NONNEGATIVE MULTIPLICATIVE CONTROLLABILITY FOR SEMILINEAR MULTIDIMENSIONAL REACTION-DIFFUSION EQUATIONS

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ABSTRACT. In this paper we consider a multidimensional semilinear reaction-diffusion equation and we obtain at any arbitrary time an approximate controllability result between nonnegative states using as control term the reaction coefficient, that is via multiplicative controls.

Keywords: Multidimensional semilinear reaction-diffusion equations; approximate controllability; multiplicative controls; nonnegative states.
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1. Introduction

Let $n \in \mathbb{N}$, Ω a bounded open subset of $\mathbb{R}^n$, $T > 0$, $Q_T := \Omega \times (0,T)$, and let denote by (x,t) the generic element of the Cartesian product Q_T. Let consider the following semilinear parabolic Cauchy-Dirichlet boundary value problem

$$\begin{cases} u_t = \Delta u + v(x,t)u + f(u) & \text{in } Q_T = \Omega \times (0,T), \\ u\mid_{\partial\Omega} = 0, & t \in (0,T), \\ u\mid_{t=0} = u_0 \in L^2(\Omega), \end{cases} \tag{1.1}$$

where $v \in L^\infty(Q_T)$, and the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is supposed to be a Lipschitz function, that is, there exists a positive constant L such that

$$\mid f(u_1) - f(u_2) \mid \leq L \mid u_1 - u_2 \mid, \quad \forall u_1, u_2 \in \mathbb{R}, \tag{1.2}$$

moreover, we assume that

$$f(0) = 0. \tag{1.3}$$

In this paper we study the global approximate controllability properties of the semilinear problem (1.1), where the control function, that is the variable coefficient through which we can act on the process, is the reaction coefficient $v(x,t)$, that in literature is called *multiplicative control* (see, e.g., [13], [6], [9], and [11]).

Let us recall briefly the classical *well-posedness* of the system (1.1). So, we need to consider the standard Sobolev spaces:

$$H^1(\Omega) = \{\phi \in L^2(\Omega) \mid \ \phi_x \in L^2(\Omega)\}$$
$$H^1_0(\Omega) = \{\phi \in H^1(\Omega) \mid \phi|_{\partial\Omega} = 0\}$$
$$H^2(\Omega) = \{\phi \in H^1(\Omega) \mid \ \phi_{x_i x_i} \in L^2(\Omega),\, i = 1, \ldots, n\}.$$

By classical well-posedness results (see, for instance, Theorem 6.1 in [14], pp. 466-467) problem (1.1) with initial data $u_0 \in L^2(\Omega)$ admits a unique solution

$$u \in L^2(0,T; H^1_0(\Omega)) \cap C([0,T]; L^2(\Omega)).$$

Furthermore, if $u_0 \in H_0^1(\Omega)$, then the solution u of problem (1.1) satisfies

$$u \in H^1(0,T;L^2(\Omega)) \cap C([0,T];H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)).$$

The above functional spaces are equipped with the standard norms. Moreover, in this paper we will use $\|\cdot\|$, $\|\cdot\|_\infty$ and $\langle\cdot,\cdot\rangle$, instead of the norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^\infty(Q_T)}$, and the inner product $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$, respectively.

Now we can present the main result in Theorem 1.2, where we prove that the system (1.1) is *nonnegatively globally approximately controllable in* $L^2(\Omega)$ *at any time* $T > 0$, *by means of multiplicative controls* v. We will see that the multiplicative controls v have a simple structure, that is v are piecewise static functions, in the sense of the following definition.

Definition 1.1. *We say that a function $v \in L^\infty(Q_T)$ is piecewise static (or a simple function with respect to the variable t), if there exist $m \in \mathbb{N}$, $v_k(x) \in L^\infty(Q_T)$ and $t_k \in [0,T]$, $t_{k-1} < t_k$, $k = 1,\dots,m$ with $t_0 = 0$ and $t_m = T$, such that*

$$v(x,t) = v_1(x)\mathcal{X}_{[t_0,t_1]}(t) + \sum_{k=2}^{m} v_k(x)\mathcal{X}_{(t_{k-1},t_k]}(t), \tag{1.4}$$

where $\mathcal{X}_{[t_0,t_1]}$ and $\mathcal{X}_{(t_{k-1},t_k]}$ are the indicator function of $[t_0,t_1]$ and $(t_{k-1},t_k]$, respectively. Sometime, for clarity purposes, we will call the function v in (1.4) *a m-steps piecewise static function*

Finally, we can give the main result.

