Nonperturbative RG for the weak interaction corrections to the magnetic moment

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We analyze, by rigorous renormalization group methods, a Fermi model for weak forces with a single family of leptons, one massless and the other with mass $m = Me^{-\beta}$, with M the gauge boson mass, a quartic nonlocal interaction with coupling λ^2, and a momentum cutoff Λ. The magnetic moment is written as a series in λ^2, with n-th coefficients bounded by $C^n(\frac{m^2}{M^2})\beta^{2n}(\frac{\Lambda^2}{M^2})^{(1+0^+)(n-1)}$ if C is a constant; this implies convergence and provides nonperturbative bounds on the higher order contributions. The fact that the magnetic moment is associated with a dimensionally irrelevant quantity requires the implementation of cancellations in the multiscale analysis.

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I. INTRODUCTION AND MAIN RESULTS

The anomalous magnetic moment $\frac{g-2}{2}$ plays a central role in physics since the beginning of quantum field theory [1] and it is nowadays attracting a renewed interest [2,3]. Its theoretical value can be computed in the Standard Model with very high precision and comparison with experiments provides a stringent test on the completeness of the theory.

The contributions to the anomalous magnetic moment can be divided into the ones involving also strong forces and the ones considering only electroweak ones. In the first case, the nonperturbative nature of the low energy strong interactions requires numerical lattice or data driven approaches, see, e.g., [4–6]. In the second, an analytical perturbative approach is, in principle, justified by the smallness (in adimensional units) of the couplings involved; that is, $\alpha = 1/137,\ldots$ and $\lambda^2 = 4\pi\alpha/\sin\theta_W^2$ ($\sin^2\theta_W = 0,2231\ldots$).

The electroweak theory allows us to write the magnetic moment as a series $\sum \alpha^n \lambda^{2m} A_{n,m}$ with coefficients $A_{n,m}$ expressed by the sum of Feynman graphs. Perturbative renormalizability [7] (see also [8]) ensures that the ultraviolet divergences present in the graphs can be exactly compensated by a suitable choice of the bare parameters, so that each coefficient $A_{n,m}$ is finite, removing the cutoffs, typically with a factorial growth in the order.

The coefficients $A_{n,m}$ can be explicitly computed and their evaluation becomes more and more challenging increasing the order. In the case of the pure QED contributions $A_{n,0}$ the first order was computed in [1] $A_{1,0} = 1/2\pi$, the second in [9], and more recent computations were done up to $n = 5$, see, e.g., [10,11] and the review [6]. Such coefficients are universal numbers (if a single lepton is considered). In contrast, the weak-interaction corrections depend on the lepton masses; in particular, see [12–14], $A_{0,1} = \frac{5}{24\sqrt{2}\pi^2}\frac{m^2}{M_W^2}(1+\frac{1}{5}(1-4s^2)^2)$, where m is the lepton mass and M_W is the W mass. The smallness of the ratio $(m/M_W)^2$ says that the weak contributions are suppressed with respect to the electromagnetic (e.m.) ones.

The above predictions are done by "truncating" the series expansion at a certain order n, and the effect of higher orders is estimated to be α^{n+1} in the case of QED or $\frac{m^2}{M_W^2}\lambda^{2(n+1)}$ for weak forces, up to a constant C^n with C of size suggested by lowest orders. However, such series are not convergent [if so the error would be indeed $O(C^n\varepsilon^n)$ if ε is the coupling], so that the truncation cannot be done at arbitrary order; if asymptotic the error would be $O(C^n n!\varepsilon^n)$ (and the truncation could be done only up to a finite order), but it is likely that, at least if one restricts to the electroweak sector, even this is not the case [15,16] due to the triviality phenomenon, rigorously established for ϕ^4 [17,18]. Other sources of nonperturbative errors in the truncation are in [19,20].

One can compare the anomalous magnetic moment with the Hall conductivity [21,22], as both quantities were used as an experimental input to get the value of the fine-structure constant. However, for the latter there is no theoretical uncertainty due to truncation: even if, in principle, it could acquire corrections due the presence of many body interactions, all higher orders are *exactly vanishing* due to topological protections, as recently rigorously established [23,24]. This is, however, *not* the case for the anomalous magnetic moment, and an estimate on the higher order terms neglected in the perturbative approach is required. In more recent times, the independent measurements of the parameters, like the fine-structure constant from the atomic physics, with a precision competitive with the magnetic moment of the electron, had the effect that such a quantity can be used to test the Standard Model, see, e.g., [25], and this again requires bounds on truncation error or estimates of higher perturbative orders in the electroweak sector.

A "nonperturbative" framework is obtained by expressing the magnetic moment in terms of functional integrals regularized with a finite ultraviolet cutoff Λ in the Euclidean setting, which is suitable for the magnetic moment [20]. The cutoff must be much larger than the experiment scale, so that the results are expected to be cutoff independent. On the other hand, the cutoff cannot be taken arbitrarily high, at least if one considers only electroweak forces, due to the triviality. There is a well-known relation between the renormalizability properties and the maximal allowed cutoff. In a renormalizable model, like the electroweak sector, one expects, in principle, that a cutoff at least exponentially high in the inverse coupling can be reached, ensuring, due to the smallness of the coupling, that cutoff corrections are negligible. This, however, requires a nonperturbative formulation of the electroweak theory and there are well-known difficulties for a chiral gauge theory like that [26–28].

We consider, therefore, lower values of the cutoff Λ where the weak forces can be described by Fermi interactions, and we restrict to a single family of leptons l, ν. The "effective potential" is given by

$$e^{V^e_\Lambda(A,\phi)} = \int P(d\psi) e^{V(\psi+\phi)+\bar{B}(A,\psi+\phi)} \tag{1}$$

where $\psi_{x,i}, \bar{\psi}_{x,i}$ are Grassmann variables, $i = l, \nu$ is the particle index, $x \in (0, L]^4$ and periodic boundary conditions are imposed, $\psi_{x,i} = (\psi^-_{x,i,L}, \psi^-_{x,i,R})$, $\bar{\psi}_{x,i} = (\psi^+_{x,i,R}, \psi^+_{x,i,L})$,

