot{h}_{ij} + \eta \tau_\Pi \ddot{h}_{ij} - \frac{\kappa}{2} (\ddot{h}_{ij} + L^2 \Delta_L h_{ij}), \quad (2)$$

where  $\Delta_L$  is the Lichnerowicz operator and  $L$  is the AdS<sub>5</sub> length. We choose a harmonic time dependence for the perturbation and we expand it in hyperspherical harmonics. We now extract the retarded Green function for the spatial components of the stress-energy tensor  $T^{ij}$  in the tensor channel and, from Eq. (2), we read

$$G_{T^{ij}T^{ij}}^R(\omega, \ell) = -P - i\omega\eta - \omega^2\eta\tau_\Pi - \frac{\kappa}{2} (\omega^2 + L^2\gamma), \quad (3)$$

where  $\gamma \equiv \ell(\ell + 2) - 2$  is the eigenvalue of the Lichnerowicz operator and  $\ell = 1, 2, 3, \dots$  is the first number associated with the hyperspherical harmonic expansion. Equation (3) allows us to derive a Kubo formula for the analogue of the shear viscosity  $\tilde{\eta}$  for a relativistic QFT on a spatial spherical background as

$$\tilde{\eta} = - \lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im} G_{T^{ij}T^{ij}}^R(\omega, \ell \rightarrow \ell_0), \quad (4)$$

where  $\ell_0$  is the smallest eigenvalue of the Lichnerowicz operator and  $\omega$  is the frequency of the perturbation. Notice that the only difference of Eq. (4) with the planar case is the evaluation of the retarded Green function in  $\ell \rightarrow \ell_0$  instead of wavenumber  $k \rightarrow 0$ .

It is important to stress that, with respect to the planar case, we have an additional contribution to the stress-energy tensor (2) ruled by the transport coefficient  $\kappa$ . This is rather expected in view of the breaking of translational invariance. However, these additional contributions do not contribute to the shear viscosity.

### 3. Black hole solutions in five dimensions

We consider static and spherically symmetric five-dimensional BHs,

$$ds^2 = g_{ab}^{(0)} dx^a dx^b = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\bar{\Omega}_3^2, \quad (5)$$

where  $d\bar{\Omega}_3^2$  is the line element of the 3-sphere. For the AdS-RN BHs the metric function is

$$f_{\text{RN}}(r) = 1 + \frac{r^2}{L^2} - \frac{8G_5 M}{3\pi r^2} + \frac{4\pi G_5 Q^2}{3r^4}, \quad (6)$$

while, in the branch that allows for BH solutions, the metric function for GB gravity is [22–24]

$$f_{\text{GB}}(r) = 1 + \frac{r^2}{2\lambda L^2} \left[ 1 - \sqrt{1 - 4\lambda L^2 \left( \frac{1}{L^2} - \frac{8G_5 M}{3\pi r^4} + \frac{4\pi G_5 Q^2}{3r^6} \right)} \right]. \quad (7)$$

In Eqs. (6) and (7),  $L$  is the AdS length,  $M$  and  $Q$  are, respectively, the BH mass and charge. To have asymptotically AdS-GB BH solutions, the dimensionless GB coupling constant  $\lambda$  must be smaller than  $1/4$ . Its value is also constrained by the unitarity bounds for the dual QFT [7, 25, 26], so that in this work we consider the range  $0 < \lambda \leq 9/100$ .

