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On sharp estimates for Schrödinger groups of fractional powers of nonnegative self-adjoint operators

The Anh Bui [a,*], Piero D'Ancona [b], Xuan Thinh Duong [a]

[a] School of Mathematical and Physical Sciences, Macquarie University, NSW 2109, Australia
[b] Dipartimento di Matematica, Sapienza Università di Roma, Piazzale A. Moro 2, 00185 Roma, Italy

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Abstract

Let L be a non negative, selfadjoint operator on $L^2(X)$, where X is a metric space endowed with a doubling measure. Consider the Schrödinger group for fractional powers of L. If the heat flow e^{-tL} satisfies suitable conditions of Davies–Gaffney type, we obtain the following estimate in Hardy spaces associated to L:

$$\big\|(I+L)^{-\beta/2}e^{i\tau L^{\gamma/2}}f\big\|_{H^p_L(X)} \le C(1+|\tau|)^{n\mathfrak{s}_p}\|f\|_{H^p_L(X)}$$

where $p\in(0,1]$, $\gamma\in(0,1]$, $\beta/\gamma = n|\frac{1}{2}-\frac{1}{p}| = n\mathfrak{s}_p$ and $\tau\in\mathbb{R}$.

If in addition e^{-tL} satisfies a localized $L^{p_0}\to L^2$ polynomial estimate for some $p_0\in[1,2)$, we obtain

$$\big\|(I+L)^{-\beta/2}e^{i\tau L^{\gamma/2}}f\big\|_{p_0,\infty} \le C(1+|\tau|)^{n\mathfrak{s}_{p_0}}\|f\|_{p_0}, \quad \forall\tau\in\mathbb{R},$$

provided $0<\gamma\neq 1$, $\beta/\gamma = n|\frac{1}{2}-\frac{1}{p}| = n\mathfrak{s}_p$ and $\tau\in\mathbb{R}$. By interpolation, the second estimate implies also, for all $p\in(p_0,p_0')$, the strong (p,p) type estimate

$$\big\|(I+L)^{-\beta/2}e^{i\tau L^{\gamma/2}}f\big\|_{p} \le C(1+|\tau|)^{n\mathfrak{s}_p}\|f\|_{p}.$$

* Corresponding author.
E-mail addresses: the.bui@mq.edu.au (T.A. Bui), dancona@mat.uniroma1.it (P. D'Ancona), xuan.duong@mq.edu.au (X.T. Duong).

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The applications of our theory span a diverse spectrum, ranging from the Schrödinger operator with an inverse square potential to the Dirichlet Laplacian on open domains. It showcases the effectiveness of our theory across various settings.

Keywords: Sharp L^p estimate; Heat kernel; Fractional Schrödinger equation

1. Introduction

The Schrödinger flow $e^{i\tau(-\Delta)^{\gamma/2}}$ with $\gamma > 0$ is a group of isometries on $L^2(\mathbb{R}^n)$ but is unbounded on every other L^p space with $p \neq 2$. It is well known, however, that boundedness in L^p can be recovered at the price of a loss of derivatives. More precisely, for $0 < \gamma \neq 1$, $1 < p < \infty$ and $\tau > 0$ one has

$$\|(I-\Delta)^{-\beta/2} e^{i\tau(-\Delta)^{\gamma/2}} f\|_p \lesssim (1+|\tau|)^{n\mathfrak{s}_p} \|f\|_p, \qquad \mathfrak{s}_p = \left|\frac{1}{2} - \frac{1}{p}\right|, \quad \beta = \gamma n \mathfrak{s}_p \tag{1}$$

while for $0 < p \leq 1$, $\tau \in \mathbb{R}$ one has

$$\|(I-\Delta)^{-\beta/2} e^{i\tau(-\Delta)^{\gamma/2}} f\|_{H^p(\mathbb{R}^n)} \lesssim (1+|\tau|)^{n\mathfrak{s}_p} \|f\|_{H^p(\mathbb{R}^n)}, \qquad \mathfrak{s}_p = \left|\frac{1}{2} - \frac{1}{p}\right|, \quad \beta = \gamma n \mathfrak{s}_p, \tag{2}$$

where $H^p(\mathbb{R}^n)$ denotes the classical Hardy spaces (see [37]). A similar situation holds for the half–wave flow $e^{it(-\Delta)^{1/2}}$ corresponding to $\gamma = 1$, but with a loss $(n-1)\mathfrak{s}_p$, see for example [27,28,26,32]. These results can be regarded as instances of general L^p estimates for Fourier integral operators ([34,29,30]) and are strongly connected to the Schrödinger equation with a fractional Laplacian

$$\begin{cases} i\dfrac{\partial u}{\partial \tau} + (-\Delta)^{\gamma/2} u = 0, \\ u(x,0) = f(x). \end{cases} \tag{3}$$

Indeed, estimate (1) implies that any solution u of equation (3) satisfies

$$\|u(x,\tau)\|_p \lesssim (1+|\tau|)^{n\mathfrak{s}_p} \|f\|_{W^{\beta,p}(\mathbb{R}^n)}, \quad 1 < p < \infty, \quad \beta = \gamma n \mathfrak{s}_p.$$

Since the flows $e^{itL^{\gamma/2}}$ are well defined for arbitrary non negative selfadjoint operators via spectral calculus, it is natural to investigate possible extensions of the previous results beyond the case of the Laplacian. The study of Schrödinger flows beyond the Laplacian case is an interesting topic and has attracted a great deal of attention, see [29,30,6,17,16,7,13,14,9].

To introduce our results, we give a brief overview of related research. In the case $\gamma = 2$, Lohoué [24] (see also [1]) proved a similar result to (1) for $\beta > 2n\mathfrak{s}_p$ on Lie groups with polynomial growth and manifolds with nonnegative curvature. In [12], Carron, Coulhon and Ouhabaz prove the following inequality

$$\|(I+L)^{-s-\epsilon} e^{i\tau L} f\|_p \lesssim (1+|\tau|)^{s+\epsilon} \|f\|_p, \quad 1 < p < \infty \tag{4}$$

for $s = n\mathfrak{s}_p$ and $\epsilon > 0$, where L is a nonnegative self-adjoint operator satisfying the Gaussian upper bound on spaces of homogeneous type space. It is easy to see that in comparison with the classical case (1) the above estimate is not sharp.