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma_j = \begin{pmatrix} 0 & i\sigma_j \\ -i\sigma_j & 0 \end{pmatrix},$$

and $P(d\psi)$ is the fermionic integration with propagator, $i = l, \nu$,

$$g_i(x-y) = \frac{1}{L^4} \sum_k e^{ik(x-y)} \frac{\chi_N(k)}{k + m_i}, \tag{2}$$

where $\chi_N(k) = \chi(\gamma^{-N} k)$, with $\chi(k)$ as a cutoff function such that $\chi = 1$ for $|k| \leq 1/\gamma$ and $\chi = 0$ for $|k| \geq 1$ and $\Lambda = \gamma^N$, with N a positive integer, and where $\gamma > 1$ is a scaling parameter. Moreover $\sigma^L_\mu = (\sigma_0, i\sigma)$ and $\sigma^R_\mu = (\sigma_0, -i\sigma)$ with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The interaction is given by

$$V = \frac{\lambda^2}{2} \int dx dy [v_W(x,y) j^{+W}_{\mu,x} j^{-W}_{\mu,y} + v_Z(x,y) j^Z_{\mu,x} j^Z_{\mu,y}] \tag{3}$$

with $\hat{v}_W(k) = \frac{\chi_N(k)}{k^2+M_W^2}$ and $\hat{v}_Z(k) = \frac{\chi_N(k)}{k^2+M_Z^2}$. The charged currents are $j^{+W}_{\mu,x} = \psi^+_{l,L,x} \sigma^L_\mu \psi^-_{\nu,L,x}$ and $j^{+W}_{\mu,x} = \psi^+_{\nu,L,x} \sigma^L_\mu \psi^-_{l,L,x}$ and the neutral current is $s = L, R$,

$$j^Z_{\mu,x} = \sum_{i,s} (\varepsilon_s - \sin^2 \theta_W Q_i) \psi^+_{x,i,s} \sigma^s_\mu \psi^-_{x,i,s}, \tag{4}$$

with $s = L$, R $Q_l = Q$, $Q_\nu = 0$, and $\varepsilon_L = -\varepsilon_R = 1$. Note that the interaction is nonlocal in space and it decays with inverse rate M_W, M_Z.

The source term is given by $\bar{B}(A,\psi) = \int dx A_\mu j^{e.m.}_{\mu,x}$ with $j^{e.m.}_{\mu,x}$ the total e.m. current $j^{e.m.}_x = \sum_s \mathcal{Z}_s Q \psi^+_{x,l,s} \sigma^s_\mu \psi^-_{x,l,s}$. The fermion l is massive and the fermion ν massless, $m_l = m$ and $m_\nu = 0$; moreover, we define $M_W = M, M_Z = \cos\theta_W M$ with $\cos\theta_W \sim 0,881\ldots$ and $m/M = e^{-\beta}$ with $\beta \sim 3$ for muons and ~ 6 for electrons, and $\Lambda \geq M$.

The "vertex function" is given by $\Gamma_{\mu,i,s,s'}(z;x,y) = \frac{\partial^3 V^e_\Lambda}{\partial A_{\mu,z} \partial \phi^-_{x,i,s} \phi^+_{y,i,s'}}\Big|_0$. If $\hat{\Gamma}_{\mu,i,s,s'}(k_1,k_2)$ denotes its Fourier transform, the "anomalous magnetic moment," corresponding to a term $\frac{Q}{2m} \varepsilon_{\mu,\nu} p_\nu A_\mu \sigma_{\mu\nu}$ in the Dirac action, is obtained from $G_{\mu,\nu} = m\partial_\nu \hat{\Gamma}_{\mu,l,R,L}(k_1,k_2)|_0$, while the dressed charge is related to $\hat{\Gamma}_{\mu,l,s,s}(0,0)$.

The dressed charge can be expressed by a series expansion in λ^2 with nth coefficients $O(C^n(\lambda\Lambda/M)^{2n})$, see [29,30]; the series is therefore convergent provided that $\lambda\Lambda/M$ is small. There is nontrivial charge renormalization, due to the fact that the Ward identities are violated at finite cutoff, and $\mathcal{Z}_s$ has to be chosen so that the value of the dressed charge is just Q. A similar convergent expansion holds for the wave function renormalization, the chiral anomaly, or the two-point correlations. Such quantities are associated with terms that are relevant or marginal in the renormalization group (RG) sense; that is, connected to terms with positive or vanishing scaling dimension ($D = 4 - \frac{3}{2} n_\psi - n_A - p$, if p is the order of derivatives in coordinates space). They are therefore directly running coupling constants, as in the case of the dressed charge, or with a dominant part depending only on relevant or marginal terms.

In contrast, the magnetic moment is associated with an "irrelevant" term with dimension $D = -1$. The derivative of $\hat{\Gamma}_{\mu,l,s,s'}$ produces an extra factor $1/m$, so a naive dimensional estimate for the nth order of $\partial_\nu \hat{\Gamma}_{\mu,l,R,L}$ is $O(m^{-1} C^n (\lambda\Lambda/M)^{2n})$; this is of no use for estimating the error done truncating the series, as explicit computations of lowest orders are $\frac{m}{M^2}$. One needs, therefore, to improve the dimensional bounds by implementing suitable cancellations at any order in the convergent expansion.

Theorem. Given (1) with $\Lambda > M$ and $M/m = e^\beta$, $\beta > 0$ we can write $G_{\mu,\nu} = \sum_{n=1}^{\infty} G_{\mu,\nu}^{(n)} \lambda^{2n}$ with, in the limit $L \to \infty$,

$$|G_{\mu,\nu}^{(n)}| \leq \frac{m^2}{M^2} \beta^{2n} C^{2n} \left(\frac{\Lambda^2}{M^2}\right)^{n-1} \left(\log \frac{\Lambda}{M}\right)^{2n} \tag{5}$$

and C is a constant independent of M, m, Λ.

The above result proves analyticity of the magnetic moment for $\lambda C\beta(\Lambda/M)^{1+0^+} < 1$. Note the presence of the small factor $\frac{m^2}{M^2}$ on the rhs of (5), which is obtained implementing cancellations in the expansion. C is an $O(1)$ constant whose value can be obtained collecting all constants in the bounds below. In particular, it is found that $G_{\mu,\nu} = \frac{m^2}{M^2} \lambda^2 (A^\Lambda_{\mu\nu} + O(\lambda^2 (\frac{\Lambda^2}{M^2})^{1+0^+}))$ with $A^\Lambda_{\mu\nu} = O(1)$, from which an upper and lower bound follows (there is no extra $\log \frac{\Lambda}{m}$ in the lowest order). One can therefore exclude nonperturbative effects and justify truncation providing a rigorous estimate of the error. Note also that $A^\Lambda_{\mu\nu} = A^\infty_{\mu\nu}(1 + O(\frac{M^2}{\Lambda^2}))$ so that the result is nonsensitive to the cutoff for $\frac{M}{\Lambda} \ll 1$. Similar considerations can be done for higher order truncation.

The rest of the paper is organized in the following way. In Sec. II we perform an RG integration, in which the main novelty is that certain irrelevant terms are renormalized to improve the scaling dimension of the theory thanks to cancellations. In Sec. III we introduce the tree expansion and we get a bound for the effective potential. In Sec. IV we show that the expansion for the anomalous magnetic factor has suitable cancellations allowing us to get the bound (5). Finally, in Sec. V the conclusions are presented.