The temperature of these BHs can be expressed in terms of the horizon radius  $r_+$

$$T(r_+) = \left( \frac{r_+}{\pi L^2} + \frac{1}{2\pi r_+} - \frac{2G_5 Q^2}{3r_+^5} \right) \left( 1 + \frac{2\lambda L^2}{r_+^2} \right)^{-1}. \quad (8)$$

For RN BHs, as the charge of the BH decreases to the critical charge  $Q_c = L^2/6\sqrt{5\pi}$ , the BH undergoes a second-order phase transition. Below  $Q_c$  the system is characterised by the presence of two stable states (small and large BHs) connected through a meta-stable region of intermediate BHs. The phase transition small/large BHs is a first-order one [27]. The picture is analogous for GB BHs [28, 29]. This is very similar to a Van der Waals-like fluid behaviour with second-order phase transition controlled by a critical parameter ( $\lambda$  and/or  $Q$  for GB BHs,  $Q$  for RN BHs) [27, 30], first-order one (controlled by  $T$ ) and metastabilities (small/large BH region separated by metastable region of intermediate BHs).

### 4. Linear perturbations

Following the rules of the AdS/CFT correspondence, to compute the spherical analogue viscosity to entropy density ratio  $\tilde{\eta}/s$  for the QFT dual to a five-dimensional spherically symmetric charged AdS BH, we consider transverse and traceless perturbations about the background metric (5),  $g_{ab} = g_{ab}^{(0)} + h_{ab}$ . In particular,  $h_{ab} = 0$  unless  $(a, b) = (i, j)$  and  $h_{ij}(r, t, x) = r^2 \phi(r, t) \bar{h}_{ij}(x)$  being  $\bar{h}_{ij}$  the eigentensor of the Lichnerowicz operator built on the background 3-sphere, whose eigenvalues are  $\ell(\ell + 2) - 2$  with  $\ell = 1, \dots$ . Such perturbations are gauge-invariant and by linearising the Einstein field equations, the angular part decouples [31, 32]. By assuming a harmonic time dependence for the perturbation,  $\phi(r, t) = \psi(r) e^{-i\omega t}$ , one finds the linear second-order differential equation for  $\psi(r)$

$$\frac{1}{r^3} \frac{d}{dr} \left[ r^3 f(r) F(r) \frac{d\psi(r)}{dr} \right] + \left[ \frac{F(r)}{f(r)} \omega^2 - m^2(r) \right] \psi(r) = 0, \quad (9)$$

where  $f(r)$  is given by Eq. (6) for AdS-RN BHs or Eq. (7) for GB BHs,  $F(r) \equiv 1 - \lambda L^2 f'(r)/r$ . The mass term for the perturbation is  $m^2(r) = (4 - \ell(\ell + 2)) (1 - \lambda L^2 f''(r))/r^2$  and  $\ell$  are the

eigenvalues of the Lichnerowicz operator on the 3-sphere. The presence of a non-vanishing mass term in Eq. (9) is a consequence of the breaking of translational symmetry due to the spherical geometry of the horizon. The general solution of Eq. (9) does not exist in analytical form, but in the  $r \rightarrow \infty$  limit, Eq. (9) with  $\omega = 0$  admits as solutions a non-normalisable and a normalisable mode that behave asymptotically as

$$\psi_0(r) = 1 - \frac{\lambda L^2}{2(1 - \sqrt{1 - 4\lambda})r^2} + \mathcal{O}(\log r/r^4), \quad \psi_1(r) = \frac{1}{r^4} + \mathcal{O}(1/r^6). \quad (10)$$

The near-horizon behaviour of the non-normalisable mode is different for non-extremal and extremal BHs. For the former case, it is given by a power-series expansion, i.e.  $\psi_0(r) = \psi_0(r_+) + \mathcal{O}(r - r_+)$ , while in the latter  $\psi_0(r) = (r - r_+)^\nu$  being  $\nu$  an appropriate index.

