

OPTIMAL EIGENVALUE ESTIMATES FOR THE ROBIN LAPLACIAN ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We consider the first eigenvalue $\lambda_1(\Omega, \sigma)$ of the Laplacian with Robin boundary conditions on a compact Riemannian manifold Ω with smooth boundary, $\sigma \in \mathbf{R}$ being the Robin boundary parameter. When $\sigma > 0$ we give a positive, sharp lower bound of $\lambda_1(\Omega, \sigma)$ in terms of an associated one-dimensional problem depending on the geometry through a lower bound of the Ricci curvature of Ω , a lower bound of the mean curvature of $\partial\Omega$ and the inradius. When the boundary parameter is negative, the lower bound becomes an upper bound. In particular, explicit bounds for mean-convex Euclidean domains are obtained, which improve known estimates.

Then, we extend a monotonicity result for $\lambda_1(\Omega, \sigma)$ obtained in Euclidean space by Giorgi and Smits [10], to a class of manifolds of revolution which include all space forms of constant sectional curvature. As an application, we prove that $\lambda_1(\Omega, \sigma)$ is uniformly bounded below by $\frac{(n-1)^2}{4}$ for all bounded domains in the hyperbolic space of dimension n , provided that the boundary parameter $\sigma \geq \frac{n-1}{2}$ (McKean-type inequality). Asymptotics for large hyperbolic balls are also discussed. ¹

1. INTRODUCTION

1.1. Definition and some known facts. Let (Ω^n, g) be a compact Riemannian manifold of dimension n with smooth boundary $\partial\Omega$, and let Δ be the Laplacian associated to the metric g . The sign convention is that, on \mathbf{R}^n :

$$\Delta = - \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}.$$

We are interested in the first eigenvalue of the Robin problem:

$$(1) \quad \begin{cases} \Delta u = \lambda u & \text{on } \Omega \\ \frac{\partial u}{\partial N} = \sigma u & \text{on } \partial\Omega \end{cases}$$

where $\sigma \in \mathbf{R}$ is a parameter and N is the inner unit normal. The eigenvalues form a discrete sequence diverging to infinity:

$$\lambda_1(\Omega, \sigma) < \lambda_2(\Omega, \sigma) \leq \dots;$$

it is known that the first eigenvalue $\lambda_1(\Omega, \sigma)$ is simple and that any first eigenfunction does not change sign, so that we can assume that it is positive.

The first eigenvalue $\lambda_1(\Omega, \sigma)$ models heat diffusion with absorbing ($\sigma > 0$) or radiating ($\sigma < 0$) boundary; it can also be seen as the fundamental tone of an elastically supported membrane.

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1.2. Some features of $\lambda_1(\Omega, \sigma)$. It is immediately seen that for $\sigma = 0$ we recover the classical Neumann problem; in particular $\lambda_1(\Omega, 0) = 0$, the associated eigenfunctions being the constants. Hence we can assume $\sigma \neq 0$. The Rayleigh min-max principle reads:

$$(2) \quad \lambda_1(\Omega, \sigma) = \inf_{u \in H^1(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 + \int_{\partial\Omega} \sigma u^2}{\int_{\Omega} u^2}$$

and one can see easily that $\lambda_1(\Omega, \sigma)$ is positive for all $\sigma > 0$, and negative when $\sigma < 0$. Moreover $\lambda_1(\Omega, \sigma)$ is an increasing function of σ which tends to $\lambda_1(\Omega, \infty) \doteq \lambda_1^D(\Omega)$ (the first Dirichlet eigenvalue of Ω) when $\sigma \rightarrow +\infty$. In particular:

$$\lambda_1(\Omega, \sigma) < \lambda_1^D(\Omega)$$

for all $\sigma \in \mathbf{R}$.

• Problem (1) continues to make sense and admits a discrete spectrum when σ is a continuous function on $\partial\Omega$, and not just a constant. The min-max principle (2) makes clear that then:

$$\lambda_1(\Omega, \inf_{\partial\Omega} \sigma) \leq \lambda_1(\Omega, \sigma) \leq \lambda_1(\Omega, \sup_{\partial\Omega} \sigma).$$

However in this paper we tacitly assume that σ is a real constant.

The behaviour when the boundary parameter $\sigma \rightarrow -\infty$ is quite interesting. In that case the first eigenfunction concentrates near the boundary and, for domains in \mathbf{R}^n having C^∞ -smooth boundary, one has the following asymptotic expansion as $\sigma \rightarrow -\infty$:

$$\lambda_1(\Omega, \sigma) = -\sigma^2 + (n-1)H_{\max}\sigma + o(\sqrt{\sigma}),$$

where H_{\max} denotes the maximum value of the mean curvature of $\partial\Omega$. This was first proved by Pankrashkin [20] in dimension 2 and later generalized by Pankrashkin and Popoff [21]. The presence of corners affects the first term (see [17]).

Domain monotonicity is an essential feature of the Dirichlet problem: if $\Omega_1 \subseteq \Omega_2$ then $\lambda_1^D(\Omega_1) \geq \lambda_1^D(\Omega_2)$. This is an important tool in estimating eigenvalues. As observed in [10] domain monotonicity does not hold in full generality when $\sigma < +\infty$, even for convex domains. However, it does hold in Euclidean space \mathbf{R}^n when the outer domain is a ball (see [10] Theorem 1). We will in fact extend the argument in [10] to prove a similar monotonicity result for a certain class of revolution manifolds, in particular, for the other space forms \mathbf{H}^n and \mathbf{S}^n . This will be applied to generalize the classical McKean inequality [19] to the Robin Laplacian.

1.3. Some known eigenvalue estimates. When $\sigma > 0$, a Faber-Krahn type inequality has been proved by Bossel [5] for domains in \mathbf{R}^2 ; the result was extended to domains in \mathbf{R}^n by Daners [6]. The conclusion is that, among all Euclidean domains with fixed volume, the ball minimises $\lambda_1(\Omega, \sigma)$ for any fixed $\sigma > 0$.

When $\sigma < 0$, it was conjectured by Bareket [3] that the ball would be, instead, a maximiser. As shown in [7] this is true for domains which are close to a ball. But in fact the Bareket conjecture is false in general, as shown by Freitas and Krejcirik [8], who showed that when σ is large negative annuli with the same volume have larger first eigenvalue. In dimension 2, they also showed that there exists a critical parameter $\sigma^* < 0$, depending only on the area, such that for any $\sigma \in [\sigma^*, 0]$ the ball is a maximiser.

Finally, let us mention the following explicit upper and lower bounds for convex domains in \mathbf{R}^n , proved by Kovarik in [15] :

$$(3) \quad \frac{\sigma}{4R + 4R^2\sigma} \leq \lambda_1(\Omega, \sigma) \leq \frac{2K_n\sigma}{R + R^2\sigma},$$

where K_n is an explicit constant. Here R is the *inradius* of Ω , that is, the largest radius of a ball included in Ω . Note that the lower bound in (3) is sometimes much better than the Faber Krahn inequality (think of a convex domain which has small inradius and fixed volume). By passing to the limit as $\sigma \rightarrow \infty$ the lower bound becomes:

$$(4) \quad \lambda_1^D(\Omega) \geq \frac{1}{4R^2};$$

the author observes in Remark 4.6 of [15] that, due to the method used (Hardy inequality), (4) cannot be sharp; in fact the sharp bound in terms of the inradius would be

$$(5) \quad \lambda_1^D(\Omega) \geq \frac{\pi^2}{4R^2},$$

as proved by Hersch in [12]. The lower bound (5) was later shown to hold for a wider class of Riemannian manifolds in [18] and [14] (and later by the author in [24], by different methods). We will in fact prove a sharp lower bound (resp. upper bound) for all $\sigma > 0$ (resp. $\sigma < 0$) in the Riemannian case, by adapting the method of Laplacian comparison to the Robin boundary conditions; when applied to convex Euclidean domains, this will improve the lower bound in (3) and yield in the limit the sharp estimate (5).

The scope of this paper is twofold: we first prove a comparison theorem for a general Riemannian manifold (see Theorem 1 and Theorem 2) and then we prove a monotonicity result for a large class of revolution manifolds (Theorem 5). Both these methods will produce sharp bounds (in particular, a McKean-type inequality, Theorem 6).

The paper is structured as follows. In Section 2 we state our main results and in Section 3 we prove some preliminary facts. Section 4 is devoted to the proof of the comparison theorem, while in Sections 5 and 6 we prove domain monotonicity and the McKean-type inequality. Finally in the Appendix we describe the model domains for our comparison theorem.

2. MAIN RESULTS

2.1. Comparison theorem. We will compare $\lambda_1(\Omega, \sigma)$ with the first eigenvalue of a one-dimensional problem on the interval $[0, R]$, where R is the inradius of Ω :

$$(6) \quad \begin{cases} u'' + \frac{\Theta'}{\Theta}u' + \lambda u = 0 \\ u'(0) = \sigma u(0) \\ u'(R) = 0 \end{cases}$$

This problem carries a weight $\Theta = \Theta(r)$ depending explicitly on the geometry of Ω , as follows. We say that Ω has *curvature data* (K, H) if:

- the Ricci curvature of Ω is bounded below by $(n - 1)K$,
- the mean curvature of $\partial\Omega$ is bounded below by H .