Theorem 1.2. *For any nonnegative $u_0, u^* \in L^2(\Omega)$ with $u_0 \neq 0_{L^2(\Omega)}$, for every $\varepsilon > 0$, and any $T > 0$ there exists a piecewise static multiplicative control $v = v(\varepsilon, T, u_0, u^*)$, $v \in L^\infty(Q_T)$, such that for the corresponding solution $u(x,t)$ of* (1.1) *we obtain*

$$\|u(\cdot,T) - u^*\|_{L^2(\Omega)} < \varepsilon.$$

Theorem 1.2 is proved in Section 3.

It is useful the following remark.

Remark 1.3. We note that as a consequence of the assumptions (1.2) and (1.3) on the nonlinear function f the following inequality holds

$$|f(u)| \leq L|u|, \quad \forall u \in \mathbb{R}, \tag{1.5}$$

where L is the Lipschitz constant in (1.2). □

We observe that the nonnegative control result, given in Theorem 1.2, is consistent with the constraints given by the PDE in (1.1). Indeed, it also holds

$$\frac{f(u)}{u} \in L^\infty(Q_T),$$

that follows from (1.5), thus we can extend the strong maximum principle from linear parabolic PDEs (see, e.g, Chapter 2 in [12], p. 34) to semilinear parabolic problem (1.1), since the terms $v(x,t)u + f(u(x,t))$ can be written as $\widetilde{v}(x,t)u(x,t)$, where $\widetilde{v} := v + \dfrac{f(u)}{u} \in L^\infty(Q_T)$.
So, the strong maximum principle implies that the system (1.1) cannot be steered anywhere from $u_0 \equiv 0$, and if $u_0(x) \geq 0$ in Ω, then the corresponding solution to (1.1) remains nonnegative at any moment of time, regardless of the possible

choice of the multiplicative control v. This means the constraint that the system (1.1) cannot be steered from any such nonnegative $u_0 \in L^2(\Omega)$ to any target state $u^* \in L^2(\Omega)$ which is negative on a nonzero measure set in the space domain.

To prove Theorem 1.2 we need an intermediate and crucial controllability result given in Theorem 1.4, obtained under further regularity assumptions and constraints on the initial and target states.

Theorem 1.4. *Let u_0, $u^* \in C^2(\Omega)$ such that $u_0(x) \neq 0$ for every $x \in \Omega$, and*

$$\exists \nu > 0 : \ \nu \leq \frac{u^*(x)}{u_0(x)} \leq 1 \quad \forall x \in \Omega . \tag{1.6}$$

Then, for every $\varepsilon > 0$ and any $T > 0$ there exists a static multiplicative control $v = v(\varepsilon, T, u_0, u^*) \in L^\infty(Q_T)$, $v = v(x)$, such that

$$\|u(\cdot, T) - u^*(\cdot)\|_{L^2(\Omega)} \leq \varepsilon , \tag{1.7}$$

where u is the corresponding solution to (1.1) on Q_T. Moreover, the static multiplicative control v is the following function

$$v(x,t) = \frac{v_0^*(x)}{T} \quad \forall (x,t) \in Q_T,$$

with $v_0^*(x) := \ln\left(\frac{u^*(x)}{u_0(x)}\right)$, for every $x \in \Omega$.

Remark 1.5. Theorem 1.4 includes the case u_0 and u^* both strictly negative on Ω, under the condition (1.6). So, for this result it is just important that both initial and target state have the same sign. □

Structure of the paper. We prove the main result, that is Theorem 1.2, in Section 3. The proof of Theorem 1.2 need the intermediate and crucial result given in Theorem 1.4, that is proved in Section 2 together with some useful PDE estimates.

State of the art. The nonnegative approximate controllability results for reaction-diffusion equations are consistent with the strong maximum principle constraints. In literature, for this kind of results we have to refer to the pioneering papers by A.Y. Khapalov, contained in the book [13], where it is obtained for reaction diffusion equations the nonnegative approximate controllability via multiplicative controls, before in large time, and after in small time under very strong assumptions on the initial and target states. In this paper, we are able to remove those strong constraints on the data in our general Theorem 1.2. Moreover, the proof of that theorem permits us to obtain the nonnegative controllability first in *arbitrary small time*, then at any time by an iteration argument. This proof is inspired by the recent paper [10] of the author, where a similar result is proved in the unidimensional setting for degenerate reaction-diffusion equations. About nonnegative controllability results for degenerate parabolic equations we also mention [9]. Moreover, in [15] Vancostenoble proved a nonnegative controllability result in large time for a linear parabolic equation with singular potential, following the approach of [4] and [5]. For completeness we need recall some recent results about the approximate multiplicative controllability for unidimensional reaction-diffusion equations between sign-changing states, see [6], by the author with Cannarsa and Khapalov regarding a semilinear uniformly parabolic system, and [11], by the author with

Nitsch and Trombetti, about degenerate parabolic equations.
Recently, it is also studied the exact controllability for evolution equations via bilinear controls. See, e.g., [1] and [2] by Alabau-Boussouira, Cannarsa and Urbani, [7] by Cannarsa and Urbani, and [8] by Duprez and Lissy.
Finally, an interesting work in progress, related to this paper is the problem of the approximate controllability via multiplicative control for nonlocal operators, applied to the fractional heat equation studied in [3] by Biccari, Warma and Zuazua.