In [16], given a selfadjoint, non-negative operator L on $L^2(\mathbb{R}^n)$ whose heat kernel satisfies a mild smoothness effect and a mild off-diagonal decay, the following estimate is proved for $k \in \mathbb{Z}$, $\tau > 0$ and a suitable $p \in (1, \infty)$:

$$\left\| e^{i\tau L}\phi\big(2^{-k}L^{1/2}\big)f\right\|_p \lesssim (1+2^{2k}|\tau|)^{n\mathfrak{s}_p}\|f\|_p \qquad \text{where} \qquad \mathfrak{s}_p = \left|\frac{1}{2}-\frac{1}{p}\right|, \tag{5}$$

where $\phi \in C_c^\infty(\mathbb{R}^n)$ is a cut-off function. This result includes the case of Schrödinger operators with Kato class electromagnetic potentials. Note that (5) implies the estimate (4). The results in [16] were extended in [7] to the very general setting of metric measure spaces with a doubling measure (homogeneous spaces). This goes far beyond the Euclidean case and includes Riemannian manifolds, homogeneous groups, sublaplacians on Heisenberg groups, and operators with singular potentials. Hence, it is natural to raise a question on the validity of a sharp estimate for $s = n\mathfrak{s}_p$

$$\|(I+L)^{-s}e^{i\tau L}f\|_p \lesssim (1+|\tau|)^s\|f\|_p, \qquad 1 < p < \infty. \tag{6}$$

Recently, the sharp estimate (6) was proved in [13] (see also [19]) under the assumption of Gaussian upper bounds of order $m \geq 2$ on the heat kernel of L. The estimate was later extended to Hardy type spaces, see for example [14,4]. However, to the best of our knowledge, only partial results are known for the general flows $e^{itL^{\gamma/2}}$. This leads to our purpose to establish sharp estimates for the flows $e^{itL^{\gamma/2}}$ with $0 < \gamma \neq 1$ generated by arbitrary fractional powers of the operator L.

We now introduce the setting of our results. Let (X, d, μ) be a metric space with distance d, endowed with a nonnegative Borel measure μ. Denote by $B(x, r)$ the open ball of radius $r > 0$ and center $x \in X$, and by $V(x, r) = \mu(B(x, r))$ its volume. We say that (X, d, μ) is a space of *homogeneous type* (in the sense of Coifman and Weiss [15]) if it satisfies the doubling property, i.e. there exists a constant $C > 0$ such that

$$V(x, 2r) \leq CV(x, r) \tag{7}$$

for all $x \in X$ and $r > 0$. Notice that the doubling property (7) implies the following properties:

$$V(x, \lambda r) \leq C\lambda^n V(x, r); \tag{8}$$

and

$$V(x, r) \leq C\Big(1+\frac{d(x, y)}{r}\Big)^n V(y, r), \tag{9}$$

for all $x, y \in X$ and $r > 0$. A direct consequence of (9) is that $V(x, r) \approx V(y, r)$ when $d(x, y) \leq r$.

Let L be a non-negative, self-adjoint operator L on $L^2(X)$ which generates an analytic semigroup $\{e^{-tL}\}_{t>0}$. If $E_L(\lambda)$ is the spectral decomposition of L and $F : [0, \infty) \to \mathbb{C}$ is any bounded Borel function, we denote by $F(\sqrt{\lambda})$ the bounded operator on $L^2(X)$ defined as

$$F(\sqrt{L}) = \int_0^\infty F(\lambda) dE_L(\lambda).$$

We shall consider two different assumptions on the semigroup. We say that e^{-tL} satisfies the *Davies–Gaffney estimates* if there exist constants C, $c > 0$ such that for any open subsets $U_1, U_2 \subset X$,

$$|\langle e^{-tL} f_1, f_2\rangle| \le C \exp\Big(-\frac{\mathrm{dist}(U_1, U_2)^2}{ct}\Big)\|f_1\|_{L^2(X)}\|f_2\|_{L^2(X)}, \quad \forall t > 0, \tag{10}$$

for every $f_i \in L^2(X)$ with supp $f_i \subset U_i$, $i = 1, 2$, where $\mathrm{dist}(U_1, U_2) := \inf_{x\in U_1, y\in U_2} d(x, y)$.

We also consider the following condition: there exist $p_0 \in [1, 2)$ and constants C and $c > 0$ such that for all balls $B \subset X$ and $t > 0$,

$$\|1_B e^{-tL}\|_{p_0\to 2} + \|e^{-tL} 1_B\|_{p_0\to 2} \lesssim \sup_{x\in B} \frac{1}{V(x, \sqrt{t})^{1/p_0-1/2}}. \tag{11}$$

Note that assumption (11) is implied by the following generalized Gaussian condition GGE(p_0): there exist constants $C, c > 0$ such that

$$\|1_{B(x,\sqrt{t})} e^{-tL} 1_{B(y,\sqrt{t})}\|_{p_0\to 2} \lesssim \frac{1}{V(x, \sqrt{t})^{1/p_0-1/2}} \exp\Big(-\frac{d(x, y)^2}{ct}\Big) \tag{12}$$

for all $t > 0$ and $x, y \in X$. It is important to note that the generalized Gaussian GGE(1) is equivalent to the Gaussian upper bound, i.e., there exist constants $C, c > 0$ such that

$$|e^{-tL}(x, y)| \lesssim \frac{1}{V(x, \sqrt{t})} \exp\Big(-\frac{d(x, y)^2}{ct}\Big)$$

for all $x, y \in X$ and $t > 0$.

Recalling the notation

$$\mathfrak{s}_p = \Big|\frac{1}{2} - \frac{1}{p}\Big| \qquad \text{for} \qquad p \in (0, \infty],$$

the main results of this paper are the following two theorems.

Theorem 1.1. *Let L be a nonnegative self-adjoint operator on $L^2(X)$. Assume that L satisfies (10) and (11). Then, for $0 < \gamma \neq 1$ and $\beta = \gamma n \mathfrak{s}_p$, we have*

$$\big\|(I + L)^{-\beta/2} e^{i\tau L^{\gamma/2}} f\big\|_{p_0,\infty} \le C(1 + |\tau|)^{n\mathfrak{s}_{p_0}} \|f\|_{p_0}, \quad \forall \tau \in \mathbb{R}. \tag{13}$$

By interpolation, we obtain

$$\big\|(I + L)^{-\beta/2} e^{i\tau L^{\gamma/2}} f\big\|_p \le C(1 + |\tau|)^{n\mathfrak{s}_{p_0}} \|f\|_p \tag{14}$$

for all $\tau \in \mathbb{R}$ and $p \in (p_0, p_0')$.

The Hardy space $H^p_L(X)$ mentioned in the next statement is defined in Section 2:

Theorem 1.2. *Let L be a nonnegative self-adjoint operator on $L^2(X)$. Assume that L satisfies (10). For $0 < \gamma \neq 1$ and $\beta = \gamma n \mathfrak{s}_p$, we have*

$$\big\|(I+L)^{-\beta/2}e^{i\tau L^{\gamma/2}}f\big\|_{H^p_L(X)} \le C(1+|\tau|)^{n\mathfrak{s}_p}\|f\|_{H^p_L(X)} \tag{15}$$

for all $\tau \in \mathbb{R}$ and $p \in (0,1]$, where $H^p_L(X)$ is the Hardy space associated to the operator L.

Moreover, if L satisfies (11) additionally, then the estimate (14) holds true by the interpolation.