We stress that K and H may assume any real value. Our convention on the mean curvature is the following. Let S be the shape operator of the immersion of $\partial\Omega$ into Ω , with respect to the inner unit normal N : this is the self-adjoint operator acting on $T\partial\Omega$ and defined by $S(X) = -\nabla_X N$, for all $X \in T\partial\Omega$. Then the mean curvature is

$$\mathcal{H} = \frac{1}{n-1} \operatorname{tr} S.$$

The sign convention is such that \mathcal{H} is positive, and equal to $\frac{1}{R}$, on the boundary of the ball of radius R in \mathbf{R}^n . As usual, we denote by R the inradius of Ω . Introduce the function on $[0, R]$:

$$(7) \quad s_k(r) = \begin{cases} \frac{1}{\sqrt{K}} \sin(r\sqrt{K}), & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\sqrt{|K|}} \sin(r\sqrt{|K|}), & \text{if } K < 0. \end{cases}$$

We now define what we will call the *weight function* $\Theta : [0, R] \rightarrow \mathbf{R}$ by:

$$(8) \quad \Theta(r) = \left(s'_K(r) - H s_k(r) \right)^{n-1}.$$

Note that Θ depends on K and H , and that $\Theta(0) = 1$. As a consequence of Theorem A in [13] (see also Proposition 14 in [24]) we have that Θ is positive on $[0, R]$, and moreover $\Theta(R) = 0$ if and only if Ω is a geodesic ball in the space form M_K , that is, the simply connected manifold with constant sectional curvature K .

Here is a general comparison theorem.

Theorem 1. *Let Ω be a compact manifold with smooth boundary having curvature data (K, H) and inradius R . If $\sigma > 0$, then:*

$$\lambda_1(\Omega, \sigma) \geq \lambda_1(R, \Theta, \sigma),$$

where $\lambda_1(R, \Theta, \sigma)$ is the first eigenvalue of problem (6), and Θ is defined in (8). If $\sigma < 0$ the inequality is reversed:

$$\lambda_1(\Omega, \sigma) \leq \lambda_1(R, \Theta, \sigma).$$

In other words one has, for all $\sigma \in \mathbf{R}$:

$$|\lambda_1(\Omega, \sigma)| \geq |\lambda_1(R, \Theta, \sigma)|.$$

For the proof, see section 4. The estimate is sharp in every dimension: see section 2.2 below. The eigenvalue $\lambda_1(R, \Theta, \sigma)$ is always positive and, when $\sigma > 0$, the theorem gives a positive lower bound for any compact Riemannian manifold with boundary. For the Dirichlet problem ($\sigma = +\infty$) the result is due to Kasue [14].

A particularly simple situation is when $K = H = 0$, so that $\Theta(r) = 1$. We obtain the following fact.

Theorem 2. *Let $\sigma > 0$. Assume that both the Ricci curvature of Ω and the mean curvature of $\partial\Omega$ are non-negative. Let R be the inradius of Ω . Then:*

$$\lambda_1(\Omega, \sigma) \geq \lambda_1([0, 2R], \sigma),$$

where on the right we have the first Robin eigenvalue of the interval $[0, 2R]$. The estimate is sharp in any dimension. Precisely, if Ω is any flat cylinder (that is, a Riemannian product

$[0, 2R] \times \Sigma^{n-1}$, where Σ^{n-1} is a closed Riemannian manifold of dimension $n - 1$) then equality holds.

If $\sigma < 0$ the inequality is reversed and sharp as well.

Explicit evaluation of the right-hand side using the Becker-Starck inequality implies the following estimate.

Corollary 3. *If both the Ricci curvature of Ω and the mean curvature of $\partial\Omega$ are non-negative (in particular, for mean-convex Euclidean domains) we have, if $\sigma > 0$:*

$$\lambda_1(\Omega, \sigma) > \frac{\pi^2 \sigma}{\pi^2 R + 4R^2 \sigma},$$

while if $\sigma < 0$, then $\lambda_1(\Omega, \sigma) < -\sigma^2$.

(The proof is given in section 4.4). The estimate applies to any mean-convex (in particular, convex) domain in \mathbf{R}^n , and it improves the bound (3) for all $\sigma > 0$ and R . As $\sigma \rightarrow +\infty$ it gives the expected sharp bound:

$$\lambda_1^D(\Omega) \geq \frac{\pi^2}{4R^2}.$$

2.2. Sharpness, method of proof. Theorem 1 is sharp in all dimensions. In fact, for any $R > 0$ and for any curvature data (K, H) we will construct a *model domain* $\bar{\Omega} \doteq \bar{\Omega}(K, H, R)$ of dimension n with two boundary components: $\partial\bar{\Omega} = \Gamma_1 \cup \Gamma_2$ such that :

$$\lambda_1(\bar{\Omega}, \sigma) = \lambda_1(R, \Theta, \sigma)$$

where on the left we have the first eigenvalue of $\bar{\Omega}$ with Robin conditions on Γ_1 and Neumann conditions on Γ_2 . For the definition of $\bar{\Omega}$ we refer to the Appendix.

In some cases the model domain can be a ball in a space-form M_K and we have an equality case:

Theorem 4. *Let Ω be a domain with curvature data (K, H) , and assume one of the following three cases: a) $K > 0$ and $H \in \mathbf{R}$, b) $K = 0$ and $H > 0$, c) $K < 0$ and $H > \sqrt{|K|}$. Then:*

- a) *There is a unique ball $\tilde{\Omega}$ in M_K with mean curvature equal to H .*
- b) *The radius \tilde{R} of $\tilde{\Omega}$ satisfies $\tilde{R} \geq R$.*
- c) *One has $\lambda_1(\Omega, \sigma) \geq \lambda_1(\tilde{\Omega}, \sigma)$ with equality if and only if Ω is isometric to $\tilde{\Omega}$.*

For the proof, see the Appendix. The proof of Theorem 1 is by Laplacian comparison, and is obtained by extending the methods in [14] and [24] to the Robin Laplacian.

2.3. Domain monotonicity on revolution manifolds. As remarked before, domain monotonicity does not hold in full generality for the Robin Laplacian, and the first aim is here to extend the monotonicity result of [10] from Euclidean space to other manifolds.

We focus here on the class of *revolution manifolds with pole x_0* : these are manifolds $M \doteq [0, T] \times \mathbf{S}^{n-1}$ (T could be $+\infty$) with metric

$$g = dr^2 + \Phi(r)^2 g_{\mathbf{S}^{n-1}},$$

where $g_{\mathbf{S}^{n-1}}$ is the canonical metric on the sphere \mathbf{S}^{n-1} . Here $r \in [0, T]$ is the geodesic distance to the pole and $\Phi(r)$ is the *warping function*; this is a smooth, positive function on $[0, T]$ satisfying the conditions:

$$\Phi(0) = \Phi''(0) = 0, \quad \Phi'(0) = 1,$$

which are imposed in order to have a C^2 -metric at the pole. However, if we make the stronger assumptions that Φ has vanishing even derivatives at zero then the metric is C^∞ -smooth everywhere.

- The geodesic ball centered at the pole of M having radius $R \leq T$, that is, $B(x_0, R)$, is evidently a revolution manifold itself and will then be regarded as such. We also remark that the function

$$H(r) \doteq \frac{\Phi'}{\Phi}(r)$$

expresses the mean curvature of the geodesic sphere $\partial B(x_0, r)$ with respect to the inner unit normal $N = -\nabla r$.

Recall that the *space-form of constant curvature* K , denoted M_K , is : Euclidean space if $K = 0$, the round sphere of radius $\frac{1}{\sqrt{K}}$ if $K > 0$, and the hyperbolic space \mathbf{H}_K^n of (constant) curvature K if $K < 0$. The space-form M_K is a revolution manifold around any of its points, the warping function being $\Phi(r) = s_K(r)$. Precisely:

$$\begin{aligned} \Phi(r) &= r \quad \text{if } K = 0 \\ \Phi(r) &= \frac{1}{\sqrt{K}} \sin(r\sqrt{K}) \quad \text{if } K > 0, \\ \Phi(r) &= \frac{1}{\sqrt{-K}} \sinh(r\sqrt{-K}) \quad \text{if } K < 0. \end{aligned}$$

We will be interested in the situation where the warping function is log-concave, that is $(\log \Phi)'' < 0$. This is equivalent to asking that the mean curvature of $\partial B(x_0, r)$ is a decreasing function of r (the distance to the pole). It is clear that the condition is satisfied by the warping function of all space-forms M_K .