2. Some PDE estimates and an intermediate goal

We prove the crucial intermediate goal given in Theorem 1.4 in Section 2.2. For the proof of Theorem 1.4 and for that of the main result, that is Theorem 1.2, we need before some general PDE estimates for the solution of the problem (1.1), that we give before in Section 2.1.

2.1. **Some PDE estimates.** Let us start this section by the statement of Proposition 2.1 that we prove immediately below.

Proposition 2.1. *Let $T \in (0, \frac{1}{4L}]$. Let $u_0 \in H_0^1(\Omega)$, $v \in C^2(Q_T)$ with $v(x,t) \leq 0$ on Q_T, and let u the corresponding unique solution to* (1.1). *Then, we have*

$$f(u) \in C([0,T]; L^2(\Omega))$$

and the following estimates hold:

$$\begin{aligned}
\star \qquad \|u\|_{C([0,T];L^2(\Omega))} &\leq \sqrt{2}\|u_0\|_{L^2(\Omega)},\\
\star \quad \|f(u)\|_{C([0,T];L^2(\Omega))} &\leq \sqrt{2}L\|u_0\|_{L^2(\Omega)},\\
\star \qquad\qquad \|\Delta u\|_{L^2(Q_T)} &\leq C(L,T,v)\|u_0\|_{H_0^1(\Omega)},
\end{aligned}$$

where L is the Lipschitz constant in (1.2) and

$$C(L,T,v) := \sqrt{1 + 2T \max_{x\in\overline{\Omega}} |\Delta v| + 2L^2 T}\,.$$

Proof. We start this proof by evaluating $\|u\|_{C([0,T];L^2(\Omega))}$. Since $v(x,t) \leq 0$ for a.e. $(x,t) \in Q_T$, multiplying by u the equation in (1.1), integrating by parts and using (1.5) yields

$$\begin{aligned}
\frac{1}{2}\int_0^t \int_\Omega (u^2)_t dxds &= \int_0^t \int_\Omega u\, u_t dxds\\
&= \int_0^t \int_\Omega u\,\Delta u\, dxds + \int_0^t \int_\Omega v\, u^2 dxds + \int_0^t \int_\Omega f(u)u\, dxds\\
&\leq -\int_0^t \int_\Omega |\nabla u|^2\, dxds + L\int_0^T \int_\Omega u^2 dxdt \leq L\int_0^T \int_\Omega u^2 dxdt,
\end{aligned}$$

where L is the Lipschitz constant in (1.2). Then, since $T \in \left(0, \frac{1}{4L}\right)$ we deduce

$$\begin{aligned}
\int_\Omega u^2(x,t)dx &\leq \int_\Omega u_0^2(x)dx + 2L\int_0^T \int_\Omega u^2 dxdt\\
&\leq \int_\Omega u_0^2(x)dx + 2LT\|u\|^2_{C([0,T];L^2(\Omega))}\\
&\leq \|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{C([0,T];L^2(\Omega))}, \quad t \in (0,T),
\end{aligned}$$

thus,

$$\|u\|_{C([0,T];L^2(\Omega))} \;\le\; \sqrt{2}\|u_0\|_{L^2(\Omega)}, \tag{2.1}$$

that proves the first estimate in the statement of Proposition 2.1.

Now, we evaluate $\|f(u)\|_{C([0,T];L^2(\Omega))}$. From (1.5) and (2.1) it easy follows that

$$f(u) \in C([0,T];L^2(\Omega)),$$

and the following estimate holds

$$\|f(u)\|_{C([0,T];L^2(\Omega))} \;\le\; L\|u\|_{C([0,T];L^2(\Omega))} \le \sqrt{2}L\|u_0\|_{L^2(\Omega)}, \tag{2.2}$$

that proves the second estimate in Proposition 2.1.

Finally, we evaluate $\|\Delta u\|_{L^2(Q_T)}$. Multiplying by Δu the equation in (1.1), integrating over Q_T, and applying Young's inequality we obtain

$$\begin{aligned}
\| \Delta u \|^2_{L^2(Q_T)} &= \int_0^T\int_\Omega u_t \Delta u dx dt - \int_0^T\int_\Omega vu\Delta u dx dt - \int_0^T\int_\Omega f(u)\Delta u\, dx dt \\
&\le \int_0^T\int_\Omega u_t \Delta u dx dt - \int_0^T\int_\Omega vu\Delta u dx dt \\
&\qquad + \frac{1}{2}\int_0^T\int_\Omega f^2(u) dx dt + \frac{1}{2}\int_0^T\int_\Omega |\Delta u|^2 dx dt\,.
\end{aligned}$$

