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CONTACT INTERACTIONS FOR MANY-PARTICLE QUANTUM SYSTEMS IN DIMENSION THREE

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«One, remember to look up at the stars and not down at your feet.
Two, never give up work. Work gives you meaning and purpose and life is empty without it.
Three, if you are lucky enough to find love, remember it is there and don't throw it away.»

Stephen Hawking

Daniele Ferretti

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Introduction

Hamiltonians with *zero-range interactions* (also known as contact or point interactions) are often used in Quantum Mechanics as toy models to describe low energy behaviors of a (non-relativistic) particle system. The advantage in exploiting this kind of interaction is due to the simplicity of its structure, which allows in many cases to explicitly compute the quantities of main interest. At least formally, the two-body zero-range interaction is characterized by a single physical parameter known as two-body *scattering length* $\mathfrak{a} \in \mathbb{R}$ defined by

$$\mathfrak{a} := -\lim_{|\boldsymbol{k}|\to 0} f_0(\boldsymbol{k}).$$

Here $\boldsymbol{k}$ denotes the wave number and f_0 is the s-wave[1] scattering amplitude (associated with angular momentum $\ell = 0$) of the two-body scattering process, which corresponds to an isotropic differential cross section, i.e. $|f_0(\boldsymbol{k})|^2 = |f_0(|\boldsymbol{k}|)|^2$. It is well known that for a quantum gas at low temperature, the average thermal wavelength (behaving as $T^{-\frac{1}{2}}$ at low temperatures T for massive particles) is much larger than the typical range of a pairwise short-range interaction. In this situation *universal behaviors* are very likely to appear, namely, some phenomenon could rise with the peculiarity of not being dependent on the specific structure of the two-body interaction but only on the parameter $\mathfrak{a}$. Indeed, since the details of the two-body potential cannot be resolved, it is reasonable to expect that a n-particle system in dimension d is effectively described by the following formal Hamiltonian

$$\tilde{\mathcal{H}} = -\sum_{i=1}^{n} \frac{\hbar^2}{2m_i}\Delta_{\boldsymbol{x}_i} + \sum_{1\leq i<j\leq n} \nu_{ij}\, \delta(\boldsymbol{x}_i - \boldsymbol{x}_j)$$

where $\hbar$ stands for the reduced Planck's constant (we set $\hbar = 1$ henceforth), $\boldsymbol{x}_i \in \mathbb{R}^d$ represents the position of the i-th particle and m_i its mass, while ν_{ij} is a coupling constant related to the two-body scattering length associated with the subsystem composed of the particles i and j. These kind of Hamiltonians are widely used in the physical literature in describing several low energy phenomena ranging from nuclear physics to condensed matter physics. We list here some important examples.

- In a three-body system, the first theoretical analysis of the *Efimov effect* [14], that is, roughly speaking, the emergence of an infinite sequence of three-body bound states accumulating

[1]In the fermionic case the low energy behavior is described by a scattering process in the p-wave, i.e. $\ell = 1$ and the differential cross section generally depends on the scattering angle.

at the threshold of the essential spectrum, was carried out adopting (formal) zero-range interactions and a suitable three-body regularization.

- The *Kronig–Penney model* describes an electron in a one-dimensional periodic lattice of positive ions, whose potential can be represented by a delta-periodic pseudo-potential [34].
- The first derivation of the *Lee-Huang-Yang formula* [23] for the ground state energy per unit volume of a diluted Bose gas was obtained using (formal) zero-range interactions.

However, the mathematical construction of such Hamiltonians as self-adjoint (s.a.) and, possibly, lower bounded operators in a proper Hilbert space requires some care.
In general, they are constructed as s.a. extensions of the free Hamiltonian restricted on the space of H^2-functions vanishing on the coincidence hyperplanes, namely those hyperplanes in which the contact interaction occurs

$$\pi := \bigcup_{1\leq i<j\leq n} \{(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)\in\mathbb{R}^{dn} \mid \boldsymbol{x}_i=\boldsymbol{x}_j\}.$$

STATE OF THE ART

In the case of a particle scattering against a fixed point through a zero-range interaction, namely the one-body case ($n=1$), a complete theory is available ([1]). As an example, since in this thesis we shall focus on dimension three, let us consider a system composed of two (distinguishable) spinless particles interacting via a zero-range interaction in $\mathbb{R}^3$. Clearly, this represents a one-body problem with scattering point at the origin, once the center of mass reference frame is adopted. The free Hamiltonian is simply given by

$$h=-\frac{1}{2\mu}\Delta_{\boldsymbol{r}}\,,\qquad \mu=\frac{m_1m_2}{m_1+m_2}$$

with $\boldsymbol{r}$ the relative coordinate and μ the reduced mass. The restriction of h on the domain of smooth functions vanishing in the origin has defect indices $(1,1)$ and a one-parameter family of s.a. extensions $\{h_\alpha\}_{\alpha\in\mathbb{R}}$ can be explicitly constructed. In particular h_α all act the same as h in $H^2_0(\mathbb{R}^3\smallsetminus\{\mathbf{0}\})$ and $\psi\in\mathscr{D}(h_\alpha)\subset H^2(\mathbb{R}^3\smallsetminus\{\mathbf{0}\})$ satisfies the following singular boundary condition

$$\psi(\boldsymbol{r})=\frac{q}{|\boldsymbol{r}|}+\alpha q+o(1)\,,\qquad |\boldsymbol{r}|\longrightarrow 0^+$$

where $q\in\mathbb{C}$ is a constant depending on the specific choice of ψ and α is related to the two-body scattering length $\mathfrak{a}$ via the relation

$$\mathfrak{a}=-\frac{1}{\alpha}\,.$$

Notice that the strength of the point interaction goes to zero if $|\alpha|\to+\infty$. In the simple case of h_α all the spectral properties can be characterized.
The situation is in general much more difficult in the n-body case, since the dimension of the

aforementioned defect space becomes infinite and the dimensionality of the space d plays a crucial role.
In particular, the case $d = 1$ is relatively simple, since perturbation theory applies and the model is well understood (see, e.g., [4], [22] for recent contributions).
In dimension two, using a more thorough version of the strategy adopted in the one-body case, the s.a. and bounded from below Hamiltonian can be constructed (see [11], [13]) and analysed in detail (for instance, in [21] the zero-range model is achieved as an approximation of Schrödinger operators with suitably rescaled potentials in the norm-resolvent sense).
In dimension three new difficulties come to light concerning the energetic stability of the system. Let us consider for simplicity the case $n = 3$ for three identical spinless bosons of mass $\frac{1}{2}$. Adopting the *Jacobi coordinates* (see (1.1) for more details) in the center of mass reference frame, the free Hamiltonian is given by

$$H = -\Delta_{\boldsymbol{x}} - \frac{3}{4}\Delta_{\boldsymbol{y}}\,.$$

The problem consists in defining a rigorous Hamiltonian encoding a contact interaction as a perturbation of H supported on the coincidence hyperplanes π. However, as already mentioned above, the defect space of the restriction of H on the space of regular functions vanishing on π is infinite-dimensional. This means that one has to choose properly the s.a. extension so that the right physical properties are satisfied. A first attempt, based on the analogy with the one-body case, has been made by Ter-Martirosyan and Skornyakov in [35], where they defined a Hamiltonian H_α acting the same as H outside the coincidence hyperplanes and whose domain fulfills the so-called Ter-Martirosyan Skornyakov (TMS) boundary condition

$$\psi(\boldsymbol{x},\boldsymbol{y}) = \frac{\xi(\boldsymbol{y})}{|\boldsymbol{x}|} + \alpha\,\xi(\boldsymbol{y}) + o(1), \qquad \text{for } |\boldsymbol{x}| \longrightarrow 0^+ \text{ and } \boldsymbol{y} \neq \boldsymbol{0}$$

where ξ is a function depending on $\psi \in \mathscr{D}(H_\alpha)$. Unfortunately, as first observed by Danilov [10] and then rigorously analysed by Minlos and Faddeev ([26, 27]), the operators H_α dictated by imposing the TMS boundary condition turn out to be symmetric but not s.a. and all their s.a. extensions need to take into account an additional boundary condition associated with the *triple coincidence point* $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{0}$. However, all these s.a. extensions are unbounded from below, hence the Hamiltonians H_α defined in [26] are unsatisfactory from the physical point of view. Such instability property is known in physical literature as *Thomas effect*, or Thomas collapse, and it refers to the situation in which one has an infinite sequence of bound states whose energy accumulates at $-\infty$. In particular, this fact has been proved in [27] for the TMS Hamiltonian together with the presence of the Efimov effect. It is known, since the '30s when the paper [36] was published, that the aforementioned ultraviolet singularity is due to the fact that the interaction becomes too strongly attractive when all the three particles coincide (see also the recent papers [25], [19] and the references therein) unless the system is entirely composed of (half-integer spin) fermions, in which case the anti-symmetry provides stability[2].

[2]The situation is different for systems made of two species of (spinless) fermions, for which the TMS Hamiltonian turns out to be s.a. and bounded from below for certain regimes of the mass ratio (see, e.g., [8, 9], [17], [29, 30]).

Nevertheless, Minlos and Faddeev suggested at the end of [26] how to construct a regularized version of the Hamiltonian for a system of three bosons. Roughly speaking, the idea is to introduce an effective scattering length which vanishes when the position of two particles coincides and the third particle is getting closer to the common position of the first two, whereas the usual two-body point interaction is restored as soon as this third particle is far enough. In this sense, one introduces a three-body repulsion that reduces to zero the strength of the interaction between two particles "only" when the third particle is getting closer (actually it turns out that the range of such a three-body repulsion can be chosen arbitrarily short). More precisely, Minlos and Faddeev claimed that a lower bounded Hamiltonian can be obtained by replacing $\alpha \longmapsto \alpha_M$ in the TMS boundary condition with

$$(\alpha_M \hat{\xi})(\boldsymbol{p}) := \alpha \hat{\xi}(\boldsymbol{p}) + (\mathcal{K}\hat{\xi})(\boldsymbol{p})$$

where $\mathcal{K}$ is a given convolution operator with kernel $K(\boldsymbol{p} - \boldsymbol{p}')$ having the asymptotic behavior

$$K(\boldsymbol{q}) \sim \frac{\gamma}{|\boldsymbol{q}|^2}, \qquad \text{for } |\boldsymbol{q}| \longrightarrow +\infty$$

provided γ large enough. Unfortunately, the proof of this statement has never been disclosed by the authors. Later, in the early '80s, Albeverio et al. proposed at the end of [3] another way of constructing a lower semi-bounded Hamiltonian encoding a contact interaction, again replacing α with a new position-dependent parameter given by

$$\alpha_A(\boldsymbol{y}) = \frac{\delta}{|\boldsymbol{y}|} + \mathcal{O}(1)\,, \qquad \text{for } \boldsymbol{y} \longrightarrow \boldsymbol{0}$$

with $\delta > 0$ to be chosen large enough. Also in this case a proof of this claim is lacking since they postponed the discussion of this problem in a forthcoming paper that has never been published. However, we can observe that comparing the idea introduced by Minlos and Faddeev with theirs, the former is essentially the Fourier transform of the latter. Nevertheless, in the position-space representation the heuristic meaning of the replacement is more intuitive. Indeed, as the third particle approaches the other two interacting particles we have $\boldsymbol{y} \longrightarrow \boldsymbol{0}$ and the effective scattering length, hence the interaction, is vanishing in this limit.

The suggestion made by Minlos and Faddeev was finally developed (taking account of the the effective scattering length in its position representation) by some recent works ([18], [25, section 9], [5], [19, section 6]) dealing with a three-boson system with zero-range interactions. The common result is that the regularization needed to heal the ultraviolet singularity consists in a sort of renormalization of the coupling constant. More precisely, the "bare" constant α, henceforth denoted by $\alpha_0 = -\frac{1}{\mathfrak{a}}$ in order to avoid confusion, is replaced with a sort of "running coupling constant", which is a function depending on the position of the particles. Such a function is characterized by the asymptotic behavior pointed out by Albeverio et al., with a proportionality constant γ, to which we refer to as the strength of the regularization. It turns out that γ must be larger than a certain threshold parameter γ_c in order to obtain a bounded from below Hamiltonian.

It is worth to mention that Albeverio et al. proposed in [2] an alternative method to construct lower bounded zero-range n-body Hamiltonians in dimension three based on the theory of *Dirichlet forms*. The method is relatively simple and allows one to define a s.a. extension of the free Hamiltonian restricted to smooth functions vanishing on the coincidence hyperplanes, thus, a n-body Hamiltonian with contact interactions. Such a method has also the advantage of providing the infimum of the spectrum as a preassigned non-positive value. Still, the construction of this Hamiltonian is rather implicit and it is not clear from their analysis which boundary condition the elements of the operator domain should satisfy on the coincidence hyperplanes. In other words, the domain of the Hamiltonian is not explicitly characterized and therefore it is not evident what kind of s.a. extension is being constructed. A further intrinsic limitation of the method is the fact that the two-body scattering length (i.e., when all the other particles are far away) must be non-negative. This fact is intuitively clear by analogy with the simple case of a particle subject to a point interaction placed at the origin.

MAIN RESULTS

In this thesis we shall further develop the Minlos-Faddeev regularization applied to different problems, solving different physical situations (examples of which are given below), involving zero-range interactions.
More precisely, in chapter 1 the three-boson system is taken into account, recalling known facts from literature in order to introduce the notation and the required machinery and, afterwards, two new results are proven. Firstly, we show that the threshold parameter $\gamma_c^{3\mathrm{b}}$ found in [5] is optimal in a proper sense. As a second result, we provide an alternative proof of the lower semi-boundedness of the Hamiltonian that is much simpler and more intuitive, even if a little price in terms of generality needs to be paid since we have to require a slightly larger threshold parameter. This approach has the additional advantage of isolating an explicit negative contribution (containing the singularity) of the quadratic form associated to the energy of the system and, therefore, the choice of the regularizing term necessary to compensate such a negative contribution becomes manifest. In other words, this method justifies the asymptotic behavior of the effective scattering length at the triple coincidence point.
In chapter 2, we consider a system composed of N identical non-interacting bosons interacting only with a different particle. We shall see that one can find a bijection between the quadratic form associated to this system and the analogous quadratic form of [5] describing 2 bosons interacting with the impurity. We take advantage of this fact to reproduce step by step analogous results to the ones only briefly mentioned in chapter 1. Moreover, we stress that the existence of such a correspondence implies that a three-body regularization is sufficient to heal also the singularity of this problem and further n-body repulsions, with $n > 3$ are not necessary. However, it turns out that the analysis adopted in the three-body case cannot be satisfactorily applied in our situation and we provide the solution with the development of new techniques. More precisely, the novelty introduced here is an order by order approximation (the content of section 2.4) carried out for all

angular momenta needed in order to obtain the boundedness from below of the Hamiltonian for any value of N. In short, the approach exploited in [5] takes account only of the lower angular momenta and, although it works fine for three bosons, some of the adopted estimates fail as soon as N grows large enough (see the end of section 2.1 for more details).
In chapter 3 we analyze the case of an interacting Bose gas via regularized zero-range interactions. In this framework, a new kind of singularity emerges associated with the coincidence of two pairs of interacting particles (corresponding to the *quadruple coincidence point*). Adopting the intuitive approach developed in section 1.3 we are able to prove stability by taking into account an additional four-body regularization, required to handle this new singular behavior. Furthermore, we compare the results obtained in [2] with ours and we show that the class of Hamiltonians defined there via Dirichlet forms is a particular case of the much wider family of s.a. extensions constructed in this thesis (see proposition 3.6 in section 3.4).
In appendix A we recall some abstract facts exploited throughout the text concerning the theory of s.a. extensions.
In appendix B we state some useful technical fact.

Possible Physical Applications

Here we provide some physical motivations and possible applications of the mathematical results obtained in this thesis.

In chapter 1 we take into account a three-body system that is a rather peculiar problem in non-relativistic Quantum Mechanics since, in suitable conditions of the interaction, a very counter-intuitive universal phenomenon could rise, i.e. the Efimov effect ([14]) which is a purely quantum phenomenon. Despite the absence of bound states in the two-particle subsystems, infinitely many *Borromean*[3] *bound states* can emerge in the system of three particles, if at least two of the three two-particle subsystems exhibit a *zero-energy resonance*. Such a crucial condition corresponds, in short, to the existence of a solution of the two-body Schrödinger equation at zero energy that is only locally integrable and decays at infinity too weakly to be an eigenfunction of the two-body Hamiltonian. Heuristically, these non-integrable solutions contribute to the emergence of an effective long-range potential regardless the detail of the two-body interaction, whose range, on the other hand, can be remarkably chosen arbitrarily short. Furthermore, this kind of universal effective long-range potential is responsible for the infinite sequence of eigenvalues accumulating at zero with the asymptotic geometric law

$$\lim_{n\to+\infty}\frac{E_{n+1}}{E_n}=e^{-\frac{2\pi}{\sigma}}$$

with $\sigma>0$ depending on the mass ratio of the particles and the statistics[4] of the problem. How-

[3] A three-body Borromean bound state is characterized by the fact that if any of the three particles were removed, no bound states would be left.

[4] The Efimov effect does not take place in case of three identical fermions.

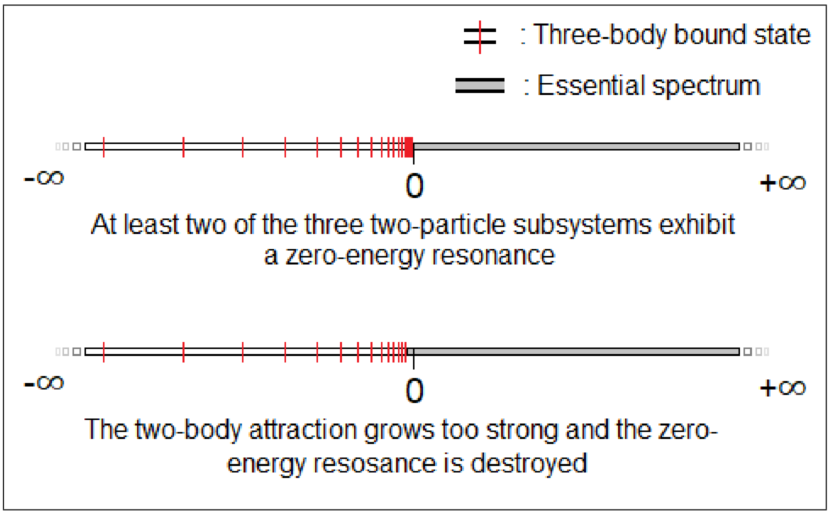

Figure 1: Infinitely many three-body bound states fall into the continuum spectrum as soon as the attractive pair interaction grows strong enough to destroy the zero-energy resonance.

ever, a rigorous proof of this behavior for usual two-body potentials is still lacking. In case of contact interactions, such asymptotic geometric law can be explicitly obtained when $\alpha_0 = 0$ (corresponding to the resonant case), as pointed out e.g. in [3] and [18], still the Thomas collapse also occurs and therefore the result is unsatisfactory. In principle, the meaning of the Minlos-Faddeev regularization is to heal the ultraviolet singularity leaving untouched the infrared structure of the spectrum. This means that the Hamiltonian discussed in chapter 1 is a reasonable candidate for trying to prove the geometrical law for regularized contact interactions in an energetically stable system.

In chapter 2, we consider a gas of bosons that interact only with another particle. Impurity problems are often studied in condensed matter physics as toy models aimed at obtaining preliminary information about a more complicated situation. Furthermore, they also find concrete applications from semiconductors to polarons. In the latter case we mention the situation in which one has a low-energy light particle (e.g. an electron) interacting with the quasi-particles (e.g. phonons) of a polar crystal whose structure is locally deformed by the induced polarization of the field. The entity composed of the interaction carrier and such induced polarization of the field is called “Polaron” and it is characterize by an effective mass and its response to external solicitations (e.g. a magnetic field). In case of a large (Fröhlich) polaron the lattice can be approximated with the continuum, since the de Broglie wave length is much larger than the lattice space of the medium. Alternatively, when the self-induced polarization becomes of the order of the lattice parameter, a small (Holstein) polaron can arise. In this situation, one may be interested in evaluating the ground state energy of the system in a proper

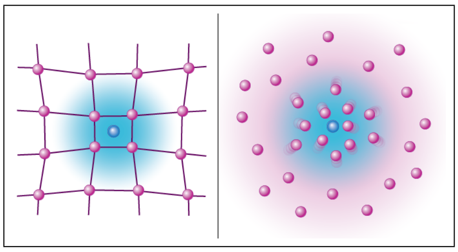

Figure 2: Illustration of the Holstein (left) and Fröhlich (right) polaron.

limit regime. For instance, one can take account of a strongly-coupled (quasi) classical limit[5] that tends to treat the field variables classically and leaves untouched the quantum nature of the light particle ([33]). Typically, this kind of problems are solvable either in a strongly-coupled regime or in a weakly-interacting one and a unified theoretical description valid for arbitrary coupling strengths constitutes a challenging task. In this sense, defining a proper polaron model with a regularized contact interaction may be a promising simplification aimed at looking for further results in this direction.

In chapter 3 we study a gas of interacting bosons. Probably the most popular phenomenon concerning a Bose gas is *Bose-Einstein condensation* (BEC), that is roughly speaking a transition phase occurring at extremely low temperatures in which the lowest quantum state is occupied by a macroscopic number of particles. More precisely there exists a critical temperature below which the condensate appears and lower the temperature, larger the number of particles occupying the ground state. Such a ground state ψ is normalized in such a way that $\|\psi\|^2_{L^2(\mathbb{R}^3)}$ represents the number of particles in the condensate. One finds out, using mean-field theory, that ψ solves the Gross-Pitaevskii equation (GPE) at temperature $T = 0$

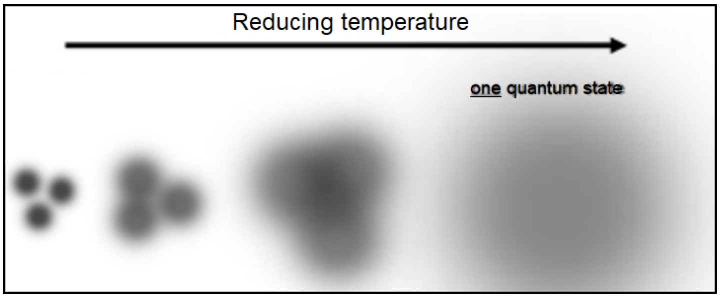

Figure 3: The emerging of the condensate.

$$\left(-\frac{1}{2m}\Delta_{\boldsymbol{x}} + V(\boldsymbol{x}) + \frac{4\pi\mathfrak{a}}{m}|\psi(\boldsymbol{x})|^2\right)\psi(\boldsymbol{x}) = \mu\psi(\boldsymbol{x})$$

where V and μ stand for the external and the chemical potential, respectively. However mean-field theory does not take into account correlations that play actually a crucial role in BEC as soon as $T > 0$ and some correction is therefore required (in the dilute Bose gas Lee-Huang-Yang formula applies [23]). In particular, many results are known concerning a confined Bose gas in the mean-field regime (frequent, weak interactions) and the Gross-Pitaevskii regime (rare, strong interactions), however, studying in general a confined interacting Bose gas in the *thermodynamic-limit*, i.e. sending the number of particles and the size of the confining box to infinity keeping constant their ratio (a macroscopic quantity, namely the density), is still an open problem and only partial results are known. The hope is that a regularized contact interaction may represent a simplified version of the problem similarly to what has been done in [28] for a Fermi gas. Moreover, in the thermodynamic-limit the scale of the regularization should be negligible, hence we believe that in principle it is possible to define a proper scaling of the regularization so that it vanishes in the limit.

[5]The difference between the semi-classical limit morally consists in sending to zero the commutators associated to the field variables instead of the Planck's constant.

Notation

For the reader's convenience, we collect here some of the notations adopted in the thesis.

- Given the Euclidean space $(\mathbb{R}^d, \cdot)$, $\boldsymbol{x}$ is a vector in $\mathbb{R}^d$, $x = |\boldsymbol{x}|$ is its magnitude and, if $\boldsymbol{x} \neq \boldsymbol{0}$ we set $\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{x} \in \mathbb{S}^{d-1}$.
- $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions.
- For any $p \geq 1$ and Ω open set in $\mathbb{R}^d$, $L^p(\Omega, \mu)$ is the Banach space of p-integrable functions with respect to the Borel measure μ. We use $L^p(\Omega)$ in case μ is the Lebesgue measure and we denote $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R}^d)}$.
- If $\mathfrak{H}$ is a complex Hilbert space, we denote by $\langle\cdot,\cdot\rangle_{\mathfrak{H}}$, $\|\cdot\|_{\mathfrak{H}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathfrak{H}}}$ the inner product and the induced norm, respectively.
- if $\mathfrak{H} = L^2(\mathbb{R}^d)$, we simply denote by $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ the inner product and the norm.
- $\mathcal{F} : \psi \longmapsto \hat{\psi}$ is the Fourier transform of $\psi \in L^2(\mathbb{R}^d)$.
- $H^s(\mathbb{R}^d)$ is the standard Hilbert-Sobolev space of order $s > 0$ in $\mathbb{R}^d$.
- $f|_\pi \in H^s(\mathbb{R}^{dn})$ is the trace of $f \in H^{s+\frac{d}{2}}(\mathbb{R}^{d(n+1)})$ on the hyperplane π of codimension d.
- Given X and Y Hilbert spaces, $\mathscr{L}(X,Y)$ and $\mathscr{B}(X,Y)$ denote the Banach spaces of the linear operators and the linear, bounded operators from X to Y, respectively.
- Moreover, $\mathscr{L}(X) := \mathscr{L}(X,X)$ and $\mathscr{B}(X) := \mathscr{B}(X,X)$.
- Given an integral operator $\mathcal{A} \in \mathscr{L}(L^2(Y), L^2(X))$, we denote by $A\binom{x}{y}$ its kernel, with $x \in X$ and $y \in Y$.
- Given X and Y two Hilbert spaces, if $A : \mathscr{D}(A) \subseteq X \longrightarrow Y$ is a densely defined, linear and closed operator, $\rho(A)$ denotes its resolvent set and $\mathcal{R}_A(z) \in \mathscr{B}(Y, \mathscr{D}(A))$ its resolvent operator, with $z \in \rho(A)$.

1. Three-Boson System

The aim of this chapter is to describe the construction of a Hamiltonian encoding a regularized point interaction following the approach developed in [5] and also to prove two further results ([16]). More precisely, in section 1.1 we introduce the notation and we formulate the main result of [5], essentially based on the analysis of a suitable quadratic form Q. Such analysis shall be resumed in detail in the next chapter, with proper slight modifications due to the different system taken into account.
In section 1.2 we prove that the threshold value $\gamma_c^{3\mathrm{b}}$ obtained in [5] is optimal, in the sense that for $\gamma < \gamma_c^{3\mathrm{b}}$ the quadratic form Q is unbounded from below and therefore no Hamiltonian can be defined.
In section 1.3 we give a different proof of the lower boundedness of the regularized Hamiltonian based on a new approach in position space. This method loses a bit of generality in the sense that it is valid only for γ strictly greater than a certain $\bar{\gamma}_c^{3\mathrm{b}}$, with $\bar{\gamma}_c^{3\mathrm{b}} > \gamma_c^{3\mathrm{b}}$, nevertheless, it has the advantage to be considerably easier and to isolate an explicit negative contribution (containing the singularity) of the quadratic form Q (see proposition 1.5). This fact shows that the choice of the three-body force is not arbitrary but it is manifestly dictated by the inherent singularity of the problem.

1.1 Regularized Hamiltonian

Let us consider a system composed of three identical spinless bosons of mass $\frac{1}{2}$ in three dimensions and let us fix the center of mass reference frame so that $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3 = -\boldsymbol{x}_1 - \boldsymbol{x}_2$ represent the Cartesian coordinates of the three particles. We also introduce the Jacobi coordinates

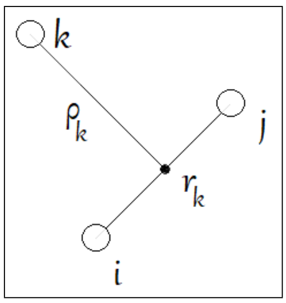

Figure 1.1: Jacobi coordinates associated with the pair (i, j).

$$\begin{cases} \boldsymbol{r}_k := \frac{1}{2}\sum_{i,j=1}^{3} \epsilon_{ijk}(\boldsymbol{x}_i - \boldsymbol{x}_j), \\ \boldsymbol{\rho}_k := \frac{3}{2}\,\boldsymbol{x}_k - \frac{1}{2}\sum_{\ell=1}^{3} \boldsymbol{x}_\ell, \end{cases} \qquad k \in \{1, 2, 3\} \tag{1.1}$$

where ϵ_{ijk} is the Levi-Civita symbol, so that one has the following identities

$$\begin{cases} \boldsymbol{r}_{k\pm1} = -\frac{1}{2}\boldsymbol{r}_k \mp \boldsymbol{\rho}_k, \\ \boldsymbol{\rho}_{k\pm1} = \pm\frac{3}{4}\boldsymbol{r}_k - \frac{1}{2}\boldsymbol{\rho}_k, \end{cases} \qquad k \in \mathbb{Z}\diagup_{\{3\}}. \tag{1.2}$$

Denoting by $\boldsymbol{x} = \boldsymbol{r}_1 = \boldsymbol{x}_2 - \boldsymbol{x}_3$ and $\boldsymbol{y} = \boldsymbol{\rho}_1 = \boldsymbol{x}_1 - \frac{\boldsymbol{x}_2 + \boldsymbol{x}_3}{2}$, the Hilbert space of the system is

$$\mathscr{H}_{3\mathrm{b}} := \left\{ \psi \in L^2(\mathbb{R}^6) \,\middle|\, \psi(\boldsymbol{x},\boldsymbol{y}) = \psi(-\boldsymbol{x},\boldsymbol{y}) = \psi\big(\tfrac{1}{2}\,\boldsymbol{x} + \boldsymbol{y}, \tfrac{3}{4}\,\boldsymbol{x} - \tfrac{1}{2}\,\boldsymbol{y}\big) \right\}. \tag{1.3}$$

Indeed, notice that the symmetry conditions in (1.3) corresponds to the exchange of particles $2, 3$ and $1, 2$ that implies also the condition $\psi(\boldsymbol{x},\boldsymbol{y}) = \psi\big(\frac{1}{2}\,\boldsymbol{x} - \boldsymbol{y}, -\frac{3}{4}\,\boldsymbol{x} - \frac{1}{2}\,\boldsymbol{y}\big)$, associated with the exchange of particles $3, 1$. If the bosons interact via zero-range forces, then the system is described, at least formally, by the Hamiltonian

$$\tilde{\mathcal{H}} = -\Delta_{\boldsymbol{x}} - \tfrac{3}{4}\,\Delta_{\boldsymbol{y}} + \nu\,\delta(\boldsymbol{x}) + \nu\,\delta(\boldsymbol{y} + \tfrac{1}{2}\,\boldsymbol{x}) + \nu\,\delta(\boldsymbol{y} - \tfrac{1}{2}\,\boldsymbol{x}) \tag{1.4}$$

where $\nu \in \mathbb{R}$ is a coupling constant (related to the two-body scattering length) and we denote by $\mathcal{H}_0$ the free Hamiltonian of the system, i.e.

$$\mathcal{H}_0 = -\Delta_{\boldsymbol{x}} - \tfrac{3}{4}\,\Delta_{\boldsymbol{y}}, \qquad \mathscr{D}(\mathcal{H}_0) = H^2(\mathbb{R}^6) \cap \mathscr{H}_{3\mathrm{b}}\,. \tag{1.5}$$

As already mentioned in the introduction, in order to define a rigorous counterpart of $\tilde{\mathcal{H}}$, one needs to build a perturbation of the free Hamiltonian supported on the coincidence hyperplanes

$$\pi_k := \left\{ (\boldsymbol{r}_k, \boldsymbol{\rho}_k) \in \mathbb{R}^6 \,\middle|\, \boldsymbol{r}_k = \boldsymbol{0} \right\}, \qquad \pi := \textstyle\bigcup_{k=1}^{3} \pi_k \tag{1.6a}$$

or, equivalently,

$$\begin{gathered} \pi_1 := \left\{ (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^6 \,\middle|\, \boldsymbol{x} = \boldsymbol{0} \right\}, \quad \pi_2 := \left\{ (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^6 \,\middle|\, \boldsymbol{y} = -\tfrac{1}{2}\,\boldsymbol{x} \right\}, \\ \pi_3 := \left\{ (\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^6 \,\middle|\, \boldsymbol{y} = \tfrac{1}{2}\,\boldsymbol{x} \right\}. \end{gathered} \tag{1.6b}$$

In other words, we look for a s.a. and bounded from below extension in $\mathscr{H}_{3\mathrm{b}}$ of the following symmetric, densely defined and closed (with respect to the graph norm of $\mathcal{H}_0$) operator

$$\dot{\mathcal{H}}_0 := \mathcal{H}_0\big|_{\mathscr{D}(\dot{\mathcal{H}}_0)}, \qquad \mathscr{D}(\dot{\mathcal{H}}_0) := H^2_0(\mathbb{R}^6 \setminus \pi) \cap \mathscr{H}_{3\mathrm{b}}\,. \tag{1.7}$$

In particular, we are interested in the family of s.a. extensions studied in [5] whose domain, at least formally, are characterized by the boundary condition

$$\psi(\boldsymbol{x},\boldsymbol{y}) = \frac{\xi(\boldsymbol{y})}{x} + \alpha(\boldsymbol{y})\xi(\boldsymbol{y}) + o(1)\,, \qquad x \longrightarrow 0^+, \tag{1.8}$$

where α is a position dependent parameter given by

$$\begin{aligned} \alpha \colon \mathbb{R}^3 &\longrightarrow \mathbb{R}, \\ \boldsymbol{y} &\longmapsto \alpha_0 + \frac{\gamma}{y}\,\theta(y) \end{aligned} \tag{1.9}$$

with γ a positive parameter representing the strength of the regularization and $\theta : \mathbb{R}_+ \longrightarrow \mathbb{R}$ an essentially bounded function satisfying

$$1 - \tfrac{r}{b} \leq \theta(r) \leq 1 + \tfrac{r}{b}, \quad \text{for some } b > 0. \tag{1.10}$$

We observe that the function θ, by assumption (1.10), is positive in a neighborhood of the origin and it is continuous at zero, with $\theta(0) = 1$. We also stress that the simplest choices for the function θ could be

- the identically constant function,
- the characteristic function $\mathbb{1}_b(\cdot)$ of the ball of radius b centered in the origin,
- the exponentially decaying function $\exp\Big(-\frac{|\cdot|}{b}\Big)$.

Furthermore, due to the symmetry constraints of $\mathscr{H}_{3\text{b}}$, boundary condition (1.8) implies

$$\psi(\boldsymbol{x},\boldsymbol{y}) = \frac{\xi\big(\frac{3}{4}\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\big)}{\big|\boldsymbol{y}+\frac{1}{2}\boldsymbol{x}\big|} + \alpha\big(\tfrac{3}{4}\boldsymbol{x}-\tfrac{1}{2}\boldsymbol{y}\big)\,\xi\big(\tfrac{3}{4}\boldsymbol{x}-\tfrac{1}{2}\boldsymbol{y}\big) + o(1)\,, \qquad \boldsymbol{y}\longrightarrow -\tfrac{1}{2}\boldsymbol{x},$$
$$\psi(\boldsymbol{x},\boldsymbol{y}) = \frac{\xi\big(-\frac{3}{4}\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}\big)}{\big|\boldsymbol{y}-\frac{1}{2}\boldsymbol{x}\big|} + \alpha\big(-\tfrac{3}{4}\boldsymbol{x}-\tfrac{1}{2}\boldsymbol{y}\big)\,\xi\big(-\tfrac{3}{4}\boldsymbol{x}-\tfrac{1}{2}\boldsymbol{y}\big) + o(1)\,, \qquad \boldsymbol{y}\longrightarrow \tfrac{1}{2}\boldsymbol{x}.$$

Observe that for $\gamma = 0$ equation (1.8) reduces to the standard TMS boundary condition, which leads to the Thomas effect. Then, for $\gamma > 0$ we are introducing a three-body repulsion meant to regularize the ultraviolet singularity occurring when the positions of all particles coincide. However, if one additionally assumes θ compactly supported, the usual two-body point interaction is restored when the third particle is far enough.

The procedure adopted in [5] for the rigorous construction of the Hamiltonian is the following: one first introduces the quadratic form Q in $\mathscr{H}_{3\text{b}}$ describing, at least formally, the expectation value of the energy of our three-body system. Then one defines a suitable form domain $\mathscr{D}(Q)$ and proves that $Q, \mathscr{D}(Q)$ is closed and bounded from below. Finally, the Hamiltonian is defined as the unique s.a and bounded from below operator associated to such a quadratic form. In other words, one properly defines the energy form describing (at least heuristically) the desired properties of the system in order to construct an associated Hamiltonian (see [5, section 2] or the analogous case discussed in appendix 2.B).

To this end, we first need to introduce the hermitian quadratic form Φ^λ in $L^2(\mathbb{R}^3)$ given by [5, equation (3.1)] for $\lambda > 0$, namely

$$\Phi^\lambda := \Phi^\lambda_{\text{diag}} + \Phi^\lambda_{\text{off}} + \Phi_{\text{reg}} + \Phi_0, \qquad \mathscr{D}(\Phi^\lambda) = H^{1/2}(\mathbb{R}^3), \tag{1.11}$$

where

$$\Phi^\lambda_{\text{diag}}[\xi] := 12\pi \int_{\mathbb{R}^3} d\boldsymbol{p}\, \sqrt{\tfrac{3}{4}p^2+\lambda}\; |\hat{\xi}(\boldsymbol{p})|^2, \tag{1.12a}$$

$$\Phi^\lambda_{\text{off}}[\xi] := -\frac{12}{\pi}\int_{\mathbb{R}^6} d\boldsymbol{p}d\boldsymbol{q}\; \frac{\overline{\hat{\xi}(\boldsymbol{p})}\,\hat{\xi}(\boldsymbol{q})}{p^2+q^2+\boldsymbol{p}\cdot\boldsymbol{q}+\lambda}, \tag{1.12b}$$

$$\Phi_{\text{reg}}[\xi] := \frac{6\gamma}{\pi}\int_{\mathbb{R}^6} d\boldsymbol{p}d\boldsymbol{q}\; \frac{\overline{\hat{\xi}(\boldsymbol{p})}\,\hat{\xi}(\boldsymbol{q})}{|\boldsymbol{p}-\boldsymbol{q}|^2}, \tag{1.12c}$$

$$\Phi_0[\xi] := 12\pi\int_{\mathbb{R}^3} d\boldsymbol{y}\; \beta(\boldsymbol{y})|\xi(\boldsymbol{y})|^2, \qquad \beta : \boldsymbol{y} \longmapsto \alpha_0 + \gamma\,\frac{\theta(y)-1}{y}. \tag{1.12d}$$

By assumption (1.10), one has $\beta \in L^\infty(\mathbb{R}^3)$ and therefore Φ_0 is bounded. This means that Φ_0 cannot play any role in the compensation of the singularity contained in $\Phi^\lambda_{\text{off}}$. The proof of the fact that Φ^λ is well defined in $H^{1/2}(\mathbb{R}^3)$ is relatively standard and it is given in [5, proposition 3.1].

The more relevant point concerning Φ^λ is that it is coercive for λ large enough as long as $\gamma > \gamma_c^{3b}$, with

$$\gamma_c^{3b} := \frac{4}{3} - \frac{\sqrt{3}}{\pi} \approx 0.782004. \tag{1.13}$$

The proof is given in [5, proposition 3.6] and it is based on a rather long and non trivial analysis performed in the momentum representation (a similar discussion is provided in chapter 2). The conclusion is that there exists $\lambda_0 > 0$ such that Φ^λ is closed and bounded from below by a positive constant for each $\lambda > \lambda_0$ and $\gamma > \gamma_c^{3b}$. Therefore one can uniquely define a s.a., positive and invertible operator Γ^λ in $L^2(\mathbb{R}^3)$ such that

$$\Phi^\lambda[\xi] = \langle \xi, \, \Gamma^\lambda \xi \rangle, \qquad \forall \, \xi \in D \tag{1.14}$$

with $D = \mathscr{D}(\Gamma^\lambda)$ a dense subspace independent of λ.
Next, defining the continuous[1] operator (see proposition B.4)

$$\begin{aligned} \tau : \mathscr{D}(\mathcal{H}_0) &\longrightarrow L^2(\mathbb{R}^3), \\ \varphi &\longmapsto 12\pi \, \varphi|_{\pi_1} \end{aligned} \tag{1.15}$$

satisfying $\operatorname{ran}(\tau) = H^{\frac{1}{2}}(\mathbb{R}^3)$ and $\ker(\tau) = \mathscr{D}(\dot{\mathcal{H}}_0)$, one can check that[2] the injective operator $G(z) := (\tau R_{\mathcal{H}_0}(\bar{z}))^* \in \mathscr{B}(L^2(\mathbb{R}^3), \mathscr{H}_{3b})$ with $z \in \rho(\mathcal{H}_0)$ is represented in the Fourier space by

$$\widehat{(G(z)\xi)}(\boldsymbol{k}, \boldsymbol{p}) = \sqrt{\frac{2}{\pi}} \, \frac{\hat{\xi}(\boldsymbol{p}) + \hat{\xi}\big(\boldsymbol{k} - \frac{1}{2}\boldsymbol{p}\big) + \hat{\xi}\big(-\boldsymbol{k} - \frac{1}{2}\boldsymbol{p}\big)}{k^2 + \frac{3}{4}p^2 - z}. \tag{1.16}$$

Indeed, let $(\boldsymbol{k}, \boldsymbol{p})$ be the conjugate coordinates of $(\boldsymbol{x}, \boldsymbol{y})$ and $\mathscr{H}_{3b}$ in the space of momenta reads

$$\mathcal{F}\mathscr{H}_{3b} = \Big\{ \hat{\psi} \in L^2(\mathbb{R}^6) \, \Big| \, \hat{\psi}(\boldsymbol{k}, \boldsymbol{p}) = \hat{\psi}(-\boldsymbol{k}, \boldsymbol{p}) = \hat{\psi}\big(\tfrac{1}{2}\boldsymbol{k} + \tfrac{3}{4}\boldsymbol{p}, \boldsymbol{k} - \tfrac{1}{2}\boldsymbol{p}\big) \Big\}$$

that encodes the bosonic symmetry under the exchange of particles $2, 3$ and $1, 2$, that clearly also implies the symmetry by exchange of particles $3, 1$, i.e. $\hat{\psi}(\boldsymbol{k}, \boldsymbol{p}) = \hat{\psi}\big(\frac{1}{2}\boldsymbol{k} - \frac{3}{4}\boldsymbol{p}, -\boldsymbol{k} - \frac{1}{2}\boldsymbol{p}\big)$.
We are now in position to introduce the quadratic form in $\mathscr{H}_{3b}$ ([5, definition 2.1])

$$\begin{aligned} \mathscr{D}(Q) &:= \Big\{ \psi \in \mathscr{H}_{3b} \, \Big| \, \psi = \phi_\lambda + G(-\lambda)\xi, \; \phi_\lambda \in H^1(\mathbb{R}^6), \; \xi \in H^{\frac{1}{2}}(\mathbb{R}^3), \; \lambda > 0 \Big\}, \\ Q[\psi] &:= \|\mathcal{H}_0^{1/2}\phi_\lambda\|^2 + \lambda\|\phi_\lambda\|^2 - \lambda\|\psi\|^2 + \Phi^\lambda[\xi]. \end{aligned} \tag{1.17}$$

Using the properties of Φ^λ and $G(-\lambda)$, it is now easy to show that the above quadratic form is closed and bounded from below if $\gamma > \gamma_c^{3b}$. Hence it uniquely defines a s.a. and lower semi-bounded operator $\mathcal{H}$ which, by definition, is the Hamiltonian of the three-boson system.

Following an equivalent approach (that shall be adopted in chapter 3), one can consider the densely defined and closed operator $\Gamma(z) : D \subset L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3)$, given by

$$\Gamma(z) := \Gamma^\lambda - (\lambda + z) G(\bar{z})^* G(-\lambda), \qquad \lambda > \lambda_0, \; z \in \rho(\mathcal{H}_0) \tag{1.18}$$

[1] Here $\mathscr{D}(\mathcal{H}_0)$ must be intended as a Hilbert subspace of $\mathscr{H}_{3b}$ endowed with the graph norm of $\mathcal{H}_0$.
[2] A similar computation to refer to has been carried out in appendix 2.A.

which represents a sort of analytic continuation of Γ^λ, D. Actually, one can prove that $\Gamma(z)$ fulfils

$$\Gamma(z)^* = \Gamma(\bar{z}), \qquad \forall\, z \in \rho(\mathcal{H}_0), \tag{1.19a}$$

$$\Gamma(z) - \Gamma(w) = (z-w)\,G(\bar{z})^* G(w), \qquad \forall\, w, z \in \rho(\mathcal{H}_0), \tag{1.19b}$$

$$\forall\, z \in \mathbb{C}:\ \operatorname{Re}(z) < -\lambda_0 \ \vee\ \operatorname{Im}(z) > 0, \quad 0 \in \rho(\Gamma(z)). \tag{1.19c}$$

These properties imply, according to e.g. [31] (see also [7, theorem 2.19]), that for any $z \in \mathbb{C}$ such that $\Gamma(z)$ has a bounded inverse, the operator

$$R(z) = \mathcal{R}_{\mathcal{H}_0}(z) + G(z)\Gamma(z)^{-1}G(\bar{z})^* \tag{1.20}$$

defines the resolvent of a s.a. and bounded from below operator which coincides with the Hamiltonian $\mathcal{H}$ obtained with the approach based on the quadratic form and

$$\left\{z \in \mathbb{C} \,\middle|\, \operatorname{Re}(z) < -\lambda_0 \ \vee\ \operatorname{Im}(z) > 0\right\} \subseteq \rho(\mathcal{H}).$$

Moreover, one can verify that $\mathcal{H}$ coincides with $\mathcal{H}_0$ on $\mathscr{D}(\dot{\mathcal{H}}_0)$, satisfies boundary condition (1.8) in the L^2 sense (see [5, remark 4.1]) and it is characterized by

$$\begin{aligned} &\mathscr{D}(\mathcal{H}) = \left\{\psi \in \mathscr{D}(Q) \,\middle|\, \phi_z \in \mathscr{D}(\mathcal{H}_0),\ \xi \in D,\ \Gamma(z)\xi = \tau\phi_z\right\}, \\ &\mathcal{H}\psi = \mathcal{H}_0\phi_z + zG(z)\xi \end{aligned} \tag{1.21}$$

provided $\operatorname{Im} z = 0 \ \vee\ \phi_z \perp G(z)\xi$. More details about this abstract setting are discussed in appendix A.

1.2 Optimality of $\gamma_c^{3\mathrm{b}}$

In this section we prove the optimality of the threshold parameter $\gamma_c^{3\mathrm{b}}$ defined by (1.13). More precisely our goal is to prove the following theorem.

Theorem 1.1.
Whenever $\gamma < \gamma_c^{3\mathrm{b}}$, the quadratic form Q given by (1.17) *is unbounded from below.*

In order to achieve the result, we shall adapt the ideas contained in [17, section 5]. Denote for short $G^\lambda := G(-\lambda)$ for any $\lambda > 0$ and let $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(Q)$ be a sequence of trial functions given by

$$u_n(\boldsymbol{x}, \boldsymbol{y}) = (G^\lambda \eta_n)(\boldsymbol{x}, \boldsymbol{y}), \tag{1.22}$$

$$\eta_n(\boldsymbol{y}) = n^2 f(n\boldsymbol{y}), \qquad f \in H^{\frac{1}{2}}(\mathbb{R}^3). \tag{1.23}$$

We stress that, by an explicit estimate due to (1.16), one finds

$$\inf_{n\in\mathbb{N}} \|G^\lambda \eta_n\|_{\mathscr{H}_{3\mathrm{b}}} \gneq 0. \tag{1.24}$$

Indeed,

$$\begin{aligned}\|G^\lambda\eta_n\|^2_{\mathscr{H}_{3\mathrm{b}}} &= \frac{2}{\pi}\int_{\mathbb{R}^6} d\boldsymbol{k}d\boldsymbol{p}\,\frac{1}{n^2}\frac{\left|\hat f\left(\frac{\boldsymbol{p}}{n}\right)+\hat f\left(\frac{\boldsymbol{k}}{n}-\frac{\boldsymbol{p}}{2n}\right)+\hat f\left(\frac{\boldsymbol{k}}{n}+\frac{\boldsymbol{p}}{2n}\right)\right|^2}{\left(k^2+\frac34 p^2+\lambda\right)^2}\\ &= \frac{2}{\pi}\int_{\mathbb{R}^6} d\boldsymbol{\kappa}d\boldsymbol{q}\,\frac{\left|\hat f(\boldsymbol{q})+\hat f\left(\boldsymbol{\kappa}-\frac{\boldsymbol{q}}{2}\right)+\hat f\left(\boldsymbol{\kappa}+\frac{\boldsymbol{q}}{2}\right)\right|^2}{\left(\kappa^2+\frac34 q^2+\frac{\lambda}{n^2}\right)^2} > \|G^\lambda f\|^2_{\mathscr{H}_{3\mathrm{b}}}.\end{aligned}$$

Our goal is to show that whenever γ is smaller than the threshold value $\gamma_c^{3\mathrm{b}}$ given by (1.13), one has

$$\lim_{n\to+\infty} Q[u_n] = -\infty. \tag{1.25}$$

According to (1.17), we have

$$Q[u_n] = -\lambda\,\|G^\lambda\eta_n\|_{\mathscr{H}_{3\mathrm{b}}} + \Phi^\lambda[\eta_n] \leq -\lambda\,\|G^\lambda f\|_{\mathscr{H}_{3\mathrm{b}}} + \Phi^\lambda[\eta_n] \tag{1.26}$$

and therefore the theorem is proven as soon as we exhibit some $f\in H^{\frac12}(\mathbb{R}^3)$ such that

$$\lim_{n\to+\infty}\Phi^\lambda[\eta_n] = -\infty. \tag{1.27}$$

In the following lemma we are going to show that the leading order of $\Phi^\lambda[\eta_n]$ as n goes to infinity does not depend on λ and, therefore, we are reducing the problem to the study of the hermitian quadratic form evaluated in $\lambda = 0$ which is, as we shall see, diagonalizable.

Lemma 1.2. *Let Φ^λ and η_n be given by* (1.11) *and* (1.23)*, respectively. Then, one has*

$$\Phi^\lambda[\eta_n] = n^2(\Phi^0_{\mathrm{diag}} + \Phi^0_{\mathrm{off}} + \Phi_{\mathrm{reg}})[f] + \mathcal{O}(n).$$

Proof. First of all, we can neglect the bounded component Φ_0, since

$$\begin{aligned}\Phi_0[\eta_n] &= 12\pi n^4\int_{\mathbb{R}^3} d\boldsymbol{y}\,\beta(y)\,|f(n\boldsymbol{y})|^2 = 12\pi n\int_{\mathbb{R}^3} d\boldsymbol{t}\,\beta\left(\tfrac{t}{n}\right)|f(\boldsymbol{t})|^2\\ &\leq 12\pi n\,\|\beta\|_\infty\|f\|^2 \implies \Phi_0[\eta_n] = \mathcal{O}(n), \qquad n\to+\infty.\end{aligned}$$

Next, rescaling properly the variables in computing $\Phi^\lambda_{\mathrm{diag}}[\eta_n]$, one gets

$$\begin{aligned}\Phi^\lambda_{\mathrm{diag}}[\eta_n] &= 12\pi n\int_{\mathbb{R}^3} d\boldsymbol{\kappa}\,\sqrt{\tfrac34\,n^2\kappa^2+\lambda}\,|\hat f(\boldsymbol{\kappa})|^2\\ &= 6\sqrt{3}\,\pi n^2\int_{\mathbb{R}^3} d\boldsymbol{\kappa}\,|\boldsymbol{\kappa}|\,|\hat f(\boldsymbol{\kappa})|^2 + 12\pi n\int_{\mathbb{R}^3} d\boldsymbol{\kappa}\left(\sqrt{\tfrac34\,n^2\kappa^2+\lambda}-\sqrt{\tfrac34}\,n\kappa\right)|\hat f(\boldsymbol{\kappa})|^2\\ &= n^2\Phi^0_{\mathrm{diag}}[f] + o(n)\,.\end{aligned}$$

Indeed, exploiting the elementary inequality $\sqrt{a^2+b^2}-|a|\leq|b|$, we can use the dominated convergence theorem to obtain

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} d\boldsymbol{\kappa}\left(\sqrt{\tfrac34\,n^2\kappa^2+\lambda}-\sqrt{\tfrac34}\,n\kappa\right)|\hat f(\boldsymbol{\kappa})|^2 = 0.$$

Concerning the regularizing contribution, one simply has

$$\begin{aligned}\Phi_{\mathrm{reg}}[\eta_n] &= \frac{6\gamma}{\pi} n^2 \int_{\mathbb{R}^3} d\boldsymbol{p} \int_{\mathbb{R}^3} d\boldsymbol{q} \; \frac{\overline{\hat{f}(\boldsymbol{p})}\, \hat{f}(\boldsymbol{q})}{|\boldsymbol{p}-\boldsymbol{q}|^2} \\ &= n^2 \Phi_{\mathrm{reg}}[f].\end{aligned}$$

Finally, we compute $\Phi_{\mathrm{off}}^{\lambda}[\eta_n]$

$$\begin{aligned}\Phi_{\mathrm{off}}^{\lambda}[\eta_n] = &-\frac{12}{\pi} n^2 \int_{\mathbb{R}^6} d\boldsymbol{p} d\boldsymbol{q} \; \frac{\overline{\hat{f}(\boldsymbol{p})}\, \hat{f}(\boldsymbol{q})}{p^2+q^2+\boldsymbol{p}\cdot\boldsymbol{q}+\frac{\lambda}{n^2}} \\ = &-\frac{12}{\pi} n^2 \int_{\mathbb{R}^6} d\boldsymbol{p} d\boldsymbol{q} \; \frac{\overline{\hat{f}(\boldsymbol{p})}\, \hat{f}(\boldsymbol{q})}{p^2+q^2+\boldsymbol{p}\cdot\boldsymbol{q}} + \\ &+ \frac{12}{\pi} \lambda \int_{\mathbb{R}^6} d\boldsymbol{p} d\boldsymbol{q} \; \frac{\overline{\hat{f}(\boldsymbol{p})}\, \hat{f}(\boldsymbol{q})}{\left(p^2+q^2+\boldsymbol{p}\cdot\boldsymbol{q}+\frac{\lambda}{n^2}\right)\left(p^2+q^2+\boldsymbol{p}\cdot\boldsymbol{q}\right)}.\end{aligned}$$

Defining the integral operator in $L^2(\mathbb{R}^3)$ given by

$$(\mathcal{P}_n \hat{\varphi})(\boldsymbol{p}) := \frac{12}{\pi} \lambda \int_{\mathrm{R}^3} d\boldsymbol{q} \; \frac{\hat{\varphi}(\boldsymbol{q})}{\left(p^2+q^2+\boldsymbol{p}\cdot\boldsymbol{q}+\frac{\lambda}{n^2}\right)\left(p^2+q^2+\boldsymbol{p}\cdot\boldsymbol{q}\right)}, \tag{1.28}$$

we can write

$$\Phi_{\mathrm{off}}^{\lambda}[\eta_n] = n^2 \Phi_{\mathrm{off}}^{0}[f] + \int_{\mathrm{R}^3} d\boldsymbol{p} \; \overline{\hat{f}(\boldsymbol{p})}\, (\mathcal{P}_n \hat{f})(\boldsymbol{p}).$$

We notice that $\mathcal{P}_n$ is a Hilbert-Schmidt operator and

$$\begin{aligned}\|\mathcal{P}_n\|^2_{\mathscr{B}(L^2(\mathbb{R}^3))} &\leq \frac{124\,\lambda^2}{\pi^2} \int_{\mathbb{R}^6} d\boldsymbol{p} d\boldsymbol{q} \; \frac{1}{\left(p^2+q^2+\boldsymbol{p}\cdot\boldsymbol{q}+\frac{\lambda}{n^2}\right)^2\left(p^2+q^2+\boldsymbol{p}\cdot\boldsymbol{q}\right)^2} \\ &\leq \frac{124\lambda^2}{\pi^2} \int_{\mathbb{R}^6} d\boldsymbol{p} d\boldsymbol{q} \; \frac{4}{\left(\frac{p^2+q^2}{2}+\frac{\lambda}{n^2}\right)^2\left(p^2+q^2\right)^2} = 496\,\pi\lambda^2 \int_0^{+\infty} dk \; \frac{k}{\left(\frac{k^2}{2}+\frac{\lambda}{n^2}\right)^2} \\ &= 496\,\pi\lambda n^2.\end{aligned}$$

Owing to the Cauchy–Schwarz inequality, the above estimate implies

$$\Phi_{\mathrm{off}}^{\lambda}[\eta_n] = n^2 \Phi_{\mathrm{off}}^{0}[f] + \mathcal{O}(n)$$

and the lemma is proven.

