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Volterra-renewal Integral Equations: a Combined Simplified Numerical Approach

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Abstract

This paper presents a novel numerical approach for solving Volterra-renewal integral equations, which arise in various fields, including biology, engineering, and economics. Traditional treatment of the Volterra-renewal integral equations systems of this type utilises computationally large iterative algorithms. In order to tackle these limitations, we propose a hybrid method that is numerical in nature whereby an analytic and discretisation techniques are used to obtain accurate and efficient solutions. Several test problems have been solved and the peculiarities of this method demonstrated, its possibility to solve a wide class of renewal-type problems is also presented. We have addressed a number of obstacles, such as managing nonlinearities and memory effects over extended periods of time, by using the Picard approach to provide a

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more accurate numerical approximation. We illustrate the benefits of the proposed methods over closed-form solutions using numerical examples.

Mathematics Subject Classification:45D05, 35B05, 45G15

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1 Introduction

Volterra integral equations have proved crucial for simulating dynamic systems, especially when the system's whole past affects its current state. One important family of these is Volterra-renewal integral equations (VRIEs), which have the ability to model renewal-type processes, where the system "resets" or renews at intervals. The equations in question, which have originated in Volterra's publications, differ from ordinary differential equations in that they account for memory implications and integrate prior conditions across a variable time period [6]. VRIEs are especially applicable in circumstances when past occurrences have an equal impact on current and future conduct. This feature has made VRIEs an important mathematical tool in the fields of population evolving and engineering reliability [9]. The mathematical bases of VRIEs are found in a renewal theory, a particular field of probability theory that focusses on processes that "renew" following certain events. Combining this with Volterra's integral technique provides equations that precisely describe systems with deterministic or random resets over time. In population biology, for instance, VRIEs are used for modelling age-structured populations where births are renewal events that affect population growth and decline in accordance with historical survival rates [5]. VRIEs have been used in engineering to predict component failures and improve maintenance plans in systems that undergo wear and tear because of the probability of failure connected to the whole operating history [8]. The ability of VRIEs to handle both discrete and continuous effects offer substantial advantages for accurately capturing complex, time-dependent events. The mathematical complexity associated with VRIEs presents significant challenges in finding analytical and numerical solutions, considering their wide range of applications, particularly in nonlinear or time-dependent scenarios. Classical methods like as Laplace transforms, resolvent kernels, and iterative algorithms provide a foundation for fundamental VRIEs, but they are frequently unsuccessful or require modification for more complicated, nonlinear forms. Advances in numerical approaches have helped to address these issues to some extent by allowing more efficient approximation calculations. Nonetheless, it is crucial to enhance computational approaches in order to make VRIEs more approachable and applicable to a greater variety of scientific and technical problems. The paper is organized as follows: we present our algorithm in Section 2, Section 3 is devoted to some numerical example numerical experiments while Section 4 concludes the contribution.

2 Main result

In this paper we consider a Volterra integral equation of the following form

$$u(x,t) = f(x,t) + \int_0^t k(t-\eta)u(g(x,t,\eta),\eta)\,d\eta \tag{1}$$

where $k(t)$, $f(x,t)$, $g(x,t,\eta)$, are known continuous functions. In addition k and f are continuous non-negative functions in all their arguments, $x \in I = [a,b]$ (a closed real interval)and $t > 0$. More precisely, equation (1) is a Volterra type integral equation respect to the variable t. As discussed in Annunziato et al. [2] this integral equation models the distribution probability function of a class of piecewise deterministic processes resulting from a semi-Markov process (see Feller [4]), the kernel synthesizes the memory of the process and $g : [a,b] \times [a,b] \times [0,\infty]$ is the flux.

Following Alturk [1], we report the weighted mean value theorem for integrals whose proof may be found, for example, in Apostol [3]. Let us consider a closed real interval $I = [a,b]$.

Theorem 2.1 *Le be h_1 and h_2 functions that are continuous in I. If h_2 never changes sign in I, then exists a number $\xi \in I$ such that*

$$\int_a^b h_1(t)h_2(t)\,dt = h_1(\xi)\int_a^b h_2(t)\,dt.$$

Let us consider equation (1) and assume the integrand appearing in equation (1) satisfies the hypotheses of Theorem 2.1. For any positive integer n, let us consider a partition of the interval I into n equally-spaced sub-interval of length $\Delta = \frac{b-a}{n}$, i. e. $x_0 = a < x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} = b$.