Thus, integrating by parts, keeping in mind the boundary condition in (1.1), taking into account that the reaction term v is such that $v(x,t) \le 0$ for every $(x,t) \in Q_T$, and using the estimates (2.2) and (2.1) it follows that

$$\begin{aligned}
\|\Delta u\|^2_{L^2(Q_T)} &\le 2\int_0^T\int_\Omega u_t \Delta u dx dt \;-\; 2\int_0^T\int_\Omega v\, u\, \Delta u dx dt \;+\; \int_0^T\int_\Omega f^2(u) dx dt \\
&\le \;-\int_0^T\int_\Omega (|\nabla u|^2)_t dx dt \;+\; 2\int_0^T\int_\Omega v|\nabla u|^2 dx dt \\
&\qquad + \int_0^T\int_\Omega \nabla v\cdot\nabla(u^2) dx dt \;+\; T\,\|f(u)\|^2_{C([0,T];L^2(\Omega))} \\
&\le \int_\Omega |\nabla u_0|^2 dx \;-\; \int_0^T\int_\Omega \Delta v\, u^2 dx dt \;+ 2L^2T\|u_0\|^2_{L^2(\Omega)} \\
&\le \int_\Omega |\nabla u_0|^2 dx + \max_{x\in\overline{\Omega}}|\Delta v| \int_0^T\int_\Omega u^2 dx dt + 2L^2T\|u_0\|^2_{L^2(\Omega)} \\
&\le \|\nabla u_0\|^2_{L^2(\Omega)} + \max_{x\in\overline{\Omega}}|\Delta v| \int_0^T \|u\|^2_{C([0,T];L^2(\Omega))} dt + 2L^2T\|u_0\|^2_{L^2(\Omega)} \\
&\le \left(1 + 2T\max_{x\in\overline{\Omega}}|\Delta v| + 2L^2T\right)\|u_0\|^2_{H^1_0(\Omega)}.
\end{aligned}$$

□

In the following Proposition 2.2, we generalize the first estimate in Proposition 2.1 to the case of a general reaction coefficient $v \in L^\infty(Q_T)$.

Proposition 2.2. *Let $T > 0$. Let $u_0 \in H^1_0(\Omega)$, $v \in L^\infty(Q_T)$, and let u the corresponding unique solution to* (1.1)*. Then, we have*

$$\|u\|_{C([0,T];L^2(\Omega))} \;\le\; e^{(L+\|v^+\|_\infty)T}\|u_0\|_{L^2(\Omega)},$$

where L is the Lipschitz constant in (1.2) and $v^+(x,t) = \max\{v(x,t), 0\}$ is the positive part of v.

Proof. Proceeding as in the proof of Proposition 2.1, that is multiplying by u the equation in (1.1), integrating by parts and using (1.5) we obtain

$$\begin{aligned}\frac{1}{2}\int_0^t\int_\Omega (u^2)_t dxds = &\int_0^t\int_\Omega u\,\Delta u\,dxds + \int_0^t\int_\Omega v\,u^2 dxds + \int_0^t\int_\Omega f(u)u\,dxds\\ &\leq -\int_0^t\int_\Omega |\nabla u|^2\,dxds + \int_0^t\int_\Omega v^+\,u^2 dxds + L\int_0^t\int_\Omega u^2 dxdt\\ &\leq (L+\|v^+\|_\infty)\int_0^t\int_\Omega u^2 dxds, \quad \forall t\in(0,T),\end{aligned}$$

then

$$\int_\Omega u^2(x,t)dx \leq \int_\Omega u_0^2(x)dx + (L+\|v^+\|_\infty)\int_0^t\int_\Omega u^2 dxds, \quad \forall\, t\in(0,T).$$

Thus, applying Grönwall's inequality we deduce

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq e^{2(L+\|v^+\|_\infty)T}\|u_0\|^2_{L^2(\Omega)},\ t\in(0,T),$$

from which it follows the conclusion of the proof. $\square$

From Proposition 2.2 we can easy obtain the following Corollary 2.3.

Corollary 2.3. *Let $T>0$. Let u_1^0, $u_2^0 \in H_0^1(\Omega)$, $v\in L^\infty(Q_T)$, and let u_1 and u_2 the unique solution of* (1.1) *corresponding to u_1^0 and u_2^0, respectively. Then, we have*

$$\|u_1-u_2\|_{C([0,T];L^2(\Omega))} \ \leq\ e^{(L+\|v^+\|_\infty)T}\|u_1^0-u_2^0\|_{L^2(\Omega)},$$

where L is the Lipschitz constant in (1.2) and $v^+(x,t) = \max\{v(x,t),0\}$ is the positive part of v.