□

In light of lemma 1.2, it is straightforward to see that (1.27) is achieved as soon as we exhibit a function $f \in H^{\frac{1}{2}}(\mathbb{R}^3)$ such that, whenever $\gamma < \gamma_c^{3\mathrm{b}}$, there holds

$$\Phi_{\mathrm{diag}}^{0}[f] + \Phi_{\mathrm{off}}^{0}[f] + \Phi_{\mathrm{reg}}[f] < 0.$$

In the next lemma we exhibit a trial function that allows us to prove such a result.

Lemma 1.3. *Let γ_c^{3b} be defined by* (1.13)*, assume $\gamma < \gamma_c^{3b}$ and let us consider the family of trial functions $f_\beta \in \mathcal{S}(\mathbb{R}^3) \subset H^{\frac{1}{2}}(\mathbb{R}^3)$ such that*

$$\hat{f}_\beta(\boldsymbol{p}) = \tfrac{1}{p^2} \exp\Big(-\tfrac{p^\beta + p^{-\beta}}{2}\Big), \qquad \beta > 0.$$

Then there exists $\beta_0 > 0$ such that for any $\beta \in (0, \beta_0)$ we have

$$\big(\Phi^0_{\text{diag}} + \Phi^0_{\text{off}} + \Phi_{\text{reg}}\big)[f_\beta] < 0.$$

Proof. We stress that our trial functions are entirely lying in the s-wave subspace, therefore we have

$$\Phi^0_{\text{diag}}[f_\beta] = 48\pi^2 \sqrt{\tfrac{3}{4}} \int_0^{+\infty} dp\, p^3\, |\hat{f}_\beta(p)|^2, \tag{1.29a}$$

$$\Phi^0_{\text{off}}[f_\beta] = -96\pi \int_0^{+\infty} dp\, p \int_0^{+\infty} dq\, q\, \overline{\hat{f}_\beta(p)}\, \hat{f}_\beta(q) \ln\left(\frac{p^2+q^2+pq}{p^2+q^2-pq}\right), \tag{1.29b}$$

$$\Phi_{\text{reg}}[f_\beta] = 24\pi\gamma \int_0^{+\infty} dp\, p \int_0^{+\infty} dq\, q\, \overline{\hat{f}_\beta(p)}\, \hat{f}_\beta(q) \ln\left(\frac{p^2+q^2+2pq}{p^2+q^2-2pq}\right), \tag{1.29c}$$

where we have used the identity[3]

$$\int_{\mathbb{R}^6} d\boldsymbol{p} d\boldsymbol{q}\; g\Big(p, q, \tfrac{\boldsymbol{p}\cdot\boldsymbol{q}}{pq}\Big) = 8\pi^2 \int_0^{+\infty} dp\, p^2 \int_0^{+\infty} dq\, q^2 \int_{-1}^{1} du\, g(p,q,u) \tag{1.30}$$

holding for any integrable function $g : \mathbb{R}^2_+ \times [-1, 1] \longrightarrow \mathbb{C}$. According to, e.g. [5, lemma 3.4], the quantities in equations (1.29) can be diagonalized through the transformation

$$\begin{aligned} \mathcal{M} : L^2(\mathbb{R}_+, p^2\sqrt{p^2+1}\, dp) &\longrightarrow L^2(\mathbb{R}), \\ \psi \longmapsto \psi^\sharp(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dt\, e^{-itx} e^{2t} \psi(e^t) \end{aligned} \tag{1.31}$$

yielding (see [5, lemmata 3.4, 3.5] and section 2.3 for more details)

$$\Phi^0_{\text{diag}}[f_\beta] = 48\sqrt{\tfrac{3}{4}}\, \pi^2 \int_{\mathbb{R}} dx\, |\hat{f}^\sharp_\beta(x)|^2, \tag{1.32a}$$

$$\Phi^0_{\text{off}}[f_\beta] = -48\pi^2 \int_{\mathbb{R}} dx\, |\hat{f}^\sharp_\beta(x)|^2\, \frac{4\sinh\big(\frac{\pi}{6}x\big)}{x\cosh\big(\frac{\pi}{2}x\big)}, \tag{1.32b}$$

$$\Phi_{\text{reg}}[f_\beta] = 48\pi^2 \int_{\mathbb{R}} dx\, |\hat{f}^\sharp_\beta(x)|^2\, \frac{\gamma\tanh\big(\frac{\pi}{2}x\big)}{x}. \tag{1.32c}$$

Let us introduce the bounded and continuous (except at the point $x = 0$) function

$$S(x) := \frac{\sqrt{3}}{2} + \frac{\gamma\sinh\big(\frac{\pi}{2}x\big) - 4\sinh\big(\frac{\pi}{6}x\big)}{x\cosh\big(\frac{\pi}{2}x\big)} \tag{1.33}$$

so that we have

$$\big(\Phi^0_{\text{diag}} + \Phi^0_{\text{off}} + \Phi_{\text{reg}}\big)[f_\beta] = 48\pi^2 \int_{\mathbb{R}} dx\, |\hat{f}^\sharp_\beta(x)|^2\, S(x), \tag{1.34}$$

[3]Equation (1.30) is due to the addition formula for the spherical harmonics in the s-wave (see proposition B.2).

with

$$\lim_{x\to 0} S(x) = \frac{\sqrt{3}}{2} - \frac{2\pi}{3} + \frac{\pi}{2}\gamma = \frac{\pi}{2}(\gamma - \gamma_c^{3b}) < 0. \tag{1.35}$$

Roughly speaking, the integral in (1.34) is negative if we choose the trial function such that the support of $\hat{f}_\beta^\sharp$ is sufficiently concentrated in a neighborhood of zero. More precisely, considering the explicit expression of $\hat{f}_\beta$, we have[4]

$$\hat{f}_\beta^\sharp(x) = \frac{1}{\beta}\,\hat{h}\Big(\tfrac{x}{\beta}\Big) \tag{1.36}$$

where $h : p \longmapsto e^{-\cosh p} \in \mathcal{S}(\mathbb{R})$. Then

$$\begin{aligned}\int_{\mathbb{R}} dx\, |\hat{f}_\beta^\sharp(x)|^2 S(x) &= \frac{1}{\beta^2}\int_{\mathbb{R}} dx\, |\hat{h}(x/\beta)|^2\, S(x)\\ &= \frac{1}{\beta}\int_{\mathbb{R}} dx\, |\hat{h}(x)|^2\, S(\beta x).\end{aligned}$$

By dominated convergence theorem we obtain

$$\lim_{\beta\to 0^+} \int_{\mathbb{R}} dx\, |\hat{h}(x)|^2\, S(\beta x) = \|h\|^2 \lim_{x\to 0} S(x) < 0.$$

Hence, the lemma is proven by noticing that the previous integral is continuous in $\beta > 0$ and therefore, the quadratic form $\big(\Phi^0_{\text{diag}} + \Phi^0_{\text{off}} + \Phi_{\text{reg}}\big)[f_\beta]$ is negative for any β small enough.

□

Proof of theorem 1.1. Let $\hat{f}_\beta$ be the trial function given in lemma 1.3 with $\beta < \beta_0$ and consider the following sequence of charges

$$\hat{\eta}_n^\beta(\boldsymbol{p}) = \frac{1}{n}\,\hat{f}_\beta\Big(\frac{\boldsymbol{p}}{n}\Big)\,.$$

By lemma 1.2, we know that

$$\Phi^\lambda[\eta_n^\beta] = n^2(\Phi^0_{\text{diag}} + \Phi^0_{\text{off}} + \Phi_{\text{reg}})[f_\beta] + \mathcal{O}(n), \qquad n \to +\infty$$

and then $\Phi^\lambda[\eta_n^\beta] \longrightarrow -\infty$ as n grows to infinity.

□

1.3 Analysis in Position Space

In this section, we give a different proof of the coercivity of Φ^λ, defined in (1.11), based on its representation in position space. In particular, with proposition 1.5 we identify a negative contribution of the quadratic form $\Phi^\lambda_{\text{off}}$ that is not bounded in L^2 (hence it contains the singularity

[4]We stress that $\hat{f}_\beta^\sharp(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\beta}\,\mathrm{K}_{ix/\beta}(1)$ because of the integral representation for the Macdonald function K_ν given by [20, p. 384, 3.547 4].

of the problem) and therefore the choice of the regularization term Φ_{reg} needed to compensate such a negative contribution is justified.
In the next proposition we write the quadratic form Φ^λ in the position-space representation.

Proposition 1.4. *For any* $\xi \in H^{\frac{1}{2}}(\mathbb{R}^3)$ *and* $\lambda > 0$ *one has*

$$\Phi^\lambda_{\text{diag}}[\xi] = 12\pi\sqrt{\lambda}\,\|\xi\|^2 + \frac{2\sqrt{3}\lambda}{\pi}\int_{\mathbb{R}^6} d\boldsymbol{x}d\boldsymbol{y}\;\frac{|\xi(\boldsymbol{x})-\xi(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^2}\,\mathrm{K}_2\Big(\sqrt{\tfrac{4\lambda}{3}}\,|\boldsymbol{x}-\boldsymbol{y}|\Big), \tag{1.37a}$$

$$\Phi^\lambda_{\text{off}}[\xi] = -\frac{8\sqrt{3}\lambda}{\pi}\int_{\mathbb{R}^6} d\boldsymbol{x}d\boldsymbol{y}\;\frac{\overline{\xi(\boldsymbol{x})}\,\xi(\boldsymbol{y})}{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\mathrm{K}_2\Big(\sqrt{\tfrac{4\lambda}{3}}\,\sqrt{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\Big), \tag{1.37b}$$

$$\Phi_{\text{reg}}[\xi] = 12\pi\gamma\int_{\mathbb{R}^3} d\boldsymbol{x}\;\frac{|\xi(\boldsymbol{x})|^2}{|\boldsymbol{x}|} \tag{1.37c}$$

where $\mathrm{K}_\mu : \mathbb{R}_+ \longrightarrow \mathbb{C}$ *is the modified Bessel function of the second kind (also known as Macdonald function) and order* $\mu \in \mathbb{C}$*.*

Proof. Identity (1.37a) is a consequence of (1.12a) and [24, section 7.12, (5)], while (1.37c) is obtained by comparing (1.12c) with the identity

$$\int_{\mathbb{R}^3} d\boldsymbol{r}\;\frac{|f(\boldsymbol{r})|^2}{r} = \frac{1}{2\pi^2}\int_{\mathbb{R}^6} d\boldsymbol{p}d\boldsymbol{q}\;\frac{\overline{\hat{f}(\boldsymbol{p})}\,\hat{f}(\boldsymbol{q})}{|\boldsymbol{p}-\boldsymbol{q}|^2}, \qquad \forall\, f \in H^{1/2}(\mathbb{R}^3). \tag{1.38}$$

Concerning the proof of (1.37b), we consider (1.12b) for $\varphi \in \mathcal{S}(\mathbb{R}^3)$ and we observe that we have uniformly in $\lambda > 0$

$$\boldsymbol{\sigma} \longmapsto \frac{1}{\sigma^2+\tau^2+\boldsymbol{\tau}\cdot\boldsymbol{\sigma}+\lambda} \in L^2(\mathbb{R}^3, d\boldsymbol{\sigma}), \qquad \text{for } \tau \neq 0.$$

Therefore, by Plancherel's theorem we find

$$\Phi^\lambda_{\text{off}}[\varphi] = -\frac{12}{\pi}\int_{\mathbb{R}^3} d\boldsymbol{\tau}\;\overline{\hat{\varphi}(\boldsymbol{\tau})}\int_{\mathbb{R}^3} d\boldsymbol{x}\;\frac{\varphi(\boldsymbol{x})}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} d\boldsymbol{\sigma}\;\frac{e^{-i\boldsymbol{x}\cdot\boldsymbol{\sigma}}}{\tau^2+\sigma^2+\boldsymbol{\tau}\cdot\boldsymbol{\sigma}+\lambda}.$$

Using the change of coordinates $\boldsymbol{\sigma} = \boldsymbol{q} - \frac{\boldsymbol{\tau}}{2}$, we obtain

$$\begin{aligned}
\Phi^\lambda_{\text{off}}[\varphi] &= -\frac{12}{\pi}\int_{\mathbb{R}^3} d\boldsymbol{\tau}\;\overline{\hat{\varphi}(\boldsymbol{\tau})}\int_{\mathbb{R}^3} d\boldsymbol{x}\;\frac{e^{\frac{\boldsymbol{\tau}\cdot\boldsymbol{x}}{2}i}}{(2\pi)^{3/2}}\,\varphi(\boldsymbol{x})\int_{\mathbb{R}^3} d\boldsymbol{q}\;\frac{e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}}{\frac{3}{4}\tau^2+q^2+\lambda} \\
&= -\frac{24\pi}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} d\boldsymbol{\tau}\;\overline{\hat{\varphi}(\boldsymbol{\tau})}\int_{\mathbb{R}^3} d\boldsymbol{x}\;\frac{\varphi(\boldsymbol{x})}{|\boldsymbol{x}|}\,e^{\frac{\boldsymbol{\tau}\cdot\boldsymbol{x}}{2}i-\sqrt{\frac{3}{4}\tau^2+\lambda}\,|\boldsymbol{x}|} \\
&= -\frac{24\pi}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} d\boldsymbol{x}\;\frac{\varphi(\boldsymbol{x})}{|\boldsymbol{x}|}\int_{\mathbb{R}^3} d\boldsymbol{\tau}\;\overline{\hat{\varphi}(\boldsymbol{\tau})}\,e^{\frac{\boldsymbol{\tau}\cdot\boldsymbol{x}}{2}i-\sqrt{\frac{3}{4}\tau^2+\lambda}\,|\boldsymbol{x}|}.
\end{aligned}$$

Since uniformly in $\lambda > 0$

$$\boldsymbol{\tau} \longmapsto e^{\frac{\boldsymbol{\tau}\cdot\boldsymbol{x}}{2}\,i-\sqrt{\frac{3}{4}\tau^2+\lambda}\,|\boldsymbol{x}|} \in L^2(\mathbb{R}^3, d\boldsymbol{\tau}), \qquad \text{for } x \neq 0$$

we use again Plancherel's theorem to get

$$\Phi^\lambda_{\text{off}}[\varphi] = -\frac{24\pi}{(2\pi)^3}\int_{\mathbb{R}^3} d\boldsymbol{x}\;\frac{\varphi(\boldsymbol{x})}{|\boldsymbol{x}|}\int_{\mathbb{R}^3} d\boldsymbol{y}\;\overline{\varphi(\boldsymbol{y})}\int_{\mathbb{R}^3} d\boldsymbol{\tau}\;e^{i\boldsymbol{\tau}\cdot\left(\boldsymbol{y}+\frac{\boldsymbol{x}}{2}\right)-\sqrt{\frac{3}{4}\tau^2+\lambda}\,|\boldsymbol{x}|}$$

$$= -\frac{12}{\pi}\int_{\mathbb{R}^3} d\boldsymbol{x}\ \frac{\varphi(\boldsymbol{x})}{|\boldsymbol{x}|}\int_{\mathbb{R}^3} d\boldsymbol{y}\ \frac{\overline{\varphi(\boldsymbol{y})}}{|\boldsymbol{y}+\frac{\boldsymbol{x}}{2}|}\int_0^{+\infty} d\tau\, \tau \sin\big(\tau|\boldsymbol{y}+\tfrac{\boldsymbol{x}}{2}|\big)\, e^{-\sqrt{\frac{3}{4}\tau^2+\lambda}\,|\boldsymbol{x}|}.$$

The last integral can be explicitly computed using the formula (see, e.g., [20, p. 491, 3.914.6])

$$\int_0^{+\infty} dx\ x\sin(bx)\, e^{-\beta\sqrt{x^2+\gamma^2}} = \frac{b\beta\gamma^2}{\beta^2+b^2}\,\mathrm{K}_2\Big(\gamma\sqrt{\beta^2+b^2}\Big), \quad \forall\ b\in\mathbb{R} \text{ and } \beta,\gamma>0 \tag{1.39}$$

and therefore identity (1.37b) is proven for $\varphi\in\mathcal{S}(\mathbb{R}^3)$. By a density argument[5] the result is extended to any $\varphi\in H^{\frac{1}{2}}(\mathbb{R}^3)$.

□

Before proceeding, let us briefly recall some elementary properties of $\mathrm{K}_\mu(\cdot)$, $\mu\geq 0$:

$$z^\mu\,\mathrm{K}_\mu(z) \text{ is decreasing in } z\in\mathbb{R}_+, \tag{1.40a}$$

$$\mathrm{K}_\mu(z) = \frac{2^{\mu-1}\Gamma(\mu)}{z^\mu} + o\big(z^{-\mu}\big), \qquad \text{for } z\longrightarrow 0^+,\ \mu>0, \tag{1.40b}$$

$$\mathrm{K}_\mu(z) = \sqrt{\tfrac{\pi}{2z}}\, e^{-z}\Big[1+\tfrac{4\mu^2-1}{8z}+\mathcal{O}\big(z^{-2}\big)\Big], \qquad \text{for } z\longrightarrow+\infty \tag{1.40c}$$

where $\Gamma:\ \mathbb{R}\smallsetminus -\mathbb{N}_0\longrightarrow\mathbb{R}$ denotes the Euler Gamma function. In particular, notice that (1.40a) and (1.40b) imply

$$z^\mu\,\mathrm{K}_\mu(z)\leq 2^{\mu-1}\Gamma(\mu), \qquad \forall\ z\geq 0,\ \mu>0. \tag{1.40d}$$

In the next proposition we show the relevant fact that a negative contribution of $\Phi^\lambda_{\text{off}}$ that is not bounded in $L^2(\mathbb{R}^3)$ can be explicitly isolated.

Proposition 1.5. *For any $\varphi\in H^{\frac{1}{2}}(\mathbb{R}^3)$ and $\lambda>0$ one has*

$$\begin{aligned}\Phi^\lambda_{\text{off}}[\varphi] = &-24\pi\int_{\mathbb{R}^3} d\boldsymbol{x}\ \frac{e^{-\sqrt{\lambda}\,|\boldsymbol{x}|}}{|\boldsymbol{x}|}\,|\varphi(\boldsymbol{x})|^2+\\ &+\frac{4\sqrt{3}\lambda}{\pi}\int_{\mathbb{R}^6} d\boldsymbol{x}d\boldsymbol{y}\ \frac{|\varphi(\boldsymbol{x})-\varphi(\boldsymbol{y})|^2}{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\mathrm{K}_2\Big(\sqrt{\tfrac{4\lambda}{3}}\,\sqrt{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\Big).\end{aligned} \tag{1.41}$$

Proof. Let us decompose the expression given by (1.37b) as follows

$$\begin{aligned}\Phi^\lambda_{\text{off}}[\varphi] = -\frac{8\sqrt{3}\lambda}{\pi}\Bigg[&\int_{\mathbb{R}^3} d\boldsymbol{y}\ |\varphi(\boldsymbol{y})|^2\int_{\mathbb{R}^3} d\boldsymbol{x}\ \frac{\mathrm{K}_2\Big(\sqrt{\frac{4\lambda}{3}}\,\sqrt{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\Big)}{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}+\\ &+\int_{\mathbb{R}^3} d\boldsymbol{y}\ \overline{\varphi(\boldsymbol{y})}\int_{\mathbb{R}^3} d\boldsymbol{x}\ \frac{\varphi(\boldsymbol{x})-\varphi(\boldsymbol{y})}{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\mathrm{K}_2\Big(\sqrt{\tfrac{4\lambda}{3}}\,\sqrt{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\Big)\Bigg],\end{aligned}$$

Then, we evaluate the first term in the right hand side. In proposition 1.4 we have seen that the function

$$\begin{aligned}\hat{f}^\lambda_x:\mathbb{R}^3&\longrightarrow\mathbb{R}_+, \qquad x,\lambda\in\mathbb{R}_+,\\ \boldsymbol{\tau}&\longmapsto\frac{e^{-x\sqrt{\frac{3}{4}\tau^2+\lambda}}}{x}\end{aligned} \tag{1.42}$$

[5] Because of propositions 1.5 and 1.6, one can obtain a control in the $H^{1/2}$ norm.

is such that

$$f^{\lambda}_{|\boldsymbol{x}|}\Big(\boldsymbol{y}+\frac{\boldsymbol{x}}{2}\Big)=\sqrt{\frac{8}{3\pi}}\,\lambda\,\frac{\mathrm{K}_2\Big(\sqrt{\frac{4\lambda}{3}}\,\sqrt{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\Big)}{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}. \tag{1.43}$$

Notice the symmetry in the exchange $\boldsymbol{x}\longleftrightarrow\boldsymbol{y}$. Then,

$$\int_{\mathbb{R}^3}\!d\boldsymbol{x}\;f^{\lambda}_{|\boldsymbol{x}|}\big(\boldsymbol{y}+\tfrac{\boldsymbol{x}}{2}\big)=\int_{\mathbb{R}^3}\!d\boldsymbol{x}\;f^{\lambda}_{|\boldsymbol{y}|}\big(\boldsymbol{x}+\tfrac{\boldsymbol{y}}{2}\big)=\int_{\mathbb{R}^3}\!d\boldsymbol{z}\;f^{\lambda}_{|\boldsymbol{y}|}(\boldsymbol{z})=(2\pi)^{3/2}\hat{f}^{\lambda}_{|\boldsymbol{y}|}(0).$$

Therefore we find

$$\frac{\lambda}{\sqrt{3}\pi^2}\int_{\mathbb{R}^3}\!d\boldsymbol{x}\;\frac{\mathrm{K}_2\Big(\sqrt{\frac{4\lambda}{3}}\,\sqrt{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\Big)}{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}=\frac{e^{-\sqrt{\lambda}|\boldsymbol{y}|}}{|\boldsymbol{y}|},\qquad\forall\;\lambda>0,\;\boldsymbol{y}\in\mathbb{R}^3\smallsetminus\{\boldsymbol{0}\}. \tag{1.44}$$

According to (1.44), we obtain

$$\begin{aligned}\Phi^{\lambda}_{\text{off}}[\varphi]=&-24\pi\int_{\mathbb{R}^3}\!d\boldsymbol{y}\;|\varphi(\boldsymbol{y})|^2\,\frac{e^{-\sqrt{\lambda}|\boldsymbol{y}|}}{|\boldsymbol{y}|}+\\&+\frac{8\sqrt{3}\lambda}{\pi}\int_{\mathbb{R}^6}\!d\boldsymbol{x}d\boldsymbol{y}\;\frac{\overline{\varphi(\boldsymbol{y})}\,[\varphi(\boldsymbol{y})-\varphi(\boldsymbol{x})]}{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\mathrm{K}_2\Big(\sqrt{\tfrac{4\lambda}{3}}\,\sqrt{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\Big).\end{aligned} \tag{1.45}$$

It is now sufficient to notice that the symmetry in exchanging $\boldsymbol{x}\longleftrightarrow\boldsymbol{y}$ allows us to write

$$\begin{aligned}\int_{\mathbb{R}^3}\!d\boldsymbol{y}\;\overline{\varphi(\boldsymbol{y})}\int_{\mathbb{R}^3}\!d\boldsymbol{x}\;\frac{\varphi(\boldsymbol{y})-\varphi(\boldsymbol{x})}{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\mathrm{K}_2\Big(\sqrt{\tfrac{4\lambda}{3}}\,\sqrt{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\Big)=&\\=\frac{1}{2}\int_{\mathbb{R}^6}\!d\boldsymbol{x}d\boldsymbol{y}\;\frac{|\varphi(\boldsymbol{y})-\varphi(\boldsymbol{x})|^2}{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\mathrm{K}_2\Big(\sqrt{\tfrac{4\lambda}{3}}\,\sqrt{y^2+x^2+\boldsymbol{x}\cdot\boldsymbol{y}}\,\Big).&\end{aligned}$$

and the proposition is proved.

□

In the next proposition we are going to provide lower and upper bounds for Φ^{λ} exploiting the representation given by propositions 1.4 and 1.5. To this end, it is convenient to introduce the Gagliardo semi-norm of the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^d)$, defined as

$$[u]^2_{\frac{1}{2}}:=\int_{\mathbb{R}^{2d}}\!d\boldsymbol{x}d\boldsymbol{y}\;\frac{|u(\boldsymbol{x})-u(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+1}},\qquad u\in H^{\frac{1}{2}}(\mathbb{R}^d), \tag{1.46}$$

so that $\|u\|^2_{H^{1/2}(\mathbb{R}^d)}=\|u\|^2+[u]^2_{\frac{1}{2}}$. In terms of the Fourier transform of u we also have (see e.g., [12, proposition 3.4] or [24, section 7.12 (4)])

$$[u]^2_{\frac{1}{2}}=\frac{2\pi^{\frac{d+1}{2}}}{\Gamma\big(\frac{d+1}{2}\big)}\int_{\mathbb{R}^d}\!d\boldsymbol{k}\;|\boldsymbol{k}||\hat{u}(\boldsymbol{k})|^2. \tag{1.47}$$

Proposition 1.6. *For any given $\varphi\in H^{\frac{1}{2}}(\mathbb{R}^3)$, one has*

$$\Phi^{\lambda}[\varphi]\geq\Phi_0[\varphi]+\Phi^{\lambda}_{\text{diag}}[\varphi]+3\min\{0,\gamma-2\}[\varphi]^2_{\frac{1}{2}}, \tag{1.48a}$$

$$\Phi^{\lambda}[\varphi]\leq\Phi_0[\varphi]+\Phi^{\lambda}_{\text{diag}}[\varphi]+\Big(3\gamma+\tfrac{96\sqrt{3}}{\pi}\Big)[\varphi]^2_{\frac{1}{2}}. \tag{1.48b}$$

Proof. The lower bound is obtained by neglecting the positive part of $\Phi^{\lambda}_{\text{off}}$

$$\Phi^{\lambda}_{\text{off}}[\varphi] + \Phi_{\text{reg}}[\varphi] \geq 12\pi \int_{\mathbb{R}^3} d\boldsymbol{y}\, \frac{|\varphi(\boldsymbol{y})|^2}{|\boldsymbol{y}|} \left(\gamma - 2e^{-\sqrt{\lambda}|\boldsymbol{y}|}\right)$$

and by considering the following inequalities

$$\inf_{\boldsymbol{y}\in\mathbb{R}^3} \left\{\gamma - 2e^{-\sqrt{\lambda}|\boldsymbol{y}|}\right\} \geq \min\{0, \gamma - 2\}, \tag{1.49}$$

$$\int_{\mathbb{R}^3} d\boldsymbol{x}\, \frac{|\varphi(\boldsymbol{x})|^2}{|\boldsymbol{x}|} \leq \frac{1}{4\pi} [\varphi]^2_{\frac{1}{2}}. \tag{1.50}$$

Notice that (1.50) is a consequence of the comparison between (1.47) and the (sharp) Hardy-Rellich inequality (see[6] [37])

$$\int_{\mathbb{R}^d} d\boldsymbol{x}\, \frac{|u(\boldsymbol{x})|^2}{|\boldsymbol{x}|^{2s}} \leq \frac{1}{2^{2s}} \frac{\Gamma^2\left(\frac{d}{4} - \frac{s}{2}\right)}{\Gamma^2\left(\frac{d}{4} + \frac{s}{2}\right)} \int_{\mathbb{R}^d} d\boldsymbol{k}\, |\boldsymbol{k}|^{2s} |\hat{u}(\boldsymbol{k})|^2, \qquad \forall\, u \in H^s(\mathbb{R}^d),\ s < \tfrac{d}{2} \tag{1.51}$$

in case $s = \frac{1}{2}$ and $d = 3$. In order to obtain the upper bound, we recall (1.46) to get

$$\int_{\mathbb{R}^6} d\boldsymbol{x} d\boldsymbol{y}\, \frac{|\varphi(\boldsymbol{y}) - \varphi(\boldsymbol{x})|^2}{y^2 + x^2 + \boldsymbol{x}\cdot\boldsymbol{y}} \mathrm{K}_2\left(\sqrt{\tfrac{4\lambda}{3}}\, \sqrt{y^2 + x^2 + \boldsymbol{x}\cdot\boldsymbol{y}}\right) \leq$$
$$\leq [\varphi]^2_{\frac{1}{2}} \sup_{(\boldsymbol{x},\,\boldsymbol{y})\in\mathbb{R}^6} \frac{|\boldsymbol{x} - \boldsymbol{y}|^4}{y^2 + x^2 + \boldsymbol{x}\cdot\boldsymbol{y}} \mathrm{K}_2\left(\sqrt{\tfrac{4\lambda}{3}}\, \sqrt{y^2 + x^2 + \boldsymbol{x}\cdot\boldsymbol{y}}\right).$$

We make use of (1.40d) and get rid of the dependence on the angles in evaluating the $\sup$, since

$$\begin{cases} x^2 + y^2 - 2\,\boldsymbol{x}\cdot\boldsymbol{y} \leq 2(x^2 + y^2), \\ x^2 + y^2 + \boldsymbol{x}\cdot\boldsymbol{y} \geq \frac{x^2+y^2}{2}, \end{cases} \implies \frac{|\boldsymbol{x} - \boldsymbol{y}|^2}{x^2 + y^2 + \boldsymbol{x}\cdot\boldsymbol{y}} \leq 4.$$

Hence,

$$\int_{\mathbb{R}^6} d\boldsymbol{x} d\boldsymbol{y}\, \frac{|\varphi(\boldsymbol{y}) - \varphi(\boldsymbol{x})|^2}{y^2 + x^2 + \boldsymbol{x}\cdot\boldsymbol{y}} \mathrm{K}_2\left(\sqrt{\tfrac{4\lambda}{3}}\, \sqrt{y^2 + x^2 + \boldsymbol{x}\cdot\boldsymbol{y}}\right) \leq$$
$$\leq \frac{3}{2\lambda} [\varphi]^2_{\frac{1}{2}} \sup_{(\boldsymbol{x},\,\boldsymbol{y})\in\mathbb{R}^6} \frac{|\boldsymbol{x} - \boldsymbol{y}|^4}{\left(y^2 + x^2 + \boldsymbol{x}\cdot\boldsymbol{y}\right)^2} = \frac{24}{\lambda} [\varphi]^2_{\frac{1}{2}}.$$

We stress that in the last step we have an equality, since the argument of the supremum in $\mathbb{R}^6$ attains the previous upper bound along the hyperplane $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{0}$.
So far, we have obtained for any $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^3)$

$$\Phi^{\lambda}_{\text{off}}[\varphi] \leq -24\pi \int_{\mathbb{R}^3} d\boldsymbol{y}\, \frac{|\varphi(\boldsymbol{y})|^2}{|\boldsymbol{y}|} e^{-\sqrt{\lambda}|\boldsymbol{y}|} + \frac{96\sqrt{3}}{\pi} [\varphi]^2_{\frac{1}{2}}. \tag{1.52}$$

We complete the proof simply by neglecting the negative contribution.

□

[6] There is a typo in [37, equation (1.4)]: the power 2 on the Euler Gamma function in the numerator is missing.

The major difficulties in the proof of the coercivity of Φ^λ in momentum space obtained in [5] lie in the search of a lower bound. On the other hand, in position space such estimate, provided in proposition 1.6, turns out to be much easier. However, some accuracy is lost in this framework. Indeed, adopting (1.48a) to obtain an estimate from below for Φ^λ, one gets

$$\Phi^\lambda[\xi] \geq \Phi^\lambda_{\text{diag}}[\xi] - 3\max\{0, 2-\gamma\}[\xi]^2_{\frac{1}{2}} + 12\pi \operatorname*{essinf}_{\boldsymbol{y}\in\mathbb{R}^3} \beta(\boldsymbol{y})\, \|\xi\|^2 \tag{1.53}$$

or, equivalently

$$\begin{aligned}\Phi^\lambda[\xi] &\geq \int_{\mathbb{R}^3} d\boldsymbol{p}\left[12\pi\sqrt{\tfrac{3}{4}p^2+\lambda} - 6\pi^2\max\{0,2-\gamma\}p + 12\pi \operatorname*{essinf}_{\boldsymbol{y}\in\mathbb{R}^3}\beta(\boldsymbol{y})\right]|\hat{\xi}(\boldsymbol{p})|^2 \\ &= 12\pi\int_{\mathbb{R}^3} d\boldsymbol{p}\left[1 - \frac{\pi}{2}\max\{0,2-\gamma\}\frac{p}{\sqrt{3/4\,p^2+\lambda}} + \frac{1}{\sqrt{3/4\,p^2+\lambda}}\operatorname*{essinf}_{\boldsymbol{y}\in\mathbb{R}^3}\beta(\boldsymbol{y})\right]\sqrt{\tfrac{3}{4}p^2+\lambda}\,|\hat{\xi}(\boldsymbol{p})|^2.\end{aligned} \tag{1.54}$$

First assume $\operatorname*{essinf}_{\boldsymbol{y}\in\mathbb{R}^3}\beta(\boldsymbol{y}) \geq 0$. Then, one simply has

$$\Phi^\lambda[\xi] \geq \left(1 - \frac{\pi}{\sqrt{3}}\max\{0,2-\gamma\}\right)\Phi^\lambda_{\text{diag}}[\xi], \qquad \forall\,\lambda > 0$$

and the right hand side is positive as soon as

$$\gamma > 2 - \frac{\sqrt{3}}{\pi} =: \bar{\gamma}_c^{\text{3b}}. \tag{1.55}$$

On the other hand, let $\operatorname{essinf}\beta$ be negative. In this case, the function in square brackets in (1.54) attains its minimum at

$$p_{\text{min}} = \frac{2\pi\lambda\max\{0,2-\gamma\}}{3|\operatorname{essinf}\beta|},$$

Plugging the value of p_{min} in the right hand side of (1.54), one gets

$$\Phi^\lambda[\xi] \geq \left(1 - \sqrt{\tfrac{\pi^2}{3}\max\{0,2-\gamma\}^2 + \tfrac{(\operatorname{essinf}\beta)^2}{\lambda}}\right)\Phi^\lambda_{\text{diag}}[\xi] \tag{1.56}$$

which is positive provided

$$1 - \tfrac{\pi}{\sqrt{3}}\max\{0,2-\gamma\} > 0 \iff \gamma > 2 - \frac{\sqrt{3}}{\pi} = \bar{\gamma}_c^{\text{3b}} \tag{1.57}$$

and

$$\lambda > \frac{3(\operatorname{essinf}\beta)^2}{3 - \pi^2\max\{0,2-\gamma\}^2}. \tag{1.58}$$

Thus, regardless the sign of $\operatorname{essinf}\beta$ we have obtained the coercivity by assuming

$$\gamma > \bar{\gamma}_c^{\text{3b}} \qquad \wedge \qquad \lambda > \frac{3\min\{0,\operatorname{essinf}\beta\}^2}{3-\pi^2\max\{0,2-\gamma\}^2} =: \lambda_0\,.$$

As mentioned above, $\bar{\gamma}_c^{\text{3b}} \approx 1.44867$ is not optimal, since $\bar{\gamma}_c^{\text{3b}} > \gamma_c^{\text{3b}}$.

Exploiting (1.10), we are able to estimate $\inf \sigma(\mathcal{H})$ as follows

$$\inf \sigma(\mathcal{H}) > -\lambda_0 \geq \begin{cases} 0, & \alpha_0 \geq \dfrac{\gamma}{b}, \\ -\dfrac{3(\gamma/b - \alpha_0)^2}{3 - \pi^2 \max\{0, 2-\gamma\}^2}, & \alpha_0 < \dfrac{\gamma}{b}. \end{cases} \tag{1.59}$$

We conclude this chapter with the following observation. The coercivity is obtained in $H^{\frac{1}{2}}(\mathbb{R}^3)$ for any λ large enough so that the closedness of Φ^λ easily follows for those values of λ. However, the closedness of Φ^λ can be proved for any $\lambda > 0$, provided $\gamma > \gamma_c^{3\mathrm{b}}$ (see proposition 2.1 in the next chapter, whose proof is given in section 2.6). Furthermore, notice that in [5] the lower bound of the infimum of the spectrum is negative regardless the value of α_0. In this sense estimate (1.59) is more detailed (clearly the same estimates exploited here can be adopted also in the momentum-space representation in order to obtain an even better lower bound for the spectrum involving the optimal threshold value $\gamma_c^{3\mathrm{b}}$).

2. Bose Gas with an Impurity

In this chapter we exploit the Minlos-Faddeev regularization discussed in the introduction to construct a regularized zero-range Hamiltonian in dimension three for a gas of bosons interacting with an impurity. More precisely, we consider a quantum system of N identical spinless bosons of mass m and we assume that the bosons interact only with a different particle of mass m_0, via a zero-range interaction. Let us denote by

$$\mathscr{H}_{N\mathrm{b}+1} := L^2(\mathbb{R}^3) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3N}) \subset L^2(\mathbb{R}^{3(N+1)}), \quad N \geq 2 \tag{2.1}$$

the Hilbert space of the system. At a formal level, the Hamiltonian reads

$$\tilde{\mathcal{H}} = -\frac{1}{2m_0}\Delta_{\boldsymbol{x}_0} - \frac{1}{2m}\sum_{i=1}^{N}\Delta_{\boldsymbol{x}_i} + \nu\sum_{i=1}^{N}\delta(\boldsymbol{x}_i - \boldsymbol{x}_0), \tag{2.2}$$

where ν is a coupling constant and we denote by $\mathcal{H}_0$ the free Hamiltonian, given by

$$\mathcal{H}_0 := -\frac{1}{2m_0}\Delta_{\boldsymbol{x}_0} - \frac{1}{2m}\sum_{i=1}^{N}\Delta_{\boldsymbol{x}_i}, \qquad \mathscr{D}(\mathcal{H}_0) = \mathscr{H}_{N\mathrm{b}+1} \cap H^2(\mathbb{R}^{3(N+1)}). \tag{2.3}$$

We want to define a rigorous counterpart of the formal operator (2.2) as a s.a. and bounded from below operator $\mathcal{H}$ in $\mathscr{H}_{N\mathrm{b}+1}$. By definition, such an operator must be a proper singular perturbation of $\mathcal{H}_0$ supported on the coincidence hyperplanes

$$\pi := \bigcup_{i=1}^{N}\pi_i, \qquad \pi_i := \left\{(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_N) \in \mathbb{R}^{3(N+1)} \,\middle|\, \boldsymbol{x}_i = \boldsymbol{x}_0\right\}. \tag{2.4}$$

In particular, $\mathcal{H}$ must satisfy the property

$$\mathcal{H}\psi = \mathcal{H}_0\psi \qquad \forall\, \psi \in \mathscr{D}(\mathcal{H}_0) \text{ s.t. } \psi|_\pi = 0 \tag{2.5}$$

and, given the operator

$$\dot{\mathcal{H}}_0 := \mathcal{H}_0|_{\mathscr{D}(\dot{\mathcal{H}}_0)}, \qquad \mathscr{D}(\dot{\mathcal{H}}_0) := \mathscr{H}_{N\mathrm{b}+1} \cap H^2_0(\mathbb{R}^{3(N+1)} \setminus \pi), \tag{2.6}$$

which is symmetric and closed according to the graph norm of $\mathcal{H}_0$, we want to find the Hamiltonian $\mathcal{H}$ as a s.a. and bounded from below extension of $\dot{\mathcal{H}}_0$.

The TMS class of extensions is obtained by requiring that an element ψ in the domain of $\mathcal{H}$ satisfies on each hyperplane π_i

$$\begin{aligned}\psi(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_N) = {} & \frac{\xi\left(\frac{m\boldsymbol{x}_i + m_0\boldsymbol{x}_0}{m+m_0}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_i \dots, \boldsymbol{x}_N\right)}{|\boldsymbol{x}_i - \boldsymbol{x}_0|} + \\ & + \alpha_0\, \xi(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_i \dots, \boldsymbol{x}_N) + o(1), \quad \text{for } \boldsymbol{x}_i \longrightarrow \boldsymbol{x}_0,\end{aligned} \tag{2.7}$$

for some $\xi \in \mathscr{H}_{(N-1)\mathrm{b}+1}$, where the notation $\check{\boldsymbol{x}}_i$ indicates the omission of the variable $\boldsymbol{x}_i$.
The s.a. extensions obtained by requiring boundary condition (2.7) lead to the Hamiltonian unbounded from below studied in [26]. As already mentioned, in order to obtain an energetically stable system, we introduce a suitable regularization in (2.7). More precisely, we replace the parameter α_0 by a new, position dependent, coupling constant on each coincidence plane π_i

$$\alpha_0 \longmapsto \alpha(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_i \dots, \boldsymbol{x}_N),$$

where the function $\alpha\colon \mathbb{R}^3 \otimes \mathbb{R}^{3(N-1)} \longrightarrow \mathbb{R}$ is given by

$$\alpha : (\boldsymbol{z}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{N-1}) \longmapsto \alpha_0 + \gamma \sum_{j=1}^{N-1} \frac{\theta(|\boldsymbol{y}_j - \boldsymbol{z}|)}{|\boldsymbol{y}_j - \boldsymbol{z}|}, \tag{2.8}$$

with $\gamma > 0$ and $\theta\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ an essentially bounded function satisfying (1.10).
With the above replacement, we define the modified boundary condition

$$\begin{aligned} \psi(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_N) = {} & \frac{\xi\left(\frac{m\boldsymbol{x}_i + m_0\boldsymbol{x}_0}{m+m_0}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_i \dots, \boldsymbol{x}_N\right)}{|\boldsymbol{x}_i - \boldsymbol{x}_0|} + \\ & + (\Gamma^i_{\mathrm{reg}}\xi)(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_i \dots, \boldsymbol{x}_N) + o(1), \quad \text{for } \boldsymbol{x}_i \to \boldsymbol{x}_0, \end{aligned} \tag{2.9}$$

where Γ^i_{reg} acts as follows

$$\Gamma^i_{\mathrm{reg}} : \xi \longmapsto \alpha(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_i \dots, \boldsymbol{x}_N)\xi(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_i \dots, \boldsymbol{x}_N). \tag{2.10}$$

Notice that $\Gamma^i_{\mathrm{reg}}\xi$ is symmetric under the exchange of any couple $\boldsymbol{x}_k \longleftrightarrow \boldsymbol{x}_\ell$ with $\ell \neq k \in \{1, \dots, N\} \smallsetminus \{i\}$. In analogy with (2.7), boundary condition (2.9) characterizes the point interaction between the impurity and the i-th boson. The function $\alpha(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_i \dots, \boldsymbol{x}_N)$ diverges if $\boldsymbol{x}_j \to \boldsymbol{x}_0$, for any $j \neq i$ and this means that the strength of the point interaction between the impurity and the i-th boson decreases to zero when a third particle, in our case another boson, approaches the common position of the first two particles. In other words, as already pointed out, we are introducing a three-body interaction meant to regularize the ultraviolet singular behavior occurring when the positions of more than two particles coincide. We also stress that θ can be chosen compactly supported, so that the usual two-body point interaction between the impurity and the i-th boson is restored when the other particles are far enough.
The content of this chapter refers to [15] and its goal is to show that boundary condition (2.9) allows to give a rigorous construction of a s.a. and bounded from below Hamiltonian $\mathcal{H}$.
The approach is based on the theory of quadratic forms. More precisely, by a heuristic procedure[1] based on the conditions (2.5), (2.9), we compute the expectation value of the energy of the system by exploiting the formal Hamiltonian (2.2), so that we arrive at the definition of a quadratic form in $\mathscr{H}_{N\mathrm{b}+1}$, given by (2.16), which is the starting point of the rigorous analysis. Our main result is the proof that the quadratic form is closed and bounded from below for any γ larger than a threshold

[1] In this regard, refer to appendix 2.B.

value $\gamma_c^{N\mathrm{b}+1}(M)$. Such a threshold value is explicitly given by (2.19) and it is uniformly bounded in N and M. Furthermore, we characterize the s.a. and bounded from below operator $\mathcal{H}$ uniquely defined by the quadratic form. Such operator, by construction, is our Hamiltonian for the boson gas interacting with an impurity via regularized zero-range interactions.

2.1 Results and Strategy of the Proof

In this section we introduce some definitions in order to formulate our main results. Then, an outline of the strategy adopted to prove such results is provided.
Let us define the bounded operator $G^\lambda : \mathscr{H}_{(N-1)\mathrm{b}+1} \longrightarrow L^2(\mathbb{R}^{3(N+1)})$ whose Fourier representation is given by

$$(\widehat{G^\lambda \xi})(\boldsymbol{p}, \boldsymbol{k}_1, \dots, \boldsymbol{k}_N) := \frac{1}{\mu}\frac{1}{\sqrt{2\pi}}\frac{\sum_{j=1}^N \hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_j, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_j \dots, \boldsymbol{k}_N)}{\frac{1}{2m_0}p^2 + \frac{1}{2m}\sum_{n=1}^N k_n^2 + \lambda}, \tag{2.11}$$

where $\lambda > 0$ and

$$\mu := \frac{m_0 m}{m_0 + m} \tag{2.12}$$

denotes the reduced mass of the two-particle subsystem composed by a boson and the impurity. We shall refer to $G^\lambda \xi$ as the potential produced by the charge ξ distributed on π. A more detailed discussion on the properties of the potential is postponed to appendix 2.A. Here we only mention that G^λ is injective, $\mathrm{ran}(G^\lambda) \subset \mathscr{H}_{N\mathrm{b}+1}$ and $G^\lambda \xi \notin H^1(\mathbb{R}^{3(N+1)})$ (see remarks 2.6, 2.7).
Next, let us define the following hermitian quadratic form in $L^2(\mathbb{R}^{3N})$

$$\mathscr{D}(\Phi^\lambda) = H^{\frac{1}{2}}(\mathbb{R}^{3N}), \qquad \Phi^\lambda := \Phi^\lambda_{\mathrm{diag}} + \Phi^\lambda_{\mathrm{off}} + \Phi_{\mathrm{reg}} + \Phi_0, \tag{2.13}$$

where

$$\Phi^\lambda_{\mathrm{diag}}[\xi] := \tfrac{4\pi N}{\sqrt{2\mu}} \int_{\mathbb{R}^{3N}} d\boldsymbol{p}\, d\boldsymbol{k}_1 \cdots d\boldsymbol{k}_{N-1} \sqrt{\tfrac{p^2}{2(m_0+m)} + \sum_{n=1}^{N-1} \tfrac{k_n^2}{2m} + \lambda}\, |\hat{\xi}(\boldsymbol{p}, \boldsymbol{k}_1, \dots, \boldsymbol{k}_{N-1})|^2, \tag{2.14a}$$

$$\Phi^\lambda_{\mathrm{off}}[\xi] := -\tfrac{N(N-1)}{2\pi\,\mu^2} \int_{\mathbb{R}^{3(N+1)}} d\boldsymbol{p}\, d\boldsymbol{k}_1 \cdots d\boldsymbol{k}_N \, \frac{\overline{\hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_1, \boldsymbol{k}_2, \dots, \boldsymbol{k}_N)}\,\hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}_3, \dots, \boldsymbol{k}_N)}{\frac{1}{2m_0}p^2 + \frac{1}{2m}\sum_{n=1}^N k_n^2 + \lambda}, \tag{2.14b}$$

$$\Phi_{\mathrm{reg}}[\xi] := \tfrac{2\pi N(N-1)\gamma}{\mu} \int_{\mathbb{R}^{3N}} d\boldsymbol{y}\, d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_{N-1} \, \frac{|\xi(\boldsymbol{y}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1})|^2}{|\boldsymbol{y}-\boldsymbol{x}_1|}, \tag{2.14c}$$

$$\Phi_0[\xi] := \tfrac{2\pi N}{\mu} \int_{\mathbb{R}^{3N}} d\boldsymbol{y}\, d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_{N-1} \, \beta(|\boldsymbol{y}-\boldsymbol{x}_1|)\, |\xi(\boldsymbol{y}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1})|^2 \tag{2.14d}$$

with

$$\beta : r \longmapsto \alpha_0 + (N-1)\gamma\, \frac{\theta(r) - 1}{r}, \qquad r > 0. \tag{2.15}$$

Notice that assumption (1.10) implies $\beta \in L^\infty(\mathbb{R}_+)$ and therefore the quadratic form Φ_0 is bounded in $L^2(\mathbb{R}^{3N})$. We are now in position to introduce the main object of our analysis, i.e., the quadratic form in $\mathscr{H}_{N\mathrm{b}+1}$ given by

$$\mathscr{D}(Q) := \Big\{ \psi \in \mathscr{H}_{N\mathrm{b}+1} \,\Big|\, \psi = w^\lambda + G^\lambda \xi,\ w^\lambda \in H^1(\mathbb{R}^{3(N+1)}),\ \xi \in \mathscr{H}_{(N-1)\mathrm{b}+1} \cap H^{\frac{1}{2}}(\mathbb{R}^{3N}) \Big\},$$

$$Q[\psi] := \mathscr{F}_\lambda[w^\lambda] - \lambda\,\|\psi\|^2 + \Phi^\lambda[\xi] \tag{2.16}$$

where

$$\begin{aligned} \mathscr{F}_\lambda \colon H^1(\mathbb{R}^{3(N+1)}) &\longrightarrow \mathbb{R}_+, \\ \varphi &\longmapsto \|\mathcal{H}_0^{\frac{1}{2}}\varphi\|^2 + \lambda\,\|\varphi\|^2. \end{aligned} \tag{2.17}$$

The heuristic motivation leading to the above definition is discussed in appendix 2.B. Preliminarily, we observe that $\mathscr{D}(Q)$ is an extension of the form domain of $\mathcal{H}_0$, since $H^1(\mathbb{R}^{3(N+1)}) \cap \mathscr{H}_{N\mathrm{b}+1}$ is a proper subset of $\mathscr{D}(Q)$ and

$$Q[\psi] = \langle \mathcal{H}_0^{\frac{1}{2}}\psi,\, \mathcal{H}_0^{\frac{1}{2}}\psi\rangle, \qquad \text{for } \psi \in H^1(\mathbb{R}^{3(N+1)}) \cap \mathscr{H}_{N\mathrm{b}+1}\,. \tag{2.18}$$

This is due to the injectivity of G^λ that implies $\psi \in \mathscr{D}(Q) \cap H^1(\mathbb{R}^{3(N+1)})$ if and only if $\xi \equiv 0$.
Moreover, for any fixed $M := \frac{m_0}{m} > 0$ and $N \geq 2$, we introduce the critical parameter

$$\gamma_c^{N\mathrm{b}+1}(M) := \frac{2(M+1)}{\pi}\arcsin\!\left(\tfrac{1}{M+1}\right) - \frac{2\sqrt{M(M+2)}}{\pi(N-1)(M+1)}. \tag{2.19}$$

It is easy to check that (see figure 2.1 at page 34) $\gamma_c^{N\mathrm{b}+1}$ is positive and

$$\inf_{M>0}\gamma_c^{N\mathrm{b}+1}(M) = \frac{2}{\pi}\frac{N-2}{N-1}, \qquad\qquad \sup_{M>0}\gamma_c^{N\mathrm{b}+1}(M) = 1.$$

We notice that $\gamma_c^{N\mathrm{b}+1}$ is uniformly bounded in $N \geq 2$ and $M > 0$. We also stress that our results hold for any $\gamma > \gamma_c^{N\mathrm{b}+1}$.
Moreover, we observe that the Born-Oppenheimer regime is achieved when M is chosen small enough, since the positions of the N bosons would be approximately fixed with respect to the impurity which would play the role of a light particle. In particular one has

$$\lim_{M\to 0^+}\gamma_c^{N\mathrm{b}+1}(M) = 1. \tag{2.20}$$

In the special case $M = 1$, $N = 2$ we have $\gamma_c^{2\mathrm{b}+1}(1) = \frac{2}{3} - \frac{\sqrt{3}}{\pi} \approx 0.115338$. It is worth to observe that in the case of three interacting bosons, a larger critical value $\gamma_c^{3\mathrm{b}} \approx 0.782004$ is found (see (1.13) in the previous chapter). The difference is due to the fact that, in this new framework, the two bosons are non interacting and therefore the singular negative contribution to be compensated, contained in (2.14b), is smaller by a factor 2.
Our first result concerns the quadratic form Φ^λ and it is formulated in the next proposition.

Proposition 2.1.
i) For any $\gamma > 0$ and $\lambda > 0$ one has

$$\Phi^\lambda[\xi] \,\leq\, C_1\,\Phi^\lambda_{\mathrm{diag}}[\xi], \qquad \xi \in H^{\frac{1}{2}}(\mathbb{R}^{3N}) \tag{2.21}$$

where C_1 is a positive constant.
ii) Let us assume $\gamma > \gamma_c^{N\mathrm{b}+1}$. Then, Φ^λ is closed for any $\lambda > 0$ and there exists $\lambda_0 > 0$ s.t. for any $\lambda > \lambda_0$ one has

$$\Phi^\lambda[\xi] \,\geq\, C_2\,\|\xi\|^2_{L^2(\mathbb{R}^{3N})}, \qquad \xi \in H^{\frac{1}{2}}(\mathbb{R}^{3N}) \tag{2.22}$$

where C_2 is a positive constant.

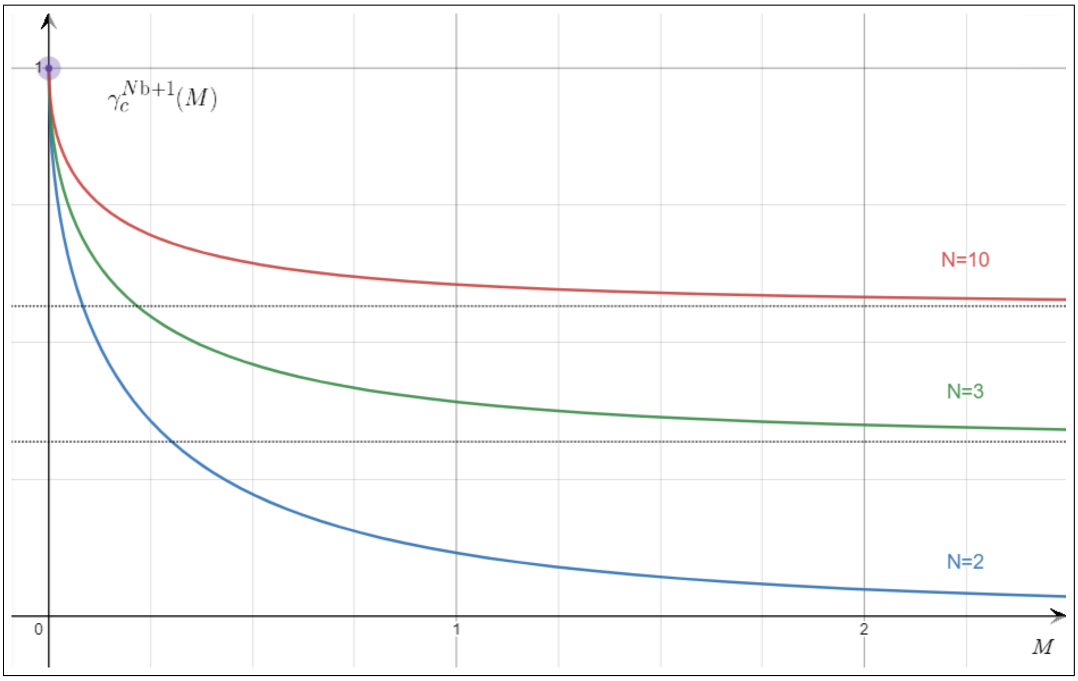

Figure 2.1: Plot of the threshold parameter in terms of the mass ratio.

