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Differentiating Siegel Modular Forms and the Moving Slope of $\mathcal{A}_g$

Samuel Grushevsky[1,*], Tomoyoshi Ibukiyama[2], Gabriele Mondello[3], and Riccardo Salvati Manni[3]

[1]Department of Mathematics and Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY 11794-3651, USA, [2]Department of Mathematics, Graduate School of Mathematics, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka, 560-0043 Japan, and [3]Università “La Sapienza”, Dipartimento di Matematica Guido Castelnuovo, Piazzale A. Moro 2, I-00185, Roma, Italy

**Correspondence to be sent to: e-mail: sam@math.stonybrook.edu*

We study the cone of moving divisors on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties. Partly motivated by the generalized Rankin–Cohen bracket, we construct a non-linear holomorphic differential operator that sends Siegel modular forms to Siegel modular forms, and we apply it to produce new modular forms. Our construction recovers the known divisors of minimal moving slope on $\mathcal{A}_g$ for $g \le 4$, and gives an explicit upper bound for the moving slope of $\mathcal{A}_5$ and a conjectural upper bound for the moving slope of $\mathcal{A}_6$.

1 Introduction

1.1 Moduli of principally polarized abelian varieties and compactifications

Denote $\mathcal{A}_g$ the moduli space of complex principally polarized abelian varieties (ppav), which is the quotient of its (orbifold) universal cover, the Siegel upper half-space $\mathbb{H}_g$, by the action of the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. Let $\mathcal{A}_g^*$ denote the Satake–Baily–Borel

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compactification, and recall that the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{A}_g^*) = \mathbb{Q}\lambda$ is one-dimensional, generated by the class λ of the line bundle $\mathcal{L} \to \mathcal{A}_g^*$ of Siegel modular forms of weight one, which is ample on $\mathcal{A}_g^*$.

Let $\mathcal{A}'_g$ be Mumford's partial compactification of $\mathcal{A}_g$, so that $\partial\mathcal{A}'_g = \mathcal{X}_{g-1}/\pm 1$, where $\pi : \mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ denotes the universal family of ppav of dimension $g-1$.

All toroidal compactifications of $\mathcal{A}_g$ contain $\mathcal{A}'_g$. The boundary of the perfect cone toroidal compactification $\overline{\mathcal{A}}_g$ (we use this notation as no other toroidal compactification will appear) is an irreducible Cartier divisor D, and $\partial\mathcal{A}'_g$ is dense within D. The compactification $\overline{\mathcal{A}}_g$ is $\mathbb{Q}$-factorial, with $\mathrm{Pic}_{\mathbb{Q}}\,\overline{\mathcal{A}}_g = \mathbb{Q}\lambda \oplus \mathbb{Q}\delta$, where δ denotes the class of D. The Picard group $\mathrm{Pic}_{\mathbb{Q}}\,\mathcal{A}'_g$ is generated by the restrictions of the classes λ and δ from $\overline{\mathcal{A}}_g$ to $\mathcal{A}'_g$. Philosophically, in what follows, the definition of the slope of divisors takes place on $\mathcal{A}'_g$, though to formally make sense of it we work on $\overline{\mathcal{A}}_g$ (and refer to [27, Appendix] for a discussion of why this notion is the same for any other toroidal compactification).

1.2 The ample and effective slopes

Given a divisor E on $\overline{\mathcal{A}}_g$ such that its class in the Picard group is $[E] = a\lambda - b\delta$, its *slope* is defined to be $s(E) := a/b$. The slope of a cone of divisors on $\overline{\mathcal{A}}_g$ is defined as the infimum of the slopes of divisors contained in the cone. Shepherd-Barron [48] proved that the *ample slope* of $\mathcal{A}_g$, that is the slope of the cone of ample divisors is equal to 12, namely,

$$s_{\mathrm{Amp}}(\overline{\mathcal{A}}_g) := \inf\left\{s(E)\colon E \in \mathrm{Amp}(\overline{\mathcal{A}}_g)\right\} = 12\,.$$

The *effective slope*, that is the slope of the cone of effective divisors

$$s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g) := \inf\left\{s(E)\colon E \in \mathrm{Eff}(\overline{\mathcal{A}}_g)\right\}\,,$$

has attracted a lot of attention, in particular because

$$s(K_{\overline{\mathcal{A}}_g}) = s\left((g+1)\lambda - \delta\right) = g+1\,,$$

so that the inequality $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g) < g+1$ would imply that $\mathcal{A}_g$ is of general type. Freitag [24] used the theta-null divisor θ_{null}, of slope $s(\theta_{\mathrm{null}}) = 8 + 2^{3-g}$, to show that $\mathcal{A}_g$ is of general type for $g \geq 8$, Mumford [40] used the Andreotti–Mayer divisor N_0, of slope $s(N_0) = 6 + \frac{12}{g+1}$, to show that $\mathcal{A}_g$ is of general type for $g \geq 7$, while recently the fourth author with collaborators [11] showed that $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_6) \leq 7$, which implies that the Kodaira dimension of $\mathcal{A}_6$ is non-negative.

It is known that $\mathcal{A}_g$ is unirational for $g \le 5$ (see [39], [7], [13], [52] for the harder cases of $g = 4, 5$). In fact, $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g)$ is known explicitly for $g \le 5$: the computation of $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_5)$ is one of the main results of [18], and the lower genera cases are reviewed below.

On the other hand, the slope $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g)$ is not known for any $g \ge 6$. While the techniques of Tai [50] show that $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g) = O(1/g)$ for $g \to \infty$ (as explained in [25]), not a single explicit effective divisor E on $\mathcal{A}_g$, for any g, with $s(E) \le 6$ is known.

The analogous notion of effective slope for the moduli space of curves $\mathcal{M}_g$ has been investigated in many papers, in particular for its similar link with the Kodaira dimension of $\mathcal{M}_g$, starting with [30] [28], [15], and with continuing recent progress such as [19].

1.3 The moving slope

Recall that an effective divisor E is called *moving* if $h^0(E) > 1$ and if moreover the base locus of its linear system $|E|$ has codimension at least two. The *moving slope* is the slope of the cone Mov of moving divisors

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) := \inf\{s(E) \colon E \in \mathrm{Mov}(\overline{\mathcal{A}}_g)\}\,.$$

Since the moving cone is contained in the effective cone, we have $s_{\mathrm{Eff}}(\mathcal{A}_g) \le s_{\mathrm{Mov}}(\mathcal{A}_g)$. We first observe that if the effective slope is in fact an infimum but not a minimum, then $s_{\mathrm{Eff}}(\mathcal{A}_g) = s_{\mathrm{Mov}}(\mathcal{A}_g)$ since there is an infinite sequence of effective divisors of strictly decreasing slopes converging to this infimum (see Lemma 2.2(iii) for a precise statement and proof). Thus, investigating the moving slope is only of interest if there exists an effective divisor $E \subset \mathcal{A}_g$ of slope $s(E) = s_{\mathrm{Eff}}(\mathcal{A}_g)$.

While the moving slope of $\overline{\mathcal{A}}_g$ is less well-studied than the effective slope, it is also important in attempting to determine the structure of the ring of Siegel modular forms, and in attempting to run the log-MMP for $\mathcal{A}_g$ and determine its interesting birational models: in fact, the pull-back of an ample divisor on a normal projective variety X via a non-constant rational map $f : \overline{\mathcal{A}}_g \dashrightarrow X$ is a moving divisor, as remarked in [4, Section 1.2].

The moving slope of $\overline{\mathcal{A}}_g$ is known for $g \le 4$, as we will review below, and Tai's results also imply that $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) = O(1/g)$ as $g \to \infty$. While the original published version of the paper [17] claimed an upper bound for $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_5)$, there was a numerical error, and the corrected (arXiv) version [18] does not allow to deduce any statement on $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_5)$. For $g = 6$ the knowledge of the moving slope of $\overline{\mathcal{A}}_6$ would help determining the Kodaira dimension of $\mathcal{A}_6$, if it turns out that $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_6) = 7 = s(K_{\overline{\mathcal{A}}_6})$. As in the case $g = 5$, though, the moving slope of $\overline{\mathcal{A}}_g$ remains unknown at present for every $g \ge 6$.

1.4 Context

Our paper revolves around the problem of constructing, from a given modular form, or from given modular forms, new modular forms of controlled slope. In particular, given a modular form of minimal slope, such procedure can provide other interesting modular forms of low slope: for example, for $2 \leq g \leq 4$, it does provide a modular form of minimal moving slope (Corollary C). Our construction(s) will consist in applying certain holomorphic differential operators to Siegel modular forms, so as to yield Siegel modular forms again (Theorem A).

For motivation, recall the definition of two such well-known operators for $g = 1$. The first one is the *Serre derivative* (credited by Serre [47, Theorem 4] to Ramanujan [42]): it sends modular forms of weight a to modular forms of weight $a+2$, and is defined as $\mathcal{S}_a(F) := \frac{dF}{d\tau} - \frac{\pi i a}{6} E_2 \cdot F$, where E_2 is the Eisenstein series of weight 2 (see also [56, Section 5] and [49, Lemma 3]). The second one is the second *Rankin–Cohen bracket* (see [43] and [8]), which sends a modular form of weight a to a modular form of weight $2a+4$, and is defined as $[F, F]_{2,a} := aF\frac{d^2F}{d\tau^2} - (a+1)\left(\frac{dF}{d\tau}\right)^2$. Note that $\mathcal{S}_a$ is a 1-homogeneous (i.e., multiplying F by a constant λ multiplies $\mathcal{S}_a(F)$ by λ^1) differential operator in τ with non-constant coefficients, while $[\cdot,\cdot]_{2,a}$ is 2-homogeneous, of pure order 2 (meaning that all summands involve the derivative $\frac{d}{d\tau}$ twice), with constant coefficients. There are also $2n$-th Rankin–Cohen brackets $[\cdot,\cdot]_{2n,a}$, which are $2n$-homogeneous, of pure order $2n$, with constant coefficients, and send modular forms of weight a to modular forms of weight $2a+4n$.

The holomorphic differential operators that we will produce for $g \geq 2$ are, on one hand, analogous to $\mathcal{S}_a$, as they will be g-homogeneous, of order g; on the other hand, they share some similarities with the even Rankin–Cohen brackets, as they will be pure of order g (meaning that each summand involves exactly g partial derivatives), with constant coefficients.

1.5 Main results

In order to formulate our main result, given a holomorphic function $F : \mathbb{H}_g \to \mathbb{C}$, we assemble the coefficients of its differential dF into the matrix

$$\partial F := \begin{pmatrix} \frac{\partial F}{\partial \tau_{11}} & \frac{\partial F}{2\partial \tau_{12}} & \cdots & \frac{\partial F}{2\partial \tau_{1g}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F}{2\partial \tau_{g1}} & \frac{\partial F}{2\partial \tau_{g2}} & \cdots & \frac{\partial F}{\partial \tau_{gg}} \end{pmatrix},$$

and we consider the holomorphic function $\det(\partial F) : \mathbb{H}_g \to \mathbb{C}$.

Suppose now that F is a modular form of weight a, with vanishing order b along the boundary $\partial\overline{\mathcal{A}}_g$ (this will be defined formally in the next section). The determinant $\det(\partial F)$ is in general not a modular form, but its restriction to the zero locus $\{F = 0\}$ behaves as a modular form of weight $ga + 2$ (a more intrinsic approach to $\det(\partial F)$ will be given in Remark 4.6). Our main result is the following construction.

Theorem A. For every $g \geq 2$ and every integer $a \geq \frac{g}{2}$ there exists a differential operator $\mathfrak{D}_{g,a}$ acting on the space of genus g Siegel modular forms of weight a that satisfies the following properties:

(i) if F is a genus g Siegel modular form of weight a and vanishing order b along the boundary, then $\mathfrak{D}_{g,a}(F)$ is a Siegel modular form of weight $ga + 2$ and of vanishing order $\beta \geq gb$ along the boundary;

(ii) the restriction of $\mathfrak{D}_{g,a}(F)$ to the zero locus of F is equal to the restriction of $\det(\partial F)$.

Remark 1.1. In Theorem A, it is possible to deal with Siegel modular forms F with character with respect to $\mathrm{Sp}(2g, \mathbb{Z})$, which occur only for $g = 2$ only. Since $\mathfrak{D}_{2,a}$ is quadratic, $\mathfrak{D}_{2,a}(F)$ will then still be a modular form (with trivial character).

What we will actually prove is a more precise version of this statement. In Theorem 6.2, we construct for every $g \geq 2$ and $a \geq \frac{g}{2}$ a holomorphic differential operator $\mathcal{D}_{Q_{g,a}}$ in the τ_{ij} with constant coefficients and we define $\mathfrak{D}_{g,a}(F) := \mathcal{D}_{Q_{g,a}}(F)/g!$ for every Siegel modular forms F of genus g and weight a. Thus, $\mathfrak{D}_{g,a}(F)$ is always polynomial in F and its partial derivatives, though its coefficients depend on the weight a. Though the operator $\mathfrak{D}_{g,a}$ need not be unique, properties (i–ii) in Theorem A force the Siegel modular form $\mathfrak{D}_{g,a}(F)$ to be unique up to adding modular forms divisible by F (which would thus vanish on the zero locus of F). The construction is explicit, and in Section 6.3 we will give the formulas for $\mathfrak{D}_{2,a}$ and $\mathfrak{D}_{3,a}$ explicitly.

A priori $\mathfrak{D}_{g,a}(F)$ could have a common factor with F, or could even be identically zero. In order to prevent such behavior, we will apply Theorem A only to modular forms F that satisfy what will be our *main condition:*

$$\det(\partial F) \textit{ does not vanish identically} \atop \textit{on any irreducible component of } \{F = 0\}. \qquad (\star)$$

Our main application is an immediate consequence of Theorem A.