Note that Theorem 1.2 gives only a weak type estimate at the endpoint p_0. Hence, Theorem 1.1 is not a consequence of Theorem 1.2. In the special case $L = -\Delta$, the Hardy space $H^p_L(\mathbb{R}^n)$ turns out to be the classical Hardy space $H^p(\mathbb{R}^n)$ (see [11]), and hence our estimate recovers the classical estimate (2).

We emphasize that Theorems 1.1 and 1.2 are non-trivial extensions of the classical results. Indeed, the Fourier transform is not available in the setting of metric spaces. Moreover, while the proof of the classical cases relies heavily on the Calderón-Zygmund theory, it seems that this theory might not be applicable in our setting due to the mild assumption on the main operator L, hence new ideas and techniques are required. As a consequence, our paper not only extends, but also provides new proofs for the classical results. Further comments on Theorems 1.1 and 1.2 will be given after Corollary 1.4 below.

As it is well-known, Theorem 1.1 is closely connected with the Schrödinger equation

$$\begin{cases} i\dfrac{\partial u}{\partial \tau} + L^{\gamma/2}u = 0, \\ u(x,0) = f(x), \end{cases} \tag{16}$$

where $0 < \gamma \neq 1$. Indeed, from Theorem 1.1 we have:

Corollary 1.3. *Let L be a nonnegative self-adjoint operator on $L^2(X)$. Assume that L satisfies (10) and (11). If $u(x,\tau)$ is a solution to (16) with $0 < \gamma \neq 1$, then for $p \in (p_0, p_0')$ and $\beta = \gamma n \mathfrak{s}_p$,*

$$\|u(\cdot,t)\|_p \le C(1+|\tau|)^{n\mathfrak{s}_p}\|(I+L)^{\beta/2}f\|_p, \quad \tau \in \mathbb{R}.$$

Another application of Theorems 1.1 Theorem 1.2 concerns the Riesz means associated to L, defined via the following operator:

$$I_{s,t}(L) = st^{-s}\int_0^t (t-\lambda)^{-s-1}e^{-i\lambda L^{\gamma/2}}d\lambda, \quad s,t>0,$$

while $I_{s,t}(L) = \bar{I}_{s,-t}(L)$ for $t < 0$. See [27,36] for the study of these operators in the case L is the standard Laplacian on $\mathbb{R}^n$ and [24,1] for extensions to more general contexts. From Theorems 1.1 and 1.2, by standard arguments we obtain the following:

Corollary 1.4. *Let L be a nonnegative self-adjoint operator on $L^2(X)$. Assume that L satisfies* (10)*. Then for $p \in (0,1]$, $0 < \gamma \neq 1$ and $s = \gamma n \mathfrak{s}_p$,*

$$\left\| I_{s,t}(L) f \right\|_{H^p_L(X)} \le C \| f \|_{H^p_L(X)} \quad \text{if } t > 0.$$

If in addition L satisfies (11)*, then for $p \in (p_0, p_0')$ we have*

$$\left\| I_{s,t}(L) f \right\|_p \le C \| f \|_p \quad \text{if } t > 0.$$

Some comments regarding Theorems 1.1 and 1.2 are in order:

(i) By a careful examination of our proofs, the results in Theorem 1.1, Theorem 1.2, Corollary 1.3 and Corollary 1.4 hold true also for $\beta \ge \gamma n \mathfrak{s}_p$. Our approach could be modified to study the case $\gamma = 1$, but the resulting estimate is not sharp in comparison with the classical cases (1) and (2). Hence, we do not pursue the case $\gamma = 1$ here.

(ii) As mentioned above, boundedness of the Schrödinger group, corresponding to the case $\gamma = 2$, has been studied extensively. L^p-boundedness of the Schrödinger groups when $\gamma = 2$ under the assumption of Gaussian upper bounds of order $m \ge 2$ was proved in [13]. Boundedness on the Hardy spaces $H^p_L(X)$ was obtained in [14,4]. Boundedness under the generalized Gaussian estimates of order $m \ge 2$ was obtained in [19].

(iii) Much less was known for $\gamma \neq 2$. In the special case of the Hermite operator, boundedness on $L^p(X)$ and on Hardy spaces associated to the operator (with the exception of the weak type estimate) was obtained in [8].

(iv) Theorem 1.2 is new even when $\gamma = 2$. Boundedness on the Hardy space $H^1_L(X)$ was obtained in [14], but the approach there does not work for the case $0 < p < 1$. Although Theorem 1.2 also implies the boundedness of $(I + L)^{-\beta/2} e^{i\tau L^{\gamma/2}}$ on L^p for $p_0 < p < p_0'$, the weak type boundedness (p_0, p_0) in Theorem 1.1 is unique.

(v) We emphasize that the techniques in [13] do not work in our setting. The approach in [13] relies heavily on estimates for operators of the form $e^{-tL} e^{i\tau L}$, leading to Besov norm estimates of the function $e^{-(t-\tau)\cdot}$. This approach fails completely if we replace the flow $e^{i\tau L}$ by the general flow $e^{i\tau L^{\gamma/2}}$. To overcome this problem, we need to establish new operator estimates in Lemma 2.10, which play a crucial role in the proofs of our main results. Our approach can be used to obtain sharp estimates for imaginary powers of L, and this will be the topic of an upcoming paper.

Our theory is highly comprehensive, encompassing a broad range of significant operators in harmonic analysis and partial differential equations (PDEs). Notable examples include Schrödinger operators with inverse-square potentials [5,25], the Kohn-Laplacian on pseudoconvex manifolds of finite type, as studied by Nagel-Stein [31], and the Laplace-Beltrami operators on doubling manifolds [2]. Additional operators of interest can be found in references such as [7,10] and the references therein. To demonstrate the practical applications of our theory, we present two compelling instances in the realm of PDEs.

Schrödinger operators with inverse-square potentials. Consider the following Schrödinger operators with inverse square potential on $\mathbb{R}^n$, $n \ge 3$:

$$\mathcal{L}_a = -\Delta + \frac{a}{|x|^2} \quad \text{with} \quad a \ge -\Big(\frac{n-2}{2}\Big)^2. \tag{17}$$

Set

$$\sigma := \frac{n-2}{2} - \frac{1}{2}\sqrt{(n-2)^2 + 4a}.$$

The Schrödinger operator $\mathcal{L}_a$ is understood as the Friedrichs extension of $-\Delta + \frac{a}{|x|^2}$ defined initially on $C_c^\infty(\mathbb{R}^n\backslash\{0\})$. The condition $a \ge -\Big(\frac{n-2}{2}\Big)^2$ guarantees that $\mathcal{L}_a$ is nonnegative. It is well-known that $\mathcal{L}_a$ is self–adjoint and the extension may not be unique as $-\Big(\frac{n-2}{2}\Big)^2 \le a < 1 - \Big(\frac{n-2}{2}\Big)^2$. For further details, we refer the readers to [20,33,38]. Set $n_\sigma = n/\sigma$ if $\sigma > 0$ and $n_\sigma = \infty$ if $\sigma \le 0$. It was proved in [3, Theorem 3.1], for any $n'_\sigma < p \le q < n_\sigma$ there exist $C, c > 0$ such that for every $t > 0$, any measurable subsets $E, F \subset \mathbb{R}^n$, and all $f \in L^p(E)$, we have:

$$\Big\| e^{-t\mathcal{L}_a} f \Big\|_{L^q(F)} \le C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} e^{-\frac{d(E,F)^2}{ct}} \|f\|_{L^p(E)}. \tag{18}$$