Proposition 2.1 implies that Φ^λ uniquely defines a s.a. and invertible operator Γ^λ in $L^2(\mathbb{R}^{3N})$ for any $\lambda > \lambda_0$, as long as $\gamma > \gamma_c^{N\mathrm{b}+1}$. Moreover, we shall see that its domain D is independent of $\lambda > \lambda_0$ and we shall also extend the definition of Γ^λ, D to all $\lambda \in -\rho(\mathcal{H}_0)$, preserving its invertibility for any $\lambda \in \mathbb{C} \smallsetminus (-\infty, \lambda_0]$.
Using the above proposition, we can prove our main results that are summarized in the following two theorems.

Theorem 2.2.
Let us assume $\gamma > \gamma_c^{N\mathrm{b}+1}$. Then, the quadratic form Q, $\mathscr{D}(Q)$ in $\mathscr{H}_{N\mathrm{b}+1}$ is closed and bounded from below. In particular, $Q \geq -\lambda_0$.

The second theorem is an elementary consequence of the previous one, and it consists in the characterization of the Hamiltonian $\mathcal{H}$.

Theorem 2.3.
Let us assume $\gamma > \gamma_c^{N\mathrm{b}+1}$. Then, the quadratic form Q, $\mathscr{D}(Q)$ uniquely defines the s.a. and bounded from below operator $\mathcal{H}$, $\mathscr{D}(\mathcal{H})$ characterized as follows

$$\begin{aligned} &\mathscr{D}(\mathcal{H}) = \left\{ \psi \in \mathscr{D}(Q) \,\middle|\, w^\lambda \in H^2(\mathbb{R}^{3(N+1)}),\ \xi \in D,\ \Gamma^\lambda \xi = \tfrac{2\pi N}{\mu} w^\lambda\big|_{\pi_N},\ \lambda > 0 \right\}, \\ &\mathcal{H}\psi = \mathcal{H}_0 w^\lambda - \lambda G^\lambda \xi. \end{aligned} \tag{2.23}$$

Moreover, the resolvent is given by

$$\mathcal{R}_{\mathcal{H}}(z)\psi = \mathcal{R}_{\mathcal{H}_0}(z)\psi + G^{-z}\xi, \qquad \forall\, z \in \mathbb{C} \smallsetminus [-\lambda_0, +\infty), \tag{2.24}$$

where $\psi \in \mathscr{H}_{N\mathrm{b}+1}$ and $\xi \in D$ solves the equation

$$\Gamma^{-z}\xi = \tfrac{2\pi N}{\mu}\Big(\mathcal{R}_{\mathcal{H}_0}(z)\psi\Big)\Big|_{\pi_N}. \tag{2.25}$$

Let us comment on our lower bound $-\lambda_0$ of the quadratic form Q (i.e. the infimum of the spectrum of $\mathcal{H}$). In the course of the proof of proposition 2.1 in section 2.6, we explicitly find

$$\sigma(\mathcal{H}) \geq -\lambda_0 := \begin{cases} 0, & \alpha_0 \geq (N-1)\frac{\gamma}{b}, \\ -\dfrac{[(N-1)\gamma/b - \alpha_0]^2}{2\mu[1-\Lambda_\gamma^2(N,M)]}, & \alpha_0 < (N-1)\frac{\gamma}{b}. \end{cases} \tag{2.26}$$

where

$$\Lambda_\gamma(N,M) = \max\left\{0, 1 - \tfrac{\pi(N-1)}{2} \tfrac{M+1}{\sqrt{M(M+2)}} \left[\gamma - \gamma_c^{N\mathrm{b}+1}(M)\right]\right\} \in [0,1). \tag{2.27}$$

We notice that $\Lambda_\gamma(N,M) \longrightarrow 1^-$ whenever γ approaches $\gamma_c^{N\mathrm{b}+1}(M)$ from above for any choice of $N \geq 2$ and $M > 0$ and therefore we have $-\lambda_0 \longrightarrow -\infty$ for $\gamma \longrightarrow (\gamma_c^{N\mathrm{b}+1})^+$ if α_0 is not large enough.

We conclude this section with an outline of the strategy of the proof. We stress that the main technical point is proposition 2.1, where we estimate the quadratic form Φ^λ. It is worth to notice that the hard part of the work is devoted to finding the estimate from below.

In section 2.2 we introduce suitable changes of coordinates and we rewrite the quadratic form Φ^λ in $L^2(\mathbb{R}^{3N})$ in terms of the quadratic form Θ^ζ in $L^2(\mathbb{R}^3)$ (see (2.39), (2.40) and (2.41)), that is of the type studied in [5, section 3] for the three-particle case. This allows us to reduce the analysis to the estimate of Θ^ζ. As a first result, we prove an estimate from above for Θ^ζ (see proposition 2.5).

In section 2.3 we consider the expressions (2.40) of Θ^ζ in the Fourier space and we expand the quadratic form in partial waves (see (2.45), (2.46)). We also recall some known results about the terms F_ℓ^ζ, $\ell \in \mathbb{N}_0$ of this expansion.

In section 2.4 we prove some estimates that are crucial to control F_ℓ^ζ. We stress that we perform a careful analysis for each value of $\ell \in \mathbb{N}_0$ that leads to a detailed control of the upper bound and the lower bound. In particular, the result of lemma 2.10 allows us to prove proposition 2.1 by introducing the threshold value $\gamma_c^{N\mathrm{b}+1}(M)$ that is uniformly bounded in $M > 0$ and $N \geq 2$. It is worth to mention that [5, lemma 3.5], which provides an analogous result for the s-wave, fails in proving the stability of the system for a region of values of M and N. More precisely, they consider the case $\ell = 0$ and they manage to control higher momenta in terms of the $\ell = 2$ contribution. However, in our framework such a result is not sufficient to obtain a uniform control from below of the quadratic form, since the estimates associated with $\ell = 2$ do not work for all values of M and N and therefore one would be forced to assume a further ad hoc constraint on M (depending on N). Owing to the new techniques developed in this section we are able to avoid this problem and to remove such a technical constraint.

In section 2.5 we use the above order by order approximation to obtain the key estimate from below of Θ^ζ (see (2.105), (2.106)). We also prove another estimate from above of Θ^ζ (see (2.107), (2.108)) which improves the result in proposition 2.5.

In section 2.6 we show that the estimates of Θ^ζ imply those on Φ^λ and thus we conclude the proof of proposition 2.1. Then, following a standard procedure, we also prove theorems 2.2 and 2.3.

We conclude the chapter with an appendix. In appendix 2.A we discuss some useful properties of the potential G^λ. In appendix 2.B we give a heuristic derivation of the quadratic form Q.

2.2 Reduction to a Three-Body Problem

We start the study of Φ^λ, defined by (2.13), introducing suitable changes of variables that reduce the analysis to a quadratic form of the type studied in [5, section 3] for the three-particle case. In the end we shall prove that $\mathscr{D}(\Phi^\lambda) = H^{\frac{1}{2}}(\mathbb{R}^{3N})$.

Let η be the modified reduced mass of the system

$$\eta := \frac{m(m_0+m)}{m_0+2m} = \left(\frac{1}{m} + \frac{1}{m_0+m}\right)^{-1} \tag{2.28}$$

and set $\tilde{\boldsymbol{k}} = (\boldsymbol{k}_1, \dots, \boldsymbol{k}_{N-1})$. Then, we denote for short

$$\hat{\phi}(\boldsymbol{\sigma}, \tilde{\boldsymbol{k}}) := \\ = \left(\tfrac{\tilde{k}^2}{2m} + \lambda\right)^{\frac{3}{4}} \left(\tfrac{m_0}{\eta}\right)^{\frac{3}{4}} \hat{\xi}\left(\sqrt{\tfrac{\mu}{m}}\sqrt{\tfrac{\tilde{k}^2}{2m}+\lambda}\, \boldsymbol{\sigma} + \sqrt{\tfrac{m_0\eta}{m\mu}}\, \boldsymbol{k}_1, \sqrt{\tfrac{\mu\eta}{m_0 m}}\, \boldsymbol{k}_1 - \sqrt{\tfrac{\mu}{m}}\sqrt{\tfrac{\tilde{k}^2}{2m}+\lambda}\, \boldsymbol{\sigma}, \boldsymbol{k}_2, \dots, \boldsymbol{k}_{N-1}\right). \tag{2.29}$$

In the next lemma we rewrite $\Phi^\lambda[\xi]$ in terms of this new charge $\hat{\phi}$.

Lemma 2.4. *For any* $\xi \in H^{\frac{1}{2}}(\mathbb{R}^{3N})$ *one has*

$$\begin{aligned} \Phi^\lambda[\xi] = \Phi_0[\xi] + \tfrac{2\pi N}{\sqrt{m\mu}} \int_{\mathbb{R}^{3(N-1)}} d\tilde{\boldsymbol{k}} \sqrt{\tfrac{\tilde{k}^2}{2m} + \lambda} \,\Bigg[\int_{\mathbb{R}^3} d\boldsymbol{\sigma} \sqrt{\tfrac{\mu}{\eta}\sigma^2 + 2m}\, |\hat{\phi}(\boldsymbol{\sigma}, \tilde{\boldsymbol{k}})|^2 + \\ + \frac{(N-1)\gamma}{2\pi^2} \int_{\mathbb{R}^6} d\boldsymbol{\sigma} d\boldsymbol{\tau}\, \frac{\overline{\hat{\phi}(\boldsymbol{\sigma}, \tilde{\boldsymbol{k}})}\, \hat{\phi}(\boldsymbol{\tau}, \tilde{\boldsymbol{k}})}{|\boldsymbol{\sigma} - \boldsymbol{\tau}|^2} - \frac{N-1}{2\pi^2} \int_{\mathbb{R}^6} d\boldsymbol{\sigma} d\boldsymbol{\tau}\, \frac{\overline{\hat{\phi}(\boldsymbol{\sigma}, \tilde{\boldsymbol{k}})}\, \hat{\phi}(\boldsymbol{\tau}, \tilde{\boldsymbol{k}})}{\sigma^2 + \tau^2 + \frac{2m\, \boldsymbol{\sigma}\cdot\boldsymbol{\tau}}{m_0+m} + 2m} \Bigg] \end{aligned} \tag{2.30}$$

with $\hat{\phi}$ *given by* (2.29).

In order to prove the previous lemma, let us introduce the change of coordinates

$$\begin{cases} \boldsymbol{R} = \sqrt{\tfrac{\eta}{m}} \left(\sqrt{\tfrac{m_0}{\mu}}\, \boldsymbol{x}_0 + \sqrt{\tfrac{\mu}{m_0}}\, \boldsymbol{x}_i\right), \\ \boldsymbol{r} = \sqrt{\tfrac{\mu}{m}}\, (\boldsymbol{x}_0 - \boldsymbol{x}_i) \end{cases} \iff \begin{cases} \boldsymbol{x}_0 = \sqrt{\tfrac{\mu\eta}{m_0 m}}\, \boldsymbol{R} + \tfrac{\eta}{m_0}\sqrt{\tfrac{\mu}{m}}\, \boldsymbol{r}, \\ \boldsymbol{x}_i = \sqrt{\tfrac{\mu\eta}{m_0 m}}\, \boldsymbol{R} - \tfrac{\eta}{\sqrt{m\mu}}\, \boldsymbol{r}. \end{cases} \tag{2.31}$$

Such transformation is encoded by the following unitary operator

$$\begin{aligned} &\mathscr{K}_i \colon L^2(\mathbb{R}^{3N}) \longrightarrow L^2(\mathbb{R}^{3N}, d\boldsymbol{r} d\boldsymbol{R} d\boldsymbol{x}_1 \cdots d\check{\boldsymbol{x}}_i \cdots d\boldsymbol{x}_{N-1}), \\ &(\mathscr{K}_i\psi)(\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_i \dots, \boldsymbol{x}_{N-1}) := \\ &\left(\tfrac{\eta}{m_0}\right)^{\frac{3}{4}} \psi\left(\sqrt{\tfrac{\mu\eta}{m_0 m}}\, \boldsymbol{R} + \tfrac{\eta}{m_0}\sqrt{\tfrac{\mu}{m}}\, \boldsymbol{r}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{i-1}, \sqrt{\tfrac{\mu\eta}{m_0 m}}\, \boldsymbol{R} - \tfrac{\eta}{\sqrt{m\mu}}\, \boldsymbol{r}, \boldsymbol{x}_{i+1}, \dots, \boldsymbol{x}_{N-1}\right). \end{aligned} \tag{2.32a}$$

Notice that $\mathscr{K}_i$ is unitary since $\left(\frac{\eta}{m_0}\right)^{\frac{3}{2}}$ is the Jacobian associated to (2.31). Similarly, one has

$$\begin{aligned}&\mathscr{K}_i^{\,*}\colon L^2(\mathbb{R}^{3N},\, d\boldsymbol{r}d\boldsymbol{R}d\boldsymbol{x}_1\cdots d\check{\boldsymbol{x}}_i\cdots d\boldsymbol{x}_{N-1})\longrightarrow L^2(\mathbb{R}^{3N}),\\ &(\mathscr{K}_i^{\,*}\psi)(\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_{N-1})=\\ &\quad\left(\tfrac{m_0}{\eta}\right)^{\frac{3}{4}}\psi\left(\sqrt{\tfrac{\mu}{m}}\,(\boldsymbol{x}_0-\boldsymbol{x}_i),\,\sqrt{\tfrac{m_0\eta}{m\mu}}\,\boldsymbol{x}_0+\sqrt{\tfrac{\mu\eta}{m_0 m}}\,\boldsymbol{x}_i,\boldsymbol{x}_1,\dots\check{\boldsymbol{x}}_i\dots,\boldsymbol{x}_{N-1}\right).\end{aligned}\tag{2.32b}$$

Next, for any given $\xi\in H^{\frac{1}{2}}(\mathbb{R}^{3N})$, we define the modified charge $\chi\in H^{\frac{1}{2}}(\mathbb{R}^{3N})$ given by

$$\chi(\boldsymbol{r},\boldsymbol{R},\boldsymbol{x}_2,\dots,\boldsymbol{x}_{N-1}):=(\mathscr{K}_1\xi)(\boldsymbol{r},\boldsymbol{R},\boldsymbol{x}_2,\dots,\boldsymbol{x}_{N-1}).\tag{2.33}$$

With this definition we have (see equation (2.37))

$$\hat{\phi}(\boldsymbol{\sigma},\tilde{\boldsymbol{k}})=\left(\tfrac{\tilde{k}^2}{2m}+\lambda\right)^{\frac{3}{4}}\hat{\chi}\left(\sqrt{\tfrac{\tilde{k}^2}{2m}+\lambda}\,\boldsymbol{\sigma},\tilde{\boldsymbol{k}}\right).\tag{2.34}$$

Proof of lemma 2.4. Let us define a new hermitian quadratic form $\tilde{\Phi}^\lambda\colon L^2(\mathbb{R}^{3N})\longrightarrow\mathbb{R}$, given by

$$\tilde{\Phi}^\lambda:=\tilde{\Phi}^\lambda_{\text{diag}}+\tilde{\Phi}^\lambda_{\text{off}}+\tilde{\Phi}_{\text{reg}}\,,\qquad \mathscr{D}(\tilde{\Phi}^\lambda)=H^{\frac{1}{2}}(\mathbb{R}^{3N}),$$

$$\tilde{\Phi}^\lambda_{\text{diag}}[\chi]:=\tfrac{4\pi N}{\sqrt{2\mu}}\int_{\mathbb{R}^{3N}}d\boldsymbol{q}d\boldsymbol{k}_1\cdots d\boldsymbol{k}_{N-1}\sqrt{\tfrac{\mu}{2m\eta}q^2+\textstyle\sum_{n=1}^{N-1}\tfrac{k_n^2}{2m}+\lambda}\;|\hat{\chi}(\boldsymbol{q},\boldsymbol{k}_1,\dots,\boldsymbol{k}_{N-1})|^2,\tag{2.35a}$$

$$\tilde{\Phi}^\lambda_{\text{off}}[\chi]:=-\tfrac{N(N-1)}{2\pi\sqrt{m^3\mu}}\int_{\mathbb{R}^{3(N+1)}}d\boldsymbol{q}d\boldsymbol{p}d\boldsymbol{k}_1\cdots d\boldsymbol{k}_{N-1}\,\frac{\overline{\hat{\chi}(\boldsymbol{q},\boldsymbol{k}_1,\dots,\boldsymbol{k}_{N-1})}\,\hat{\chi}(\boldsymbol{p},\boldsymbol{k}_1,\dots,\boldsymbol{k}_{N-1})}{\frac{q^2+p^2}{2m}+\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{m_0+m}+\frac{1}{2m}\sum_{n=1}^{N-1}k_n^2+\lambda},\tag{2.35b}$$

$$\tilde{\Phi}_{\text{reg}}[\chi]:=\tfrac{N(N-1)\gamma}{\pi\sqrt{m\mu}}\int_{\mathbb{R}^{3(N+1)}}d\boldsymbol{q}d\boldsymbol{p}d\boldsymbol{k}_1\cdots d\boldsymbol{k}_{N-1}\,\frac{\overline{\hat{\chi}(\boldsymbol{q},\boldsymbol{k}_1,\dots,\boldsymbol{k}_{N-1})}\,\hat{\chi}(\boldsymbol{p},\boldsymbol{k}_1,\dots,\boldsymbol{k}_{N-1})}{|\boldsymbol{q}-\boldsymbol{p}|^2}.\tag{2.35c}$$

We want to show that

$$\Phi^\lambda_{\text{diag}}[\xi]=\tilde{\Phi}^\lambda_{\text{diag}}[\chi],\quad \Phi^\lambda_{\text{off}}[\xi]=\tilde{\Phi}^\lambda_{\text{off}}[\chi],\quad \Phi_{\text{reg}}[\xi]=\tilde{\Phi}_{\text{reg}}[\chi]$$

where χ is given by (2.33). Let us show that $\tilde{\Phi}_{\text{reg}}[\chi]=\Phi_{\text{reg}}[\xi]$. To this end, we adopt the change of variables associated to $\mathscr{K}_1$ in (2.14c), yielding

$$\begin{aligned}\Phi_{\text{reg}}[\xi]&=\tfrac{2\pi N(N-1)\gamma}{\mu}\int_{\mathbb{R}^{3N}}d\boldsymbol{x}_0d\boldsymbol{x}_1\cdots d\boldsymbol{x}_{N-1}\,\frac{|\xi(\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_{N-1})|^2}{|\boldsymbol{x}_1-\boldsymbol{x}_0|}\\ &=\tfrac{2\pi N(N-1)\gamma}{\mu}\int_{\mathbb{R}^{3N}}d\boldsymbol{r}d\boldsymbol{R}d\boldsymbol{x}_2\cdots d\boldsymbol{x}_{N-1}\,\sqrt{\frac{\mu}{m}}\,\frac{1}{r}\,|\chi(\boldsymbol{r},\boldsymbol{R},\boldsymbol{x}_2,\dots,\boldsymbol{x}_{N-1})|^2\\ &=\tfrac{N(N-1)\gamma}{\pi\sqrt{m\mu}}\int_{\mathbb{R}^{3(N+1)}}d\boldsymbol{q}d\boldsymbol{p}d\boldsymbol{Q}\,d\boldsymbol{k}_2\cdots d\boldsymbol{k}_{N-1}\,\frac{\overline{\hat{\chi}(\boldsymbol{q},\boldsymbol{Q},\boldsymbol{k}_2,\dots,\boldsymbol{k}_{N-1})}\,\hat{\chi}(\boldsymbol{p},\boldsymbol{Q},\boldsymbol{k}_2,\dots,\boldsymbol{k}_{N-1})}{|\boldsymbol{q}-\boldsymbol{p}|^2}\\ &=\tilde{\Phi}_{\text{reg}}[\chi].\end{aligned}$$

Notice that in the last step we have used identity (1.38). In order to prove the same result for $\tilde{\Phi}^\lambda_{\text{diag}}$ and $\tilde{\Phi}^\lambda_{\text{off}}$, it is helpful to deal with $\mathscr{K}_i$ in Fourier space. If we denote with $(\boldsymbol{q},\,\boldsymbol{Q})$ the couple of conjugate variables of $(\boldsymbol{r},\,\boldsymbol{R})$, while $(\boldsymbol{p},\,\boldsymbol{k}_i)$ are conjugate to $(\boldsymbol{x}_0,\,\boldsymbol{x}_i)$, in the space of momenta (2.31) reads

$$\begin{cases}\boldsymbol{Q}=\sqrt{\frac{\mu\eta}{m_0 m}}(\boldsymbol{p}+\boldsymbol{k}_i),\\ \boldsymbol{q}=\frac{\eta}{m_0}\sqrt{\frac{\mu}{m}}\,\boldsymbol{p}-\frac{\eta}{\sqrt{m\mu}}\,\boldsymbol{k}_i\end{cases}\iff\begin{cases}\boldsymbol{p}=\sqrt{\frac{\mu}{m}}\,\boldsymbol{q}+\sqrt{\frac{m_0\eta}{m\mu}}\,\boldsymbol{Q},\\ \boldsymbol{k}_i=-\sqrt{\frac{\mu}{m}}\,\boldsymbol{q}+\sqrt{\frac{\mu\eta}{m_0 m}}\,\boldsymbol{Q}.\end{cases}\tag{2.36}$$

In particular, one can verify that

$$\hat{\chi}(\boldsymbol{q},\boldsymbol{Q},\boldsymbol{k}_2,\dots,\boldsymbol{k}_{N-1}) = \left(\tfrac{m_0}{\eta}\right)^{\frac{3}{4}}\hat{\xi}\Big(\sqrt{\tfrac{\mu}{m}}\,\boldsymbol{q}+\sqrt{\tfrac{m_0\eta}{m\mu}}\,\boldsymbol{Q},\,\sqrt{\tfrac{\mu\eta}{m_0 m}}\,\boldsymbol{Q}-\sqrt{\tfrac{\mu}{m}}\,\boldsymbol{q},\boldsymbol{k}_2,\dots,\boldsymbol{k}_{N-1}\Big). \tag{2.37}$$

Hence, considering that $\frac{p^2}{2(m_0+m)}+\frac{k_1^2}{2m}=\frac{Q^2}{2m}+\frac{\mu}{2m\eta}q^2$, one has

$$\begin{aligned}\Phi^\lambda_{\text{diag}}[\xi] &= \tfrac{4\pi N}{\sqrt{2\mu}}\int_{\mathbb{R}^{3N}} d\boldsymbol{p}\,d\boldsymbol{k}_1\cdots d\boldsymbol{k}_{N-1}\sqrt{\tfrac{p^2}{2(m_0+m)}+\tfrac{1}{2m}\textstyle\sum_{n=1}^{N-1}k_n^2+\lambda}\;|\hat{\xi}(\boldsymbol{p},\boldsymbol{k}_1,\dots,\boldsymbol{k}_{N-1})|^2\\ &= \tfrac{4\pi N}{\sqrt{2\mu}}\int_{\mathbb{R}^{3N}} d\boldsymbol{Q}\,d\boldsymbol{q}\,d\boldsymbol{k}_2\cdots d\boldsymbol{k}_{N-1}\sqrt{\tfrac{Q^2}{2m}+\tfrac{\mu}{2m\eta}q^2+\tfrac{1}{2m}\textstyle\sum_{n=2}^{N-1}k_n^2+\lambda}\;|\hat{\chi}(\boldsymbol{q},\boldsymbol{Q},\boldsymbol{k}_2,\dots,\boldsymbol{k}_{N-1})|^2\\ &= \tilde{\Phi}^\lambda_{\text{diag}}[\chi].\end{aligned}$$

In the remaining case, we need the change of coordinates

$$\begin{cases}\boldsymbol{p}=\sqrt{\frac{\mu}{m}}(\boldsymbol{q}_1+\boldsymbol{q}_2)+\sqrt{\frac{\mu\eta m_0}{m^3}}\,\boldsymbol{Q},\\ \boldsymbol{k}_1=-\sqrt{\frac{\mu}{m}}\,\boldsymbol{q}_2+\sqrt{\frac{\mu\eta}{m_0 m}}\,\boldsymbol{Q},\\ \boldsymbol{k}_2=-\sqrt{\frac{\mu}{m}}\,\boldsymbol{q}_1+\sqrt{\frac{\mu\eta}{m_0 m}}\,\boldsymbol{Q}\end{cases}\implies\begin{cases}\boldsymbol{p}+\boldsymbol{k}_1=\sqrt{\frac{\mu}{m}}\boldsymbol{q}_1+\sqrt{\frac{m_0\eta}{m\mu}}\,\boldsymbol{Q},\\ \boldsymbol{p}+\boldsymbol{k}_2=\sqrt{\frac{\mu}{m}}\boldsymbol{q}_2+\sqrt{\frac{m_0\eta}{m\mu}}\,\boldsymbol{Q},\\ \frac{p^2}{2m_0}+\frac{k_1^2}{2m}+\frac{k_2^2}{2m}=\frac{q_1^2}{2m}+\frac{q_2^2}{2m}+\frac{Q^2}{2m}+\frac{1}{m_0+m}\boldsymbol{q}_1\cdot\boldsymbol{q}_2\,.\end{cases}$$

Indeed, notice that this substitution of Jacobian $\left(\frac{m_0\mu}{m\eta}\right)^{\frac{3}{2}}$, together with (2.37) lead to

$$\begin{aligned}\Phi^\lambda_{\text{off}}[\xi] &= -\tfrac{N(N-1)}{2\pi\,\mu^2}\int_{\mathbb{R}^{3(N+1)}} d\boldsymbol{p}\,d\boldsymbol{k}_1\cdots d\boldsymbol{k}_N\;\frac{\overline{\hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_1,\boldsymbol{k}_2,\dots,\boldsymbol{k}_N)}\,\hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_2,\boldsymbol{k}_1,\boldsymbol{k}_3,\dots,\boldsymbol{k}_N)}{\frac{1}{2m_0}p^2+\frac{1}{2m}\sum_{n=1}^N k_n^2+\lambda}\\ &= -\tfrac{N(N-1)}{2\pi\sqrt{m^3\mu}}\int_{\mathbb{R}^{3(N+1)}} d\boldsymbol{q}_1 d\boldsymbol{q}_2\,d\boldsymbol{Q}\,d\boldsymbol{k}_3\cdots d\boldsymbol{k}_N\;\frac{\overline{\hat{\chi}(\boldsymbol{q}_1,\boldsymbol{Q},\boldsymbol{k}_3,\dots,\boldsymbol{k}_N)}\,\hat{\chi}(\boldsymbol{q}_2,\boldsymbol{Q},\boldsymbol{k}_3,\dots,\boldsymbol{k}_N)}{\frac{q_1^2}{2m}+\frac{q_2^2}{2m}+\frac{\boldsymbol{q}_1\cdot\boldsymbol{q}_2}{m_0+m}+\frac{Q^2}{2m}+\frac{1}{2m}\sum_{n=3}^N k_n^2+\lambda}\\ &= \tilde{\Phi}^\lambda_{\text{off}}[\chi].\end{aligned}$$

Next, let us define the unitary scaling operator in $L^2(\mathbb{R}^{3N})$

$$\begin{aligned}U\colon L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^{3(N-1)}) &\longrightarrow L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R}^{3(N-1)}),\\ \psi(\boldsymbol{p},\boldsymbol{k}_1,\dots,\boldsymbol{k}_{N-1}) &\longmapsto \Big(\textstyle\sum_{n=1}^{N-1}\frac{k_n^2}{2m}+\lambda\Big)^{\frac{3}{4}}\psi\Big(\sqrt{\textstyle\sum_{n=1}^{N-1}\frac{k_n^2}{2m}+\lambda}\;\boldsymbol{p},\boldsymbol{k}_1,\dots,\boldsymbol{k}_{N-1}\Big).\end{aligned} \tag{2.38}$$

We stress that $\operatorname{ran}(U|_{H^{1/2}(\mathbb{R}^{3N})})\subset H^{\frac{1}{2}}(\mathbb{R}^{3N})$. Using the scaling defined by $\hat{\phi}=U\hat{\chi}$ in $\tilde{\Phi}^\lambda$, we conclude the proof.

□

Formula (2.30) suggests to define the hermitian quadratic form Θ^ζ, for $\zeta\geq 0$, with

$$\mathscr{D}(\Theta^\zeta)=H^{\frac{1}{2}}(\mathbb{R}^3),\quad \Theta^\zeta:=\Theta^\zeta_{\text{diag}}+\Theta^\zeta_{\text{off}}+\Theta_{\text{reg}}\,, \tag{2.39}$$

where, for a given $\varphi\in H^{\frac{1}{2}}(\mathbb{R}^3)$, we have (recalling $M=\frac{m_0}{m}$)

$$\Theta^\zeta_{\text{diag}}[\varphi]:=\int_{\mathbb{R}^3} d\boldsymbol{\sigma}\sqrt{\tfrac{\mu}{\eta}\sigma^2+\zeta}\;|\hat{\varphi}(\boldsymbol{\sigma})|^2, \tag{2.40a}$$

$$\Theta_{\mathrm{reg}}[\varphi] := \frac{(N-1)\gamma}{2\pi^2}\int_{\mathbb{R}^6} d\boldsymbol{\sigma} d\boldsymbol{\tau}\, \frac{\overline{\hat{\varphi}(\boldsymbol{\sigma})}\,\hat{\varphi}(\boldsymbol{\tau})}{|\boldsymbol{\sigma}-\boldsymbol{\tau}|^2}, \tag{2.40b}$$

$$\Theta_{\mathrm{off}}^{\zeta}[\varphi] := -\frac{N-1}{2\pi^2}\int_{\mathbb{R}^6} d\boldsymbol{\sigma} d\boldsymbol{\tau}\, \frac{\overline{\hat{\varphi}(\boldsymbol{\sigma})}\,\hat{\varphi}(\boldsymbol{\tau})}{\sigma^2+\tau^2+\frac{2\boldsymbol{\sigma}\cdot\boldsymbol{\tau}}{M+1}+\zeta}. \tag{2.40c}$$

Observe that

$$\Phi^{\lambda}[\xi] = \Phi_0[\xi] + \tfrac{2\pi N}{\sqrt{m\mu}}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{k}_1\cdots d\boldsymbol{k}_{N-1}\sqrt{\textstyle\sum_{n=1}^{N-1}\frac{k_n^2}{2m}+\lambda}\;\Theta^{2m}[\phi](\boldsymbol{k}_1,\dots,\boldsymbol{k}_{N-1}) \tag{2.41}$$

where ϕ is given by (2.29). Equation (2.41) shows that the study of Φ^{λ} in $L^2(\mathbb{R}^{3N})$ can be reduced to the analysis of Θ^{ζ} in $L^2(\mathbb{R}^3)$. Indeed, notice that Θ^{ζ} is pretty similar to the quadratic form discussed in the three-body case given in (1.12). In the rest of this section and in sections 2.3, 2.4 and 2.5 we shall concentrate on the estimate from above and from below of Θ^{ζ}.
Clearly, $\Theta^{\zeta}_{\mathrm{diag}}[\varphi] < +\infty$ if and only if $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^3)$. With the following proposition we prove that the whole quadratic form Θ^{ζ} is well defined in $H^{\frac{1}{2}}(\mathbb{R}^3)$.

Proposition 2.5. *Given $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{3N})$, $\gamma > 0$ and $\zeta \geq 0$, we have*

$$|\Theta^{\zeta}[\varphi]| \leq \left[1 + \tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}}\left(\tfrac{M+1}{M}+\tfrac{\pi}{2}\gamma\right)\right]\Theta^{\zeta}_{\mathrm{diag}}[\varphi]. \tag{2.42}$$

Proof. The statement is a revised version of [5, proposition 3.1] and the strategy is the same. Concerning $\Theta^{\zeta}_{\mathrm{off}}$, we have

$$|\Theta^{\zeta}_{\mathrm{off}}[\varphi]| \leq \frac{N-1}{2\pi^2}\int_{\mathbb{R}^6} d\boldsymbol{p} d\boldsymbol{q}\, \frac{|\hat{\varphi}(\boldsymbol{p})|\,|\hat{\varphi}(\boldsymbol{q})|}{p^2+q^2+\frac{2\boldsymbol{p}\cdot\boldsymbol{q}}{M+1}} \leq \frac{(N-1)(M+1)}{2M\pi^2}\int_{\mathbb{R}^6} d\boldsymbol{p} d\boldsymbol{q}\, \frac{|\hat{\varphi}(\boldsymbol{p})|\,|\hat{\varphi}(\boldsymbol{q})|}{p^2+q^2},$$

where we have used

$$x^2+y^2-\tfrac{2}{M+1}\,xy \geq (x^2+y^2)\big(1-\tfrac{1}{M+1}\big) = \tfrac{M}{M+1}(x^2+y^2).$$

Let us define the integral operator $\mathcal{Q}\colon L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3)$ acting as

$$(\mathcal{Q}\,\psi)(\boldsymbol{p}) := \int_{\mathbb{R}^3} d\boldsymbol{k}\, \frac{\psi(\boldsymbol{k})}{\sqrt{p}\,(p^2+k^2)\sqrt{k}}. \tag{2.43}$$

Thanks to [17, lemma 2.1], we know that $\mathcal{Q}$ is bounded with norm $2\pi^2$.
Hence, denoted $g(\boldsymbol{k}) := \sqrt{k}\,|\hat{\varphi}(\boldsymbol{k})| \in L^2(\mathbb{R}^3)$, we have

$$\begin{aligned}|\Theta^{\zeta}_{\mathrm{off}}[\varphi]| &\leq \frac{(N-1)(M+1)}{2M\pi^2}\langle g,\,\mathcal{Q}\,g\rangle \leq \frac{(N-1)(M+1)}{M}\,\|g\|^2\\ &= \frac{(N-1)(M+1)}{M}\int_{\mathbb{R}^3} d\boldsymbol{k}\; |\boldsymbol{k}||\hat{\varphi}(\boldsymbol{k})|^2 \leq \frac{(N-1)\,(M+1)^2}{M\sqrt{M(M+2)}}\,\Theta^{\zeta}_{\mathrm{diag}}[\varphi].\end{aligned}$$

Let us consider Θ_{reg}. Using equation (1.38), inequality (1.51) in our case reads

$$\frac{1}{2\pi^2}\int_{\mathbb{R}^6} d\boldsymbol{p} d\boldsymbol{q}\, \frac{\overline{\hat{u}(\boldsymbol{p})}\,\hat{u}(\boldsymbol{q})}{|\boldsymbol{p}-\boldsymbol{q}|^2} = \int_{\mathbb{R}^3} d\boldsymbol{x}\, \frac{|u(\boldsymbol{x})|^2}{|\boldsymbol{x}|} \leq \frac{\pi}{2}\int_{\mathbb{R}^3} d\boldsymbol{k}\; |\boldsymbol{k}||\hat{u}(\boldsymbol{k})|^2.$$

Therefore

$$0 \leq \Theta_{\mathrm{reg}}[\varphi] \leq \frac{\pi}{2}(N-1)\,\gamma\,\sqrt{\frac{\eta}{\mu}}\,\Theta^{\zeta}_{\mathrm{diag}}[\varphi]. \tag{2.44}$$

and the proof is complete. □

2.3 Partial-Wave Decomposition

In order to establish a lower bound for Θ^ζ, in this section we start by following the first steps of [5, section 3] properly adapted to our case, i.e., we study the quadratic form decomposed in partial waves. Given $\hat{\varphi} \in L^2(\mathbb{R}^3,\, \sqrt{p^2+1}\, d\boldsymbol{p})$, one has

$$\hat{\varphi}(\boldsymbol{p}) = \sum_{\ell \in \mathbb{N}_0} \sum_{m=-\ell}^{\ell} \hat{\varphi}_{\ell,m}(p)\, Y_\ell^m(\hat{\boldsymbol{\omega}}). \tag{2.45}$$

Here, $Y_\ell^m : \mathbb{S}^2 \longrightarrow \mathbb{C}$ denotes the Spherical Harmonic of order ℓ, m, while $(p, \hat{\boldsymbol{\omega}}) \in \mathbb{R}_+ \times \mathbb{S}^2$ represents $\boldsymbol{p} \in \mathbb{R}^3$ in spherical coordinates and $\hat{\varphi}_{\ell,m} \in L^2(\mathbb{R}_+,\, p^2\sqrt{p^2+1}\, dp)$ are the Fourier coefficients of $\hat{\varphi}$. Accordingly, we decompose the quadratic form Θ^ζ for any $\zeta \geq 0$

$$\Theta^\zeta[\varphi] = \sum_{\ell \in \mathbb{N}_0} \sum_{m=-\ell}^{\ell} F_\ell^\zeta[\hat{\varphi}_{\ell,m}], \tag{2.46}$$

$$F_\ell^\zeta \colon L^2(\mathbb{R}_+,\, p^2 dp) \longrightarrow \mathbb{R}, \qquad \mathscr{D}(F_\ell^\zeta) = L^2(\mathbb{R}_+,\, p^2\sqrt{p^2+1}\, dp). \tag{2.47}$$

As usual, we consider the three-components

$$F_\ell^\zeta := F^\zeta_{\mathrm{diag}} + F^\zeta_{\mathrm{off};\,\ell} + F_{\mathrm{reg};\,\ell}\,, \tag{2.48}$$

each of which is going to be computed in the following lemma. From now on, we denote by $P_\ell(y) = \frac{1}{2^\ell \ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell}(y^2-1)^\ell$ the Legendre polynomial of degree $\ell \in \mathbb{N}_0$.

Lemma 2.6. *For any $\psi \in L^2(\mathbb{R}_+,\, p^2\sqrt{p^2+1}\, dp)$, taking into account decomposition* (2.46) *and definition* (2.48)*, we have the following expressions for any $\zeta \geq 0,\ \ell \in \mathbb{N}_0$*

$$F^\zeta_{\mathrm{diag}}[\psi] = \int_0^{+\infty} \! dk\; k^2 \sqrt{\tfrac{\mu}{\eta} k^2 + \zeta}\; |\psi(k)|^2, \tag{2.49a}$$

$$F_{\mathrm{reg};\,\ell}[\psi] = \tfrac{(N-1)\gamma}{\pi} \int_0^{+\infty} \! dp\; p^2 \int_0^{+\infty} \! dq\; q^2\, \overline{\psi(p)}\, \psi(q) \int_{-1}^{1} \! dy\; \frac{P_\ell(y)}{p^2 + q^2 - 2pq\, y}\,, \tag{2.49b}$$

$$F^\zeta_{\mathrm{off};\,\ell}[\psi] = -\tfrac{N-1}{\pi} \int_0^{+\infty} \! dp\; p^2 \int_0^{+\infty} \! dq\; q^2\, \overline{\psi(p)}\, \psi(q) \int_{-1}^{1} \! dy\; \frac{P_\ell(y)}{p^2 + q^2 + \frac{2}{M+1} pq\, y + \zeta}\,. \tag{2.49c}$$

Proof. The result is proved in [8, lemma 3.1] and here we give some details for reader's convenience. First we consider the diagonal contribution. Using spherical coordinates in (2.40a) so that any $\hat{\varphi}(\boldsymbol{p}) \in L^2(\mathbb{R}^3,\, \sqrt{p^2+1}\, d\boldsymbol{p})$ can be replaced in $\Theta^\zeta_{\mathrm{diag}}[\varphi]$ via decomposition (2.45), one gets

$$\Theta^\zeta_{\mathrm{diag}}[\varphi] = \sum_{\ell \in \mathbb{N}_0} \sum_{m=-\ell}^{\ell} F^\zeta_{\mathrm{diag}}[\hat{\varphi}_{\ell,m}], \tag{2.50}$$

thanks to the orthonormality of Y_ℓ^m, with F^ζ_{diag} given by (2.49a).
Regarding the regularizing contribution, the same procedure yields

$$\begin{aligned}\Theta_{\mathrm{reg}}[\varphi] = \sum_{\substack{\ell_1 \in \mathbb{N}_0 \\ \ell_2 \in \mathbb{N}_0}} \sum_{m_1=-\ell_1}^{\ell_1} \sum_{m_2=-\ell_2}^{\ell_2} \tfrac{(N-1)\gamma}{2\pi^2} \int_0^{+\infty} \! dp_1\; p_1^2 \int_0^{+\infty} \! dp_2\; p_2^2\, \overline{\hat{\varphi}_{\ell_1,m_1}(p_1)}\, \hat{\varphi}_{\ell_2,m_2}(p_2) \times \\ \times \int_{\mathbb{S}^2} \! d\hat{\boldsymbol{\omega}}_1 \int_{\mathbb{S}^2} \! d\hat{\boldsymbol{\omega}}_2\; \frac{\overline{Y_{\ell_1}^{m_1}(\hat{\boldsymbol{\omega}}_1)}\, Y_{\ell_2}^{m_2}(\hat{\boldsymbol{\omega}}_2)}{p_1^2 + p_2^2 - 2p_1 p_2\, \hat{\boldsymbol{\omega}}_1 \cdot \hat{\boldsymbol{\omega}}_2}\,.\end{aligned} \tag{2.51}$$

Decomposing the function $f_{p_1 p_2} : x \longmapsto \frac{1}{p_1^2+p_2^2-2\,p_1p_2\,x}$ in terms of Legendre polynomials, one obtains for almost every $(p_1, p_2) \in \mathbb{R}_+ \times \mathbb{R}_+$

$$f_{p_1 p_2}(x) = \sum_{\ell\in\mathbb{N}_0} \tfrac{2\ell+1}{2} \langle P_\ell\,,\, f_{p_1 p_2}\rangle_{L^2(-1,\,1)}\, P_\ell(x), \qquad \text{for almost every } x \in [-1,\, 1]. \tag{2.52}$$

Making use of the addition formula for the Spherical Harmonics

$$P_\ell(\hat{\boldsymbol{\omega}}_1 \cdot \hat{\boldsymbol{\omega}}_2) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{\boldsymbol{\omega}}_1)\, \overline{Y_\ell^m(\hat{\boldsymbol{\omega}}_2)}, \tag{2.53}$$

one has the following decomposition for almost every value of $\hat{\boldsymbol{\omega}}_1 \cdot \hat{\boldsymbol{\omega}}_2 \in [-1, 1]$

$$f_{p_1 p_2}(\hat{\boldsymbol{\omega}}_1 \cdot \hat{\boldsymbol{\omega}}_2) = \sum_{\ell\in\mathbb{N}_0} 2\pi \int_{-1}^{1} dy\, \frac{P_\ell(y)}{p_1^2 + p_2^2 - 2p_1p_2\, y} \sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{\boldsymbol{\omega}}_1)\, \overline{Y_\ell^m(\hat{\boldsymbol{\omega}}_2)}. \tag{2.54}$$

Replacing the previous expression in (2.51), we find

$$\Theta_{\mathrm{reg}}[\varphi] = \sum_{\ell\in\mathbb{N}_0} \sum_{m=-\ell}^{\ell} F_{\mathrm{reg};\,\ell}[\hat{\varphi}_{\ell,m}], \tag{2.55}$$

with $F_{\mathrm{reg};\,\ell}$ given by (2.49b).
The computation related to the off-diagonal contribution is exactly the same.

□

In the next lemma we characterize the sign of $F^{\zeta}_{\mathrm{off};\,\ell}$ and $F_{\mathrm{reg};\,\ell}$.

Lemma 2.7. *Let* $\psi \in L^2(\mathbb{R}_+,\, p^2\sqrt{p^2+1}\, dp)$. *Then, for all* $\ell \in \mathbb{N}_0$ *one has* $F_{\mathrm{reg};\,\ell} \geq 0$ *and*

$$\begin{cases} F^0_{\mathrm{off};\,\ell}[\psi] \geq F^{\zeta}_{\mathrm{off};\,\ell}[\psi] \geq 0, & \textit{if } \ell \textit{ is odd,} \\ F^0_{\mathrm{off};\,\ell}[\psi] \leq F^{\zeta}_{\mathrm{off};\,\ell}[\psi] \leq 0, & \textit{if } \ell \textit{ is even.} \end{cases}$$

Proof. The result concerning $F^{\zeta}_{\mathrm{off};\,\ell}$ has been proved in [8, lemma 3.3] and the same procedure can be adapted to obtain $F_{\mathrm{reg};\,\ell} \geq 0$. Indeed, following the strategy proposed in [8, lemma 3.3], considering that $\frac{2}{M+1}\, pq \lneq p^2 + q^2$, we can use the geometric series to write an absolutely convergent series expansion

$$\frac{1}{p^2+q^2+\frac{2}{M+1}\,pq\,y+\zeta} = \sum_{j\in\mathbb{N}_0} \frac{(-1)^j}{(p^2+q^2+\zeta)^{j+1}} \left(\frac{2pq}{M+1}\, y\right)^{j}. \tag{2.56}$$

Therefore, adopting (2.56) in (2.49c) and exchanging the sum with the integration, one gets

$$\begin{aligned} F^{\zeta}_{\mathrm{off};\,\ell}[\psi] &= -\tfrac{N-1}{\pi} \sum_{j\in\mathbb{N}_0} \left(\tfrac{-2}{M+1}\right)^j \int_0^{+\infty} dp\, p^{2+j} \int_0^{+\infty} dq\, q^{2+j}\, \frac{\overline{\psi(p)}\,\psi(q)}{(p^2+q^2+\zeta)^{j+1}} \int_{-1}^{1} dy\, y^j P_\ell(y) \\ &= -\tfrac{N-1}{\pi 2^\ell \ell!} \sum_{j\in\mathbb{N}_0} \left(\tfrac{-2}{M+1}\right)^j \int_0^{+\infty} dp \int_0^{+\infty} dq\, \frac{p^{2+j}\,\overline{\psi(p)}\, q^{2+j}\, \psi(q)}{(p^2+q^2+\zeta)^{j+1}} \int_{-1}^{1} dy\, y^j \frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell} \big(y^2-1\big)^\ell \end{aligned}$$

$$= -\tfrac{N-1}{\pi 2^\ell \ell!}\sum_{j\in\mathbb{N}_0}\big(\tfrac{-2}{M+1}\big)^j\int_0^{+\infty}dp\int_0^{+\infty}dq\;\frac{p^{2+j}\,\overline{\psi(p)}\,q^{2+j}\,\psi(q)}{(p^2+q^2+\zeta)^{j+1}}\int_{-1}^{1}dy\left(\frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell}y^j\right)\big(1-y^2\big)^\ell,$$

where we integrated by parts ℓ times in the last step. Next, exploiting the identity

$$\frac{1}{a^{j+1}} = \frac{1}{j!}\int_0^{+\infty}ds\; s^j e^{-a\,s},\quad \forall\, a>0 \tag{2.57}$$

with $a = p^2+q^2+\zeta$, we obtain

$$F^{\zeta}_{\mathrm{off};\,\ell}[\psi] = \tfrac{N-1}{\pi}\sum_{j=\ell}^{+\infty}B_{j\ell}\int_0^{+\infty}ds\; s^j e^{-\zeta s}\left|\int_0^{+\infty}dk\; k^{2+j}\psi(k)\,e^{-k^2 s}\right|^2, \tag{2.58}$$

where we have introduced the coefficients

$$B_{j\ell} := -\frac{1}{2^\ell\,\ell!\,j!}\left(\frac{-2}{M+1}\right)^j\int_{-1}^{1}dy\left(\frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell}y^j\right)\big(1-y^2\big)^\ell. \tag{2.59}$$

It is straightforward to see that $B_{j\ell}=0$ for each $j<\ell$ and whenever j and ℓ don't share the same parity. In all remaining cases, $B_{j\ell}$ is positive if j and ℓ are odd, while it is negative as long as j and ℓ are even. Using this fact in (2.58), the statement for $F^{\zeta}_{\mathrm{off};\,\ell}$ has been proved.
For the sake of completeness we underline that for all $j,\ell\in\mathbb{N}_0$ s.t. $j\geq\ell$ and $j-\ell$ is even,

$$\int_{-1}^{1}dy\left(\frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell}y^j\right)\big(1-y^2\big)^\ell = \frac{2^{2\ell+1}j!\,\ell!}{(j+\ell+1)!}\,\frac{\big(\frac{j+\ell}{2}\big)!}{\big(\frac{j-\ell}{2}\big)!}. \tag{2.60}$$

Therefore, whenever $\frac{j-\ell}{2}\in\mathbb{N}_0$

$$B_{j\ell} = (-1)^{j+1}\frac{2^{j+\ell+1}}{(j+\ell+1)!}\,\frac{1}{(1+M)^j}\,\frac{\big(\frac{j+\ell}{2}\big)!}{\big(\frac{j-\ell}{2}\big)!}. \tag{2.61}$$

With the same procedure we can complete the proof, since

$$\frac{1}{p^2+q^2-2\,pq\,y} = \sum_{j\in\mathbb{N}_0}\frac{1}{(p^2+q^2)^{j+1}}\,(2pq\,y)^j. \tag{2.62}$$

Therefore, we obtain the following representation for $F_{\mathrm{reg};\,\ell}$

$$F_{\mathrm{reg};\,\ell}[\psi] = \tfrac{(N-1)\gamma}{\pi}\sum_{j=\ell}^{+\infty}C_{j\ell}\int_0^{+\infty}ds\; s^j\left|\int_0^{+\infty}dk\; k^{2+j}\psi(k)\,e^{-k^2 s}\right|^2, \tag{2.63}$$

with

$$C_{j\ell} := \begin{cases} 0, & \text{if } j-\ell \text{ is odd,}\\[4pt] \dfrac{2^{j+\ell+1}}{(j+\ell+1)!}\,\dfrac{\big(\frac{j+\ell}{2}\big)!}{\big(\frac{j-\ell}{2}\big)!}, & \text{if } j-\ell \text{ is even.}\end{cases} \tag{2.64}$$

□

Notice that, thanks to lemma 2.7, for the sake of a lower bound, we can neglect $F^{\zeta}_{\text{off};\,\ell}$ with ℓ odd and focus on $F^{0}_{\text{off};\,\ell}$ that represents a lower estimates for $F^{\zeta}_{\text{off};\,\ell}$ in case ℓ is even.
Moreover, comparing (2.61) with (2.64) one finds out that

$$F_{\text{reg};\,\ell} + F^{\zeta}_{\text{off};\,\ell} \geq 0, \qquad \text{provided } \gamma \geq 1$$

and this is a proof of the regularization of the ultraviolet instability. However, one can go beyond this result dropping the assumption $\gamma \geq 1$ in search for a lower (and possibly optimal) threshold parameter for γ. In particular, this is the content of section 2.4.
We conclude this section by presenting the diagonalization of the quadratic form F^{0}_{ℓ}.

Lemma 2.8. *Given $\psi \in L^2(\mathbb{R}_+,\, k^2\sqrt{k^2+1}\, dk)$, let $\psi^\sharp \in L^2(\mathbb{R})$ be defined by*

$$\psi^\sharp(p) := (\mathcal{M}\psi)(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dt\; e^{-ipt} e^{2t} \psi(e^t) \tag{2.65}$$

with $\mathcal{M}$ defined in (1.31). *Then, considering the quantities computed in lemma 2.6, one has*

$$F^{0}_{\text{diag}}[\psi] = \sqrt{\frac{\mu}{\eta}} \int_{\mathbb{R}} dp\; |\psi^\sharp(p)|^2, \tag{2.66a}$$

$$F^{0}_{\text{off};\,\ell}[\psi] = \frac{N-1}{2} \int_{\mathbb{R}} dp\; |\psi^\sharp(p)|^2\, S_{\text{off};\,\ell}(p), \tag{2.66b}$$

$$F_{\text{reg};\,\ell}[\psi] = \frac{N-1}{2} \int_{\mathbb{R}} dp\; |\psi^\sharp(p)|^2\, S_{\text{reg};\,\ell}(p), \tag{2.66c}$$

where

$$S_{\text{off};\,\ell}(p) = \begin{cases} -\displaystyle\int_{-1}^{1} dy\; P_\ell(y)\, \frac{\cosh\big(p \arcsin \frac{y}{M+1}\big)}{\sqrt{1-\frac{y^2}{(M+1)^2}}\, \cosh\big(\frac{\pi}{2}p\big)}, & \textit{if } \ell \textit{ is even}, \\[2ex] \displaystyle\int_{-1}^{1} dy\; P_\ell(y)\, \frac{\sinh\big(p \arcsin \frac{y}{M+1}\big)}{\sqrt{1-\frac{y^2}{(M+1)^2}}\, \sinh\big(\frac{\pi}{2}p\big)}, & \textit{if } \ell \textit{ is odd}. \end{cases} \tag{2.67a}$$

$$S_{\text{reg};\,\ell}(p) = \begin{cases} \gamma\displaystyle\int_{-1}^{1} dy\; P_\ell(y)\, \frac{\cosh(p \arcsin y)}{\sqrt{1-y^2}\, \cosh\big(\frac{\pi}{2}p\big)}, & \textit{if } \ell \textit{ is even}, \\[2ex] \gamma\displaystyle\int_{-1}^{1} dy\; P_\ell(y)\, \frac{\sinh(p \arcsin y)}{\sqrt{1-y^2}\, \sinh\big(\frac{\pi}{2}p\big)}, & \textit{if } \ell \textit{ is odd}. \end{cases} \tag{2.67b}$$

Moreover,

$$\begin{cases} S_{\text{off};\,\ell}(p) \leq S_{\text{off};\,\ell+2}(p) \leq 0, & \textit{if } \ell \textit{ is even}; \\ S_{\text{off};\,\ell}(p) \geq S_{\text{off};\,\ell+2}(p) \geq 0, & \textit{if } \ell \textit{ is odd}, \end{cases} \tag{2.68a}$$

while

$$S_{\text{reg};\,\ell}(p) \geq S_{\text{reg};\,\ell+2}(p) \geq 0, \quad \forall\, \ell \in \mathbb{N}_0. \tag{2.68b}$$

Proof. The result about the diagonal and off-diagonal terms are proved in [8, lemma 3.4], whereas the statements regarding the regularizing contribution have been shown in [5, lemma 3.4]. In the following, we provide the details. Notice that, if we apply the substitution $k \longmapsto e^t$ in (2.49a), the unitary map $\mathcal{M}$ is obtained by performing the Fourier transform in $L^2(\mathbb{R})$. Therefore, equation (2.66a) is a consequence of Plancherel's theorem.