Corollary B. Suppose that the effective slope $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g) = a/b$ is realized by a modular form F of weight $a \geq \frac{g}{2}$ that satisfies Condition ($\star$). Then

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) \leq s(\mathfrak{D}_{g,a}(F)) \leq s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g) + \frac{2}{bg}\,. \tag{1}$$

We first note that if the zero locus of F in Corollary B is not irreducible, that is, if $F = F_1 + F_2$ with F_1, F_2, effective, then since $s(F) \leq s(F_1), s(F_2)$, it follows that $s(F) = s(F_1) = s(F_2)$. As a consequence, $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) = s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g)$ (as will be proven carefully in Lemma 2.2(i)), and so the statement becomes trivial. Thus, we can assume that the zero locus of F is irreducible.

We stress that the inequality (1) for the moving slope depends on the actual class $[F]$, not just on the slope $s(F)$. Moreover, Condition ($\star$) forces F to be square-free. For every $g \leq 5$, it is known that a reduced effective divisor on $\overline{\mathcal{A}}_g$ of minimal slope exists and is unique. For $g \leq 4$, the machinery of Corollary B produces an (already known) divisor that realizes the moving slope.

Corollary C. For $2 \leq g \leq 4$, the modular form F of minimal slope on $\overline{\mathcal{A}}_g$ satisfies Condition ($\star$) and has weight $a \geq \frac{g}{2}$. Moreover, $\mathfrak{D}_{g,a}(F)$ realizes the moving slope of $\mathcal{A}_g$.

For $g = 5$, in [18], it was proven that the Andreotti–Mayer divisor N_0' (whose definition will be recalled in Section 3.3) is the unique effective divisor of minimal slope on $\overline{\mathcal{A}}_5$. Since we will show in Proposition 3.2 that N_0' satisfies Condition ($\star$), as a consequence of Corollary B, we obtain the following:

Corollary D. The moving slope of $\mathcal{A}_5$ is bounded above by $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_5) \leq \frac{271}{35}$, and the slope 271/35 is achieved by a moving effective divisor.

In the following table, we collect what is thus known about the effective and moving slopes of $\overline{\mathcal{A}}_g$:

	$s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g)$	$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g)$
$g=1$	12	
$g=2$	10	12
$g=3$	9	$28/3 = 9.333\ldots$
$g=4$	8	$17/2 = 8.500\ldots$
$g=5$	$54/7 = 7.714\ldots$	$\leq 271/35 = 7.742\ldots$
$g=6$	$[\frac{53}{10},\ 7]$	$(?) \leq 43/6 = 7.166\ldots$
$g \gg 1$	$O(1/g)$	$O(1/g)$

where the upper bound $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_6) \leq 7$ is provided by the Siegel modular form $\theta_{L,h,2}$ of class $14\lambda - 2\delta$ constructed in [11]. The question mark in the above table marks a conjectural upper bound $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_6) \leq 43/6$, which is a consequence of the following.

Corollary E. The form $\theta_{L,h,2}$ on $\mathcal{A}_6$ is prime, that is, not a product of non-constant Siegel modular forms. Moreover, if $\theta_{L,h,2}$ satisfies Condition ($\star$), then

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_6) \leq \frac{43}{6}.$$

The Torelli map $\tau_g : \mathcal{M}_g \to \mathcal{A}_g$ sending a curve to its Jacobian is an injection of coarse moduli spaces, but for $g \geq 3$ is 2-to-1 as a map of stacks. We denote by $\mathcal{J}_g$ the closure of $\tau_g(\mathcal{M}_g)$ inside $\mathcal{A}_g$, which is called the locus of Jacobians. For $g \leq 3$, we have $\mathcal{J}_g = \mathcal{A}_g$, while $\mathcal{J}_4 \subset \mathcal{A}_4$ is the zero locus of the Schottky modular form S_4, which has weight 8. Since (even) theta constants always vanish on curves with even multiplicity, this implies that $\theta_{\mathrm{null}} \cap \mathcal{J}_g = 2\Theta_{\mathrm{null}}$ for $g \geq 3$, where $\Theta_{\mathrm{null}} \subset \mathcal{J}_g$ is an integral divisor. As a byproduct of our analysis, we also obtain the following result on Jacobians:

Corollary F. The form $\mathfrak{D}_{4,8}(S_4)$ restricts on $\mathcal{J}_4$ to Θ_{null}.

Beyond these results, we investigate the applications of both Rankin–Cohen brackets and of differential operators acting on Siegel modular forms to constructing new effective divisors. Our results above go essentially one step in this direction, by applying the differentiation technique to a modular form of lowest slope. This construction can be iterated or varied to apply it to a tuple of different modular forms: it would be interesting to investigate the collection of modular forms thus produced, and to see in particular if this sheds any further light on the generators of the ring of Siegel modular forms in any genus $g \geq 4$, where they are not fully known.

1.6 Structure of the paper

The paper is organized as follows. In Section 2, we set the notation and review the relation between effective divisors on $\mathcal{A}_g$ and Siegel modular forms. In Section 3, we recall the construction and the slopes of the theta-null divisor θ_{null} and of the Andreotti–Mayer divisor N_0', and we show that both satisfy Condition ($\star$). In Section 4, we define the Rankin–Cohen bracket and prove a weaker version of Corollary B. In Section 5, we review the computation of the effective and moving slopes for $g \leq 4$, derive Corollaries C–D–E from Theorem A, and prove Corollary F. Finally, in Section 6, we introduce a remarkable class of differential operators acting on Siegel modular forms, we define $\mathcal{D}_{Q_{g,a}}$ and we prove Theorem A.

2 Siegel Modular Forms and Compactifications of $\mathcal{A}_g$

We briefly recall the standard notions on Siegel modular forms, referring to [24] for a more detailed introduction. Unless specified otherwise, we assume $g \geq 2$.

2.1 The Siegel space and the moduli space of ppav

The Siegel upper half-space $\mathbb{H}_g$ is the space of complex symmetric $g \times g$ matrices τ with positive definite imaginary part.

An element γ of the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$, written as $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $g \times g$ block form, acts on $\mathbb{H}_g$ via

$$\gamma \cdot \tau := (A\tau + B)(C\tau + D)^{-1}.$$

The action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathbb{H}_g$ is properly discontinuous, with finite stabilizers. The quotient $\mathcal{A}_g = \mathbb{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$ is the moduli space of ppav—it is a quasi-projective variety that can be given the structure of an orbifold (or a Deligne–Mumford stack). We denote by $\pi : \mathcal{X}_g \to \mathcal{A}_g$ the universal family of principally polarized abelian varieties (ppav in short), considered as a stack.

2.2 Divisors and Siegel modular forms

A holomorphic function $F : \mathbb{H}_g \to \mathbb{C}$ is called a holomorphic *Siegel modular form of weight k* with respect to $\mathrm{Sp}(2g, \mathbb{Z})$ if

$$F(\gamma \cdot \tau) = \det(C\tau + D)^k F(\tau)$$

for all $\tau \in \mathbb{H}_g$ and for all $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$ (for $g = 1$, there is an additional regularity condition that, by Koecher principle, is unnecessary for $g \geq 2$).

This automorphy property with respect to $\mathrm{Sp}(2g, \mathbb{Z})$ defines the line bundle

$$\mathcal{L}^{\otimes k} \longrightarrow \mathcal{A}_g$$

of Siegel modular forms of weight k on $\mathcal{A}_g$.

Remark 2.1. While in our paper we focus on Siegel modular forms for $\mathrm{Sp}(2g,\mathbb{Z})$, the holomorphic differential operator that we consider is defined for any holomorphic functions on $\mathbb{H}_g$, and will preserve suitable automorphy properties. It can thus also be applied to Siegel modular forms with multiplier systems for subgroups of $\mathrm{Sp}(2g,\mathbb{Z})$. In particular, we will apply it to a Siegel modular forms with a character, namely the theta-null T_2 in genus two, discussed in Section 3.1.

2.3 Satake compactification

The Satake–Baily–Borel compactification $\mathcal{A}_g^*$ can be defined as

$$\mathcal{A}_g^* := \mathbf{Proj}\left(\oplus_{n\geq 0} H^0(\mathcal{A}_g,\mathcal{L}^{\otimes n})\right).$$

What this means is that sections of a sufficiently high power of $\mathcal{L}$ embed $\mathcal{A}_g$ into a projective space, and $\mathcal{A}_g^*$ is the closure of the image of $\mathcal{A}_g$ under such an embedding. Since $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{A}_g^*) = \mathbb{Q}\lambda$, where λ denotes the class of $\mathcal{L}$, this implies that any effective $\mathbb{Z}$-divisor on $\mathcal{A}_g$ is the zero locus of a Siegel modular form.

2.4 Partial and perfect cone toroidal compactifications

Set-theoretically, $\mathcal{A}_g^*$ is the union of locally closed strata

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_0.$$

The partial (aka Mumford, or rank one) toroidal compactification

$$\mathcal{A}'_g := \mathcal{A}_g \sqcup \partial\mathcal{A}'_g$$

is obtained by blowing up the partial Satake compactification $\mathcal{A}_g \sqcup \mathcal{A}_{g-1}$ along its boundary $\mathcal{A}_{g-1}$, and the exceptional divisor $\partial\mathcal{A}'_g$ can then be identified with $\mathcal{X}_{g-1}/\pm 1$.

Any toroidal compactification contains $\mathcal{A}'_g$ and admits a blowdown morphism to $\mathcal{A}_g^*$. The perfect cone toroidal compactification $\overline{\mathcal{A}}_g$ has the property that the complement $\overline{\mathcal{A}}_g \setminus \mathcal{A}'_g$ is of codimension 2 inside $\overline{\mathcal{A}}_g$. The boundary

$$D := \partial\overline{\mathcal{A}}_g$$

is an irreducible Cartier divisor, which is the closure of $\partial\mathcal{A}'_g$. We denote by $p : \overline{\mathcal{A}}_g \to \mathcal{A}_g^*$ the blowdown map.

2.5 Effective divisors on $\mathcal{A}_g$

The effective and moving slope is computed on effective divisors in $\mathcal{A}_g$, or, equivalently, on effective divisors in $\overline{\mathcal{A}}_g$, whose support does not contain D. We will call such divisors *internal*. For clarity and completeness, we explain how to associate an internal divisor to a Siegel modular form.

A Siegel modular form F of weight a, thought of as a section of $\mathcal{L}^{\otimes a}$ on $\mathcal{A}_g^*$, can be pulled back to a section of $p^*\mathcal{L}^{\otimes a}$ on $\overline{\mathcal{A}}_g$. If the vanishing order $\mathrm{ord}_D(p^*F)$ of p^*F along the divisor D is b, this means that the zero locus of p^*F on $\overline{\mathcal{A}}_g$ is the union of an effective divisor not containing D in its support, which we will denote by (F) and call the zero divisor of the modular form, and of the divisor D with multiplicity b. Since by definition the zero locus $\{F = 0\} \subset \mathcal{A}_g^*$ has class $a\lambda$, its preimage in $\overline{\mathcal{A}}_g$ has class $ap^*\lambda$ (or $a\lambda$ in our notation abuse), it follows that the class of the zero divisor of a modular form is

$$[(F)] = a\lambda - b\delta \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{A}}_g)$$

with $a > 0$ and $b \geq 0$.

To summarize the above discussion, we see that internal effective divisors on $\overline{\mathcal{A}}_g$ correspond bijectively to Siegel modular forms up to multiplication by a constant, and from now on we will talk about them interchangeably, additionally suppressing the adjective “internal” as we will never need to deal with effective divisors on $\overline{\mathcal{A}}_g$ whose support contains D.

We thus define the slope $s(F)$ of a modular form F to be the slope of the corresponding (internal) effective divisor (F). We will write F for the modular form considered on $\overline{\mathcal{A}}_g$, and stress that the notation $[F] := [(F)]$ for the class of the zero divisor of a Siegel modular form on $\overline{\mathcal{A}}_g$ does *not* signify the class of the pullback p^*F, which would be simply equal to $a\lambda$.

Every effective divisor $E \subset \overline{\mathcal{A}}_g$ can be uniquely written as $E = \sum c_i E_i$ for suitable $c_i > 0$ and pairwise distinct, irreducible, reduced divisors E_i. We say that two divisors $E = \sum c_i E_i$ and $E' = \sum d_j E'_j$ have distinct supports if $E_i \neq E'_j$ for all i, j.

Similarly, a Siegel modular form F can be uniquely written as a product $F = \prod F_i^{c_i}$ for suitable $c_j > 0$ and pairwise distinct, *prime* Siegel modular forms F_i (i.e., forms that cannot be factored as products of non-constant modular forms). Two modular forms $F = \prod F_i^{c_i}$ and $F' = \prod (F'_j)^{d_j}$ are said to *not have a common factor* if $F_i \neq F'_j$ for all i, j.

2.6 Fourier–Jacobi expansion

The vanishing order of a Siegel modular form F at D can be computed using the Fourier–Jacobi expansion, which we briefly recall for further use. Writing an element $\tau \in \mathbb{H}_g$ as

$$\tau = \begin{pmatrix} \tau' & z \\ z^t & w \end{pmatrix} \in \mathbb{H}_g$$

with $\tau' \in \mathbb{H}_{g-1}$, $z \in \mathbb{C}^{g-1}$, $w \in \mathbb{C}^*$, and setting $q := \exp(2\pi i w)$, we expand F in power series in q:

$$F(\tau) = \sum_{r \geq 0} f_r(\tau', z) q^r. \quad (2)$$

Then the vanishing order $\mathrm{ord}_D F$ (which we will often denote b) of F along D is detected by the Fourier–Jacobi expansion as

$$\mathrm{ord}_D F = \min\{r \geq 0\, f_r(\tau', z) \not\equiv 0\}\,. \quad (3)$$

The form F is called a *cusp form* if it vanishes identically on D; equivalently, if $f_0(\tau', 0) = 0$, that is if $\mathrm{ord}_D F > 0$.

2.7 First properties of the moving slope

Here we record some properties of the moving slope, showing that one should only focus on the case when there exists an effective divisor of minimal slope, and furthermore that one should only focus on irreducible effective divisors. These are general properties that we state for $\overline{\mathcal{A}}_g$, but hold on any projective variety.