Theorem 5. *Let Ω be a domain of a revolution manifold M with pole x_0 , whose warping function Φ is log-concave : $(\log \Phi)'' < 0$. Assume that $\Omega \subseteq B(x_0, R)$. If $\sigma > 0$ then*

$$\lambda_1(\Omega, \sigma) \geq \lambda_1(B(x_0, R), \sigma),$$

while if $\sigma < 0$ then the opposite inequality holds:

$$\lambda_1(\Omega, \sigma) \leq \lambda_1(B(x_0, R), \sigma).$$

In particular, the above monotonicity holds true in any space-form M_K .

For the proof, see section 5. When $M = \mathbf{R}^n$ the result is due to Giorgi and Smits [10].

2.4. McKean-type inequality. It is well-known that the first Dirichlet eigenvalue of any bounded domain in \mathbf{H}^n satisfies the bound

$$\lambda_1^D(\Omega) \geq \frac{(n-1)^2}{4},$$

known as McKean inequality (see [19]). The remarkable fact here is that the inequality holds regardless of the size of Ω (volume, diameter, etc.). By domain monotonicity, which is valid in \mathbf{H}^n thanks to Theorem 5, and by explicit calculations for geodesic balls, we can extend McKean inequality to the Robin Laplacian, as follows.

Theorem 6. *Let Ω be a domain in \mathbf{H}^n and let $\sigma > 0$. Then:*

$$\lambda_1(\Omega, \sigma) \geq \begin{cases} \frac{(n-1)^2}{4} & \text{if } \sigma \geq \frac{n-1}{2} \\ (n-1)\sigma - \sigma^2 & \text{if } 0 < \sigma \leq \frac{n-1}{2} \end{cases}$$

If instead we assume $\sigma < 0$, then $\lambda_1(\Omega, \sigma) \leq -\sigma^2 + (n-1)\sigma$.

For $\sigma > \frac{n-1}{2}$, the estimate is sharp, because if B_R is any hyperbolic ball of radius R , we have:

$$\lim_{R \rightarrow \infty} \lambda_1(B_R, \sigma) = \frac{(n-1)^2}{4}.$$

This will be clear from the next Theorem 7. Note that the lower bound is independent of σ and also on R when σ is large enough. We will in fact refine the estimate for hyperbolic balls by taking into account the value R of the radius.

In the case $\sigma = +\infty$ (Dirichlet problem) it was proved in [23], Theorem 5.6, that

$$(9) \quad \frac{(n-1)^2}{4} + \frac{\pi^2}{R^2} - \frac{4\pi^2}{(n-1)R^3} \leq \lambda_1^D(B_R) \leq \frac{(n-1)^2}{4} + \frac{\pi^2}{R^2} + \frac{C}{R^3},$$

where $C = \frac{\pi^2(n^2-1)}{2} \int_0^\infty \frac{r^2}{\sinh^2 r} dr$.

For the Robin problem, and for σ sufficiently large, we obtain the following calculation.

Theorem 7. *Let B_R be the ball of radius R in the hyperbolic space \mathbf{H}^n and let $\lambda_1(B_R, \sigma)$ be the first eigenvalue of the Robin Laplacian with parameter $\sigma > \frac{n-1}{2}$.*

Then there are positive constants R_0, c_0 depending only on σ and n such that, for all $R \geq R_0$ one has:

$$\frac{(n-1)^2}{4} + \frac{\pi^2}{R^2} - \frac{c_0}{R^3} \leq \lambda_1(B_R, \sigma) \leq \frac{(n-1)^2}{4} + \frac{\pi^2}{R^2} + \frac{C}{R^3},$$

where $C = \frac{\pi^2(n^2-1)}{2} \int_0^\infty \frac{r^2}{\sinh^2 r} dr$. (The upper bound holds for all R and σ).

Consequently, for all $\sigma \in (\frac{n-1}{2}, \infty]$ one has the following two-term asymptotic expansion as $R \rightarrow \infty$:

$$\lambda_1(B_R, \sigma) \sim \frac{(n-1)^2}{4} + \frac{\pi^2}{R^2} + O\left(\frac{1}{R^3}\right).$$

We see that when the radius is large and the parameter σ is greater than $\frac{n-1}{2}$, the eigenvalues $\lambda_1(B_R, \sigma)$ and $\lambda_1^D(B_R)$ are very very close; in fact, the boundary parameter makes its appearance only in the remainder term, which we find a bit surprising. The constants R_0 and c_0 will be explicated in the proof.

When $\sigma = +\infty$ (Dirichlet problem) the lower bound in (9) was improved in [2] and the two-term expansion has been refined in Theorem 1.1 of [16].

For simplicity we state the estimates in constant negative curvature -1 . However the above estimates easily extend to arbitrary constant negative curvature, as follows: let Ω be any domain

in $\mathbf{H}_{-\kappa^2}^n$, the hyperbolic space of constant curvature $-\kappa^2$ (we assume $\kappa > 0$). If $\sigma > 0$ then

$$\lambda_1(\Omega, \sigma) \geq \begin{cases} \frac{(n-1)^2}{4}\kappa^2 & \text{if } \sigma \geq \frac{n-1}{2}\kappa \\ (n-1)\kappa\sigma - \sigma^2 & \text{if } 0 < \sigma \leq \frac{n-1}{2}\kappa \end{cases}$$

If instead we assume $\sigma < 0$, then $\lambda_1(\Omega, \sigma) \leq -\sigma^2 + (n-1)\kappa\sigma$. Moreover, if B_R denotes the ball of radius R in $\mathbf{H}_{-\kappa^2}^n$, then one has a two-term asymptotic expansion as $R \rightarrow \infty$:

$$\lambda_1(B_R, \sigma) \sim \frac{(n-1)^2}{4}\kappa^2 + \frac{\pi^2}{R^2} + O\left(\frac{1}{R^3}\right).$$

Observe that the second term is in fact independent on the ambient curvature κ and on the boundary parameter σ .

Theorems 6 and 7 are proved in section 6.2.

3. PRELIMINARY FACTS

In this section we first prove some properties of eigenfunctions of the model one-dimensional problem, then we explain the method of proof of the comparison theorem, based on interior parallels and laplacian comparison. The exposition is based on [24], but see also the Appendix in [23]. We chose to be here as self-contained as possible.

3.1. One-dimensional model problem. We will compare $\lambda_1(\Omega, \sigma)$ with the first eigenvalue $\lambda_1(R, \Theta, \sigma)$ of the following one-dimensional mixed problem on the interval $[0, R]$:

$$(10) \quad \begin{cases} u'' + \frac{\Theta'}{\Theta}u' + \lambda u = 0 \\ u'(0) = \sigma u(0) \\ u'(R) = 0 \end{cases}$$

with the weight function $\Theta(r)$ as in (8). Note that the boundary conditions are: Robin at $r = 0$, Neumann at $r = R$; with these boundary conditions the spectrum of (10) is the spectrum of the operator

$$Lu \doteq -u'' - \frac{\Theta'}{\Theta}u'$$

acting on the weighted space $L^2([0, R], \mu)$ for the measure $\mu = \Theta(r) dr$. Then, L is self-adjoint and the spectrum is discrete:

$$\lambda_1(R, \Theta, \sigma) \leq \lambda_2(R, \Theta, \sigma) \leq \dots \rightarrow +\infty.$$

The min-max principle reads

$$(11) \quad \lambda_1(R, \Theta, \sigma) = \inf_{u \in H^1[0, R]} \left\{ \frac{\int_0^R u'(r)^2 \Theta(r) dr + \sigma u(0)^2}{\int_0^R u(r)^2 \Theta(r) dr} \right\}.$$

If $\sigma = 0$ we have a Neumann weighted problem, the non-zero constants are eigenfunctions and then:

$$\lambda_1(\Omega, \sigma) = \lambda_1(R, \Theta, 0) = 0.$$

Therefore, we can assume $\sigma \neq 0$.

Lemma 8. *Let u be a positive first eigenfunction of (10).*

- a) *If $\sigma > 0$ then $u' > 0$ on $[0, R)$.*
- b) *If $\sigma < 0$ then $u' < 0$ on $[0, R)$.*
- c) *If $\sigma > 0$ and $\bar{R} < R$ then $\lambda_1(R, \Theta, \sigma) < \lambda_1(\bar{R}, \Theta, \sigma)$.*

In d) and e), we let $\sigma(r) = \frac{u'(r)}{u(r)}$ and assume that Θ is strictly log-concave (that is, $(\log \Theta)'' < 0$ on $[0, R)$).