Hence, the operator $\mathcal{L}_a$ satisfies (10) and (11) with $p_0 = n'_\sigma$. Therefore, from Theorem 1.1, for $0 < \gamma \ne 1$ and $\beta = \gamma n \mathfrak{s}_p$, we have

$$\big\| (I + \mathcal{L}_a)^{-\beta/2} e^{i\tau \mathcal{L}_a^{\gamma/2}} f \big\|_p \le C(1 + |\tau|)^{n\mathfrak{s}_{p_0}} \|f\|_p$$

for all $\tau \in \mathbb{R}$ and $p \in (n'_\sigma, n_\sigma)$.

Dirichlet Laplacians on open domains. Let Ω be an open subset of $\mathbb{R}^n$. Note that Ω may not satisfy the doubling condition. Let $-\Delta_D$ be Dirichlet Laplacian on the domain Ω. It is well known that the semigroup kernel $e^{t\Delta_D}(x, y)$ of $e^{t\Delta_D}$ satisfies the Gaussian upper bound

$$e^{t\Delta_D}(x, y) \le \frac{1}{(4\pi t)^{n/2}} \exp\Big(- \frac{|x-y|^2}{4t}\Big),$$

for all $t > 0$ and all $x, y \in \Omega$. By the extension argument as in [18], we can obtain the estimates for the Schrödinger group associated to the fractional Laplacian $(-\Delta_D)^{\gamma/2}$. More precisely, we have for $0 < \gamma \ne 1$ and $\beta = \gamma n \mathfrak{s}_p$, we have

$$\big\| (I - \Delta_D)^{-\beta/2} e^{i\tau(-\Delta_D)^{\gamma/2}} f \big\|_p \le C(1 + |\tau|)^{n\mathfrak{s}_{p_0}} \|f\|_p$$

for all $\tau \in \mathbb{R}$ and $p \in (1, \infty)$. To the best of our knowledge, this result is new.

In the next Section, we recall the properties of Hardy spaces associated to the operator L, and some estimates for functions of the operator. The proof of Theorems 1.1 and 1.2 is given in Section 3.

Notation. Throughout this paper, we use C to denote positive constants, which are independent of the main parameters involved and whose values may vary at every occurrence. By writing $f \lesssim g$, we mean that $f \le Cg$. We also use $f \sim g$ to denote that $C^{-1}g \le f \le Cg$.

To simplify notation, we will often just use B for $B(x_B, r_B)$ and $V(E)$ for $\mu(E)$ for any measurable subset $E \subset X$. Also given $\lambda > 0$, we will write λB for the $B(x_B, \lambda r_B)$. For each ball $B \subset X$ we set

$$S_0(B) = 0, \quad S_j(B) = 2^j B \backslash 2^{j-1} B \quad \text{for } j \in \mathbb{N}.$$

2. Preliminaries

2.1. Hardy spaces associated to the operator L

In this section, we assume that the operator L is a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimates (10). We first recall from [22,23] the definition of the Hardy spaces associated to an operator. Let L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Gaussian upper bound (A2). Let $0 < p \le 2$. Then the Hardy space $H^p_L(X)$ is defined as the completion of

$$\{f \in L^2(X) : S_L f \in L^p(X)\}$$

under the norm $\|f\|_{H^p_L(X)} = \|S_L f\|_{L^p}$, where the square function S_L is defined as

$$S_L f(x) = \Big(\int_0^\infty \int_{d(x,y)<t} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y)dt}{tV(x,t)}\Big)^{1/2}.$$

Definition 2.1 *([22,23]).* Let $0 < p \le 1$ and $M \in \mathbb{N}$. A function $a(x)$ supported in a ball $B \subset X$ of radius r_B is called a $(p, 2, M, L)$-atom if there exists a function $b \in D(L^M)$ such that

(i) $a = L^M b$;
(ii) $\operatorname{supp} L^k b \subset B,\ k = 0, 1, \dots, M$;
(iii) $\|L^k b\|_{L^2(X)} \le r_B^{2(M-k)} V(B)^{1/2-1/p},\ k = 0, 1, \dots, M$.

Definition 2.2 *(Atomic Hardy spaces for L).* Given $0 < p \le 1$ and $M \in \mathbb{N}$, we say that $f = \sum \lambda_j a_j$ is an atomic $(p, 2, M, L)$-representation if $\{\lambda_j\}_{j=0}^\infty \in \ell^p$, each a_j is a $(p, 2, M, L)$-atom, and the sum converges in $L^2(X)$. The space $H^p_{L,at,M}(X)$ is then defined as the completion of

$$\Big\{f \in L^2(X) : f \text{ has an atomic } (p, 2, M, L)\text{-representation}\Big\},$$

with the norm given by

$$\|f\|^p_{H^p_{L,at,M}(X)} = \inf\Big\{\sum |\lambda_j|^p : f = \sum \lambda_j a_j \text{ is an atomic } (p, 2, M, L)\text{-representation}\Big\}.$$

Theorem 2.3 *([23]). Let $p \in (0,1]$ and $M > \frac{n}{2}(\frac{1}{p} - 1)$. Then the Hardy spaces $H^p_{L,at,M}(X)$ and $H^p_L(X)$ coincide and have equivalent norms.*

We note that if $L = -\Delta$ on $L^2(\mathbb{R}^n)$, then $H^p_L(\mathbb{R}^n)$ reduces to the standard Hardy space $H^p(\mathbb{R}^n)$ on $\mathbb{R}^n$ for $p \in (0,1]$. In general, depending on the choice of the operator L, it may happen that either $H^p(\mathbb{R}^n) \subset H^p_L(\mathbb{R}^n)$, or $H^p_L(\mathbb{R}^n) \subset H^p(\mathbb{R}^n)$, or $H^p(\mathbb{R}^n) \neq H^p_L(\mathbb{R}^n)$ without inclusions. See for example [11].

Proposition 2.4 *([21]). Let $1 \le p_0 < p < p_1 \le 2$ and L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimates (10). Then we have*

$$\big[H^{p_0}_L(X), H^{p_1}_L(X)\big]_\theta = H^p_L(X) \tag{19}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

and $[\cdot,\cdot]_\theta$ stands for the complex interpolation brackets.

Proposition 2.5 *(Theorem 3.7, [21]). Let L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimates (10) and generalized Gaussian GGE(p_0) (12) for some $p_0 \in [1,2)$. Then we have*

$$H^p_L(X) = L^p(X)$$

for all $p_0 < p < p_0'$.