Lemma 2.2. The moving slope satisfies the following properties:

(i) if $E \neq E'$ are irreducible reduced effective divisors, then

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) \leq \max\{s(E), s(E')\}\,;$$

(ii) if $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) = s(E)$ for *some* moving divisor E, then there exists an *irreducible* moving divisor E' such that $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) = s(E')$;

(iii) if there does *not* exist an effective divisor E such that $s(E) = s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g)$, then $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g) = s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g)$.

Proof. (i) Let $[E] = a\lambda - b\delta$ and $[E'] = a'\lambda - b'\delta$, and suppose that $s(E) \le s(E')$. Then the linear system $|aE'|$ contains $a'E$, and its base locus is contained inside $E \cap E'$, which has codimension at least two. It follows that aE' is a moving divisor. Since $[aE'] = a(a'\lambda - b'\delta)$, we obtain $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) \le s(aE') = s(E') = a'/b'$.

(ii) If the general element of the linear system $|E|$ is irreducible, then we can choose E' to be any such element. Otherwise, a general element $E_t \in |E|$ can be written as a sum $E_t = E_t^1 + \cdots + E_t^m$ of m distinct effective divisors. Since E is moving, the base locus of $|E|$ has codimension at least two, thus each E_t^i is moving.

Moreover $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) \le \min_i s(E_t^i) \le s(E_t) = s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g)$, we conclude that $s(E_t^i) = s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g)$ for all i. Hence, it is enough to take $E' = E_t^i$ for any i.

(iii) Consider a sequence (E_n) of effective divisors on $\mathcal{A}_g$ whose slopes are strictly decreasing and converging to $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g)$. Up to replacing E_n by the irreducible component of E_n with smallest slope, and up to passing to a subsequence, we can assume that all E_n are irreducible. Since the slopes are strictly decreasing, the E_n are all distinct. Applying (i) to the pair E_{n-1}, E_n, we have $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) \le s(E_n)$. The conclusion follows, since $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g) \le s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) \le s(E_n) \to s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g)$. ∎

3 Some Relevant Modular Forms

In this section, we briefly recall the definitions and the main properties of theta constants, of the Schottky form, and of Andreotti–Mayer divisors.

3.1 Theta functions and theta constants

For $\varepsilon, \delta \in \{0,1\}^g$ the *theta function with characteristic* $\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right]$ is the function $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right] : \mathbb{H}_g \times \mathbb{C}^g \to \mathbb{C}$ defined as

$$\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau, z) := \sum_{n\in\mathbb{Z}^g} \exp \pi i \left[\left(n+\tfrac{\varepsilon}{2}\right)^t \tau \left(n+\tfrac{\varepsilon}{2}\right) + 2\left(n+\tfrac{\varepsilon}{2}\right)^t \left(z+\tfrac{\delta}{2}\right)\right].$$

Characteristics $\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right]$ are called *even* or *odd* depending on the parity of the standard scalar product $\langle\varepsilon, \delta\rangle$. This is the same as the parity of θ as a function of $z \in \mathbb{C}^g$ for fixed $\tau \in \mathbb{H}_g$, and there are $2^{g-1}(2^g+1)$ even characteristics and $2^{g-1}(2^g-1)$ odd ones. The *theta constant* is the evaluation of the theta function at $z = 0$, which is thus a function $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau) : \mathbb{H}_g \to \mathbb{C}$. By the above, all odd theta constants vanish identically, while an even theta constants are modular form of weight $1/2$ (meaning that a suitable square root of the automorphic factor $\det(C\tau + D)$ is taken) with respect to a certain finite index subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$. The product of all even theta constants

$$T_g := \prod_{\left[\substack{\varepsilon\\ \delta}\right] \text{ even}} \theta\left[\substack{\varepsilon\\ \delta}\right]$$

turns out to be a modular form for the full symplectic group, for $g \geq 3$, called the *theta-null* modular form, and its zero locus is called the theta-null divisor θ_{null}. It has class

$$[T_g] = 2^{g-2}(2^g+1)\lambda - 2^{2g-5}\delta\,, \qquad \text{and so} \quad s(T_g) = s(\theta_{\text{null}}) = 8 + 2^{3-g}\,. \tag{4}$$

The case $g = 2$ is slightly different since T_2 has a character, meaning that it satisfies

$$T_2(\gamma\cdot\tau) = \pm\det(C\tau+D)^5 T_2(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}(4,\mathbb{Z})$. Hence, T_2^2 is a well-defined modular form.

3.2 The Schottky form

The *Schottky form* is the weight 8 modular form on $\mathcal{A}_g$ given by the following degree 16 polynomial in theta constants:

$$S_g := \frac{1}{2^g}\sum_{\varepsilon,\delta}\theta^{16}\left[\substack{\varepsilon\\ \delta}\right] - \frac{1}{2^{2g}}\left(\sum_{\varepsilon,\delta}\theta^{8}\left[\substack{\varepsilon\\ \delta}\right]\right)^2 .$$

The Schottky form is a modular form for $\mathrm{Sp}(2g,\mathbb{Z})$, and is natural because it can alternatively be expressed as $S_g = \theta_{D_{16}^+} - \theta_{E8\oplus E8}$ as the difference of the lattice theta functions associated to the only two even, unimodular lattices in $\mathbb{R}^{16}$ (see [34] or [35]). It is known that S_g vanishes identically on $\mathcal{A}_g$ if and only if $g \leq 3$, and moreover that the zero locus of S_4 is the locus of Jacobians $\mathcal{J}_4 \subset \mathcal{A}_4$. The form S_4 vanishes identically to first order along D, and thus

$$[S_4] = 8\lambda - \delta\,, \quad \text{and} \quad s(S_4) = 8\,, \tag{5}$$

while for $g \geq 5$ the form S_g is not a cusp form (and so it has infinite slope).

3.3 Andreotti–Mayer divisor

The *Andreotti–Mayer divisor* [2] is defined to be the locus N_0 of ppav whose theta divisor is singular.

It is known that N_0 is a divisor that has for $g \geq 4$ precisely two irreducible components: $N_0 = \theta_{\text{null}} \cup N_0'$ (see [40],[10]), while for $g = 2, 3$ the Andreotti–Mayer divisor is simply $N_0 = \theta_{\text{null}}$.

Remark 3.1. For a generic point of θ_{null}, the unique singularity of the theta divisor of the corresponding ppav is the double point at the two-torsion point of the ppav corresponding to the characteristic of the vanishing theta constant. It is known that generically this singular point is an ordinary double point (i.e., that the Hessian matrix, of the second derivatives of the theta function with respect to z at this point is non-degenerate). For a generic point of N_0', the theta divisor of the corresponding ppav has precisely two opposite singular points, both of which are generically ordinary double points again, see [26] for a detailed study.

In this short section, we prove the following.

Proposition 3.2. The form T_g for $g \geq 2$ and the form I_g for $g \geq 4$ satisfy Condition ($\star$).

The genus restrictions in this statement are simply to ensure that the forms are well-defined and not identically zero.

As we already know, θ_{null} is the zero locus of the modular form T_g that is the product of all even theta constants, and we know the class of the corresponding divisor by (4). The modular form, which we denote I_g, defining the effective divisor N_0' is not known explicitly for any $g \geq 5$ (see [37]), while the Riemann theta singularity implies that in genus 4 we have $N_0' = \mathcal{J}_4$, and thus $I_4 = S_4$. The class of the divisor N_0' was computed by Mumford [40]:

$$[N_0'] = [I_g] = (g!\,(g+3)/4 - 2^{g-3}(2^g+1))\lambda - ((g+1)!\,/24 - 2^{2g-6})\delta\,, \tag{6}$$

$$\text{and so} \quad s(I_g) = 6 \cdot \frac{1 + 2/(g+1) - 2^{g-1}(2^g+1)/(g+1)!}{1 - 3 \cdot 2^{2g-3}/(g+1)!} > 6\,.$$

Before proving Proposition 3.2, we recall that theta functions satisfy the heat equation

$$d_\tau \theta = 2\pi i \cdot \text{Hess}_z\, \theta\,, \tag{7}$$

where Hess_z denotes the Hessian, that is the matrix of the second partial derivatives of the theta function with respect to $z_1, \ldots, z_g$. It will follow from Lemma 3.3 below, the differentials dT_g and dI_g are related to the Hessian of the theta function in the z-variables. Indeed, even though an equation for N_0' is not known, a precise description of its tangent space is provided by Lemma 3.3, which is a special case of results proven in [2] (see also [3]).

Lemma 3.3. Let Z be θ_{null} or N_0' and call $\widetilde{Z}$ its preimage in $\mathbb{H}_g$. For every general smooth point τ_0 of $\widetilde{Z}$, and every ordinary double point $z_0 \in \mathbb{C}^g$ of $\theta(\tau_0, \cdot) = 0$, the tangent space $T_{\tau_0}\widetilde{Z}$ has equation $d_\tau \theta(\tau_0, z_0) = 0$ inside $T_{\tau_0}\mathbb{H}_g$.

Using the above considerations, we can now prove the main proposition of this subsection.

Proof of Proposition 3.2. If τ_0 is a smooth point of θ_{null}, then the theta divisor $\Theta_{\tau_0} \subset X_{\tau_0} = \mathbb{C}^g/(\mathbb{Z}^g \oplus \tau_0\mathbb{Z}^g)$ is singular at a unique 2-torsion point, and such a singularity is ordinary if and only if $\det(dT_g) \neq 0$ at τ_0 by (7) and Lemma 3.3.

Similarly, if τ_0 is a generic point of N_0', then the singular locus of Θ_{τ_0} consists of two opposite non-2-torsion singular points $\pm z_0$; moreover, $\pm z_0$ are ordinary double points of Θ_{τ_0} if and only if $\det(dI_g) \neq 0$ at τ_0 by (7) and Lemma 3.3.

The conclusion follows from Remark 3.1. ■

4 Rankin–Cohen Bracket

Our method to bound the moving slope of $\mathcal{A}_g$ from above is by constructing new Siegel modular forms starting from a given known modular form. For example, starting from the known Siegel modular form minimizing the slope of the effective cone, we will try to construct another Siegel modular form, with which it has no common factor, and which has a slightly higher slope. In this section, we do this using the Rankin–Cohen bracket (of two different modular forms), which will allow us to prove the main application Corollary B, but only under the assumption that the moving slope is achieved.

While our construction of the differential operators $\mathfrak{D}_{g,a}$ in Theorem A yields a stronger result, we now give the details of the geometrically motivated construction using the Rankin–Cohen brackets. These were defined in [43] and [8] for $g = 1$ (see also [56]); a vector-valued version appears in [46] and a scalar-valued version appears in [55].

For further use, we define the symmetric $g \times g$ matrix-valued holomorphic differential operator acting on functions on $\mathbb{H}_g$

$$\partial_\tau := \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \tau_{ij}} \right)_{1 \le i,j \le g}. \tag{8}$$

When no confusion is possible, we will sometimes denote this differential operator simply by ∂.

4.1 Vector-valued bracket

Let F and G be genus g Siegel modular forms of weights k and h, respectively.

Definition 4.1 ([46]). The *vector-valued Rankin–Cohen bracket* of F and G is

$$\{F, G\} := \frac{G^{k+1}}{F^{h-1}} \cdot d\left(\frac{F^h}{G^k}\right).$$

where $d = d_\tau$ is the differential of a function of $\tau \in \mathbb{H}_g$.

Lemma 4.2. The vector-valued bracket

$$\{F, G\} = -\{G, F\} = hG\,dF - kF\,dG$$

is a $\mathcal{L}^{\otimes(h+k)}$-valued holomorphic $(1,0)$-form on $\mathcal{A}_g$. Moreover $\{F, G\} \not\equiv 0$ unless F^h and G^k are constant multiples of each other.

Proof. Since F^h/G^k is a meromorphic function on $\mathbb{H}_g$, its differential is a meromorphic $(1,0)$-form. Moreover, G^{k+1}/F^{h-1} is a meromorphic Siegel modular form of weight $h+k$ (i.e., it is a meromorphic function on $\mathbb{H}_g$ that satisfies the transformation property). It is immediate to check that $\{F, G\} = hG\,dF - kF\,dG$, which shows that $\{F, G\}$ is thus a holomorphic Siegel-modular-form-valued $(1,0)$ form. Since F and G are non-zero, the bracket vanishes identically if and only if $d(F^k/G^h)$ is identically zero, which is equivalent to this ratio being a constant. ■

Another way to state Lemma 4.2 is that, writing $\{F, G\}$ as a $g \times g$ matrix, this matrix satisfies the transformation law

$$\{F, G\}(\gamma \cdot \tau) = \det(C\tau + D)^{k+h}(C\tau + D)^t \cdot \{F, G\}(\tau) \cdot (C\tau + D)$$

for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\mathrm{Sp}(2g, \mathbb{Z})$.

4.2 Scalar-valued bracket

Let $\mathbb{E} \to \mathcal{A}_g$ denote the holomorphic rank g Hodge bundle of $(1, 0)$-holomorphic forms on ppav, namely $\mathbb{E} = \pi_*\Omega^{1,0}_\pi$ (where we recall that $\pi : \mathcal{X}_g \to \mathcal{A}_g$ denotes the universal family of ppav). Recall that the cotangent bundle $T^*\mathcal{A}_g$ can be identified with $\mathrm{Sym}^2\,\mathbb{E} \subset \mathrm{Hom}(\mathbb{E}^\vee, \mathbb{E})$. Since

$$\det : \mathrm{Hom}(\mathbb{E}^\vee, \mathbb{E}) \to (\det \mathbb{E})^{\otimes 2} \subset \Lambda^g(\mathrm{Sym}^2\,\mathbb{E}) \cong \Omega^{g,0}\mathcal{A}_g$$

and $\det \mathbb{E} \cong \mathcal{L}$, it follows that det restricts to a map $\det : T^*\mathcal{A}_g \to \mathcal{L}^{\otimes 2}$, which is homogeneous of degree g. If f is a meromorphic function defined on $\mathcal{A}_g$, then $\det(df)$ is a meromorphic section of $\mathcal{L}^{\otimes 2}$.