Proof. Let set $w := u_1 - u_2$, we note that w satisfy the following Cauchy-Dirichlet

$$\left\{\begin{array}{ll} w_t \ =\ \Delta w +\ v(x,t)w\ +\ f(u_1)-f(u_2) & \text{in}\quad Q_T=\Omega\times(0,T)\,,\\ w\,|_{\partial\Omega}=0, & t\in(0,T),\\ w\,|_{t=0} = u_1^0-u_2^0\in H_0^1(\Omega). & \end{array}\right. \tag{2.3}$$

Following the same idea of the proof of Proposition 2.2, that is multiplying by w the equation in (2.3), integrating by parts and using (1.2) we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega w^2(x,t)dx =&\frac{1}{2}\int_\Omega w^2(x,0)dx + \int_0^t\int_\Omega w\,\Delta w\,dxds\\ &+\int_0^t\int_\Omega v\,w^2 dxds + \int_0^t\int_\Omega (f(u_1)-f(u_2))\,w\,dxds\\ \leq&\frac{1}{2}\|u_1^0-u_2^0\|^2_{L^2(\Omega)} + (L+\|v^+\|_\infty)\int_0^t\int_\Omega w^2 dxds, \quad \forall t\in(0,T),\end{aligned}$$

Then, applying Grönwall's inequality we obtain

$$\|u_1(\cdot,t)-u_2(\cdot,t)\|^2_{L^2(\Omega)} \leq e^{2(L+\|v^+\|_\infty)T}\|u_1^0-u_2^0\|^2_{L^2(\Omega)}, \quad \forall t\in(0,T),$$

from which the conclusion. $\square$

2.2. **The proof of Theorem 1.4.** Now, we are ready to give the proof of Theorem 1.4 .

Proof of Theorem 1.4. Let us set $v_0^*(x) := \ln\left(\dfrac{u^*(x)}{u_0(x)}\right)$, for every $x \in \Omega$. From (1.6) we note that $v_0^* \in L^\infty(\Omega)$ and for the static multiplicative control v we have

$$v(x,t) := \frac{v_0^*(x)}{T} \leq 0 \quad \text{for every } (x,t) \in Q_T. \tag{2.4}$$

Then, let us compute the solution u of (1.1) at time T, corresponding to the previous choice of the reaction coefficient v, using the following representation.

$$u(x,T) \;=\; u^*(x) \;+\; \int_0^T e^{v_0^*(x)\frac{(T-t)}{T}}(\Delta u(x,\tau) + f(u(x,\tau)))dt, \quad \forall x \in \Omega. \tag{2.5}$$

The formula (2.5) is obtained in the following way. For every fixed $\bar{x} \in \Omega$, let us consider the non-homogeneous first-order ODE

$$u'(\bar{x},t) = \frac{v_0^*(\bar{x})}{T} u(\bar{x},t) \;+ \big(\Delta u(\bar{x},t) + f(u(\bar{x},t))\big) \quad t \in (0,T),$$

associated to (1.1). Then, we easy obtain that the solution u has the following representation formula

$$u(x,t) \;=\; e^{v_0^*(x)\frac{t}{T}} u_0(x) \;+\; \int_0^t e^{v_0^*(x)\frac{(t-\tau)}{T}}(\Delta u(x,\tau) + f(u(x,\tau)))d\tau, \quad \forall (x,t) \in Q_T,$$

so for $t = T$ we obtain (2.5).

From (2.5), using (2.4) and Hölder's inequality, and applying the estimates in Proposition 2.1, we deduce the following inequalities

$$\begin{aligned}
\|u(x,T) - u^*(x)\|^2_{L^2(\Omega)} &= \int_\Omega \left(\int_0^T e^{v_0^*(x)\frac{(T-\tau)}{T}}(\Delta u(x,\tau) + f(u(x,\tau)))d\tau\right)^2 dx \\
&\leq T\|\Delta u + f(u)\|^2_{L^2(Q_T)} \\
&\leq 2\,T\left(\|\Delta u\|^2_{L^2(Q_T)} + \|f(u)\|^2_{L^2(Q_T)}\right) \\
&\leq 2\,T\left[\left(1 + 2T\max_{x\in\overline{\Omega}}|\Delta v| + 2L^2T\right) + 2L^2T\right]\|u_0\|^2_{H^1_0(\Omega)}. \\
&= 2\,T\left(1 + 2\max_{x\in\overline{\Omega}}|\Delta v_0^*| + 4L^2T\right)\|u_0\|^2_{H^1_0(\Omega)}\,.
\end{aligned} \tag{2.6}$$

We note that in the last equality we have done the replacement $v = \frac{v_0^*}{T}$ in the constant $C(L,T,v)$ of Proposition 2.1 and for the sequel it is crucial that constant (and consequently the last member of (2.6)) it is still bounded, as $T \to 0^+$, also with that choice of the "*singular*" (in T) reaction coefficient v.