In addition let t varies in the interval $I_T = [0, T]$ and let us consider a partition of such interval in m equally-spaced sub-interval o length $\Delta_T = \frac{T}{m}$, i. e. $t_0 = 0 < t_1 < t_2 < t_3 < \cdots < t_n < t_{m+1} = T$. For each x_i, $i = 0, 1, 2, 3, \ldots, n$, applying Theorem 2.1 to discretized version of equation (1), we obtain, for all x_i

$$u(x_i, t_j) = f(x_i, t_j) + c(x_i, t_j) \int_0^{t_j} k(t_j - \eta)\, d\eta \tag{2}$$

where we assume $u(g(x_i, t_j, \xi), \xi) = c(x_i, t_j)$ is a constant. Then, replacing equation (2) in both the left and the right hand side of equation (1), we obtain

$$c(x_i, t_j) \int_0^{t_i} k(t_i - \eta)\, d\eta = \int_0^{t_i} k(t_i - \eta)(f(g(x_i, t_i, \eta), \eta) + c(x_i, t_j) \int_0^{\eta} k(\zeta - \eta)\, d\zeta)\, d\eta \tag{3}$$

For each x_i, for $i = 1, 2, \ldots, n+1$, and considering $t_0 = 0$ it results that $u(x_i, 0) = f(x_i, 0)$ because, when $t_i = 0$ the integral appearing in the right side of equation (1) disappears. Instead, for each x_i, when $i = 1, 2, \ldots, n+1$, quantities $c(x_i, t_j)$ are calculated solving n triangular non-linear system in m equations and m unknown that are built considering equation (3). Then, in order to obtain an accurate numerical approximation, following Martire and Oliva [7] we apply the Picard method using as starting quantities $c(x_i, t_j)$.

3 Numerical results

In this section we show the effectiveness of our algorithm considering two different example also considered in Annunziato et al. [2]. All experiments are numerically solved using the Matlab software. In all experiment we choose $n = m = 40$. In both cases the Picard method is iterated 30 times.
As a first example we consider, with $T = 1$,

$$u(x, t) = f(x, t) + \int_0^t e^{-(t-\eta)} u(g(x, t, \eta), \eta)\, d\eta \tag{4}$$

where

$$f(x, t) = e^{-2t}((x-1)(2x^3 + e^t(-1 - x - tx^2 + (t-2)x^3)) + x^4 sinh(t))$$

and the flux

$$g(x, t, \eta) = x - x^2(1 - e^{-(x-(t-\eta))})$$

with $(x, t, \eta) \in ([0, 1] \times [0, 1] \times [0, 1])$.