Definition 4.3. The *scalar Rankin–Cohen bracket* of Siegel modular forms F, G is defined as

$$[F, G] := \det\{F, G\}\,.$$

The scalar Rankin–Cohen bracket seems not to have been systematically studied in the literature. Here we collect some of its basic properties.

Lemma 4.4. Let F, G be Siegel modular forms, of classes

$$[F] = k\lambda - x\delta; \qquad [G] = h\lambda - y\delta.$$

Then $[F, G]$ is a Siegel modular form of class

$$\Big[[F, G]\Big] = (g(k + h) + 2)\lambda - \beta\delta\,,$$

where

(i) $\beta > 0$ (i.e., $[F, G]$ is a cusp form, even if F and G are not);

(ii) $\beta \geq g(x+y)$;
(iii) for any integer $n > 0$, $[F, F^n] = 0$;
(iv) if H is another modular form, then $[H^2F, G]$ and $[HF, HG]$ are divisible by H^g;
(v) if F, G do not have any common factors, and F satisfies Condition ($\star$), then F and $[F, G]$ do not have any common factors.

Proof. (i) Recall that $\{F, G\} = (G^{k+1}/F^{h-1}) \cdot d\left(F^h/G^k\right)$ and so $\det\{F, G\} = G^{g(k+1)}/F^{g(h-1)} \det(d\left(F^h/G^k\right))$. It follows that $\det\{F, G\}$ is a modular form of weight $gh(k+1) - gk(h-1) + 2 = g(h+k) + 2$ and, from the local expression of $\{F, G\}$, it follows that $[F, G]$ is holomorphic.

Consider then the Fourier–Jacobi expansions

$$F(\tau) = F_0(\tau', 0) + \sum_{r>0} F_r(\tau', z)q^r, \quad G(\tau) = G_0(\tau', 0) + \sum_{r>0} G_r(\tau', z)q^r,$$

at $\tau = \begin{pmatrix} \tau' & z \\ z^t & w \end{pmatrix}$. We have

$$dF = \begin{pmatrix} d_{\tau'}F & d_zF \\ (d_zF)^t & d_wF \end{pmatrix}, \quad dG = \begin{pmatrix} d_{\tau'}G & d_zG \\ (d_zG)^t & d_wG \end{pmatrix}.$$

Recall that $q = \exp(2\pi i w)$, so that $\partial(q^r)/\partial w = 2\pi r i q^r$. It is immediate to check that the last columns of dF and dG are divisible by q. It follows that $[F, G]$ is divisible by q, and so is a cusp form.

(ii) Writing dF and dG as above, it is immediate that $\operatorname{ord}_D dF = \operatorname{ord}_D F$ and $\operatorname{ord}_D dG = \operatorname{ord}_D G$. Hence $\operatorname{ord}_D\{F, G\} = \operatorname{ord}_D F + \operatorname{ord}_D G$, and the conclusion follows.

(iii) By direct computation $\{F, F^n\} = (nk)F^n dF - kF(nF^{n-1})dF = 0$.

(iv) Let ℓ be the weight of H; we compute directly

$$\begin{aligned}\{H^2F, G\} &= hG(H^2\, dF + 2HF\, dH) - (2\ell + k)H^2F\, dG \\ &= H(hHG\, dF + 2hF\, dH - (2\ell + k)HF\, dG)\end{aligned}$$

and

$$\begin{aligned}\{HF, HG\} &= (\ell + h)HG(H\, dF + F\, dH) - (\ell + k)HF(H\, dG + G\, dH) \\ &= H(H\{F, G\} + \ell H(G\, dF - F\, dG) + (h-k)FG\, dH)\,.\end{aligned}$$

(v) Note first that Condition ($\star$) implies that F is square-free. Evaluating $[F, G]$ along the zero divisor of F, we obtain

$$[F, G]|_{F=0} = h^g G^g \det(\partial F)\,. \tag{9}$$

Since F and G do not have common factors, $[F, G]$ is identically zero along a component of $\{F = 0\}$ if and only if $\det(\partial F)$ is. ■

Remark 4.5. It is possible that the strict inequality $\beta > g(x + y)$ holds in (ii) above: for example, (i) implies that $\beta \geq 1$ for $x = y = 0$.

Remark 4.6. Statement (v) above is one instance where we see the key importance of Condition ($\star$), and of $\det(\partial F)$. A more intrinsic description of the function $\det(\partial F)$ is as follows. If F is a modular form of weight k, its differential is not well-defined on $\mathcal{A}_g$, but the restriction of dF to the zero divisor $E = \{F = 0\}$ of F is. Thus $dF|_E$ is a section of $\mathcal{L}^{\otimes k} \otimes \mathrm{Sym}^2\, \mathbb{E}|_E$, and $\det(dF)$ is a section of $\mathcal{L}^{\otimes(kg+2)}|_E$. In other words, the restriction of $\det(\partial F)$ to the zero locus of F behaves as a modular form of weight $gk + 2$, as mentioned in the introduction.

4.3 The bracket and the moving slope

In this section, we apply the scalar Rankin–Cohen bracket to two modular forms of low slope in order to produce another modular form of low slope. This will allow us to prove the following weaker version of Corollary B—it is weaker only in that it assumes that the moving slope is achieved, that is, is a minimum rather than infimum.

Proposition 4.7. Assume that the effective slope $s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g) = a/b$ is realized by a Siegel modular form F of class $a\lambda - b\delta$ that satisfies Condition ($\star$). Suppose moreover that the moving slope $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) = a'/b'$ is achieved by a Siegel modular form G of class $a'\lambda - b'\delta$. Then

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) \leq s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g) + \frac{2}{bg}\,.$$

Proof. If F is a product of at least two distinct prime factors, then each of them realizes the effective slope. Hence, $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) = s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g)$ by Lemma 2.2(i), and so the conclusion trivially holds. Hence, we can assume that F is a prime Siegel modular form.

Up to replacing G by a general element in its linear system, we can assume that F does not divide G. By Lemma 4.4(v), the form $[F,G]$ is not divisible by F, and so in particular $[F,G]$ does not identically vanish. It follows from Lemma 4.4(ii) that

$$\frac{a'}{b'} = s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) \le s([F,G]) \le \frac{g(a+a')+2}{g(b+b')}\,,$$

which can be rewritten as

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_g) = \frac{a'}{b'} \le \frac{a}{b} + \frac{2}{bg} = s_{\mathrm{Eff}}(\overline{\mathcal{A}}_g) + \frac{2}{bg}\,.$$

■

Both the scalar Rankin–Cohen bracket and $\mathfrak{D}_{g,a}$ (which will be introduced in Section 6) are holomorphic differential operators of degree g, but their relationship is not clear, and deserves a further investigation.

5 Effective and Moving Slopes for Small g

In this section, we recall what is known about the effective and moving slopes of $\mathcal{A}_g$ for $2 \le g \le 5$. In all these cases, the effective slopes are achieved, and we analyze what we obtain by applying Theorem A (whose proof is postponed till Section 6) to such effective divisors of minimal slope, and we prove Corollaries C–D–E–F.

5.1 Case $g = 2$

In genus 2, the unique effective divisor of minimal slope is the closure of the locus $\mathcal{A}_2^{\mathrm{dec}}$ of decomposable abelian varieties inside $\overline{\mathcal{A}}_2$. Set-theoretically, this locus is simply equal to the theta-null divisor θ_{null}. We thus obtain

$$s_{\mathrm{Eff}}(\overline{\mathcal{A}}_2) = s(\mathcal{A}_2^{\mathrm{dec}}) = s(\theta_{\mathrm{null}}) = s(5\lambda - \delta/2) = 10\,.$$

Remark 5.1. Note that the class $[T_2] = \frac{1}{2}(10\lambda - \delta)$ in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{A}}_g)$ is not integral, though its double is. From the stacky point of view, this is a manifestation of the fact that $\mathcal{A}_2^{\mathrm{dec}} \cong (\mathcal{A}_1 \times \mathcal{A}_1)/S_2$ and so the general element of $\mathcal{A}_2^{\mathrm{dec}}$ has an automorphism group $\{\pm 1\} \times \{\pm 1\}$, of order 4, whereas the general genus 2 ppav has automorphism group $\{\pm 1\}$, of order 2.

As mentioned in the introduction, Theorem A can be applied to T_2, even though T_2 is a modular form with character. Since T_2 satisfies Condition $(\star)$ by Proposition 3.2, we obtain a cusp form $\mathfrak{D}_{2,5}(T_2)$ of weight 12 that is not identically zero on θ_{null}. As in Corollary B, it follows that

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_2) \leq s\left(\mathfrak{D}_{2,5}(T_2)\right) = 12\,.$$

Proof of Corollary C for $g = 2$. It is known [24] that the ideal of cusp forms inside the ring of genus 2 Siegel modular forms is generated by two modular forms $\chi_{10} := T_2^2$ and χ_{12}, which has class $[\chi_{12}] = 12\lambda - \delta$. It then follows that $\mathfrak{D}_{2,5}(T_2)$ and χ_{12} are proportional, and so $\mathfrak{D}_{2,5}(T_2)$ realizes the moving slope. ■

Since T_2 satisfies Condition ($\star$) by Proposition 3.2, and since T_2 and χ_{12} are square-free and without common factors, Lemma 4.4(v) ensures that the cusp form $[T_2, \chi_{12}]$ does not vanish identically along θ_{null}. By Lemma 4.4, it follows that

$$\left[[T_2, \chi_{12}]\right] = 36\lambda - 3\delta\,,$$

and so $[T_2, \chi_{12}]$ is another Siegel modular form that achieves the moving slope.

5.2 Jacobian and hyperelliptic loci

As mentioned in the introduction, it is possible to define a slope for effective divisors in the moduli space $\mathcal{M}_g$ of projective curves of genus g. We denote

$$\tau_g : \mathcal{M}_g \longrightarrow \mathcal{A}_g$$

the Torelli map that sends a smooth projective curve of genus g to its Jacobian.

The Torelli map is known to extend to a morphism $\overline{\tau}_g : \overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g$ from the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ to $\overline{\mathcal{A}}_g$ [1], but for our purposes, it will suffice to work with the well-known partial extension

$$\tau'_g : \mathcal{M}'_g \longrightarrow \mathcal{A}'_g$$

from the moduli space $\mathcal{M}'_g$ of irreducible stable curves of genus g with at most one node. The partial compactification $\mathcal{M}'_g$ is the union of $\mathcal{M}_g$ and the boundary divisor $\Delta' = \partial\mathcal{M}'_g$ consisting of singular curves with only one node, which is non-separating. It is well-known that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}'_g) = \mathbb{Q}\lambda_1 \oplus \mathbb{Q}\delta'$ for $g \geq 3$, and that the map induced by τ'_g on Picard groups is

$$(\tau'_g)^*\lambda = \lambda_1, \quad (\tau'_g)^*\delta = \delta' .$$

The slope for a divisor $a\lambda_1 - b\delta'$ on $\mathcal{M}'_g$ is defined to be a/b, and the slopes of cones of divisors on $\mathcal{M}'_g$ are defined analogously to $\mathcal{A}'_g$.

Remark 5.2. The standard definition of slope for an effective divisor in $\overline{\mathcal{M}}_g$ involves the vanishing order at all the boundary divisors of $\overline{\mathcal{M}}_g$. It follows from [20] that, if we limit ourselves to divisors of slopes less than 29/3 for $g \leq 5$, then the two definitions are equivalent.

As a consequence of the above discussion, we have obtained the following

Lemma 5.3. Let $g \geq 4$ and let E be an (internal) effective divisor on $\mathcal{A}_g$.

(i) If E does not contain the Jacobian locus $\mathcal{J}_g$, then $(\tau'_g)^{-1}(E)$ is an effective divisor in $\mathcal{M}'_g$ of slope $s(E)$;

(ii) If $s(E) < s_{\mathrm{Eff}}(\mathcal{M}_g)$, then E contains the Jacobian locus $\mathcal{J}_g$.

Intersecting an effective divisor with the locus $\mathcal{HJ}_g$ of hyperelliptic Jacobians can also provide a constraint for the slope. The closure $\mathcal{H}'_g$ of the locus of hyperelliptic curves $\mathcal{H}_g$ inside $\mathcal{M}'_g$ is obtained by adding the locus $\partial\mathcal{H}'_g$ consisting of curves with one non-disconnecting node, obtained from smooth hyperelliptic curves of genus $g-1$ by identification of two points that are exchanged by the hyperelliptic involution, cf. [9].

Call the restriction of λ_1 to $\mathcal{H}'_g$ from $\overline{\mathcal{M}}_g$ still λ_1, and denote ξ_0 the class of $\partial\mathcal{H}'_g$ (which is also the restriction of δ_0 from $\overline{\mathcal{M}}_g$). It is known that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{H}'_g)$ has dimension 1 and is generated by λ_1, ξ_0 with the relation $(8g+4)\lambda_1 = g\xi_0$ (see [9, Proposition 4.7]). The map $\overline{\tau}_g$ restricts to $\mathcal{H}'_g \to \mathcal{A}'_g$ and sends $\partial\mathcal{H}'_g$ to the boundary of $\mathcal{A}'_g$.

The following result was proven by Weissauer [53] (see [45] for details). Here we present a different argument.

Proposition 5.4. For every $g \geq 3$, modular forms of slope strictly less than $8 + \frac{4}{g}$ must contain $\mathcal{HJ}_g$.

Proof. Let F be a modular form on $\mathcal{A}_g$ with class $[F] = a\lambda - b\delta$, and suppose that F does not vanish on the entire $\mathcal{HJ}_g$. We want to show that $s(F) = \frac{a}{b} \geq 8 + \frac{4}{g}$.

The pullback of F on $\mathcal{H}_g$ vanishes on an effective divisor V of class $[V] = a\lambda_1 - \beta\xi_0$ with $\beta \geq b$. Using the relation in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{H}'_g)$, we obtain $[V] = \beta\left(\frac{a}{\beta} - \left(8 + \frac{4}{g}\right)\right)\lambda_1$.