Considering the same change of variables $p = e^{x_0}, q = e^{x_1}$ in (2.49c), we can rewrite $F^0_{\mathrm{off};\,\ell}$ as

$$\begin{aligned} F^0_{\mathrm{off};\,\ell}[\psi] &= -\frac{N-1}{\pi}\int_{-\infty}^{+\infty} dx_0\, e^{3x_0}\int_{-\infty}^{+\infty} dx_1\, e^{3x_1}\,\overline{\psi(e^{x_0})}\,\psi(e^{x_1})\int_{-1}^{1} dy\; \frac{P_\ell(y)}{e^{2x_0}+e^{2x_1}+\frac{2}{M+1}\,y\,e^{x_0+x_1}} \\ &= -\frac{N-1}{2\pi}\int_{\mathbb{R}} dx_0\, e^{2x_0}\overline{\psi(e^{x_0})}\int_{\mathbb{R}} dx_1\, e^{2x_1}\psi(e^{x_1})\int_{-1}^{1} dy\; \frac{P_\ell(y)}{\cosh(x_0-x_1)+\frac{1}{M+1}\,y}. \end{aligned}$$

The previous expression is manifestly an inner product in $L^2(\mathbb{R}, dx_0)$ between a function and a convolution involving the same function. Thus, thanks to the properties of the Fourier transform,

$$F^0_{\mathrm{off};\,\ell}[\psi] = \frac{N-1}{2}\int_{\mathbb{R}} dp\; |\psi^\sharp(p)|^2 S_{\mathrm{off};\,\ell}(p),$$

where

$$S_{\mathrm{off};\,\ell}(p) := -\frac{1}{\pi}\int_{\mathbb{R}} dx\; e^{-ipx}\int_{-1}^{1} dy\; \frac{P_\ell(y)}{\cosh(x)+\frac{y}{M+1}}. \tag{2.69}$$

Since the function $(x, y) \longmapsto \frac{1}{\cosh(x)+\frac{y}{M+1}} \in L^1(\mathbb{R}\times[-1,1],\, dx\,dy)$ uniformly in $M \geq 0$, we can use Fubini's theorem and [20, p. 511, 3.983.1] to get

$$S_{\mathrm{off};\,\ell}(p) = -\frac{1}{\pi}\int_{-1}^{1} dy\, P_\ell(y)\int_{\mathbb{R}} dx\; \frac{e^{-ipx}}{\cosh(x)+\frac{y}{M+1}} = -2\int_{-1}^{1} dy\, P_\ell(y)\; \frac{\sinh\!\big(p\arccos\frac{y}{M+1}\big)}{\sqrt{1-\frac{y^2}{(M+1)^2}}\,\sinh(\pi p)}. \tag{2.70}$$

Proceeding in the same way for $F_{\mathrm{reg};\,\ell}$, we obtain

$$F_{\mathrm{reg};\,\ell}[\psi] = \frac{N-1}{2}\int_{\mathbb{R}} dp\; |\psi^\sharp(p)|^2 S_{\mathrm{reg};\,\ell}(p),$$

where

$$S_{\mathrm{reg};\,\ell}(p) := \frac{\gamma}{\pi}\int_{-1}^{1} dy\; P_\ell(y)\int_{\mathbb{R}} dx\; \frac{e^{-ipx}}{\cosh(x)-y} = 2\gamma\int_{-1}^{1} dy\; P_\ell(y)\, \frac{\sinh(p\arccos(-y))}{\sqrt{1-y^2}\,\sinh(\pi p)}. \tag{2.71}$$

Next, since $\arccos a = \frac{\pi}{2} - \arcsin a$ for all $a \in [-1,1]$, we have

$$\sinh(p\arccos a) = \sinh\!\big(\tfrac{\pi}{2}p\big)\cosh(p\arcsin a) - \cosh\!\big(\tfrac{\pi}{2}p\big)\sinh(p\arcsin a), \tag{2.72}$$

that, together with the fact that P_ℓ is a polynomial of the same parity as ℓ, identities (2.67a) and (2.67b) are recovered.

To conclude the proof, we can make use of proposition B.1 to study the monotonicity of $S_{\mathrm{off};\,\ell}$ and $S_{\mathrm{reg};\,\ell}$. In particular, we need to verify that the functions $f_{1,p}, f_{2,p}\colon (-1,1) \longrightarrow \mathbb{R}$ given by

$$f_{1,p} : a \longmapsto \frac{\sinh(p\arcsin a)}{\sqrt{1-a^2}}, \qquad f_{2,p} : a \longmapsto \frac{\cosh(p\arcsin a)}{\sqrt{1-a^2}} \tag{2.73}$$

have expansions in power series with radius of convergence 1 and positive coefficients for all $p \in \mathbb{R}$. To this end, let $\mathscr{A}$ be the set of all analytic functions in one variable whose Taylor expansion is made of positive coefficients. It is straightforward that $\mathscr{A}$ is closed under multiplications, compositions and dilations. Furthermore, given a function in $\mathscr{A}$, its derivative still belongs to $\mathscr{A}$. Therefore, since $\sinh(\cdot), \cosh(\cdot), \arcsin(\cdot) \in \mathscr{A}$, one has

$$\frac{\mathrm{d}}{\mathrm{d}a} \arcsin a = \frac{1}{\sqrt{1-a^2}} \in \mathscr{A} \implies f_{1,p}, f_{2,p} \in \mathscr{A}.$$

Moreover, notice that the radius of convergence of the power series expansions of both $f_{1,p}$ and $f_{2,p}$ is simply 1.

□

2.4 Order by Order Approximation

In the following, we obtain some key estimates useful to control F_ℓ^ζ. We first provide the upper bound.

Lemma 2.9. *For any $\psi \in L^2(\mathbb{R}_+,\, k^2\sqrt{k^2+1}\,dk)$ and $\ell \in \mathbb{N}_0$, one has*

$$F_{\mathrm{reg};\,\ell}[\psi] \leq \begin{cases} \frac{\pi\gamma}{2} \frac{(N-1)(M+1)}{\sqrt{M(M+2)}} F^0_{\mathrm{diag}}[\psi], & \text{if } \ell \text{ is even}, \\ \frac{2\gamma}{\pi} \frac{(N-1)(M+1)}{\sqrt{M(M+2)}} F^0_{\mathrm{diag}}[\psi], & \text{if } \ell \text{ is odd}, \end{cases} \tag{2.74a}$$

$$F^0_{\mathrm{off};\,\ell}[\psi] \leq \begin{cases} 0, & \text{if } \ell \text{ is even}, \\ \frac{2(N-1)(M+1)^2}{\pi} \left[\frac{1}{\sqrt{M(M+2)}} - \arcsin \frac{1}{1+M} \right] F^0_{\mathrm{diag}}[\psi], & \text{if } \ell \text{ is odd}. \end{cases} \tag{2.74b}$$

Proof. Let us first consider the case ℓ even. According to (2.68a) we have $S_{\mathrm{off};\,\ell} \leq 0$ while, from (2.67b) and (2.68b), we know that

$$S_{\mathrm{reg};\,\ell}(p) \leq S_{\mathrm{reg};\,0}(p) = 2\gamma \, \frac{\tanh\left(\frac{\pi}{2}p\right)}{p} \leq \pi\gamma.$$

Hence, it is straightforward to see that

$$F_{\mathrm{reg};\,\ell}[\psi] \leq \tfrac{N-1}{2} \sqrt{\tfrac{\eta}{\mu}}\, S_{\mathrm{reg};\,0}(0)\, F^0_{\mathrm{diag}}[\psi] = \tfrac{\pi\gamma}{2} \tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}} F^0_{\mathrm{diag}}[\psi].$$

Next, we consider ℓ odd. In light of (2.68), both $S_{\mathrm{reg};\,\ell}(p)$ and $S_{\mathrm{off};\,\ell}(p)$ are maximised at $\ell = 1$, uniformly in $p \in \mathbb{R}$. Furthermore, thanks to the parity of the integrand, we get

$$S_{\mathrm{off};\,1}(p) = 2\int_0^1 dy \, \frac{y \sinh\left(p \arcsin \frac{y}{1+M}\right)}{\sqrt{1 - \frac{y^2}{(1+M)^2}} \sinh\left(\frac{\pi}{2}p\right)} = \frac{2(M+1)^2}{\sinh\left(\frac{\pi}{2}p\right)} \int_0^{\arcsin \frac{1}{M+1}} du \, \sin u \, \sinh(pu),$$

$$S_{\mathrm{reg};\,1}(p) = 2\gamma \int_0^1 dy \, \frac{y \sinh(p \arcsin y)}{\sqrt{1-y^2} \sinh\left(\frac{\pi}{2}p\right)} = \frac{2\gamma}{\sinh\left(\frac{\pi}{2}p\right)} \int_0^{\frac{\pi}{2}} du \, \sin u \, \sinh(pu).$$

In both cases, we are dealing with decreasing functions in $p > 0$. Indeed, one has that the function $p \longmapsto \frac{\sinh(pu)}{\sinh\left(\frac{\pi}{2}p\right)}$ is non-increasing, uniformly in $u \in [0, \frac{\pi}{2}]$. Thus, performing the derivatives

$\frac{\mathrm{d}}{\mathrm{d}p} S_{\mathrm{off};\,1}(p)$ and $\frac{\mathrm{d}}{\mathrm{d}p} S_{\mathrm{reg};\,1}(p)$, one gets negative quantities for any $p > 0$, since the integrands are negative for almost every $u \in [0, \frac{\pi}{2}]$.
Hence, since $S_{\mathrm{off};\,1}(p)$ and $S_{\mathrm{reg};\,1}(p)$ are even and decreasing in $p > 0$, the maximum is attained at the point $p = 0$. Therefore,

$$S_{\mathrm{off};\,\ell}(p) \leq S_{\mathrm{off};\,1}(0) = \frac{4(M+1)^2}{\pi} \int_0^{\arcsin \frac{1}{1+M}} du\; u \sin(u) = \frac{4(M+1)}{\pi} \left[1 - \sqrt{M(M+2)} \arcsin \frac{1}{1+M} \right],$$
$$S_{\mathrm{reg};\,\ell}(p) \leq S_{\mathrm{reg};\,1}(0) = \frac{4\gamma}{\pi} \int_0^{\frac{\pi}{2}} du\; u \sin(u) = \frac{4\gamma}{\pi}.$$

In conclusion, for ℓ odd we have

$$F_{\mathrm{reg};\,\ell}[\psi] \leq \tfrac{N-1}{2} \sqrt{\tfrac{\eta}{\mu}}\, S_{\mathrm{reg};\,1}(0)\, F^0_{\mathrm{diag}}[\psi] = \tfrac{2\gamma}{\pi} \tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}} F^0_{\mathrm{diag}}[\psi];$$
$$F_{\mathrm{off};\,\ell}[\psi] \leq \tfrac{N-1}{2} \sqrt{\tfrac{\eta}{\mu}}\, S_{\mathrm{off},\,1}(0)\, F^0_{\mathrm{diag}}[\psi] = \tfrac{2(N-1)(M+1)^2}{\pi} \left[\tfrac{1}{\sqrt{M(M+2)}} - \arcsin \tfrac{1}{1+M} \right] F^0_{\mathrm{diag}}[\psi].$$

□

Let us introduce some further notation. For any $a \in \mathbb{C}$ and $n \in \mathbb{N}_0$, let $(a)_n$ be the Pochhammer symbol, also known as rising factorial, given by

$$(a)_n := \begin{cases} a(a+1)\cdots(a+n-1), & \text{if } n \in \mathbb{N}, \\ 1, & \text{if } n = 0. \end{cases} \tag{2.75a}$$

It is easy to see that for any $n \in \mathbb{N}_0$

$$(a)_n = \begin{cases} (-1)^n n! \binom{|a|}{|a|-n}, & \text{if } a \in -\mathbb{N}_0, \\ \frac{\Gamma(a+n)}{\Gamma(a)}, & \text{otherwise.} \end{cases} \tag{2.75b}$$

In particular, notice that if $a \in -\mathbb{N}_0$, then $(a)_n = 0$ for all $n > |a|$. Next, we recall the definition of the Gauss hypergeometric function

$${}_2F_1(a, b; c; z) := \sum_{k \in \mathbb{N}_0} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}. \tag{2.76}$$

Representation (2.76) is well defined for $a, b \in \mathbb{C}$, $c \in \mathbb{C} \setminus -\mathbb{N}_0$ and its radius of convergence is 1. However, if a or b is a non-positive integer, then the Gauss hypergeometric function reduces to a polynomial in z. In this case, c can also assume non-positive integer values, provided that $|c|$ is greater than or equal to the degree of the polynomial. We also remind the Gauss' summation theorem

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \text{if } \operatorname{Re}(c-a-b) > 0. \tag{2.77}$$

In proposition B.3 we give the explicit computation of the integrals appearing in equations (2.67a) and (2.67b) for ℓ even in terms of the Gauss hypergeometric function.

Remark 2.1. *We point out that the integral evaluated in proposition B.3 considerably simplifies in case $x = 1$ or $p = 0$. Indeed, making use of* (2.77) *and* (B.4)*, one gets in case $x = 1$*

$$\int_{-1}^{1} dy\, P_\ell(y)\, \frac{\cosh(p \arcsin y)}{\sqrt{1-y^2}} = \frac{2^{\ell+1}\sqrt{\pi}\, \ell!\, \Gamma\big(\ell+\frac{3}{2}\big)}{(2\ell+1)!\, \big|\Gamma\big(\frac{\ell+2+ip}{2}\big)\big|^2} \prod_{n=1}^{\frac{\ell}{2}} \big[p^2 + (2n-1)^2\big]$$

$$= \frac{2^{\ell}\sqrt{\pi}\, \ell!\, \Gamma\big(\ell+\frac{1}{2}\big)}{(2\ell)!\, \big|\Gamma\big(\frac{\ell+2+ip}{2}\big)\big|^2} \prod_{n=1}^{\frac{\ell}{2}} \big[p^2 + (2n-1)^2\big] = \frac{\pi}{2^{\ell}\, \big|\Gamma\big(\frac{\ell+2+ip}{2}\big)\big|^2} \prod_{n=1}^{\frac{\ell}{2}} \big[p^2 + (2n-1)^2\big].$$

Exploiting the identity

$$|\Gamma(n+1+ib)|^2 = \frac{\pi b}{\sinh(\pi b)} \prod_{k=1}^{n} (k^2 + b^2), \qquad \forall\, b \in \mathbb{R},\ n \in \mathbb{N}_0, \tag{2.78}$$

one obtains

$$\int_{-1}^{1} dy\, P_\ell(y)\, \frac{\cosh(p \arcsin y)}{\sqrt{1-y^2}} = \frac{2\sinh\big(\frac{\pi}{2}p\big)}{p} \prod_{k=1}^{\frac{\ell}{2}} \frac{p^2 + (2k-1)^2}{p^2 + 4k^2}. \tag{2.79}$$

Let us consider the case $p = 0$. Taking into account that

$$\prod_{k=1}^{\frac{\ell}{2}} (2k-1)^2 = (\ell-1)!!^2 = \frac{\ell!^2}{2^{\ell}\, \big(\frac{\ell}{2}\big)!^2},$$

where $(\cdot)!!$ denotes the double factorial, i.e.

$$n!! := \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n-2k) = \begin{cases} 2^{\frac{n}{2}} \big(\frac{n}{2}\big)!, & \text{if } n \text{ is even}, \\ \dfrac{(n+1)!}{2^{\frac{n+1}{2}} \big(\frac{n+1}{2}\big)!}, & \text{if } n \text{ is odd}, \end{cases} \tag{2.80}$$

one obtains

$$\int_{-1}^{1} dy\, \frac{P_\ell(y)}{\sqrt{1-x^2y^2}} = \frac{2x^{\ell}\, \ell!^3}{(2\ell+1)!\, \big(\frac{\ell}{2}\big)!^2}\, {}_2F_1\big(\tfrac{\ell+1}{2}, \tfrac{\ell+1}{2}; \ell+\tfrac{3}{2}; x^2\big). \tag{2.81}$$

In the special case $x = 1$ and $p = 0$, one has

$$\int_{-1}^{1} dy\, \frac{P_\ell(y)}{\sqrt{1-y^2}} = \frac{2\sqrt{\pi}\, \ell!^3\, \Gamma\big(\ell+\frac{3}{2}\big)}{(2\ell+1)!\, \big(\frac{\ell}{2}\big)!^4} = \frac{\sqrt{\pi}\, \ell!^3\, \Gamma\big(\ell+\frac{1}{2}\big)}{(2\ell)!\, \big(\frac{\ell}{2}\big)!^4} = \frac{\pi\, \ell!^2}{2^{2\ell}\, \big(\frac{\ell}{2}\big)!^4} \tag{2.82}$$

where we have used (2.77) *and* (B.4).

The next lemma is the key technical ingredient for the proof of proposition 2.1. It involves a sequence of new auxiliary quadratic forms $\Xi^{\zeta}_{\ell,\, s_\ell}$ defined on $L^2(\mathbb{R}_+,\, p^2 dp)$ for any given $\ell \in \mathbb{N}_0$ and for some parameter $s_\ell \in (0,1)$ as follows

$$\Xi^{\zeta}_{\ell,\, s_\ell} := s_\ell F^{\zeta}_{\mathrm{diag}} + F^{0}_{\mathrm{off};\, \ell} + F_{\mathrm{reg};\, \ell}, \qquad \mathscr{D}(\Xi^{\zeta}_{\ell,\, s_\ell}) = L^2(\mathbb{R}_+,\, p^2\sqrt{p^2+1}\, dp). \tag{2.83}$$

As we shall see, these quadratic forms will be useful to obtain a lower bound for F^{ζ}_{ℓ}.

Lemma 2.10. *Let $\psi \in L^2(\mathbb{R}_+,\, p^2\sqrt{p^2+1}\, dp)$ and $\gamma_c^{N\mathrm{b}+1}$ given by* (2.19)*. Then, for $\gamma > \gamma_c^{N\mathrm{b}+1}$, there exists $\{s^*_\ell\}_{\ell \in \mathbb{N}_0} \subset (0,1)$ such that each quadratic form $\Xi^{\zeta}_{\ell,\, s^*_\ell}$ defined by* (2.83)*, is non-negative for any $\zeta \geq 0$ and $\ell \in \mathbb{N}_0$.*

Proof. Taking into account the diagonalization derived in lemma 2.8, one has

$$\Xi^{\zeta}_{\ell,\,s_\ell}[\psi] \geq \big(s_\ell F^{0}_{\mathrm{diag}} + F^{0}_{\mathrm{off};\,\ell} + F_{\mathrm{reg};\,\ell}\big)[\psi] = \int_{\mathbb{R}} dp\ |\psi^\sharp(p)|^2 \Big[s_\ell\sqrt{\tfrac{\mu}{\eta}} + \tfrac{N-1}{2}\,(S_{\mathrm{off};\,\ell} + S_{\mathrm{reg};\,\ell})(p)\Big]$$
$$=: \int_{\mathbb{R}} dp\ |\psi^\sharp(p)|^2 f^N_{\ell,\,s_\ell}(p).$$

The lemma is proved if we show that for each order ℓ, there exists $s_\ell \in (0,1)$ such that the function $f^N_{\ell,\,s_\ell}$ is non-negative uniformly in $N \geq 2$. Notice that this is actually the case for ℓ odd, in light of (2.68), so from now on we focus on the case ℓ even.
In particular, the uniform non-negativity of the function is eventually achieved since

$$\lim_{p\to+\infty} f^N_{\ell,\,s_\ell}(p) = s_\ell\sqrt{\tfrac{\mu}{\eta}} > 0.$$

We notice that $S_{\mathrm{off};\,\ell}$ and $S_{\mathrm{reg};\,\ell}$, and then $f^N_{\ell,\,s_\ell}$, are written in terms of the Gauss hypergeometric function ${}_2F_1$ (see (2.67a), (2.67b) and proposition B.3) and therefore the main point is a careful control from below of such a function.
The proof will be constructed in two steps: first we show that $f^N_{\ell,\,s_\ell}$ evaluated at zero is positive uniformly in $N \geq 2$ for a proper choice of $\{s_\ell\}_{\ell\in\mathbb{N}_0} \subset (0,1)$, then we prove that $f^N_{\ell,\,s_\ell}$ is bounded from below by a monotonic function $h^N_{\ell,\,s_\ell}$ that shares the same values with $f^N_{\ell,\,s_\ell}$ at zero and infinity. Once these statements are proven, we will have $f^N_{\ell,\,s_\ell} \geq h^N_{\ell,\,s_\ell} > 0$ as long as s_ℓ is such that $f^N_{\ell,\,s_\ell}(0) > 0$ for all $\ell \in \mathbb{N}_0$ and uniformly in $N \geq 2$.

Step 1. We observe that $f^N_{\ell,\,s_\ell}(0)$ is positive if and only if

$$\begin{aligned} s_\ell &> -\tfrac{N-1}{2}\sqrt{\tfrac{\eta}{\mu}}\,(S_{\mathrm{off};\,\ell} + S_{\mathrm{reg};\,\ell})(0) \\ &= \tfrac{N-1}{2}\sqrt{\tfrac{\eta}{\mu}}\left[\int_{-1}^{1} dy\,\frac{P_\ell(y)}{\sqrt{1-\frac{y^2}{(1+M)^2}}} - \gamma\int_{-1}^{1} dy\,\frac{P_\ell(y)}{\sqrt{1-y^2}}\right]. \end{aligned} \tag{2.84}$$

The requirement $s_\ell \in (0,1)$ implies a constraint for the parameter γ, since we need the right hand side of (2.84) to be strictly less than 1. Therefore

$$\gamma > \gamma^\ell_M := \left[\int_{-1}^{1} dy\,\frac{P_\ell(y)}{\sqrt{1-y^2}}\right]^{-1}\left[\int_{-1}^{1} dy\,\frac{P_\ell(y)}{\sqrt{1-\frac{y^2}{(1+M)^2}}} - \frac{2}{N-1}\sqrt{\frac{\mu}{\eta}}\right]. \tag{2.85}$$

Let us show that

$$\gamma_c^{N\mathrm{b}+1}(M) = \max_{k\in\mathbb{N}_0}\{\gamma^{2k}_M\} = \gamma^0_M. \tag{2.86}$$

Taking into account equations (2.81) and (2.82), condition (2.85) reads

$$\gamma > \gamma^\ell_M = \gamma^\ell_{M,1} - \gamma^\ell_{M,2}\,,$$

with

$$\gamma^\ell_{M,1} := \frac{2^{2\ell+1}\ell!\,\big(\frac{\ell}{2}\big)!^2}{\pi\,(2\ell+1)!\,(M+1)^\ell}\,{}_2F_1\Big(\tfrac{\ell+1}{2}, \tfrac{\ell+1}{2}; \ell+\tfrac{3}{2}; \tfrac{1}{(M+1)^2}\Big), \tag{2.87}$$

$$\gamma^{\ell}_{M,2} := \frac{2^{2\ell+1}\big(\frac{\ell}{2}\big)!^4\sqrt{M(M+2)}}{\pi\,\ell!^2(N-1)(M+1)}. \tag{2.88}$$

We observe that $\gamma^{\ell}_{M,2}$ is increasing in ℓ, since

$$\frac{2^{2(\ell+2)}\big(\frac{\ell+2}{2}\big)!^4}{(\ell+2)!^2} = \frac{2^{2\ell+4}\big(\frac{\ell}{2}+1\big)!^4}{(\ell+2)!^2} = \frac{2^{2\ell}\,2^4\big(\frac{\ell}{2}+1\big)^4\big(\frac{\ell}{2}\big)!^4}{(\ell+2)^2(\ell+1)^2\ell!^2} = \frac{2^{2\ell}(\ell+2)^2\big(\frac{\ell}{2}\big)!^4}{(\ell+1)^2\ell!^2} > \frac{2^{2\ell}\big(\frac{\ell}{2}\big)!^4}{\ell!^2}.$$

Therefore

$$\gamma^{\ell}_{M} < \gamma^{\ell}_{M,1} - \gamma^{0}_{M,2} = \gamma^{\ell}_{M,1} - \frac{2}{\pi(N-1)}\frac{\sqrt{M(M+2)}}{M+1}. \tag{2.89}$$

Let us consider $\gamma^{\ell}_{M,1}$. Using the Euler's integral representation of the Gauss hypergeometric function

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt\,\frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}, \qquad \mathrm{Re}(c) > \mathrm{Re}(b) > 0, \tag{2.90}$$

one has for any $x \in [0,1]$

$$\begin{aligned}{}_2F_1\big(\tfrac{\ell+1}{2},\tfrac{\ell+1}{2};\ell+\tfrac{3}{2};x^2\big) &= \frac{\Gamma\big(\ell+\frac{3}{2}\big)}{\Gamma\big(\frac{\ell+1}{2}\big)\big(\frac{\ell}{2}\big)!}\int_0^1 dt\,\frac{t^{\frac{\ell-1}{2}}(1-t)^{\frac{\ell}{2}}}{(1-x^2t)^{\frac{\ell+1}{2}}} = \frac{2\,\Gamma\big(\ell+\frac{3}{2}\big)}{\Gamma\big(\frac{\ell+1}{2}\big)\big(\frac{\ell}{2}\big)!}\int_0^1 du\,\frac{u^{\ell}(1-u^2)^{\frac{\ell}{2}}}{(1-x^2u^2)^{\frac{\ell+1}{2}}}\\ &= \frac{2^{\ell}(2\ell+1)\,\Gamma\big(\ell+\frac{1}{2}\big)}{\sqrt{\pi}\,\ell!}\int_0^1 du\,\frac{u^{\ell}(1-u^2)^{\frac{\ell}{2}}}{(1-x^2u^2)^{\frac{\ell+1}{2}}} = \frac{(2\ell+1)!}{2^{\ell}\,\ell!^2}\int_0^1 du\,\frac{u^{\ell}(1-u^2)^{\frac{\ell}{2}}}{(1-x^2u^2)^{\frac{\ell+1}{2}}}.\end{aligned}$$

Exploiting the trivial inequality $1-u^2 \leq 1-x^2u^2$, we obtain an estimate from above

$${}_2F_1\big(\tfrac{\ell+1}{2},\tfrac{\ell+1}{2};\ell+\tfrac{3}{2};x^2\big) \leq \frac{(2\ell+1)!}{2^{\ell}\,\ell!^2}\int_0^1 du\,\frac{u^{\ell}}{\sqrt{1-x^2u^2}}, \tag{2.91}$$

where equality holds if $\ell = 0 \vee x = 1$. From (2.91) one gets

$$\gamma^{\ell}_{M,1} \leq \frac{2^{\ell+1}\big(\frac{\ell}{2}\big)!^2}{\pi\,\ell!\,(M+1)^{\ell}}\int_0^1 du\,\frac{u^{\ell}}{\sqrt{1-\frac{u^2}{(1+M)^2}}} =: \bar{\gamma}^{\ell}_{M}, \tag{2.92}$$

where equality holds if $\ell = 0 \vee M = 0$. Hence, in particular we know that

$$\gamma^{0}_{M,1} = \bar{\gamma}^{0}_{M} = \tfrac{2(M+1)}{\pi}\arcsin\big(\tfrac{1}{M+1}\big). \tag{2.93}$$

In the following computations we set $x = \frac{1}{M+1}$ for the sake of notation. Let us prove that $\big\{\bar{\gamma}^{\ell}_{M}\big\}_{\ell\in 2\mathbb{N}_0}$ is a decreasing sequence for all fixed $M > 0$. We have

$$\begin{aligned}\bar{\gamma}^{\ell}_{M} - \bar{\gamma}^{\ell+2}_{M} &= \frac{2^{\ell+1}x^{\ell}\big(\frac{\ell}{2}\big)!^2}{\pi\,\ell!}\int_0^1 du\,\frac{u^{\ell}}{\sqrt{1-x^2u^2}}\left[1-\frac{4x^2\big(\frac{\ell}{2}+1\big)^2u^2}{(\ell+2)(\ell+1)}\right]\\ &= \frac{2^{\ell+1}x^{\ell}\big(\frac{\ell}{2}\big)!^2}{\pi\,\ell!}\int_0^1 du\,\frac{u^{\ell}}{\sqrt{1-x^2u^2}}\left[1-\frac{(\ell+2)x^2u^2}{\ell+1}\right].\end{aligned}$$

Our goal is to show that the last integral is positive for any given $x \in (0,1)$ and ℓ even, so that $\bar{\gamma}^{\ell+2}_{M} < \bar{\gamma}^{\ell}_{M}$. To this end, we first point out that the integral is manifestly positive at $x = 0$, whereas the evaluation of the integral at $x = 1$ yields

$$\int_0^1 du\,\frac{2u^{\ell}}{\sqrt{1-u^2}}\left[1-\frac{\ell+2}{\ell+1}\,u^2\right] = \frac{\pi\,\ell!}{2^{\ell}\big(\frac{\ell}{2}\big)!^2} - \frac{\ell+2}{\ell+1}\,\frac{\pi\,(\ell+2)!}{2^{\ell+2}\big(\frac{\ell}{2}+1\big)!^2} = 0.$$

We observe that, in order to obtain $\inf\{\bar{\gamma}_M^\ell - \bar{\gamma}_M^{\ell+2} \mid M>0\} \geq 0$, it is sufficient to prove that the integral is a monotonic decreasing function in x for any ℓ. In other words, we want to show

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 du\, \frac{u^\ell}{\sqrt{1-x^2u^2}}\left[1-\frac{\ell+2}{\ell+1}\,x^2u^2\right] < 0, \qquad \forall\, x\in(0,1),\ \ell \text{ even.} \tag{2.94}$$

By the Leibniz integral rule, the derivative with respect to x can be computed inside the integral. Therefore, for any $x\in(0,1), u\in[0,1]$ and ℓ even, one has

$$\begin{aligned}\frac{\partial}{\partial x}\,\frac{u^\ell}{\sqrt{1-x^2u^2}}\left[1-\frac{\ell+2}{\ell+1}\,x^2u^2\right] &= \frac{x\,u^{\ell+2}}{(1-x^2u^2)^{\frac32}}\left[1-\frac{\ell+2}{\ell+1}\,x^2u^2\right] - \frac{\ell+2}{\ell+1}\,\frac{2x\,u^{\ell+2}}{\sqrt{1-x^2u^2}}\\ = \frac{x\,u^{\ell+2}}{(1-x^2u^2)^{\frac32}}\left[1-\frac{\ell+2}{\ell+1}\left(x^2u^2+2-2x^2u^2\right)\right] &= \frac{x\,u^{\ell+2}}{(1-x^2u^2)^{\frac32}}\left[\frac{\ell+2}{\ell+1}\,x^2u^2-\frac{\ell+3}{\ell+1}\right] < 0.\end{aligned}$$

Since the integral of a negative function obviously yields a negative quantity, (2.94) is proven. This means that $\{\bar{\gamma}_M^\ell\}_{\ell\in 2\mathbb{N}_0}$ is decreasing for any fixed $M>0$. Thus, taking into account (2.92) and (2.93), we finally get

$$\gamma_{M,1}^\ell \leq \bar{\gamma}_M^\ell \leq \bar{\gamma}_M^0 = \gamma_{M,1}^0, \qquad \forall\,\ell \text{ even.}$$

Hence, thanks to (2.89), equation (2.86) is proved.

Step 2. Let us define the following function

$$h_{\ell,s_\ell}^N(p) := s_\ell\,\frac{\sqrt{M(M+2)}}{M+1} + (N-1)(\gamma-\gamma_{M,1}^\ell)\,\frac{\tanh\left(\frac{\pi}{2}p\right)}{p}\prod_{k=1}^{\frac{\ell}{2}}\frac{p^2+(2k-1)^2}{p^2+4k^2}, \tag{2.95}$$

where $\gamma_{M,1}^\ell$ has been defined in (2.87). We shall prove that h_{ℓ,s_ℓ}^N satisfies

$$h_{\ell,s_\ell}^N \leq f_{\ell,s_\ell}^N, \tag{2.96a}$$

$$h_{\ell,s_\ell}^N(0) = f_{\ell,s_\ell}^N(0), \tag{2.96b}$$

$$\lim_{p\to+\infty} h_{\ell,s_\ell}^N(p) = \lim_{p\to+\infty} f_{\ell,s_\ell}^N(p), \tag{2.96c}$$

$$h_{\ell,s_\ell}^N(p) \text{ is monotonic in } p\in\mathbb{R}_+. \tag{2.96d}$$

Starting with (2.96a), we take into account proposition B.3 and equation (2.79) to obtain an explicit expression for f_{ℓ,s_ℓ}^N

$$\begin{aligned} f_{\ell,s_\ell}^N(p) &= s_\ell\,\frac{\sqrt{M(M+2)}}{M+1} + \frac{N-1}{2}\,(S_{\text{off};\,\ell}+S_{\text{reg};\,\ell})(p)\\ &= s_\ell\,\frac{\sqrt{M(M+2)}}{M+1} + \frac{(N-1)\,\bar{h}_\ell(p)}{\cosh\left(\frac{\pi}{2}p\right)}\prod_{k=1}^{\frac{\ell}{2}}\left[p^2+(2k-1)^2\right],\end{aligned} \tag{2.97}$$

where we have introduced, for the sake of notation, the function

$$\bar{h}_\ell(p) := \gamma\,\frac{\sinh\left(\frac{\pi}{2}p\right)}{p}\prod_{k=1}^{\frac{\ell}{2}}\frac{1}{p^2+4k^2} - \frac{2^\ell\,\ell!\,{}_2F_1\left(\frac{\ell+1+ip}{2},\frac{\ell+1-ip}{2};\ell+\frac32;\frac{1}{(1+M)^2}\right)}{(2\ell+1)!\,(M+1)^\ell}. \tag{2.98}$$

To achieve the result, consider the Euler's transformation formula

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z) \tag{2.99} $$

and the inequality

$$ |\Gamma(a+ib)|^2 \leq |\Gamma(a)|^2\,, \quad \forall\, a,b\in\mathbb{R}. \tag{2.100} $$

Indeed, one can write

$$ \begin{aligned} {}_2F_1\big(\tfrac{\ell+1+ip}{2},\tfrac{\ell+1-ip}{2};\ell+\tfrac{3}{2};x^2\big) &= \sqrt{1-x^2}\,{}_2F_1\big(\tfrac{\ell+2-ip}{2},\tfrac{\ell+2+ip}{2};\ell+\tfrac{3}{2};x^2\big) \\ &= \sqrt{1-x^2}\sum_{k\in\mathbb{N}_0}\frac{x^{2k}}{k!}\frac{\big(\frac{\ell+2-ip}{2}\big)_k\big(\frac{\ell+2+ip}{2}\big)_k}{\big(\ell+\frac{3}{2}\big)_k} \leq \frac{\sqrt{1-x^2}\,\big(\frac{\ell}{2}\big)!^2}{\big|\Gamma\big(\frac{\ell+2+ip}{2}\big)\big|^2}\sum_{k\in\mathbb{N}_0}\frac{x^{2k}}{k!}\frac{\big(\frac{\ell}{2}+1\big)_k^2}{\big(\ell+\frac{3}{2}\big)_k} \\ &= \frac{\sqrt{1-x^2}\,\big(\frac{\ell}{2}\big)!^2}{\big|\Gamma\big(\frac{\ell+2+ip}{2}\big)\big|^2}\,{}_2F_1\big(\tfrac{\ell}{2}+1,\tfrac{\ell}{2}+1;\ell+\tfrac{3}{2};x^2\big), \end{aligned} $$

where we have used $\Gamma(\bar{z}) = \overline{\Gamma(z)}$ and inequality (2.100), according to which

$$ \frac{\big|\Gamma\big(\frac{\ell+2+ip}{2}+k\big)\big|^2}{\big|\Gamma\big(\frac{\ell+2+ip}{2}\big)\big|^2} \leq \frac{\Gamma^2\big(\frac{\ell}{2}+1\big)}{\big|\Gamma\big(\frac{\ell+2+ip}{2}\big)\big|^2}\frac{\Gamma^2\big(\frac{\ell}{2}+1+k\big)}{\Gamma^2\big(\frac{\ell}{2}+1\big)} = \frac{\big(\frac{\ell}{2}\big)!^2}{\big|\Gamma\big(\frac{\ell+2+ip}{2}\big)\big|^2}\big(\tfrac{\ell}{2}+1\big)_k^2. $$

Using again (2.99) to the right hand side, one obtains

$$ {}_2F_1\big(\tfrac{\ell+1+ip}{2},\tfrac{\ell+1-ip}{2};\ell+\tfrac{3}{2};x^2\big) \leq \frac{\big(\frac{\ell}{2}\big)!^2}{\big|\Gamma\big(\frac{\ell+2+ip}{2}\big)\big|^2}\,{}_2F_1\big(\tfrac{\ell+1}{2},\tfrac{\ell+1}{2};\ell+\tfrac{3}{2};x^2\big). \tag{2.101} $$

Making use of identity (2.78) in the previous inequality, one has

$$ \begin{aligned} \bar{h}_\ell(p) &\geq \frac{\sinh\big(\frac{\pi}{2}p\big)}{p}\prod_{k=1}^{\frac{\ell}{2}}\frac{1}{p^2+4k^2}\left[\gamma - \frac{2^{2\ell+1}\,\ell!\,\big(\frac{\ell}{2}\big)!^2}{\pi\,(2\ell+1)!\,(M+1)^\ell}\,{}_2F_1\Big(\tfrac{\ell+1}{2},\tfrac{\ell+1}{2};\ell+\tfrac{3}{2};\tfrac{1}{(1+M)^2}\Big)\right] \\ &= \frac{\sinh\big(\frac{\pi}{2}p\big)}{p}\,(\gamma-\gamma^\ell_{M,1})\prod_{k=1}^{\frac{\ell}{2}}\frac{1}{p^2+4k^2}\,. \end{aligned} $$

Exploiting this lower bound in (2.97), one finds out that h^N_{ℓ,s_ℓ} satisfies condition (2.96a). Furthermore, we stress that we have obtained this estimate by using only inequality (2.100), according to which the equality sign holds in case $p=0$. In other words, we have also proved (2.96b).
Next, we show (2.96c). Since $\frac{p^2+(2k-1)^2}{p^2+4k^2} < 1$ for all k,

$$ \left| h^N_{\ell,s_\ell}(p) - s_\ell\,\frac{\sqrt{M(M+2)}}{M+1}\right| \leq (N-1)|\gamma-\gamma^\ell_{M,1}|\,\frac{\tanh\big(\frac{\pi}{2}p\big)}{p} $$

where the right hand side vanishes as p goes to infinity. Therefore,

$$ \lim_{p\to+\infty} h^N_{\ell,s_\ell}(p) = s_\ell\,\frac{\sqrt{M(M+2)}}{M+1} = \lim_{p\to+\infty} f^N_{\ell,s_\ell}(p). $$

It remains to prove the monotonicity of h^N_{ℓ,s_ℓ} in $\mathbb{R}_+$. In particular, it suffices to show that the function

$$p \longmapsto \frac{\tanh\left(\frac{\pi}{2}p\right)}{p} \prod_{k=1}^{\ell/2} \frac{p^2+(2k-1)^2}{p^2+4k^2} \tag{2.102}$$

is decreasing in $\mathbb{R}_+$. Let us remind the product representation of the hyperbolic tangent

$$\tanh(z) = z \prod_{k\in\mathbb{N}} \frac{1+\frac{z^2}{\pi^2 k^2}}{1+\frac{4z^2}{\pi^2(2k-1)^2}}\,. \tag{2.103}$$

Denoting $z = \frac{\pi}{2}p$, one has

$$\begin{aligned}
\frac{\tanh\left(\frac{\pi}{2}p\right)}{p} \prod_{k=1}^{\frac{\ell}{2}} \frac{p^2+(2k-1)^2}{p^2+4k^2} &= \frac{\pi}{2} \prod_{k=1}^{\frac{\ell}{2}} \frac{1+\frac{p^2}{4k^2}}{1+\frac{p^2}{(2k-1)^2}} \frac{p^2+(2k-1)^2}{p^2+4k^2} \prod_{k=\frac{\ell}{2}+1}^{+\infty} \frac{1+\frac{p^2}{4k^2}}{1+\frac{p^2}{(2k-1)^2}} \\
&= \frac{\pi}{2} \prod_{k=1}^{\frac{\ell}{2}} \frac{(2k-1)^2}{4k^2} \prod_{k=\frac{\ell}{2}+1}^{+\infty} \frac{1+\frac{p^2}{4k^2}}{1+\frac{p^2}{(2k-1)^2}} = \frac{\pi}{2} \frac{(\ell-1)!!^2}{\ell!!^2} \prod_{k=\frac{\ell}{2}+1}^{+\infty} \frac{1+\frac{p^2}{4k^2}}{1+\frac{p^2}{(2k-1)^2}}\,.
\end{aligned}$$

In order to prove that the function (2.102) is decreasing, we consider

$$\begin{aligned}
\ln\left(\frac{\tanh\left(\frac{\pi}{2}p\right)}{p} \prod_{k=1}^{\frac{\ell}{2}} \frac{p^2+(2k-1)^2}{p^2+4k^2}\right) &= \ln\left(\frac{\pi}{2}\right) + 2\ln\left[\frac{(\ell-1)!!}{\ell!!}\right] + \\
&\quad + \sum_{k=\frac{\ell}{2}+1}^{+\infty} \ln\left(1+\frac{p^2}{4k^2}\right) - \ln\left[1+\frac{p^2}{(2k-1)^2}\right].
\end{aligned} \tag{2.104}$$

We notice that for $p > 0$

$$\begin{aligned}
\frac{\partial}{\partial p}\left\{\ln\left(1+\frac{p^2}{4k^2}\right) - \ln\left[1+\frac{p^2}{(2k-1)^2}\right]\right\} &= \frac{2p}{p^2+4k^2} - \frac{2p}{p^2+(2k-1)^2} \\
&= \frac{2p\,(1-4k)}{(p^2+4k^2)[p^2+(2k-1)^2]} < 0, \quad \forall\, k \geq 1.
\end{aligned}$$

Hence, (2.104) is decreasing in $p > 0$ since it is a sum of decreasing functions. Therefore, also (2.102) is decreasing and (2.96d) is proven.

In conclusion, we know that whenever $\gamma > \gamma_c^{N\mathrm{b}+1}$, there exists $s^*_\ell \in (0,1)$ for any $\ell \in \mathbb{N}_0$, such that $f^N_{\ell,s^*_\ell}(0) > 0$ uniformly in $N \geq 2$. Since we also know that f^N_{ℓ,s_ℓ} is eventually positive, conditions (2.96) imply that $f^N_{\ell,s^*_\ell} \geq h^N_{\ell,s^*_\ell} > 0$ and the proof is completed.

□

Remark 2.2. *In lemma 2.10, we have shown that, if $\gamma \geq \gamma^\ell_{M,1}$, any $s^*_\ell \in (0,1)$ is such that $f^N_{\ell,s^*_\ell} \geq 0$, whereas in case $\gamma \in \left(\gamma_c^{N\mathrm{b}+1},\, \gamma^\ell_{M,1}\right)$, the function f^N_{ℓ,s^*_ℓ} is still non negative for all s^*_ℓ s.t.*

$$\frac{\pi\,\ell!^2}{2^{2\ell+1}\left(\frac{\ell}{2}\right)!^4} \frac{(N-1)(M+1)}{\sqrt{M(M+2)}} \left(\gamma^\ell_{M,1}-\gamma\right) < s^*_\ell < 1.$$

Notice that the lower bound is non-increasing in ℓ, hence the sequence $\{s^*_\ell\}$ that makes $\Xi^\zeta_{\ell,s^*_\ell}$ non-negative for all $\zeta \geq 0$ and $\ell \in \mathbb{N}_0$ can be chosen within an interval that does not depend on ℓ, namely

$$\max\left\{0,\, \frac{\pi}{2} \frac{(N-1)(M+1)}{\sqrt{M(M+2)}} \left(\gamma^0_{M,1}-\gamma\right)\right\} < s^*_\ell < 1, \qquad \forall\, \ell \in \mathbb{N}_0\,.$$

2.5 Estimate of Θ^ζ

Collecting the results obtained in the previous two sections, we can now establish detailed estimates for Θ^ζ. Indeed, in the next proposition we prove a lower bound, which is the crucial ingredient for the proof of our main results. We also prove an upper bound, which improves the result already obtained in proposition 2.5.

Proposition 2.11. *Given $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^3)$ and $\zeta \geq 0$, we have*

$$\Theta^\zeta[\varphi] \geq [1 - \Lambda_\gamma(N,M)]\, \Theta^\zeta_{\text{diag}}[\varphi], \qquad \text{for } \gamma > \gamma_c^{N\text{b}+1}, \tag{2.105}$$

where

$$\Lambda_\gamma(N,M) := \max\left\{0, 1 - \tfrac{\pi(N-1)}{2} \tfrac{M+1}{\sqrt{M(M+2)}} \left[\gamma - \gamma_c^{N\text{b}+1}(M)\right]\right\} \in [0,1). \tag{2.106}$$

Moreover

$$\Theta^\zeta[\psi] \leq \Theta^\zeta_{\text{diag}}[\psi] + \Lambda'_\gamma(N,M)\, \Theta^0_{\text{diag}}[\psi]\,, \qquad \text{for } \gamma > 0, \tag{2.107}$$

where

$$\Lambda'_\gamma(N,M) := \tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}} \max\left\{ \tfrac{\pi}{2}\gamma,\ \tfrac{1}{2} S_{\text{off};\,1}(0) + \tfrac{2}{\pi}\gamma \right\}. \tag{2.108}$$

Proof. Let $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^3)$ and consider decomposition (2.46) and lemma 2.7. Then,

$$\begin{aligned}
\Theta^\zeta[\varphi] &= \sum_{\ell\in\mathbb{N}_0} \sum_{m=-\ell}^{\ell} F^\zeta_\ell[\hat{\varphi}_{\ell,m}] = \sum_{\ell\in\mathbb{N}_0} \sum_{m=-\ell}^{\ell} \Big(F^\zeta_{\text{diag}} + F^\zeta_{\text{off};\,\ell} + F_{\text{reg};\,\ell}\Big)[\hat{\varphi}_{\ell,m}] \\
&\geq \sum_{\substack{\ell\in\mathbb{N} \\ \ell \text{ odd}}} \sum_{m=-\ell}^{\ell} F^\zeta_{\text{diag}}[\hat{\varphi}_{\ell,m}] + \sum_{\substack{\ell\in\mathbb{N}_0 \\ \ell \text{ even}}} \sum_{m=-\ell}^{\ell} \Big(F^\zeta_{\text{diag}} + F^0_{\text{off};\,\ell} + F_{\text{reg};\,\ell}\Big)[\hat{\varphi}_{\ell,m}].
\end{aligned}$$

Taking account of definition (2.83), for any choice of $\{s_\ell\}_{\ell\in\mathbb{N}_0} \subset (0,1)$, the previous inequality reads

$$\Theta^\zeta[\varphi] \geq \sum_{\substack{\ell\in\mathbb{N} \\ \ell \text{ odd}}} \sum_{m=-\ell}^{\ell} F^\zeta_{\text{diag}}[\hat{\varphi}_{\ell,m}] + \sum_{\substack{\ell\in\mathbb{N}_0 \\ \ell \text{ even}}} \sum_{m=-\ell}^{\ell} (1-s_\ell) F^\zeta_{\text{diag}}[\hat{\varphi}_{\ell,m}] + \Xi^\zeta_{\ell,\,s_\ell}[\hat{\varphi}_{\ell,m}].$$

According to lemma 2.10, there exists a sequence $\{s^*_\ell\}_{\ell\in\mathbb{N}_0} \subset (0,1)$ such that $\Xi^\zeta_{\ell,\,s^*_\ell} \geq 0$, hence

$$\begin{aligned}
\Theta^\zeta[\varphi] &\geq \sum_{\substack{\ell\in\mathbb{N} \\ \ell \text{ odd}}} \sum_{m=-\ell}^{\ell} F^\zeta_{\text{diag}}[\hat{\varphi}_{\ell,m}] + \sum_{\substack{\ell\in\mathbb{N}_0 \\ \ell \text{ even}}} \sum_{m=-\ell}^{\ell} (1-s^*_\ell) F^\zeta_{\text{diag}}[\hat{\varphi}_{\ell,m}] \\
&\geq \sum_{\ell\in\mathbb{N}_0} \sum_{m=-\ell}^{\ell} (1-s^*_\ell) F^\zeta_{\text{diag}}[\hat{\varphi}_{\ell,m}] \geq \inf_{k\in\mathbb{N}_0} (1-s^*_k) \sum_{\ell\in\mathbb{N}_0} \sum_{m=-\ell}^{\ell} F^\zeta_{\text{diag}}[\hat{\varphi}_{\ell,m}] \\
&= \inf_{k\in\mathbb{N}_0} (1-s^*_k)\, \Theta^\zeta_{\text{diag}}[\varphi]
\end{aligned}$$

where, according to remark 2.2, each s_k^* can be arbitrarily chosen within an interval in $(0,1)$ that does not shrink as k varies. Exploiting this fact, we can optimize the inequality by choosing

$$s_k^* = \tfrac{\pi(N-1)}{2}\tfrac{M+1}{\sqrt{M(M+2)}}\max\big\{0,\ \gamma_{M,1}^0-\gamma\big\} = \Lambda_\gamma(N,M), \qquad \forall\, k\in\mathbb{N}_0$$

so that $\Theta^\zeta[\varphi]\geq[1-\Lambda_\gamma(N,M)]\,\Theta^\zeta_{\text{diag}}[\varphi]$, where $\Lambda_\gamma(N,M)$ is given by (2.106).
Let us consider the upper bound. By lemmata 2.7 and 2.9, we have

$$\begin{aligned}
\Theta^\zeta[\varphi] &= \sum_{\ell\in\mathbb{N}_0}\sum_{m=-\ell}^{\ell} F_\ell^\zeta[\hat\varphi_{\ell,m}] = \sum_{\ell\in\mathbb{N}_0}\sum_{m=-\ell}^{\ell}\Big(F^\zeta_{\text{diag}}+F^\zeta_{\text{off};\,\ell}+F_{\text{reg};\,\ell}\Big)[\hat\varphi_{\ell,m}]\\
&\leq \sum_{\substack{\ell\in\mathbb{N}_0\\ \ell\text{ even}}}\sum_{m=-\ell}^{\ell}\Big(F^\zeta_{\text{diag}}+F_{\text{reg};\,\ell}\Big)[\hat\varphi_{\ell,m}] + \sum_{\substack{\ell\in\mathbb{N}\\ \ell\text{ odd}}}\sum_{m=-\ell}^{\ell}\Big(F^\zeta_{\text{diag}}+F^0_{\text{off};\,\ell}+F_{\text{reg};\,\ell}\Big)[\hat\varphi_{\ell,m}]\\
&\leq \sum_{\ell\in\mathbb{N}_0}\sum_{m=-\ell}^{\ell} F^\zeta_{\text{diag}}[\hat\varphi_{\ell,m}] + \tfrac{N-1}{2}\sqrt{\tfrac{\eta}{\mu}}\max\Big\{S_{\text{reg};\,0}(0),\ S_{\text{off};\,1}(0)+S_{\text{reg};\,1}(0)\Big\}F^0_{\text{diag}}[\hat\varphi_{\ell,m}]\\
&= \Theta^\zeta_{\text{diag}}[\varphi]+\Lambda'_\gamma(N,M)\,\Theta^0_{\text{diag}}[\varphi]
\end{aligned}$$

where $\Lambda'_\gamma(N,M)$ is given by (2.108).

□

We end the section with a couple of observations.

Remark 2.3. *We stress that the upper bound obtained in* (2.107) *is an improvement of estimate* (2.42)*, since* $\Theta^0_{\text{diag}}\leq\Theta^\zeta_{\text{diag}}$ *and*

$$1+\Lambda'_\gamma(N,M)\leq 1+\tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}}\left(\tfrac{M+1}{M}+\tfrac{\pi}{2}\gamma\right). \tag{2.109}$$

Indeed, in case $\gamma\geq\frac{\pi}{\pi^2-4}S_{\text{off};\,1}(0)$ *one has*

$$\tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}}\left(\tfrac{M+1}{M}+\tfrac{\pi}{2}\gamma\right)\geq\tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}}\tfrac{\pi}{2}\gamma=\Lambda'_\gamma(N,M).$$

On the other hand, consider $0<\gamma<\frac{\pi}{\pi^2-4}S_{\text{off};\,1}(0)$*. Taking into account the following elementary estimate*

$$\arcsin(t)\geq\ t\geq t\sqrt{\frac{1-t}{1+t}},\qquad 0\leq t\leq 1, \tag{2.110}$$

we have $\frac{1}{t}\sqrt{1-t^2}\,\arcsin(t)\geq 1-t$*, or* $1-\frac{1}{t}\sqrt{1-t^2}\,\arcsin(t)\leq t$*. Then*

$$1-\sqrt{x^2-1}\,\arcsin\tfrac{1}{x}\leq\tfrac{1}{x},\qquad x\geq 1. \tag{2.111}$$

Therefore, exploiting (2.111) *with* $x=M+1$*, one obtains*

$$\begin{aligned}
\tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}}\left(\tfrac{M+1}{M}+\tfrac{\pi}{2}\gamma\right)&\geq\tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}}\left(1+\tfrac{2}{\pi}\gamma\right)\\
&\geq\tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}}\Big[(M+1)\Big(1-\sqrt{M(M+2)}\,\arcsin\tfrac{1}{M+1}\Big)+\tfrac{2}{\pi}\gamma\Big]\\
&\geq\tfrac{2}{\pi}\tfrac{(N-1)(M+1)}{\sqrt{M(M+2)}}\Big[(M+1)\Big(1-\sqrt{M(M+2)}\,\arcsin\tfrac{1}{M+1}\Big)+\gamma\Big]=\Lambda'_\gamma(N,M).
\end{aligned}$$

Remark 2.4. *We observe that a different kind of estimate from below can be performed for Θ^ζ. In particular, one has for any $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^3)$*

$$\begin{aligned}\Theta^\zeta[\varphi] &\geq \sum_{\substack{\ell\in\mathbb{N}\\ \ell\ odd}} \sum_{m=-\ell}^{\ell} F^\zeta_{\mathrm{diag}}[\hat{\varphi}_{\ell,m}] + \sum_{\substack{\ell\in\mathbb{N}_0\\ \ell\ even}} \sum_{m=-\ell}^{\ell} \Big(F^\zeta_{\mathrm{diag}} - s_\ell F^0_{\mathrm{diag}} + \Xi^0_{\ell,\,s_\ell}\Big)[\hat{\varphi}_{\ell,m}] \\ &\geq \sum_{\substack{\ell\in\mathbb{N}\\ \ell\ odd}} \sum_{m=-\ell}^{\ell} F^\zeta_{\mathrm{diag}}[\hat{\varphi}_{\ell,m}] + \sum_{\substack{\ell\in\mathbb{N}_0\\ \ell\ even}} \sum_{m=-\ell}^{\ell} F^\zeta_{\mathrm{diag}}[\hat{\varphi}_{\ell,m}] - s^*_\ell F^0_{\mathrm{diag}}[\hat{\varphi}_{\ell,m}] \\ &\geq \sum_{\ell\in\mathbb{N}_0} \sum_{m=-\ell}^{\ell} F^\zeta_{\mathrm{diag}}[\hat{\varphi}_{\ell,m}] - s^*_\ell F^0_{\mathrm{diag}}[\hat{\varphi}_{\ell,m}] \geq \Theta^\zeta_{\mathrm{diag}}[\varphi] - \sup_{k\in\mathbb{N}_0}\{s^*_k\}\,\Theta^0_{\mathrm{diag}}[\varphi].\end{aligned}$$

Again, this inequality is optimized by choosing s^*_k identically equal to $\Lambda_\gamma(N,M)$, so that one has

$$\Theta^\zeta[\varphi] \geq \int_{\mathbb{R}^3} d\boldsymbol{\sigma}\, \left(\sqrt{\tfrac{\mu}{\eta}\sigma^2 + \zeta} - \Lambda_\gamma(N,M)\sqrt{\tfrac{\mu}{\eta}}\,\sigma\right)|\hat{\varphi}(\boldsymbol{\sigma})|^2.$$

The quantity in parenthesis in the previous integrand has a minimum in σ, since $\Lambda_\gamma(N,M) < 1$ (namely $\gamma > \gamma_c^{N\mathrm{b}+1}$), i.e.

$$\sigma_{\min} = \Lambda_\gamma(N,M)\sqrt{\frac{\eta\,\zeta}{\mu[1-\Lambda^2_\gamma(N,M)]}}\,,$$

obtaining

$$\Theta^\zeta[\varphi] \geq \sqrt{1-\Lambda^2_\gamma(N,M)}\,\sqrt{\zeta}\,\|\varphi\|^2_{L^2(\mathbb{R}^3)}\,. \tag{2.112}$$

We stress that this estimate is in principle better than (2.105)*, in the sense that, during the calculations, we have neglected $\Xi^0_{\ell,\,s_\ell}$, whereas in proposition 2.11 we neglected $\Xi^\zeta_{\ell,\,s_\ell} \geq \Xi^0_{\ell,\,s_\ell}$.*

2.6 Proof of the Main Results

In this section we complete the proof of the results stated in section 2.1.