Finally, fixed $\varepsilon > 0$, since the last side of (2.6) goes to zero as $T \to 0^+$, there exists $T_0^* \in (0, \frac{1}{4L})$, $T_0^* = T_0^*(\varepsilon, v_0^*)$ such that for every $T \in (0, T_0^*]$ we obtain

$$\|u(\cdot,T) - u^*\|^2_{L^2(\Omega)} < \varepsilon,$$

that is the approximate controllability at any time $T \in (0, T^*]$.
Furthermore, if $T > T_0^*$ we can prove the approximate controllability at time T, using an argumentation introduced by the author in [10] (see the proof of Theorem 1.4 in that paper) for unidimensional degenerate reaction-diffusion equations. So,

we first obtain the approximate controllability at time T_0^*. Then, we restart at time T_0^* from a state close to u^*, and we stabilize the system into the neighborhood of u^*, applying the above strategy overall n times, for some $n \in \mathbb{N}$, on n small time intervals by measure $\frac{T-T_0^*}{n}$, steering the system in any interval from a suitable approximation of u^* to u^*. □

3. The proof of the main result

Let us give the proof of the main result of this paper, Theorem 1.2.

Proof. (Proof of Theorem 1.2). Let fix $\varepsilon > 0$. Since u_0, $u^* \in L^2(\Omega)$ by approximating there exist u_0^ε, $u_\varepsilon^* \in C^2(\overline{\Omega})$ such that:

- ⋆ $u_0^\varepsilon,\ u_\varepsilon^* > 0$ on Ω, and the quotient function $\dfrac{u_\varepsilon^*}{u_0^\varepsilon}$ is bounded on Ω, that is,

$$\exists\, M_\varepsilon := M(\varepsilon, u_0, u^*) > 0 \ :\ 0 < \frac{u_\varepsilon^*(x)}{u_0^\varepsilon(x)} \le M_\varepsilon, \qquad \forall x \in \Omega, \tag{3.1}$$

  for the following is useful to choose directly an upper bound $M_\varepsilon > 1$;
- ⋆ u_ε^* and u_0^ε satisfy the following approximation conditions

$$\|u_\varepsilon^* - u^*\| < \frac{\varepsilon}{4} \quad \text{and} \quad \|u_0^\varepsilon - u_0\| < \frac{\varepsilon}{16 e^L M_\varepsilon}\,, \tag{3.2}$$

  where L is the Lipschitz constant in (1.2).

For every $x \in \Omega$, let us define $\operatorname{dist}(x, \partial\Omega) := \inf\limits_{y \in \partial\Omega} |x - y|\,(> 0)$. Let $\eta > 0$ such that the following set

$$\Omega_\eta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \eta\}$$

is a not empty open set, then we have $\overline{\Omega}_\eta \subset \Omega$.
From (3.1) we deduce that

$$0 < \min_{x \in \overline{\Omega}_\eta} \left\{ \frac{u_\varepsilon^*(x)}{u_0^\varepsilon(x)} \right\} \le \frac{u_\varepsilon^*(x)}{u_0^\varepsilon(x)} \le M_\varepsilon, \qquad \forall x \in \overline{\Omega}_\eta,$$

then, there exists $\nu(\eta) > 0$ $\left(\nu(\eta) := \dfrac{1}{M_\varepsilon} \min\limits_{x \in \overline{\Omega}_\eta} \left\{ \dfrac{u_\varepsilon^*(x)}{u_0^\varepsilon(x)} \right\} \right)$ such that

$$\nu(\eta) \le \frac{u_\varepsilon^*(x)}{M_\varepsilon\, u_0^\varepsilon(x)} \le 1, \qquad \forall x \in \overline{\Omega}_\eta\,. \tag{3.3}$$

Let note that (3.3) has the same structure of the assumption (1.6) in Theorem 1.4, so it is natural to proceed by the following strategy consisting in two control actions: in the first step we drive the system from the initial state u_0 to the intermediate target state $M_\varepsilon u_0^\varepsilon$, then in the second step we steer the system from this intermediate state to u^*, using the crucial Theorem 1.4.

For that we consider a further approximation of u^*, indeed there exist $\eta = \eta(\varepsilon) > 0$ and $u_\eta^*, v_0^\eta \in C^2(\overline{\Omega})$ with $v_0^\eta \le 0$ such that:

$$(i)\ \ u_\eta^*(x) \ = \ \begin{cases} u_\varepsilon^*(x), & x \in \Omega_\eta, \\ 0, & x \in \overline{(\Omega \backslash \Omega_{\frac{\eta}{2}})}, \end{cases} \qquad \text{and}$$

$$\|u_\eta^* - u_\varepsilon^*\|_{L^2(\Omega)} < \frac{\varepsilon}{4}, \tag{3.4}$$

$$(ii)\ \ v_0^\eta(x) = \begin{cases} \ln\left(\frac{u_\varepsilon^*(x)}{M_\varepsilon\, u_0^\varepsilon(x)}\right), & x \in \Omega_\eta, \\ 0, & x \in \overline{(\Omega \backslash \Omega_{\frac{\eta}{2}})}\,, \end{cases}$$

where $\Omega_{\frac{\eta}{2}} := \left\{ x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \dfrac{\eta}{2} \right\}$.