Consider now the double cover C_t of $\mathbb{P}^1$ branched at $\lambda_1, \ldots, \lambda_{2g+1}, t$, and fix distinct $\lambda_1, \ldots, \lambda_{2g+1}, t_0$ such that $C_{t_0} \notin V$. Then $(C_t)_{t\in\mathbb{P}^1}$ induces a map $\mathbb{P}^1 \to \mathcal{H}'_g$, whose image is a complete, irreducible curve $B \subset \mathcal{H}'_g$ not contained in $V \cup \partial\mathcal{H}'_g$. It follows that $\deg_B(V) \geq 0$ and $\deg_B(\lambda_1) > 0$, and so $\frac{a}{\beta} - \left(8 + \frac{4}{g}\right) \geq 0$. The conclusion follows, since $s(F) \geq \frac{a}{\beta}$. ■

5.3 Case $g = 3$

The moduli space $\mathcal{A}_3$ has a meaningful effective divisor, namely (the closure of) the locus $\mathcal{HJ}_3$ of hyperelliptic Jacobians.

Proof of Corollary C for $g = 3$. By Proposition 5.4, a divisor in $\mathcal{A}_3$ with slope smaller than $\frac{28}{3}$ must contain $\mathcal{HJ}_3$. This implies that the only effective divisor that could be of slope under $\frac{28}{3}$ is (the closure of) the hyperelliptic locus itself, and so $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_3) \geq \frac{28}{3}$. Since the closure of $\mathcal{HJ}_3$ coincides with the theta-null divisor, we obtain from (4)

$$s(\mathcal{HJ}_3) = s(T_3) = s(18\lambda - 2\delta) = 9 < \frac{28}{3}\,.$$

It follows that

$$s_{\mathrm{Eff}}(\overline{\mathcal{A}}_3) = s(\mathcal{HJ}_3) = 9 \quad \text{and} \quad s_{\mathrm{Mov}}(\overline{\mathcal{A}}_3) \geq \frac{28}{3}\,.$$

Since T_3 satisfies Condition ($\star$) by Proposition 3.2, Theorem A provides a modular form $\mathfrak{D}_{3,18}(T_3)$ of class $56\lambda - \beta\delta$ with $\beta \geq 6$. If $\beta \geq 7$, then the slope of $\mathfrak{D}_{3,18}(T_3)$ would be $s(\mathfrak{D}_{3,18}(T_3)) \leq 56/7$, which is less than 9, contradicting the knowledge of effective slope. Thus $\beta = 6$, and

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_3) \leq s(\mathfrak{D}_{3,18}(T_3)) = \frac{56}{6} = \frac{28}{3}\,.$$

This proves that the moving slope is equal to $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_3) = \frac{28}{3}$, and is realized by $\mathfrak{D}_{3,18}(T_3)$. ■

There are also other constructions of Siegel modular forms in $\mathcal{A}_3$ of slope $\frac{28}{3}$:

- Let M be the set of all octuplets of even characteristics that are cosets of some three-dimensional vector space of characteristics, and define

$$\chi_{28} := \sum_{M \text{ even octuplet coset}} \left(\frac{T_3}{\prod_{m \in M} \theta_m} \right)^2 .$$

  This can be checked to be a modular form of class $[\chi_{28}] = 28\lambda - 3\delta$, see [51]. We verify that χ_{28} cannot be divisible by T_3, as otherwise χ_{28}/T_3 would be a holomorphic cusp form of weight 10, which does not exist by [38, 51].
- Alternatively, one can consider the forms

$$\chi_{140} := \sum_{m \text{ even}} \left(\frac{T_3}{\theta_m} \right)^8 ,$$

  which can be shown to have class $[\chi_{140}] = 140\lambda - 15\delta$. We remark that the decomposable locus $\mathcal{A}_3^{\text{dec}}$ can be described by the equations $T_3 = \chi_{140} = 0$, since $\mathcal{A}_3^{\text{dec}}$ is simply the locus where at least two theta constants vanish. Since $\mathcal{A}_3^{\text{dec}}$ has codimension 2 within $\mathcal{A}_3$, this confirms that the forms T_3 and χ_{140} could not have a common factor.
- Since T_3 satisfies Condition ($\star$) by Proposition 3.2, and since T_3 and χ_{28} are square-free and without common factors, Lemma 4.4(v) ensures that the Rankin–Cohen bracket $[T_3, \chi_{28}]$ does not vanish identically along θ_{null}. By Lemma 4.4, $[T_3, \chi_{28}]$ has weight 140, and vanishes to order $\beta \geq 15$ along the boundary. However, if it were to vanish to order 16 or higher, then its slope would be at most $\frac{140}{16} = 8.75$, which is impossible since $s_{\text{Eff}}(\overline{\mathcal{A}}_3) = 9$. Thus, we must have $\beta = 15$, so that

$$\Big[[T_3, \chi_{28}]\Big] = 140\lambda - 15\delta$$

  is a Siegel modular form that also realizes $s_{\text{Mov}}(\overline{\mathcal{A}}_3) = \frac{140}{15} = \frac{28}{3}$. It is not clear whether $[T_3, \chi_{28}]$ and χ_{140} are proportional; however, it is is easy to see that χ_{140} lies in the ideal generated by T_3 and $[T_3, \chi_{28}]$.

5.4 Case $g = 4$

The locus of Jacobians $\mathcal{J}_4$ is a divisor in $\mathcal{A}_4$, which is known to be the unique effective divisor on $\overline{\mathcal{A}}_4$ of minimal slope, see [44]. It is known that the effective slope of $\mathcal{M}_4$ is $s_{\mathrm{Eff}}(\mathcal{M}_4) = \frac{17}{2}$ (see [16] and [20]),

By Riemann's theta singularity theory, theta divisors of Jacobians are singular, and in fact $\mathcal{J}_4 = N_0'$. Since $I_4 = S_4$ and since $[I_4] = 8\lambda - \delta$ as recalled in Section 3.3, this reconfirms the equality

$$s_{\mathrm{Eff}}(\overline{\mathcal{A}}_4) = s(I_4) = s(8\lambda - \delta) = 8\,.$$

As I_4 satisfies Condition ($\star$) by Proposition 3.2, Theorem A applied to I_4 produces a modular form $\mathfrak{D}_{4,8}(I_4)$ not divisible by I_4, of class $[\mathfrak{D}_{4,8}(I_4)] = 34\lambda - \beta\delta$, with $\beta \geq 4$. Again, if β were at least 5, the slope would be at most $34/5 < 8$, contradicting the effective slope, and thus we must have $\beta = 4$.

Proof of Corollary C for $g = 4$. From the above discussion, it follows that

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_4) \leq s(\mathfrak{D}_{4,8}(I_4)) = s(34\lambda - 4\delta) = \tfrac{17}{2}\,.$$

On the other hand, Lemma 5.3 implies that any effective divisor in $\mathcal{A}_4$ that does not contain the locus of Jacobians has slope at least $s_{\mathrm{Eff}}(\mathcal{M}_4) = \frac{17}{2}$. It follows that $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_4) \geq \frac{17}{2}$.

We thus conclude that $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_4) = \frac{17}{2} = s(\mathfrak{D}_{4,8}(I_4))$. ■

There are at least two other modular forms in $\mathcal{A}_4$ that realize the moving slope.

- The first one is T_4, whose class is $[T_4] = 68\lambda - 8\delta$ by (4).
- The second one is the Rankin–Cohen bracket $[I_4, T_4]$. Since I_4 satisfies Condition ($\star$) by Proposition 3.2, and since I_4 and T_4 are square-free and without common factors, Lemma 4.4 ensures that $[I_4, T_4]$ does not vanish identically along $N_0' = \mathcal{J}_4$, and has class $306\lambda - \beta\delta$, with $\beta \geq 36$. Since the effective slope of $\mathcal{A}_4$ is 8 and the moving slope of $\mathcal{A}_4$ is $\frac{17}{2}$, we must have $\beta = 36$ and

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_4) = \tfrac{17}{2} = \tfrac{306}{36} = s([I_4, T_4])\,.$$

As mentioned in the introduction, the pullback $\tau_g^* T_g$ via the Torelli map gives $2\Theta_{\text{null}}$ on $\mathcal{M}_g$, that is, $\sqrt{T_g}$ is not a modular form, but its restriction to $\mathcal{J}_g$ is a Teichmüller modular form. For $g = 4$, we exhibit a Siegel modular form that intersects $\mathcal{J}_4$ in the divisor Θ_{null}, with multiplicity 1.

Proof of Corollary F. Recall that $S_4 = I_4$. By Corollary C for $g = 4$ proven above, $\mathfrak{D}_{4,8}(S_4)$ realizes the moving slope of $\mathcal{A}_4$, and so it does not contain the divisor of minimal slope, namely the Schottky divisor. Thus, $\tau_4^* \mathfrak{D}_{4,8}(S_4)$ is an effective divisor on $\mathcal{M}_4'$ of class $34\lambda_1 - 4\delta'$, which thus realizes the effective slope $s_{\text{Eff}}(\overline{\mathcal{M}}_4) = \frac{17}{2}$. Thus, the pullbacks $\tau_4^* T_4$ and $\tau_4^* \mathfrak{D}_{4,8}(S_4)$ must have the same support. Since $[T_4] = 2[\mathfrak{D}_{4,8}(S_4)]$, we conclude that $\tau_4^* \mathfrak{D}_{4,8}(S_4) = \Theta_{\text{null}}$. ■

Remark 5.5. For the sake of completeness, we recall that the moving slope of $\mathcal{M}_4$ is $s_{\text{Mov}}(\mathcal{M}_4) = 60/7$, see [22]. We can exhibit a modular form (analogous to χ_{140} for $g = 3$) with this slope, namely

$$\phi_{540} := \sum_{m \text{ even}} \left(\frac{T_4}{\theta_m}\right)^8 .$$

The Siegel modular form ϕ_{540} has class $540\lambda - 63\delta$, see [34], and hence $\tau_4^* \phi_{540}$ gives an effective divisor on $\mathcal{M}_4'$ that realizes the moving slope $s_{\text{Mov}}(\mathcal{M}_4')$. Finally, we observe that both T_4 and ϕ_{540} have slope less than 9, and the equations $T_4 = \phi_{540} = 0$ define, set theoretically, the hyperelliptic locus $\mathcal{H}_4 \subset \mathcal{M}_4$, as discussed in [23].

5.5 Case $g = 5$

We recall that one of the main results of [18] was the proof that the divisor N_0' in $\overline{\mathcal{A}}_5$ has minimal slope:

$$s_{\text{Eff}}(\overline{\mathcal{A}}_5) = s(I_5) = s(108\lambda - 14\delta) = \tfrac{54}{7} = 7.714\ldots$$

Since I_5 satisfies Condition ($\star$) by Proposition 3.2, by Theorem A

$$[\mathfrak{D}_{5,108}(I_5)] = 542\lambda - \beta\delta \quad \text{with} \quad \beta \geq 70$$

is a modular form that does not vanish identically on N_0'.

Proof of Corollary B. If $\beta \geq 71$, then the slope of $\mathfrak{D}_{5,108}(I_5)$ would be at most

$$542/71 = 7.633\cdots < 7.714\cdots = 54/7 = s_{\mathrm{Eff}}(\overline{\mathcal{A}}_5)\,,$$

which is a contradiction. Thus $\beta = 70$, and

$$s_{\mathrm{Mov}}(\overline{\mathcal{A}}_5) \leq s(\mathfrak{D}_{5,108}(I_5)) = \tfrac{271}{35} = 7.742\ldots \qquad \blacksquare$$

5.6 Case $g = 6$

For genus 6, the slope is bounded from below $s(\overline{\mathcal{A}}_6) \geq \frac{53}{10}$ by [21]. Moreover, an interesting Siegel modular form $\theta_{L,h,2}$ of class $14\lambda - 2\delta$ was constructed in [11], showing that $s(\overline{\mathcal{A}}_6) \leq 7$ and that the Kodaira dimension of $\mathcal{A}_6$ is non-negative.

Proof of Corollary E. In light of the classification of modular forms in low genus and weight in [5] and [6], in genus 6, there are no cusp forms in weight $7, 8, 9, 11, 13$. Now, in genus 6, there are no Siegel modular forms of weight 2 and we have seen above that $s(\overline{\mathcal{A}}_6) \geq \frac{53}{10}$. Hence, the unique (up to multiple) cusp form of weight 10 vanishes with multiplicity one along D (and so does a possible cusp form in weight 6). As $\mathrm{ord}_D\, \theta_{L,h,2} = 2$, the form $\theta_{L,h,2}$ must thus be prime.

As for the second claim, there are two possibilities:

(a) there exists a Siegel modular form of slope at most 7, not divisible by $\theta_{L,h,2}$: in this case, from Lemma 2.2 it follows that $s_{\mathrm{Mov}}(\overline{\mathcal{A}}_6) \leq 7$;

(b) $\theta_{L,h,2}$ is the unique genus 6 Siegel modular form of slope 7 (up to taking powers): the claim then follows from Corollary B, since $s(\mathfrak{D}_{6,14}(\theta_{L,h,2})) = 7 + \frac{2}{2\cdot 6} = \frac{43}{6}$ (as usual, if it happened that $\mathfrak{D}_{6,14}(\theta_{L,h,2})$ were to vanish to order strictly higher than $6 \cdot 2 = 12$, then its slope would be at most $\frac{86}{13} < 7$).

In either case, the result is proven. $\blacksquare$

In the above case (a), the moduli space $\mathcal{A}_6$ would have Kodaira dimension at least 1, in case (b), it would have Kodaira dimension 0.

6 Pluriharmonic Differential Operators

In this section, we introduce a suitable differential operator on the space of modular forms and prove Theorem A using a general result of [32]. Before introducing the relevant notions, we explain the outline of what is to be done.

We are looking for an operator $\mathfrak{D}_{g,a}$ that will map a genus g Siegel modular form F of weight a to another modular form satisfying certain properties: more precisely, $\mathfrak{D}_{g,a}(F)$ will be a polynomial in F and its partial derivatives. There are various motivations for looking for $\mathfrak{D}_{g,a}$ of such a form, which are discussed for the general setup for the problem in [32], [33], [14].