II. RENORMALIZATION GROUP ANALYSIS

It is convenient to introduce the "generating function"

$$e^{W_\Lambda(A,\phi)} = \int P(d\psi) e^{V(\psi) + B(A,\psi)} \tag{6}$$

with $B(A,\psi) = \int dx A_\mu j^{e.m.}_{\mu,x} + \int dx (\bar\psi_x \phi_x + \psi_x \bar\phi_x)$. The two-point Schwinger function is $S^\Lambda_{i,s,s'}(x,y) = \frac{\partial^2 W_\Lambda}{\partial \phi^-_{x,i,s} \phi^+_{y,i,s'}}\Big|_0$ and the three point is

$$S^\Lambda_{\mu,i,s,s'}(z;x,y) = \frac{\partial^3 W_\Lambda}{\partial A_{\mu,z} \partial \phi^-_{x,i,s} \phi^+_{y,i,s'}}. \tag{7}$$

The Fourier transform is defined as $\hat S^\Lambda_{i,s,s'}(k)$, $\hat S^\Lambda_{\mu,i,s,s'}(k_1,k_2)$ with $k = \frac{2\pi}{L} n$, with n an integer vector. Using that $-V^e_\Lambda(A, g*\phi) + (\phi, g*\phi) = W_\Lambda(A,\phi)$, obtained from the change of variables $\psi + g*\phi \to \tilde\psi$ if $g*\phi = \int dy g(x,y)\phi_y$, we can write

$$\hat\Gamma_{\mu,l,s,s'}(k_1,k_2) = \hat g^{-1}_{l,s}(k_1) \hat S_{\mu,l,s,s'} \hat g^{-1}_{l,s'}(k_2). \tag{8}$$

We compute the correlations by an exact renormalization group analysis. The cutoff function is written as

$$\chi_N(k) = \sum_{h=-\infty}^{N} f_h(k) f_h(k) = \chi(\gamma^{-h}k) - \chi(\gamma^{-h+1}k) \tag{9}$$

so that $f_h(k)$ is a smooth cutoff function selecting momenta $\gamma^{h-1} \leq |k| \leq \gamma^{h+1}$; we also call $\chi_h(k) = \sum_{j=-\infty}^{h} f_j(k)$ the cutoff function selecting momenta $|k| \leq \gamma^h$. The generic integration step can be inductively defined in the following way. If $\mathcal{V}^{(N)} = V + B$ and assume that we have integrated the fields $\psi^{(N)}, \psi^{(N-1)}, \ldots, \psi^{(h+1)}$, then

$$\int P(d\psi) e^{\mathcal{V}^{(N)}(\psi,A,\phi)} = \int P(d\psi^{(\leq h)}) e^{\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},A,\phi)} \tag{10}$$

with $\mathcal{V}^{(h)}(A,\psi^{(\leq h)}) =$

$$\sum_{l,m=0}^{\infty} \int d\underline{x} d\underline{y} \sum_{\underline{s},\underline{\varepsilon}} W^{(h)}_{l,m}(\underline{x},\underline{y}) \prod_{j=1}^{l} \psi^{\varepsilon_j,(\leq h)}_{s_j,i_j,x_j} \prod_{j=1}^{m} A^{\varepsilon_j}_{\mu_j,y_j} \tag{11}$$

with ψ including also the ϕ fields and $P(d\psi^{(\leq h)})$ has propagator $g^{(\leq h)}_i(x,y) =$

$$\frac{1}{L^4} \sum_k e^{ik(x-y)} \chi_h(k) \begin{pmatrix} Z^L_{h,i} \sigma^L_\mu k_\mu & m_{h,i} \\ m_{h,i} & Z^R_{h,i} \sigma^R_\mu k_\mu \end{pmatrix}^{-1}. \tag{12}$$

The single scale propagator is bounded by

$$|g^{(h)}(x)| \leq C\gamma^{3h} e^{-(\gamma^h |x|)^{\frac{1}{2}}}, \tag{13}$$

hence $\int dx |g^{(h)}(x)| \leq C\gamma^{-h}$; moreover, $\int |v_W(x)| \leq C/M^2_W$ and $\int |v_Z(x)| \leq C/M^2_Z$.

$\mathcal{V}^{(h)}$ is the sum of monomials of any degree in the fields, with scaling dimension $D = 4 - \frac{3}{2} l - m$. We introduce a renormalization procedure extracting from V^h not only the terms with scaling dimension ≥ 0 (that is, only the relevant or marginal term), but also the irrelevant terms with scaling dimension -1.

We write, therefore,

$$\int P(d\psi^{(\leq h)}) e^{\mathcal{L}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},A,\phi) + \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},A,\phi)}, \tag{14}$$

where $\mathcal{R} = 1 - \mathcal{L}$ is the renormalization operation and $\mathcal{L}$ acts on the monomials in V^h in the following way:

$$\mathcal{L}\hat{W}_{2,1;s,s}(k_1,k_2) = \hat{W}_{2,1;s,s}(0,0) + k_1\partial_1\hat{W}_{2,1;s,s}(0,0) + k_2\partial_2\hat{W}_{2,1;s,s}(0,0)$$
$$\mathcal{L}\hat{W}_{2,1;L,R}(k_1,k_2) = \hat{W}_{2,1;L,R}(0,0) + k_1\partial_1\hat{W}_{2,1;L,R}(0,0)|_{m=0} + k_2\partial_2\hat{W}_{2,1;L,R}(0,0)|_{m=0}$$
$$\mathcal{L}\hat{W}_{2;s,s}(k) = \hat{W}_{2;s,s}(0) + k\partial\hat{W}_{2;s,s}(0) + \frac{1}{2}k^2\partial^2\hat{W}_{2;s,s}(0)$$
$$\mathcal{L}\hat{W}_{2;L,R}(k) = \hat{W}_{2;L,R}(0) + k\partial\hat{W}_{2;L,R}(0) + \frac{1}{2}k^2\partial^2\hat{W}_{2;L,R,s}(0)\Big|_{m=0}. \quad (15)$$

Note that the propagators involving the same chirality are odd in the exchange $k \to -k$ and the ones involving different chiralities are even. Therefore,

$$\hat{W}_{2;s,s}(0) = \partial^2\hat{W}_{2;s,s}(0) = 0 \quad (16)$$

as is given by graphs with an odd number of diagonal propagators and an even number of nondiagonal ones; $\partial\hat{W}_{2;L,R}(0,0) = 0$ as is given by graphs with an even number of diagonal propagators and an odd number of nondiagonal ones; $\partial^2\hat{W}_{2;L,R}(0,0)_{m=0} = 0$ as they require a nondiagonal propagator to be nonvanishing; $\hat{W}_{2,1;L,R}(0,0) = 0$ as there is an odd number of nondiagonal propagators and an odd number of diagonal ones; $\partial\hat{W}_{2,1;L,R}(0,0)_{m=0} = 0$ as they require a nondiagonal propagator to be nonvanishing; $\partial\hat{W}_{2,1;s,s}(0,0) = 0$ as is given by graphs with an odd number of diagonal propagators and an even number of nondiagonal ones.

We can write, therefore,

$$\int \tilde{P}(d\psi^{(\leq h)})e^{\tilde{\mathcal{L}}\mathcal{V}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)},A,\phi)+\mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)},A,\phi)} \quad (17)$$

where for $h \leq N-1$,

$$\mathcal{L}V^h(\sqrt{Z_{h-1}}\psi^{(\leq h)},A,\phi) = \sum_s \int dx Z^A_{h,s}A_\mu \psi^+_{x,l,s}\sigma^s_\mu \psi^+_{x,l,s} \quad (18)$$

and $\tilde{P}(d\psi^{(\leq h)})$ has a propagator given by (12) with $Z_{h,i,s}$ replaced by $Z_{h-1,i,s} = Z_{h,i,s} + \partial W^h_{2,s,s}(0)$, $m_{h-1} = m_h + W^h_{2,R,L}(0)$, and $Z^A_{h-1,s} = \frac{Z_{h-1,l,s}}{Z_{h,l,s}}(Z^A_{h,s} + W_{2;1,s,s}(0,0))$. In the case of $\psi\phi$ or $A\psi\phi$ we use the fact that the $\mathcal{L}$ part is vanishing, as there is surely a propagator $g^h(0) = 0$; hence there is no running coupling constant associated.