Proof of proposition 2.1. Let us recall that, for any charge $\xi \in H^{\frac{1}{2}}(\mathbb{R}^{3N})$, we have defined a rescaled charge $\phi \in H^{\frac{1}{2}}(\mathbb{R}^{3N})$ given by (2.29). According to equations (2.41) and (2.105), we can deduce a lower bound for the quadratic form Φ^λ

$$\begin{aligned}\Phi^\lambda[\xi] &= \Phi_0[\xi] + \tfrac{2\pi N}{\sqrt{m\mu}}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{k}_2\cdots d\boldsymbol{k}_N\,\sqrt{\tfrac{1}{2m}\textstyle\sum_{j=2}^N k_j^2 + \lambda}\;\Theta^{2m}[\phi](\boldsymbol{k}_2,\ldots,\boldsymbol{k}_N) \\ &\geq \Phi_0[\xi] + [1-\Lambda_\gamma(N,M)]\,\tfrac{2\pi N}{\sqrt{m\mu}}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{k}_2\cdots d\boldsymbol{k}_N\,\sqrt{\tfrac{1}{2m}\textstyle\sum_{j=2}^N k_j^2 + \lambda}\;\Theta^{2m}_{\mathrm{diag}}[\phi](\boldsymbol{k}_2,\ldots,\boldsymbol{k}_N) \\ &= \Phi_0[\xi] + [1-\Lambda_\gamma(N,M)]\,\Phi^\lambda_{\mathrm{diag}}[\xi], \qquad \forall\,\lambda>0,\ \gamma > \gamma_c^{N\mathrm{b}+1}.\end{aligned}$$

Recalling definition (2.14d) and assumption (1.10) (which implies β essentially bounded), we have

$$\Phi_0[\xi] \geq \tfrac{2\pi N}{\mu}\inf_{\mathbb{R}_+}\{\beta\}\,\langle\xi,\,\xi\rangle_{L^2(\mathbb{R}^{3N})} = \tfrac{2\pi N}{\mu}\Big(\alpha_0 - \tfrac{(N-1)\,\gamma}{b}\Big)\|\xi\|^2_{L^2(\mathbb{R}^{3N})}$$

$$
\begin{aligned}
&\geq \left(\min\{0,\alpha_0\} - \tfrac{(N-1)\,\gamma}{b}\right) \tfrac{2\pi N}{\mu}\, \|\hat{\xi}\|^2_{L^2(\mathbb{R}^{3N})} \\
&\geq \frac{\min\{0,\alpha_0\} - \frac{(N-1)\gamma}{b}}{\sqrt{2\lambda\mu}}\, \frac{4\pi N}{\sqrt{2\mu}} \int_{\mathbb{R}^{3N}} d\boldsymbol{k}_1 \cdots d\boldsymbol{k}_N \sqrt{\tfrac{k_1^2}{2(m_0+m)} + \textstyle\sum_{j=2}^N \tfrac{k_j^2}{2m} + \lambda}\; |\hat{\xi}(\boldsymbol{k}_1,\dots,\boldsymbol{k}_N)|^2 \\
&= \frac{\min\{0,\ \alpha_0 b\} - (N-1)\,\gamma}{b\sqrt{2\lambda\mu}}\, \Phi^\lambda_{\text{diag}}[\xi].
\end{aligned}
$$

Collecting the results obtained so far, we get

$$
\Phi^\lambda[\xi] \geq \left[1 - \Lambda_\gamma(N,M) - \tfrac{\max\{(N-1)\,\gamma,\ (N-1)\,\gamma - \alpha_0\, b\}}{b\sqrt{2\lambda\mu}}\right] \Phi^\lambda_{\text{diag}}[\xi]. \tag{2.113}
$$

The last expression is positive if λ is large enough, i.e. if $\lambda > \lambda_0^*$, with

$$
\lambda_0^* := \begin{cases} \dfrac{(N-1)^2\,\gamma^2}{2\mu[1-\Lambda_\gamma(N,M)]^2\, b^2}, & \text{if } \alpha_0 \geq 0, \\[2ex] \dfrac{[(N-1)\,\gamma/b + |\alpha_0|]^2}{2\mu[1-\Lambda_\gamma(N,M)]^2}, & \text{if } \alpha_0 < 0. \end{cases} \tag{2.114}
$$

Concerning the upper bound, we proceed in the same way and we find

$$
\Phi^\lambda[\xi] \leq [1 + \Lambda'_\gamma(N,M)]\, \Phi^\lambda_{\text{diag}}[\xi] + \Phi_0[\xi], \qquad \forall\, \lambda > 0,\ \gamma > 0. \tag{2.115}
$$

Moreover,

$$
\begin{aligned}
\Phi_0[\xi] &\leq \tfrac{2\pi N}{\mu} \sup_{\mathbb{R}_+}\{\beta\}\, \|\xi\|^2_{L^2(\mathbb{R}^{3N})} = \left(\alpha_0 + \tfrac{(N-1)\,\gamma}{b}\right) \tfrac{2\pi N}{\mu} \|\hat{\xi}\|^2_{L^2(\mathbb{R}^{3N})} \\
&\leq \frac{\max\{0,\alpha_0\} + \frac{(N-1)\gamma}{b}}{\sqrt{2\lambda\mu}}\, \frac{4\pi N}{\sqrt{2\mu}} \int_{\mathbb{R}^{3N}} d\boldsymbol{k}_1 \cdots d\boldsymbol{k}_N \sqrt{\tfrac{k_1^2}{2(m_0+m)} + \textstyle\sum_{j=2}^N \tfrac{k_j^2}{2m} + \lambda}\; |\hat{\xi}(\boldsymbol{k}_1,\dots,\boldsymbol{k}_N)|^2 \\
&= \frac{\max\{0,\ \alpha_0\, b\} + (N-1)\,\gamma}{b\sqrt{2\lambda\mu}}\, \Phi^\lambda_{\text{diag}}[\xi].
\end{aligned}
$$

Using the last estimate, we obtain

$$
\Phi^\lambda[\xi] \leq \left[1 + \Lambda'_\gamma(N,M) + \tfrac{\max\{(N-1)\,\gamma,\ (N-1)\,\gamma + \alpha_0\, b\}}{b\sqrt{2\lambda\mu}}\right] \Phi^\lambda_{\text{diag}}[\xi]. \tag{2.116}
$$

Hence, Φ^λ is closed for any $\lambda > \lambda_0^*$, since it is equivalent to the $H^{\frac{1}{2}}$-norm. Now let us give an alternative lower bound for Φ^λ taking into account inequality (2.112), identity (2.41) and the lower bound for θ in (1.10)

$$
\Phi^\lambda[\xi] \geq \tfrac{2\pi N}{\mu} \left[\sqrt{1 - \Lambda^2_\gamma(N,M)}\, \sqrt{2\mu\lambda} + \alpha_0 - (N-1)\,\tfrac{\gamma}{b}\right] \|\xi\|^2_{L^2(\mathbb{R}^{3N})}. \tag{2.117}
$$

In particular, Φ^λ is coercive for any $\lambda > \lambda_0$ with

$$
\lambda_0 := \begin{cases} 0, & \alpha_0 \geq (N-1)\,\tfrac{\gamma}{b}, \\[1ex] \dfrac{[(N-1)\,\gamma/b - \alpha_0]^2}{2\mu[1 - \Lambda^2_\gamma(N,M)]}, & \alpha_0 < (N-1)\,\tfrac{\gamma}{b}. \end{cases} \tag{2.118}
$$

Notice that $\lambda_0 < \lambda_0^*$.

Next, we want to show that Φ^λ is closed for any $\lambda > 0$. To this end, consider the quantity

$$\|\cdot\|^2_{\Phi^\lambda} := \Phi^\lambda[\cdot] + \tfrac{2\pi N}{\mu}\Big[\sqrt{2\mu\lambda}\,\Big(1-\sqrt{1-\Lambda^2_\gamma(N,M)}\Big) - \alpha_0 + (N-1)\tfrac{\gamma}{b}\Big]\|\cdot\|^2_{L^2(\mathbb{R}^{3N})} \tag{2.119}$$

that defines a norm in $L^2(\mathbb{R}^{3N})$ for all $\lambda > 0$. Indeed notice that, owing to (2.117) one has

$$\|\cdot\|^2_{\Phi^\lambda} \geq 2\pi N\sqrt{\tfrac{2\lambda}{\mu}}\,\|\cdot\|^2_{L^2(\mathbb{R}^{3N})},$$

hence $\|\xi\|_{\Phi^\lambda} = 0$ implies $\xi = 0$. In order to prove the closedness of Φ^λ we need to show that its domain is complete according to $\|\cdot\|_{\Phi^\lambda}$. To this end, let $\{\xi_n\}_{n\in\mathbb{N}} \subset H^{\frac{1}{2}}(\mathbb{R}^{3N})$ be a Cauchy sequence for $\|\cdot\|_{\Phi^\lambda}$ such that $\|\xi_n - \xi\|_{L^2(\mathbb{R}^{3N})} \longrightarrow 0$ for some $\xi \in L^2(\mathbb{R}^{3N})$. Since for all $\lambda > 0$ there holds

$$\Phi^\lambda[\cdot] \geq [1-\Lambda_\gamma(N,M)]\,\Phi^\lambda_{\text{diag}}[\cdot] + \tfrac{2\pi N}{\mu}\left[\alpha_0 - (N-1)\tfrac{\gamma}{b}\right]\|\cdot\|^2_{L^2(\mathbb{R}^{3N})}, \tag{2.120}$$

provided $\gamma > \gamma_c^{N\mathrm{b}+1}$, we also have

$$[1-\Lambda_\gamma(N,M)]\,\Phi^\lambda_{\text{diag}}[\xi_n-\xi] + 2\pi N\sqrt{\tfrac{2\lambda}{\mu}}\left[1-\sqrt{1-\Lambda^2_\gamma(N,M)}\right]\|\xi_n-\xi\|^2_{L^2(\mathbb{R}^{3N})} \leq \|\xi_n-\xi\|^2_{\Phi^\lambda} \longrightarrow 0.$$

Since both terms in the left hand side are positive, we observe that we have obtained

$$\Phi^\lambda_{\text{diag}}[\xi_n-\xi] \longrightarrow 0.$$

Hence, $\{\xi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence also for the $H^{\frac{1}{2}}$-norm so that $\xi \in H^{\frac{1}{2}}(\mathbb{R}^{3N})$ and the proof is complete.

□

We have shown that, provided $\gamma > \gamma_c^{N\mathrm{b}+1}$, the quadratic form Φ^λ is closed for any $\lambda > 0$ and it is also bounded from below by a positive constant for any $\lambda > \lambda_0$. Thus, under these assumptions, Φ^λ uniquely defines a s.a. and positive operator Γ^λ in $L^2(\mathbb{R}^{3N})$ for all $\lambda > \lambda_0$. Such operator is characterized as follows

$$\begin{aligned}&\mathscr{D}(\Gamma^\lambda) = \Big\{\xi \in H^{\frac{1}{2}}(\mathbb{R}^{3N}) \;\big|\; \exists\, g \in L^2(\mathbb{R}^{3N}) \text{ s.t. } \Phi^\lambda[\varphi,\xi] = \langle\varphi, g\rangle, \quad \forall\,\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{3N})\Big\},\\ &\Gamma^\lambda\xi = g, \quad \forall\,\xi \in \mathscr{D}(\Gamma^\lambda)\end{aligned} \tag{2.121}$$

where $\Phi^\lambda[\cdot,\cdot]$ is the sesquilinear form associated to $\Phi^\lambda[\cdot]$ via the polarization identity. Moreover, Γ^λ is invertible for all $\lambda > \lambda_0$.
We are now in position to conclude the proof of theorem 2.2.

Proof of theorem 2.2. Taking into account proposition 2.1, Q is bounded from below, since for any $\psi \in \mathscr{D}(Q)$, one has

$$Q[\psi] = \mathscr{F}_\lambda[w^\lambda] - \lambda\,\|\psi\|^2 + \Phi^\lambda[\xi] \geq -\lambda\,\|\psi\|^2, \quad \forall\,\lambda > \lambda_0.$$

Now, let us fix $\lambda > \lambda_0$. By construction Q is hermitian, hence, the associated sesquilinear form $Q[\cdot,\cdot]$ is symmetric. In particular, this means that the sesquilinear form $s[\cdot,\cdot]$ given by

$$s[\psi,\varphi] := Q[\psi,\varphi] + (1+\lambda)\langle\psi,\,\varphi\rangle, \quad \forall\,\psi,\varphi\in\mathscr{D}(Q)$$

defines a scalar product in $\mathscr{H}_{N\mathrm{b}+1}$. Therefore, we equip $\mathscr{D}(Q)\subset\mathscr{H}_{N\mathrm{b}+1}$ with the norm

$$\|\psi\|_Q^2 := Q[\psi] + (1+\lambda)\,\|\psi\|^2 = \mathscr{F}_\lambda[w^\lambda] + \Phi^\lambda[\xi] + \|\psi\|^2. \tag{2.122}$$

We prove that Q is closed by showing the completeness of $\mathscr{D}(Q)$ with respect to $\|\cdot\|_Q$. To this end, let $\{\psi_n\}\subset\mathscr{D}(Q)$ and $\psi\in\mathscr{H}_{N\mathrm{b}+1}$ be respectively a sequence and a vector s.t. $\|\psi_n-\psi_m\|_Q\longrightarrow 0$ as n,m go to infinity and $\|\psi_n-\psi\|\longrightarrow 0$. By (2.122), we have

$$\mathscr{F}_\lambda[w_n^\lambda - w_m^\lambda] + \Phi^\lambda[\xi_n-\xi_m]\longrightarrow 0 \tag{2.123}$$

and, since both $\mathscr{F}_\lambda$ and Φ^λ are closed and positive, (2.123) implies

$$\mathscr{F}_\lambda[w_n^\lambda - w_m^\lambda]\longrightarrow 0, \qquad \Phi^\lambda[\xi_n-\xi_m]\longrightarrow 0, \qquad \text{as } n,m\,\rightarrow\,+\infty.$$

Hence, we have obtained that $\{w_n^\lambda\}$ and $\{\xi_n\}$ are Cauchy sequences in $H^1(\mathbb{R}^{3(N+1)})\cap\mathscr{H}_{N\mathrm{b}+1}$ and $H^{\frac{1}{2}}(\mathbb{R}^{3N})\cap\mathscr{H}_{(N-1)\mathrm{b}+1}$, respectively. Thus, there exist $w^\lambda\in H^1(\mathbb{R}^{3(N+1)})\cap\mathscr{H}_{N\mathrm{b}+1}$ and $\xi\in H^{\frac{1}{2}}(\mathbb{R}^{3N})\cap\mathscr{H}_{(N-1)\mathrm{b}+1}$ such that

$$\left\|w_n^\lambda - w^\lambda\right\|_{H^1(\mathbb{R}^{3(N+1)})}\longrightarrow 0, \qquad \|\xi_n-\xi\|_{H^{1/2}(\mathbb{R}^{3N})}\longrightarrow 0, \qquad \text{as } n\,\rightarrow\,+\infty.$$

Furthermore, since G^λ defined in (2.11) is bounded for all $\lambda>0$, one has that $\psi_n = w_n^\lambda + G^\lambda\xi_n$ converges in $\mathscr{H}_{N\mathrm{b}+1}$ to the vector $w^\lambda + G^\lambda\xi$. By uniqueness of the limit, $\psi = w^\lambda + G^\lambda\xi$ and thus, $\psi\in\mathscr{D}(Q)$. We have shown that $(\mathscr{D}(Q),\|\cdot\|_Q)$ is a Banach space, hence Q is closed.

□

From theorem 2.2, we know that Q uniquely defines a self-adjoint and bounded from below Hamiltonian $\mathcal{H}$, $\mathscr{D}(\mathcal{H})$ in the Hilbert space $\mathscr{H}_{N\mathrm{b}+1} = L^2(\mathbb{R}^3)\otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3N})$. We conclude this section with the proof of theorem 2.3 which characterizes domain and action of $\mathcal{H}$.

Proof of theorem 2.3. Let us assume that $\psi = w^\lambda + G^\lambda\xi\in\mathscr{D}(\mathcal{H})$, with $\lambda>\lambda_0$. Then, there exists $f\in\mathscr{H}_{N\mathrm{b}+1}$ such that the sesquilinear form $Q[\cdot,\cdot]$ associated to $Q[\cdot]$ via the polarization identity satisfies

$$Q[v,\psi] = \langle v,\,f\rangle, \quad \forall\,v = w_v^\lambda + G^\lambda\xi_v\in\mathscr{D}(Q) \tag{2.124}$$

where $f =: \mathcal{H}\psi$. By definition one has

$$Q[v,\psi] = \langle\mathcal{H}_0^{\frac{1}{2}}w_v^\lambda,\,\mathcal{H}_0^{\frac{1}{2}}w^\lambda\rangle + \lambda\langle w_v^\lambda,\,w^\lambda\rangle - \lambda\langle v,\,\psi\rangle + \Phi^\lambda[\xi_v,\xi]. \tag{2.125}$$

Let us consider $v\in H^1(\mathbb{R}^{3(N+1)})\cap\mathscr{H}_{N\mathrm{b}+1}$, so that $\xi_v\equiv 0$ by injectivity of G^λ. Then

$$\langle\mathcal{H}_0^{\frac{1}{2}}v,\,\mathcal{H}_0^{\frac{1}{2}}w^\lambda\rangle + \lambda\langle v,\,w^\lambda\rangle - \lambda\langle v,\,\psi\rangle = \langle v,\,f\rangle, \quad \forall\,v\in H^1(\mathbb{R}^{3(N+1)})\cap\mathscr{H}_{N\mathrm{b}+1}.$$

Hence, $w^\lambda \in H^2(\mathbb{R}^{3(N+1)}) \cap \mathscr{H}_{N\mathrm{b}+1}$ and

$$(\mathcal{H}_0+\lambda)w^\lambda - \lambda\psi = f \tag{2.126}$$

which is equivalent to

$$\mathcal{H}\psi = \mathcal{H}_0 w^\lambda - \lambda G^\lambda \xi. \tag{2.127}$$

Now, let $v \in \mathscr{D}(Q)$. Taking account of (2.126), we have

$$\langle v,\, f+\lambda\psi\rangle = \langle w_v^\lambda,\, (\mathcal{H}_0+\lambda)w^\lambda\rangle + \langle G^\lambda \xi_v,\, (\mathcal{H}_0+\lambda)w^\lambda\rangle.$$

On the other hand, recalling (2.124) and (2.125),

$$\langle v,\, f+\lambda\psi\rangle = Q[v,\psi] + \lambda\langle v,\, \psi\rangle = \langle w_v^\lambda,\, (\mathcal{H}_0+\lambda)w^\lambda\rangle + \Phi^\lambda[\xi_v,\xi],$$

hence,

$$\Phi^\lambda[\xi_v,\xi] = \langle G^\lambda\xi_v,\, (\mathcal{H}_0+\lambda)w^\lambda\rangle = \tfrac{2\pi N}{\mu}\langle \xi_v,\, w^\lambda|_{\pi_N}\rangle_{L^2(\mathbb{R}^{3N})}, \quad \forall\, \xi_v \in H^{\frac{1}{2}}(\mathbb{R}^{3N}) \cap \mathscr{H}_{(N-1)\mathrm{b}+1}$$

where we have used definition (2.131) in the last step. Therefore, we conclude that $\xi \in D$ and $\Gamma^\lambda\xi = \frac{2\pi N}{\mu} w^\lambda\big|_{\pi_N}$. Comparing this identity with equation (2.142), the following relation holds

$$\Gamma^\lambda\xi = \tfrac{2\pi N}{\mu}(\Gamma_{\mathrm{diag}}^{N,\lambda} + \Gamma_{\mathrm{off}}^{N,\lambda} + \Gamma_{\mathrm{reg}}^{N})\xi, \qquad \forall\, \xi \in D. \tag{2.128}$$

Therefore, from equation (2.144), we have $\Gamma^\lambda = \overline{\Gamma}_\lambda$. This means that, owing to (2.145), Γ^λ satisfies the assumptions of proposition A.2, hence its definition can be extended to all $\lambda \in -\rho(\mathcal{H}_0)$ so that its extension fulfils conditions (A.2). Then, denoting this operator by $\Gamma(z)$, with $z \in \rho(\mathcal{H}_0)$, we have obtained $\rho(\mathcal{H}) \supseteq \mathbb{C} \smallsetminus [-\lambda_0, +\infty)$ and

$$\mathcal{R}_{\mathcal{H}_0}(z) + G(z)\Gamma(z)^{-1}G(\bar{z})^* = (\mathcal{H}-z)^{-1},$$

according to the theory discussed[2] in appendix A. □

Remark 2.5. *We stress that we have finally achieved a rigorous definition of the operator $\mathcal{H}$ as a lowered semi-bounded s.a. extension of $\dot{\mathcal{H}}_0$ defined by* (2.6)*. In particular, this means that for any $\psi \in \mathscr{D}(\dot{\mathcal{H}}_0) \subset \mathscr{D}(\mathcal{H})$ (roughly speaking, for all functions vanishing on π), we have $\mathcal{H}\psi = \mathcal{H}_0\psi$. Moreover, $\psi = w^\lambda + G^\lambda\xi \in \mathscr{D}(\mathcal{H})$ satisfies boundary condition* (2.9) *at least in the weak topology. Indeed, because of equation* (2.143)*, we have for any $\xi \in D$*

$$\Gamma^\lambda\xi = \tfrac{2\pi N}{\mu}\Gamma_{\mathrm{reg}}^N\xi + \tfrac{2\pi N}{\mu}\,\underset{\boldsymbol{x}_N\to\boldsymbol{x}_0}{\mathrm{w-lim}}\left[\frac{\xi\left(\frac{m\boldsymbol{x}_N+m_0\boldsymbol{x}_0}{m+m_0},\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{N-1}\right)}{|\boldsymbol{x}_N-\boldsymbol{x}_0|} - (\psi-w^\lambda)(\boldsymbol{x}_0,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)\right],$$

therefore,

$$\Gamma^\lambda\xi = \tfrac{2\pi N}{\mu}\Gamma_{\mathrm{reg}}^N\xi + \tfrac{2\pi N}{\mu}w^\lambda|_{\pi_N} + \tfrac{2\pi N}{\mu}\,\underset{\boldsymbol{x}_N\to\boldsymbol{x}_0}{\mathrm{w-lim}}\left[\frac{\xi\left(\frac{m\boldsymbol{x}_N+m_0\boldsymbol{x}_0}{m+m_0},\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{N-1}\right)}{|\boldsymbol{x}_N-\boldsymbol{x}_0|} - \psi(\boldsymbol{x}_0,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)\right]$$

[2] Actually, exploiting the results described in appendix A and proposition A.3, one can equivalently characterize the s.a. and bounded from below Hamiltonian of the system in an alternative way, as we shall see in the next chapter.

according to proposition B.5. Hence, owing to the boundary condition $\Gamma^\lambda \xi = \frac{2\pi N}{\mu} w^\lambda|_{\pi_N}$ *one gets*

$$\mathrm{w-}\lim_{\boldsymbol{x}_N \to \boldsymbol{x}_0} \left[\psi(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) - \frac{\xi\left(\frac{m\boldsymbol{x}_N + m_0 \boldsymbol{x}_0}{m+m_0}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\right)}{|\boldsymbol{x}_N - \boldsymbol{x}_0|} \right] = (\Gamma^N_{\mathrm{reg}} \xi)(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}).$$

In other words, requiring $\Gamma^\lambda \xi = \frac{2\pi N}{\mu} w^\lambda|_{\pi_N}$ *in the domain of the Hamiltonian* $\mathcal{H}$ *is equivalent to impose boundary condition* (2.9) *on the coincidence hyperplanes at least in weak-topology*[3] *sense.*

[3]In order to obtain the same result in the L^2-topology, more information on D is required.

Addendum

2.A Potential and Boundary Condition

In the next chapter, we will exploit the abstract setting described in Appendix A in a more extensive and systematic way. In order to provide a preliminary hint, let us establish the connection of such abstract theory with the problem dealt in this chapter.
The Hilbert space $\mathfrak{H}$ is given by $\mathscr{H}_{N\mathrm{b}+1}$, while the free Hamiltonian $\mathcal{H}_0$ plays the role of the operator A and S corresponds to $\dot{\mathcal{H}}_0$. The auxiliary Hilbert space $\mathscr{X}$ in our case is $\mathscr{H}_{(N-1)\mathrm{b}+1}$. Define

$$\mathfrak{X}_i := L^2(\mathbb{R}^3, d\boldsymbol{x}_0) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3(N-1)}, d\boldsymbol{x}_1 \cdots d\check{\boldsymbol{x}}_i \cdots d\boldsymbol{x}_N) \tag{2.129}$$

and let $T_i \colon \mathscr{D}(\mathcal{H}_0) \longrightarrow \mathfrak{X}_i$ be the trace operator $f \longmapsto \frac{2\pi}{\mu} f|_{\pi_i}$. Observe that each $\mathfrak{X}_i$ is isomorphic to $\mathscr{H}_{(N-1)\mathrm{b}+1}$. Denoting with $\mathcal{C}_i$ the isomorphism sending $\mathscr{H}_{(N-1)\mathrm{b}+1}$ to $\mathfrak{X}_i$, the continuous map τ is represented by the operator T, given by

$$\begin{gathered} T \colon \mathscr{D}(\mathcal{H}_0) \subset \mathscr{H}_{N\mathrm{b}+1} \longrightarrow \mathscr{H}_{(N-1)\mathrm{b}+1}, \\ T : \psi \longmapsto \sum_{i=1}^{N} \mathcal{C}_i^* T_i \psi = \tfrac{2\pi}{\mu} \sum_{i=1}^{N} \mathcal{C}_i^* \psi|_{\pi_i} = \tfrac{2\pi N}{\mu}\, \mathcal{C}_j^* \psi|_{\pi_j}, \qquad \text{for any } j \in \{1, \dots, N\}. \end{gathered} \tag{2.130}$$

The reason behind the constant $\frac{2\pi}{\mu}$ is clarified in remark 2.8. Notice that T is bounded since the trace operators $T_i : f \longmapsto f|_{\pi_i}$ are continuous between $H^{\frac{3}{2}+s}(\mathbb{R}^{3(N+1)})$ and $H^s(\mathbb{R}^{3N})$ for any $s > 0$ (proposition B.4). Moreover, $\operatorname{ran}(T)$ is dense and $\mathscr{D}(\dot{\mathcal{H}}_0) = \ker(T)$ by construction.
Finally, the operator $\mathcal{G}(z)$ is represented by $(T\mathcal{R}_{\mathcal{H}_0}(\bar{z}))^* =: G(z) \in \mathscr{B}\big(\mathscr{H}_{(N-1)\mathrm{b}+1}, \mathscr{H}_{N\mathrm{b}+1}\big)$.
Morally, our efforts in the previous sections have been devoted to the construction of the invertible operator $\Gamma(z)$ that encodes the ultraviolet regularization and, together with T, fully characterizes the lower semi-bounded Hamiltonian $\mathcal{H}$.

In the following, we show that the operator $G(-\lambda)$ can be identified with the potential G^λ, for $\lambda > 0$, whose definition has been provided in (2.11) and then we discuss some of its properties.
Denote by $\xi_i \in \mathfrak{X}_i$ the "charge" associated to the i-th coincidence hyperplane π_i and the operator $G_i(z) : \mathfrak{X}_i \longrightarrow L^2_{\mathrm{sym}}(\mathbb{R}^6, d\boldsymbol{x}_0 d\boldsymbol{x}_i) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3(N-1)}, d\boldsymbol{x}_1 \cdots d\check{\boldsymbol{x}}_i \cdots d\boldsymbol{x}_N)$ the potential generated by the i-th charge. Since the particles interacting with the impurity are all indistinguishable from each other, all the charges must be equal, namely $\xi_i = \mathcal{C}_i \xi$. In other words, given $\xi \in \mathscr{H}_{(N-1)\mathrm{b}+1}$ we have

$$G(z)\xi = (T\mathcal{R}_{\mathcal{H}_0}(\bar{z}))^* \xi = \textstyle\sum_{i=1}^{N} G_i(z) \mathcal{C}_i \xi \tag{2.131}$$

with the action of $G_i(z)$ yet to be determined. Next, we claim that the continuous operator $G(-\lambda)$ coincides with the definition of the potential G^λ given by (2.11) for any $\lambda > 0$. Let us prove this assertion. For any $\xi \in \mathscr{H}_{(N-1)\mathrm{b}+1}$, $\psi \in \mathscr{H}_{N\mathrm{b}+1}$, by definition,

$$\langle G(-\lambda)\xi,\, \psi\rangle = \langle \xi,\, T\mathcal{R}_{\mathcal{H}_0}(-\lambda)\psi\rangle_{L^2(\mathbb{R}^{3N})} = \langle \hat{\xi},\, \widehat{T\mathcal{R}_{\mathcal{H}_0}(-\lambda)\psi}\rangle_{L^2(\mathbb{R}^{3N})}.$$

From (2.130), for any $\phi \in H^s(\mathbb{R}^{3(N+1)})$ with $s > \frac{3}{2}$, one obtains

$$\widehat{T_i\phi}(\boldsymbol{P}, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_i \dots, \boldsymbol{k}_N) = \frac{1}{\mu}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^3} d\boldsymbol{Q}\; \hat{\phi}(\boldsymbol{Q}, \boldsymbol{k}_1, \dots, \boldsymbol{k}_{i-1}, \boldsymbol{P}-\boldsymbol{Q}, \boldsymbol{k}_{i+1}, \dots, \boldsymbol{k}_N). \tag{2.132}$$

Therefore, in our case,

$$\langle \hat{\xi},\, \widehat{T\mathcal{R}_{\mathcal{H}_0}(-\lambda)\psi}\rangle_{L^2(\mathbb{R}^{3N})} = \frac{1}{\sqrt{2\pi}\,\mu}\sum_{i=1}^N \int_{\mathbb{R}^{3N}} d\boldsymbol{P} d\boldsymbol{k}_1 \cdots d\check{\boldsymbol{k}}_i \cdots d\boldsymbol{k}_N\; \overline{\hat{\xi}(\boldsymbol{P}, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_i \dots, \boldsymbol{k}_N)}\, \times$$
$$\times \int_{\mathbb{R}^3} d\boldsymbol{Q}\; \frac{\hat{\psi}(\boldsymbol{Q}, \boldsymbol{k}_1, \dots, \boldsymbol{k}_{i-1}, \boldsymbol{P}-\boldsymbol{Q}, \boldsymbol{k}_{i+1}, \dots, \boldsymbol{k}_N)}{\frac{1}{2m_0}Q^2 + \frac{1}{2m}|\boldsymbol{P}-\boldsymbol{Q}|^2 + \frac{1}{2m}\sum_{j\neq i} k_j^2 + \lambda}.$$

Finally, adopting the substitution $\boldsymbol{P} \longmapsto \boldsymbol{Q} + \boldsymbol{k}_i$,

$$\langle \widehat{T\mathcal{R}_{\mathcal{H}_0}(-\lambda)\psi},\, \hat{\xi}\rangle_{L^2(\mathbb{R}^{3N})} = \frac{1}{\sqrt{2\pi}\,\mu}\int_{\mathbb{R}^{3(N+1)}} d\boldsymbol{Q} d\boldsymbol{k}_1 \cdots d\boldsymbol{k}_N\; \overline{\hat{\psi}(\boldsymbol{Q}, \boldsymbol{k}_1, \dots, \boldsymbol{k}_N)}\, \times$$
$$\times \sum_{i=1}^N \frac{\hat{\xi}(\boldsymbol{Q}+\boldsymbol{k}_i, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_i \dots, \boldsymbol{k}_N)}{\frac{1}{2m_0}Q^2 + \frac{1}{2m}\sum_{j=1}^N k_j^2 + \lambda},$$

from which the following expression for $G(-\lambda)$ in the space of momenta comes out

$$\begin{aligned}(\widehat{G(-\lambda)\xi})(\boldsymbol{p}, \boldsymbol{k}_1, \dots, \boldsymbol{k}_N) &= \frac{1}{\sqrt{2\pi}\,\mu}\sum_{i=1}^N \frac{\hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_i, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_i \dots, \boldsymbol{k}_N)}{\frac{1}{2m_0}p^2 + \frac{1}{2m}\sum_{j=1}^N k_j^2 + \lambda} \\ &=: \sum_{i=1}^N (\widehat{G_i(-\lambda)\mathcal{C}_i\xi})(\boldsymbol{p}, \boldsymbol{k}_1, \dots, \boldsymbol{k}_N).\end{aligned} \tag{2.133}$$

Notice that definition (2.11) has been recovered.

Remark 2.6. *We stress that, since G^λ has been shown to be equal to $G(-\lambda)$, its properties are inherited, for instance,* $\mathrm{ran}(G^\lambda) \subset \mathscr{H}_{N\mathrm{b}+1}$ *and* $\ker(G^\lambda) = \{0\}$.

Remark 2.7. *Recall, from remark A.1, that* $\mathrm{ran}(G^\lambda)$ *does not share non-trivial elements with* $H^2(\mathbb{R}^{3(N+1)})$*. Moreover, from* (2.133)*, one can verify that* $\mathrm{ran}(G^\lambda) \cap H^1(\mathbb{R}^{3(N+1)}) = \{0\}$ *as well. This fact is remarkable in defining the quadratic form Q in* (2.16)*.*

Next, our goal is to extract the asymptotic behaviour of the potential in a neighborhood of the coincidence hyperplanes, in the position representation. We compute such asymptotic behavior for a regular charge $\xi \in \mathcal{S}(\mathbb{R}^{3N}) \cap \mathfrak{X}_j$. From (2.133), we get

$$\big(G_j^\lambda\xi\big)(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_N) = \frac{2\pi}{\mu}\int_{\mathbb{R}^{3(N+1)}} d\boldsymbol{q} d\boldsymbol{k}_1 \cdots d\boldsymbol{k}_N\; \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_0 + i\sum\limits_{n=1}^N \boldsymbol{k}_n\cdot\boldsymbol{x}_n}}{(2\pi)^{\frac{3}{2}(N+2)}}\; \frac{\hat{\xi}(\boldsymbol{q}+\boldsymbol{k}_j, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_j \dots, \boldsymbol{k}_N)}{\frac{1}{2m_0}q^2 + \frac{1}{2m}\sum_{n=1}^N k_n^2 + \lambda}$$

$$= \frac{2\pi}{\mu}\int_{\mathbb{R}^{3N}} d\boldsymbol{p}\, d\boldsymbol{k}_1 \cdots d\check{\boldsymbol{k}}_j \cdots d\boldsymbol{k}_N \; \frac{e^{i\boldsymbol{p}\cdot\left(\frac{m\boldsymbol{x}_j + m_0\boldsymbol{x}_0}{m+m_0}\right) + i\sum\limits_{n\neq j}\boldsymbol{k}_n\cdot\boldsymbol{x}_n}}{(2\pi)^{\frac{3}{2}(N+2)}}\; \hat{\xi}(\boldsymbol{p}, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_j \dots, \boldsymbol{k}_N) \times$$
$$\times \frac{(m_0 m)^{\frac{3}{2}}}{(m_0+m)^3}\int_{\mathbb{R}^3} d\boldsymbol{\kappa}\; \frac{e^{i\frac{\sqrt{m_0 m}}{m_0+m}\boldsymbol{\kappa}\cdot(\boldsymbol{x}_0 - \boldsymbol{x}_j)}}{\frac{\kappa^2+p^2}{2(m_0+m)} + \frac{1}{2m}\sum_{n\neq j} k_n^2 + \lambda},$$

where a change of variables of Jacobian $\left(\frac{\sqrt{m_0 m}}{m_0+m}\right)^3$ has occurred in the last step, where

$$\begin{cases} \boldsymbol{p} = \boldsymbol{q} + \boldsymbol{k}_j\,, \\ \boldsymbol{\kappa} = \sqrt{\frac{m}{m_0}}\,\boldsymbol{q} - \sqrt{\frac{m_0}{m}}\,\boldsymbol{k}_j \end{cases} \iff \begin{cases} \boldsymbol{q} = \frac{m_0}{m_0+m}\Big(\boldsymbol{p} + \sqrt{\frac{m}{m_0}}\,\boldsymbol{\kappa}\Big), \\ \boldsymbol{k}_j = \frac{m}{m_0+m}\big(\boldsymbol{p} - \sqrt{\frac{m_0}{m}}\,\boldsymbol{\kappa}\big). \end{cases} \tag{2.134}$$

The last integral in $d\boldsymbol{\kappa}$ is well known, since, given $a > 0$, one has

$$\int_{\mathbb{R}^3} d\boldsymbol{k}\; \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{k^2 + a^2} = \frac{2\pi^2}{|\boldsymbol{x}|}\, e^{-a|\boldsymbol{x}|}\,, \quad \forall\, \boldsymbol{x} \neq \boldsymbol{0}. \tag{2.135}$$

Hence,

$$\begin{aligned}\big(G_j^\lambda\xi\big)(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_N) = &\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{k}_1 \cdots d\check{\boldsymbol{k}}_j \cdots d\boldsymbol{k}_N \; \frac{e^{i\sum\limits_{n\neq j}\boldsymbol{k}_n\cdot\boldsymbol{x}_n}}{(2\pi)^{\frac{3}{2}(N-1)}}\int_{\mathbb{R}^3} d\boldsymbol{p}\; \frac{e^{i\boldsymbol{p}\cdot\left(\frac{m\boldsymbol{x}_j + m_0\boldsymbol{x}_0}{m+m_0}\right)}}{(2\pi)^{\frac{3}{2}}} \times \\ &\times \frac{\hat{\xi}(\boldsymbol{p}, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_j \dots, \boldsymbol{k}_N)}{|\boldsymbol{x}_0 - \boldsymbol{x}_j|}\, e^{-\sqrt{2\mu}\,|\boldsymbol{x}_0 - \boldsymbol{x}_j|\sqrt{\frac{p^2}{2(m_0+m)} + \frac{1}{2m}\sum_{n\neq j} k_n^2 + \lambda}}. \end{aligned} \tag{2.136}$$

From the last equation, notice that the term $G_j^\lambda\xi$ is regular in $\mathbb{R}^{3(N+1)} \setminus \pi_j$. Furthermore, since we are working with $\hat{\xi} \in \mathcal{S}(\mathbb{R}^{3N})$, with a Taylor expansion of the exponential, we can easily expand in terms of powers of $|\boldsymbol{x}_j - \boldsymbol{x}_0|$

$$\begin{aligned}\big(G_j^\lambda\xi\big)(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_N) = &\frac{\xi\Big(\frac{m\boldsymbol{x}_j + m_0\boldsymbol{x}_0}{m+m_0}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_j \dots, \boldsymbol{x}_N\Big)}{|\boldsymbol{x}_0 - \boldsymbol{x}_j|} + \\ &- \sqrt{2\mu}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{k}_1 \cdots d\check{\boldsymbol{k}}_j \cdots d\boldsymbol{k}_N \; \frac{e^{i\sum\limits_{n\neq j}\boldsymbol{k}_n\cdot\boldsymbol{x}_n}}{(2\pi)^{\frac{3}{2}(N-1)}}\int_{\mathbb{R}^3} d\boldsymbol{p}\; \frac{e^{i\boldsymbol{p}\cdot\left(\frac{m\boldsymbol{x}_j + m_0\boldsymbol{x}_0}{m+m_0}\right)}}{(2\pi)^{\frac{3}{2}}} \times \\ &\times \sqrt{\frac{p^2}{2(m_0+m)} + \frac{1}{2m}\sum_{n\neq j} k_n^2 + \lambda}\; \hat{\xi}(\boldsymbol{p}, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_j \dots, \boldsymbol{k}_N) + \\ &+ \mathcal{O}(|\boldsymbol{x}_0 - \boldsymbol{x}_j|)\,. \end{aligned}$$

Therefore, one obtains an explicit behavior of the potential near π_j

$$\big(G_j^\lambda\xi\big)(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_N) = \frac{\xi\Big(\frac{m\boldsymbol{x}_j + m_0\boldsymbol{x}_0}{m+m_0}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_j \dots, \boldsymbol{x}_N\Big)}{|\boldsymbol{x}_0 - \boldsymbol{x}_j|} - \Gamma_{\text{diag}}^{j,\lambda}\xi + o(1)\,, \tag{2.137}$$

where

$$\begin{aligned}(\Gamma_{\text{diag}}^{j,\lambda}\xi)(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_j \dots, \boldsymbol{x}_N) = &\sqrt{2\mu}\int_{\mathbb{R}^{3N}} d\boldsymbol{p}\, d\boldsymbol{k}_1 \cdots d\check{\boldsymbol{k}}_j \cdots d\boldsymbol{k}_N \; \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}_0 + i\sum\limits_{n\neq j}\boldsymbol{k}_n\cdot\boldsymbol{x}_n}}{(2\pi)^{\frac{3}{2}N}} \times \\ &\times \sqrt{\frac{p^2}{2(m_0+m)} + \frac{1}{2m}\sum_{n\neq j} k_n^2 + \lambda}\; \hat{\xi}(\boldsymbol{p}, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_j \dots, \boldsymbol{k}_N)\,. \end{aligned} \tag{2.138}$$

A similar asymptotic expansion holds for G^λ in a neighborhood of π_j

$$(G^\lambda\xi)(\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)=\frac{\xi\Big(\frac{m\boldsymbol{x}_j+m_0\boldsymbol{x}_0}{m+m_0},\boldsymbol{x}_1,\dots\check{\boldsymbol{x}}_j\dots,\boldsymbol{x}_N\Big)}{|\boldsymbol{x}_0-\boldsymbol{x}_j|}-(\Gamma^{j,\lambda}_{\text{diag}}+\Gamma^{j,\lambda}_{\text{off}})\xi+o(1)\,, \quad (2.139)$$

with $\Gamma^{j,\lambda}_{\text{off}}$ representing the contribution of all other potentials $\{G^\lambda_i\}_{i\neq j}$ evaluated on π_j, i.e.

$$(\Gamma^{j,\lambda}_{\text{off}}\xi)(\boldsymbol{x}_0,\boldsymbol{x}_1,\dots\check{\boldsymbol{x}}_j\dots,\boldsymbol{x}_N)=-\tfrac{1}{4\pi^2\mu}\int_{\mathbb{R}^3}d\boldsymbol{p}\int_{\mathbb{R}^{3N}}d\boldsymbol{k}_1\cdots d\boldsymbol{k}_N\,\frac{e^{i(\boldsymbol{p}+\boldsymbol{k}_j)\cdot\boldsymbol{x}_0+i\sum\limits_{\ell\neq j}\boldsymbol{k}_\ell\cdot\boldsymbol{x}_\ell}}{(2\pi)^{\frac{3}{2}N}}\times$$
$$\times\sum_{\substack{n=1\\n\neq j}}^{N}\frac{\hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_n,\boldsymbol{k}_1,\dots\check{\boldsymbol{k}}_n\dots,\boldsymbol{k}_N)}{\frac{1}{2m_0}p^2+\frac{1}{2m}\sum_{\ell=1}^N k_\ell^2+\lambda}. \quad (2.140)$$

Remark 2.8. *Notice that, in light of* (2.139)*, we have obtained precisely the same singular behavior around* π_j *for both* $G^\lambda\xi$ *and* $\psi\in\mathscr{H}_{N\text{b}+1}\cap H^2(\mathbb{R}^{3(N+1)}\setminus\pi)$ *satisfying Minlos-Faddeev boundary condition* (2.9)*. Therefore, this suggests to write the vector* ψ *satisfying boundary condition* (2.9) *as a sum of two terms, namely*

$$\psi=G^\lambda\xi+w^\lambda, \quad (2.141)$$

where, taking into account definition (2.10)*,* $w^\lambda\in\mathscr{H}_{N\text{b}+1}\cap H^2(\mathbb{R}^{3(N+1)})$ *fulfils*

$$w^\lambda|_{\pi_j}=(\Gamma^{j,\lambda}_{\text{diag}}+\Gamma^{j,\lambda}_{\text{off}}+\Gamma^{j}_{\text{reg}})\xi\,,\qquad\forall j\in\{1,\dots,N\}. \quad (2.142)$$

Next, let us define the symmetric operator $\Gamma_\lambda:\mathcal{S}(\mathbb{R}^{3N})\cap\mathscr{H}_{(N-1)\text{b}+1}\longrightarrow\mathscr{H}_{(N-1)\text{b}+1}$, given by

$$\Gamma_\lambda:\xi\longrightarrow\tfrac{2\pi N}{\mu}\,\underset{\boldsymbol{x}_N\to\boldsymbol{x}_0}{\text{w}-\lim}\left[\frac{\xi\Big(\frac{m\boldsymbol{x}_N+m_0\boldsymbol{x}_0}{m+m_0},\boldsymbol{x}_1,\dots,\boldsymbol{x}_{N-1}\Big)}{|\boldsymbol{x}_N-\boldsymbol{x}_0|}-G^\lambda\xi(\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)\right]+\tfrac{2\pi N}{\mu}\Gamma^N_{\text{reg}}\xi. \quad (2.143)$$

Therefore, following (2.139), we can observe that

$$\tfrac{2\pi N}{\mu}(\Gamma^{N,\lambda}_{\text{diag}}+\Gamma^{N,\lambda}_{\text{off}}+\Gamma^N_{\text{reg}})=\Gamma_\lambda. \quad (2.144)$$

Furthermore, owing to proposition A.1, $G^{\lambda_1}-G^{\lambda_2}\in\mathscr{D}(\mathcal{H}_0)$ for all $\lambda_1,\lambda_2>0$, hence

$$\Gamma_{\lambda_1}-\Gamma_{\lambda_2}=T(G^{\lambda_2}-G^{\lambda_1}) \quad (2.145)$$

because of proposition B.5. This means that the s.a. extension of Γ_λ (that has been proved to be Γ^λ characterized by (2.121)) is a good candidate on which testing the assumptions of proposition A.2.

2.B Heuristic Derivation of the Quadratic Form

Here we provide a heuristic discussion meant to justify the definition of the quadratic form Q given in (2.16). Given $\psi\in\mathscr{H}_{N\text{b}+1}\cap H^2(\mathbb{R}^{3(N+1)}\setminus\pi)$ a vector fulfilling boundary condition (2.9), our aim is to construct the energy form $\langle\psi,\, \tilde{\mathcal{H}}\psi\rangle$ associated to the formal Hamiltonian $\tilde{\mathcal{H}}$ discussed at the beginning of this chapter. Recalling that the Hamiltonian acts as $\mathcal{H}_0$ outside π, given $\epsilon > 0$, let $D_\epsilon := \big\{(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_N) \in \mathbb{R}^{3(N+1)} \,\big|\, \min_{1\le i\le N} |\boldsymbol{x}_i - \boldsymbol{x}_0| > \epsilon\big\}$. Then

$$\langle\psi,\, \tilde{\mathcal{H}}\psi\rangle = \lim_{\epsilon\to 0}\int_{D_\epsilon} d\boldsymbol{x}_0 d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_N\, \overline{\psi(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_N)}\, (\mathcal{H}_0\psi)(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_N). \tag{2.146}$$

Introducing the decomposition of remark 2.8, $\psi = w^\lambda + G^\lambda\xi$, equation (2.146) reads

$$\langle\psi,\, \tilde{\mathcal{H}}\psi\rangle = \langle w^\lambda,\, (\mathcal{H}_0 + \lambda) w^\lambda\rangle - \lambda\, \|\psi\|^2 + \langle G^\lambda\xi,\, (\mathcal{H}_0 + \lambda) w^\lambda\rangle, \tag{2.147}$$

since $\mathscr{D}(\mathcal{H}_0) \cap \operatorname{ran}(G^\lambda) = \{0\}$. The last term can be simplified using (2.131)

$$\begin{aligned}\langle\psi,\, \tilde{\mathcal{H}}\psi\rangle &= \langle w^\lambda,\, (\mathcal{H}_0 + \lambda) w^\lambda\rangle - \lambda\, \|\psi\|^2 + \langle \xi,\, T w^\lambda\rangle_{L^2(\mathbb{R}^{3N})}\\ &= \mathscr{F}_\lambda[w^\lambda] - \lambda\, \|\psi\|^2 + \tfrac{2\pi}{\mu}\textstyle\sum_{i=1}^N \langle \mathcal{C}_i\xi,\, w^\lambda|_{\pi_i}\rangle_{\mathfrak{X}_i}.\end{aligned}$$

In the last step, the first contribution has been rewritten using definition (2.17), while replacing $w^\lambda|_{\pi_i}$ via (2.142) in the last expression, one gets

$$\langle\psi,\, \tilde{\mathcal{H}}\psi\rangle = \mathscr{F}[w^\lambda] - \lambda\, \|\psi\|^2 + \tfrac{2\pi}{\mu}\textstyle\sum_{i=1}^N \langle \mathcal{C}_i\xi,\, (\Gamma^{i,\lambda}_{\text{diag}} + \Gamma^{i,\lambda}_{\text{off}} + \Gamma^{i}_{\text{reg}})\xi\rangle_{\mathfrak{X}_i}. \tag{2.148}$$

Exploiting the symmetry, one obtains

$$\langle\psi,\, \tilde{\mathcal{H}}\psi\rangle = \mathscr{F}[w^\lambda] - \lambda\, \|\psi\|^2 + \tfrac{2\pi N}{\mu} \langle \xi,\, (\Gamma^{N,\lambda}_{\text{diag}} + \Gamma^{N,\lambda}_{\text{off}} + \Gamma^{N}_{\text{reg}})\xi\rangle_{L^2(\mathbb{R}^{3N})}. \tag{2.149}$$

Now, we want to show that the last term is equal to $\Phi^\lambda[\xi]$, given by (2.13). To this end, we consider separately each component of the inner product in (2.149). Firstly, according to (2.138), we change the order of integration, obtaining

$$\begin{aligned}\tfrac{2\pi N}{\mu}\langle\xi,\, \Gamma^{N,\lambda}_{\text{diag}}\xi\rangle_{\mathfrak{X}_N} &= \tfrac{4\pi N}{\sqrt{2\mu}}\int_{\mathbb{R}^{3N}} d\boldsymbol{p}\, d\boldsymbol{k}_1\cdots d\boldsymbol{k}_{N-1}\, \sqrt{\tfrac{p^2}{2(m_0+m)} + \textstyle\sum_{n=1}^{N-1}\tfrac{k_n^2}{2m} + \lambda}\, |\hat{\xi}(\boldsymbol{p}, \boldsymbol{k}_1, \dots, \boldsymbol{k}_{N-1})|^2\\ &= \Phi^\lambda_{\text{diag}}[\xi].\end{aligned}$$

Indeed, notice that definition (2.14a) is recovered. Similarly, taking into account (2.140),

$$\begin{aligned}\tfrac{2\pi N}{\mu}\langle\xi,\, \Gamma^{N,\lambda}_{\text{off}}\xi\rangle_{\mathfrak{X}_N} = -\tfrac{N}{2\pi\,\mu^2}\int_{\mathbb{R}^{3(N+1)}} & d\boldsymbol{p}\, d\boldsymbol{k}_1\cdots d\boldsymbol{k}_N\, \overline{\hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_N, \boldsymbol{k}_1, \dots, \boldsymbol{k}_{N-1})}\,\times\\ &\times \sum_{n=1}^{N-1} \frac{\hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_n, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_n \dots, \boldsymbol{k}_N)}{\frac{1}{2m_0}p^2 + \frac{1}{2m}\sum_{\ell=1}^N k_\ell^2 + \lambda}.\end{aligned}$$

Exchanging the role of $\boldsymbol{k}_1$ and $\boldsymbol{k}_N$, since $\xi \in \mathscr{H}_{(N-1)\mathrm{b}+1}$, one gets

$$\begin{aligned}\tfrac{2\pi N}{\mu}\langle\xi,\, \Gamma^{N,\lambda}_{\text{off}}\xi\rangle_{\mathfrak{X}_N} = -\tfrac{N}{2\pi\,\mu^2}\sum_{n=2}^{N}\int_{\mathbb{R}^{3(N+1)}} & d\boldsymbol{p}\, d\boldsymbol{k}_1\cdots d\boldsymbol{k}_N\, \overline{\hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_1, \boldsymbol{k}_2, \dots, \boldsymbol{k}_N)}\,\times\\ &\times \frac{\hat{\xi}(\boldsymbol{p}+\boldsymbol{k}_n, \boldsymbol{k}_1, \dots \check{\boldsymbol{k}}_n \dots, \boldsymbol{k}_N)}{\frac{1}{2m_0}p^2 + \frac{1}{2m}\sum_{\ell=1}^N k_\ell^2 + \lambda}.\end{aligned}$$

In the last integral, we can change the variables setting $\boldsymbol{\kappa}_{\sigma_n(i)} = \boldsymbol{k}_i$, for all $i \in \{1, \ldots, N\}$, where the permutation of N elements σ_n is given by

$$\sigma_n := \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n & n+1 & \ldots & N \\ 1 & 3 & 4 & \ldots & n & 2 & n+1 & \ldots & N \end{pmatrix}, \quad \text{for } n > 2, \tag{2.150}$$

while, σ_2 is the identity permutation. Applying this change of variables, one obtains

$$\begin{aligned} \tfrac{2\pi N}{\mu} \langle \xi,\, \Gamma_{\text{off}}^{N,\lambda} \xi \rangle_{\mathfrak{X}_N} = -\tfrac{N}{2\pi\,\mu^2} \sum_{n=2}^{N} \int_{\mathbb{R}^{3(N+1)}} d\boldsymbol{q} d\boldsymbol{\kappa}_1 \cdots d\boldsymbol{\kappa}_N\, \overline{\hat{\xi}(\boldsymbol{q} + \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_{\sigma_n(2)}, \ldots, \boldsymbol{\kappa}_{\sigma_n(N)})} \times \\ \times \frac{\hat{\xi}(\boldsymbol{q} + \boldsymbol{\kappa}_2, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_3 \ldots, \boldsymbol{\kappa}_N)}{\frac{1}{2m_0} q^2 + \frac{1}{2m} \sum_{\ell=1}^{N} \kappa_\ell^2 + \lambda}. \end{aligned}$$

Notice that the symmetry properties of $\xi \in \mathscr{H}_{(N-1)\mathrm{b}+1}$ make the integrand actually independent of n. Therefore, the expression in definition (2.14b) is achieved

$$\begin{aligned} \tfrac{2\pi N}{\mu} \langle \xi, \Gamma_{\text{off}}^{N,\lambda} \xi \rangle_{\mathfrak{X}_N} &= -\tfrac{N(N-1)}{2\pi\,\mu^2} \int_{\mathbb{R}^{3(N+1)}} d\boldsymbol{p} d\boldsymbol{\kappa}_1 \cdots d\boldsymbol{\kappa}_N\, \overline{\hat{\xi}(\boldsymbol{p} + \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \ldots, \boldsymbol{\kappa}_N)}\, \frac{\hat{\xi}(\boldsymbol{p} + \boldsymbol{\kappa}_2, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_3, \ldots, \boldsymbol{\kappa}_N)}{\frac{1}{2m_0} p^2 + \frac{1}{2m} \sum_{\ell=1}^{N} \kappa_\ell^2 + \lambda} \\ &= \Phi_{\text{off}}^{\lambda}[\xi]. \end{aligned}$$

Finally, from (2.10) and (2.8),

$$\begin{aligned} \tfrac{2\pi N}{\mu} \langle \xi, \Gamma_{\text{reg}}^{N} \xi \rangle_{\mathfrak{X}_N} &= \tfrac{2\pi N}{\mu} \int_{\mathbb{R}^{3N}} d\boldsymbol{x}_0 d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_{N-1} \left[\alpha_0 + \gamma \sum_{n=1}^{N-1} \frac{\theta(|\boldsymbol{x}_n - \boldsymbol{x}_0|)}{|\boldsymbol{x}_n - \boldsymbol{x}_0|} \right] |\xi(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})|^2 \\ &= \tfrac{2\pi N}{\mu} \int_{\mathbb{R}^{3N}} d\boldsymbol{x}_0 d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_{N-1} \left[\alpha_0 + (N-1)\,\gamma\, \frac{\theta(|\boldsymbol{x}_1 - \boldsymbol{x}_0|) - 1}{|\boldsymbol{x}_1 - \boldsymbol{x}_0|} \right] |\xi(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})|^2 + \\ &\quad + \tfrac{2\pi N(N-1)\,\gamma}{\mu} \int_{\mathbb{R}^{3N}} d\boldsymbol{x}_0 d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_{N-1}\, \frac{|\xi(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1})|^2}{|\boldsymbol{x}_1 - \boldsymbol{x}_0|} \\ &= \Phi_0[\xi] + \Phi_{\text{reg}}[\xi]. \end{aligned}$$

We have shown that, for a sufficiently regular charge $\xi \in \mathscr{H}_{(N-1)\mathrm{b}+1}$, equation (2.149) reduces to the action of Q defined in (2.16), since by definition (2.13) $\Phi^\lambda = \Phi_0 + \Phi_{\text{diag}}^{\lambda} + \Phi_{\text{off}}^{\lambda} + \Phi_{\text{reg}}$.