In our situation, motivated by the occurrence of $\det(\partial F)$ in our treatment of Rankin–Cohen bracket (see Remark 4.6 and Proposition 4.7), we will want $\mathfrak{D}_{g,a}(F)$ to restrict to $\det(\partial F)$ along the zero locus $\{F = 0\}$. Note that $\det(\partial F)$ is homogeneous in F of degree g, in the sense that each monomial involves a product of g different partial derivatives of F, and moreover it is a purely g'th order differential operator, in the sense that each monomial involves precisely g differentiations. Hence, we will look for a $\mathfrak{D}_{g,a}$ that shares these two properties.

Besides $F \mapsto \det(\partial F)$, another operator with the above properties is $F \mapsto F^{g-1}(\det\partial)F$, where each monomial is F^{g-1} multiplied by a suitable g'th order partial derivative of F. Of course $\mathfrak{D}_{g,a}(F)$ cannot be defined either as $\det(\partial F)$ or as $F^{g-1}(\det\partial)F$, as these are not modular forms. But a wished-for $\mathfrak{D}_{g,a}$ can be constructed explicitly: in order to do so, we will use the general machinery of [32], which implies that a differential operator with constant coefficients maps a non-zero modular form to a modular form if the corresponding polynomial is pluriharmonic and satisfies a suitable transformation property under the action of $\mathrm{GL}(g,\mathbb{C})$.

We now begin by reviewing the general notation, before stating a particular case of [32, Thm. 2] that allows the construction of $\mathfrak{D}_{g,a}$.

6.1 Polynomials and differential operators

Let $R_1,\dots,R_g$ be a g-tuple of $g\times g$ *symmetric* matrices, and denote the entries of R_h by $(r_{h;ij})$. Denote

$$\mathbb{C}[R_1,\dots,R_g] := \mathbb{C}[\{r_{h;ij}\}]$$

the space of polynomials in the entries of these matrices. The group $\mathrm{GL}(g,\mathbb{C})$ naturally acts by congruence on each symmetric matrix R_h (namely, via $R_h \mapsto AR_hA^t$ for every $A\in\mathrm{GL}(g,\mathbb{C})$), and so on the space $\mathbb{C}[R_1,\dots,R_g]$. For every integer $v\ge 0$, we denote by $\mathbb{C}[R_1,\dots,R_g]_v\subset\mathbb{C}[R_1,\dots,R_g]$ the vector subspace of those polynomials $P\in\mathbb{C}[R_1,\dots,R_g]$ that satisfy

$$P(AR_1A^t,\dots,AR_gA^t) = \det(A)^v P(R_1,\dots,R_g)$$

for all $A\in\mathrm{GL}(g,\mathbb{C})$.

For every polynomial $Q \in \mathbb{C}[R_1, \ldots, R_g]$, we define

$$Q_\partial := Q(\partial_1, \ldots, \partial_g), \quad \text{where as usual } (\partial_h)_{ij} := \frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \tau_{h;ij}}.$$

Such Q_∂ is then a holomorphic differential operator with constant coefficients acting on holomorphic functions in the variables $\tau_{h;ij}$. We further define the holomorphic differential operator $\mathcal{D}_Q$ that sends a g-tuple of holomorphic functions $F_1(\tau_1), \ldots, F_g(\tau_g)$ on $\mathbb{H}_g$ to another holomorphic function on the Siegel space given by

$$\mathcal{D}_Q(F_1, \ldots, F_g)(\tau) := Q_\partial(F_1(\tau_1) \cdots F_g(\tau_g))\Big|_{\tau_1 = \cdots = \tau_g = \tau}.$$

What this means is that applying each ∂_h takes the suitable partial derivatives of F_h with respect to the entries of the period matrix τ_h, and then once the polynomial in such partial derivatives is computed, it is evaluated at the point $\tau_1 = \cdots = \tau_g = \tau$.

While the general theory of applying $\mathcal{D}_Q$ to a g-tuple of modular forms is very interesting, we will focus on the case $F = F_1 = \cdots = F_g$, denoting then simply $\mathcal{D}_Q(F) := \mathcal{D}_Q(F, \ldots, F)$.

Example 6.1. It is immediate to check that the following polynomial (the general notation $\mathfrak{R}$ will be introduced in Section 6.2 below)

$$\mathfrak{R}(1, \ldots, 1) = \sum_{\sigma \in S_g} \operatorname{sgn}(\sigma) \prod_{\{a_1, \ldots, a_n\} = \{1, \ldots, n\}} r_{a_1;1,\sigma(1)} \cdots r_{a_n;n,\sigma(n)}$$

induces the differential operator $\mathcal{D}_{\mathfrak{R}(1,\ldots,1)}(F) = g! \det(\partial F)$. For example, for $g = 2$, this operator first gives

$$\frac{\partial F(\tau_1)}{\partial \tau_{1;1,1}} \frac{\partial F(\tau_2)}{\partial \tau_{2;2,2}} + \frac{\partial F(\tau_1)}{\partial \tau_{1:2,2}} \frac{\partial F(\tau_2)}{\partial \tau_{2;1,1}} - 2 \frac{\partial F(\tau_1)}{\partial \tau_{1;1,1}} \frac{\partial F(\tau_2)}{\partial \tau_{2;1,2}}$$

and then restricting to $\tau_1 = \tau_2$ yields $2 \det(\partial F)$. On the other hand, the polynomial

$$\mathfrak{R}(g, \underbrace{0, \ldots, 0}_{g-1 \text{ times}}) := \sum_{\sigma \in S_g} \operatorname{sgn}(\sigma) \prod_{j=1}^{g} r_{1;j,\sigma(j)}$$

induces the differential operator $\mathcal{D}_{\mathfrak{R}(g,0,\ldots,0)}(F) = F^{g-1} (\det \partial) F$. We stress that while each term of $\mathcal{D}_{\mathfrak{R}(1,\ldots,1)}$ is a product of g first-order partial derivatives of F, each term of $\mathcal{D}_{\mathfrak{R}(g,0,\ldots,0)}$ is equal to F^{g-1} multiplied by one g'th order partial derivative of F.

Our main result is the following Theorem 6.2, which is a refined version of Theorem A: indeed, to obtain Theorem A from it, we just need to set

$$\mathfrak{D}_{g,a}(F) := \tfrac{1}{g!}\mathcal{D}_{Q_{g,a}}(F)$$

for every modular form F of genus $g \geq 2$ and weight $a \geq \frac{g}{2}$ (the constant factor $g!$ is introduced only for notational convenience).

In order to motivate the statement below, recall that we want $\mathcal{D}_{Q_{g,a}}(F)$ to be equal to $g!\det(\partial F)$ modulo F. Since $g!\det(\partial F) = \mathcal{D}_{\mathfrak{R}(1^g)}(F)$ as in Example 6.1, and since $\mathfrak{R}(1^g)$ belongs to $\mathbb{C}[R_1,\dots,R_g]_2$, it is rather natural to look for $Q_{g,a}$ inside $\mathbb{C}[R_1,\dots,R_g]_2$.

Theorem 6.2. For every $g \geq 2$ and every $a \geq \frac{g}{2}$, there exists a polynomial $Q_{g,a} \in \mathbb{C}[R_1,\dots,R_g]_2$ such that the following properties hold for every genus g Siegel modular form F of weight a:

(i) $\mathcal{D}_{Q_{g,a}}(F)$ is a Siegel modular form of weight $ga+2$;

(ii) if $\operatorname{ord}_D F = b$, then $\operatorname{ord}_D \mathcal{D}_{Q_{g,a}}(F) \geq gb$;

(iii) the restriction of $\mathcal{D}_{Q_{g,a}}(F)$ to the zero locus $\{F=0\}$ of F is equal to $g!\cdot\det(\partial F)$.

Moreover, for any other polynomial $Q'_{g,a} \in \mathbb{C}[R_1,\dots,R_g]_2$ such that $\mathcal{D}_{Q'_{g,a}}$ satisfies properties (i) and (iii), the difference $\mathcal{D}_{Q_{g,a}}(F) - \mathcal{D}_{Q'_{g,a}}(F)$ is a Siegel modular form divisible by F.

The above differential operator $\mathcal{D}_{Q_{g,a}}$, which is homogeneous of degree g, can be also applied to modular forms with a character, which only occur for $g=2$: in this case, the output is a modular form (with trivial character).

Remark 6.3. As a consequence of Theorem 6.2(iii), if a modular form F of genus g and weight $a \geq \frac{g}{2}$ satisfies Condition ($\star$), then $\mathcal{D}_{Q_{g,a}}(F)$ does not vanish identically on the zero divisor of F.

The reason we are able to construct $Q_{g,a}$ explicitly is that we can use a lot of prior work, especially by the second author and collaborators, on differential operators acting on modular forms. In particular, by Theorem 6.10 the operator $\mathcal{D}_{Q_{g,a}}$ will map modular forms to modular forms if $Q_{g,a}$ is pluriharmonic—this essential notion will be recalled in Section 6.5.

Thus to prove Theorem 6.2, it will suffice to construct a pluriharmonic $Q_{g,a} \in \mathbb{C}[R_1,\dots,R_g]_2$. Property (i) will rely on Theorem 6.10 and (ii) will be easily seen to hold. Up to rescaling, we will also check (iii), and the last claim will follow.

6.2 A basis of $\mathbb{C}[R_1,\ldots,R_g]_2$

Consider the g-tuple of symmetric $g\times g$ matrices $R_1,\ldots,R_g$. We set

$$\mathfrak{R} := t_1R_1+\cdots+t_gR_g\,, \tag{10}$$

and denote by $\mathfrak{R}(\mathbf{n})\in\mathbb{C}[R_1,\ldots,R_g]$ the coefficients of the expansion of the determinant

$$\det(\mathfrak{R}) = \sum_{\mathbf{n}\in\mathbf{N}_g}\mathfrak{R}(\mathbf{n})t_1^{n_1}\ldots t_g^{n_g}$$

as a polynomial in the variables $t_1,\ldots,t_g$, where

$$\mathbf{N}_g := \{\mathbf{n}=(n_1,\ldots,n_g)\in\mathbb{N}^g \mid n_h\geq 0 \text{for all} h,\ \sum n_h = g\}\,.$$

The importance of the polynomials $\mathfrak{R}(\mathbf{n})$ for us relies on the fact that they clearly belong to $\mathbb{C}[R_1,\ldots,R_g]_2$, simply because $\det(A\mathfrak{R}A^t)=\det(A)^2\det(\mathfrak{R})$ for all $A\in\mathrm{GL}(g,\mathbb{C})$.

The following lemma, of a very classical flavor, was communicated to us by Claudio Procesi.

Lemma 6.4. The set of polynomials $\{\mathfrak{R}(\mathbf{n})\}_{\mathbf{n}\in\mathbf{N}}$ is a basis of $\mathbb{C}[R_1,\ldots,R_g]_2$.

Proof. Let V be a complex g-dimensional vector space and let $\mathrm{GL}(V)$ naturally act on $\mathrm{Sym}^2(V^*)^{\oplus g}$ via

$$A\cdot((\phi_1\otimes\phi_1),\ldots,(\phi_g\otimes\phi_g)) := ((\phi_1A)\otimes(\phi_1A),\ldots,(\phi_gA)\otimes(\phi_gA))$$

for $A\in\mathrm{GL}(V)$. Consider the $\mathbb{C}$-algebra $\mathcal{I}(V,g)$ of $\mathrm{SL}(V)$-invariants inside $\mathrm{Sym}^2(V^*)^{\oplus g}$. The quotient $\mathrm{GL}(V)/\mathrm{SL}(V)\cong\mathbb{C}^*$ acts on $\mathcal{I}(V,g)$ and, under this action, the algebra of invariants decomposes as

$$\mathcal{I}(V,g)=\bigoplus_d\mathcal{I}(V,g)_d,\ \text{where}\ \mathcal{I}(V,g)_d := \{P\in\mathcal{I}(V,g)\mid A\cdot P=\det(A)^{2d}P\}.$$

Clearly, $\mathcal{I}(V,g)_d$ is simply the subspace of $\mathcal{I}(V,g)$ consisting of invariant polynomial maps $P:\mathrm{Sym}^2(V)^{\oplus g}\to\mathbb{C}$ of total degree $d\cdot\dim(V)$ with respect to the above $\mathbb{C}^*$-action.

The subspace $\mathcal{I}(V,g)_1$, which by definition is $\mathbb{C}[R_1,\dots,R_g]_2$, decomposes as

$$\mathcal{I}(V,g)_1 = \bigoplus_{\mathbf{n}\in\mathbf{N}_g} \mathcal{I}(V,g)_{\mathbf{n}}$$

where $\mathcal{I}(V,g)_{\mathbf{n}} := \bigotimes_{i=1}^{g} Sym^{n_i}(Sym^2(V^*))$ denotes the subspace of invariant polynomial functions $\mathrm{Sym}^2(V)^{\oplus g} \to \mathbb{C}$ of multi-degree $\mathbf{n}$.

Since it is easy to check that $\mathfrak{R}(\mathbf{n}) \in \mathcal{I}(V,g)_{\mathbf{n}}$, it is enough to show that $\mathcal{I}(V,g)_{\mathbf{n}}$ has dimension 1 for all $\mathbf{n} \in \mathbf{N}_g$. Moreover, $\mathcal{I}(V,g)_{\mathbf{n}}$ is isomorphic to $\mathcal{I}(V,g)_{(1,\dots,1)} = (\mathrm{Sym}^2(V^*)^{\otimes g})^{\mathrm{SL}(V)}$ as an $\mathrm{SL}(V)$-module for all $\mathbf{n} \in \mathbf{N}_g$, and so it is enough to show that $(\mathrm{Sym}^2(V^*)^{\otimes g})^{\mathrm{SL}(V)}$ has dimension at most 1 (and in fact it will have dimension 1, since $\mathfrak{R}(\mathbf{n}) \not\equiv 0$).