- d) *If $\sigma > 0$ then $0 \leq \sigma(r) \leq \sigma$ for all $r \in [0, R]$.*
- e) *If $\sigma < 0$ then $0 \geq \sigma(r) \geq \sigma$ for all $r \in [0, R]$.*

Proof. a) Let $\sigma > 0$ and let \bar{R} be the first zero of u' . Note that $\bar{R} > 0$ because $u'(0) = \sigma u(0) > 0$. We reason by contradiction and assume that $\bar{R} < R$. Then, by restriction, u is also an eigenfunction of the same problem on $[0, \bar{R}]$; as u is positive, it must be the first, hence:

$$\lambda_1(\bar{R}, \Theta, \sigma) = \lambda_1(R, \Theta, \sigma).$$

We show that this is impossible. Consider the function $w \in H^1[0, R]$ defined as follows:

$$(12) \quad w(r) = \begin{cases} u(r) & r \in [0, \bar{R}] \\ u(\bar{R}) & r \in [\bar{R}, R]. \end{cases}$$

We use w as a test-function for the problem (10) on $[0, R]$ and therefore, by the min-max principle (11), we have :

$$(13) \quad \lambda_1(R, \Theta, \sigma) \int_0^R w^2 \Theta \leq \int_0^R w'^2 \Theta + \sigma w(0)^2$$

Now:

$$(14) \quad \int_0^R w^2 \Theta = \int_0^{\bar{R}} u^2 \Theta + u(\bar{R})^2 \int_{\bar{R}}^R \Theta > \int_0^{\bar{R}} u^2 \Theta$$

because Θ is positive on $[0, R)$ by assumption. On the other hand

$$(15) \quad \begin{aligned} \int_0^R w'^2 \Theta + \sigma w(0)^2 &= \int_0^{\bar{R}} u'^2 \Theta + \sigma u(0)^2 \\ &= \int_0^{\bar{R}} u Lu \cdot \Theta \\ &= \lambda_1(\bar{R}, \Theta, \sigma) \int_0^{\bar{R}} u^2 \Theta \end{aligned}$$

Putting together (13), (14) and (15) we get $\lambda_1(R, \Theta, \sigma) < \lambda_1(\bar{R}, \Theta, \sigma)$ which is a contradiction as asserted.

b) Let $\sigma < 0$ and let \bar{R} be the first zero of u' ; note that $\bar{R} > 0$. Assume $\bar{R} < R$. Let v be the restriction of u to $[\bar{R}, R]$. Then v is a Neumann eigenfunction of $[\bar{R}, R]$ which implies $\lambda_1(R, \Theta, \sigma) \geq 0$ because the Neumann spectrum is non-negative. This is impossible because, as $\sigma < 0$, we have $\lambda_1(R, \Theta, \sigma) < 0$ by the min-max principle (take $w = 1$ as test-function).

c) One prolongs an eigenfunction of $[0, \bar{R}]$ on $[0, R]$ by a constant and uses the min-max principle. The construction is exactly as in part a).

d) Recall that u satisfies $u'' + \frac{\Theta'}{\Theta}u' + \lambda u = 0$; an easy calculation shows that

$$\sigma' = \left(\frac{u'}{u}\right)' = -\frac{\Theta'}{\Theta}\sigma - \lambda - \sigma^2.$$

Now σ is positive on $(0, R)$ by a); moreover $\sigma(0) = \sigma > 0$ and $\sigma(R) = 0$. It is then enough to show that σ has no relative maximum in the open interval $(0, R)$. Assume by contradiction that \bar{r} is one such. Then we would have

$$(16) \quad \sigma'(\bar{r}) = 0, \quad \sigma''(\bar{r}) \leq 0.$$

Now

$$\sigma'' = -\left(\frac{\Theta'}{\Theta}\right)'\sigma - \frac{\Theta'}{\Theta}\sigma' - 2\sigma\sigma',$$

hence:

$$\sigma''(\bar{r}) = -(\log \Theta)''(\bar{r})\sigma(\bar{r}) > 0$$

because σ is strictly positive on $(0, R)$, which contradicts (16). Hence the assertion.

e) We proceed in a similar way. We know that $u' < 0$ on $[0, R)$, hence $\sigma(r) < 0$ on that interval. It is enough to show that σ has no relative minimum in the open interval $(0, R)$. We assume that $\bar{r} \in (0, R)$ is such a relative minimum and find a contradiction as before. \square

3.2. Distance to the boundary and cut-locus. For complete details we refer to [24]. Let Ω be a compact domain with smooth boundary and let $\rho : \Omega \rightarrow \mathbf{R}$ be the distance function to the boundary:

$$\rho(x) = \text{dist}(x, \partial\Omega).$$

The function ρ is Lipschitz and, as $\partial\Omega$ is smooth, it is smooth on a small tubular neighborhood of the boundary; moreover, ρ is singular precisely on the *cut-locus* $\text{Cut}_{\partial\Omega}$, which is a closed set of measure zero in Ω . Let us recall its definition.

Let N_x be the inner unit normal at $x \in \partial\Omega$. Consider the unit speed geodesic starting at x and going inside Ω , in the direction normal to the boundary, that is, $\gamma_x(t) = \exp_x(tN_x)$. The *cut-radius* at x is the positive number $c(x)$ defined as follows:

- the geodesic $\gamma_x(t)$ minimizes distance to $\partial\Omega$ if and only if $t \in [0, c(x)]$.

Thus, we obtain the map $c : \partial\Omega \rightarrow [0, +\infty)$ which is known to be continuous; it is positive because $\partial\Omega$ is smooth (in fact, $\inf_{\partial\Omega} c$ is also called the *injectivity radius* of the normal exponential map). The cut-locus $\text{Cut}_{\partial\Omega}$ is the closed subset of Ω defined by

$$\text{Cut}_{\partial\Omega} = \{\exp_x(c(x)N_x) : x \in \partial\Omega\}.$$

It is known that a point on the cut-locus is either a focal point along a normal geodesic, or is a point which can be joined to the boundary by at least two distinct minimizing geodesics. The cut locus has zero measure in Ω ; we let

$$\Omega_{\text{reg}} \doteq \Omega \setminus \text{Cut}_{\partial\Omega},$$

and call it the set of *regular points* of ρ . In fact, ρ is C^∞ -smooth on Ω_{reg} and there one has $|\nabla\rho| = 1$. In conclusion, we have a disjoint union

$$\Omega = \Omega_{\text{reg}} \cup \text{Cut}_{\partial\Omega}.$$

Note that each point $p \in \Omega_{\text{reg}}$ can be joined to the boundary by a unique geodesic segment minimizing distance.

3.3. Normal coordinates. We now consider the set:

$$U \doteq \{(r, x) \in [0, \infty) \times \partial\Omega : 0 \leq r < c(x)\}$$

and see that the exponential map $\Phi : U \rightarrow \Omega_{\text{reg}}$ defined by $\Phi(r, x) = \exp_x(rN_x)$ is actually a diffeomorphism. The pair (r, x) gives rise to the *normal coordinates* of a regular point. We pull-back the Riemannian volume form by Φ , and we write

$$\Phi^* dv_n(r, x) = \theta(r, x) dr dv_{n-1}(x),$$

where the Jacobian $\theta(r, x)$ can then be seen as the density of the Riemannian measure in normal coordinates. Obviously θ is positive on U and $\theta(0, x) = 1$ for all $x \in \partial\Omega$. Any integrable function f on Ω can be integrated in normal coordinates, as follows:

$$(17) \quad \int_{\Omega} f dv_n = \int_{\Omega_{\text{reg}}} f dv_n = \int_{\partial\Omega} \int_0^{c(x)} f(r, x) \theta(r, x) dr dv_{n-1}(x).$$

where we identify a regular point of Ω with its normal coordinates. The map $r \mapsto \theta(r, x)$ is smooth and extends by continuity on $[0, c(x)]$:

$$\theta(c(x), x) = \lim_{r \rightarrow c(x)-} \theta(r, x).$$

Note that $\theta(c(x), x)$ could be zero; it is zero precisely at the focal points of the boundary. The function

$$\Delta_{\text{reg}} \rho \doteq \Delta(\rho|_{\Omega_{\text{reg}}})$$

is the Laplacian of ρ restricted to the regular points. It is then smooth on Ω_{reg} , and it is also in $L^1(\Omega)$ (see [24]). In normal coordinates it has the following expression:

$$(18) \quad \Delta_{\text{reg}} \rho(r, x) = -\frac{\theta'(r, x)}{\theta(r, x)}$$

where for simplicity θ' refers to differentiation with respect to r . For a proof see [9] p. 40. Note also that $\Delta_{\text{reg}} \rho(r, x)$ is $(n-1)$ -times the mean curvature at (r, x) of the level set $\rho = r$. By the classical Heintze-Karcher volume estimates we see that all regular points (r, x) one has the inequality:

$$(19) \quad \Delta_{\text{reg}} \rho(r, x) \geq -\frac{\Theta'(r)}{\Theta(r)}$$

where Θ has been defined in (8).

3.4. Distributional Laplacian and main technical lemma. As ρ is only Lipschitz, we will define its Laplacian in the distributional sense, as the pairing

$$(20) \quad (\Delta \rho, \phi) \doteq \int_{\Omega} \rho \Delta \phi$$

for all smooth functions ϕ which are compactly supported in the interior of Ω . We have the following lemma.