3. INTERACTING BOSE GAS

The aim of this chapter is to construct a regularized zero-range Hamiltonian for a Bose gas in dimension three adopting the Minlos-Faddeev regularization. In particular, we consider[1] a non-relativistic quantum system of N identical spinless bosons of mass $\frac{1}{2}$ interacting with each other via a regularized contact interaction. The Hilbert space of the system is therefore given by

$$\mathscr{H}_{N\mathrm{b}} := L^2_{\mathrm{sym}}(\mathbb{R}^{3N}), \quad N \geq 3. \tag{3.1}$$

We denote by $\mathscr{P}$ the following set

$$\mathscr{P} := \big\{\{i,j\} \,\big|\, i,j \in \{1,\dots,N\},\, i \neq j\big\}, \qquad |\mathscr{P}| = \frac{N(N-1)}{2} \tag{3.2}$$

so that, the formal Hamiltonian associated with the two-body point interaction is

$$\tilde{\mathcal{H}} = -\sum_{i=1}^{N} \Delta_{\boldsymbol{x}_i} + \nu \sum_{\mathscr{P} \ni \sigma = \{i,j\}}^{N} \delta(\boldsymbol{x}_i - \boldsymbol{x}_j), \tag{3.3}$$

where ν is a coupling constant. Clearly, in this case the free Hamiltonian is

$$\mathcal{H}_0 := -\sum_{i=1}^{N} \Delta_{\boldsymbol{x}_i}, \qquad \mathscr{D}(\mathcal{H}_0) = \mathscr{H}_{N\mathrm{b}} \cap H^2(\mathbb{R}^{3N}). \tag{3.4}$$

In order to rigorously define the Hamiltonian $\mathcal{H}$ associated with the formal operator (3.3) as a s.a. and lower semi-bounded operator in $\mathscr{H}_{N\mathrm{b}}$, we need to construct a singular perturbation of $\mathcal{H}_0$ supported on the coincidence hyperplanes

$$\pi := \bigcup_{\sigma \in \mathscr{P}} \pi_\sigma, \qquad \pi_{ij} := \big\{(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N) \in \mathbb{R}^{3N} \,\big|\, \boldsymbol{x}_i = \boldsymbol{x}_j\big\}. \tag{3.5}$$

This means that, as already discussed in the previous chapters, we require $\mathcal{H}$ to coincide with the free Hamiltonian on the space of H^2-functions whose traces vanish along π

$$\mathcal{H}\psi = \mathcal{H}_0\psi, \qquad \forall\, \psi \in \mathscr{H}_{N\mathrm{b}} \cap H^2(\mathbb{R}^{3N}) \text{ s.t. } \psi|_\pi = 0. \tag{3.6}$$

Ultimately, $\mathcal{H}$ can be defined as a proper s.a. and bounded from below extension of the operator

$$\dot{\mathcal{H}}_0 := \mathcal{H}_0|_{\mathscr{D}(\dot{\mathcal{H}}_0)}, \qquad \mathscr{D}(\dot{\mathcal{H}}_0) := \mathscr{H}_{N\mathrm{b}} \cap H^2_0(\mathbb{R}^{3N} \setminus \pi) \tag{3.7}$$

which is symmetric and closed according to the graph norm of $\mathcal{H}_0$.

[1]The results of this chapter are going to appear in a forthcoming paper.

By construction, one has $\mathscr{D}(\mathcal{H}) \subset \mathscr{H}_{N\mathrm{b}} \cap H^2(\mathbb{R}^{3N} \setminus \pi)$ and a class of extensions of $\dot{\mathcal{H}}_0$ can be characterized by imposing a proper boundary condition to a vector $\psi \in \mathscr{H}_{N\mathrm{b}} \cap H^2(\mathbb{R}^{3N} \setminus \pi)$ on each coincidence hyperplane π_σ in some topology. More precisely, we are interested in obtaining an energetically stable system by exploiting the Minlos-Faddeev regularization. To this end, let us consider the function

$$\begin{aligned}\alpha \colon \mathbb{R}^3 \otimes \mathbb{R}^{3(N-2)} &\longrightarrow \mathbb{R}, \\ (\boldsymbol{z}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{N-2}) &\longmapsto \alpha_0 + \gamma \sum_{k=1}^{N-2} \frac{\theta(|\boldsymbol{y}_k - \boldsymbol{z}|)}{|\boldsymbol{y}_k - \boldsymbol{z}|} + \frac{\gamma}{2} \sum_{1 \le k < \ell \le N-2} \frac{\theta(|\boldsymbol{y}_k - \boldsymbol{y}_\ell|)}{|\boldsymbol{y}_k - \boldsymbol{y}_\ell|}\end{aligned} \tag{3.8}$$

with $\gamma > 0$ representing the strength of the regularization and $\theta : \mathbb{R}_+ \longrightarrow \mathbb{R}$ an essentially bounded function satisfying (1.10). The effective scattering length depending on the positions of each particle, associated with the coincidence hyperplane π_σ shall be given by

$$\alpha\big(\tfrac{\boldsymbol{x}_i + \boldsymbol{x}_j}{2}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_\sigma \dots, \boldsymbol{x}_N\big)$$

where $\check{\boldsymbol{x}}_{\sigma=\{i,j\}}$ denotes the omission of the variables $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. Setting $\boldsymbol{x} = \frac{\boldsymbol{x}_i + \boldsymbol{x}_j}{2}$, one has on π_σ

$$\alpha : (\boldsymbol{x}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_\sigma \dots, \boldsymbol{x}_N) \longmapsto \alpha_0 + \gamma \sum_{\substack{1 \le k \le N \\ k \notin \sigma}} \frac{\theta(|\boldsymbol{x}_k - \boldsymbol{x}|)}{|\boldsymbol{x}_k - \boldsymbol{x}|} + \frac{\gamma}{2} \sum_{\substack{k=1 \\ k \notin \sigma}}^{N-1} \sum_{\substack{\ell = k+1 \\ \ell \notin \sigma}}^{N} \frac{\theta(|\boldsymbol{x}_k - \boldsymbol{x}_\ell|)}{|\boldsymbol{x}_k - \boldsymbol{x}_\ell|}. \tag{3.9}$$

Heuristically, we are therefore describing a repulsive force that weakens the contact interaction. In particular, two kind of repulsions are described by (3.9): the first term represents a three-body force that makes the usual two-body point interaction weaker and weaker as a third particle is approaching the common position of the two interacting particles of the couple σ, while the second term represents a four-body repulsion meant to regularize the singular ultraviolet behavior associated with the situation in which two other different particles compose an interacting couple ν, with $\nu \cap \sigma = \emptyset$, that is getting closer to the interacting couple σ. Clearly, this latter kind of singularity occurs in a smaller set, i.e. $\pi_\sigma \cap \pi_\nu$.
With the above considerations, we characterize the regularized point interaction by introducing the boundary condition (to be fulfilled in a suitable topology)

$$\begin{aligned}\psi(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N) = \; & \frac{\xi\big(\frac{\boldsymbol{x}_i + \boldsymbol{x}_j}{2}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_\sigma \dots, \boldsymbol{x}_N\big)}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} + \\ & + (\Gamma^\sigma_{\mathrm{reg}} \xi)\big(\tfrac{\boldsymbol{x}_i + \boldsymbol{x}_j}{2}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_\sigma \dots, \boldsymbol{x}_N\big) + o(1), \quad \text{for } |\boldsymbol{x}_i - \boldsymbol{x}_j| \longrightarrow 0,\end{aligned} \tag{3.10}$$

with $\xi \in L^2(\mathbb{R}^3) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3(N-2)})$ some vector depending on ψ and $\Gamma^\sigma_{\mathrm{reg}}$ acting as follows

$$\Gamma^\sigma_{\mathrm{reg}} : \xi \longmapsto \alpha(\boldsymbol{x}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_\sigma \dots, \boldsymbol{x}_N) \xi(\boldsymbol{x}, \boldsymbol{x}_1, \dots \check{\boldsymbol{x}}_\sigma \dots, \boldsymbol{x}_N). \tag{3.11}$$

In this chapter we shall prove that the Hamiltonian constructed as a s.a. extension of $\dot{\mathcal{H}}_0$ satisfying (3.10) at least in the weak topology, is bounded from below if one assumes γ larger than a proper critical value $\gamma_c^{N\mathrm{b}}$ given by (3.19). The result is obtained by making a broader use of the abstract theory of s.a. extensions recovered in appendix A with respect to the previous chapters. In particular, we shall see that the analysis is reduced to the study of the closedness and coercivity of a densely defined hermitian quadratic form Φ^λ defined by (3.18). We are going to adopt the approach developed in section 1.3 in order to isolate the singularities contained in an unbounded (in L^2 sense) negative contribution of Φ^λ in its position-representation. In this way the choice of the needed regularizing term associated with the effective scattering length introduced in (3.9) shall be manifestly justified. The result is surely not optimal, since in chapter 1 a more detailed discussion yields a smaller critical parameter $\gamma_c^{3\mathrm{b}} \approx 0.782004$ (proved to be optimal in section 1.2) whereas in this framework the case $N=3$ provides $\gamma_c^{N\mathrm{b}} \approx 1.54984$.

3.1 Main Results

In this section we introduce some notation and the main objects of our analysis in order to state the obtained results. Define the isomorphic spaces

$$\mathfrak{X}_\sigma := L^2(\mathbb{R}^3, d\boldsymbol{x}) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3(N-2)}, d\boldsymbol{x}_1 \cdots d\check{\boldsymbol{x}}_\sigma \cdots d\boldsymbol{x}_N), \tag{3.12a}$$

$$\mathfrak{X} := L^2(\mathbb{R}^3) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3(N-2)}) \tag{3.12b}$$

and let $\mathcal{C}_\sigma$ be the unitary transformation sending $\mathfrak{X}$ to $\mathfrak{X}_\sigma$. Then, given $T_\sigma : \mathscr{D}(\mathcal{H}_0) \longrightarrow \mathfrak{X}_\sigma$ the linear bounded[2] operator (according to proposition B.4) whose action in $\mathcal{S}(\mathbb{R}^{3N})$ is explicitly given by

$$T_{\sigma=\{i,j\}} : f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \longmapsto 8\pi f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)|_{\boldsymbol{x}_i = \boldsymbol{x}_j = \boldsymbol{x}}, \tag{3.13}$$

we define a trace operator $T \in \mathscr{B}(\mathscr{D}(\mathcal{H}_0), \mathfrak{X})$ as follows

$$T : \psi \longmapsto \sum_{\sigma \in \mathscr{P}} \mathcal{C}_\sigma^* T_\sigma \psi = \tfrac{N(N-1)}{2} \mathcal{C}_\nu^* T_\nu \psi, \qquad \forall \nu \in \mathscr{P}. \tag{3.14}$$

Lastly, we introduce the injective operator $\mathcal{G}^\lambda \in \mathscr{B}(\mathfrak{X}, \mathscr{H}_{N\mathrm{b}})$ given by

$$\mathcal{G}^\lambda := (T \mathcal{R}_{\mathcal{H}_0}(-\lambda))^*, \qquad \lambda > 0. \tag{3.15}$$

More details about this operator are postponed to section 3.A. We are now in position to highlight the connection of the abstract setting discussed in appendix A with our problem. First of all, the Hilbert space $\mathfrak{H}$ is given by $\mathscr{H}_{N\mathrm{b}}$, while the s.a. operator A represents the free Hamiltonian $\mathcal{H}_0$ and S corresponds to $\dot{\mathcal{H}}_0$. Concerning the operator τ, it is plain to see that it is associated in our framework with T defined in (3.14). Indeed, notice that T satisfies the required properties: it is bounded since the trace operators $f \longmapsto f|_{\pi_\sigma}$ are continuous between $H^{\frac{3}{2}+s}(\mathbb{R}^{3N})$ and $H^s(\mathbb{R}^{3(N-1)})$ for any $s > 0$, $\mathrm{ran}(T)$ is dense in $\mathfrak{X}$ and lastly $\ker(T) = \mathscr{D}(\dot{\mathcal{H}}_0)$, by construction.

[2]Here we are considering $\mathscr{D}(\mathcal{H}_0)$ as a Hilbert subspace of $\mathscr{H}_{N\mathrm{b}}$ endowed with the graph norm of $\mathcal{H}_0$.

We stress that (see proposition B.5) if there exists $s>0$ such that $f\in\mathscr{H}_{N\mathrm{b}}\cap H^{\frac{3}{2}+s}(\mathbb{R}^{3N})$, then the operator T_σ given by (3.13) can be rewritten as

$$(T_\sigma f)(\boldsymbol{x},\boldsymbol{X}_\sigma)=8\pi\,\mathrm{w}\!-\!\lim_{r\to 0^+}(U_\sigma f)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma), \tag{3.16}$$

where we denote for short $\boldsymbol{X}_\sigma:=(\boldsymbol{x}_1,\dots\check{\boldsymbol{x}}_\sigma\dots,\boldsymbol{x}_N)\in\mathbb{R}^{3(N-2)}$ and U_σ is the following unitary operator

$$\begin{aligned}U_\sigma\colon\mathscr{H}_{N\mathrm{b}}&\longrightarrow L^2(\mathbb{R}^6,d\boldsymbol{r}d\boldsymbol{x})\otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3(N-2)},d\boldsymbol{X}_\sigma),\qquad\sigma\in\mathscr{P},\\(U_\sigma f)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma)&=f\big(\boldsymbol{x}+\tfrac{\boldsymbol{r}}{2},\boldsymbol{x}-\tfrac{\boldsymbol{r}}{2},\boldsymbol{X}_\sigma\big).\end{aligned} \tag{3.17}$$

The reason behind the constant 8π in the definition of T_σ will be clarified by remark 3.2.
According to appendix A, in order to characterize the lower semi-bounded Hamiltonian $\mathcal{H}$ we need to construct a proper continuous map $\Gamma\colon\mathbb{C}\smallsetminus\mathbb{R}_+\longrightarrow\mathscr{L}(\mathfrak{X})$ satisfying properties (A.2) and encoding the ultraviolet regularization described above. To this end, we focus our analysis on the densely defined quadratic form in $\mathfrak{X}$ given by

$$\Phi^\lambda[\xi]:=8\pi\sum_{\sigma\in\mathscr{P}}\Big[\langle\mathcal{C}_\sigma\xi,\,\Gamma^\sigma_{\mathrm{reg}}\xi\rangle_{\mathfrak{X}_\sigma}+\lim_{r\to0}\langle\mathcal{C}_\sigma\xi,\,\tfrac{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}{r}-(U_\sigma\mathcal{G}^\lambda\xi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma)\rangle_{\mathfrak{X}_\sigma}\Big],\qquad\lambda>0 \tag{3.18}$$

and we shall see that such a map Γ will be uniquely deduced by Φ^λ. In this way, we will prove that characterization (A.9) holds for the Hamiltonian of the system.
In section 3.2 we motivate definition (3.18) of the quadratic form showing that it is associated with a regularized zero-range Hamiltonian fulfilling boundary condition (3.10) at least in weak topology (hence we are taking account of the Minlos-Faddeev regularization). We shall also prove that Φ^λ is hermitian with $\mathscr{D}(\Phi^\lambda)=H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$. Defining for any fixed $N\geq 3$ the threshold parameter

$$\gamma_c^{N\mathrm{b}}:=2-\frac{4\sqrt{2}}{\pi(N-2)(N+1)} \tag{3.19}$$

our claim is that stability is achieved with the assumption $\gamma>\gamma_c^{N\mathrm{b}}$. More precisely, the technical difficulties lie in the following proposition.

Proposition 3.1.
Let Φ^λ be the hermitian quadratic form in $\mathfrak{X}$ defined by (3.18) *and assume $\gamma>\gamma_c^{N\mathrm{b}}$. Then Φ^λ is closed for any $\lambda>0$ and satisfies*

$$\Phi^\lambda[\xi]\geq 4\pi N(N-1)\left[\sqrt{\tfrac{\lambda}{2}}\,\sqrt{1-\Lambda_N^2}+\alpha_0-\tfrac{(N+1)(N-2)\gamma}{4b}\right]\|\xi\|^2,\quad\forall\,\lambda>0,\ \xi\in\mathfrak{X},$$

where

$$\Lambda_N=\max\Big\{0,1-\tfrac{(N+1)(N-2)\,\pi}{4\sqrt{2}}\,(\gamma-\gamma_c^{N\mathrm{b}})\Big\}\in[0,1). \tag{3.20}$$

Moreover, Φ^λ is coercive for any $\lambda>\lambda_0$ with

$$\lambda_0:=\frac{2\max\{0,-\alpha_0+(N+1)(N-2)\,\gamma/(4b)\}^2}{1-\Lambda_N^2}. \tag{3.21}$$

We observe that proposition 3.1 provides the definition of a s.a. operator Γ^λ associated to Φ^λ for any $\lambda > 0$, which is positive and invertible whenever $\lambda > \lambda_0$ since Φ^λ is closed and bounded from below by a positive constant under this condition. Actually one can show that the domain D of Γ^λ does not depend on λ. These properties make the assumptions of proposition A.2 fulfilled. In particular, in proposition A.2 we show how to construct the continuous map Γ starting from this s.a. operator Γ^λ and we also prove that it is associated to a bounded from below Hamiltonian (proposition A.3).

Theorem 3.2.
Let $\gamma_c^{N\mathrm{b}}$ and λ_0 be respectively defined by (3.19) *and* (3.21). *Then, assuming $\gamma > \gamma_c^{N\mathrm{b}}$, the operator characterized by*

$$\begin{aligned}&\mathscr{D}(\mathcal{H}) = \Big\{\psi \in \mathscr{H}_{N\mathrm{b}} \,\Big|\, \psi = \phi_\lambda + \mathcal{G}^\lambda\xi,\ \phi_\lambda \in H^2(\mathbb{R}^{3N}),\ \xi \in D,\ \Gamma^\lambda\xi = T\phi_\lambda,\ \lambda > 0\Big\},\\ &\mathcal{H}\psi = \mathcal{H}_0\phi_\lambda - \lambda\mathcal{G}^\lambda\xi.\end{aligned} \tag{3.22}$$

is a s.a. extension of the operator $\dot{\mathcal{H}}_0$ defined by (3.7) *and the elements of its domain satisfy boundary condition* (3.10) *in the weak topology. Moreover, there holds $\mathcal{H} \geq -\lambda_0$.*

We stress that theorem 3.2 provides the following estimate for the infimum of the spectrum of the Hamiltonian

$$\inf\sigma(\mathcal{H}) \geq \begin{cases} 0, & \text{if } \alpha_0 \geq \frac{(N+1)(N-2)\,\gamma}{4b},\\ -\dfrac{[(N+1)(N-2)\,\gamma - 4b\,\alpha_0]^2}{8b^2(1-\Lambda_N^2)}, & \text{otherwise.}\end{cases} \tag{3.23}$$

The proof of the previous results is postponed to section 3.3.
In the end, in section 3.4, we take into account the approach based on the theory of Dirichlet forms developed by Albeverio et al. in [2] and we will prove that the class of regularized zero-range Hamiltonians there constructed is a special case of ours.

3.2 The Quadratic Form of the Charges

In this section we clarify the reason behind definition (3.18) of the quadratic form of the charges and we provide some of its properties.
Let us consider a singular perturbation of $\mathcal{H}_0$ supported on π which, according to appendix A, is characterized by the choice of a continuous map $\Gamma : \mathbb{C} \smallsetminus \mathbb{R}_+ \longrightarrow \mathscr{L}(\mathfrak{X})$ fulfilling conditions (A.2). By construction, this operator coincides with $\mathcal{H}_0$ on $\mathscr{H}_{N\mathrm{b}} \cap H_0^2(\mathbb{R}^{3N} \setminus \pi)$ and, according to (A.9), any element of its domain ψ can be decomposed as $\psi = \phi_\lambda + \mathcal{G}^\lambda\xi$ with $\phi_\lambda \in \mathscr{D}(\mathcal{H}_0)$ and ξ in the domain of $\Gamma(-\lambda)$ satisfying the boundary condition $\Gamma(-\lambda)\xi = T\phi_\lambda$. In order for this singular perturbation to be the Hamiltonian of our system, we need to isolate the proper operator

$\Gamma(-\lambda)$ that encodes the Minlos-Faddeev ultraviolet regularization discussed at the beginning of this chapter. To this end, we identify the action of the quadratic form associated to $\Gamma(-\lambda)$ by imposing on ψ boundary condition (3.10) in the weak topology, i.e.

$$\underset{r\to 0^+}{\mathrm{w-lim}}\left[(U_\nu\psi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\nu)-\tfrac{\xi(\boldsymbol{x},\boldsymbol{X}_\nu)}{r}\right]=(\Gamma^\nu_{\mathrm{reg}}\xi)(\boldsymbol{x},\boldsymbol{X}_\nu),\qquad \forall\,\nu\in\mathscr{P},\tag{3.24}$$

that, in particular, implies

$$\langle\mathcal{C}_\nu\xi,\,\Gamma^\nu_{\mathrm{reg}}\xi\rangle_{\mathfrak{X}_\nu}+\lim_{r\to 0^+}\langle\mathcal{C}_\nu\xi,\left[\tfrac{\xi(\boldsymbol{x},\boldsymbol{X}_\nu)}{r}-U_\nu(\mathcal{G}^\lambda\xi+\phi_\lambda)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\nu)\right]\rangle_{\mathfrak{X}_\nu}=0,\qquad \forall\,\nu\in\mathscr{P}.$$

Therefore, since $\phi_\lambda\in H^2(\mathbb{R}^{3N})$, taking into account (3.16) one gets for any $\nu\in\mathscr{P}$

$$\langle\mathcal{C}_\nu\xi,\,\Gamma^\nu_{\mathrm{reg}}\xi\rangle_{\mathfrak{X}_\nu}+\lim_{r\to 0^+}\langle\mathcal{C}_\nu\xi,\left[\tfrac{\xi(\boldsymbol{x},\boldsymbol{X}_\nu)}{r}-(U_\nu\mathcal{G}^\lambda\xi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\nu)\right]\rangle_{\mathfrak{X}_\nu}-\tfrac{1}{8\pi}\langle\mathcal{C}_\nu\xi,\,T_\nu\phi_\lambda\rangle_{\mathfrak{X}_\nu}=0,$$

namely, exploiting (3.14)

$$\langle\mathcal{C}_\nu\xi,\,\Gamma^\nu_{\mathrm{reg}}\xi\rangle_{\mathfrak{X}_\nu}+\lim_{r\to 0^+}\langle\mathcal{C}_\nu\xi,\left[\tfrac{\xi(\boldsymbol{x},\boldsymbol{X}_\nu)}{r}-(U_\nu\mathcal{G}^\lambda\xi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\nu)\right]\rangle_{\mathfrak{X}_\nu}=\tfrac{1}{4\pi N(N-1)}\langle\xi,\,T\phi_\lambda\rangle_{\mathfrak{X}}.$$

Thus, the boundary condition $\Gamma(-\lambda)\xi=T\phi_\lambda$ provides the expression of the quadratic form associated to the s.a. operator $\Gamma(-\lambda)$

$$\langle\xi,\,\Gamma(-\lambda)\xi\rangle_{\mathfrak{X}}=4\pi N(N-1)\left\{\langle\mathcal{C}_\nu\xi,\,\Gamma^\nu_{\mathrm{reg}}\xi\rangle_{\mathfrak{X}_\nu}+\lim_{r\to 0^+}\langle\mathcal{C}_\nu\xi,\left[\tfrac{\xi(\boldsymbol{x},\boldsymbol{X}_\nu)}{r}-(U_\nu\mathcal{G}^\lambda\xi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\nu)\right]\rangle_{\mathfrak{X}_\nu}\right\},$$

whose expression coincides with (3.18) because of the bosonic symmetry.
Therefore, let Φ^λ be the densely defined quadratic form in $\mathfrak{X}$ given by (3.18). Our goal is to show that Φ^λ is associated to a s.a. operator Γ^λ satisfying the hypotheses of propositions A.2 and A.3 so that a s.a. and bounded from below extension of $\dot{\mathcal{H}}_0$ can be defined by (A.9). This s.a. extension shall be the Hamiltonian $\mathcal{H}$ of our system, since, by construction, it will coincide with $\mathcal{H}_0$ on $\mathscr{H}_{N\mathrm{b}}\cap H^2_0(\mathbb{R}^{3N}\setminus\pi)$ and any element $\psi\in\mathscr{D}(\mathcal{H})$ satisfies boundary condition (3.10) at least in the weak topology.

Remark 3.1. *Notice that definition* (3.18) *satisfies for any* $\lambda_1,\lambda_2>0$

$$\begin{aligned}\Phi^{\lambda_1}[\xi]-\Phi^{\lambda_2}[\xi]&=8\pi\sum_{\sigma\in\mathscr{P}}\lim_{r\to 0^+}\langle\mathcal{C}_\sigma\xi,\,(U_\sigma\mathcal{G}^{\lambda_2}\xi-U_\sigma\mathcal{G}^{\lambda_1}\xi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma)\rangle_{\mathfrak{X}_\sigma}\\&=8\pi\sum_{\sigma\in\mathscr{P}}\langle\mathcal{C}_\sigma\xi,\,\underset{r\to 0^+}{\mathrm{w-lim}}\,(U_\sigma\mathcal{G}^{\lambda_2}\xi-U_\sigma\mathcal{G}^{\lambda_1}\xi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma)\rangle_{\mathfrak{X}_\sigma}\\&=\sum_{\sigma\in\mathscr{P}}\langle\mathcal{C}_\sigma\xi,\,T_\sigma(\mathcal{G}^{\lambda_2}-\mathcal{G}^{\lambda_1})\xi\rangle_{\mathfrak{X}_\sigma}\end{aligned}$$

thanks to (3.16)*, since* $\mathrm{ran}(\mathcal{G}^{\lambda_2}-\mathcal{G}^{\lambda_1})\subset\mathscr{D}(\mathcal{H}_0)$. *Hence, according to* (3.14)*, we have obtained*

$$\Phi^{\lambda_1}[\xi]-\Phi^{\lambda_2}[\xi]=\langle\xi,\,T\big(\mathcal{G}^{\lambda_2}-\mathcal{G}^{\lambda_1}\big)\xi\rangle_{\mathfrak{X}}.\tag{3.25}$$

Next, let us decompose Φ^λ as follows

$$\Phi^\lambda=4\pi N(N-1)\big(\Phi^\lambda_{\mathrm{diag}}+\Phi^\lambda_{\mathrm{off};0}+\Phi^\lambda_{\mathrm{off};1}+\Phi_{\mathrm{reg}}\big),\tag{3.26}$$

where, considering $\sigma \in \mathscr{P}$, we have defined

$$\Phi^\lambda_{\text{diag}}[\xi] := \lim_{r\to 0^+} \langle \mathcal{C}_\sigma \xi,\, \Big[\tfrac{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}{r} - (U_\sigma \mathcal{G}^\lambda_\sigma \mathcal{C}_\sigma \xi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma)\Big]\rangle_{\mathfrak{X}_\sigma}, \tag{3.27a}$$

$$\Phi^\lambda_{\text{off};\sharp}[\xi] := -\sum_{\substack{\nu\in\mathscr{P}:\\ |\sigma\cap\nu|=\sharp}} \lim_{r\to 0^+} \langle \mathcal{C}_\sigma \xi,\, (U_\sigma \mathcal{G}^\lambda_\nu \mathcal{C}_\nu \xi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma)\rangle_{\mathfrak{X}_\sigma}, \quad \sharp\in\{0,1\}, \tag{3.27b}$$

$$\Phi_{\text{reg}}[\xi] := \langle \mathcal{C}_\sigma \xi,\, \Gamma^\sigma_{\text{reg}}\xi\rangle_{\mathfrak{X}_\sigma}. \tag{3.27c}$$

We point out that for any given $\sigma \in \mathscr{P}$

$$\big|\{\nu\in\mathscr{P} \,\big|\, |\nu\cap\sigma|=0\}\big| = \tfrac{(N-2)(N-3)}{2}, \qquad \big|\{\nu\in\mathscr{P} \,\big|\, |\nu\cap\sigma|=1\}\big| = 2(N-2).$$

Before proceeding, let us state the following proposition in which we ensure that Φ^λ is hermitian.

Proposition 3.3. *Let $\xi \in \mathcal{S}(\mathbb{R}^{3(N-1)}) \cap \mathfrak{X}$ and Φ^λ be the quadratic form given by* (3.18). *Then, the components of Φ^λ, defined by* (3.27) *can be represented as follows*

$$\Phi^\lambda_{\text{diag}}[\xi] = \sqrt{\tfrac{\lambda}{2}}\,\|\xi\|^2 + 4\pi\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{X}_\sigma \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{Y}_\sigma\, |\xi(\boldsymbol{y},\boldsymbol{Y}_\sigma)-\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)|^2\, G^\lambda\begin{pmatrix}\boldsymbol{x}, & \boldsymbol{x}, & \boldsymbol{X}_\sigma\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma\end{pmatrix}, \tag{3.28a}$$

$$\begin{aligned}\Phi^\lambda_{\text{off};0}[\xi] = {}& -4\pi(N-2)(N-3)\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}_1\cdots d\boldsymbol{x}_{N-2}\, \overline{\xi(\boldsymbol{x},\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{N-2})}\,\times\\ &\times\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{y}_1\cdots d\boldsymbol{y}_{N-2}\, \xi(\boldsymbol{y},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_{N-2})\, G^\lambda\begin{pmatrix}\boldsymbol{x}_1, & \boldsymbol{x}_2, & \boldsymbol{x}, & \boldsymbol{x}, & \boldsymbol{x}_3, & \ldots, & \boldsymbol{x}_{N-2}\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{y}_1, & \boldsymbol{y}_2, & \boldsymbol{y}_3, & \ldots, & \boldsymbol{y}_{N-2}\end{pmatrix},\end{aligned} \tag{3.28b}$$

$$\begin{aligned}\Phi^\lambda_{\text{off};1}[\xi] = {}& -16\pi(N-2)\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}_1\cdots d\boldsymbol{x}_{N-2}\, \overline{\xi(\boldsymbol{x},\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{N-2})}\,\times\\ &\times\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{y}_1\cdots d\boldsymbol{y}_{N-2}\, \xi(\boldsymbol{y},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_{N-2})\, G^\lambda\begin{pmatrix}\boldsymbol{x}, & \boldsymbol{x}_1, & \boldsymbol{x}, & \boldsymbol{x}_2, & \ldots, & \boldsymbol{x}_{N-2}\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{y}_1, & \boldsymbol{y}_2, & \ldots, & \boldsymbol{y}_{N-2}\end{pmatrix},\end{aligned} \tag{3.28c}$$

where the kernel G^λ is defined by equation (3.70) *in section 3.A. Furthermore, Φ^λ is hermitian.*

Proof. Concerning the diagonal term, in light of equation (3.75), we have

$$\begin{aligned}\Phi^\lambda_{\text{diag}}[\xi] = \lim_{r\to 0^+}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{X}_\sigma\, \overline{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}\Bigg[&\frac{1-e^{-\sqrt{\frac{\lambda}{2}}\,r}}{r}\,\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)+\\ &-8\pi\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{Y}_\sigma\, [\xi(\boldsymbol{y},\boldsymbol{Y}_\sigma)-\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)]\, G^\lambda\begin{pmatrix}\boldsymbol{x}+\frac{\boldsymbol{r}}{2}, & \boldsymbol{x}-\frac{\boldsymbol{r}}{2}, & \boldsymbol{X}_\sigma\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma\end{pmatrix}\Bigg].\end{aligned} \tag{3.29}$$

The first term has the integrable majorant $\sqrt{\frac{\lambda}{2}}\,|\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)|^2$, therefore

$$\begin{aligned}\Phi^\lambda_{\text{diag}}[\xi] = \sqrt{\tfrac{\lambda}{2}}\,\|\xi\|^2 - 8\pi\lim_{r\to 0^+}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{X}_\sigma\, \overline{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{Y}_\sigma\, [\xi(\boldsymbol{y},\boldsymbol{Y}_\sigma)-\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)]\,\times\\ \times G^\lambda\begin{pmatrix}\boldsymbol{x}+\frac{\boldsymbol{r}}{2}, & \boldsymbol{x}-\frac{\boldsymbol{r}}{2}, & \boldsymbol{X}_\sigma\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma\end{pmatrix}.\end{aligned}$$

Notice that the quantity $|\boldsymbol{x}+\frac{\boldsymbol{r}}{2}-\boldsymbol{y}|^2+|\boldsymbol{x}-\frac{\boldsymbol{r}}{2}-\boldsymbol{y}|^2+|\boldsymbol{X}_\sigma-\boldsymbol{Y}_\sigma|^2$ is symmetric under the exchange $(\boldsymbol{x},\boldsymbol{X}_\sigma)\longleftrightarrow(\boldsymbol{y},\boldsymbol{Y}_\sigma)$. Hence,

$$\Phi^\lambda_{\text{diag}}[\xi]=\sqrt{\tfrac{\lambda}{2}}\,\|\xi\|^2+4\pi\lim_{r\to 0^+}\int_{\mathbb{R}^{3(N-1)}}d\boldsymbol{x}d\boldsymbol{X}_\sigma\int_{\mathbb{R}^{3(N-1)}}d\boldsymbol{y}d\boldsymbol{Y}_\sigma\;|\xi(\boldsymbol{y},\boldsymbol{Y}_\sigma)-\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)|^2\times$$
$$\times G^\lambda\begin{pmatrix}\boldsymbol{x}+\frac{\boldsymbol{r}}{2}, & \boldsymbol{x}-\frac{\boldsymbol{r}}{2}, & \boldsymbol{X}_\sigma\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma\end{pmatrix}.$$

Then, exploiting the decrease of G^λ and according to the elementary inequality

$$|\boldsymbol{x}+\tfrac{\boldsymbol{r}}{2}-\boldsymbol{y}|^2+|\boldsymbol{x}-\tfrac{\boldsymbol{r}}{2}-\boldsymbol{y}|^2\geq\tfrac{1}{2}|\boldsymbol{x}+\tfrac{\boldsymbol{r}}{2}-\boldsymbol{y}+\boldsymbol{x}-\tfrac{\boldsymbol{r}}{2}-\boldsymbol{y}|^2=2|\boldsymbol{x}-\boldsymbol{y}|^2,$$

one gets

$$G^\lambda\begin{pmatrix}\boldsymbol{x}+\frac{\boldsymbol{r}}{2}, & \boldsymbol{x}-\frac{\boldsymbol{r}}{2}, & \boldsymbol{X}_\sigma\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma\end{pmatrix}\leq G^\lambda\begin{pmatrix}\boldsymbol{x}, & \boldsymbol{x}, & \boldsymbol{X}_\sigma\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma\end{pmatrix},\qquad\forall\,\boldsymbol{r}\in\mathbb{R}^3.$$

Let us show that we have found an integrable majorant so that dominated convergence theorem shall apply. Adopting the change of coordinates $(\boldsymbol{z},\boldsymbol{Z}_\sigma)=(\boldsymbol{y}-\boldsymbol{x},\boldsymbol{Y}_\sigma-\boldsymbol{X}_\sigma)$, one gets

$$\int_{\mathbb{R}^{3(N-1)}}d\boldsymbol{x}d\boldsymbol{X}_\sigma\int_{\mathbb{R}^{3(N-1)}}d\boldsymbol{y}d\boldsymbol{Y}_\sigma\;|\xi(\boldsymbol{y},\boldsymbol{Y}_\sigma)-\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)|^2\,G^\lambda\begin{pmatrix}\boldsymbol{x}, & \boldsymbol{x}, & \boldsymbol{X}_\sigma\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma\end{pmatrix}$$
$$=\int_{\mathbb{R}^{3(N-1)}}d\boldsymbol{x}d\boldsymbol{X}_\sigma\int_{\mathbb{R}^{3(N-1)}}d\boldsymbol{z}d\boldsymbol{Z}_\sigma\;|\xi(\boldsymbol{z}+\boldsymbol{x},\boldsymbol{Z}_\sigma+\boldsymbol{X}_\sigma)-\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)|^2\,G^\lambda\begin{pmatrix}\boldsymbol{0}, & \boldsymbol{0}, & \boldsymbol{0}\\ \boldsymbol{z}, & \boldsymbol{z}, & \boldsymbol{Z}_\sigma\end{pmatrix}.$$

Clearly, because of (1.40c), the behavior at infinity is rapidly decaying, thus let us focus on a compact $K\subset\mathbb{R}^{3(N-1)}$ containing the origin, so that

$$\int_K d\boldsymbol{x}d\boldsymbol{X}_\sigma\int_K d\boldsymbol{z}d\boldsymbol{Z}_\sigma\;|\xi(\boldsymbol{z}+\boldsymbol{x},\boldsymbol{Z}_\sigma+\boldsymbol{X}_\sigma)-\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)|^2\,G^\lambda\begin{pmatrix}\boldsymbol{0}, & \boldsymbol{0}, & \boldsymbol{0}\\ \boldsymbol{z}, & \boldsymbol{z}, & \boldsymbol{Z}_\sigma\end{pmatrix}$$
$$\leq C|K|\int_K d\boldsymbol{z}d\boldsymbol{Z}_\sigma\;(z^2+Z_\sigma^2)\,G^\lambda\begin{pmatrix}\boldsymbol{0}, & \boldsymbol{0}, & \boldsymbol{0}\\ \frac{\boldsymbol{z}}{\sqrt{2}}, & \frac{\boldsymbol{z}}{\sqrt{2}}, & \boldsymbol{Z}_\sigma\end{pmatrix}$$

since each smooth function is at least locally Lipschitz continuous. According to (1.40b), the previous integrand behaves at the origin as $(z^2+Z_\sigma^2)^{2-\frac{3N}{2}}$ that is an integrable contribution and therefore, identity (3.28a) is obtained. In particular, $\Phi^\lambda_{\text{diag}}\geq 0$.
Next, take into account $\Phi^\lambda_{\text{off};0}$. Thanks to the symmetry in $\mathfrak{X}$, one has

$$\Phi^\lambda_{\text{off};0}[\xi]=-4\pi(N-2)(N-3)\lim_{r\to 0^+}\int_{\mathbb{R}^{3(N-1)}}d\boldsymbol{x}d\boldsymbol{x}_1\cdots d\boldsymbol{x}_{N-2}\;\overline{\xi(\boldsymbol{x},\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{N-2})}\times$$
$$\times\int_{\mathbb{R}^{3(N-1)}}d\boldsymbol{y}d\boldsymbol{y}_1\cdots d\boldsymbol{y}_{N-2}\;\xi(\boldsymbol{y},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_{N-2})\,G^\lambda\begin{pmatrix}\boldsymbol{x}_1, & \boldsymbol{x}_2, & \boldsymbol{x}+\frac{\boldsymbol{r}}{2}, & \boldsymbol{x}-\frac{\boldsymbol{r}}{2}, & \boldsymbol{x}_3, & \ldots, & \boldsymbol{x}_{N-2}\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{y}_1, & \boldsymbol{y}_2, & \boldsymbol{y}_3, & \ldots, & \boldsymbol{y}_{N-2}\end{pmatrix}. \tag{3.30}$$

Observe that exploiting equation (3.75), one obtains the following inequality

$$|\Phi^\lambda_{\text{off};0}[\xi]|\leq\tfrac{(N-2)(N-3)}{2}\,\|\xi\|_\infty\int_{\mathbb{R}^{3(N-1)}}d\boldsymbol{x}d\boldsymbol{x}_1\cdots d\boldsymbol{x}_{N-2}\;|\xi(\boldsymbol{x},\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{N-2})|\;\frac{e^{-\sqrt{\frac{\lambda}{2}}\,|\boldsymbol{x}_1-\boldsymbol{x}_2|}}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}<+\infty.$$

Therefore, since we can exhibit an integrable majorant uniformly in $\boldsymbol{r} \in \mathbb{R}^3$, identity (3.28b) is proven. Furthermore, notice that (3.28b) implies $\overline{\Phi^\lambda_{\text{off};0}[\xi]} = \Phi^\lambda_{\text{off};0}[\xi]$ thanks to the symmetry in exchanging $(\boldsymbol{x}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-2}) \longleftrightarrow (\boldsymbol{y}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{N-2})$.
Lastly, we take account of $\Phi^\lambda_{\text{off};1}$. In analogy with (3.30), we have

$$\Phi^\lambda_{\text{off};1}[\xi] = -16\pi(N-2) \lim_{r\to 0^+} \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_{N-2}\, \overline{\xi(\boldsymbol{x}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-2})} \times$$
$$\times \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{y}_1 \cdots d\boldsymbol{y}_{N-2}\, \xi(\boldsymbol{y}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{N-2})\, G^\lambda \begin{pmatrix} \boldsymbol{x} + \frac{\boldsymbol{r}}{2}, & \boldsymbol{x}_1, & \boldsymbol{x} - \frac{\boldsymbol{r}}{2}, & \boldsymbol{x}_2, & \dots, & \boldsymbol{x}_{N-2} \\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{y}_1, & \boldsymbol{y}_2, & \dots, & \boldsymbol{y}_{N-2} \end{pmatrix}. \tag{3.31}$$

Exploiting again equation (3.75), one finds

$$|\Phi^\lambda_{\text{off};1}[\xi]| \leq 2(N-2)\, \|\xi\|_\infty \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_{N-2} \sup_{z \in \mathbb{R}^3} |\xi(\boldsymbol{z}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-2})|\, \frac{e^{-\sqrt{\frac{\lambda}{2}}\, |\boldsymbol{x} - \boldsymbol{x}_1|}}{|\boldsymbol{x} - \boldsymbol{x}_1|} < +\infty.$$

Notice that the dependence on $\boldsymbol{r}$ has been first moved to the argument of ξ and then removed by taking the supremum. Hence, computing the limit inside the integral, identity (3.28c) has been recovered. Again, the symmetry by exchange yields $\overline{\Phi^\lambda_{\text{off};1}[\xi]} = \Phi^\lambda_{\text{off};1}[\xi]$.
To conclude the proof, notice that the operator $\mathcal{C}^*_\sigma \Gamma^\sigma_{\text{reg}}$ is symmetric, since it is a multiplication by a real function and therefore Φ_{reg} is also hermitian.

□

Remark 3.2. *We stress that the cancellation of the singular leading order from equation* (3.27a) *to* (3.28a) *is allowed by the properly tuned constant* 8π *in definition* (3.13)*, in accordance with proposition 3.7.*

Remark 3.3. *A negative part of the off-diagonal contributions can be highlighted since*

$$\Phi^\lambda_{\text{off};0}[\xi] = -\tfrac{(N-2)(N-3)}{2} \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{x}''d\boldsymbol{X}\, \frac{e^{-\sqrt{\frac{\lambda}{2}}\, |\boldsymbol{x}' - \boldsymbol{x}''|}}{|\boldsymbol{x}' - \boldsymbol{x}''|}\, |\xi(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}'', \boldsymbol{X})|^2 + \tag{3.32a}$$
$$+ 2\pi(N-2)(N-3) \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{x}''d\boldsymbol{X} \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{y}'d\boldsymbol{y}''d\boldsymbol{Y}\, |\xi(\boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \boldsymbol{Y}) - \xi(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}'', \boldsymbol{X})|^2 \times$$
$$\times G^\lambda \begin{pmatrix} \boldsymbol{x}', & \boldsymbol{x}'', & \boldsymbol{x}, & \boldsymbol{x}, & \boldsymbol{X} \\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{y}', & \boldsymbol{y}'', & \boldsymbol{Y} \end{pmatrix},$$

$$\Phi^\lambda_{\text{off};1}[\xi] = -2(N-2) \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{X}\, \frac{e^{-\sqrt{\frac{\lambda}{2}}\, |\boldsymbol{x} - \boldsymbol{x}'|}}{|\boldsymbol{x} - \boldsymbol{x}'|}\, |\xi(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{X})|^2 + \tag{3.32b}$$
$$+ 8\pi(N-2) \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{X} \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{y}'d\boldsymbol{Y}\, |\xi(\boldsymbol{y}, \boldsymbol{y}', \boldsymbol{Y}) - \xi(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{X})|^2\, G^\lambda \begin{pmatrix} \boldsymbol{x}, & \boldsymbol{x}', & \boldsymbol{x}, & \boldsymbol{X} \\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{y}', & \boldsymbol{Y} \end{pmatrix}.$$

In the following, we prove that Φ^λ is bounded in the dense subspace $H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$ so that representations (3.28) for the components defined by (3.27), actually hold for the wider class of functions $\mathfrak{X} \cap H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$.

Proposition 3.4. *Let* Φ^λ *be the quadratic form given by* (3.18)*. Then, there exists* $C > 0$ *such that*

$$|\Phi^\lambda[\xi]| \leq C\, \|\xi\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})}, \qquad \forall\, \xi \in H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)}).$$

Proof. Let us consider one by one all the terms of Φ^λ given by (3.26).

The regularizing contribution. First, owing to (3.11) and (3.27c), we have

$$\begin{aligned}\Phi_{\rm reg}[\xi] = \alpha_0\,\|\xi\|^2 + (N-2)\gamma\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{X}\;\frac{\theta(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,|\xi(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{X})|^2 + \\ +\;\frac{(N-2)(N-3)}{4}\gamma\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{x}''d\boldsymbol{X}\;\frac{\theta(\boldsymbol{x}'-\boldsymbol{x}'')}{|\boldsymbol{x}'-\boldsymbol{x}''|}\,|\xi(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}'',\boldsymbol{X})|^2.\end{aligned} \tag{3.33}$$

Since translations preserve the Gagliardo semi-norm $[\cdot]_{\frac12}$ introduced in (1.46), we can define

$$\eta_0(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}'',\boldsymbol{X}) := \xi(\boldsymbol{x},\boldsymbol{x}'+\boldsymbol{x}'',\boldsymbol{x}'',\boldsymbol{X}), \qquad \eta_1(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{X}) := \xi(\boldsymbol{x}+\boldsymbol{x}',\boldsymbol{x}',\boldsymbol{X})$$

so that $\|\eta_0\|_{H^{1/2}(\mathbb{R}^{3(N-1)})} = \|\eta_1\|_{H^{1/2}(\mathbb{R}^{3(N-1)})} = \|\xi\|_{H^{1/2}(\mathbb{R}^{3(N-1)})}$. Therefore, one has

$$\begin{aligned}|\Phi_{\rm reg}[\xi]| \le \alpha_0\,\|\xi\|^2 + (N-2)\gamma\,\|\theta\|_\infty\Bigg[\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{X}\;\frac{|\eta_1(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{X})|^2}{x} + \\ +\;\frac{N-3}{4}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{x}''d\boldsymbol{X}\;\frac{|\eta_0(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}'',\boldsymbol{X})|^2}{x'}\Bigg].\end{aligned}$$

Adopting Hardy-Rellich inequality (1.51) (with $s=1/2$ and $d=3$) and (1.47), we infer

$$|\Phi_{\rm reg}[\xi]| \le \alpha_0\,\|\xi\|^2 + \frac{(N+1)(N-2)\gamma\,\Gamma\big(\frac{3N}{2}-1\big)}{16\pi^{\frac{3N}{2}-2}}\,\|\theta\|_\infty\,[\xi]^2_{\frac12}. \tag{3.34}$$

The diagonal contribution. Taking into account (3.28a) and (1.46), one obtains

$$\Phi^\lambda_{\rm diag}[\xi] \le \sqrt{\tfrac{\lambda}{2}}\,\|\xi\|^2 + 4\pi\,[\xi]^2_{\frac12}\sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3\\ \boldsymbol{X}_\sigma,\boldsymbol{Y}_\sigma\in\mathbb{R}^{3(N-2)}}}\left\{\big(|\boldsymbol{x}-\boldsymbol{y}|^2+|\boldsymbol{X}_\sigma-\boldsymbol{Y}_\sigma|^2\big)^{\frac{3N-2}{2}} G^\lambda\begin{pmatrix}\boldsymbol{x}, & \boldsymbol{x}, & \boldsymbol{X}_\sigma\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma\end{pmatrix}\right\}.$$

Hence, exploiting (1.40d), one has

$$\begin{aligned}\Phi^\lambda_{\rm diag}[\xi] &\le \sqrt{\tfrac{\lambda}{2}}\,\|\xi\|^2 + \frac{\Gamma\big(\frac{3N}{2}-1\big)}{\pi^{\frac{3N}{2}-1}}\,[\xi]^2_{\frac12}\sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3\\ \boldsymbol{X}_\sigma,\boldsymbol{Y}_\sigma\in\mathbb{R}^{3(N-2)}}}\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2+|\boldsymbol{X}_\sigma-\boldsymbol{Y}_\sigma|^2}{2|\boldsymbol{x}-\boldsymbol{y}|^2+|\boldsymbol{X}_\sigma-\boldsymbol{Y}_\sigma|^2}\right)^{\frac{3N}{2}-1} \\ &\le \sqrt{\tfrac{\lambda}{2}}\,\|\xi\|^2 + \tfrac{\Gamma(\frac{3N}{2}-1)}{\pi^{3N/2-1}}\,[\xi]^2_{\frac12},\end{aligned}$$

namely

$$\Phi^\lambda_{\rm diag}[\xi] \le \max\left\{\sqrt{\tfrac{\lambda}{2}},\, \tfrac{\Gamma(\frac{3N}{2}-1)}{\pi^{3N/2-1}}\right\}\|\xi\|^2_{H^{1/2}(\mathbb{R}^{3(N-1)})}. \tag{3.35}$$

The off-diagonal sharing term. Analogously, in order to estimate $\Phi^\lambda_{\rm off;1}$ we can use (3.32b) to achieve

$$\Phi^\lambda_{\rm off;1}[\xi] \le 8\pi(N-2)[\xi]^2_{\frac12}\sup_{\substack{\boldsymbol{x},\boldsymbol{y},\boldsymbol{x}',\boldsymbol{y}'\in\mathbb{R}^3\\ \boldsymbol{X},\boldsymbol{Y}\in\mathbb{R}^{3(N-3)}}}\left\{\big(|\boldsymbol{x}-\boldsymbol{y}|^2+|\boldsymbol{x}'-\boldsymbol{y}'|^2+|\boldsymbol{X}-\boldsymbol{Y}|^2\big)^{\frac{3N-2}{2}} G^\lambda\begin{pmatrix}\boldsymbol{x}, & \boldsymbol{x}', & \boldsymbol{x} & \boldsymbol{X}\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{y}', & \boldsymbol{Y}\end{pmatrix}\right\}.$$

Clearly, we have to ensure that

$$s_N := \sup_{\substack{\boldsymbol{x},\boldsymbol{y},\boldsymbol{x}',\boldsymbol{y}'\in\mathbb{R}^3\\ \boldsymbol{X},\boldsymbol{Y}\in\mathbb{R}^{3(N-3)}}} \left\{ \left(|\boldsymbol{x}-\boldsymbol{y}|^2+|\boldsymbol{x}'-\boldsymbol{y}'|^2+|\boldsymbol{X}-\boldsymbol{Y}|^2\right)^{\frac{3N-2}{2}} G^\lambda\begin{pmatrix}\boldsymbol{x}, & \boldsymbol{x}', & \boldsymbol{x} & \boldsymbol{X}\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{y}', & \boldsymbol{Y}\end{pmatrix}\right\} \tag{3.36}$$

is finite for all $N\geq 3$. A first simplification is obtained exploiting again (1.40d), i.e.

$$s_N = \frac{\Gamma\left(\frac{3N}{2}-1\right)}{4\pi^{\frac{3N}{2}}} \sup_{\substack{\boldsymbol{x},\boldsymbol{y},\boldsymbol{x}',\boldsymbol{y}'\in\mathbb{R}^3\\ \boldsymbol{X},\boldsymbol{Y}\in\mathbb{R}^{3(N-3)}}} \left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2+|\boldsymbol{x}'-\boldsymbol{y}'|^2+|\boldsymbol{X}-\boldsymbol{Y}|^2}{|\boldsymbol{x}-\boldsymbol{y}|^2+|\boldsymbol{x}'-\boldsymbol{y}|^2+|\boldsymbol{x}-\boldsymbol{y}'|^2+|\boldsymbol{X}-\boldsymbol{Y}|^2}\right)^{\frac{3N}{2}-1}.$$

In order to evaluate this supremum, let c be the positive solution to the equation

$$x^4-\tfrac{8}{3}x^2-\tfrac{2}{9}=0, \tag{3.37}$$

namely $c=\sqrt{\frac{4}{3}+\sqrt{2}}$. Then, consider the following change of coordinates

$$\begin{cases}\boldsymbol{x}=c\boldsymbol{r}_1+\frac{\boldsymbol{R}_1}{3c}\,,\\ \boldsymbol{x}'=-2c\boldsymbol{r}_1+\frac{\boldsymbol{R}_1}{3c}\,,\end{cases}\qquad \begin{cases}\boldsymbol{y}=c\boldsymbol{r}_2+\frac{\boldsymbol{R}_2}{3c}\,,\\ \boldsymbol{y}'=-2c\boldsymbol{r}_2+\frac{\boldsymbol{R}_2}{3c}\,,\end{cases} \tag{3.38}$$

so that, one has

$$\begin{aligned}|\boldsymbol{x}-\boldsymbol{y}|^2+|\boldsymbol{x}'-\boldsymbol{y}'|^2 &\longmapsto \tfrac{2}{9c^2}|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+5c^2|\boldsymbol{r}_1-\boldsymbol{r}_2|^2-\tfrac{2}{3}(\boldsymbol{R}_1-\boldsymbol{R}_2)\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2),\\ |\boldsymbol{x}-\boldsymbol{y}|^2+|\boldsymbol{y}'-\boldsymbol{x}|^2+|\boldsymbol{x}'-\boldsymbol{y}|^2 &\longmapsto \tfrac{1}{3c^2}|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+6c^2(r_1^2+r_2^2+\boldsymbol{r}_1\cdot\boldsymbol{r}_2).\end{aligned}$$

Therefore, exploiting elementary estimates one has for the denominator

$$\tfrac{1}{3c^2}|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+6c^2(r_1^2+r_2^2+\boldsymbol{r}_1\cdot\boldsymbol{r}_2)\geq \tfrac{1}{3c^2}|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+3c^2(r_1^2+r_2^2),$$

whereas in the numerator one gets

$$\begin{aligned}&\tfrac{2}{9c^2}|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+5c^2|\boldsymbol{r}_1-\boldsymbol{r}_2|^2-\tfrac{2}{3}(\boldsymbol{R}_1-\boldsymbol{R}_2)\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\leq\\ &\quad\leq\left(\tfrac{2}{9c^2}+\tfrac{1}{3}\right)|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+\left(5c^2+\tfrac{1}{3}\right)|\boldsymbol{r}_1-\boldsymbol{r}_2|^2\\ &\quad\leq\left(\tfrac{2}{9c^2}+\tfrac{1}{3}\right)|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+\left(10c^2+\tfrac{2}{3}\right)(r_1^2+r_2^2)\\ &\quad=\left(\tfrac{2}{3}+c^2\right)\tfrac{1}{3c^2}|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+\left(\tfrac{10}{3}+\tfrac{2}{9c^2}\right)3c^2(r_1^2+r_2^2)\\ &\quad=\left(\tfrac{2}{3}+c^2\right)\left[\tfrac{1}{3c^2}|\boldsymbol{R}_1-\boldsymbol{R}_2|^2+3c^2(r_1^2+r_2^2)\right]\end{aligned}$$

since c solves equation (3.37). Hence,

$$s_N=\frac{\Gamma\left(\frac{3N}{2}-1\right)}{4\pi^{\frac{3N}{2}}}(2+\sqrt{2})^{\frac{3N}{2}-1}. \tag{3.39}$$

Indeed, the supremum attains its maximum value along the hyperplanes $\boldsymbol{X}=\boldsymbol{Y}$, $\boldsymbol{R}_1-\boldsymbol{R}_2=2\boldsymbol{r}_2$ and $\boldsymbol{r}_1+\boldsymbol{r}_2=0$. Therefore, we have obtained

$$\Phi^\lambda_{\text{off};1}[\xi]\leq 2(N-2)\,\Gamma\left(\tfrac{3N}{2}-1\right)\left(\tfrac{2+\sqrt{2}}{\pi}\right)^{\frac{3N}{2}-1}[\xi]^2_{\frac{1}{2}}. \tag{3.40}$$

The off-diagonal non-sharing term. Concerning $\Phi^{\lambda}_{\text{off};0}$, we proceed by taking account of (3.28b) in its Fourier representation. In particular, thanks to proposition 3.8 one has

$$\Phi^{\lambda}_{\text{off};0}[\xi] = -\tfrac{(N-2)(N-3)}{2\pi^2}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\,d\boldsymbol{p}_1 d\boldsymbol{p}_2\, d\boldsymbol{P}\; \overline{\hat{\xi}(\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{P})}\int_{\mathbb{R}^3} d\boldsymbol{q}\; \frac{\hat{\xi}\big(\boldsymbol{p}_1+\boldsymbol{p}_2,\frac{\boldsymbol{p}}{2}+\boldsymbol{q},\frac{\boldsymbol{p}}{2}-\boldsymbol{q},\boldsymbol{P}\big)}{\frac{1}{2}p^2+2q^2+p_1^2+p_2^2+P^2+\lambda}$$

namely, replacing $(\boldsymbol{p},\boldsymbol{q}) \longmapsto (\boldsymbol{p}_3+\boldsymbol{p}_4, \frac{\boldsymbol{p}_3-\boldsymbol{p}_4}{2})$ and setting $\boldsymbol{P}=(\boldsymbol{p}_5,\dots,\boldsymbol{p}_N)$

$$\Phi^{\lambda}_{\text{off};0}[\xi] = -\tfrac{(N-2)(N-3)}{2\pi^2}\int_{\mathbb{R}^{3N}} d\boldsymbol{p}_1\cdots d\boldsymbol{p}_N\; \frac{\overline{\hat{\xi}(\boldsymbol{p}_3+\boldsymbol{p}_4,\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_5,\dots,\boldsymbol{p}_N)}\,\hat{\xi}(\boldsymbol{p}_1+\boldsymbol{p}_2,\boldsymbol{p}_3,\boldsymbol{p}_4,\boldsymbol{p}_5,\dots,\boldsymbol{p}_N)}{\sum_{j=1}^N p_j^2+\lambda} \tag{3.41}$$

that is controlled by the $H^{\frac{1}{2}}$-norm of ξ. Indeed, consider

$$f_\xi(\boldsymbol{p}_4,\boldsymbol{P}) := \sqrt{\int_{\mathbb{R}^6} d\boldsymbol{p}_1 d\boldsymbol{p}_3\; |\hat{\xi}(\boldsymbol{p}_1,\boldsymbol{p}_3,\boldsymbol{p}_4,\boldsymbol{P})|^2}, \tag{3.42a}$$

$$g_\xi(\boldsymbol{p}_2,\boldsymbol{P}) := \sqrt{\int_{\mathbb{R}^6} d\boldsymbol{p}_1 d\boldsymbol{p}_3\; |\hat{\xi}(\boldsymbol{p}_3,\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{P})|^2} \tag{3.42b}$$

so that one has

$$\|f_\xi\|_{H^{1/2}(\mathbb{R}^{3(N-3)})} \le \|\xi\|_{H^{1/2}(\mathbb{R}^{3(N-1)})}, \qquad \|g_\xi\|_{H^{1/2}(\mathbb{R}^{3(N-3)})} \le \|\xi\|_{H^{1/2}(\mathbb{R}^{3(N-1)})}.$$

Notice that, exploiting the Cauchy–Schwarz inequality in the $d\boldsymbol{p}_1 d\boldsymbol{p}_3$ integration, one obtains

$$\begin{aligned}|\Phi^{\lambda}_{\text{off};0}[\xi]| &\le \tfrac{(N-2)(N-3)}{2\pi^2}\int_{\mathbb{R}^{3(N-2)}} d\boldsymbol{p}_2\, d\boldsymbol{p}_4\, d\boldsymbol{P}\; \frac{\overline{g_\xi(\boldsymbol{p}_2,\boldsymbol{P})}\, f_\xi(\boldsymbol{p}_4,\boldsymbol{P})}{p_2^2+p_4^2+P^2+\lambda}\\ &\le \tfrac{(N-2)(N-3)}{2\pi^2}\int_{\mathbb{R}^{3(N-2)}} d\boldsymbol{q}_1\, d\boldsymbol{q}_2\, d\boldsymbol{P}\; \frac{\overline{g_\xi(\boldsymbol{q}_1,\boldsymbol{P})}\, f_\xi(\boldsymbol{q}_2,\boldsymbol{P})}{q_1^2+q_2^2}.\end{aligned}$$

The previous integral can be rewritten as the scalar product in $L^2(\mathbb{R}^{3(N-3)})$ of $\sqrt{\cdot}\,g_\xi(\cdot)$ against the action on $\sqrt{\cdot}\,f_\xi$ of the integral operator $\mathcal{Q}$ defined in (2.43) that is bounded in $L^2(\mathbb{R}^3)$ with norm $2\pi^2$. Hence, since

$$\sqrt{q_1}\,g_\xi(\boldsymbol{q}_1,\boldsymbol{P}) \in L^2(\mathbb{R}^{3(N-3)}, d\boldsymbol{q}_1 d\boldsymbol{P}), \qquad \sqrt{q_2}\,f_\xi(\boldsymbol{q}_2,\boldsymbol{P}) \in L^2(\mathbb{R}^{3(N-3)}, d\boldsymbol{q}_2\, d\boldsymbol{P}),$$

clearly one has

$$|\Phi^{\lambda}_{\text{off};0}[\xi]| \le \frac{(N-2)(N-3)\,\Gamma\big(\frac{3N}{2}-1\big)}{2\pi^{\frac{3N}{2}-1}}\,[\xi]^2_{\frac{1}{2}}. \tag{3.43}$$

□

Notice that estimate (3.43) implies that decomposition (3.32a) is valid for all $\xi \in H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$ since the negative term is controlled in the $H^{\frac{1}{2}}$-norm and, therefore, so must be the positive contribution. We end the section by observing that proposition 3.4 implies $\mathscr{D}(\Phi^{\lambda}) \supseteq H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$, but since $\mathscr{D}(\Phi^{\lambda}_{\text{diag}}) = H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$ (see equation (3.46)), we also have the opposite inclusion.