Remark 2.6. Let $\psi \in C^\infty_c(\mathbb{R})$ be an even function such that $\operatorname{supp}\psi \subset \{\xi : 1/4 \le |\xi| \le 4\}$ and $\psi = 1$ on $\{\xi : 1/2 < |\xi| < 2\}$. Define

$$S_{\psi,L} f(x) = \Big(\int_0^\infty \int_{d(x,y)<t} |\psi(t\sqrt{L}) f(y)|^2 \frac{d\mu(y)dt}{tV(x,t)}\Big)^{1/2}.$$

By a careful examination of the proofs in the papers [22,23], it can be verified that the Hardy space $H^p_L(X)$ can be defined by using $S_{\psi,L}$ instead of S_L.

2.2. Some estimates on the functional calculus

Assume that the operator L is a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimates (10). It is well-known that the kernel $K_{\cos(t\sqrt{L})}$ of $\cos(t\sqrt{L})$ satisfies

$$\operatorname{supp} K_{\cos(t\sqrt{L})} \subset \{(x,y) \in X \times X : d(x,y) \le t\}. \tag{20}$$

See for example [10]. We first recall the following result in [15, Lemma 1].

Lemma 2.7. *Let L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimates (10). If F is an even bounded Borel function with* $\operatorname{supp}\widehat{F} \subset [-r, r]$ *for some $r > 0$, then*

$$\operatorname{supp} K_{F(\sqrt{L})} \subset \{(x, y) \in X \times X : d(x, y) \le r\}.$$

Lemma 2.8. *Let L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimates (10). Assume that the operator L satisfies (11) for some $p_0 \in [1, 2)$ additionally. Then for $p \in [p_0, 2]$ and $N > n\mathfrak{s}_p/2$, we have*

$$\|(I + tL)^{-N} 1_B\|_{p\to 2} + \|1_B (I + tL)^{-N}\|_{p\to 2} \lesssim \sup_{x\in B} \frac{1}{V(x, \sqrt{t})^{\mathfrak{s}_p}}$$

for any ball $B \subset X$.

Proof. For any $N \in \mathbb{N}$,

$$(I + tL)^{-N} = \frac{1}{N!} \int_0^\infty s^{N-1} e^{-s} e^{-stL} ds.$$

Note that under the Davies-Gaffney estimates (10) and (11), by the interpolation we have, for any $p \in [p_0, 2]$,

$$\|1_B e^{-tL}\|_{p\to 2} + \|e^{-tL} 1_B\|_{p\to 2} \lesssim \sup_{x\in B} \frac{1}{V(x, \sqrt{t})^{\mathfrak{s}_p}}.$$

Therefore,

$$\|(I + tL)^{-N} 1_B\|_{p\to 2} \lesssim \int_0^\infty \sup_{x\in B} \frac{1}{V(x, \sqrt{st})^{\mathfrak{s}_p}} s^{N-1} e^{-s} ds.$$

This, together with (8),

$$\begin{aligned} \|(I + tL)^{-N} 1_B\|_{p\to 2} &\lesssim \sup_{x\in B} \frac{1}{V(x, \sqrt{t})^{\mathfrak{s}_p}} \int_0^\infty \Big(1 + \frac{1}{\sqrt{s}}\Big)^{n\mathfrak{s}_p} s^{N-1} e^{-s} ds \\ &\lesssim \sup_{x\in B} \frac{1}{V(x, \sqrt{t})^{\mathfrak{s}_p}}, \end{aligned}$$

as long as $N > n\mathfrak{s}_p/2$.

Similarly,

$$\|1_B (I + tL)^{-N}\|_{p\to 2} \lesssim \sup_{x\in B} \frac{1}{V(x, \sqrt{t})^{\mathfrak{s}_p}},$$

as long as $N > n\mathfrak{s}_p/2$.

Hence, this completes our proof. □

We have the following useful lemma.

Lemma 2.9. *Let L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimates (10). Let $\varphi \in C_0^\infty(\mathbb{R})$ be an even function with $\operatorname{supp}\varphi \subset (-1,1)$ and $\int \varphi = 2\pi$. Denote by Φ the Fourier transform of φ. Then the kernel $K_{\Phi(t\sqrt{L})}$ of $\Phi(t\sqrt{L})$ satisfies*

$$\operatorname{supp} K_{\Phi(t\sqrt{L})} \subset \{(x,y) \in X \times X : d(x,y) \le t\} \tag{21}$$

for all $x, y \in X$ and $t > 0$.

If the operator L satisfies (11) for some $p_0 \in [1,2)$ additionally, then for $p \in [p_0, 2]$ we have

$$\|1_B \Phi(t\sqrt{L})\|_{p\to 2} + \|\Phi(t\sqrt{L})1_B\|_{p\to 2} \lesssim \sup_{x\in B} \frac{1}{V(x,t)^{\mathfrak{s}_p}}. \tag{22}$$

Proof. For (21), we refer to [35, Lemma 2].

For the inequality (22), we fix $p \in [p_0, 2]$ and $N \in \mathbb{N}$, $N > n\mathfrak{s}_p$, and then we write

$$\Phi(t\sqrt{L}) = (I + t^2L)^{-N}(I + t^2L)^N \Phi(t\sqrt{L}).$$

It follows that

$$\|1_B \Phi(t\sqrt{L})\|_{p\to 2} \le \|1_B(I + t^2L)^{-N}\|_{p\to 2} \|(I + t^2L)^N \Phi(t\sqrt{L})\|_{2\to 2}.$$

Since $\|(I + t^2L)^N \Phi(t\sqrt{L})\|_{2\to 2} \le \|(1 + t^2\xi^2)^N \Phi(t\xi)\|_\infty \le$ constant, by using Lemma 2.8 we have

$$\begin{aligned}\|1_B \Phi(t\sqrt{L})\|_{p\to 2} &\lesssim \|1_B(I + t^2L)^{-N}\|_{p\to 2} \\ &\lesssim \sup_{x\in B} \frac{1}{V(x,t)^{\mathfrak{s}_p}}.\end{aligned}$$

Similarly,

$$\|\Phi(t\sqrt{L})1_B\|_{p\to 2} \lesssim \sup_{x\in B} \frac{1}{V(x,t)^{\mathfrak{s}_p}}.$$