Using that $\tilde{P}(d\psi^{(\leq h)}) = P(d\psi^{(\leq h-1)})P(d\psi^{(h)})$ with $g^{(h)}$ given by (12) with χ_h replaced f_h, we get that (17) can be written as the rhs of (10) with $h-1$ replacing h and, see Fig 1

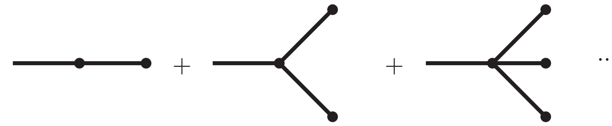

FIG. 1. Graphical representation of (19), that is, $\mathcal{E}^T_h(\tilde{\mathcal{V}}^{(h)}) + \frac{1}{2}\mathcal{E}^T_h(\tilde{\mathcal{V}}^{(h)};\tilde{\mathcal{V}}^{(h)}) + \frac{1}{3!}\mathcal{E}^T_h(\tilde{\mathcal{V}}^{(h)};\tilde{\mathcal{V}}^{(h)};\tilde{\mathcal{V}}^{(h)}) + \ldots$ with $\tilde{\mathcal{V}}^{(h)} = \tilde{\mathcal{L}}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$.

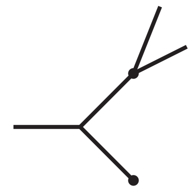

FIG. 2. Graphical representation of $\frac{1}{2}\mathcal{E}^T_h(\tilde{\mathcal{L}}\mathcal{V}^{(h)}; \mathcal{R}\frac{1}{2}\mathcal{E}^T_{h+1}(\mathcal{V}^{h+1};\mathcal{V}^{h+1}))$.

$$\mathcal{V}^{h-1} = \sum_{n=0}^\infty \frac{1}{n!}\mathcal{E}^T_h(\tilde{\mathcal{L}}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)};\ldots;\tilde{\mathcal{L}}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}), \quad (19)$$

where $\mathcal{E}^T_h$ are the fermionic truncated expectations; that is, $\mathcal{E}^T_h(O;n) = \frac{\partial^n}{\partial\lambda^n}\log\int P(d\psi^h)e^{\lambda O}|_0$.

The procedure can be then iterated up the scale of the fermionic mass defined as

$$\gamma^{h^*} = m_{h^*}. \quad (20)$$

At this point, one can write $\int P(d\psi^{\leq h^*}_l)e^{V^{h^*}(\psi_l,\psi_\nu,A,\phi)} = e^{V^{h^*}(\psi_\nu,A,\phi)}$ using that $|g^{\leq h^*}(x)| \leq C\gamma^{3h^*}e^{-(\gamma^{h^*}|x|)^{\frac{1}{2}}}$. The integration of the remaining scales is done as above, the only difference being that only the fields ψ_ν remain; the ψ_l has been already integrated out. Note that $m_{h,\nu} = 0$ by symmetry.

In order to write explicitly the effective potential $\mathcal{V}^{(h-1)}$ one has to express the $\mathcal{R}\mathcal{V}^h$ on the rhs of (19) in terms of the sum of truncated expectations, while no further expansion is done in the $\mathcal{L}\mathcal{V}^h$; a graphical representation of a term is in Fig. 2.

This procedure can be iterated up to the scale N, resulting in a tree expansion described below.

III. RENORMALIZED EXPANSION

Iterating (19) we get an expansion for $\mathcal{V}^h$ in terms of "trees" [31], see Fig. 3,

$$\mathcal{V}^{(h)}(\psi^{(\leq h)},A,\phi) = \sum_{n=1}^\infty \sum_{\tau\in\mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau), \quad (21)$$

with τ a tree, constructed by joining a point, the root r, with an ordered set of $n \geq 1$ end points and associating a label

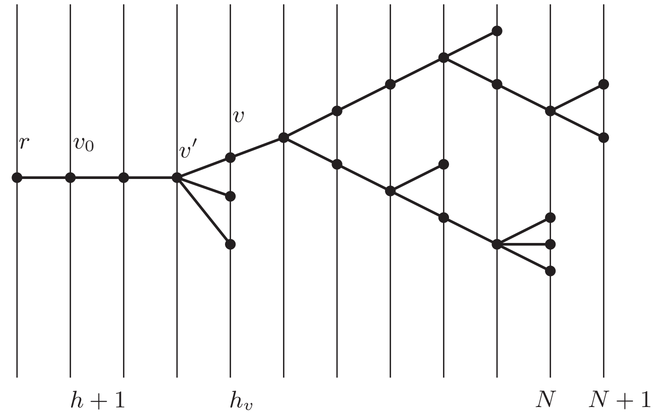

FIG. 3. A labeled tree.

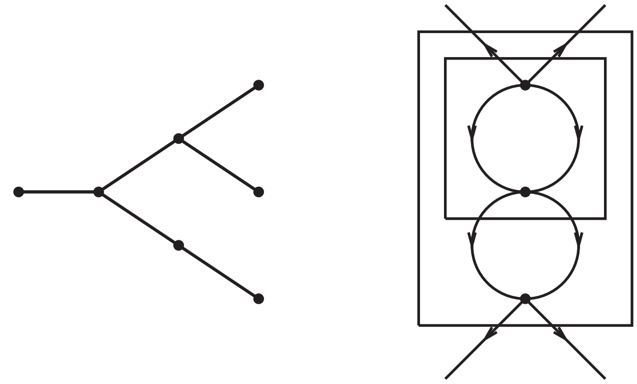

FIG. 4. A graph with its clusters and the corresponding tree; the smaller cluster has scale h_1 and the larger $h+1$.

$h \le N-1$ with the root; moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, N+1]$ intersecting all the nontrivial vertices, the end points, and other points called trivial vertices. To each vertex, v is associated with a scale h_v; they are partially ordered and, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$; moreover, given v there are S_v points following v. The first vertex has scale $h+1$. The end points v can be (1) λ points to which is associated $V(\psi)$, or ϕ points to which is associated $B(\psi, 0, \phi)$; in this case, the scale is $h_v = N+1$. (2) Z points, which are associated with $\mathcal{L}\mathcal{V}^{h_v-1}(\psi^{(\le h_v-1)}, A)$ and in this case the scale is $h_v \le N+1$ and there is the constraint that $h_v = h_{v'}+1$ if v' is the nontrivial vertex preceding v. Given a vertex v we call m_v^λ the number of λ points following v, m_v^ϕ the number of ϕ points following v, and m_v^A the number of Z points following v.