Now, keeping in mind that $M_\varepsilon > 1$, we can select the positive constant reaction coefficient

$$v_1 := \frac{\log M_\varepsilon}{T_1} > 0, \qquad \forall\,(x,t) \in \overline{\Omega} \times (0, T_1), \quad \text{for some } T_1 > 0. \tag{3.5}$$

Then, let us choose the 2-steps piecewise static (see Definition 1.1) multiplicative control

$$v(x,t) \ = \begin{cases} v_1\,, & (x,t) \in \overline{\Omega} \times (0, T_1), \\ \\ \dfrac{v_0^\eta(x)}{T - T_1}\,, & (x,t) \in \overline{\Omega} \times (T_1, T)\,, \end{cases} \tag{3.6}$$

where T_1 and T will be determined below.
Let u be the solution to (1.1) corresponding to the above choice of the multiplicative control v and to the initial state u_0.

Steering the system from u_0^ε to $M_\varepsilon u_0^\varepsilon$, at some time $T_1 > 0$. Let us denote by $u^\varepsilon(x,t)$ the solution of (1.1) with initial state u_0^ε. Thus, the solution $u^\varepsilon(x,t)$, at some time T_1, is represented in Fourier series in the following way

$$u^\varepsilon(x, T_1) = \ e^{v_1 T_1} \sum_{k=1}^{\infty} e^{-\lambda_k T_1} \langle u_0^\varepsilon, \varphi_k \rangle \varphi_k(x) + F_\varepsilon(x, T_1) \tag{3.7}$$

with $\ F_\varepsilon(x,T_1) := \displaystyle\sum_{k=1}^{\infty} \left[\int_0^{T_1} e^{(v_1 - \lambda_k)(T_1 - t)} \langle f(u^\varepsilon(\cdot, t)), \varphi_k \rangle dt \right] \varphi_k(x),$
where $\{-\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvalues of the Laplacian operator $A_0 u := \Delta u$ (we note that $\lambda_k \geq 0$ and $\lambda_k \leq \lambda_{k+1}$, for every $k \in \mathbb{N}$, and $\lambda_k \to +\infty$, as $k \to +\infty$), and $\{\varphi_k\}_{k\in\mathbb{N}}$ are the corresponding eigenfunctions that form a complete orthonormal system in $L^2(\Omega)$. We trivially remark that the eigenvalues of the operator $Au := \Delta u + v_1 u$ are obtained by a shift, that is we have $\{-\lambda_k + v_1\}_{k\in\mathbb{N}}$, and the corresponding eigenfunctions are the same $\{\varphi_k\}_{p\in\mathbb{N}}$.
By the strong continuity semigroup property of the heat equation, we deduce

$$\sum_{k=1}^{\infty} e^{-\lambda_k T_1} \langle u_0^\varepsilon, \varphi_k \rangle \varphi_k(x) \longrightarrow u_0^\varepsilon \quad \text{ in } L^2(-1,1) \quad \text{as} \quad T_1 \to 0.$$

So, there exists a small time $T_1' \in (0,1)$, $T_1' = T_1'(\varepsilon, u_0, u^*)$, such that, keeping also in mind that $e^{v_1 T_1} = M_\varepsilon$ by (3.5), we deduce

$$\|u^\varepsilon(\cdot, T_1) - M_\varepsilon u_0^\varepsilon(\cdot)\| < \frac{\varepsilon}{32} + \|F_\varepsilon(\cdot, T_1)\|, \qquad \forall\, T_1 \in (0, T_1']. \tag{3.8}$$

Using Hölder's inequality, Parseval's identity, the inequality (1.5), and Proposition 2.2 we deduce

$$
\begin{aligned}
(3.9)\qquad \|F_\varepsilon(x,T_1)\|^2 &= \sum_{k=1}^{\infty}\left|\int_0^{T_1} e^{(v_1-\lambda_k)(T_1-t)}\langle f(u^\varepsilon(\cdot,t)),\varphi_k\rangle dt\right|^2 \\
&\le \sum_{k=1}^{\infty}\left(\int_0^{T_1} e^{2(v_1-\lambda_k)(T_1-t)}dt\right)\cdot\left(\int_0^{T_1}|\langle f(u^\varepsilon(\cdot,t)),\varphi_k\rangle|^2 dt\right) \\
&\le e^{2v_1T_1}T_1\int_0^{T_1}\sum_{k=1}^{\infty}|\langle f(u^\varepsilon(\cdot,t)),\varphi_k\rangle|^2 dt = M_\varepsilon^2 T_1\int_0^{T_1}\|f(u^\varepsilon(\cdot,t))\|^2 dt \\
&\le M_\varepsilon^2 T_1\int_0^{T_1}\|u^\varepsilon\|^2 dt \le c(T_1)LM_\varepsilon^2 T_1\|u_0^\varepsilon\|^2_{L^2(\Omega)}\,,
\end{aligned}
$$