This completes our proof. □

Lemma 2.10. *Let L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Davies-Gaffney estimates (10) and (11) for some $p_0 \in [1,2)$. Let Φ be the function in Lemma 2.9 and let $\psi \in C_c^\infty(\mathbb{R})$ be an even function such that $\operatorname{supp}\psi \subset \{\xi : 1/4 \le |\xi| \le 4\}$ and $\psi = 1$ on $\{\xi : 1/2 < |\xi| < 2\}$. Let E, F be two measurable sets in X such that $d(E,F) > 0$ and $\beta, \gamma > 0$. Then for every $p \in [p_0, 2]$, $s > 0$, $\tau \in \mathbb{R}$, $M, k_0 \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, we have*

$$\|\psi(2^{-\ell}\sqrt{L})(I+L)^{-\beta/2}e^{i\tau L^{\gamma/2}}(I-\Psi(s\sqrt{L}))^M\|_{L^p(E)\to L^2(F)}$$
$$\lesssim \sup_{z\in E}\frac{C}{V(z,2^{-\ell})^{\mathfrak{s}_p}}d(E,F)^{-k_0}2^{-\ell k_0}(1+2^{\ell\gamma(k_0-\beta/\gamma)})(1+|\tau|)^{k_0}\min\{1,(2^\ell s)^{2M}\} \tag{23}$$

and

$$\|\psi(2^{-\ell}\sqrt{L})(I+L)^{-\beta/2}e^{i\tau L^{\gamma/2}}(I-\Psi(s\sqrt{L}))^M\|_{L^p(E)\to L^2(X)}$$
$$\lesssim \sup_{z\in E}\frac{C}{V(z,2^{-\ell})^{\mathfrak{s}_p}}(1+2^\ell)^{-\beta}\min\{1,(2^\ell s)^{2M}\}, \tag{24}$$

where C is a constant independent of s, τ, ℓ and the sets E, F.

The inequalities (23) and (24) still hold true if we replace $(I-\Psi(s\sqrt{L}))^M$ by $(I-e^{-s^2L})^M$.

In the case $p=2$, the condition (11) can be omitted.

Proof. We need only to give the proof for the case $p\neq 2$ since the case $p=2$ can be done similarly.

Proof of (23).

Set $\mathbf{d}:=d(E,F)$ and

$$F_{\ell,s}(t)=\psi(2^{-\ell}t)(1+t^2)^{-\beta/2}e^{i\tau|t|^\gamma}(I-\Psi(st))^M. \tag{25}$$

Let $\varphi\in C_c^\infty(\mathbb{R})$ be an even function supported in $[-1/2,1/2]$ such that

$$\varphi=1 \text{ on } [-1/4,1/4].$$

Then the Fourier transforms of $F_{\ell,s}$ and $F_{\ell,s}*(\delta-\widehat{\varphi_{\mathbf{d}}})=:H_{\ell,s}$ agree on $\{\xi:|\xi|\geq \mathbf{d}/2\}$, where δ is the Dirac mass at 0 and $\varphi_{\mathbf{d}}(\xi)=\varphi(\xi/\mathbf{d})$. This, along with Lemma 2.7, implies that

$$K_{F_{\ell,s}(\sqrt{L})}(x,y)=K_{H_{\ell,s}(\sqrt{L})}(x,y),$$

whenever $d(x,y)>\mathbf{d}/2$.

Consequently,

$$\|F_{\ell,s}(\sqrt{L})\|_{L^p(E)\to L^2(F)}=\|H_{\ell,s}(\sqrt{L})\|_{L^p(E)\to L^2(F)}.$$

From Lemma 2.8, we have for a fixed $N>n/2$,

$$\begin{aligned}\|H_{\ell,s}(\sqrt{L})\|_{L^p(E)\to L^2(F)}&\leq \|(I+2^{2\ell}L)^{-N}\|_{L^p(E)\to L^2(X)}\|(I+2^{2\ell}L)^N H_{\ell,s}(\sqrt{L})\|_{2\to 2}\\&\lesssim \sup_{z\in E}\frac{1}{V(z,2^{-\ell})^{\mathfrak{s}_p}}\|(I+2^{2\ell}L)^N H_{\ell,s}(\sqrt{L})\|_{2\to 2}\\&\lesssim \sup_{z\in E}\frac{1}{V(z,2^{-\ell})^{\mathfrak{s}_p}}\|(1+2^{2\ell}t^2)^N H_{\ell,s}(t)\|_{\infty}.\end{aligned} \tag{26}$$

It reduces to estimate the L^∞ norm of the function

$$t \mapsto (1 + 2^{-2\ell} t^2)^N H_{\ell,s}(t).$$

Since $\operatorname{supp} H_{\ell,s} \subset \{t : 2^{\ell-2} \le |t| \le 2^{\ell+2}\}$, it suffices to estimate $\|H_{\ell,s}\|_\infty$. To do this, we borrow some ideas in [35]. Recall that

$$H_{\ell,s} := F_{\ell,s} * (\delta - \widehat{\varphi_{\mathbf{d}}}),$$

which follows that for any $k_0 \in \mathbb{N}$,

$$H_{\ell,s} = \mathbf{d}^{-k_0} F^{(k_0)}_{\ell,s} * \widehat{\eta_{\mathbf{d}}},$$

where $\eta(\xi) = (i\xi)^{-k_0}(1 - \varphi(\xi))$ and $\eta_{\mathbf{d}}(\xi) = \eta(\xi/\mathbf{d})$.

It follows that

$$H_{\ell,s}(t) = \mathbf{d}^{-k_0} \int_{\mathbb{R}} F^{(k_0)}_{\ell,s}(t-u)\widehat{\eta_{\mathbf{d}}}(u)du.$$

Recall that

$$F_{\ell,s}(t) = \psi(2^{-\ell}t)(1+t^2)^{-\beta/2} e^{i\tau|t|^\gamma}(1 - \Phi(st))^M.$$

Since Φ is an even Schwartz function with $\Phi'(0) = 0$, $1 - \Phi(t) \simeq t^2$ as $t \to 0$. Hence, it is easy to verify that

$$|F^{(k_0)}_{\ell,s}(t)| \lesssim 2^{-\ell k_0}(1 + |\tau|)^{k_0}(1 + 2^{\ell\gamma(k_0 - \beta/\gamma)}) \min\{1, (2^\ell s)^{2M}\}.$$

This, along with (26), implies (23).

This completes our proof of (23).

Proof of (24).

From (25), we have

$$\begin{aligned}
\|F_{\ell,s}(\sqrt{L})\|_{L^p(E)\to L^2(X)} &\le \|(I + 2^{2\ell}L)^{-N}\|_{L^p(E)\to L^2(X)} \|(I + 2^{2\ell}L)^N F_{\ell,s}(\sqrt{L})\|_{2\to 2} \\
&\lesssim \sup_{z\in E} \frac{1}{V(z, 2^{-\ell})^{\mathfrak{s}_p}} \|(I + 2^{2\ell}L)^N F_{\ell,s}(\sqrt{L})\|_{2\to 2} \\
&\lesssim \sup_{z\in E} \frac{1}{V(z, 2^{-\ell})^{\mathfrak{s}_p}} \|(I + 2^{2\ell}t^2)^N F_{\ell,s}(t)\|_\infty \\
&\lesssim \sup_{z\in E} \frac{1}{V(z, 2^{-\ell})^{\mathfrak{s}_p}} (1 + 2^\ell)^{-\beta} \min\{1, (2^\ell s)^M\}.
\end{aligned}$$

Similarly, it can be verified that the inequalities (23) and (24) still hold true if we replace $(I - \Psi(s\sqrt{L}))^M$ by $(I - e^{-s^2L})^M$.