With the above definitions, the value of $\mathcal{V}^{(h)}(\tau)$ is obtained iteratively by the relations

$$\mathcal{V}^{(h)}(\tau) = \frac{(-1)^{s+1}}{s!}\mathcal{E}_{h+1}^T[\bar{\mathcal{V}}^{(h+1)}(\tau_1); ..; \bar{\mathcal{V}}^{(h+1)}(\tau_s)], \qquad (22)$$

where $\tau_1, \ldots, \tau_s$ are the subtrees with root in v, $\bar{\mathcal{V}}^{(h+1)}(\tau) = \mathcal{R}\mathcal{V}^{(h+1)}(\tau)$ if the subtree τ_i contains more then one end point, while if τ_i contains only one end point $\bar{\mathcal{V}}^{(h+1)}(\tau)$ is $V(\psi^{(\le N)}, 0, \phi)$ if $h = N$ or if $h \le N$ is $\mathcal{L}\mathcal{V}^{h+1}(A, \psi^{(\le h+1)})$.

By (22) we see that $\mathcal{V}^{(h)}(\tau) = \sum_P \mathcal{V}^{(h)}(\tau, P)$, where P is the set of all P_v associated with the vertices of the tree, corresponding to subsets of the labels of the fields associated with the end points following v. We call V the vertices such that P_v is different with respect to the preceding one.

The $\mathcal{V}^{(h)}(\tau, P)$ can be represented as sum of renormalized Feynman graphs. The difference with respect to the usual Feynman graphs is that the scale labels of the tree τ, corresponding to vertices $v \in V$, can be represented as a set of clusters enclosing the end points. To each point is associated an element of V or $\mathcal{L}V^h$, represented graphically as a point with half lines to be contracted. To each line is associated a scale, and there is the constraint that all the lines inside a cluster v have scale $\le h_v$, and at least one of them is at scale h_v. The $\mathcal{R}$ operation is applied on the clusters depending on the number of the external lines.

Each graph is finite but one needs that the sum over the scale labels is finite. Let us consider, for instance, the graph in Fig. 4; one can bound the sum over the scales by up to a constant, if $N \ge h_1 \ge h+1$, $(\frac{\lambda^2}{M^2})^3 \sum_{h_1} \gamma^{2h_1}\gamma^{2h}$. In this example, there is no $\mathcal{R}$ operation in the subgraphs. In contrast, the $\mathcal{R}$ operation is present in the graph in Fig. 5.

The effect of the $\mathcal{R}$ operation can be written as

$$\mathcal{R}\hat{W}_{2;s,s}^{(h_v)}(k) = k^3 \int_0^1 dt \partial^3 \hat{W}_{2;s,s}^{(h_v)}(tk). \qquad (23)$$

Therefore, the effect of $\mathcal{R}$ is to produce an extra k^3; to the external lines of $\hat{W}^{(h_v)}$ is associated a propagator $g^{(h_{v'})}(k)$, if v' is the vertex $\in V$ following v, with a cutoff function restricting the value of k to $\sim\gamma^{h_{v'}}$. Similarly, the derivatives on $\hat{W}^{(h_v)}$ are applied on propagators with scale $\ge h_v$. Therefore, the effect of the $\mathcal{R}$ operation is to produce an extra $\sim\gamma^{3(h_{v'}-h_v)}$ factor. Regarding the terms $W_{2;R,L}(k)$, in addition to such term there is also a contribution of the form

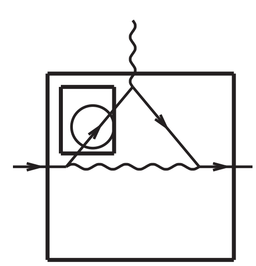

FIG. 5. A graph requiring renormalization; the smallest cluster has scale h_1 and the larger $h+1$.

$$\frac{1}{2}k^2\partial^2 W^{h_v}_{2;L,R}(k)|_{m=0} - \frac{1}{2}k^2\partial^2 W^{h_v}_{2;L,R}(k). \quad (24)$$

Such terms are present only for $i = l$; by phase symmetry when $i = \nu$ there are no terms L, R, by invariance under $\psi_{s,\nu} \to e^{i\alpha_s}\psi_{s,\nu}$. The bound for the propagator involving L and R fields has an extra factor $\frac{m_{h_v}}{\gamma^{h_v}}$ and, for $h_v \geq h^*$ (20),

$$\frac{m_{h_v}}{\gamma^{h_v}} = \frac{m_{h_v}}{m_{h^*}}\frac{m_{h^*}}{\gamma^{h_v}} \leq \gamma^{h_{v'}-h_v}, \quad (25)$$

as the smallest scale of the external fields of type $i = l$ is h^*, that is, $h_{v'} \geq h^*$.

Without the $\mathcal{R}$ operation, the graph in Fig. 5 is bounded by $h_1 \geq h_2$ $(\frac{\lambda^2}{M^2})^3 \sum_{h_1} \gamma^{5h_1}\gamma^h$; the $\mathcal{R}$ operation adds an extra $\gamma^{3(h-h_1)}$.

A finite bound for any graph is still not sufficient for getting convergence; the number of graphs has a factorial growth. Therefore, it is convenient to represent the truncated expectation as [32] $\mathcal{E}^T_h(\tilde\psi^{(h)}(P_1);\tilde\psi^{(h)}(P_2);\ldots;\tilde\psi^{(h)}(P_s)) =$

$$\sum_T \prod_{l\in T} g^{(h)}(x_l - y_l)\int dP_T \det G^{h,T}, \quad (26)$$

where $\tilde\psi^{(h)}(P)$ are monomials in the ψ, T is a set of lines forming an "anchored tree graph" connecting the set $P_1,\ldots,P_s$, and $\det G^{h,T}$ is a matrix containing the fields not belonging to T. The crucial point is that, using Gram inequality, the $\det G^{h,T}$ can be bounded by a constant times the number of fields. This is a way to implement the well-known fact that fermionic series expansion can be convergent, in contrast to bosonic ones. Using (26) one gets for a class of graph with a chosen tree τ and P the same bound as for a single graph, without factorials. If $\|V(\tau,P)\|$ denotes the integral of the modulus over all the coordinates except one, then (see, e.g., [30]) $\|V(\tau,P)\| \leq$

$$C^n \prod_{v\in V}\gamma^{4h_v(S_v-1)}\gamma^{-3h_v n_v}\prod_{v\in V}\gamma^{z_v(h_{v'}-h_v)}(\lambda^2/M^2)^n, \quad (27)$$

where n_v is the number of propagators in the cluster v and not in any smaller one, v' is the vertex in V preceding v, and S_v are the vertices following v (or the maximal clusters in v). The factor $\prod_v \gamma^{z_v(h_{v'}-h_v)}$ is the effect of the renormalization procedure; $z_v = 3$ in the terms $\psi\psi$ or $\psi\phi$ and $z_v = 2$ in the terms $A\psi\psi$ or $A\psi\phi$. We use the relations $\sum_{v\in V}(h_v - h)(S_v - 1) = \sum_{v\in V}(h_v - h_{v'})(m^\lambda_v + m^A_v + m^\phi_v - 1)$ and $\sum_{v\in V}(h_v - h)n_v = \sum_{v\in V}(h_v - h_{v'})(2m^\lambda_v + m^A_v + m^\phi_v - n^e_v/2)$, where n^e_v is the number of external ψ,ϕ lines from the cluster v. Therefore, the bound becomes, if $\bar D_v = 4 - 3n^e_v/2 + 2m^\lambda_v - m^A_v - m^\phi_v$, $\|V(\tau,P)\| \leq$