Lemma 9. a) *The distribution $\Delta\rho$ splits:*

$$(21) \quad \Delta\rho = \Delta_{\text{reg}}\rho + \Delta_{\text{cut}}\rho$$

where $\Delta_{\text{reg}}\rho \in L^1(\Omega)$ is as in (18), and where $\Delta_{\text{cut}}\rho$ is a distribution supported on the cut-locus $\text{Cut}_{\partial\Omega}$ and defined by:

$$(22) \quad (\Delta_{\text{cut}}\rho, \phi) = \int_{\partial\Omega} \phi(c(x), x)\theta(c(x), x) dv_{n-1}(x).$$

b) $\Delta_{\text{cut}}\rho$ is a positive distribution of order zero, hence a (positive) Radon measure; thus the pairing (20) can be extended to any continuous function ϕ on Ω .

c) As a consequence of the splitting (21), the positivity of $\Delta_{\text{cut}}\rho$ and (19) we have, in the distributional sense:

$$\Delta\rho \geq -\frac{\Theta'}{\Theta} \circ \rho,$$

where $\Theta(r)$ is our weight function, depending on the curvature data (K, H) and defined in (8).

Proof. We first observe that we have, for all $\phi \in C_c^\infty(\Omega)$:

$$(\Delta\rho, \phi) = \int_{\Omega} \rho \Delta\phi dv_n = \int_{\Omega} \langle \nabla\rho, \nabla\phi \rangle dv_n,$$

because ρ is Lipschitz hence also in $H^1(\Omega)$. Integrating in normal coordinates we see:

$$\int_{\Omega} \langle \nabla\rho, \nabla\phi \rangle dv_n = \int_{\partial\Omega} \int_0^{c(x)} \phi'(r, x)\theta(r, x) dr dv_{n-1}(x)$$

Integrating by parts, since $\phi(0, x) = 0$:

$$\int_0^{c(x)} \phi'(r, x)\theta(r, x) dr = \phi(c(x), x)\theta(c(x), x) + \int_0^{c(x)} \phi(r, x) \left(-\frac{\theta'(r, x)}{\theta(r, x)} \right) \theta(r, x) dr.$$

Integrating on $\partial\Omega$ and taking into account (18) and (22) we see:

$$\int_{\Omega} \langle \nabla\rho, \nabla\phi \rangle dv_n = (\Delta_{\text{cut}}\rho, \phi) + \int_{\Omega} \phi \Delta_{\text{reg}}\rho dv_n,$$

which shows the splitting (21). □

As a matter of notation, from now on we will write:

$$(\Delta\rho, \phi) = \int_{\Omega} \phi \Delta\rho,$$

where on the right it is understood the integral of ϕ with respect to the measure $\Delta\rho = \Delta_{\text{reg}}\rho + \Delta_{\text{cut}}\rho$. Note that ϕ can be any continuous function, not necessarily supported in the interior of Ω .

3.5. Main lemma. The lemma which follows is proved by integrating in normal coordinates, as we did in Lemma 9.

Lemma 10. *Let $u : [0, R] \rightarrow \mathbf{R}$ be smooth and consider the function $v = u \circ \rho$ on Ω . Then:*

a) v is Lipschitz.

b) One has, in the sense of distributions:

$$\Delta v = \Delta(u \circ \rho) = -u'' \circ \rho + (u' \circ \rho)\Delta\rho.$$

c) Green's formula holds: if $v = u \circ \rho$ and $f \in C^2(\Omega)$ then

$$\int_{\Omega} (f\Delta v - v\Delta f) = \int_{\partial\Omega} \left(f \frac{\partial v}{\partial N} - v \frac{\partial f}{\partial N} \right) dv_{n-1},$$

where Δv is taken in the sense of distributions, as in b).

About the proof: first observe that a) is immediate. For b), take a test-function ϕ and observe that

$$\left(\Delta(u \circ \rho), \phi \right) = \int_{\Omega} (u \circ \rho)\Delta\phi = \int_{\Omega} (u' \circ \rho)\langle \nabla\rho, \nabla\phi \rangle dv_n,$$

where the last equality holds because ρ is Lipschitz (hence in $H^1(\Omega)$), and $\nabla(u \circ \rho) = (u' \circ \rho)\nabla\rho$. We integrate the last term in normal coordinates, and then by parts in the inner integral and the equality follows. Finally, for c), we compute $\int_{\Omega} \langle \nabla f, \nabla v \rangle dv_n$ in two different ways. First:

$$(23) \quad \int_{\Omega} \langle \nabla f, \nabla v \rangle dv_n = \int_{\Omega} v\Delta f - \int_{\partial\Omega} v \frac{\partial f}{\partial N},$$

which is easily shown to hold because v is Lipschitz and f is smooth. On the other hand:

$$(24) \quad \begin{aligned} \int_{\Omega} \langle \nabla f, \nabla v \rangle dv_n &= \int_{\Omega_{\text{reg}}} \langle \nabla f, \nabla v \rangle dv_n \\ &= \int_{\Omega_{\text{reg}}} (u' \circ \rho)\langle \nabla f, \nabla\rho \rangle dv_n \\ &= \int_{\partial\Omega} \int_0^{c(x)} \left(u'(r)f'(r, x)\theta(r, x) dr \right) dx \end{aligned}$$

Now we integrate by parts in the inner integral and equate the two expressions; after some work we get the desired identity. We omit further details because they are straightforward.

4. PROOF OF THE COMPARISON THEOREM

4.1. Proof of the lower bound in Theorem 1. We assume $\sigma > 0$ and fix a positive first eigenfunction u of our one-dimensional model problem on $[0, R]$, associated to $\bar{\lambda}$. It satisfies:

$$\begin{cases} u'' + \frac{\Theta'}{\Theta}u' + \bar{\lambda}u = 0 \\ u'(R) = 0 \\ u'(0) = \sigma u(0) \end{cases}$$

Consider the pull-back function on Ω given by $v = u \circ \rho$. By Lemma 8 we know that $u' \geq 0$ on $[0, R]$. Hence, by Lemma 10b and Lemma 9c

$$\begin{aligned} \Delta v &= -u'' \circ \rho + (u' \circ \rho)\Delta\rho \\ &\geq \left(-u'' - \frac{\Theta'}{\Theta}u'\right) \circ \rho \\ &= \bar{\lambda}(u \circ \rho), \end{aligned}$$

that is,

$$(25) \quad \Delta v \geq \bar{\lambda}v.$$

Next, we consider a first positive eigenfunction f of our Robin problem

$$(26) \quad \begin{cases} \Delta f = \lambda f & \text{on } \Omega, \\ \frac{\partial f}{\partial N} = \sigma f & \text{on } \partial\Omega. \end{cases}$$

We multiply (25) by f , the first equation of (26) by v and subtract. We obtain

$$f\Delta v - v\Delta f \geq (\bar{\lambda} - \lambda)fv.$$

Now, by Lemma 10c

$$\int_{\Omega} (f\Delta v - v\Delta f) = \int_{\partial\Omega} \left(f \frac{\partial v}{\partial N} - v \frac{\partial f}{\partial N}\right) dv_{n-1} = 0,$$

simply because, on $\partial\Omega$:

$$\frac{\partial v}{\partial N} = u'(0) = \sigma u(0) = \sigma v \quad \text{and} \quad \frac{\partial f}{\partial N} = \sigma f.$$

We conclude that

$$0 \geq (\bar{\lambda} - \lambda) \int_{\Omega} fv.$$

As f and v are both positive we must have $\bar{\lambda} - \lambda \leq 0$ as asserted.

4.2. Proof of the upper bound of Theorem 1. Now assume $\sigma < 0$ and define u and v as in the previous case. Lemma 8 now says that $u' \leq 0$, hence we see $\Delta v \leq \bar{\lambda}v$. The proof goes exactly as before, with the inequalities reversed. The assertion follows.

4.3. Proof of Theorem 2. Recall that we have to show that if $K = H = 0$ then

$$\lambda_1(\Omega, \sigma) \geq \lambda_1([0, 2R], \sigma),$$

and that if Ω is a flat cylinder then equality holds.

Proof. We take $K = H = 0$ hence $\Theta(r) = 1$ for all r . Then, $\lambda_1(R, \Theta, \sigma)$ is the first eigenvalue of the problem:

$$(27) \quad \begin{cases} u'' + \lambda u = 0 \\ u'(0) = \sigma u(0) \\ u'(R) = 0 \end{cases}$$

On the other hand, $\lambda_1([0, 2R], \sigma)$ is the first eigenvalue of the following problem on $[0, 2R]$:

$$(28) \quad \begin{cases} u'' + \lambda u = 0 \\ u'(0) = \sigma u(0) \\ u'(2R) = -\sigma u(2R) \end{cases}$$

We show that problems (27) and (28) have in fact the same the first eigenvalue. First we observe that (28) is invariant under the symmetry $r \rightarrow 2R - r$. This means that, if we fix a positive first eigenfunction u of problem (28), then u must be either even or odd at $r = R$: as u is positive it must be even at R , so that $u'(R) = 0$. Hence u is also an eigenfunction of problem (27), necessarily the first (again, because it is positive). In conclusion:

$$\lambda_1(R, \Theta, \sigma) = \lambda_1([0, 2R], \sigma),$$

as asserted.