3.3 Proof of the Main Results

Before proving the results of section 3.1, let us state the following.

Proposition 3.5. *Let Φ^λ be the hermitian quadratic form defined in $\mathfrak{X}$ by* (3.18)*. Then, considering Λ_N given by* (3.20)*, one has for all $\lambda > 0$*

$$\Phi^\lambda[\xi] \geq 4\pi N(N-1)\left[1 - \Lambda_N - \max\left\{0, -\tfrac{\sqrt{2}\,\alpha_0}{\sqrt{\lambda}}\right\} - \tfrac{(N+1)(N-2)\gamma}{\sqrt{8\lambda}\,b}\right]\Phi^\lambda_{\text{diag}}[\xi]. \tag{3.44}$$

Furthermore, Φ^λ is coercive in $H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$ whenever $\lambda > \lambda_0^*$ with

$$\lambda_0^* := \frac{2\left[\max\{0, -\alpha_0\} + (N+1)(N-2)\,\gamma/(4b)\right]^2}{(1-\Lambda_N)^2} \tag{3.45}$$

provided $\gamma > \gamma_c^{N\mathrm{b}}$ defined by (3.19).

Proof. First, we claim that $\Phi^\lambda_{\text{diag}}$ has the following Fourier representation

$$\Phi^\lambda_{\text{diag}}[\xi] = \frac{1}{\sqrt{2}}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\,d\boldsymbol{P}_\sigma\,\sqrt{\tfrac{1}{2}p^2 + P_\sigma^2 + \lambda}\,|\hat{\xi}(\boldsymbol{p},\boldsymbol{P}_\sigma)|^2. \tag{3.46}$$

To show this, consider identity (3.28a) that can be manipulated as follows

$$\begin{aligned}
\Phi^\lambda_{\text{diag}}[\xi] &= \sqrt{\tfrac{\lambda}{2}}\,\|\xi\|^2 + 4\pi\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{z}d\boldsymbol{Z}_\sigma \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{Y}_\sigma\,|\xi(\boldsymbol{y}+\boldsymbol{z}, \boldsymbol{Y}_\sigma + \boldsymbol{Z}_\sigma) - \xi(\boldsymbol{y},\boldsymbol{Y}_\sigma)|^2\,G^\lambda\begin{pmatrix}\boldsymbol{0}, & \boldsymbol{0}, & \boldsymbol{0}\\ \boldsymbol{z}, & \boldsymbol{z}, & \boldsymbol{Z}_\sigma\end{pmatrix}\\
&= \sqrt{\tfrac{\lambda}{2}}\,\|\xi\|^2 + 4\pi\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{z}d\boldsymbol{Z}_\sigma\,G^\lambda\begin{pmatrix}\boldsymbol{0}, & \boldsymbol{0}, & \boldsymbol{0}\\ \boldsymbol{z}, & \boldsymbol{z}, & \boldsymbol{Z}_\sigma\end{pmatrix}\|\xi(\boldsymbol{y}+\boldsymbol{z}, \boldsymbol{Y}_\sigma + \boldsymbol{Z}_\sigma) - \xi(\boldsymbol{y},\boldsymbol{Y}_\sigma)\|^2_{L^2(\mathbb{R}^{3(N-1)},\, d\boldsymbol{y}d\boldsymbol{Y}_\sigma)}\\
&= \sqrt{\tfrac{\lambda}{2}}\,\|\hat{\xi}\|^2 + 4\pi\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{z}d\boldsymbol{Z}_\sigma \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}d\boldsymbol{P}_\sigma\,|e^{-i\boldsymbol{p}\cdot\boldsymbol{z} - i\boldsymbol{P}_\sigma\cdot\boldsymbol{Z}_\sigma} - 1|^2\,|\hat{\xi}(\boldsymbol{p},\boldsymbol{P}_\sigma)|^2\,G^\lambda\begin{pmatrix}\boldsymbol{0}, & \boldsymbol{0}, & \boldsymbol{0}\\ \boldsymbol{z}, & \boldsymbol{z}, & \boldsymbol{Z}_\sigma\end{pmatrix}
\end{aligned}$$

where in the last step we have exploited Plancherel's theorem. Thanks to Tonelli's theorem, we can exchange the order of integration, obtaining

$$\begin{aligned}
\Phi^\lambda_{\text{diag}}[\xi] &= \sqrt{\tfrac{\lambda}{2}}\,\|\hat{\xi}\|^2 + 8\pi\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}d\boldsymbol{P}_\sigma\,|\hat{\xi}(\boldsymbol{p},\boldsymbol{P}_\sigma)|^2 \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{z}\,d\boldsymbol{Z}_\sigma\,[1 - \cos(\boldsymbol{p}\cdot\boldsymbol{z} + \boldsymbol{P}_\sigma\cdot\boldsymbol{Z}_\sigma)]\,G^\lambda\begin{pmatrix}\boldsymbol{0}, & \boldsymbol{0}, & \boldsymbol{0}\\ \boldsymbol{z}, & \boldsymbol{z}, & \boldsymbol{Z}_\sigma\end{pmatrix}\\
&= \sqrt{\tfrac{\lambda}{2}}\,\|\hat{\xi}\|^2 + 8\pi\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}d\boldsymbol{P}_\sigma\,|\hat{\xi}(\boldsymbol{p},\boldsymbol{P}_\sigma)|^2 \lim_{r\to 0}\frac{e^{-\sqrt{\frac{\lambda}{2}}\,r} - e^{-\sqrt{\frac{1}{2}p^2 + P_\sigma^2 + \lambda}\,\frac{r}{\sqrt{2}}}}{8\pi\,r}
\end{aligned}$$

according to proposition 3.7. Evaluating the limit, equation (3.46) is recovered.
Taking account of remark 3.3, we get

$$\begin{aligned}
(\Phi_{\text{reg}} + \Phi^\lambda_{\text{off;0}} + \Phi^\lambda_{\text{off;1}})[\xi] \geq{}& (N-2)\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{X}\,\frac{\gamma - 2e^{-\sqrt{\frac{\lambda}{2}}\,|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,|\xi(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{X})|^2 +\\
&+ \frac{(N-2)(N-3)}{4}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{x}''d\boldsymbol{X}\,\frac{\gamma - 2e^{-\sqrt{\frac{\lambda}{2}}\,|\boldsymbol{x}'-\boldsymbol{x}''|}}{|\boldsymbol{x}'-\boldsymbol{x}''|}\,|\xi(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}'',\boldsymbol{X})|^2 +\\
&+ \left[\alpha_0 - \frac{(N+1)(N-2)\gamma}{4\,b}\right]\|\xi\|^2,
\end{aligned}$$

where we have made use of the lower bound of assumption (1.10). Exploiting Hardy-Rellich inequality (1.51)

$$(\Phi_{\text{reg}}+\Phi^\lambda_{\text{off};0}+\Phi^\lambda_{\text{off};1})[\xi]\geq \frac{(N+1)(N-2)\,\pi}{8}\min\{0,\gamma-2\}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\,d\boldsymbol{p}'d\boldsymbol{P}\;p'\,|\hat{\xi}(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{P})|^2+ \\ +\left[\alpha_0-\frac{(N+1)(N-2)\,\gamma}{4b}\right]\|\xi\|^2.$$

Moreover,

$$(\Phi_{\text{reg}}+\Phi^\lambda_{\text{off};0}+\Phi^\lambda_{\text{off};1})[\xi]\geq -\Lambda_N\,\Phi^\lambda_{\text{diag}}[\xi]+\left[\alpha_0-\tfrac{(N+1)(N-2)\,\gamma}{4\,b}\right]\|\xi\|^2 \tag{3.47}$$

that implies

$$(\Phi_{\text{reg}}+\Phi^\lambda_{\text{off};0}+\Phi^\lambda_{\text{off};1})[\xi]\geq -\Lambda_N\,\Phi^\lambda_{\text{diag}}[\xi]+\left[\min\{0,\alpha_0\}-\tfrac{(N+1)(N-2)\,\gamma}{4\,b}\right]\sqrt{\tfrac{2}{\lambda}}\,\Phi^\lambda_{\text{diag}}[\xi]$$

which proves the first statement. Concerning the coercivity, the result can be obtained by noticing that $\Phi^\lambda_{\text{diag}}$ defines an equivalent norm to $H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$ so that inequality (3.44) provides the result by assuming $\Lambda_N<1$ (that holds if and only if $\gamma>\gamma_c^{N\text{b}}$) and λ large enough.

□

Proof of proposition 3.1. We first show the lower bound for Φ^λ. In the course of proposition 3.5 we have seen that

$$\Phi^\lambda[\xi]\geq \tfrac{4\pi N(N-1)}{\sqrt{2}}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\,d\boldsymbol{P}\left[\sqrt{\tfrac{1}{2}p^2+P^2+\lambda}-\Lambda_N\sqrt{\tfrac{1}{2}p^2+P^2}+\sqrt{2}\,\alpha_0-\tfrac{(N+1)(N-2)\gamma}{2\sqrt{2}\,b}\right]|\hat{\xi}(\boldsymbol{p},\boldsymbol{P})|^2.$$

One can check that, as long as $\Lambda_N<1$, the quantity in square brackets in the integrand attains its minimum along the hyperplane

$$\sqrt{\tfrac{1}{2}p^2+P^2}=\Lambda_N\sqrt{\frac{\lambda}{1-\Lambda_N^2}}\,,$$

therefore

$$\Phi^\lambda[\xi]\geq \tfrac{4\pi N(N-1)}{\sqrt{2}}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\,d\boldsymbol{P}\left[\sqrt{\lambda}\,\sqrt{1-\Lambda_N^2}+\sqrt{2}\,\alpha_0-\tfrac{(N+1)(N-2)\gamma}{2\sqrt{2}\,b}\right]|\hat{\xi}(\boldsymbol{p},\boldsymbol{P})|^2 \\ =4\pi N(N-1)\left[\sqrt{\tfrac{\lambda}{2}}\,\sqrt{1-\Lambda_N^2}+\alpha_0-\tfrac{(N+1)(N-2)\gamma}{4\,b}\right]\|\xi\|^2.$$

Hence, we have proved that Φ^λ is coercive in $\mathfrak{X}$ for any $\lambda>\lambda_0$ with

$$\lambda_0:=\frac{2\max\{0,-\alpha_0+(N+1)(N-2)\,\gamma/(4b)\}^2}{1-\Lambda_N^2}\,. \tag{3.48}$$

Observe that $\lambda_0<\lambda_0^*$ given by (3.45).

Next we focus on the closedness of the quadratic form. Owing to proposition 3.5 we trivially have that Φ^λ is closed for any $\lambda>\lambda_0^*$ thus we assume $0<\lambda\leq\lambda_0^*$. Since Φ^λ is hermitian, the quantity

$$\|\cdot\|^2_{\Phi^\lambda}:=\Phi^\lambda[\cdot]+4\pi N(N-1)\left[\sqrt{\tfrac{\lambda}{2}}\left(1-\sqrt{1-\Lambda_N^2}\right)-\alpha_0+\tfrac{(N+1)(N-2)\gamma}{4b}\right]\|\cdot\|^2_{\mathfrak{X}} \tag{3.49}$$

defines a norm in $\mathfrak{X}$ for all $\lambda > 0$. Therefore, let us show that $\mathfrak{X} \cap H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$ is closed under this newly defined norm. Let $\psi \in \mathfrak{X}$ and $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{X} \cap H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$ be respectively a vector and a Cauchy sequence for $\|\cdot\|_{\Phi^\lambda}$ such that $\|\psi_n - \psi\|_{\mathfrak{X}} \longrightarrow 0$. Therefore, since inequality (3.47) reads

$$\Phi^\lambda[\cdot] \geq 4\pi N(N-1)\Big\{(1-\Lambda_N)\,\Phi^\lambda_{\text{diag}}[\cdot] + \Big[\alpha_0 - \tfrac{(N+1)(N-2)\,\gamma}{4b}\Big]\,\|\cdot\|^2_{\mathfrak{X}}\Big\}, \qquad \lambda > 0$$

one has

$$(1-\Lambda_N)\,\Phi^\lambda_{\text{diag}}[\psi_n - \psi_m] + \sqrt{\tfrac{\lambda}{2}}\left(1 - \sqrt{1-\Lambda_N^2}\right)\|\psi_n - \psi_m\|^2_{\mathfrak{X}} \leq \frac{\|\psi_n - \psi_m\|^2_{\Phi^\lambda}}{4\pi N(N-1)} \longrightarrow 0.$$

The left hand side is composed of positive terms, the second of which is vanishing in the limit. This means that

$$\Phi^\lambda_{\text{diag}}[\psi_n - \psi_m] \longrightarrow 0.$$

In other words, $\{\psi_n\}$ is a Cauchy sequence also in $H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$ and, by uniqueness of the limit, it converges to $\psi \in \mathfrak{X} \cap H^{\frac{1}{2}}(\mathbb{R}^{3(N-1)})$.

□

Proof of theorem 3.2. Because of proposition 3.1 we can uniquely associate to Φ^λ a s.a. positive and invertible operator Γ^λ for any $\lambda > \lambda_0$ such that

$$\Phi^\lambda[\xi] = \langle \xi, \Gamma^\lambda \xi \rangle, \qquad \xi \in \mathscr{D}(\Gamma^\lambda). \tag{3.50}$$

In light of remark 3.1, we know that

$$\Gamma^{\lambda_1} - \Gamma^{\lambda_2} = T(\mathcal{G}^{\lambda_2} - \mathcal{G}^{\lambda_1}), \qquad \forall\, \lambda_1, \lambda_2 > \lambda_0. \tag{3.51}$$

This means that $\mathscr{D}(\Gamma^\lambda)$ actually does not depend on λ, because of equation (A.4) (with Γ_λ used therein set equal to $\Gamma^{-\lambda}$). Thus, we are in position to exploit proposition A.2 so that a continuous map $\Gamma\colon \mathbb{C}\smallsetminus\mathbb{R}_+ \longrightarrow \mathscr{L}(\mathfrak{X})$ satisfying conditions (A.2) can be defined. The Hamiltonian is therefore characterized by (A.9). Furthermore, because of proposition A.3, we obtain the boundedness from below, namely $\mathcal{H} \geq -\lambda_0$, since $\Gamma(-\lambda) > 0$ for any $\lambda > \lambda_0$.

□

3.4 Dirichlet Forms

In this section, we compare the known results developed in [2] via the Dirichlet forms approach with ours.

In order to exhibit the one-parameter family of regularized zero-range Hamiltonians obtained in [2] we need to introduce some notation. In particular, we set $\mathbf{x} := (\boldsymbol{x}_1, \dots, \boldsymbol{x}_N) \in \mathbb{R}^{3N}$ and $\nabla := (\nabla_{\boldsymbol{x}_1}, \dots, \nabla_{\boldsymbol{x}_N})$. Then, let us introduce the quadratic form

$$E[\psi] := \int_{\mathbb{R}^{3N}} d\mathbf{x}\, \phi^2(\mathbf{x}) |\nabla \psi(\mathbf{x})|^2, \qquad \psi \in \mathscr{H}_{N\mathrm{b}} \cap H^1(\mathbb{R}^{3N}, \phi^2(\mathbf{x})\, d\mathbf{x}), \tag{3.52}$$

where

$$\phi(\mathbf{x}) := \frac{1}{4\pi} \sum_{1 \le i < j \le N} \frac{e^{-m\, |\boldsymbol{x}_i - \boldsymbol{x}_j|}}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \in L^2_{\mathrm{loc}}(\mathbb{R}^{3N}), \qquad m \ge 0. \tag{3.53}$$

It is noteworthy that

$$\Delta \phi(\mathbf{x}) := (-\mathcal{H}_0 \phi)(\mathbf{x}) = 2m^2 \phi(\mathbf{x}), \qquad \forall\, \mathbf{x} \in \mathbb{R}^{3N} \smallsetminus \pi. \tag{3.54}$$

In [2, section 2, example 4] it is shown that the quantity

$$Q_D[\psi] := E[\psi/\phi] - 2m^2\, \|\psi\|^2_{\mathscr{H}_{N\mathrm{b}}} \tag{3.55}$$

defines a singular perturbation of $\mathcal{H}_0\,, \mathscr{H}_{N\mathrm{b}} \cap H^2(\mathbb{R}^{3N})$ supported on π. More precisely, for any non-negative value of m, the quadratic form Q_D is associated to a bounded from below operator, denoted by $-\Delta_m$ such that

$$-\Delta_m \psi = \mathcal{H}_0 \psi, \qquad \forall\, \psi \in \mathscr{H}_{N\mathrm{b}} \cap H^2_0(\mathbb{R}^{3N} \smallsetminus \pi),$$
$$-\Delta_m \ge -2m^2.$$

Thus, this approach surely defines a class of zero-range Hamiltonians with preassigned lower bound $-2m^2$ avoiding any instability issue. However, wondering what exactly is the characterization of the domain of the Hamiltonian, or equivalently, which boundary condition is satisfied in this framework, is a natural question arising at this point. Our goal is to exploit the general theory discussed in appendix A in order to rewrite Q_D in our formalism so that a comparison with our results can be made.
Since $-\Delta_m$ is a singular perturbation of the free Hamiltonian supported on the coincidence hyperplanes, according to appendix A there exists a continuous map $\Gamma_D : \mathbb{C} \smallsetminus \mathbb{R}_+ \longrightarrow \mathscr{L}(\mathfrak{X})$ fulfilling conditions (A.2) such that $-\Delta_m$ coincides with the s.a. extension $-\Delta^T_{\Gamma_D}$, defined by (A.8) with T given by (3.14). Furthermore, exploiting the characterization (A.9) of $-\Delta^T_{\Gamma_D}$, we know that $\psi \in \mathscr{D}(-\Delta^T_{\Gamma_D})$ can be uniquely decomposed as $\psi = w_z + \mathcal{G}(z)\xi$ for all $z \in \mathbb{C} \smallsetminus \mathbb{R}_+$, with $w_z \in H^2(\mathbb{R}^{3N})$ and ξ in the domain of $\Gamma_D(z)$. Moreover, there holds the following boundary condition

$$\Gamma_D(z)\, \xi = T w_z.$$

Clearly, we need to understand what kind of regularization lies underneath this map Γ_D. To this end, let us recover the energy form associated to $-\Delta^T_{\Gamma_D}$ by exploiting identity (A.11)

$$\langle \psi, -\Delta^T_{\Gamma_D} \psi \rangle_{\mathscr{H}_{N\mathrm{b}}} = z\, \|\psi\|^2_{\mathscr{H}_{N\mathrm{b}}} + \langle w_z, (\mathcal{H}_0 - z)\, w_z \rangle_{\mathscr{H}_{N\mathrm{b}}} + \langle \xi, \Gamma_D(z) \xi \rangle_{\mathfrak{X}}$$

provided $z<0 \ \vee \ w_z \perp \mathcal{G}(z)\xi$. Thus, according to (3.55), we have

$$Q_D[\psi] = \langle \psi,\, -\Delta^T_{\Gamma_D}\psi\rangle, \tag{3.56}$$

whose domain is given by those elements $\psi \in \mathscr{H}_{N\mathrm{b}}$ that can be decomposed as $\psi = w_\lambda + \mathcal{G}^\lambda \xi$ for some $w_\lambda \in H^1(\mathbb{R}^{3N})$ and $\xi \in \mathscr{D}(\Gamma_D(-\lambda)^{1/2}) \subset \mathfrak{X}$ that does not depend on $\lambda > 0$. Therefore, let us consider an element in this form domain reading as

$$\psi = \mathcal{G}^\lambda \xi, \qquad \text{for some } \lambda > 0.$$

With this position, we can isolate from (3.56) the quadratic form of the charges associated with the operator Γ_D, i.e.

$$\Phi^\lambda_D[\xi] := E[\mathcal{G}^\lambda\xi/\phi] + (\lambda - 2m^2)\left\|\mathcal{G}^\lambda\xi\right\|^2_{\mathscr{H}_{N\mathrm{b}}}. \tag{3.57}$$

In the next proposition we prove that such quadratic form corresponds to a special case of the quadratic form Φ^λ defined by (3.18).

Proposition 3.6. *Let $\mathcal{H}$ be the Hamiltonian characterized in theorem 3.2 and $-\Delta_m$ the operator defined by* (3.55)*. Then, assuming $\alpha_0 = -m \leq 0$, $\gamma = 2$ and $\theta(x) = e^{\alpha_0 x}$ one has*

$$-\Delta_m = \mathcal{H} \geq -2\alpha_0^2.$$

Proof. The proposition is proved as soon as we show that, under our assumptions, $\Phi^\lambda = \Phi^\lambda_D$ given by (3.57). To this end, we proceed in evaluating $E[\mathcal{G}^\lambda\xi/\phi]$.
Let $D^\epsilon := \left\{(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) \in \mathbb{R}^{3N} \;\middle|\; \min\limits_{1\leq i<j\leq N} |\boldsymbol{x}_i - \boldsymbol{x}_j| > \epsilon\right\}$ so that

$$\begin{aligned}
E[\mathcal{G}^\lambda\xi/\phi] &= \lim_{\epsilon\to 0^+} \int_{D^\epsilon} d\mathbf{x}\; \phi^2(\mathbf{x}) \left| \frac{\nabla \mathcal{G}^\lambda \xi}{\phi} - \mathcal{G}^\lambda\xi\, \frac{\nabla\phi}{\phi^2}\right|^2 \\
&= \lim_{\epsilon\to 0^+} \int_{D^\epsilon} d\mathbf{x} \left[|\nabla\mathcal{G}^\lambda\xi|^2 - 2\,\mathrm{Re}\,\mathcal{G}^\lambda\xi\; \nabla\mathcal{G}^\lambda\xi \cdot \frac{\nabla\phi}{\phi} + |\mathcal{G}^\lambda\xi|^2\, \frac{|\nabla\phi|^2}{\phi^2}\right] \\
&= \lim_{\epsilon\to 0^+} \int_{D^\epsilon} d\mathbf{x} \left[|\nabla\mathcal{G}^\lambda\xi|^2 - \nabla|\mathcal{G}^\lambda\xi|^2 \cdot \nabla \ln\phi + |\mathcal{G}^\lambda\xi|^2\, \frac{|\nabla\phi|^2}{\phi^2}\right].
\end{aligned}$$

Exploiting the first Green's identity, one gets

$$\begin{aligned}
E[\mathcal{G}^\lambda\xi/\phi] = \lim_{\epsilon\to 0^+} \Bigg[&\int_{\partial D^\epsilon} d\mathrm{s}\; \left(\mathcal{G}^\lambda\xi\, \partial_{\boldsymbol{n}} \mathcal{G}^\lambda\xi - |\mathcal{G}^\lambda\xi|^2 \partial_{\boldsymbol{n}} \ln\phi\right) + \\
&- \int_{D^\epsilon} d\mathbf{x} \left(\mathcal{G}^\lambda\xi\, \Delta\mathcal{G}^\lambda\xi - |\mathcal{G}^\lambda\xi|^2\, \Delta\ln\phi - |\mathcal{G}^\lambda\xi|^2\, \frac{|\nabla\phi|^2}{\phi^2}\right)\Bigg],
\end{aligned}$$

where $\partial_{\boldsymbol{n}}$ denotes the outer normal derivative. Furthermore, we stress that

$$\partial D^\epsilon = \bigcup_{\mathscr{P}\ni\sigma=\{i,j\}} \left\{(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)\in\mathbb{R}^{3N} \;\middle|\; |\boldsymbol{x}_i - \boldsymbol{x}_j| = \epsilon,\ \min_{\substack{\mathscr{P}\ni\nu=\{k,\ell\}\\ \nu\neq\sigma}} |\boldsymbol{x}_k - \boldsymbol{x}_\ell| \geq \epsilon\right\} =: \bigcup_{\mathscr{P}\ni\sigma=\{i,j\}} \partial D^\epsilon_\sigma.$$

Taking into account the following identities for any $\mathbf{x} \in \mathbb{R}^{3N} \setminus \pi$

$$(\mathcal{H}_0 + \lambda)\,\mathcal{G}^\lambda\xi(\mathbf{x}) = 0, \tag{3.58}$$

$$(\Delta \ln \phi)(\mathbf{x}) = \frac{(\Delta\phi)(\mathbf{x})}{\phi(\mathbf{x})} - \frac{|\nabla\phi(\mathbf{x})|^2}{\phi^2(\mathbf{x})}, \tag{3.59}$$

the computation in the previous integral yields

$$\begin{aligned} E[\mathcal{G}^\lambda\xi/\phi] &= (2m^2-\lambda)\left\|\mathcal{G}^\lambda\xi\right\|^2_{\mathscr{H}_{N\mathrm{b}}} + \tfrac{1}{2}\lim_{\epsilon\to 0^+}\int_{\partial D^\epsilon} d\mathbf{s}\ \phi^2(\mathbf{x})\,\partial_{\boldsymbol{n}}\tfrac{|\mathcal{G}^\lambda\xi|^2}{\phi^2} \\ &= (2m^2-\lambda)\left\|\mathcal{G}^\lambda\xi\right\|^2_{\mathscr{H}_{N\mathrm{b}}} + \frac{1}{2}\sum_{\sigma\in\mathscr{P}}\lim_{\epsilon\to 0^+}\int_{\partial D^\epsilon_\sigma} d\mathbf{s}\ \phi^2(\mathbf{x})\,\partial_{\boldsymbol{n}}\tfrac{|\mathcal{G}^\lambda\xi|^2}{\phi^2}. \end{aligned}$$

Hence, because of (3.57) and the symmetry of the integrand in exchanging any couple $\boldsymbol{x}_i \longleftrightarrow \boldsymbol{x}_j$, we obtain

$$\Phi^\lambda_D[\xi] = \tfrac{N(N-1)}{4}\lim_{\epsilon\to 0^+}\int_{\partial D^\epsilon_\sigma} d\mathbf{s}\ \phi^2(\mathbf{x})\,\partial_{\boldsymbol{n}}\tfrac{|\mathcal{G}^\lambda\xi|^2}{\phi^2}, \qquad \forall\,\sigma\in\mathscr{P}. \tag{3.60}$$

In order to have an explicit representation of the previous identity, we adopt the change of variables encoded by the unitary operator U_σ, given by (3.17). We stress that for, a generic $f \in H^2(\mathbb{R}^{3N}\setminus\pi)$, one has[3]

$$\int_{D^\epsilon} d\mathbf{x}\ |\nabla f|^2 = \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x} d\boldsymbol{X}_\sigma \int_{|\boldsymbol{r}|>\epsilon} d\boldsymbol{r}\ 2|\nabla_{\boldsymbol{r}} U_\sigma f|^2 + \tfrac{1}{2}|\nabla_{\boldsymbol{x}} U_\sigma f|^2 + |\nabla_{\boldsymbol{X}_\sigma} U_\sigma f|^2 + o(1), \quad \text{as } \epsilon \longrightarrow 0.$$

Therefore, the first Green's identity for the domain D^ϵ reads

$$\begin{aligned} \int_{D^\epsilon} d\mathbf{x}\ |\nabla f|^2 = &-\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x} d\boldsymbol{X}_\sigma \int_{|\boldsymbol{r}|>\epsilon} d\boldsymbol{r}\ (U_\sigma f)\,(2\Delta_{\boldsymbol{r}} U_\sigma f + \tfrac{1}{2}\Delta_{\boldsymbol{x}} U_\sigma f + \Delta_{\boldsymbol{X}_\sigma} U_\sigma f) + \\ &- 2\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x} d\boldsymbol{X}_\sigma \int_{|\boldsymbol{r}|=\epsilon} d\boldsymbol{r}\ U_\sigma f\left(\tfrac{\boldsymbol{r}}{\epsilon}\cdot\nabla_{\boldsymbol{r}} U_\sigma f\right) + o(1)\,, \quad \text{as } \epsilon \longrightarrow 0 \end{aligned}$$

where $-\frac{\boldsymbol{r}}{r}$ is the outer normal derivative in this framework. Hence, equation (3.60) can be represented in terms of these coordinates in the following way

$$\Phi^\lambda_D[\xi] = \tfrac{N(N-1)}{2}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x} d\boldsymbol{X}_\sigma \int_{|\boldsymbol{r}|=\epsilon} d\boldsymbol{r}\ (U_\sigma\phi)^2(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma)\left(-\tfrac{\boldsymbol{r}}{\epsilon}\cdot\nabla_{\boldsymbol{r}}\tfrac{|U_\sigma\mathcal{G}^\lambda\xi|^2}{(U_\sigma\phi)^2}\right). \tag{3.61}$$

Next, we proceed in writing the asymptotic expansion of the integrand for r small. To this end we first provide the following asymptotic behavior for ϕ

$$\begin{aligned} (U_\sigma\phi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma) &= \frac{1}{4\pi}\left(\frac{1}{r} - m + 2\sum_{\ell\notin\sigma}\frac{e^{-m|\boldsymbol{x}-\boldsymbol{x}_\ell|}}{|\boldsymbol{x}-\boldsymbol{x}_\ell|} + \sum_{\substack{1\le k<\ell\le N\\ k\notin\sigma,\,\ell\notin\sigma}}\frac{e^{-m|\boldsymbol{x}_k-\boldsymbol{x}_\ell|}}{|\boldsymbol{x}_k-\boldsymbol{x}_\ell|}\right) + o(1) \\ &=: \frac{1}{4\pi}\left[\frac{1}{r} - m + A_m(\boldsymbol{x},\boldsymbol{X}_\sigma)\right] + o(1)\,, \qquad r \longrightarrow 0^+. \end{aligned} \tag{3.62}$$

Moreover, in order to have a similar expansion for the potential, we take into account representation (3.71) and proposition (3.7), so that we can write the potential in the following way

$$(\mathcal{G}^\lambda\xi)(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) = \sum_{\mathscr{P}\ni\nu=\{k,\ell\}}\tfrac{1}{|\boldsymbol{x}_k-\boldsymbol{x}_\ell|}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p} d\boldsymbol{P}_\nu\ \tfrac{e^{i\boldsymbol{p}\cdot\frac{\boldsymbol{x}_k+\boldsymbol{x}_\ell}{2}+i\boldsymbol{P}_\nu\cdot\boldsymbol{X}_\nu}}{(2\pi)^{3(N-1)/2}}\,\hat{\xi}(\boldsymbol{p},\boldsymbol{P}_\nu)\,e^{-\sqrt{\frac{1}{2}p^2+P_\nu^2+\lambda}\,\frac{|\boldsymbol{x}_k-\boldsymbol{x}_\ell|}{\sqrt{2}}}. \tag{3.63}$$

[3]We are replacing each domain $\partial D^\epsilon_{\sigma=\{i,j\}}$ with $\{(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)\in\mathbb{R}^{3N} \mid |\boldsymbol{x}_i-\boldsymbol{x}_j| = \epsilon\}$ taking into account subleading orders vanishing as ϵ goes to zero.

Applying the change of variables U_σ in identity (3.63) with $\sigma = \{i,j\}$, we can assume ξ sufficiently regular so that one has

$$
\begin{aligned}
(U_\sigma \mathcal{G}^\lambda \xi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma) &= \frac{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}{r} - \frac{1}{\sqrt{2}}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\, d\boldsymbol{P}_\sigma\, \tfrac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}+i\boldsymbol{P}_\sigma\cdot\boldsymbol{X}_\sigma}}{(2\pi)^{3(N-1)/2}}\,\hat{\xi}(\boldsymbol{p},\boldsymbol{P}_\sigma)\sqrt{\tfrac{1}{2}p^2+P_\sigma^2+\lambda}\,+\\
&+\sum_{\substack{\mathscr{P}\ni\nu=\{i,\ell\}\\ \ell\neq j}} \tfrac{1}{|\boldsymbol{x}-\boldsymbol{x}_\ell|}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\, d\boldsymbol{P}_\nu\, \tfrac{e^{i\boldsymbol{p}\cdot\frac{\boldsymbol{x}+\boldsymbol{x}_\ell}{2}+i\boldsymbol{p}_j\cdot\boldsymbol{x}+i\boldsymbol{P}_{\nu\cup\{j\}}\cdot\boldsymbol{X}_{\nu\cup\{j\}}}}{(2\pi)^{3(N-1)/2}}\,\hat{\xi}(\boldsymbol{p},\boldsymbol{P}_\nu)\, e^{-\sqrt{\frac{1}{2}p^2+P_\nu^2+\lambda}\,\frac{|\boldsymbol{x}-\boldsymbol{x}_\ell|}{\sqrt{2}}}+\\
&+\sum_{\substack{\mathscr{P}\ni\nu=\{j,\ell\}\\ \ell\neq i}} \tfrac{1}{|\boldsymbol{x}_\ell-\boldsymbol{x}|}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\, d\boldsymbol{P}_\nu\, \tfrac{e^{i\boldsymbol{p}\cdot\frac{\boldsymbol{x}+\boldsymbol{x}_\ell}{2}+i\boldsymbol{p}_i\cdot\boldsymbol{x}+i\boldsymbol{P}_{\nu\cup\{i\}}\cdot\boldsymbol{X}_{\nu\cup\{i\}}}}{(2\pi)^{3(N-1)/2}}\,\hat{\xi}(\boldsymbol{p},\boldsymbol{P}_\nu)\, e^{-\sqrt{\frac{1}{2}p^2+P_\nu^2+\lambda}\,\frac{|\boldsymbol{x}_\ell-\boldsymbol{x}|}{\sqrt{2}}}+\\
&+\sum_{\substack{\mathscr{P}\ni\nu=\{k,\ell\}\\ \sigma\cap\nu=\emptyset}} \tfrac{1}{|\boldsymbol{x}_k-\boldsymbol{x}_\ell|}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\, d\boldsymbol{P}_\nu\, \tfrac{e^{i\boldsymbol{p}\cdot\frac{\boldsymbol{x}_k+\boldsymbol{x}_\ell}{2}+i(\boldsymbol{p}_i+\boldsymbol{p}_j)\cdot\boldsymbol{x}+i\boldsymbol{P}_{\nu\cup\sigma}\cdot\boldsymbol{X}_{\nu\cup\sigma}}}{(2\pi)^{3(N-1)/2}}\,\hat{\xi}(\boldsymbol{p},\boldsymbol{P}_\nu)\, e^{-\sqrt{\frac{1}{2}p^2+P_\nu^2+\lambda}\,\frac{|\boldsymbol{x}_k-\boldsymbol{x}_\ell|}{\sqrt{2}}}+o(1)\\
&=: \frac{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}{r} + B_\xi^\lambda(\boldsymbol{x},\boldsymbol{X}_\sigma) + o(1)\,, \qquad \text{as } \boldsymbol{r}\longrightarrow\boldsymbol{0}.
\end{aligned}
$$

Elementary calculations yield

$$
\begin{aligned}
\frac{|(U_\sigma\mathcal{G}^\lambda\xi)(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma)|^2}{(U_\sigma\phi)^2(\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma)} = (4\pi)^2\big\{&|\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)|^2 + 2r\,\mathrm{Re}\,\overline{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}B_\xi^\lambda(\boldsymbol{x},\boldsymbol{X}_\sigma)+\\
&+2r\big[m-A_m(\boldsymbol{x},\boldsymbol{X}_\sigma)\big]|\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)|^2+o(r)\big\}.
\end{aligned} \tag{3.64}
$$

Therefore, as $\boldsymbol{r}\longrightarrow\boldsymbol{0}$ one has

$$
(U_\sigma\phi)^2\Big[-\tfrac{\boldsymbol{r}}{r}\cdot\nabla_{\boldsymbol{r}}\tfrac{|(U_\sigma\mathcal{G}^\lambda\xi)|^2}{(U_\sigma\phi)^2}\Big](\boldsymbol{r},\boldsymbol{x},\boldsymbol{X}_\sigma) = \tfrac{2\big[A_m(\boldsymbol{x},\boldsymbol{X}_\sigma)-m\big]|\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)|^2-2\mathrm{Re}\,\overline{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}B_\xi^\lambda(\boldsymbol{x},\boldsymbol{X}_\sigma)}{r^2}+\mathcal{O}\big(\tfrac{1}{r}\big),
$$

hence, for sufficiently regular ξ we have obtained

$$
\Phi_D^\lambda[\xi] = 4\pi N(N-1)\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}\, d\boldsymbol{X}_\sigma\;[A_m(\boldsymbol{x},\boldsymbol{X}_\sigma)-m]|\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)|^2-\mathrm{Re}\,\overline{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}B_\xi^\lambda(\boldsymbol{x},\boldsymbol{X}_\sigma). \tag{3.65}
$$

In particular, notice that the first term in the right hand side of (3.65) coincides with $\Phi_{\text{reg}}[\xi]$. Furthermore, we claim that the second term satisfies

$$
-\mathrm{Re}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}\, d\boldsymbol{X}_\sigma\;\overline{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}B_\xi^\lambda(\boldsymbol{x},\boldsymbol{X}_\sigma) = (\Phi^\lambda_{\text{diag}}+\Phi^\lambda_{\text{off},0}+\Phi^\lambda_{\text{off},1})[\xi]. \tag{3.66}
$$

Let us prove this statement. Clearly, owing to equation (3.46), one has

$$
\frac{1}{\sqrt{2}}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}\, d\boldsymbol{X}_\sigma\;\overline{\xi(\boldsymbol{x},\boldsymbol{X}_\sigma)}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\, d\boldsymbol{P}_\sigma\;\tfrac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}+i\boldsymbol{P}_\sigma\cdot\boldsymbol{X}_\sigma}}{(2\pi)^{3(N-1)/2}}\,\hat{\xi}(\boldsymbol{p},\boldsymbol{P}_\sigma)\sqrt{\tfrac{1}{2}p^2+P_\sigma^2+\lambda} = \Phi^\lambda_{\text{diag}}[\xi].
$$

Then

$$
\begin{aligned}
&\sum_{\substack{\mathscr{P}\ni\nu:\\ |\nu\cap\sigma|=1}}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}\, d\boldsymbol{x}'\, d\boldsymbol{X}\;\overline{\xi(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{X})}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\, d\boldsymbol{p}'\, d\boldsymbol{P}\;\tfrac{e^{i\boldsymbol{p}\cdot\frac{\boldsymbol{x}'+\boldsymbol{x}}{2}+i\boldsymbol{p}'\cdot\boldsymbol{x}+i\boldsymbol{P}\cdot\boldsymbol{X}}\,\hat{\xi}(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{P})\,e^{-\sqrt{\frac{1}{2}p^2+p'^2+P^2+\lambda}\,\frac{|\boldsymbol{x}'-\boldsymbol{x}|}{\sqrt{2}}}}{(2\pi)^{3(N-1)/2}\,|\boldsymbol{x}'-\boldsymbol{x}|}\\
&\qquad = 2(N-2)\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{x}\, d\boldsymbol{x}'\, d\boldsymbol{X}\;\overline{\xi(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{X})}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}\, d\boldsymbol{y}'\, d\boldsymbol{Y}\;\xi(\boldsymbol{y},\boldsymbol{y}',\boldsymbol{Y})\,\overline{F^\lambda\begin{pmatrix}\boldsymbol{x}, & \boldsymbol{x}', & \boldsymbol{X}\\ \boldsymbol{y}, & \boldsymbol{y}', & \boldsymbol{Y}\end{pmatrix}}
\end{aligned}
$$

where, having used Plancherel's theorem, we have set

$$
F^\lambda\begin{pmatrix}\boldsymbol{x}, & \boldsymbol{x}', & \boldsymbol{X}\\ \boldsymbol{y}, & \boldsymbol{y}', & \boldsymbol{Y}\end{pmatrix} := \frac{1}{\sqrt{2}}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p}\, d\boldsymbol{p}'\, d\boldsymbol{P}\;\tfrac{e^{-i\boldsymbol{p}\cdot\left(\boldsymbol{y}-\frac{\boldsymbol{x}'+\boldsymbol{x}}{2}\right)-i\boldsymbol{p}'\cdot(\boldsymbol{y}'-\boldsymbol{x})-i\boldsymbol{P}\cdot(\boldsymbol{Y}-\boldsymbol{X})}}{(2\pi)^{3(N-1)}}\;\hat{f}^\lambda_{\frac{|\boldsymbol{x}'-\boldsymbol{x}|}{\sqrt{2}}}\big(\tfrac{\boldsymbol{p}}{\sqrt{2}},\boldsymbol{p}',\boldsymbol{P}\big), \tag{3.67}
$$

with $\hat{f}^\lambda_x$ defined by (3.73). Therefore,

$$F^\lambda\begin{pmatrix}\boldsymbol{x},\ \boldsymbol{x}',\ \boldsymbol{X}\\ \boldsymbol{y},\ \boldsymbol{y}',\ \boldsymbol{Y}\end{pmatrix} = \frac{2}{(2\pi)^{\frac{3(N-1)}{2}}}\, f^\lambda_{\frac{|\boldsymbol{x}'-\boldsymbol{x}|}{\sqrt{2}}}\Big(\sqrt{2}\,(\boldsymbol{y}-\boldsymbol{x}') - \tfrac{\boldsymbol{x}-\boldsymbol{x}'}{\sqrt{2}}, \boldsymbol{y}'-\boldsymbol{x}, \boldsymbol{Y}-\boldsymbol{X}\Big). \tag{3.68}$$

Notice that, according to (3.74) we have obtained

$$F^\lambda\begin{pmatrix}\boldsymbol{x},\ \boldsymbol{x}',\ \boldsymbol{X}\\ \boldsymbol{y},\ \boldsymbol{y}',\ \boldsymbol{Y}\end{pmatrix} = 8\pi\, G^\lambda\begin{pmatrix}\boldsymbol{x},\ \boldsymbol{x}',\ \boldsymbol{x},\ \boldsymbol{X}\\ \boldsymbol{y},\ \boldsymbol{y},\ \boldsymbol{y}',\ \boldsymbol{Y}\end{pmatrix}. \tag{3.69}$$

Hence, according to identity (3.28c), we have proved that $\Phi^\lambda_{\text{off},1}[\xi]$ corresponds to the sum of the terms in the left hand side of (3.66) involving the couples ν such that $|\nu\cap\sigma| = 1$.
With completely analogous computations one finds out that the sum of the terms involving the couples in $\mathscr{P}$ which do not share any element with σ is equal to $\Phi^\lambda_{\text{off},0}[\xi]$ and equation (3.66) is therefore proven.

□

Addendum

3.A Properties of the Potential

In this section, we focus on studying the operator $\mathcal{G}^\lambda$ introduced in section 3.1. Let us denote by G^λ the kernel of the operator $\mathcal{R}_{\mathcal{H}_0}(-\lambda) \in \mathscr{B}(\mathscr{H}_{N\mathrm{b}}, \mathscr{D}(\mathcal{H}_0))$, with $\lambda > 0$, namely the Green's function associated to the differential operator $(-\Delta+\lambda)\colon H^2(\mathbb{R}^{3N}) \longrightarrow L^2(\mathbb{R}^{3N})$. It is well known that

$$G^\lambda\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \tfrac{1}{(2\pi)^{3N/2}} \left(\tfrac{\sqrt{\lambda}}{|\boldsymbol{x}-\boldsymbol{y}|}\right)^{\frac{3N}{2}-1} \mathrm{K}_{\frac{3N}{2}-1}\Big(\sqrt{\lambda}\,|\boldsymbol{x}-\boldsymbol{y}|\Big), \qquad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^{3N}, \tag{3.70}$$

where $\mathrm{K}_\mu \colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is the modified Bessel function of the second kind of order $\mu \geq 0$.
By definition (3.15), one has

$$\begin{aligned} \langle \psi,\, \mathcal{G}^\lambda \xi\rangle_{\mathscr{H}_{N\mathrm{b}}} &= \sum_{\sigma\in\mathscr{P}} \langle T_\sigma \mathcal{R}_{\mathcal{H}_0}(-\lambda)\,\psi,\, \mathcal{C}_\sigma \xi\rangle_{\mathfrak{X}_\sigma} \\ = 8\pi \sum_{\mathscr{P}\ni\,\sigma=\{i,j\}} &\int_{\mathbb{R}^{3(N-1)}} \! d\boldsymbol{y} d\boldsymbol{Y}_\sigma\, \xi(\boldsymbol{y},\boldsymbol{Y}_\sigma) \int_{\mathbb{R}^{3N}} \! d\boldsymbol{x}_i\, d\boldsymbol{x}_j\, d\boldsymbol{X}_\sigma\, \overline{\psi(\boldsymbol{x}_i,\boldsymbol{x}_j,\boldsymbol{X}_\sigma)}\, G^\lambda \begin{pmatrix} \boldsymbol{x}_i, & \boldsymbol{x}_j, & \boldsymbol{X}_\sigma \\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma \end{pmatrix}. \end{aligned}$$

Therefore, one can deduce the explicit action of $\mathcal{G}^\lambda$

$$\begin{aligned} \big(\mathcal{G}^\lambda \xi\big)(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) &= 8\pi \sum_{\mathscr{P}\ni\,\sigma=\{i,j\}} \int_{\mathbb{R}^{3(N-1)}} \! d\boldsymbol{y} d\boldsymbol{Y}_\sigma\, \xi(\boldsymbol{y},\boldsymbol{Y}_\sigma)\, G^\lambda \begin{pmatrix} \boldsymbol{x}_i, & \boldsymbol{x}_j, & \boldsymbol{X}_\sigma \\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma \end{pmatrix} \\ &=: \sum_{\sigma\in\mathscr{P}} (\mathcal{G}^\lambda_\sigma \mathcal{C}_\sigma \xi)(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N). \end{aligned} \tag{3.71}$$

With a slight abuse of terminology, we refer to $\mathcal{G}^\lambda_\sigma \mathcal{C}_\sigma \xi \in L^2_{\mathrm{sym}}(\mathbb{R}^6, d\boldsymbol{x}_i\, d\boldsymbol{x}_j) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3(N-2)}, d\boldsymbol{X}_\sigma)$ as the "potential" generated by the "charge" $\mathcal{C}_\sigma \xi \in \mathfrak{X}_\sigma$ distributed on π_σ, while $\mathcal{G}^\lambda \xi \in \mathscr{H}_{N\mathrm{b}}$ shall be consequently called the total potential generated by π. Clearly, owing to the bosonic symmetry, each charge $\mathcal{C}_\sigma \xi_\sigma \in \mathfrak{X}_\sigma$ associated to the coincidence hyperplane π_σ is equal to any other charge $\mathcal{C}_\nu \xi_\nu \in \mathfrak{X}_\nu$ distributed along another hyperplane π_ν, in the sense that $\xi_\sigma = \xi_\nu = \xi$ for any $\sigma \neq \nu \in \mathscr{P}$.
One can check that in the Fourier space, equation (3.71) reads

$$(\widehat{\mathcal{G}^\lambda \xi})(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_N) = \sqrt{\tfrac{8}{\pi}} \sum_{\mathscr{P}\ni\,\sigma=\{i,j\}} \frac{\hat{\xi}(\boldsymbol{p}_i + \boldsymbol{p}_j, \boldsymbol{p}_1, \ldots \check{\boldsymbol{p}}_\sigma \ldots, \boldsymbol{p}_N)}{\sum_{k=1}^N p_k^2 + \lambda}. \tag{3.72}$$

Here we state a useful property of G^λ.