$$C^n\gamma^{(4-3/2l+2m^\lambda-m^A-m^\phi)h}\prod_{v\in V}\gamma^{(h_v-h_{v'})(\bar D_z - z_v)}(\lambda^2/M^2)^n, \quad (28)$$

where l are the external ψ,ϕ lines associated with $V(\tau,P)$ and $z_v = 3$ for the v with two external ψ lines and $z_v = 2$ for the v with two external ψ lines and one A line. We use now the relation $i = \phi, \lambda\gamma^{hm^i_{v_0}}\prod_{v\in V}\gamma^{(h_v-h_{v'})m^i_v} = \prod_{v\in V}\gamma^{h_v\bar m^i_v}$, where $\bar m^i_v$ is the number of end points of type i contained in v and not in any smaller cluster. Therefore, if $D_v = 4 - 3n^e_v/2 - m^A_v$,

$$\|V(\tau,P)\| \leq C^n\gamma^{h(4-3/2l-m^A)}\prod_{v\in V}\gamma^{(h_v-h_{v'})(D_v-z_v)} \times \left[\prod_{v\in W_\lambda}\gamma^{2(h_{v^*}-N)}(\lambda^2\gamma^{2N}/M^2)^n\right]\left[\prod_{v\in W_\phi}\gamma^{-h_{v^*}}\right], \quad (29)$$

where W_λ and W_ϕ are the end points of λ or ϕ type and v^* is the first nontrivial vertex preceding v. Note that if $v\in W_\phi$ then $h_{v^*} = h_k, h_k + 1$; the reason is that the corresponding contribution is of the form $g^{h_{v^*}}(k)W$, and hence is nonvanishing only for such scales.

We consider first the contribution to the effective potential when there are no ϕ end points. The scale h is fixed so that the sum over all the possible scales can be done summing over all the possible scale differences (the scale h is fixed); hence, if $\tilde D_v = D_v - z_v \geq 2$,

$$\sum_{\{h\}}\prod_v \gamma^{(h_v-h_{v'})\tilde D_v} \leq C^n\prod_v \gamma^{-|n^e_v|/4}\left(\sum_{q=1}^\infty \gamma^{-2q}\right)^{4n} \quad (30)$$

as $-\tilde D_v - \chi(n^e_v \geq 8)|n^e_v|/4 \geq 2$. The factor $\gamma^{-|n^\psi_v|/4}$ is used to sum over P. Then $\sum_\tau\sum_P |V(\tau,P)| \leq C^n\gamma^{h(4-3/2l-m)}(\lambda^2\gamma^{2N}/M^2)^n$ implying summability over n if $(\lambda^2\gamma^{2N}/M^2)$ is small enough.

As an example, the bound (29) for the graph in Fig. 4 is given by up to the factor $(\frac{\lambda^2\gamma^{2N}}{M^2})^3\,\gamma^{-2h}\sum_{h_1}\gamma^{-2(h_1-h)}\gamma^{4(h_1-N)}\gamma^{2(h-N)}$. Similarly, the bound for the graph in Fig. 5 is $\sum_{h_1}\gamma^{-(h-h_1)}\gamma^{3(h-h_1)}\gamma^{4(h_1-N)}\gamma^{2(h-N)}$.

IV. THE ANOMALOUS MAGNETIC MOMENT

The three-point function $S^\Lambda_{\mu,l,s,s'}(z;x,y)$ with external fields of type l can be written as $S^\Lambda_{\mu,l,s,s'} = \sum_\tau\sum_P S_\mu(\tau,P)$, where the sum is over all the trees with two ϕ end points and a Z end point. We choose the momentum of the external fermionic lines as $|k_1|,|k_2| \leq \gamma^{h^*}$.

We can distinguish between trees with no λ end points and at least a λ end point. In the first case, one has only a contribution to $\hat S^\Lambda_{\mu,l,s,s'}(k_1,k_2)$ of the form, see the first graph in Fig. 6,

$$\sum_{\bar s} Z^A_{h^*,\bar s}\, g^{(\leq h^*)}_{\bar s,s}(k_1)\sigma^{\bar s}_\mu g^{(\leq h^*)}_{\bar s,s'}(k_2). \quad (31)$$

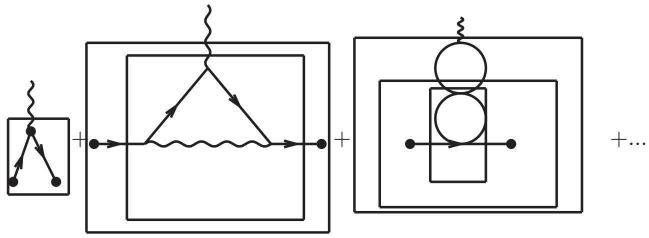

FIG. 6. Some graphs contributing to the functions Γ; the ϕ lines are not represented (they are meant as external to all clusters). In the first, the scale of the cluster is h^*; in the second, the smallest has scale h_1 and the larger h^*; in the third, the smaller h_1, the medium h^*, and the larger h.

In the second case, there is at least a λ end point, like in the second and third graph in Fig. 6. Let us consider the smallest cluster v_s containing the point v_ϕ associated with the ϕ end point; note that $h_{v_s} = h^*$ as the momentum of the external lines is assumed $\leq \gamma^{h^*}$, and the contraction of the ψ field produces a propagator with the same momentum as the external one. In the cluster v_s there is also surely a λ end point $\tilde{v}_\lambda$ (there is at least either a λ or Z point, and there is only one Z), and $\tilde{v}_s$ is the smallest cluster containing $\tilde{v}_\lambda$ contained in v_s.

In terms of trees, there is a path from $\tilde{v}_\lambda$ to v_0, v_0 being the first vertex belonging to V (or a sequence of clusters, one enclosed in the other) $v_0 < v_1 < \ldots v_s < v_{s+1} \ldots < \tilde{v}_s$, an $h_{v_0} \leq \ldots h_{v_s} < \ldots h_{\tilde{v}_s}$ with $h_{v_s} = h^*$, see Fig. 7.