### 5. The shear viscosity to entropy density ratio

The retarded Green function in Eq. (4) can be found using a modified version of the method proposed in Refs. [9,33] to include higher-order curvature corrections. The method gives a very simple and elegant way for computing correlators in a QFT dual to a gravitational bulk theory. The analogue shear viscosity to density entropy ratio is determined by the non-normalisable mode  $\psi_0(r)$  evaluated at the horizon times a function of  $\lambda$

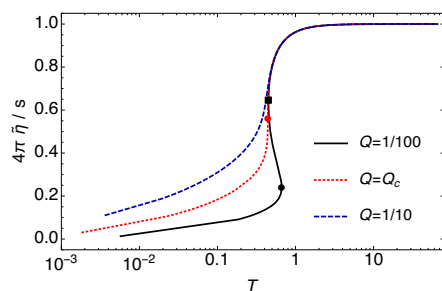
$$\frac{\tilde{\eta}}{s} = \frac{1}{4\pi} \psi_0(r_+)^2 \left[ 1 - 4\lambda \left( 1 - \frac{2\pi G_5 Q^2 L^2}{3r_+^6} \right) \right] \left( 1 + \frac{6\lambda L^2}{r_+^2} \right)^{-1}. \quad (11)$$

In the large temperature regime, by inverting Eq. (8) and using Eq. (10), Eq. (11) becomes

$$\frac{\tilde{\eta}}{s} = \frac{1 - 4\lambda}{4\pi} \left[ 1 - \frac{\lambda L^2 (7 - 6\sqrt{1 - 4\lambda})}{\pi^2 (1 - \sqrt{1 - 4\lambda}) L^4 T^2} + \mathcal{O}(1/T^4) \right]. \quad (12)$$

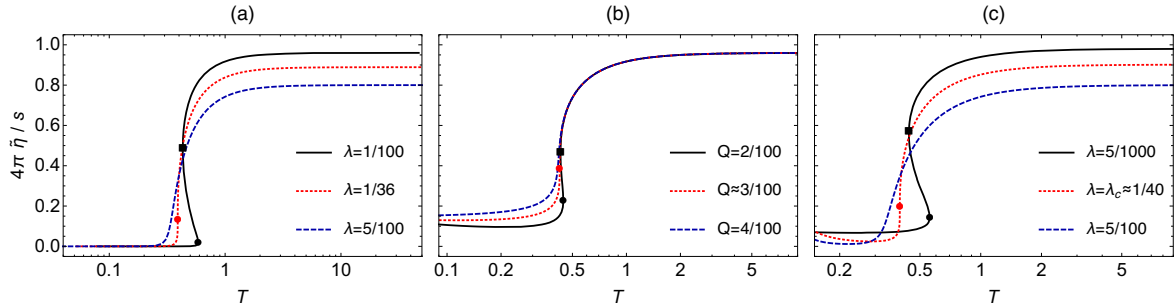
In the  $T \rightarrow \infty$  limit, the value of  $\tilde{\eta}/s$  tends to  $(1 - 4\lambda)/4\pi$  which reduces to the KSS bound for  $\lambda = 0$ . These bounds are in general violated as Eq. (12) is a decreasing function of the temperature. In the extremal case, the metric function and its first derivative vanish when evaluated on the horizon as well as the non-normalisable mode  $\psi_0(r_+)$ . This means that  $\tilde{\eta}/s$  goes to zero in the  $T = 0$  extremal limit as  $\tilde{\eta}/s \sim T^{2\nu}$ .

The global behaviour of  $\tilde{\eta}/s$  as a function of  $T$  is obtained as follows. For each value of the charge and the GB parameter, there exists a minimum mass— and hence a minimum radius. Then we numerically integrate Eq. (9) with  $\omega = 0$  outwards from the horizon to infinity supplied with a power-series boundary condition for  $\psi_0(r)$ . Next, we use a shooting method to determine  $\psi_0(r_+)$  by requiring that  $\psi_0(\infty) = 1$ . Finally, the temperature and  $\tilde{\eta}/s$  for each solution are computed with Eqs. (8) and (11). Units  $G_5 = L = 1$  are adopted.



**Figure 1.** Behaviour of  $\tilde{\eta}/s$  as a function of the temperature for dual QFTs of AdS-RN BHs. We plot  $\tilde{\eta}/s$  for selected values of the BH charge above, at and below the critical value. The dots and squares mark the critical temperatures relative to the small/large BH first-order phase transition. We have considered the smallest eigenvalue of the Lichnerowicz operator, i.e.  $\ell = \ell_0 = 1$ .