This completes our proof. $\square$

Arguing similarly (but simpler) to the proof of Lemma 2.10, we also have

Lemma 2.11. *Assume that the operator L satisfies (11) for some $p_0 \in [1,2)$ additionally. Let Φ be the function in Lemma 2.9 and let $\psi \in C_c^\infty(\mathbb{R})$ be an even function such that $\operatorname{supp}\psi \subset \{\xi : 1/4 \le |\xi| \le 4\}$ and $\psi = 1$ on $\{\xi : 1/2 < |\xi| < 2\}$. Let E, F be two measurable sets in X such that $d(E,F) > 0$ and $\beta, \gamma > 0$. Then for every $p \in [p_0, 2]$, $s > 0$, $M, k_0, \in \mathbb{N}$ and $\ell \ge 0$ we have*

$$\begin{aligned}\|\psi(2^{-\ell}\sqrt{L})(I+L)^{-\beta/2}(I-\Psi(s\sqrt{L}))^M\|_{L^p(E)\to L^2(F)}\\ \le \sup_{z\in E}\frac{C}{V(z,2^{-\ell})^{\mathfrak{s}_p}} d(E,F)^{-k_0}2^{-\ell(k_0+\beta)}\min\{1,(2^\ell s)^{2M}\}\end{aligned} \tag{27}$$

and

$$\begin{aligned}\|\psi(2^{-\ell}\sqrt{L})(I+L)^{-\beta/2}(I-\Psi(s\sqrt{L}))^M\|_{L^p(E)\to L^2(X)}\\ \le \sup_{z\in E}\frac{C}{V(z,2^{-\ell})^{\mathfrak{s}_p}} 2^{-\ell\beta}\min\{1,(2^\ell s)^{2M}\},\end{aligned} \tag{28}$$

where C is a constant independent of s, ℓ and the sets E, F.

3. Sharp estimates for the Schrödinger flows $e^{itL^{\gamma/2}}$

3.1. Sharp weak type estimates for the Schrödinger flows $e^{itL^{\gamma/2}}$

This section is dedicated to proving Theorem 1.1.

Proof of Theorem 1.1. Set $\beta = \gamma n \mathfrak{s}_p$ and $F_\tau(\xi) = (1+\xi^2)^{-\beta/2}e^{it|\xi|^\gamma}$ so that $F_\tau(\sqrt{L}) = (1+L)^{-\beta/2}e^{i\tau L^{\gamma/2}}$. Fix $p \in [p_0, 2)$ and recall $\mathfrak{s}_p = 1/p - 1/2$. Since the estimate (14) follows directly from the complex interpolation, the estimate (13) and the trivial estimate $\|e^{-\tau L^{\gamma/2}}\|_{2\to 2} = 1$, it suffices to prove the following weak type (p,p) estimate

$$\mu\Big(\Big\{|F_\tau(\sqrt{L})f| > \lambda\Big\}\Big) \lesssim (1+|\tau|)^{pn\mathfrak{s}_p}\frac{\|f\|_p^p}{\lambda^p} \tag{29}$$

for all $f \in L^p(X)$ and $\lambda > 0$.

To do this, let $f \in L^p(X)$ and $\lambda > 0$. Then by the Calderón-Zygmund decomposition [15] we can decompose $f = g + \sum_k b_k =: g + b$ such that

$$|g(x)| \lesssim \lambda \text{ for a.e. } x \in X, \quad \|g\|_p \lesssim \|f\|_p, \tag{30}$$

and

(i) $\operatorname{supp} b_k \subset B_k$ for some ball B_k;
(ii) $\|b_k\|_p^p \lesssim \lambda^p V(B_k)$;
(iii) $\sum_k V(B_k) \lesssim \frac{\|f\|_p^p}{\lambda^p}$;
(iv) $\sum_k 1_{2B_k} \lesssim 1$.

Define

$$\theta = \frac{1}{4M(1+|\tau|)^{p/2}},$$

where M is an positive integer which will be fixed later. We have

$$\mu\Big(\Big\{|F_\tau(\sqrt{L})f| > \lambda\Big\}\Big) \le \mu\Big(\Big\{|F_\tau(\sqrt{L})g| > \lambda/2\Big\}\Big) + \mu\Big(\Big\{|F_\tau(\sqrt{L})b| > \lambda/2\Big\}\Big).$$

Using Chebyshev's inequality and the L^2-boundedness of $F_\tau(\sqrt{L})$,

$$\begin{aligned}\mu\Big(\Big\{|F_\tau(\sqrt{L})g| > \lambda/2\Big\}\Big) &\lesssim \frac{\|F_\tau(\sqrt{L})g\|_2^2}{\lambda^2}\\ &\lesssim \frac{\|g\|_2^2}{\lambda^2}\\ &\lesssim \frac{\|g\|_p^p\lambda^{2-p}}{\lambda^2} = \frac{\|g\|_p^p}{\lambda^p}\\ &\lesssim \frac{\|f\|_p^p}{\lambda^p}.\end{aligned}$$

For the bad part, let Φ be the function in Lemma 2.9. Setting $\Phi_{\theta,r_{B_k}}(t) = \Phi(\theta r_{B_k} t)$, then we have

$$\begin{aligned}\mu\Big(\Big\{|F_\tau(\sqrt{L})b| > \lambda/2\Big\}\Big) &\le \mu\Big(\Big\{\Big|F_\tau(\sqrt{L})\Big[\sum_k \Big(I - \big(I - \Phi_{\theta,r_{B_k}}(\sqrt{L})\big)^M\Big)b_k\Big]\Big| > \lambda/4\Big\}\Big)\\ &\quad + \mu\Big(\Big\{\Big|\sum_k F_\tau(\sqrt{L})\big(I - \Phi_{\theta,r_{B_k}}(\sqrt{L})\big)^M b_k\Big| > \lambda/4\Big\}\Big)\\ &=: E_1 + E_2.\end{aligned}$$

We will take care of the first term E_1. Note that

$$\Psi_{\theta,r_{B_k}}(\sqrt{L}) := I - \big(I - \Phi_{\theta,r_{B_k}}(\sqrt{L})\big)^M = \sum_{k=1}^M c_k[\Phi_{\theta,r_{B_k}}(\sqrt{L})]^k,$$

where c_k are coefficients.