We get therefore the bound, if $\tilde{D}_v = D_v - z_v \leq -2$ $\|S_\mu(\tau, P)\| \leq$

$$C^n \prod_{v \in V} \gamma^{(h_v - h_{v'})\tilde{D}_v} \gamma^{2(h_{\tilde{v}_s} - N)} (\lambda^2 \gamma^{2N}/M^2)^n \gamma^{-2h^*}, \quad (32)$$

which implies

$$\|S_\mu(\tau, P)\| \leq C^n \prod_{v \in V} \gamma^{(h_v - h_{v'})(\tilde{D}_v + \theta_v)} \gamma^{\theta(h_{v_0} - h_{v_s})} \times \gamma^{\theta(h_{v_s} - h_{\tilde{v}_s})} \gamma^{2(h_{\tilde{v}_s} - N)} (\lambda^2 \gamma^{2N}/M^2)^n \gamma^{-2h^*}, \quad (33)$$

where $\theta_v = \theta < 2$ for $v = v_0, v_1, \ldots, v_s.., \tilde{v}_s$ and $\theta_v = 0$ otherwise. Therefore,

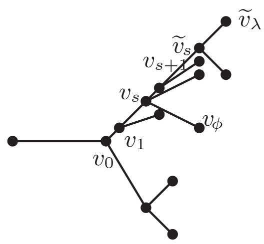

FIG. 7. A tree with the path from $\tilde{v}_\lambda$ to v_0.

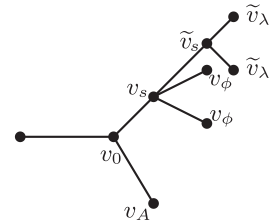

FIG. 8. The tree corresponding to the third graph in Fig. 6; $h_{v_0} = h$, $h_{v_s} = h^*$, $h_{\tilde{v}_s} = h_1$.

$$\|S_\mu(\tau, P)\| \leq \gamma^{\theta(h_{v_0} - h^*)} \gamma^{\theta(h^* - N)} \times C^n \prod_{v \in V} \gamma^{(h_v - h_{v'})(\tilde{D}_v + \theta_v)} \times \gamma^{-2h^*} (\lambda^2 \gamma^{2N}/M^2)^n. \quad (34)$$

The sum over all the scale difference is done again using the factors $\gamma^{(h_v - h_{v'})(\tilde{D}_v + \theta_v)}$; the sum over h_{v_0} is controlled by the factor $\gamma^{\theta(h_{v_0} - h^*)}$, hence the bound for the contributions to $\hat{S}^\Lambda_{\mu,l,s,s'(k_1,k_2)}$ with at least a λ term is

$$\sum_\tau \sum_P \|S(\tau, P)\| \leq C^n \gamma^{-2h^*} \gamma^{\theta(h^* - N)} (\lambda^2 \gamma^{2N}/M^2)^n. \quad (35)$$

Note that such terms are subdominant due to the extra factor $\gamma^{\theta(h^* - N)}$. The three-point function is equal to the free one with renormalized parameters, up to more regular terms containing at least an irrelevant λ interaction. A similar result holds for the two-point function.

As an example, the second graph in Fig. 6 is bounded by [up to a factor $(\lambda^2 \gamma^{2N}/M^2)$] $\gamma^{-2h^*} \sum_{h_1 \geq h^*} \gamma^{2(h^* - h_1)} \gamma^{2(h_1 - N)}$, where the factor $\gamma^{2(h^* - h_1)}$ is produced by the $\mathcal{R}$ operation: hence it is bounded by $\gamma^{-2h^*} \gamma^{\theta(h^* - N)}$.

The third graph gives [up to a factor $(\lambda^2 \gamma^{2N}/M^2)^2$] for $h_1 \geq h^* \geq h$ $\gamma^{-2h^*} \sum_{h_1,h} \gamma^{2h} \gamma^{2h_1} \gamma^{-4N}$, which can be written as $\gamma^{-2h^*} \sum_{h_1,h} \gamma^{2(h - h^*)} \gamma^{2(h^* - h_1)} \gamma^{4(h_1 - N)}$ and finally by $\gamma^{-2h^*} \gamma^{\theta(h^* - N)}$, see Fig. 8. Finally, for the graph in Fig. 5 (if the external lines are propagators, and there is an extra $\mathcal{R}$), one gets for $h_1 \geq h \geq h^*$ $\gamma^{-2h^*} \sum_{h_1,h} \gamma^{2(h^* - h)} \gamma^{2(h - h_1)} \gamma^{4(h_1 - N)} \gamma^{2(h - N)}$, which is surely smaller than $\gamma^{-2h^*} \gamma^{\theta(h^* - N)}$.

We arrive finally to the bound for $G_{\mu\nu} = m \sum_{\tau,P} \partial \Gamma_A(\tau, P)$. By (8) and (31) we see that the contribution from the dominant term to S_μ as the derivative is vanishing. One has therefore to consider the derivative of the amputated contributions with at least a λ end point. Moreover, we consider the term with $s \neq s'$, which are the only contributing at zero momentum. We get $m \|\partial \Gamma_A(\tau, P)\| \leq$

$$m^2 \gamma^{-h_{v_0}} \gamma^{-h^*} C^n \prod_{v \in V} \gamma^{(h_v - h_{v'})\tilde{D}_v} \gamma^{2(h_{\tilde{v}_s} - N)} (\lambda^2 \gamma^{2N}/M^2)^n, \quad (36)$$

where with respect to (32) there is an extra $\gamma^{-h_{v_0}}$ from the derivative and a missing γ^{-2h^*}. The above expression can be rewritten as

$$m\|\partial\Gamma_A(\tau,P)\| \leq m^2\gamma^{-h_{v_0}}\gamma^{-h^*}\gamma^{\frac{3}{2}(h_{v_0}-h^*)} \times \gamma^{2(h^*-h_{\tilde{v}_s})}C^n\prod_{v\in V}\gamma^{(h_v-h_{v'})(\tilde{D}_v+\bar{\theta}_v)} \times \gamma^{2(h_{\tilde{v}_s}-N)}(\lambda^2\gamma^{2N}/M^2)^n, \quad (37)$$

where $\bar{\theta}_v = 3/2$ for $v = v_0, v_1, \ldots, v_s$ and $\bar{\theta}_v = 2$ for $v = v_{s+1}, \ldots, \tilde{v}_s$, and $\theta_v = 0$ otherwise. Therefore,

$$\gamma^{-h_{v_0}}\gamma^{-h^*}\gamma^{3/2(h_{v_0}-h^*)}\gamma^{2(h^*-N)} \leq \gamma^{1/2(h_{v_0}-h^*)}\gamma^{-2N} \quad (38)$$

and finally we get

$$m\|\partial\Gamma_A(\tau,P)\| \leq \gamma^{1/2(h_{v_0}-h^*)}m^2\gamma^{-2N} \times C^n\prod_{v\in V}\gamma^{(h_v-h_{v'})(\tilde{D}_v+\bar{\theta}_v)}(\lambda^2\gamma^{2N}/M^2)^n. \quad (39)$$

Now the sum over h_{v_0} is done using the factor $\gamma^{1/2(h_{v_0}-h^*)}$; the sum over the difference of scales is done using $\gamma^{(h_v-h_{v'})(\tilde{D}_v+\bar{\theta}_v)}$ for any v except the ones between v_s and $\tilde{v}_s$; there are at most $2n$ of such vertices so that there is an extra factor $|N-h^*|^{2n}$. The same over P is done in the same way as we can extract a factor $\gamma^{-n_v^e/6}$ for $n_v^e \geq 8$. Therefore. the bound is $n \geq 1$

$$m\|\partial\Gamma_A(\tau,P)\| \leq \frac{m^2}{M^2}\lambda^2(\log(m/\gamma^N))^{2n}(\lambda^2\gamma^{2N}/M^2)^{n-1}. \quad (40)$$