Proposition 3.7. *Let G^λ be defined by* (3.70) *and $\sigma = \{i,j\}$. Then, provided $\boldsymbol{x}_i \neq \boldsymbol{x}_j$ one has*

$$\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y} d\boldsymbol{Y}_\sigma \, e^{-i\boldsymbol{q}\cdot\boldsymbol{y}-i\boldsymbol{Q}_\sigma\cdot\boldsymbol{Y}_\sigma} G^\lambda \begin{pmatrix} \boldsymbol{x}_i, & \boldsymbol{x}_j, & \boldsymbol{X}_\sigma \\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma \end{pmatrix} = \frac{e^{-\sqrt{\frac{q^2}{2}+Q_\sigma^2+\lambda}\,\frac{|\boldsymbol{x}_i-\boldsymbol{x}_j|}{\sqrt{2}}}}{8\pi\,|\boldsymbol{x}_i-\boldsymbol{x}_j|} e^{-i\boldsymbol{q}\cdot\frac{\boldsymbol{x}_i+\boldsymbol{x}_j}{2}-i\boldsymbol{Q}_\sigma\cdot\boldsymbol{X}_\sigma}.$$

Proof. Adopting the change of coordinates $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x}_j$, $\boldsymbol{Z}_\sigma = \boldsymbol{Y}_\sigma - \boldsymbol{X}_\sigma$, one has

$$\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y} d\boldsymbol{Y}_\sigma \, e^{-i\boldsymbol{q}\cdot\boldsymbol{y}-i\boldsymbol{Q}_\sigma\cdot\boldsymbol{Y}_\sigma} G^\lambda \begin{pmatrix} \boldsymbol{x}_i, & \boldsymbol{x}_j, & \boldsymbol{X}_\sigma \\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma \end{pmatrix} = e^{-i\boldsymbol{q}\cdot\boldsymbol{x}_j-i\boldsymbol{Q}_\sigma\cdot\boldsymbol{X}_\sigma} \times$$
$$\times \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{z} d\boldsymbol{Z}_\sigma \, e^{-i\boldsymbol{q}\cdot\boldsymbol{z}-i\boldsymbol{Q}_\sigma\cdot\boldsymbol{Z}_\sigma} G^\lambda \begin{pmatrix} \boldsymbol{x}_i-\boldsymbol{x}_j, & \boldsymbol{0}, & \boldsymbol{0} \\ \boldsymbol{z}, & \boldsymbol{z}, & \boldsymbol{Z}_\sigma \end{pmatrix}.$$

Next, take into account the following identities

$$\begin{aligned} G^\lambda \begin{pmatrix} \boldsymbol{x}_i-\boldsymbol{x}_j, & \boldsymbol{0}, & \boldsymbol{0} \\ \boldsymbol{z}, & \boldsymbol{z}, & \boldsymbol{Z}_\sigma \end{pmatrix} &= \frac{1}{(2\pi)^{3N}} \int_{\mathbb{R}^{3N}} d\boldsymbol{p}_i d\boldsymbol{p}_j d\boldsymbol{P}_\sigma \, \frac{e^{i\boldsymbol{p}_i\cdot(\boldsymbol{z}-\boldsymbol{x}_i+\boldsymbol{x}_j)+i\boldsymbol{p}_j\cdot\boldsymbol{z}+i\boldsymbol{P}_\sigma\cdot\boldsymbol{Z}_\sigma}}{p_i^2+p_j^2+P_\sigma^2+\lambda} \\ &= \frac{1}{(2\pi)^{3N}} \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p} d\boldsymbol{P}_\sigma \int_{\mathbb{R}^3} d\boldsymbol{q} \, \frac{e^{i\boldsymbol{p}\cdot\left(\sqrt{2}\boldsymbol{z}-\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{\sqrt{2}}\right)+i\boldsymbol{q}\cdot\left(\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{\sqrt{2}}\right)+i\boldsymbol{P}_\sigma\cdot\boldsymbol{Z}_\sigma}}{p^2+q^2+P_\sigma^2+\lambda} \\ &= \frac{2\sqrt{2}\pi^2}{(2\pi)^{3N}|\boldsymbol{x}_i-\boldsymbol{x}_j|} \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{p} d\boldsymbol{P}_\sigma \, e^{i\boldsymbol{p}\cdot\left(\sqrt{2}\boldsymbol{z}-\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{\sqrt{2}}\right)+i\boldsymbol{P}_\sigma\cdot\boldsymbol{Z}_\sigma} e^{-\sqrt{p^2+P_\sigma^2+\lambda}\,\frac{|\boldsymbol{x}_i-\boldsymbol{x}_j|}{\sqrt{2}}}, \end{aligned}$$

where we have set $(\boldsymbol{p}_i, \boldsymbol{p}_j) = \left(\frac{\boldsymbol{p}-\boldsymbol{q}}{\sqrt{2}}, \frac{\boldsymbol{p}+\boldsymbol{q}}{\sqrt{2}}\right)$ and exploited (2.135). Thus, if we define the function

$$\begin{aligned} \hat{f}_x^\lambda : \mathbb{R}^3 \otimes \mathbb{R}^{3(N-2)} &\longrightarrow \mathbb{R}_+, \qquad x, \lambda > 0, \\ (\boldsymbol{p}, \boldsymbol{P}) &\longmapsto \frac{e^{-\sqrt{p^2+P^2+\lambda}\,x}}{x}, \end{aligned} \tag{3.73}$$

we can write

$$G^\lambda \begin{pmatrix} \boldsymbol{x}_i-\boldsymbol{x}_j, & \boldsymbol{0}, & \boldsymbol{0} \\ \boldsymbol{z}, & \boldsymbol{z}, & \boldsymbol{Z}_\sigma \end{pmatrix} = \frac{2\pi^2}{(2\pi)^{\frac{3}{2}(N+1)}} f^\lambda_{\frac{|\boldsymbol{x}_i-\boldsymbol{x}_j|}{\sqrt{2}}} \left(\sqrt{2}\boldsymbol{z} - \tfrac{\boldsymbol{x}_i-\boldsymbol{x}_j}{\sqrt{2}}, \boldsymbol{Z}_\sigma\right). \tag{3.74}$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{z} d\boldsymbol{Z}_\sigma \, e^{-i\boldsymbol{q}\cdot\boldsymbol{z}-i\boldsymbol{Q}_\sigma\cdot\boldsymbol{Z}_\sigma} G^\lambda \begin{pmatrix} \boldsymbol{x}_i-\boldsymbol{x}_j, & \boldsymbol{0}, & \boldsymbol{0} \\ \boldsymbol{z}, & \boldsymbol{z}, & \boldsymbol{Z}_\sigma \end{pmatrix} = \\ &\qquad = \frac{\pi^2 e^{-i\boldsymbol{q}\cdot\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{2}}}{\sqrt{2}(2\pi)^{\frac{3}{2}(N+1)}} \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{s} d\boldsymbol{Z}_\sigma \, e^{-i\boldsymbol{q}\cdot\frac{\boldsymbol{s}}{\sqrt{2}}-i\boldsymbol{Q}_\sigma\cdot\boldsymbol{Z}_\sigma} f^\lambda_{\frac{|\boldsymbol{x}_i-\boldsymbol{x}_j|}{\sqrt{2}}}(\boldsymbol{s}, \boldsymbol{Z}_\sigma) \\ &\qquad = \frac{\pi^2 e^{-i\boldsymbol{q}\cdot\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{2}}}{\sqrt{2}(2\pi)^3} \hat{f}^\lambda_{\frac{|\boldsymbol{x}_i-\boldsymbol{x}_j|}{\sqrt{2}}}\left(\tfrac{\boldsymbol{q}}{\sqrt{2}}, \boldsymbol{Q}_\sigma\right) = \frac{e^{-\sqrt{\frac{q^2}{2}+Q_\sigma^2+\lambda}\,\frac{|\boldsymbol{x}_i-\boldsymbol{x}_j|}{\sqrt{2}}}}{8\pi\,|\boldsymbol{x}_i-\boldsymbol{x}_j|} e^{-i\boldsymbol{q}\cdot\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{2}}. \end{aligned}$$

□

In particular, notice that proposition 3.7 implies

$$\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y} d\boldsymbol{Y}_\sigma \, G^\lambda \begin{pmatrix} \boldsymbol{x}_i, & \boldsymbol{x}_j, & \boldsymbol{X}_\sigma \\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{Y}_\sigma \end{pmatrix} = \frac{e^{-\sqrt{\frac{\lambda}{2}}\,|\boldsymbol{x}_i-\boldsymbol{x}_j|}}{8\pi\,|\boldsymbol{x}_i-\boldsymbol{x}_j|}, \qquad \boldsymbol{x}_i \neq \boldsymbol{x}_j. \tag{3.75}$$

Lastly, let us state the following proposition.

Proposition 3.8. *Given* $\xi \in \mathcal{S}(\mathbb{R}^{3(N-1)})$ *and* $\sigma = \{i,j\} \in \mathscr{P}$ *one has*

$$I(\boldsymbol{p},\boldsymbol{p}_i,\boldsymbol{p}_j,\boldsymbol{P}_\sigma) := \mathcal{F}\int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{y}_i d\boldsymbol{y}_j\, d\boldsymbol{Y}_\sigma\; \xi(\boldsymbol{y},\boldsymbol{y}_i,\boldsymbol{y}_j,\boldsymbol{Y}_\sigma)\, G^\lambda\begin{pmatrix}\boldsymbol{x}_i, & \boldsymbol{x}_j, & \boldsymbol{x}, & \boldsymbol{x}, & \boldsymbol{X}_\sigma\\ \boldsymbol{y}, & \boldsymbol{y}, & \boldsymbol{y}_i, & \boldsymbol{y}_j, & \boldsymbol{Y}_\sigma\end{pmatrix}$$
$$= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d\boldsymbol{q}\; \frac{\hat{\xi}\big(\boldsymbol{p}_i+\boldsymbol{p}_j, \frac{\boldsymbol{p}}{2}+\boldsymbol{q}, \frac{\boldsymbol{p}}{2}-\boldsymbol{q}, \boldsymbol{P}_\sigma\big)}{\frac{1}{2}p^2 + 2q^2 + p_i^2 + p_j^2 + P_\sigma^2 + \lambda}.$$

Proof. Because of the regularity of ξ and equation (3.75), Fubini's theorem applies, therefore taking account of proposition 3.7

$$I(\boldsymbol{p},\boldsymbol{p}_i,\boldsymbol{p}_j,\boldsymbol{P}_\sigma) =$$
$$= \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{y}_i d\boldsymbol{y}_j\, d\boldsymbol{Y}_\sigma\; \frac{e^{-i\boldsymbol{p}\cdot\frac{\boldsymbol{y}_i+\boldsymbol{y}_j}{2} - i(\boldsymbol{p}_i+\boldsymbol{p}_j)\cdot\boldsymbol{y} - i\boldsymbol{P}_\sigma\cdot\boldsymbol{Y}_\sigma - \sqrt{\frac{1}{2}p^2+p_i^2+p_j^2+P_\sigma^2+\lambda}\;\frac{|\boldsymbol{y}_i-\boldsymbol{y}_j|}{\sqrt{2}}}}{8\pi\,(2\pi)^{\frac{3(N-1)}{2}}|\boldsymbol{y}_i-\boldsymbol{y}_j|}\,\xi(\boldsymbol{y},\boldsymbol{y}_i,\boldsymbol{y}_j,\boldsymbol{Y}_\sigma)$$
$$= \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{r}d\boldsymbol{s}\, d\boldsymbol{Y}_\sigma\; \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{s} - i(\boldsymbol{p}_i+\boldsymbol{p}_j)\cdot\boldsymbol{y} - i\boldsymbol{P}_\sigma\cdot\boldsymbol{Y}_\sigma - \sqrt{\frac{1}{2}p^2+p_i^2+p_j^2+P_\sigma^2+\lambda}\;\frac{r}{\sqrt{2}}}}{8\pi\,(2\pi)^{\frac{3(N-1)}{2}}\,r}\,\xi\big(\boldsymbol{y},\boldsymbol{s}+\tfrac{\boldsymbol{r}}{2},\boldsymbol{s}-\tfrac{\boldsymbol{r}}{2},\boldsymbol{Y}_\sigma\big)$$
$$= \int_{\mathbb{R}^{3(N-1)}} d\boldsymbol{y}d\boldsymbol{s}\, d\boldsymbol{Y}_\sigma \int_{\mathbb{R}^6} d\boldsymbol{q}d\boldsymbol{t}\; \frac{e^{-i\boldsymbol{q}\cdot\boldsymbol{t} - i\boldsymbol{p}\cdot\boldsymbol{s} - i(\boldsymbol{p}_i+\boldsymbol{p}_j)\cdot\boldsymbol{y} - i\boldsymbol{P}_\sigma\cdot\boldsymbol{Y}_\sigma}}{(2\pi)^{\frac{3(N-1)}{2}+3}}\;\frac{\xi\big(\boldsymbol{y},\boldsymbol{s}+\frac{\boldsymbol{t}}{2},\boldsymbol{s}-\frac{\boldsymbol{t}}{2},\boldsymbol{Y}_\sigma\big)}{2q^2+\frac{1}{2}p^2+p_i^2+p_j^2+P_\sigma^2+\lambda}$$

where in the last step we have used Plancherel's theorem and the well known identity

$$\int_{\mathbb{R}^3} d\boldsymbol{x}\; \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x} - a\,x}}{x} = \frac{4\pi}{a^2+q^2}, \qquad \forall\, a>0, \boldsymbol{q}\in\mathbb{R}^3. \tag{3.76}$$

One obtains the result by changing the coordinates $\boldsymbol{s}+\frac{\boldsymbol{t}}{2} \longmapsto \boldsymbol{u}$, $\boldsymbol{s}-\frac{\boldsymbol{t}}{2} \longmapsto \boldsymbol{v}$. Indeed

$$I(\boldsymbol{p},\boldsymbol{p}_i,\boldsymbol{p}_j,\boldsymbol{P}_\sigma) =$$
$$= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^{3(N+1)}} d\boldsymbol{q}d\boldsymbol{y}d\boldsymbol{u}\,d\boldsymbol{v}\,d\boldsymbol{Y}_\sigma\; \frac{e^{-i\left(\frac{\boldsymbol{p}}{2}+\boldsymbol{q}\right)\cdot\boldsymbol{u} - i\left(\frac{\boldsymbol{p}}{2}-\boldsymbol{q}\right)\cdot\boldsymbol{v} - i(\boldsymbol{p}_i+\boldsymbol{p}_j)\cdot\boldsymbol{y} - i\boldsymbol{P}_\sigma\cdot\boldsymbol{Y}_\sigma}\,\xi(\boldsymbol{y},\boldsymbol{u},\boldsymbol{v},\boldsymbol{Y}_\sigma)}{(2\pi)^{\frac{3(N-1)}{2}}\left(2q^2+\frac{1}{2}p^2+p_i^2+p_j^2+P_\sigma^2+\lambda\right)}$$
$$= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d\boldsymbol{q}\; \frac{\hat{\xi}\big(\boldsymbol{p}_i+\boldsymbol{p}_j, \frac{\boldsymbol{p}}{2}+\boldsymbol{q}, \frac{\boldsymbol{p}}{2}-\boldsymbol{q}, \boldsymbol{P}_\sigma\big)}{2q^2+\frac{1}{2}p^2+p_i^2+p_j^2+P_\sigma^2+\lambda}.$$

□

A. Self-Adjoint Extensions of Restrictions

In this section, we summarize some known results obtained in [31] and [32] (see [6] for further information on this abstract setting).
Let $\mathfrak{H}$ be a complex Hilbert space and $A : \mathscr{D}(A) \subseteq \mathfrak{H} \longrightarrow \mathfrak{H}$ a s.a. operator. Endowing $\mathscr{D}(A)$ with the graph norm of A, we treat $\mathscr{D}(A)$ as a Hilbert subspace. Then, we consider a densely defined, closed and symmetric operator $S : \mathscr{D}(S) \subset \mathscr{D}(A) \longrightarrow \mathfrak{H}$ such that $A|_{\mathscr{D}(S)} = S$. Under these prescriptions, we shall provide the characterization of the whole set of s.a. extensions of S.
To this end, let $\tau : \mathscr{D}(A) \longrightarrow \mathcal{X}$ be a linear and continuous operator in an auxiliary complex Hilbert space $\mathcal{X}$, such that $\overline{\operatorname{ran}(\tau)} = \mathcal{X}$ and $\ker(\tau) = \mathscr{D}(S)$.
Then, for each $z \in \rho(A)$, we define the bounded operator $\mathcal{G}(z) := (\tau \mathcal{R}_A(\bar{z}))^* \in \mathscr{B}(\mathcal{X}, \mathfrak{H})$.

Remark A.1. *The operator $\mathcal{G}(z)$ fulfils the following properties.*

i) *$\mathcal{G}(z)$ is injective, since* $\ker(\mathcal{G}(z)) = \operatorname{ran}(\tau)^\perp = \{0\}$.

ii) *Given a compact $K \subset \rho(A)$, there exists $C > 0$ s.t. $\|\mathcal{G}(z)\|_{\mathscr{L}(\mathcal{X},\mathfrak{H})} \leq C$ uniformly in $z \in K$, since τ is continuous and in general* $\|\mathcal{R}_A(w)\|_{\mathscr{L}(\mathfrak{H},\mathscr{D}(A))} \leq \operatorname{dist}(w, \sigma(A))^{-1} \quad \forall\, w \in \rho(A)$.

iii) *For any $z \in \rho(A)$,* $\operatorname{ran}(\mathcal{G}(z)) \cap \mathscr{D}(A) = \{0\}$ *due to the density of* $\ker(\tau)$ *in* $\mathscr{D}(A)$.

Indeed, given $\psi \in \ker(\tau)$, one has

$$0 = \langle \phi,\, \tau\psi \rangle_{\mathcal{X}} = \langle \phi,\, \tau \mathcal{R}_A(\bar{z})(A - \bar{z})\psi \rangle_{\mathcal{X}} = \langle \mathcal{G}(z)\phi,\, (A - \bar{z})\psi \rangle_{\mathfrak{H}}, \qquad \forall\, \phi \in \mathcal{X}.$$

By way of contradiction, assume that exists $\phi \neq 0$ such that $\mathcal{G}(z)\phi \in \mathscr{D}(A)$. Then,

$$0 = \langle (A - z)\mathcal{G}(z)\phi,\, \psi \rangle_{\mathfrak{H}}, \qquad \forall\, \psi \in \ker(\tau)$$

that means $(A - z)\mathcal{G}(z)\phi \in \ker(\tau)^\perp$, namely $(A - z)\mathcal{G}(z)\phi = 0$. Since $z \in \rho(A)$, the only solution to the previous equation is $\mathcal{G}(z)\phi = 0$ that implies $\phi = 0$, which is absurd.

Another important property of the operator $\mathcal{G}(z)$ is proved in the following proposition.

Proposition A.1. *Let $z, w \in \rho(A)$ and $\phi_1, \phi_2 \in \mathcal{X}$. Then, if* $\operatorname{ran}(\tau)$ *is dense in* $\mathcal{X}$

$$\mathcal{G}(z)\phi_1 - \mathcal{G}(w)\phi_2 \in \mathscr{D}(A) \iff \phi_1 = \phi_2 .$$

Proof. $\Leftarrow)$ Applying the operator τ in the first resolvent identity for the operator A, one gets

$$(\bar{z} - \bar{w})\mathcal{G}(w)^* \mathcal{R}_A(\bar{z}) = \mathcal{G}(z)^* - \mathcal{G}(w)^*, \tag{A.1a}$$

$$(z-w)\mathcal{R}_A(z)\mathcal{G}(w) = \mathcal{G}(z) - \mathcal{G}(w) \tag{A.1b}$$

and therefore $\operatorname{ran}(\mathcal{G}(z) - \mathcal{G}(w)) \in \mathscr{D}(A)$.

$\Rightarrow$) Rewriting $\mathcal{G}(z)$ according to equation (A.1b), one obtains

$$\mathcal{G}(z)\phi_1 - \mathcal{G}(w)\phi_2 = \mathcal{G}(w)(\phi_1 - \phi_2) + (z-w)\mathcal{R}_A(z)\mathcal{G}(w)\phi_1 .$$

By hypothesis, the left hand side belongs to $\mathscr{D}(A)$ and the same is true for the last term in the right hand side, whereas $\operatorname{ran}(\mathcal{G}(w)) \cap \mathscr{D}(A) = \{0\}$ thanks to point iii) of remark A.1. These facts imply $\phi_1 - \phi_2 \in \ker(\mathcal{G}(w)) = \{0\}$.

□

Notice that in the previous proposition the implication "$\Leftarrow$)" holds true even if $\operatorname{ran}(\tau)$ is not dense. Lastly, we consider a continuous map $\Gamma\colon \rho(A) \longrightarrow \mathscr{L}(\mathcal{X})$ satisfying

$$\Gamma : z \longmapsto \Gamma(z)\colon D \subseteq \mathcal{X} \longrightarrow \mathcal{X} \text{ is a densely defined operator,} \tag{A.2a}$$

$$\Gamma : z \longmapsto \Gamma(z) = \Gamma(\bar{z})^*, \tag{A.2b}$$

$$\Gamma(z) - \Gamma(w) = (w-z)\mathcal{G}(\bar{w})^*\mathcal{G}(z) = (w-z)\mathcal{G}(\bar{z})^*\mathcal{G}(w), \quad \forall\, w, z \in \rho(A), \tag{A.2c}$$

$$\exists\, z \in \rho(A) \;:\; 0 \in \rho(\Gamma(z)). \tag{A.2d}$$

Remark A.2. *Observe that condition* (A.2b) *makes the operator* $\Gamma(z)$ *closed for any* $z \in \rho(A)$*. Moreover, by condition* (A.2c)*, one has that the domain* D *is independent of* $z \in \rho(A)$ *and the map* Γ *is actually locally Lipschitz continuous, i.e.* (see point ii) of remark A.1)

$$\|\Gamma(w) - \Gamma(z)\|_{\mathscr{L}(\mathcal{X})} \leq C|w-z|, \qquad \forall\, w, z \in K \subset \rho(A)$$

with some K *compact and* C *positive. In particular, notice that for any* $w, z \in \rho(A)$*, one has that the difference* $\Gamma(w) - \Gamma(z) \in \mathscr{B}(\mathcal{X})$*, so that any unbounded contribution of* $\Gamma(z)$ *cannot depend on* z*.*

In the next proposition, we establish a sufficient condition to check the validity of properties (A.2).

Proposition A.2. *Let* $\lambda \in \rho(A) \cap \mathbb{R}$ *and* $\Gamma_\lambda\colon D \subset \mathcal{X} \longrightarrow \mathcal{X}$ *a s.a. operator satisfying*

i) $\Gamma_{\lambda_1} - \Gamma_{\lambda_2} = \tau(\mathcal{G}(\lambda_2) - \mathcal{G}(\lambda_1)), \qquad \forall\, \lambda_1, \lambda_2 \in \rho(A) \cap \mathbb{R}$,

ii) $\exists\, \lambda_0 \in \rho(A) \cap \mathbb{R} \;:\; \Gamma_{\lambda_0}$ *is surjective.*

Then, the map

$$z \longmapsto \Gamma(z) := \Gamma_{\lambda_0} + (\lambda_0 - z)\mathcal{G}(\lambda_0)^*\mathcal{G}(z)$$

fulfils conditions (A.2).

Proof. Condition (A.2a) is automatically satisfied due to the self-adjointness of Γ_λ.
Applying the operator τ to (A.1b), one gets for all $z, w \in \rho(A)$

$$\tau[\mathcal{G}(z) - \mathcal{G}(w)] = (z-w)\mathcal{G}(\bar{z})^*\mathcal{G}(w) = (z-w)\mathcal{G}(\bar{w})^*\mathcal{G}(z). \tag{A.3}$$

In particular, this means that

$$\Gamma_{\lambda_1} - \Gamma_{\lambda_2} = (\lambda_2 - \lambda_1)\mathcal{G}(\lambda_2)^*\mathcal{G}(\lambda_1) = (\lambda_2 - \lambda_1)\mathcal{G}(\lambda_1)^*\mathcal{G}(\lambda_2), \qquad \forall\, \lambda_1, \lambda_2 \in \rho(A) \cap \mathbb{R}, \tag{A.4}$$

and, moreover

$$\Gamma(z) = \Gamma_{\lambda_0} + (\lambda_0 - z)\mathcal{G}(\bar{z})^*\mathcal{G}(\lambda_0). \tag{A.5}$$

Thus, (A.2b) can be proven, since

$$\Gamma(\bar{z})^* - \Gamma_{\lambda_0} = (\lambda_0 - z)\mathcal{G}(\bar{z})^*\mathcal{G}(\lambda_0) = (\lambda_0 - z)\mathcal{G}(\lambda_0)^*\mathcal{G}(z) = \Gamma(z) - \Gamma_{\lambda_0}.$$

Next, concerning condition (A.2c), we use equations (A.1), the definition of $\Gamma(z)$ and identity (A.5)

$$\begin{aligned}
\Gamma(z) - \Gamma(w) &= \Gamma_\lambda + (\lambda - z)\mathcal{G}(\bar{z})^*\mathcal{G}(\lambda) - \Gamma_\lambda + (w - \lambda)\mathcal{G}(\lambda)^*\mathcal{G}(w) \\
&= (\lambda - z)\mathcal{G}(\bar{z})^*\big[\mathcal{G}(w) + (\lambda - w)\mathcal{R}_A(\lambda)\mathcal{G}(w)\big] + \\
&\quad + (w - \lambda)\big[\mathcal{G}(\bar{z})^* + (\lambda - z)\mathcal{G}(\bar{z})^*\mathcal{R}_A(\lambda)\big]\mathcal{G}(w) \\
&= (w - z)\mathcal{G}(\bar{z})^*\mathcal{G}(w).
\end{aligned}$$

Finally, we show that $\Gamma(z)$ is invertible with bounded inverse for any $z \in \mathbb{C} \smallsetminus \mathbb{R}$. According to (A.2c)

$$\Gamma(\bar{z}) - \Gamma(z) = (z - \bar{z})\mathcal{G}(z)^*\mathcal{G}(z),$$

while condition (A.2b) implies

$$\langle \xi,\, [\Gamma(\bar{z}) - \Gamma(z)]\,\xi\rangle_{\mathcal{X}} = -2i\,\mathrm{Im}\,\langle \xi,\, \Gamma(z)\xi\rangle_{\mathcal{X}}, \qquad \forall\, \xi \in D.$$

Hence, one obtains

$$\mathrm{Im}\,\langle \xi,\, \Gamma(z)\xi\rangle_{\mathcal{X}} = -\mathrm{Im}(z)\,\|\mathcal{G}(z)\xi\|^2_{\mathfrak{H}}, \qquad \forall\, \xi \in D. \tag{A.6}$$

Therefore, since $\Gamma(z) + \Gamma(\bar{z})$ is s.a., one gets for any $\xi \in D$

$$\begin{aligned}
\|\xi\|^2_{\mathcal{X}}\,\|\Gamma(z)\xi\|^2_{\mathcal{X}} \geq |\langle \xi,\, \Gamma(z)\xi\rangle_{\mathcal{X}}|^2 &= |\langle \xi,\, \tfrac{\Gamma(z)+\Gamma(\bar{z})}{2}\,\xi\rangle_{\mathcal{X}}|^2 + |\langle \xi,\, \tfrac{\Gamma(z)-\Gamma(\bar{z})}{2}\,\xi\rangle_{\mathcal{X}}|^2 \\
&= |\langle \xi,\, \tfrac{\Gamma(z)+\Gamma(\bar{z})}{2}\,\xi\rangle_{\mathcal{X}}|^2 + (\mathrm{Im}\,z)^2\,\|\mathcal{G}(z)\xi\|^4_{\mathfrak{H}} \geq (\mathrm{Im}\,z)^2\,\|\mathcal{G}(z)\xi\|^4_{\mathfrak{H}}.
\end{aligned}$$

In other words, $\Gamma(z)$ is injective for all $z \in \mathbb{C} \smallsetminus \mathbb{R}$. Moreover, $\Gamma(z)$ is also surjective because of the surjectivity of Γ_{λ_0} and therefore it is invertible with bounded inverse.

□

One of the main results of [31] and [32] is that the choice of τ and Γ identifies in an exhaustive way every s.a. extension of S, denoted by A^τ_Γ. Indeed, for any $z \in \rho(A)$ satisfying (A.2d), namely such that $\Gamma(z)^{-1} \in \mathscr{B}(\mathcal{X})$, the quantity

$$R^\tau_\Gamma(z) := \mathcal{R}_A(z) + \mathcal{G}(z)\Gamma(z)^{-1}\mathcal{G}(\bar{z})^* \tag{A.7}$$

defines the resolvent of a s.a. operator $A^\tau_\Gamma := R^\tau_\Gamma(z)^{-1} + z$ that does not depend on z and coincides with A on $\ker(\tau)$. More precisely, it is characterized by

$$\begin{cases} \mathscr{D}(A^\tau_\Gamma) := \big\{\phi \in \mathfrak{H} \,\big|\, \phi = \phi_z + \mathcal{G}(z)\Gamma(z)^{-1}\tau\phi_z\,,\ \phi_z \in \mathscr{D}(A)\big\}, \\ (A^\tau_\Gamma - z)\,\phi = (A - z)\,\phi_z\,. \end{cases} \tag{A.8}$$

Notice that, given $z \in \rho(A)$ s.t. condition (A.2d) is fulfilled, the decomposition of an element in $\mathscr{D}(A^\tau_\Gamma)$ is unique because of point $iii)$ of remark A.1.

Equivalently, the domain of the operator A^τ_Γ can be represented in terms of a proper boundary condition. Indeed, introducing the quantity $\xi := \Gamma(z)^{-1}\tau\phi_z \in \mathcal{X}$, it is straightforward to see that

$$\begin{cases} \mathscr{D}(A^\tau_\Gamma) = \big\{\phi \in \mathfrak{H} \,\big|\, \phi = \phi_z + \mathcal{G}(z)\xi,\ \Gamma(z)\xi = \tau\phi_z\,,\ \phi_z \in \mathscr{D}(A),\ \xi \in D\big\}, \\ A^\tau_\Gamma\phi = A\phi_z + z\mathcal{G}(z)\xi. \end{cases} \tag{A.9}$$

We stress that the definition of ξ does not depend on $z \in \rho(A)$. Indeed, the uniqueness of the decomposition of an element in $\mathscr{D}(A^\tau_\Gamma)$ implies

$$\phi_z - \phi_w = \mathcal{G}(w)\xi_w - \mathcal{G}(z)\xi_z\,, \qquad \forall\, w, z \in \rho(A) \ \text{satisfying (A.2d)}.$$

Since the left hand side belongs to $\mathscr{D}(A)$, according to proposition A.1 one has $\xi_z = \xi_w$, namely

$$\Gamma(z)^{-1}\tau\phi_z = \Gamma(w)^{-1}\tau\phi_w\,.$$

One can verify that an equivalent z-independent way to characterize A^τ_Γ is the following

$$\begin{cases} \mathscr{D}(A^\tau_\Gamma) = \big\{\phi \in \mathfrak{H} \,\big|\, \exists!\ \varphi \in \mathfrak{H} :\ \tfrac{\mathrm{d}}{\mathrm{d}z}\,\Gamma(z)^{-1}\tau\mathcal{R}_A(z)(\varphi - z\phi) \equiv 0\big\}, \\ A^\tau_\Gamma\phi = \varphi. \end{cases} \tag{A.10}$$

It is worth evaluating the quadratic form associated to the s.a. operator A^τ_Γ. In particular, one has for any $\phi \in \mathscr{D}(A^\tau_\Gamma)$

$$\begin{aligned} \langle\phi,\, A^\tau_\Gamma\phi\rangle_{\mathfrak{H}} &= \langle\phi,\, A\phi_z + z\,\mathcal{G}(z)\xi\rangle_{\mathfrak{H}} = z\,\|\phi\|^2_{\mathfrak{H}} + \langle\phi,\, (A - z)\,\phi_z\rangle_{\mathfrak{H}} \\ &= z\,\|\phi\|^2_{\mathfrak{H}} + \langle\phi_z,\, (A - z)\,\phi_z\rangle_{\mathfrak{H}} + \langle\mathcal{G}(z)\xi,\, (A - z)\,\phi_z\rangle_{\mathfrak{H}} \\ &= z\,\|\phi\|^2_{\mathfrak{H}} + \langle\phi_z,\, (A - z)\,\phi_z\rangle_{\mathfrak{H}} + \langle\xi,\, \tau\phi_z\rangle_{\mathcal{X}}. \end{aligned}$$

Therefore, the boundary condition implies

$$\langle\phi,\, A^\tau_\Gamma\phi\rangle_{\mathfrak{H}} = z\,\|\phi\|^2_{\mathfrak{H}} + \langle\phi_z,\, (A - z)\,\phi_z\rangle_{\mathfrak{H}} + \langle\xi,\, \Gamma(z)\xi\rangle_{\mathcal{X}}\,. \tag{A.11}$$

Remarkably, since the left hand side of the previous equation is real, one has

$$\mathrm{Im}(z)\,\|\phi_z + \mathcal{G}(z)\xi\|_{\mathfrak{H}}^2 - \mathrm{Im}(z)\,\|\phi_z\|_{\mathfrak{H}}^2 + \mathrm{Im}\langle \xi,\, \Gamma(z)\xi\rangle = 0.$$

Thus, thanks to (A.6) one obtains $\phi = \phi_z + \mathcal{G}(z)\xi \in \mathscr{D}(A_\Gamma^\tau)$ implies

$$\mathrm{Im}(z) = 0 \quad \vee \quad \phi_z \perp \mathcal{G}(z)\xi. \tag{A.12}$$

We conclude this section by providing a revised version of [31, corollary 2.1].

Proposition A.3. *Let A be a s.a. operator in $\mathfrak{H}$ and $\Gamma : \rho(A) \longrightarrow \mathscr{L}(\mathcal{X})$ a continuous map satisfying* (A.2)*. Suppose that there exist $\lambda \in \mathbb{R}$ and $\mu_0 \leq \lambda$ such that*
i) $A - \lambda > 0$*,*
ii) $\Gamma(\mu) > 0, \qquad \forall\, \mu < \mu_0.$
Then, the operator A_Γ^τ defined by (A.8) *satisfies $A_\Gamma^\tau \geq \mu_0$ for any $\tau : \mathscr{D}(A) \longrightarrow \mathcal{X}$ linear bounded operator s.t. $\overline{\mathrm{ran}(\tau)} = \mathcal{X}$ and $\overline{\ker(\tau)} = \mathscr{D}(A)$.*

Proof. Taking into account identity (A.11), one has for all $\phi \in \mathscr{D}(A_\Gamma^\tau)$ and $\mu < \mu_0$

$$\begin{aligned}\langle \phi,\, A_\Gamma^\tau \phi\rangle_{\mathfrak{H}} &= \mu\,\|\phi\|_{\mathfrak{H}}^2 + \langle \phi_\mu,\, (A-\mu)\,\phi_\mu\rangle_{\mathfrak{H}} + \langle \xi,\, \Gamma(\mu)\xi\rangle_{\mathcal{X}} \\ &> \mu\,\|\phi\|_{\mathfrak{H}}^2 + \langle \phi_\mu,\, (A-\lambda)\,\phi_\mu\rangle_{\mathfrak{H}} > \mu\,\|\phi\|_{\mathfrak{H}}^2\,.\end{aligned}$$

□

B. Technical Facts

In this appendix we prove some technical detail used in the text.

Proposition B.1. *Let* $f\colon[-1,1]\longrightarrow\mathbb{R}$ *be an analytic function whose Taylor expansion*

$$f(x)=\sum_{n\in\mathbb{N}_0}c_n x^n,\qquad c_n\geq 0$$

has a radius of convergence equal to 1*. Defining*

$$a_\ell:=\int_{-1}^{1}dy\;P_\ell(y)f(y),\qquad \ell\in\mathbb{N}_0,$$

then, for all $\ell\in\mathbb{N}_0$ *one has* $a_\ell\geq 0$ *and* $a_\ell\geq a_{\ell+2}$.

Proof. The proof can be found e.g. in [5, lemma 3.3], but we provide the details for easier reading. Notice that, since the radius of convergence is 1, one can use Fubini's theorem to get

$$a_\ell=\sum_{n\in\mathbb{N}_0}c_n\int_{-1}^{1}dy\;P_\ell(y)y^n=\frac{1}{2^\ell\,\ell!}\sum_{n\in\mathbb{N}_0}c_n\int_{-1}^{1}dy\;y^n\,\frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell}(y^2-1)^\ell.$$

Integrating by parts ℓ times one achieves the non-negativity of a_ℓ

$$a_\ell=\frac{1}{2^\ell\,\ell!}\sum_{n\in\mathbb{N}_0}c_n\int_{-1}^{1}dy\;(1-y^2)^\ell\,\frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell}y^n\geq 0.$$

Next, considering

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}y^2}(y^2-1)^{\ell+2}&=-2(\ell+2)(y^2-1)^\ell+2(\ell+2)(2\ell+3)(y^2-1)^\ell y^2\\&=4(\ell+2)(\ell+1)(y^2-1)^\ell+2(\ell+2)(2\ell+3)(y^2-1)^{\ell+1},\end{aligned}$$

the monotonicity property is recovered, since

$$\begin{aligned}a_{\ell+2}&=\frac{(-1)^\ell}{2^{\ell+2}(\ell+2)!}\sum_{n\in\mathbb{N}_0}c_n\int_{-1}^{1}dy\;\frac{\mathrm{d}^2}{\mathrm{d}y^2}(y^2-1)^{\ell+2}\left(\frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell}y^n\right)\\&=a_\ell-\frac{(2\ell+3)}{2^{\ell+1}(\ell+1)!}\sum_{n\in\mathbb{N}_0}c_n\int_{-1}^{1}dy\;(1-y^2)^{\ell+1}\frac{\mathrm{d}^\ell}{\mathrm{d}y^\ell}y^n\leq a_\ell.\end{aligned}$$

□

Proposition B.2. *Consider a function* $f\colon\mathbb{R}^2_+\times[-1,1]\longrightarrow\mathbb{C}$ *such that* *i)* $f(p,q,\cdot)\in L^2([-1,1])$ *for almost every* p *and* q *in* $\mathbb{R}_+$,

ii) the quantity $F(p,q):=\displaystyle\int_{-1}^{1}du\ f(p,q,u)$ *is in* $L^1(\mathbb{R}_+\times\mathbb{R}_+,\,p^2q^2\,dp\,dq)$.

Then, one has

$$\int_{\mathbb{R}^6}d\boldsymbol{p}\,d\boldsymbol{q}\ f\Big(p,q,\tfrac{\boldsymbol{p}\cdot\boldsymbol{q}}{pq}\Big)=8\pi^2\int_0^{+\infty}dp\,p^2\int_0^{+\infty}dq\,q^2\,F(p,q).$$

Proof. Recalling that one has $Y_0^0(\hat{\boldsymbol{\omega}})\equiv\frac{1}{\sqrt{4\pi}}$, we can use spherical coordinates to obtain

$$\int_{\mathbb{R}^6}d\boldsymbol{p}\,d\boldsymbol{q}\ f\Big(p,q,\tfrac{\boldsymbol{p}\cdot\boldsymbol{q}}{pq}\Big)=4\pi\int_0^{+\infty}dp\,p^2\int_0^{+\infty}dq\,q^2\int_{\mathbb{S}^2}d\hat{\boldsymbol{\omega}}_1\int_{\mathbb{S}^2}d\hat{\boldsymbol{\omega}}_2\ f(p,q,\hat{\boldsymbol{\omega}}_1\cdot\hat{\boldsymbol{\omega}}_2)\,\overline{Y_0^0(\hat{\boldsymbol{\omega}}_1)}\,Y_0^0(\hat{\boldsymbol{\omega}}_2).$$

Next, because of $i)$ we can decompose $f(p,q,\cdot)$ in terms of Legendre Polynomials

$$f(p,q,x)=\sum_{\ell\in\mathbb{N}_0}\tfrac{2\ell+1}{2}\langle P_\ell,\ f(p,q,\cdot)\rangle_{L^2([-1,\,1])}\,P_\ell(x),\quad\text{for almost every } x\in[-1,\ 1].\tag{B.1a}$$

Exploiting the addition formula (2.53) one gets

$$f(p,q,\hat{\boldsymbol{\omega}}_1\cdot\hat{\boldsymbol{\omega}}_2)=2\pi\sum_{\ell\in\mathbb{N}_0}\int_{-1}^{1}du\ P_\ell(u)f(p,q,u)\sum_{m=-\ell}^{\ell}Y_\ell^m(\hat{\boldsymbol{\omega}}_1)\overline{Y_\ell^m(\hat{\boldsymbol{\omega}}_2)}.\tag{B.1b}$$

Hence, the orthonormality of the spherical harmonics yields

$$\int_{\mathbb{R}^6}d\boldsymbol{p}\,d\boldsymbol{q}\ f\Big(p,q,\tfrac{\boldsymbol{p}\cdot\boldsymbol{q}}{pq}\Big)=8\pi^2\int_0^{+\infty}dp\,p^2\int_0^{+\infty}dq\,q^2\int_{-1}^{1}du\ f(p,q,u).$$

□

Proposition B.3. *For any* $x\in[0,1],p\in\mathbb{R}$ *and* ℓ *even we have*

$$\int_{-1}^{1}dy\ P_\ell(y)\,\frac{\cosh[p\arcsin(xy)]}{\sqrt{1-x^2y^2}}=\frac{2^{\ell+1}\ell!\,x^\ell}{(2\ell+1)!}\prod_{k=1}^{\frac{\ell}{2}}\big[p^2+(2k-1)^2\big]\,{}_2F_1\big(\tfrac{\ell+1+ip}{2},\tfrac{\ell+1-ip}{2};\ell+\tfrac{3}{2};x^2\big).$$

Proof. First let $z\in(-1,1)$ and take into account [20, p. 1007, 9.121.32], so that

$$\begin{aligned}\frac{\cosh(p\arcsin z)}{\sqrt{1-z^2}}&={}_2F_1\big(\tfrac{1+ip}{2},\tfrac{1-ip}{2};\tfrac{1}{2};z^2\big)=\sum_{k\in\mathbb{N}_0}\frac{\big(\frac{1+ip}{2}\big)_k\big(\frac{1-ip}{2}\big)_k}{\big(\frac{1}{2}\big)_k}\frac{z^{2k}}{k!}\\&=\sum_{k\in\mathbb{N}_0}\frac{z^{2k}}{(2k)!}\prod_{n=1}^{k}\big[p^2+(2n-1)^2\big],\end{aligned}\tag{B.2}$$

where the last identity is given by the following simple computations

$$\big(\tfrac{1+ip}{2}\big)_k\big(\tfrac{1-ip}{2}\big)_k=\big|\tfrac{1+ip}{2}\big|^2\big|\tfrac{1+ip+2}{2}\big|^2\cdots\big|\tfrac{1+ip+2k-2}{2}\big|^2=\frac{1}{2^{2k}}\prod_{n=1}^{k}\big[p^2+(2n-1)^2\big],\tag{B.3}$$

$$\big(\tfrac{1}{2}\big)_k=\frac{1}{\sqrt{\pi}}\,\Gamma\big(k+\tfrac{1}{2}\big)=\frac{1}{2^{2k}}\frac{(2k)!}{k!}.\tag{B.4}$$

Notice that (B.4) is a particular case of the Legendre's duplication formula

$$\Gamma(z)\Gamma\big(z+\tfrac{1}{2}\big) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z), \qquad z \in \mathbb{C} \smallsetminus -\tfrac{1}{2}\mathbb{N}_0. \tag{B.5}$$

Using the Rodrigues' formula for P_ℓ and integrating by parts ℓ times, one gets

$$\int_{-1}^{1} dy\; P_\ell(y)\,\frac{\cosh[p\arcsin(xy)]}{\sqrt{1-x^2y^2}} = \frac{1}{2^\ell \ell!}\int_{-1}^{1} dy\; (1-y^2)^\ell \frac{\partial^\ell}{\partial y^\ell}\frac{\cosh[p\arcsin(xy)]}{\sqrt{1-x^2y^2}}.$$

By (B.2), the function $y \longmapsto \frac{\cosh[p\arcsin(xy)]}{\sqrt{1-x^2y^2}}$ is analytic in $(-1,1)$ for all $x\in[0,1]$ and $p\in\mathbb{R}$, thus one can compute the ℓ-th derivative:

$$\frac{\partial^\ell}{\partial y^\ell}\frac{\cosh[p\arcsin(xy)]}{\sqrt{1-x^2y^2}} = \sum_{k=\frac{\ell}{2}}^{+\infty}\frac{x^{2k}}{(2k)!}\,a_k(p^2)\,\frac{(2k)!\,y^{2k-\ell}}{(2k-\ell)!} = \sum_{k\in\mathbb{N}_0}\frac{x^{\ell+2k}}{(2k)!}\,a_{k+\frac{\ell}{2}}(p^2)\,y^{2k},$$

where we have set $a_k(p^2) := \prod_{n=1}^{k}[p^2+(2n-1)^2]$ for the sake of notation. Using Tonelli's theorem to interchange the integral with the summation, one obtains

$$\int_{-1}^{1} dy\; P_\ell(y)\,\frac{\cosh[p\arcsin(xy)]}{\sqrt{1-x^2y^2}} = \frac{x^\ell}{2^\ell\ell!}\sum_{k\in\mathbb{N}_0}\frac{x^{2k}}{(2k)!}\,a_{k+\frac{\ell}{2}}(p^2)\int_{-1}^{1} dy\;(1-y^2)^\ell y^{2k}.$$

The last integral can be explicitly computed, namely

$$\int_{-1}^{1} dy\;(1-y^2)^\ell y^{2k} = \frac{\Gamma(\ell+1)\Gamma\big(k+\frac{1}{2}\big)}{\Gamma\big(\ell+k+\frac{3}{2}\big)} = \frac{2^{2\ell+2}\,\ell!\,(\ell+k+1)!\,(2k)!}{(2\ell+2k+2)!\,k!}, \tag{B.6}$$

where in the last equality we have used (B.5). Therefore,

$$\int_{-1}^{1} dy\; P_\ell(y)\,\frac{\cosh[p\arcsin(xy)]}{\sqrt{1-x^2y^2}} = 2^{\ell+2}x^\ell\sum_{k\in\mathbb{N}_0}\frac{(\ell+k+1)!}{(2\ell+2k+2)!}\,a_{k+\frac{\ell}{2}}(p^2)\,\frac{x^{2k}}{k!}.$$

Using (B.4) and (B.3), the last expression can be rewritten in terms of the Pochhammer symbols

$$\begin{aligned}\int_{-1}^{1} dy\; P_\ell(y)\,\frac{\cosh[p\arcsin(xy)]}{\sqrt{1-x^2y^2}} &= \frac{x^\ell}{2^\ell}\sum_{k\in\mathbb{N}_0}\frac{x^{2k}}{k!}\,\frac{a_{k+\frac{\ell}{2}}(p^2)}{2^{2k}\left(\frac{1}{2}\right)_{\ell+k+1}}\\ &= x^\ell\sum_{k\in\mathbb{N}_0}\frac{x^{2k}}{k!}\,\frac{\left(\frac{1+ip}{2}\right)_{k+\frac{\ell}{2}}\left(\frac{1-ip}{2}\right)_{k+\frac{\ell}{2}}}{\left(\frac{1}{2}\right)_{\ell+k+1}}.\end{aligned}$$

By definition (2.75a), one has

$$(\cdot)_{n+m} = (\cdot)_m(\,\cdot+m)_n, \qquad \forall\, n,m\in\mathbb{N}_0, \tag{B.7}$$

hence

$$\int_{-1}^{1} dy\; P_\ell(y)\,\frac{\cosh[p\arcsin(xy)]}{\sqrt{1-x^2y^2}} = \frac{x^\ell\left(\frac{1+ip}{2}\right)_{\frac{\ell}{2}}\left(\frac{1-ip}{2}\right)_{\frac{\ell}{2}}}{\left(\frac{1}{2}\right)_{\ell+1}}\sum_{k\in\mathbb{N}_0}\frac{x^{2k}}{k!}\,\frac{\left(\frac{\ell+1+ip}{2}\right)_k\left(\frac{\ell+1-ip}{2}\right)_k}{\left(\ell+\frac{3}{2}\right)_k}.$$

Using again (B.3), (B.4) and definition (2.76) one concludes the proof.

□

In order to state the following propositions, let us define the following trace operator

$$\begin{aligned} (\widehat{\tau f})(\boldsymbol{p}) &:= \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} d\boldsymbol{q} \; \hat{f}(\boldsymbol{q},\boldsymbol{p}), \\ \mathscr{D}(\tau) &= \left\{ f \in L^2(\mathbb{R}^{m+n}) \mid \int_{\mathbb{R}^n} d\boldsymbol{p} \left| \int_{\mathbb{R}^m} d\boldsymbol{q} \; \hat{f}(\boldsymbol{q},\boldsymbol{p}) \right|^2 < +\infty \right\} \end{aligned} \tag{B.8}$$

Proposition B.4. *Let τ be the operator acting in $L^2(\mathbb{R}^{m+n})$ defined by* (B.8). *Then, for any $s > 0$*

$$H^{s+\frac{m}{2}}(\mathbb{R}^{m+n}) \subset \mathscr{D}(\tau), \qquad \tau \in \mathscr{B}\big(H^{s+\frac{m}{2}}(\mathbb{R}^{m+n}), H^s(\mathbb{R}^n)\big).$$

Proof. First we verify that $H^{s+\frac{m}{2}}(\mathbb{R}^{m+n})$ is contained in the domain of τ.
Given any $f \in H^{s+\frac{m}{2}}(\mathbb{R}^{m+n})$ one has by Cauchy–Schwarz inequality

$$\begin{aligned} \int_{\mathbb{R}^n} d\boldsymbol{p} \left| \int_{\mathbb{R}^m} d\boldsymbol{q} \; \hat{f}(\boldsymbol{q},\boldsymbol{p}) \right|^2 &\leq \int_{\mathbb{R}^n} d\boldsymbol{p} \left[\int_{\mathbb{R}^m} d\boldsymbol{q} \; (1+p^2+q^2)^{s+\frac{m}{2}} |\hat{f}(\boldsymbol{q},\boldsymbol{p})|^2 \int_{\mathbb{R}^m} d\boldsymbol{k} \; \frac{1}{(1+p^2+k^2)^{s+\frac{m}{2}}} \right] \\ &= \frac{\pi^{\frac{m}{2}} \Gamma(s)}{\Gamma\big(s+\frac{m}{2}\big)} \int_{\mathbb{R}^{m+n}} d\boldsymbol{q} d\boldsymbol{p} \; \frac{(1+q^2+p^2)^{s+\frac{m}{2}}}{(1+p^2)^s} \; |\hat{f}(\boldsymbol{q},\boldsymbol{p})|^2 \\ &\leq \frac{\pi^{\frac{m}{2}} \Gamma(s)}{\Gamma\big(s+\frac{m}{2}\big)} \, \|f\|^2_{H^{s+m/2}(\mathbb{R}^{m+n})} \, . \end{aligned}$$

Hence, f is also in $\mathscr{D}(\tau)$ and the first statement is therefore proven.
Moreover, since we have shown that for any $f \in H^{s+\frac{m}{2}}(\mathbb{R}^{m+n})$ one has

$$|(\widehat{\tau f})(\boldsymbol{p})|^2 \leq \frac{\Gamma(s)}{(4\pi)^{\frac{m}{2}} \Gamma\big(s+\frac{m}{2}\big)} \frac{1}{(1+p^2)^s} \int_{\mathbb{R}^m} d\boldsymbol{q} \; (1+q^2+p^2)^{s+\frac{m}{2}} |\hat{f}(\boldsymbol{q},\boldsymbol{p})|^2,$$

we conclude that $(1+p^2)^s |(\widehat{\tau f})(\boldsymbol{p})|^2 \in L^1(\mathbb{R}^n, d\boldsymbol{p})$ or, equivalently

$$\|\tau f\|^2_{H^s(\mathbb{R}^n)} \leq \frac{\Gamma(s)}{(4\pi)^{\frac{m}{2}} \Gamma\big(s+\frac{m}{2}\big)} \, \|f\|^2_{H^{s+m/2}(\mathbb{R}^{m+n})}.$$

□

Proposition B.5. *Given $f \colon \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{C}$, suppose $f \in H^{\frac{m}{2}+s}(\mathbb{R}^{m+n})$ for some $s > 0$. Then*

$$f(\boldsymbol{x},\boldsymbol{y})|_{\boldsymbol{x}=\boldsymbol{0}} = \operatorname*{w-lim}_{\boldsymbol{x}\to\boldsymbol{0}} \; f(\boldsymbol{x},\boldsymbol{y}), \qquad \textit{for almost every } \boldsymbol{y} \in \mathbb{R}^n.$$

Proof. Let $\tau \colon H^{\frac{m}{2}+s}(\mathbb{R}^{m+n}) \longrightarrow H^s(\mathbb{R}^n)$ be the bounded operator given by (B.8) for any $s > 0$.
We want to show that

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} \int_{\mathbb{R}^n} d\boldsymbol{y} \; \overline{g(\boldsymbol{y})} f(\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbb{R}^n} d\boldsymbol{y} \; \overline{g(\boldsymbol{y})} \, (\tau f)(\boldsymbol{y}), \qquad \forall\, g \in L^2(\mathbb{R}^n), \tag{B.9a}$$

or, equivalently

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} \int_{\mathbb{R}^n} d\boldsymbol{p} \; \overline{\hat{g}(\boldsymbol{p})} \int_{\mathbb{R}^n} d\boldsymbol{y} \; \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{y}}}{(2\pi)^{\frac{n}{2}}} f(\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbb{R}^n} d\boldsymbol{p} \; \overline{\hat{g}(\boldsymbol{p})} \, (\widehat{\tau f})(\boldsymbol{p}), \qquad \forall\, g \in L^2(\mathbb{R}^n). \tag{B.9b}$$

Clearly, one has

$$\int_{\mathbb{R}^n} d\boldsymbol{y} \; \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{y}}}{(2\pi)^{\frac{n}{2}}} f(\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbb{R}^m} d\boldsymbol{q} \; \frac{e^{i\boldsymbol{x}\cdot\boldsymbol{q}}}{(2\pi)^{\frac{m}{2}}} \hat{f}(\boldsymbol{q},\boldsymbol{p}). \tag{B.10}$$

We claim that $|\hat{g}(\boldsymbol{p})||\hat{f}(\boldsymbol{q},\boldsymbol{p})|$ is an integrable majorant uniformly in $\boldsymbol{x}$, since

$$\int_{\mathbb{R}^{n+m}} d\boldsymbol{p}d\boldsymbol{q}\;|\hat{g}(\boldsymbol{p})||\hat{f}(\boldsymbol{q},\boldsymbol{p})| \leq \sqrt{\frac{\pi^{\frac{m}{2}}\Gamma(s)}{\Gamma\big(s+\frac{m}{2}\big)}}\;\|g\|\,\|f\|_{H^{m/2+s}(\mathbb{R}^{n+m})}\,. \tag{B.11}$$

Indeed, exploiting Cauchy–Schwarz inequality

$$\begin{aligned}\left(\int_{\mathbb{R}^{n+m}} d\boldsymbol{p}d\boldsymbol{q}\;|\hat{g}(\boldsymbol{p})||\hat{f}(\boldsymbol{q},\boldsymbol{p})|\right)^2 &\leq \int_{\mathbb{R}^{n+m}} d\boldsymbol{p}d\boldsymbol{q}\;\frac{|\hat{g}(\boldsymbol{p})|^2}{(p^2+q^2+1)^{\frac{m}{2}+s}}\int_{\mathbb{R}^{n+m}} d\boldsymbol{p}d\boldsymbol{q}\;(p^2+q^2+1)^{\frac{m}{2}+s}\,|\hat{f}(\boldsymbol{q},\boldsymbol{p})|^2\\ &\leq \frac{2\pi^{\frac{m}{2}}}{\Gamma\big(\frac{m}{2}\big)}\;\|f\|^2_{H^{m/2+s}(\mathbb{R}^{n+m})}\int_{\mathbb{R}^n} d\boldsymbol{p}\;|\hat{g}(\boldsymbol{p})|^2\int_0^{+\infty} dq\;\frac{q^{m-1}}{(q^2+1)^{\frac{m}{2}+s}}.\end{aligned}$$

Therefore, exploiting (B.10) and the dominated convergence theorem, one has

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}}\int_{\mathbb{R}^n} d\boldsymbol{p}\;\overline{\hat{g}(\boldsymbol{p})}\int_{\mathbb{R}^n} d\boldsymbol{y}\;\frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{y}}}{(2\pi)^{\frac{n}{2}}}\,f(\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbb{R}^n} d\boldsymbol{p}\;\overline{\hat{g}(\boldsymbol{p})}\;\frac{1}{(2\pi)^{\frac{m}{2}}}\int_{\mathbb{R}^m} d\boldsymbol{q}\;\hat{f}(\boldsymbol{q},\boldsymbol{p}).$$

and equation (B.9b) has been proved.

□

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