Now assume that $\Omega = [0, 2R] \times \Sigma$ with the product metric. Then we can separate variables and see that $L^2(\Omega)$ admits a basis of eigenfunctions of the Robin problem of type $f(r, x) = u(r)\phi(x)$ where $r \in [0, R]$ and $x \in \Sigma$. Here u is radial and satisfies (28), and ϕ is an eigenfunction of the Laplacian on the closed manifold Σ . Clearly an eigenfunction associated to $\lambda_1(\Omega, \sigma)$ must correspond to the case where $\phi(x)$ is a (non-zero) constant. Hence $f(r, x) = cu(r)$ is actually radial. This shows that equality holds for any flat cylinder, and the proof is complete. \square

4.4. Proof of Corollary 3. It is enough to show that the first eigenvalue of (27) satisfies, for $\sigma > 0$:

$$\lambda_1 > \frac{\pi^2 \sigma}{\pi^2 R + 4R^2 \sigma},$$

while, if $\sigma < 0$, one has $\lambda_1 < -\sigma$.

Proof. Case $\sigma > 0$. The spectrum is positive and any eigenfunction of problem (27) has the form

$$u(r) = a \sin(\sqrt{\lambda}r) + b \cos(\sqrt{\lambda}r)$$

for $\lambda > 0$. An easy calculation shows that the boundary conditions force $a \neq 0$ and

$$\frac{b}{a} = \frac{\sqrt{\lambda}}{\sigma} = \cot(R\sqrt{\lambda}).$$

If one sets $x = R\sqrt{\lambda}$ then $x \tan x = R\sigma$ hence x must be a positive zero of the function $\phi(x) = x \tan x - R\sigma$. In conclusion, we see that the first eigenvalue of (28) is given by :

$$\lambda_1 = \frac{c_1^2}{R^2},$$

where c_1 is the first the positive zero of $\phi(x) = x \tan x - R\sigma$. Now observe that $\phi(0)$ is negative and $\phi(x)$ gets large positive when x is close to $\frac{\pi}{2}$. This means that $c_1 \in (0, \frac{\pi}{2})$; by the Becker-Starck inequality [4]:

$$\tan x < \frac{\pi^2 x}{\pi^2 - 4x^2}, \quad \text{for all } x \in (0, \frac{\pi}{2}),$$

we see:

$$R\sigma = c_1 \tan c_1 < \frac{\pi^2 c_1^2}{\pi^2 - 4c_1^2} \quad \text{hence} \quad c_1^2 > \frac{\pi^2 R\sigma}{\pi^2 + 4R\sigma},$$

hence

$$\lambda_1 = \frac{c_1^2}{R^2} > \frac{\pi^2 \sigma}{\pi^2 R + 4R^2 \sigma},$$

which is the desired inequality.

Case $\sigma < 0$. In that case $\lambda \doteq \lambda_1 < 0$ and an associated eigenfunction is of type:

$$u(r) = a \sinh(\sqrt{|\lambda|}r) + b \cosh(\sqrt{|\lambda|}r).$$

The boundary conditions give

$$\frac{b}{a} = \frac{\sqrt{|\lambda|}}{\sigma} = -\coth(R\sqrt{|\lambda|}),$$

so that, if $x = R\sqrt{|\lambda|}$, then $R\sigma \coth x + x = 0$, which means that $R\sqrt{|\lambda_1|} = c_1$, where c_1 is the unique positive root of $\phi(x) \doteq R\sigma \coth x + x$. Now:

$$0 = \phi(c_1) = R\sigma \coth c_1 + c_1 < R\sigma + c_1,$$

hence $c_1 > -R\sigma$; consequently $\sqrt{|\lambda_1|} > -\sigma = |\sigma|$ and squaring both sides we get the assertion. \square

5. PROOF OF DOMAIN MONOTONICITY

In this section we prove Theorem 5.

5.1. The first Robin eigenvalue of a revolution manifold. Let $B(x_0, R)$ be a geodesic ball centered at the pole x_0 of a revolution manifold. It is a standard fact that, by rotational invariance, the first eigenfunction of the Robin Laplacian on $B(x_0, R)$ is radial: $v = v(r)$ where r is the distance to the pole; consequently, the first eigenvalue of $B(x_0, R)$ with parameter σ is the first eigenvalue of the following one-dimensional problem:

$$(29) \quad \begin{cases} v'' + (n-1)\frac{\Phi'}{\Phi}v' + \lambda v = 0 \\ v'(0) = 0 \\ v'(R) = -\sigma v(R) \end{cases}$$

Note that the condition $v'(0) = 0$ is imposed to have regularity at x_0 . For example, for the geodesic ball in hyperbolic space \mathbf{H}^n we have $\Phi(r) = \sinh r$ hence the problem becomes:

$$(30) \quad \begin{cases} v'' + (n-1)(\coth r)v' + \lambda v = 0 \\ v'(0) = 0 \\ v'(R) = -\sigma v(R). \end{cases}$$

It will be convenient to parametrize instead by the distance ρ to the boundary of $B(x_0, R)$. As $\rho = R - r$ we set:

$$u(r) = v(R - r), \quad \Theta(r) = \Phi(R - r)^{n-1}.$$

Note that Θ is positive on $[0, R]$; a calculation shows:

$$\frac{\Theta'(r)}{\Theta(r)} = -(n-1)\frac{\Phi'(R-r)}{\Phi(R-r)}, \quad (\log \Theta)''(r) = (n-1)(\log \Phi)''(R-r).$$

Therefore, Φ is log-concave if and only if Θ is log-concave. Problem (29) becomes the equivalent problem:

$$(31) \quad \begin{cases} u'' + \frac{\Theta'}{\Theta}u' + \lambda u = 0 \\ u'(0) = \sigma u(0) \\ u'(R) = 0. \end{cases}$$

5.2. Domain monotonicity. We now prove Theorem 5:

Theorem 11. *Let Ω be a domain of a revolution manifold M with pole x_0 , whose warping function Φ is log-concave. Assume that $\Omega \subseteq B(x_0, R)$. If $\sigma > 0$ then*

$$\lambda_1(\Omega, \sigma) \geq \lambda_1(B(x_0, R), \sigma),$$

while if $\sigma < 0$ then the opposite inequality holds: $\lambda_1(\Omega, \sigma) \leq \lambda_1(B(x_0, R), \sigma)$,

Note that the theorem applies in any space-form M_K .

Proof. Assume first $\sigma > 0$ and set for brevity $B \doteq B(x_0, R)$ and $\lambda = \lambda_1(B(x_0, R), \sigma)$. The first positive eigenfunction ϕ of B is radial, and depends only on the distance to the boundary of B , hence it is written $\phi = u \circ \rho$ where ρ is the distance function to the boundary. Therefore:

$$\nabla \phi = (u' \circ \rho) \nabla \rho$$

where u solves (31). Define a function $\sigma^* : \partial\Omega \rightarrow \mathbf{R}$ by the rule:

$$(32) \quad \sigma^*(x) = \frac{1}{\phi(x)} \frac{\partial \phi}{\partial N}(x),$$

where N is the inner unit normal to $\partial\Omega$. Set $\rho(x) = r$ and observe that, by Lemma 8, $u' \geq 0$. Then:

$$\frac{\partial \phi}{\partial N}(x) = \langle \nabla \phi(x), N(x) \rangle = u'(r) \langle \nabla \rho(x), N(x) \rangle \leq u'(r),$$

because $\nabla \rho$ is of unit length so that $\langle \nabla \rho, N \rangle \leq 1$. This means that

$$\frac{1}{\phi(x)} \frac{\partial \phi}{\partial N}(x) \leq \frac{u'(r)}{u(r)}.$$

Since Θ is log-concave, again by Lemma 8 we see:

$$\frac{u'(r)}{u(r)} \leq \sigma \quad \text{hence also} \quad \sigma^*(x) \leq \sigma,$$

for all $x \in \partial\Omega$. Now the restriction of ϕ to Ω which, by a slight abuse of language, we keep denoting by ϕ , satisfies:

$$\begin{cases} \Delta \phi = \lambda \phi & \text{on } \Omega \\ \frac{\partial \phi}{\partial N} = \sigma^* \phi & \text{on } \partial\Omega. \end{cases}$$

Since ϕ is positive on Ω , it is the first eigenfunction of that problem, hence:

$$\lambda \doteq \lambda_1(B(x_0, R), \sigma) = \lambda_1(\Omega, \sigma^*).$$

But now, as $\sigma^*(x) \leq \sigma$ we immediately see from monotonicity that $\lambda_1(\Omega, \sigma^*) \leq \lambda_1(\Omega, \sigma)$, and the assertion follows.