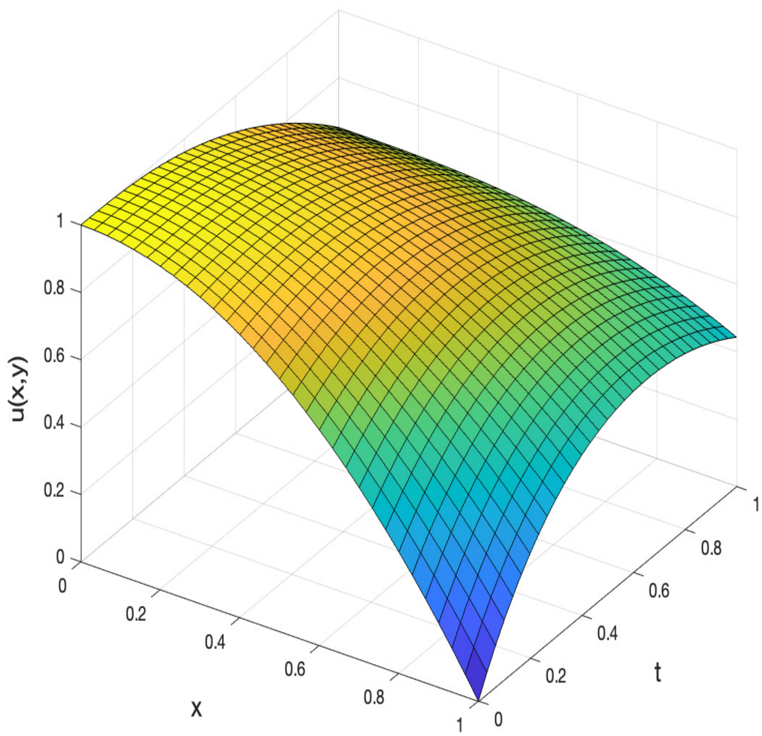

Figure 1: Numerical approximation of integral equation (4).

In figure 1 the approximated solution of integral equation (4) is depicted. As a second example we consider, with $T = 5$,

$$u(x,t) = f(x,t) + \int_0^t e^{-(t-\eta)} u(g(x,t,\eta),\eta)\, d\eta \tag{5}$$

where

$$f(x,t) = \sin(x\pi/2)e^{-(x-t)^2}$$

and the flux

$$g(x,t,\eta) = 1 - e^{-(x-(t-\eta)^2}$$

with $(x,t,\eta) \in ([0,1] \times [0,5] \times [0,5])$ and

$$f(x,t) = \sin(x\pi/2)e^{-(x-t)^2} \tag{6}$$

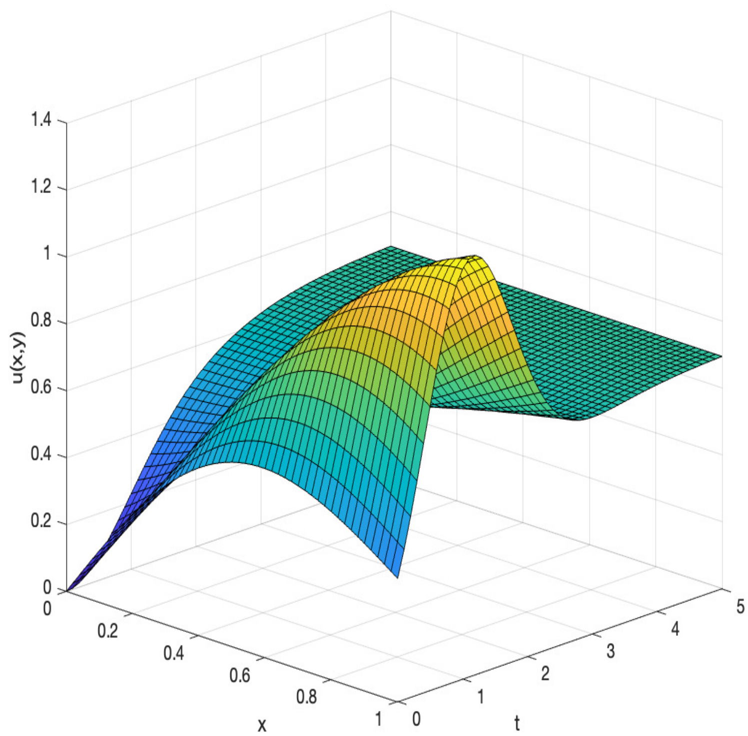

Figure 2: Numerical approximation of integral equation (5).

In figure 2 the approximated solution of integral equation (5) is depicted. The advantage of the proposed method is computational in that the proposed algorithm is very simple to implement and furthermore, for a high number of iterations of the Picardo method a parallel calculation technique can be implemented in a very simple way.

4 Conclusions

In this study, we developed and implemented a combined simplified numerical technique to solve Volterra-renewal Integral Equations with the goal of increasing the accuracy and efficiency of solutions for these difficult equations. Next, we use the Picard technique in accordance with Martire and Oliva [6] to get a more precise numerical estimate, we have been able to overcome several challenges associated with VRIEs, including handling nonlinearities, time-dependent coefficients, and memory effects over long periods of time. Our approach offers a workable solution to the computational limitations of traditional methods, which often struggle with accuracy or consistency when renewal terms are involved. In the section on numerical findings, we provide numerical examples that demonstrate the advantages of the suggested approaches over closed-form solutions.

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