In Fig. 1 we plot our results for  $\tilde{\eta}/s$  as a function of the temperature for QFTs dual to AdS-RN BHs for selected values of the charge  $Q$  and we observe that, for  $Q < Q_c$ , it exhibits a temperature-dependent hysteresis, after that the second-order Van der Waals-like phase transition occurs. In Fig. 2 we plot our numerical results for  $\tilde{\eta}/s$  as a function of  $T$  for QFTs dual to AdS-GB BHs. We show three different cases: neutral BHs; fixed  $\lambda$ ; fixed charge  $Q$ .



**Figure 2.** Behaviour of  $\tilde{\eta}/s$  as a function of the temperature for GB BHs with (a)  $Q = 0$  for selected values of the GB coupling constant above, at and below the critical value; (b)  $\lambda = 1/100$ , and selected values of charge above, at and below the critical value; (c)  $Q = 1/100$ , and selected values of GB constant above, at and below the critical value. Dots (squares) mark the maximum (minimum) of the temperature as a function of the BH horizon.

In the hydrodynamic, holographic context, a hysteretic behaviour in the shear viscosity has been already observed for AdS black branes with broken rotational symmetry and with a p-wave holographic superfluid dual [8]. Moreover, it is known that real fluids may exhibit hysteresis in the  $\eta$ - $T$  plane, this is, for instance, the case of nanofluids [34].

The mechanism that generates hysteresis in  $\tilde{\eta}/s$  is the same that is responsible for the phase transition and can be traced back to non-equilibrium thermodynamics. An unstable (meta-stable) region of intermediate BHs connects two stable regions of large and small BHs. A potential barrier prevents the evolution of the system from occurring as an equilibrium path between the two stable states [35, 36]. Equilibrium will be reached passing through a meta-stable region and a path-dependence of  $\tilde{\eta}/s$  is generated.

Notice that when the breaking of translational symmetry is generated by external fields, the symmetry may be restored or not when the system flows to the IR (as in the black brane case) [9]. In the BH case instead, because the breaking has a geometric and topological origin, translational symmetry cannot be restored in the IR.

## 6. Conclusion

We have used the AdS/CFT correspondence to obtain information about the behaviour of bulk BHs by studying the hydrodynamic properties of the dual QFTs. In particular, we have focussed on the scalar channel and defined and computed the shear viscosity to entropy ratio for QFTs holographically dual to five-dimensional AdS BH solutions of GR and GB gravity.

Our most important result is the behavior of  $\tilde{\eta}/s$  at intermediate temperatures: for AdS-RN BHs a second-order Van der Waals phase transition occurs as the system goes from large values of  $Q$  to the critical one; below the critical charge the BH undergoes a temperature-dependent first-order phase transition and  $\tilde{\eta}/s$  develops hysteresis. A similar behaviour occurs for GB gravity when we fix the charge and let the GB coupling constant  $\lambda$  to vary and viceversa.

Our definition of  $\tilde{\eta}/s$  is channel dependent. In general we have three different determinations of  $\tilde{\eta}$  for shear, sound and transverse (scalar) perturbations. In this work we have focused on



the transverse ones. However our results suggest that, for QFTs dual to bulk spherical BHs, the hydrodynamical, long wavelength modes can be described by the  $\ell \rightarrow \ell_0$  modes that probe large angles on the sphere (cfr.  $k \rightarrow 0$  on the plane). Due to the spherical topology, the hydrodynamic interpretation in terms of conserved quantities fails. However, we can still define the shear viscosity through a Kubo formula where the stress-energy tensor is only covariantly conserved and interpret it as the rate of entropy production due to a strain.

In holographic models, the shear viscosity to entropy ratio of the QFT is closely related and keeps detailed information about the thermodynamical phase structure of the dual BH background.

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