Now assume $\sigma < 0$. The proof in that case is similar, with all signs reversed; in particular $u' \leq 0$, $\sigma(r) \leq 0$ etc. One defines $\sigma^*(x)$ as before, and since $\langle \nabla \rho, N \rangle \leq 1$ we see

$$\frac{\partial \phi}{\partial N}(x) \geq u'(r)$$

hence $\sigma^*(x) \geq \sigma$. By considering the restriction of ϕ to Ω as before we conclude from monotonicity that $\lambda_1(B(x_0, r), \sigma^*) = \lambda_1(\Omega, \sigma^*) \geq \lambda_1(\Omega, \sigma)$, and the assertion follows. \square

6. PROOF OF MCKEAN-TYPE INEQUALITY

In this section we prove Theorem 6 and Theorem 7. The idea is simply to work with the ODE (30) and approximate $\coth r$ by 1.

6.1. A preparatory lemma.

Lemma 12. *Given a positive constant A , consider the following mixed Robin problem on $[0, R]$:*

$$(33) \quad \begin{cases} u'' + 2Au' + \lambda u = 0 \\ u'(0) = 0 \\ u'(R) = -\sigma u(R) \end{cases}$$

a) *One has, for all $R > 0$:*

$$\lambda_1 \geq \begin{cases} 2\sigma A - \sigma^2 & \text{if } 0 \leq \sigma \leq A \\ A^2 & \text{if } \sigma \geq A \end{cases}$$

b) *Now assume $\sigma < 0$. Then, for all $R > 0$:*

$$\lambda_1 \leq -\sigma^2 + 2A\sigma.$$

c) *Assume $\sigma > A$. There are positive constants R_0, c_0 depending only on A and σ such that, for all $R \geq R_0$ one has:*

$$\lambda_1 \geq A^2 + \frac{\pi^2}{R^2} - \frac{c_0}{R^3}.$$

Proof. We set, for brevity, $\lambda \doteq \lambda_1$ and assume, at first, that $\lambda < A^2$. Then, the solutions of the ODE:

$$(34) \quad \begin{cases} u'' + 2Au' + \lambda u = 0 \\ u'(0) = 0 \end{cases}$$

are all multiples of the function:

$$(35) \quad u(x) = e^{-Ax} \left(A \sinh(qx) + q \cosh(qx) \right)$$

where $q = \sqrt{A^2 - \lambda}$. One computes $u'(x) = -\lambda e^{-Ax} \sinh(qx)$, and the boundary condition $u'(R) = -\sigma u(R)$ gives:

$$(36) \quad \lambda = \sigma(A + q \coth(qR)).$$

a) Assume $\sigma \geq A$. As $\coth(qR) > 1$ one has $\lambda > \sigma A$. By hypothesis we have also $\lambda < A^2$, hence we see that $A^2 > \sigma A$ hence $\sigma < A$: contradiction.

The conclusion is that if $\sigma \geq A$ then $\lambda \geq A^2$.

Now assume $\sigma \leq A$. Then either $\lambda \geq A^2$ (and then, a fortiori, $\lambda \geq 2\sigma A - \sigma^2$, and we are done) or $\lambda < A^2$ and the eigenfunction is as in (35). In that case, equation (36) implies $\lambda \geq \sigma A + \sigma q$, that is:

$$\lambda - \sigma A \geq \sigma \sqrt{A^2 - \lambda}.$$

Squaring both sides we see

$$\lambda \geq 2\sigma A - \sigma^2.$$

In both cases we see that, if $\sigma \leq A$ then $\lambda \geq 2\sigma A - \sigma^2$, as asserted.

b) If $\sigma < 0$ then $\lambda < 0$; the exact expression (36) can be written:

$$|\lambda| = |\sigma|A + |\sigma|q \coth(qR).$$

Hence $|\lambda| - |\sigma|A \geq |\sigma|q$. Squaring both sides and proceeding as before we arrive at the upper bound $\lambda \leq -\sigma^2 + 2\sigma A$.

It remains to show c). By assumption $\sigma > A$ hence $\lambda > A^2$ by a). The solutions of (34) are now multiples of:

$$u(x) = e^{-Ax} \left(A \sin(qx) + q \cos(qx) \right), \quad \text{where } q = \sqrt{\lambda - A^2}.$$

Then $u'(x) = -\lambda e^{-Ax} \sin(qx)$ and the boundary condition $u'(R) = -\sigma u(R)$ gives $\lambda = \sigma A + \sigma q \cot(qR)$. Hence

$$\lambda - A^2 = \sigma q \cot(qR) + \sigma A - A^2;$$

multiplying by R^2 :

$$q^2 R^2 = (\sigma R)(qR) \cot(qR) + (\sigma A - A^2)R^2.$$

Setting $t = qR$ we see:

$$t^2 - \sigma R t \cot t - (\sigma A - A^2)R^2 = 0.$$

Conclude that the eigenvalues are given by

$$\lambda_k = A^2 + \frac{x_k^2}{R^2}$$

where $\{x_1, x_2, \dots\}$ is the sequence of positive zeroes of the function

$$\phi(x) = x^2 - \alpha x \cot x - \beta$$

where $\alpha = \sigma R$ and $\beta = (\sigma A - A^2)R^2$.

We need to estimate x_1 . Set:

$$R_0 \doteq \max\left\{\frac{2\pi}{\sqrt{\sigma A - A^2}}, \frac{4\sigma}{3(\sigma A - A^2)}\right\}.$$

If $x_1 \geq \pi$ inequality c) follows immediately. Then, in what follows we will assume $x_1 < \pi$. We first want to show that if $R \geq R_0$ then $x_1 > \frac{\pi}{2}$. In fact, in that case $\beta \doteq (\sigma A - A^2)R^2 \geq 4\pi^2$ hence, as $x_1^2 = \alpha x_1 \cot x_1 + \beta$ (by definition), one has

$$x_1^2 \geq \alpha x_1 \cot x_1 + 4\pi^2.$$

If $\cot x_1 \geq 0$ the inequality gives $x_1 > 2\pi$ and c) follows; then $\cot x_1 < 0$ so that $x_1 > \frac{\pi}{2}$ and, by our initial assumption

$$x_1 \in \left(\frac{\pi}{2}, \pi\right).$$

The definition of x_1 gives $\cot x_1 = \frac{x_1^2 - \beta}{\alpha x_1}$, or, equivalently, $\tan x_1 = \frac{\alpha x_1}{x_1^2 - \beta}$. As $\pi - x_1 \in (0, \frac{\pi}{2})$ and $\tan(\pi - x_1) = -\tan x_1$ it holds:

$$\tan(\pi - x_1) = \frac{\alpha x_1}{\beta - x_1^2}.$$

Considering that $\tan(\pi - x_1) \geq \pi - x_1$ we arrive at:

$$(37) \quad \pi - x_1 \leq \frac{\alpha x_1}{\beta - x_1^2}.$$

Since $x_1 \leq \pi$:

$$(38) \quad \frac{\alpha x_1}{\beta - x_1^2} = \frac{\sigma R x_1}{(\sigma A - A^2)R^2 - x_1^2} \leq \frac{\pi \sigma R}{(\sigma A - A^2)R^2 - \pi^2}.$$

If $R \geq R_0$ then $R \geq \frac{2\pi}{\sqrt{\sigma A - A^2}}$ and one checks that

$$(39) \quad \frac{\pi \sigma R}{(\sigma A - A^2)R^2 - \pi^2} \leq \frac{4\pi \sigma}{3(\sigma A - A^2)} \cdot \frac{1}{R}.$$

By (37), (38) and (39):

$$x_1 \geq \pi - \frac{4\pi \sigma}{3(\sigma A - A^2)} \cdot \frac{1}{R},$$

If $R \geq R_0$ the right hand side is non-negative, and squaring both sides we see that

$$x_1^2 \geq \pi^2 - \frac{c_0}{R}, \quad \text{with} \quad c_0 = \frac{8\pi \sigma}{3(\sigma A - A^2)}.$$

Eventually, when $R \geq R_0$, we obtain:

$$\frac{x_1^2}{R^2} \geq \frac{\pi^2}{R^2} - \frac{c_0}{R^3},$$

with

$$R_0 \doteq \max \left\{ \frac{2\pi}{\sqrt{\sigma A - A^2}}, \frac{4\sigma}{3(\sigma A - A^2)} \right\}, \quad \text{and} \quad c_0 = \frac{8\pi \sigma}{3(\sigma A - A^2)}.$$

The proof is complete. □

6.2. Proof of Theorems 6 and 7. Let Ω be a (bounded) domain in \mathbf{H}^n . We first assume $\sigma > 0$. Now $\Omega \subseteq B(x_0, R)$ for a suitable ball; by the monotonicity proved in Theorem 11 we see that, for $\sigma > 0$:

$$\lambda_1(\Omega, \sigma) \geq \lambda_1(B(x_0, R), \sigma).$$

Therefore, we proceed to estimate the first Robin eigenvalue of hyperbolic balls of radius R , which is the following problem on $[0, R]$ (see (30)):

$$(40) \quad \begin{cases} v'' + (n-1)(\coth r)v' + \lambda v = 0 \\ v'(0) = 0 \\ v'(R) = -\sigma v(R). \end{cases}$$