Thinking of $(\mathrm{Sym}^2(V^*)^{\otimes g})^{\mathrm{SL}(V)}$ as a subspace of $((V^*)^{\otimes 2g})^{\mathrm{SL}(V)}$, we describe a basis of $((V^*)^{\otimes 2g})^{\mathrm{SL}(V)}$; it is enough to do that for $V = \mathbb{C}^g$.

Let $\mathfrak{P}$ be the set of all permutations $(I,J) = (i_1,\dots,i_g,j_1,\dots,j_g)$ of $\{1,2,\dots,2g\}$ such that $i_h < j_h$ for all $h = 1,\dots,g$. For every $(I,J) \in \mathfrak{P}$, we denote by

$$[i_1,\dots,i_g][j_1,\dots,j_g] : V^{\otimes 2g} \longrightarrow \mathbb{C}$$

the linear map that sends $v_1 \otimes \cdots \otimes v_{2g}$ to $\det(v_I)\det(v_J)$, where v_I is the matrix whose h-th column is v_{i_h} (and similarly for v_J). It is a classical fact that the collection of $[i_1,\dots,i_g][j_1,\dots,j_g]$ with $(I,J) \in \mathfrak{P}$ is a basis of $((V^*)^{\otimes 2g})^{\mathrm{SL}(V)}$, cf. [41, pages 387, 504].

Fix now (I,J) and consider the restriction of $[i_1,\dots,i_g][j_1,\dots,j_g]$ to $\mathrm{Sym}^2(V)^{\otimes g}$, and in particular to the vectors of type

$$v_1 \otimes v_1 \otimes v_2 \otimes v_2 \otimes \cdots \otimes v_g \otimes v_g,$$

which generate $\mathrm{Sym}^2(V)^{\otimes g}$. Note that $[i_1,\dots,i_g][j_1,\dots,j_g]$ is alternating both in I and in J, and vanishes on all vectors $v_1 \otimes v_1 \otimes v_2 \otimes v_2 \otimes \cdots \otimes v_g \otimes v_g$ as soon as either I or J contains $\{2m-1, 2m\}$ for some $m = 1,\dots,g$. It follows that all elements $[i_1,\dots,i_g][j_1,\dots,j_g]$ of the above basis of $((V^*)^{\otimes 2g})^{\mathrm{SL}(V)}$ vanish on $\mathrm{Sym}^2(V)^{\otimes g}$, except possibly $[1,3,5,\dots,2g-1][2,4,6,\dots,2g]$. We conclude that $(\mathrm{Sym}^2(V^*)^{\otimes g})^{\mathrm{SL}(V)}$ is at most 1-dimensional. ■

6.3 Definition of the polynomial $Q_{g,a}$

In view of Lemma 6.4, every polynomial in $\mathbb{C}[R_1,\dots,R_g]_2$ must be a linear combination of the $\mathfrak{R}(\mathbf{n})$'s. Here we define a sought polynomial $Q_{g,a} \in \mathbb{C}[R_1,\dots,R_g]_2$ as a linear combination of the $\mathfrak{R}(\mathbf{n})$'s by providing explicit formulas for its coefficients. Pluriharmonicity of $Q_{g,a}$ will be defined and verified in Section 6.6. This $Q_{g,a}$ is not unique in general, but another choice makes no difference for proving Theorem A and Corollary C.

We define the constant

$$C(a,1) := (g-1)\prod_{i=1}^{g-1}(2a-i)\,.$$

Moreover, for every $m = 2,\dots,g$ we define the constant

$$C(a,m) := (-1)^{m-1}(m-1)!\,(2a)^{m-1}\prod_{i=m}^{g-1}(2a-i)\,,$$

where for $m = g$ the last product above is declared to be equal to 1, so that $C(a,g) = (-1)^{g-1}(g-1)!\,(2a)^{g-1}$.

By an abuse of notation, we delete the index a from $C(a,m)$ and we assume $2a \geq g \geq 2$, so that $C(1) \neq 0$, and define then

$$Q_{g,a} := \frac{1}{C(1)}\sum_{\mathbf{n}\in\mathbf{N}_g} c(\mathbf{n})\mathfrak{R}(\mathbf{n})\,,$$

where

1. $c(1,\dots,1) := C(1)$;
2. if at least two of $n_1,\dots,n_g$ are greater than 1, then we set $c(\mathbf{n}) := 0$;
3. if $n_h = m > 1$ for some h, while $0 \leq n_j \leq 1$ for any $j \neq h$, then we set $c(\mathbf{n}) := C(m)$.

Hence $c(\mathbf{n}) \neq 0$ if and only if $\mathbf{n}$ is equal to $(m,1,1,\dots,1,0,0,\dots,0)$ for some $m \geq 1$, up to permuting its components.

6.4 Explicit formulas

In order to have a more explicit expression for the polynomials $\mathfrak{R}(\mathbf{n})$, we expand the relevant determinants.

Notation 6.5. If M is a $g \times g$ matrix and $I, J \subset \{1,2,\dots,g\}$ with $|I| = |J|$, we denote by M_{IJ} the minor of M consisting of rows I and columns J, and denote by $\det_{IJ}(M)$ the determinant of M_{IJ} (if $|I| = |J| = 0$, then we formally set $\det_{IJ}(M) := 1$). Moreover, we let $\widehat{I}$ be the complement of I and, if $i \in \{1,2,\dots,g\}$, then we let $\hat{\imath} := \{1,2,\dots,g\} \setminus \{i\}$.

Applying the Laplace expansion several times yields

$$\mathfrak{R}(\mathbf{n}) = \sum_{I_\bullet, J_\bullet} \epsilon(I_\bullet, J_\bullet) \det_{I_1 J_1}(R_1) \cdots \det_{I_g J_g}(R_g)\,, \tag{11}$$

where $(I_\bullet, J_\bullet) = (I_1, \ldots, I_g, J_1, \ldots, J_g)$ and

- $I_1, \ldots, I_g, J_1, \ldots, J_g$ run over all subsets of $\{1, \ldots, g\}$ such that $|I_i| = |J_i| = n_i$ for each $i = 1, \ldots, g$ and $\sqcup_{i=1}^g I_i = \sqcup_{i=1}^g J_i = \{1, \ldots, g\}$;
- $\epsilon(I_\bullet, J_\bullet)$ is the signature of the element of S_g that maps $(I_1, \ldots, I_g)$ to $(J_1, \ldots, J_g)$, where the elements inside each subset I_i or J_i are ordered from minimum to maximum.

In order to compute $\mathcal{D}_{\mathfrak{R}(\mathbf{n})}(F, \ldots, F)$, consider $\mathbf{n} = (m, 1, \ldots, 1, 0, \ldots, 0)$. Regarding the partial sum of the parts for $|I_j| = 1$ as expansions of determinants by Laplace expansion, we have

$$\mathcal{D}_{\mathfrak{R}(\mathbf{n})}(F, \ldots, F) = F^{m-1} \sum_{|I|=|J|=m} \epsilon(I, J)(g-m)!\,(\det_{IJ} \partial)F \cdot \det_{\widehat{I},\widehat{J}}(\partial F)\,, \tag{12}$$

where we denote

$$\epsilon(I, J) := (-1)^{i_1 + \cdots + i_m + j_1 + \cdots + j_m}.$$

Thus, we have obtained the following.

Corollary 6.6.

(i) If $\mathbf{n} = (1, \ldots, 1)$, then $\mathcal{D}_{\mathfrak{R}(1,\ldots,1)}(F) = g!\, \det(\partial F)$.

(ii) If $\mathbf{n} \in \mathbf{N}_g$ and $\mathbf{n} \neq (1, \ldots, 1)$, then $\mathcal{D}_{\mathfrak{R}(\mathbf{n})}(F)$ is a multiple of F.

Proof. For (i), note that

$$\sum_{j=1}^{g} (-1)^{i+j} \partial_{ij} F \cdot \det_{\widehat{i}\,\widehat{j}}(\partial F) = \det(\partial F)$$

for every $i = 1, \ldots, g$. Then formula (12) for $\mathbf{n} = (1, 1, \ldots, 1)$ (i.e., for $m = 1$) yields

$$\mathcal{D}_{\mathfrak{R}(1,\ldots,1)}(F) = (g-1)! \sum_{i,j=1}^{g} (-1)^{i+j} \partial_{ij} F \cdot \det_{\widehat{i}\,\widehat{j}}(\partial F) = g!\, \det(\partial F)\,.$$

For (ii), we observe that since $\sum n_h = g$, and all $n_h \geq 0$, it follows that unless $\mathbf{n} = (1, \dots, 1)$, there exists at least one h such that $n_h = 0$. But then the polynomial $\mathfrak{R}(\mathbf{n})$ would contain no $r_{h;ij}$, which is to say that F_h is not differentiated at all by $\mathcal{D}_{\mathfrak{R}(\mathbf{n})}(F_1, \dots, F_g)$. This finally means that $\mathcal{D}_{\mathfrak{R}(\mathbf{n})}(F_1, \dots, F_g)$ is divisible by F_h, and thus $\mathcal{D}_{\mathfrak{R}(\mathbf{n})}(F)$ is divisible by F. ■

Example 6.7. For $g = 2, 3$ we have

$$\mathcal{D}_{Q_{2,a}}(F, F) = 2\det(\partial F) + \frac{2(2a)}{1-2a} F \cdot (\det \partial) F\,,$$

$$\mathcal{D}_{Q_{3,a}}(F, F, F) = 6\det(\partial F) + \frac{3(2a)^2}{(2a-1)(2a-2)} F^2 \cdot (\det \partial) F$$

$$- \frac{3(2a)}{(2a-1)} F \sum_{i,j=1}^{3} (\partial_{ij} F) \cdot (\det_{\hat{i}\hat{j}} \partial) F\,.$$

6.5 Pluriharmonic polynomials

Our motivation for the construction below is as follows. For any reasonable lattice L in $\mathbb{R}^m$ and a pluriharmonic, in the sense defined below, polynomial $\tilde{P}(X)$ in $n \times m$ variable matrix X satisfying $\tilde{P}(AX) = \det(A)^k \tilde{P}(X)$ for any $A \in GL(n, \mathbb{C})$, it is well-known that the theta series

$$\theta_{L,\tilde{P}}(\tau) = \sum_{x_1, \dots, x_n \in L} \tilde{P}(x_1, \dots, x_n) \exp(2\pi i \sum_{i,j} (x_i, x_j) \tau_{ij})$$

is a Siegel modular form of weight $\frac{m}{2} + k$. On the other hand, for $P = 1$ (constant 1), we have

$$\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \tau_{ij}} \theta_{L,1} = (2\pi i) \sum_{x_1, \dots, x_n \in L} (x_i, x_j) \exp(2\pi i \sum_{i,j} (x_i, x_j) \tau_{ij}).$$

This means that if $\tilde{P}(X)$ is a function of (x_i, x_j), then we can regard $\theta_{L,\tilde{P}}$ as a derivative of theta series $\theta_{L,1}$. This morally motivates Theorem 6.10 below, though the actual content and the proof of the theorem are much more subtle. Theorem 6.10 shows that the most important step toward the proof of Theorem 6.2 is checking that $Q_{g,a}$, defined in Section 6.3 as a linear combination of $\mathfrak{R}(\mathbf{n})$, is pluriharmonic. In this section, we recall the relevant setup, definitions, and statements.

Fix an $g \times k$ matrix $X = (x_{i\nu})$, and denote for $1 \le i,j \le g$

$$\Delta_{ij}(X) := \sum_{\nu=1}^{k} \frac{\partial^2}{\partial x_{i\nu} \partial x_{j\nu}} .$$

For a polynomial $P(R)$ in the entries of a symmetric $g \times g$ matrix $R = (r_{ij})$, we denote $\widetilde{P}(X) := P(XX^t)$.

Definition 6.8. The polynomial P is called *pluriharmonic* (with respect to X) if $\Delta_{ij}\widetilde{P} = 0$ for all $1 \le i,j \le g$.

To detect this pluriharmonicity in terms of R, we define the differential operator in variables (r_{ij}) by

$$D_{ij} := k \cdot \partial_{ij} + \sum_{u,w=1}^{g} r_{uw} \partial_{iu} \partial_{jw} , \tag{13}$$

where $\partial_{ij} := \frac{1+\delta_{ij}}{2} \frac{\partial}{\partial r_{ij}}$. Then a direct computation yields

$$(D_{ij}P)(XX^t) = \Delta_{ij}(\widetilde{P}(X)) , \tag{14}$$

where $P(r_{ij})$ is any polynomial, and the LHS means D_{ij} is applied to P, and then evaluated at XX^t.

This equality shows that computing the Δ_{ij} derivative of $\widetilde{P}$ (which is a second order differential operator) amounts to computing the D_{ij} applied to P, which is a differential operator that includes first and second order derivatives. Thus, pluriharmonicity is equivalent to the condition $D_{ij}(P) = 0$ for all $1 \le i,j \le g$.

Now the full setup we require is as follows. For a positive integer $k = 2a$, we consider a g-tuple of $g \times k$ matrices $X_1, \ldots, X_g$, and denote $R_h := X_h X_h^t$.

Definition 6.9. A polynomial $P \in \mathbb{C}[R_1, \ldots, R_g]$ is called *pluriharmonic* if

$$\widetilde{P}(X_1, \ldots, X_g) := P(X_1X_1^t, \ldots, X_gX_g^t) ,$$

is pluriharmonic with respect to the $g \times (gk)$ matrix $X = (X_1, \ldots, X_g)$.

The following result is a special case of [32, Theorem 2], which shows the importance of pluriharmonicity.

Theorem 6.10. For $g \geq 2$, let $P \in \mathbb{C}[R_1, \ldots, R_g]_2$ and let $F \neq 0$ be a Siegel modular form of genus g and weight $a \geq \frac{g}{2}$. Then $\mathcal{D}_P(F, \ldots, F)$ is a Siegel modular form of weight $ga + 2$ if P is pluriharmonic.

Let us first give an elementary characterization of pluriharmonicity.

Lemma 6.11. Let $P \in \mathbb{C}[R_1, \ldots, R_g]$.