For instance, the bound for the second graph in Fig. 6 is, obtaining γ^{-2h^*} by the derivatives, $m^2\frac{\lambda^2}{M^2}\sum_{h_1\geq h^*}1$; hence it is bounded by $|h^*-N|\frac{m^2\lambda^2}{M^2}$. In the case of the third graph in Fig. 6 for $h_1 \geq h^* \geq h$, it can be written as up to $m^2(\lambda^2\gamma^{2N}/M^2)^2 \quad \sum_{h_1,h}\gamma^{-h^*}\gamma^{-h}\gamma^{2(h-h^*)}\gamma^{2(h^*-h_1)}\gamma^{4(h_1-N)}$; moreover, we can write $\gamma^{-h^*+h_1}\gamma^{-h+h_1}$ as $\gamma^{-2(h^*-h_1)}\gamma^{-h+h^*}$ so that we get $\sum_{h_1,h}\gamma^{(h-h^*)}\gamma^{-2N}$ which is bounded by $|N-h^*|\gamma^{-2N}$. The contribution of the graph in Fig. 5 is $h_1 \geq h \geq h^* \quad \gamma^{-2h^*}\sum_{h_1,h}\gamma^{2(h^*-h)}\gamma^{2(h-h_1)}\gamma^{4(h_1-N)}\gamma^{2(h-N)}$ bounded by $|N-h^*|^2\gamma^{-2N}$ times $m^2(\lambda^2\gamma^{2N}/M^2)^3$.

Moreover, $(\log\Lambda/m)^{2n} = (\log\Lambda/M + \log M/m)^{2n}$ which is equal to $\sum_p \frac{2n!}{p!(2n-p)!}(\log\Lambda/M)^p(\log M/m)^{2n-p} \leq (\log\Lambda/M)^{2n}(\log M/m)^{2n}2^{2n}$.

The lowest order contribution to the magnetic moment is given by the second graph in Fig. 6 and the difference between the finite and infinite Λ is bounded by (the $\mathcal{R}$ disappears with the derivative) $m^2\int_\Lambda^\infty dk\frac{1}{k^4}\frac{1}{k^2+M_Z^2}$ which is $O(\frac{m^2}{M_Z^2}\frac{M_Z^2}{\Lambda^2})$; this follows from the nonlocality of the interaction which was not used in the bounds.

Finally, we have to study the flow of the running coupling constants. They verify recursive equations in which there is at least a λ end point so that proceeding as above we can write $\frac{Z_{i,s,h-1}}{Z_{i,s,h}} = 1+\beta_z^h$ with $\beta_z^h = O(\gamma^{\theta(h-N)}(\lambda^2\gamma^{2N}/M^2)^2)$. This implies that $\lim_{h\to\infty}Z_{i,s,h} = Z_{l,s}$ is finite and $Z_{i,s} = 1+O(\lambda^2\gamma^{2N}/M^2)$; the fermionic wave function renormalization depends on the particle and chiral index. In the same way $\lim_{h\to\infty}Z_{s,h}^A = Z_s^A$ with $Z_s^A = 1+O(\lambda^2\gamma^{2N}/M^2)$ and $m_{h^*} = m(1+O(\lambda^2\gamma^{2N}/M^2))$. Note that Z^A and Z are different, due to violation of Ward identities due to momentum regularization which produces an extra term in the Ward identities,

$$p_\mu\tilde{\Gamma}_{\mu,l,s}^\Lambda(k,k+p) = Q(S_{l,s,s}^\Lambda(k) - S_{l,s,s}^\Lambda(k+p)) + \delta\Gamma_{l,s}^\Lambda, \quad (41)$$

where $\tilde{\Gamma}_{\mu,l,s}^\Lambda$ is defined as $\Gamma_{\mu,l,s}^\Lambda$ with $\mathcal{Z}_{l,s} = 1$, and $\delta\Gamma_{l,s}^\Lambda$ is similar to $\tilde{\Gamma}_{\mu,l,s}^\Lambda$ with the current replaced by $\delta j_l = \sum_s Q\int dkdpC(k,p)\bar{\psi}_{k,l,s}\sigma_\mu^s\psi_{k+p,l,s}$, where $C(k,p) = \not{k}(\chi^{-1}(k)-1) - (\not{k}+\not{p})(\chi^{-1}(k+p)-1)$. We have therefore to choose $\mathcal{Z}_s$ to impose

$$Z_s^A/Z_{l,s} = 1. \quad (42)$$

With this condition the effective potential has the form $\int\hat{A}_{\mu,k_1-k_2}Z_{s,i}^{-1/2}Z_{s',i}^{-1/2}\bar{\phi}_{k_1,s,i}\phi_{k_2,s',i}\hat{V}_{\mu,i,s}(k_1,k_2)$ and $V_{\mu,i,s,s'} = Z_{s,i}^{1/2}Z_{s',i}^{1/2}\hat{\Gamma}_{\mu,i,s,s'}(k_1,k_2)$ with $V_{\mu,l,s,s}(0,0) = Q$ and the magnetic moment obtained by $Z_{s,i}^{1/2}Z_{s',i}^{1/2}G_{\mu,i,s,s'}$.

V. CONCLUSIONS

The series for the magnetic moment is expected to be nonconvergent and even not asymptotic, and this makes it unclear how to evaluate the error introduced by truncation. We consider a nonperturbative framework expressing the magnetic moment in terms of Euclidean functional integrals with a finite ultraviolet cutoff, considering a Fermi description for weak forces, that is integrating out the gauge bosons at tree level. The fact that the magnetic moment is associated with an irrelevant quantity in the RG sense requires careful estimates and the implementation of previously unknown cancellations. We get that the magnetic moment is expressed by series which are analytic for $\lambda(\frac{\Lambda^2}{M^2})^{(1+0^+)}$ small, with relative error due to truncation at order n $O(\lambda^{2(n-1)}(\frac{\Lambda^2}{M^2})^{(n-1)(1+0^+)})$. In addition, the lowest order coincides with its $\Lambda\to\infty$ limit up to an error term $O(\frac{M^2}{\Lambda^2})$. This excludes nonperturbative phenomena in the regime of parameters where such two errors are small, that is, $\lambda^2 \ll \frac{M^2}{\Lambda^2} \ll 1$, and it justifies the validity of truncation of the series expansion with no cutoff in this regime.

An important open question is if a similar approach based on rigorous RG and Euclidean functional integrals with cutoff can be repeated keeping the gauge interaction (and loosing analyticity), and if an estimate for the relative error due to the truncation is obtained with a weaker logarithmic dependence, that is, $O(\lambda^{2(n-1)}(\log\frac{\Lambda^2}{M^2})^{(n-1)})$, up to a constant depending on the order with some factorial. This could allow one to include larger and more realistic values of the coupling and also to include massless photons. Technical difficulties to be solved include, however, the need of an extra decomposition in the gauge boson fields and the fact that one cannot expand in the coupling, as well as the understanding of the interplay of the anomaly cancellations in a nonperturbative setting.

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