As remarked before, the change $u(r) = v(R-r)$ transforms (30) in the problem (31) of the type considered in Lemma 8: one then has $u' \geq 0$ on $[0, R]$ hence $v' \leq 0$ on $[0, R]$. As $\coth r \geq 1$ we see that

$$v'' + (n-1)v' + \lambda v \geq 0.$$

Note that if $\sigma < 0$ the inequality is reversed. The conclusion is that

Lemma 13. *If $\sigma > 0$ (resp. $\sigma < 0$) then $\lambda_1(B(x_0, R), \sigma)$ is larger than or equal to (resp. less than or equal to) the first eigenvalue of the problem:*

$$(41) \quad \begin{cases} v'' + (n-1)v' + \lambda v = 0 \\ v'(0) = 0 \\ v'(R) = -\sigma v(R). \end{cases}$$

6.3. Proof of Theorem 6. Given Lemma 13, we apply Lemma 12 for $A = \frac{n-1}{2}$. If $\sigma > 0$ we get immediately:

$$\lambda_1(\Omega, \sigma) \geq \begin{cases} \frac{(n-1)^2}{4} & \text{if } \sigma \geq \frac{n-1}{2} \\ (n-1)\sigma - \sigma^2 & \text{if } 0 < \sigma \leq \frac{n-1}{2} \end{cases}$$

If instead we assume $\sigma < 0$, then $\lambda_1(\Omega, \sigma) \leq -\sigma^2 + (n-1)\sigma$.

6.4. Proof of Theorem 7. Again, we apply Lemma 12c for $A = \frac{n-1}{2}$. We obtain:

$$\lambda_1(B_R, \sigma) \geq \frac{(n-1)^2}{4} + \frac{\pi^2}{R^2} - \frac{c_0}{R^3}$$

for $R \geq R_0$. The upper bound in Theorem 7 follows because $\lambda_1(B_R, \sigma) \leq \lambda_1^D(B_R)$ and the upper bound in (9).

7. APPENDIX : MODEL DOMAINS

We wish to construct, for any choice of K, H and R , an n -dimensional domain $\bar{\Omega} \doteq \bar{\Omega}(K, H, R)$ with boundary components Γ_1 and Γ_2 such that the first eigenvalue of $\bar{\Omega}$ with Robin conditions on Γ_1 and Neumann conditions on Γ_2 coincides with $\lambda_1(R, \Theta, \sigma)$.

Case 1. It covers three distinct situations: a) $K > 0$ and $H \in \mathbf{R}$, b) $K = 0$ and $H \neq 0$, c) $K < 0$ and $|H| > \sqrt{|K|}$.

In all these cases $\bar{\Omega}$ will be an annulus in M_K , the simply connected manifold with constant curvature K . Recall that the M_K is a revolution manifold with metric $g = dr^2 + s_K(r)^2 g_{\mathbf{S}^{n-1}}$ and that the coordinate r is geodesic distance to the pole $\{O\}$ of M . The mean curvature of the ball with center the pole and radius r , with respect to the inner unit normal $N = -\nabla r$ is

$$H(r) = \cot_K(r) \doteq \frac{s'_K(r)}{s_K(r)}$$

Let us set $A = \cot_K^{-1}(|H|)$ which is well-defined given our conditions on H . We remark that, if Ω has curvature data (K, H) , with $H > 0$, then its inner radius $R \leq A$ (see [24]). Our model domain will be the annulus defined as follows:

$$\bar{\Omega} = \bar{\Omega}(K, H, R) = \begin{cases} A \leq r \leq A + R & \text{if } H < 0 \\ A - R \leq r \leq A & \text{if } H \geq 0 \end{cases}$$

Note that the boundary of $\bar{\Omega}$ consists of two pieces; we call Γ_1 the component where $r = A$ and Γ_2 the other. One checks that the mean curvature of $\partial\bar{\Omega}$ is constant, equal to H , on Γ_1 .

Case 2. $H = 0, K = 0$. Then, we simply take the flat cylinder $\Omega = [0, R] \times \mathbf{S}^{n-1}$, and let $\Gamma_1 = \{0\} \times \mathbf{S}^{n-1}$.

Case 3. $K < 0, H \in (-\sqrt{|K|}, \sqrt{|K|})$. As ambient manifold we take the hyperbolic cylinder \tilde{M}_K , which is the rotationally invariant manifold $(-\infty, \infty) \times \mathbf{S}^{n-1}$ with metric

$$g = dr^2 + \Phi(r)^2 \cdot g_{\mathbf{S}^{n-1}}, \quad \Phi(r) \doteq s'_K(r) = \cosh(r\sqrt{|K|}).$$

The slice $\Sigma_r = \{r\} \times \mathbf{S}^{n-1}$ is isometric to the sphere of radius $\cosh(r\sqrt{|K|})$ and its mean curvature with respect to the normal ∇r is given by

$$H(r) = -\frac{\Phi'(r)}{\Phi(r)} = -\sqrt{|K|} \tanh(r\sqrt{|K|}).$$

Given $H \in (-\sqrt{|K|}, \sqrt{|K|})$, we let $A = -\frac{1}{\sqrt{|K|}} \tanh^{-1}(\frac{H}{\sqrt{|K|}})$; we define

$$\bar{\Omega} = [A, A + R] \times \mathbf{S}^{n-1},$$

and denote by Γ_1 the boundary component $\{A\} \times \mathbf{S}^{n-1}$. One checks that Γ_1 has mean curvature H with respect to the inner unit normal $N = \nabla r$ of $\bar{\Omega}$.

Case 4. $K < 0, H = \pm\sqrt{|K|}$. These are the limiting cases of Case 3 as $H \rightarrow \pm\sqrt{|K|}$.

With the above definitions, we can now state the following theorem.

Theorem 14. *Let $\lambda_1(\bar{\Omega}, \sigma)$ be the first eigenvalue of the problem*

$$(42) \quad \begin{cases} \Delta u = \lambda u & \text{on } \bar{\Omega} \\ \frac{\partial u}{\partial N} = \sigma u & \text{on } \Gamma_1, \quad \frac{\partial u}{\partial N} = 0 & \text{on } \Gamma_2. \end{cases}$$

Then $\lambda_1(\bar{\Omega}, \sigma) = \lambda_1(R, \Theta, \sigma)$.

Proof. We let $\rho : \bar{\Omega} \rightarrow \mathbf{R}$ be the distance of a point of $\bar{\Omega}$ to the component Γ_1 of $\partial\bar{\Omega}$. From its definition, the cut-locus of Γ_1 in $\bar{\Omega}$ is empty, hence ρ is C^∞ -smooth. Given the symmetries of $\bar{\Omega}$, the first eigenvalue of (42) is radial, and depends only on the distance to Γ_1 , so it can be written $u = v \circ \rho$. Now

$$\Delta u = -v'' \circ \rho + (v' \circ \rho) \Delta \rho.$$

As ρ is smooth, we have $\Delta \rho = -\frac{\Theta'}{\Theta} \circ \rho$, which is $(n-1)$ -times the mean curvature of the level set $\{\rho = r\}$. Computing the mean curvature of the level sets of $\bar{\Omega}$ one can check that in all of the above cases the formula holds with $\Theta(r) = (s'_K(r) - H s_K(r))^{n-1}$, the weight function defined in (8). Ultimately one sees that the first eigenvalue of problem (42) coincides with the first eigenvalue of the problem on $[0, R]$:

$$(43) \quad \begin{cases} v'' + \frac{\Theta'}{\Theta} v' + \lambda v = 0 \\ v'(0) = \sigma v(0), \quad v'(R) = 0 \end{cases}$$

and the assertion is proved. \square

7.1. Proof of Theorem 4. In the cases at hand the model annulus $\bar{\Omega}$ is contained in a ball $\tilde{\Omega}$ of M_K , and moreover $\partial\tilde{\Omega} = \Gamma_1$. From Theorem A in [13] (we know that $\tilde{R} \geq R$ with equality if and only if Ω is isometric to $\tilde{\Omega}$. Now the first Robin eigenvalue of $\tilde{\Omega}$ is the first eigenvalue of the problem on $[0, \tilde{R}]$:

$$\begin{cases} v'' + \frac{\Theta'}{\Theta}v' + \lambda v = 0 \\ v'(0) = \sigma v(0), \quad v'(\tilde{R}) = 0. \end{cases}$$

We compare this problem with problem (43), and as $\tilde{R} \geq R$ we see immediately from Lemma 8 that $\lambda_1(\bar{\Omega}, \sigma) \geq \lambda_1(\tilde{\Omega}, \sigma)$ with equality if and only if $\tilde{R} = R$. As $\lambda_1(\Omega, \sigma) \geq \lambda_1(\bar{\Omega}, \sigma)$ we see that, a fortiori, $\lambda_1(\Omega, \sigma) \geq \lambda_1(\tilde{\Omega}, \sigma)$ with equality iff $\tilde{R} = R$, that is, iff Ω is isometric to $\tilde{\Omega}$.

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