(i) The polynomial $\widetilde{P}(X)$ is pluriharmonic if and only if $\widetilde{P}(AX)$ is harmonic (i.e., $\sum_{i=1}^{g} \Delta_{ii}\widetilde{P}(AX) = 0$) for any $A \in \mathrm{GL}(g, \mathbb{C})$.

(ii) Assume that $\widetilde{P} \in \mathbb{C}[R_1, \ldots, R_g]_v$ for some v. Then, $\widetilde{P}(X)$ is pluriharmonic if and only if $\Delta_{11}(\widetilde{P}) = 0$.

Proof. The claim (i) is remarked in [36] and we omit the proof. In order to prove (ii), note that pluriharmonicity of $\widetilde{P}$ implies that $\Delta_{11}(\widetilde{P}) = 0$ by definition. Hence, it is enough to prove that $\Delta_{11}(\widetilde{P}) = 0$ implies pluriharmonicity. For a fixed i with $1 \leq i \leq g$, let A be the permutation matrix that exchanges the first row and the i-th row. Since

$$\begin{aligned}\Delta_{ii}(X) \cdot \widetilde{P}(X) &= \det(A)^{-v} \Delta_{ii}(X) \cdot \widetilde{P}(AX) \\ &= \det(A)^{2-v} \Delta_{11}(AX) \cdot \widetilde{P}(AX) = 0,\end{aligned}$$

the conclusion follows. ■

Denoting $D_{h;11}$ the differential operator D_{11} defined in (13) with respect to the entries of the matrix R_h, and using (14) to rewrite $\Delta_{h,ij}$ for each X_h as $D_{h;11}$, by Lemma 6.11 we have the following.

Corollary 6.12. Suppose that $\widetilde{P} \in \mathbb{C}[R_1, \ldots, R_g]_v$ for some v. Then P is pluriharmonic with respect to the $g \times (gk)$ matrix $(X_1, \ldots, X_g)$ if and only if

$$\sum_{h=1}^{g} D_{h;11} P = 0\,. \tag{15}$$

The above corollary applies to $Q_{g,a}$ and simplifies the verification of its pluriharmonicity.

6.6 Pluriharmonicity of $Q_{g,a}$

The result that we want to show is the following.

Proposition 6.13. The polynomial $Q_{g,a}$ is pluriharmonic.

Since we will be dealing with minors of the matrix $\mathfrak{R}$ defined by (10), we let

$$\mathbf{N}' := \{\mathbf{n}' = (n'_1, \ldots, n'_g) \in \mathbb{N}^g \mid n'_h \geq 0 \text{ for all } h, \ \sum n'_h = g-1\}\,, \tag{16}$$

and we denote by $\widehat{\mathfrak{R}}_{k;l}$ the determinant of the matrix $\mathfrak{R}_{\widehat{k};\widehat{l}}$, and denote by $\widehat{\mathfrak{R}}_{k;l}(\mathbf{n}')$ the polynomial appearing in the expansion

$$\widehat{\mathfrak{R}}_{k;l} = \sum_{\mathbf{n}' \in \mathbf{N}'} \widehat{\mathfrak{R}}_{k;l}(\mathbf{n}') t_1^{p_1} \cdots t_g^{p_g}\,.$$

We can now compute the derivative of $\mathfrak{R}(\mathbf{n})$ that enters into the formula (15) for pluriharmonicity.

Lemma 6.14. For any $\mathbf{n} \in \mathbf{N}_g$, we have

$$D_{h;11}\mathfrak{R}(\mathbf{n}) = 2(k - n_h + 1)\widehat{\mathfrak{R}}_{1;1}(\mathbf{n} - \mathbf{e}_h)\,,$$

where $k = 2a$, and $\{\mathbf{e}_1, \ldots, \mathbf{e}_g\}$ is the standard basis of $\mathbb{Z}^g$.

Proof. By symmetry, it is enough to prove this for $h = 1$; for simplicity, we just write r_{ij} for the entries of the symmetric matrix $r_{1;ij}$, and define ∂ by (8). We recall that $D_{1;11} = k \cdot \partial_{11} + \sum_{i,j=1}^{g} r_{ij}\partial_{1i}\partial_{1j}$.

Then by treating the cases $i = 1$ and $i \neq 1$ separately, and checking the factor of $1/2$ versus 1 appearing in the definition of ∂ for these entries, we see that

$$\partial_{1i} \det(\mathfrak{R}) = 2(-1)^{1+i} t_1 \widehat{\mathfrak{R}}_{1;i}$$

for any $i = 1, \ldots, g$. To compute the second order derivatives appearing in $D_{1;11}$, we first note that since $\widehat{\mathfrak{R}}_{1;1}$ does not depend on any r_{1i}, we have $\partial_{1i}\partial_{1j} \det(\mathfrak{R}) = 0$ if $i = 1$ or $j = 1$.

Otherwise, for $i, j \neq 1$, we compute

$$r_{ij}\partial_{1i}\partial_{1j}\det(\mathfrak{R}) = (-1)^{1+j}t_1 r_{ij}\partial_{1i}\widehat{\mathfrak{R}}_{1;j} = (-1)^{1+j+1+(i-1)}t_1^2 r_{ij}\widehat{\mathfrak{R}}_{\{1,i\};\{1,j\}}$$
$$= (-1)(-1)^{(i-1)+(j-1)}t_1^2 r_{ij}\widehat{\mathfrak{R}}_{\{1,i\};\{1,j\}}\,.$$

Summing these identities yields

$$\sum_{i,j=2}^{g} r_{ij}\partial_{1i}\partial_{1j}\det(\mathfrak{R}) = (-t_1)\sum_{i,j=2}^{g} t_1 r_{ij}(-1)^{(i-1)+(j-1)}\widehat{\mathfrak{R}}_{\{1,i\};\{1,j\}}\,.$$

Here for a fixed i, the sum $\sum_{j=2}^{g}(-1)^{(i-1)+(j-1)}r_{ij}\widehat{\mathfrak{R}}_{\{1,i\};\{1,j\}}$ is nothing but the derivative of the $(i-1)$-th row of $\widehat{\mathfrak{R}}_{1;1}$ with respect to t_1, and thus

$$\sum_{i,j=2}^{g}(-1)^{(i-1)+(j-1)}r_{ij}\widehat{\mathfrak{R}}_{\{1,i\};\{1,j\}} = \frac{\partial}{\partial t_1}\widehat{\mathfrak{R}}_{1;1}\,.$$

Recall that $\widehat{\mathfrak{R}}_{1;1} = \sum_{\mathbf{n}'\in\mathbf{N}'}\widehat{\mathfrak{R}}_{1;1}(\mathbf{n}')t_1^{p_1}\dots t_g^{p_g}$ and note that

$$t_1\frac{\partial}{\partial t_1}(t_1^{p_1}\dots t_g^{p_g}\widehat{\mathfrak{R}}_{1;1}(\mathbf{n}')) = p_1 t_1^{p_1}\dots t_g^{p_g}\widehat{\mathfrak{R}}_{1;1}(\mathbf{n}')\,.$$

Thus, the coefficient of $t_1^{n_1}\dots t_g^{n_g}$ in the expansion of $D_{1;11}\det(\mathfrak{R})$ is equal to

$$2k\widehat{\mathfrak{R}}_{1;1}(n_1-1, n_2, \dots, n_g) - 2(n_1-1)\widehat{\mathfrak{R}}_{1;1}(n_1-1, n_2, \dots, n_g)\,.$$

■

As a consequence of Lemma 6.14, we have

$$\frac{(2a-1)!}{(2a-g)!}\sum_{h=1}^{g} D_{h;11}Q_{g,a} = \sum_{h=1}^{g} c(\mathbf{n})D_{h;11}\mathfrak{R}(\mathbf{n})$$
$$= \sum_{h=1}^{g} 2(k-n_h+1)c(\mathbf{n})\widehat{\mathfrak{R}}_{1;1}(\mathbf{n}-\mathbf{e}_h)$$
$$= 2\sum_{h=1}^{g}(k-n'_h)c(\mathbf{n}'+\mathbf{e}_h)\widehat{\mathfrak{R}}_{1;1}(\mathbf{n}')\,.$$

Thus, by Corollary 6.12, to check pluriharmonicity of $Q_{g,a}$, it is enough to check that

$$\sum_{h=1}^{g}(k-n'_h)c(\mathbf{n}'+\mathbf{e}_h)=0 \tag{17}$$

for all $\mathbf{n}'\in\mathbf{N}'$.

Comparing the degrees of $\widehat{\mathfrak{R}}_{1;1}(\mathbf{n}')$ with respect to R_h, one can see that the set $\{\widehat{\mathfrak{R}}_{1,1}(\mathbf{n}')\,|\,\mathbf{n}'\in\mathbf{N}'\}$ is linearly independent over $\mathbb{C}$, and so (17) is actually equivalent to the pluriharmonicity of $Q_{g,a}$.

Proof of Proposition 6.13. It is enough to verify (17) for every $\mathbf{n}'\in\mathbf{N}'$. Up to reordering the entries of $\mathbf{n}'$, we can assume that they are non-increasing.

If $n'_1\geq n'_2>1$, then $\mathbf{n}'+\mathbf{e}_\ell$ has two entries larger than 1, and so by definition we have $c(\mathbf{n}'+\mathbf{e}_\ell)=0$ for any ℓ. It follows that all the terms in (17) are equal to zero, and the equation is trivially satisfied.

For $\mathbf{n}'=(1,\ldots,1,0)$, the LHS of (17) is

$$\begin{aligned}&k\cdot c(1,\ldots,1)+(k-1)\Big(c(2,1,\ldots,1,0)+c(1,2,\ldots,1,0)+\cdots+c(1,\ldots,2,0)\Big)\\&\quad=k\cdot C(1)+(k-1)(g-1)C(2)\,.\end{aligned}$$

By definition of $C(1)$ and $C(2)$, the terms cancel, yielding 0.

Let now $\mathbf{n}'=(m,1,\ldots,1,0,\ldots,0)$ with $g-m$ entries 1. If $2\leq\ell\leq g-m$, then $n_\ell>1$ and $c(\mathbf{n}'+\mathbf{e}_\ell)=0$ by definition. We then have

$$c(\mathbf{n}'+\mathbf{e}_1)=c(m+1,1,\ldots,1,0,\ldots,0)=C(m+1)\,.$$

If $g-m+1\leq\ell$, then $\mathbf{n}'+\mathbf{e}_\ell$ is of type $(m,1,\ldots,1,0,\ldots,0)$, $(m,1,\ldots,1,0,1,0,\ldots,0)$, $\ldots$, or $(m,1,\ldots,1,0,\ldots,0,1)$: in all these cases $\mathbf{n}'+\mathbf{e}_\ell$ has $g-m$ entries 1, and thus $c(\mathbf{n}'+\mathbf{e}_\ell)=C(m)$. So LHS of (17) is given by

$$(k-m)C(m+1)+km\cdot C(m)\,,$$

which also vanishes by our definition of the constants $C(m)$. ■

Example 6.15. In the case $g = 2$, we obtain

$$Q_{2,a} = \mathfrak{R}(1,1) - \frac{2a}{2a-1}\mathfrak{R}(2,0) - \frac{2a}{2a-1}\mathfrak{R}(0,2)\,,$$

where we have used $k = 2a$. This is a special case of the discussion in [14].

Proof of Theorem 6.2. The polynomial $Q_{g,a}$, defined in Section 6.3, belongs to $\mathbb{C}[R_1,\dots,R_g]_2$ and is pluriharmonic by Proposition 6.13. Then (i) and the first part of (ii) follow from Theorem 6.10. Moreover, since $\mathcal{D}_{Q_{g,a}}(F_1,\dots,F_g)$ is $\mathbb{C}$-linear in each F_h, and since F_h and $\frac{\partial F_h}{\partial \tau_{ij}}$ have the same vanishing order at the boundary for all h and all i,j, it follows that $\mathcal{D}_{Q_{g,a}}(F,\dots,F)$ has vanishing order $\beta \geq gb$. This completes the proof of (ii).

As for (iii), note that $2a \geq g \geq 2$ ensures that the constant $C(1)$ defined in Section 6.3 is non-zero and so, by construction,

$$Q_{g,a} = \mathfrak{R}(1,\dots,1) + \sum_{\mathbf{n}\neq(1,\dots,1)} \frac{(2a-g)!\,c(\mathbf{n})}{(2a-1)!\,(g-1)}\mathfrak{R}(\mathbf{n})\,.$$

By Corollary 6.6, it follows that

$$\mathcal{D}_{Q_{g,a}}(F,\dots,F) \equiv g!\ \det(\partial F) \qquad (\mathrm{mod}\ F)\,,$$

and so (iii) is proven. The last claim is an immediate consequence of (i) and (iii), as the modular form $\mathcal{D}_{Q_{g,a}}(F) - \mathcal{D}_{Q'_{g,a}}(F)$ vanishes along $\{F = 0\}$. ■

We make one last remark on the above proof. We are not claiming that $Q_{g,a}$ or the associated differential operator $\mathcal{D}_{Q_{g,a}}$ are unique. Since we are looking for polynomials in $\mathbb{C}[R_1,\dots,R_g]_2$, these must be linear combinations of the $\mathfrak{R}(\mathbf{n})$'s by Lemma 6.4. If $Q'_{g,a} \in \mathbb{C}[R_1,\dots,R_g]_2$ satisfies property (iii) in Theorem 6.2, then it must take the form

$$Q'_{g,a} = \mathfrak{R}(1,\dots,1) + \sum_{\mathbf{n}\neq(1,\dots,1)} c'(\mathbf{n})\mathfrak{R}(\mathbf{n})$$

by Corollary 6.6. Hence, the restrictions of $\mathcal{D}_{Q'_{g,a}}(F)$ and $\mathcal{D}_{Q_{g,a}}(F)$ to the locus $\{F = 0\}$ agree. Note that the coefficients $c'(\mathbf{n})$ may differ from the $c(\mathbf{n})$ that were defined in Section 6.3.

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