From (3.8) using (3.9) it follows that there exists $T_1^*\in(0,T_1']$, $T_1^*=T_1^*(\varepsilon,u_0,u^*)$, such that

$$
(3.10)\qquad \|u^\varepsilon(\cdot,T_1)-M_\varepsilon u_0^\varepsilon(\cdot)\| < \frac{\varepsilon}{32}+\|F_\varepsilon(\cdot,T_1)\|\le\frac{\varepsilon}{16},\qquad \forall\, T_1\in(0,T_1^*].
$$

Using Corollary 2.3 and the inequaity (3.10), keeping in mind (3.5) and (3.2), for every $T_1\in(0,T_1^*]$, we obtain

$$
\begin{aligned}
(3.11)\quad \|u(\cdot,T_1)-M_\varepsilon u_0^\varepsilon(\cdot)\| &\le \|u(\cdot,T_1)-u^\varepsilon(\cdot,T_1)\|+\|u^\varepsilon(\cdot,T_1)-M_\varepsilon u_0^\varepsilon(\cdot)\| \\
&< e^{(L+\|v_1^+\|_\infty)T_1}\|u_0-u_0^\varepsilon\|+\frac{\varepsilon}{16}\le e^{(L+v_1T_1)}\frac{\varepsilon}{16e^L M_\varepsilon}+\frac{\varepsilon}{16}=\frac{\varepsilon}{8}.
\end{aligned}
$$

Let set $\delta_0^\varepsilon(x):=u(x,T_1)-M_\varepsilon u_0^\varepsilon(x)$, by (3.11) we have

$$
(3.12)\qquad \|\delta_0^\varepsilon\| < \frac{\varepsilon}{4\sqrt{2}}\,.
$$

Steering the system from $M_\varepsilon u_0^\varepsilon+\delta_0^\varepsilon$ (at time $T_1>0$) to u^* at time T, for $T>T_1$. In this step let us restart at time T_1 from the intermediate state $M_\varepsilon u_0^\varepsilon+\delta_0^\varepsilon$ and our goal is to steer the system arbitrarily close to u^*.
Let us consider the following semilinear Dirichlet problem

$$
(3.13)\qquad \begin{cases} u_t-\Delta u=\dfrac{v_0^\eta(x)}{T-T_1}u+f(u) & \text{in } \widetilde{Q}_T:=\Omega\times(T_1,T) \\ u\mid_{\partial\Omega}=0, & t\in(T_1,T)\,, \end{cases}
$$

and we denote by $\tilde{u}(x,t)$ the unique solution to (3.13) with the initial condition $\tilde{u}(x,T_1)=M_\varepsilon u_0^\varepsilon(x)$. Of course, the restriction on $\widetilde{Q}_T$ of the solution u of (1.1), corresponding to the multiplicative control v, given in (3.6), and to the initial state u_0, solves (3.13) with the initial state $u(x,T_1)=M_\varepsilon u_0^\varepsilon(x)+\delta_0^\varepsilon(x)$. Since the inequality (3.1) holds we can apply Theorem 1.4 to steer the system (3.13) from $M_\varepsilon u_0^\varepsilon$ to the approximation u_η^*, thus, for $T>T_1$ we have

$$
(3.14)\qquad \|\widetilde{u}(\cdot,T)-u_\eta^*(\cdot)\| < \frac{\varepsilon}{4}.
$$

Then, using Corollary 2.3 (see also Proposition 2.1), from (3.14), (3.4), (3.2), and (3.12), for any T such that $T - T_1 > 0$ is sufficiently small, there exists

$$\begin{aligned}\|u(\cdot,T) - u^*(\cdot)\| &\le \|u(\cdot,T) - u^*_\eta(\cdot)\| + \|u^*_\eta - u^*\| \\ &\le \|u(\cdot,T) - \widetilde{u}(\cdot,T)\| + \|\widetilde{u}(\cdot,T) - u^*_\eta(\cdot)\| + \|u^*_\eta - u^*_\varepsilon\| + \|u^*_\varepsilon - u^*\| \\ &< \sqrt{2}\|M_\varepsilon u_0^\varepsilon + \delta_0^\varepsilon - M_\varepsilon u_0^\varepsilon\| + \frac{3}{4}\varepsilon < \varepsilon\,.\end{aligned}$$

From which it follows the conclusion and the approximate controllability at any time $T > 0$, using the same approach of the end of the proof of Theorem 1.4. □

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