From Lemma 2.9,

$$K_{\Psi_{\theta,r_{B_k}}(\sqrt{L})}(\cdot,\cdot) \subset \{(x,y) : d(x,y) < r_{B_k}/2\},$$

which implies

$$\Psi_{\theta,r_{B_k}}(\sqrt{L})b_k \subset 2B_k, \tag{31}$$

and

$$\|\Psi_{\theta,r_{B_k}}(\sqrt{L}).1_{B_k}\|_{p\to 2} \lesssim \sup_{z\in B_k} \frac{1}{V(z,\theta r_{B_k})^{\mathfrak{s}_p}}. \tag{32}$$

By the Chebyshev inequality and the L^2-boundedness of $F_\tau(\sqrt{L})$,

$$E_1 \lesssim \frac{\big\|\sum_k \Psi_{\theta,r_{B_k}}(\sqrt{L})b_k\big\|_2^2}{\lambda^2}.$$

This, along with (iv), (31), (32) and (ii), implies that

$$\begin{aligned}E_1 &\lesssim \frac{\sum_k \big\|\Psi_{\theta,r_{B_k}}(\sqrt{L})b_k\big\|_2^2}{\lambda^2}\\
&\lesssim \sum_k \frac{\|b_k\|_p^2}{\lambda^2 V(\theta B_k)^{2/p-1}}\\
&\lesssim \sum_k \frac{V(B_k)^{2/p}}{V(\theta B_k)^{2/p-1}}.\end{aligned}$$

We now apply (8) to further obtain

$$\begin{aligned}E_1 &\lesssim \sum_k \frac{\theta^{-n(2/p-1)}V(B_k)^{2/p}}{V(B_k)^{2/p-1}}\\
&\lesssim (1+|\tau|)^{n(1-p/2)}\sum_k V(B_k)\\
&\lesssim (1+|\tau|)^{pn\mathfrak{s}_p}\frac{\|f\|_p^p}{\lambda^p},\end{aligned}$$

where in the last inequality we used (iii).

It remains to estimate E_2. To do this, we set

$$F_{\theta,r_{B_k}}(t) = F_\tau(t)(1-\Phi_{\theta,r_{B_k}}(t))^M.$$

Then,

$$\begin{aligned}E_2 &\le \mu\Big(\bigcup_k 4B_k^*\Big) + \mu\Big(\Big\{x \notin \bigcup_k 4B_k^* : \Big|\sum_k F_{\theta,r_{B_k}}(\sqrt{L})b_k\Big| > \lambda/4\Big\}\Big)\\
&=: E_{21} + E_{22},\end{aligned}$$

where $B_k^* = \sigma B_k$ with $\sigma = (1+|\tau|)^{p\mathfrak{s}_p}$.

By (8) and (iii),

$$\begin{aligned}E_{21} &\le \sum_k V(4B_k^*)\\ &\lesssim \sigma^n \sum_k V(B_k)\\ &\lesssim (1+|\tau|)^{pn\mathfrak{s}_p}\frac{\|f\|_p^p}{\lambda^p}.\end{aligned}$$

Hence, it suffices to estimate the term E_{22}. Let $\psi \in C_c^\infty(\mathbb{R})$ be a even function supported in $\{\xi : 1/4 \le |\xi| \le 4\}$ and $\psi = 1$ on $\{\xi : 1/2 \le |\xi| \le 2\}$ such that

$$\sum_{\ell\in\mathbb{Z}} \psi(2^{-\ell}\xi) = 1, \qquad \xi \neq 0.$$

Then we have

$$\begin{aligned}E_{22} &= \mu\Big(\Big\{x \notin \bigcup_k 4B_k^* : \Big|\sum_k \sum_{\ell\in\mathbb{Z}} \psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})b_k\Big| > \lambda/4\Big\}\Big)\\ &\le \mu\Big(\Big\{x \notin \bigcup_k 4B_k^* : \Big|\sum_k \sum_{\ell<0} \psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})b_k\Big| > \lambda/12\Big\}\Big)\\ &\quad + \mu\Big(\Big\{x \notin \bigcup_k 4B_k^* : \Big|\sum_k \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta\ge 1}} \psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})b_k\Big| > \lambda/12\Big\}\Big)\\ &\quad + \mu\Big(\Big\{x \notin \bigcup_k 4B_k^* : \Big|\sum_k \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta< 1}} \psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})b_k\Big| > \lambda/12\Big\}\Big)\\ &=: E_{221} + E_{222} + E_{223}.\end{aligned}$$

We now take care of each term E_{221}, E_{222} and E_{223} individually.

3.1.1. Estimate of the term E_{221}

By Chebyshev's inequality,

$$\begin{aligned}E_{221} &\lesssim \frac{\Big\|\sum_k \sum_{\ell<0} \psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})b_k\Big\|_{L^1(X\backslash\cup_k 4B_k^*)}}{\lambda}\\ &\lesssim \frac{\sum_k \sum_{\ell<0} \|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})b_k\|_{L^1(X\backslash 4B_k^*)}}{\lambda}.\end{aligned} \tag{33}$$

For each k and $\ell < 0$, by Hölder's inequality, the doubling property (8) and (ii),

$$
\begin{aligned}
&\|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})b_k\|_{L^1(X\backslash 4B_k^*)} \\
&\qquad \le \sum_{j\ge 2} V(2^j B_k^*)^{1/2}\|F_{\theta,r_{B_k}}(\sqrt{L})b_k\|_{L^2(S_j(B_k^*))} \\
&\qquad \le \sum_{j\ge 2}(2^j\sigma)^{n/2}V(B_k)^{1/2}\|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k^*))}\|b_k\|_p \\
&\qquad \le \sum_{j\ge 2}\lambda(2^j\sigma)^{n/2}V(B_k)^{1/2+1/p}\|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k^*))}.
\end{aligned} \tag{34}
$$

Using (23) in Lemma 2.10, for a fixed $k_0 > n/2$ we have

$$
\begin{aligned}
&\|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k^*))} \\
&\qquad \lesssim \sup_{z\in B_k}\frac{1}{V(z,2^{-\ell})^{\mathfrak{s}_p}}(2^j\sigma r_{B_k})^{-k_0}2^{-\ell k_0}(1+|\tau|)^{k_0}\min\{1,(2^\ell\theta r_{B_k})^{2M}\} \\
&\qquad\quad \times(1+2^{\gamma\ell(k_0-n\mathfrak{s}_p)}) \\
&\qquad \lesssim V(B_k)^{-\mathfrak{s}_p}\max\{1,(2^\ell r_{B_k})^{n\mathfrak{s}_p}\}(2^j\sigma r_{B_k})^{-k_0}2^{-\ell k_0}(1+|\tau|)^{k_0}\min\{1,(2^\ell\theta r_{B_k})^{2M}\} \\
&\qquad\quad \times(1+2^{\gamma\ell(k_0-n\mathfrak{s}_p)}),
\end{aligned} \tag{35}
$$

where in the second inequality we used (8).

Consequently, we have

$$
\begin{aligned}
&\sum_{\ell<0}\sum_{j\ge 2}\lambda(2^j\sigma)^{n/2}V(B_k)^{1/2+1/p}\|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k^*))} \\
&\qquad \lesssim \sum_{\ell\in\mathbb{Z}}\sum_{j\ge 2}\lambda V(B_k)2^{j(n/2-k_0)}\max\{1,(2^\ell r_{B_k})^{n\mathfrak{s}_p}\}\sigma^{n/2-k_0}(2^\ell r_{B_k})^{-k_0} \\
&\qquad\quad \times(1+|\tau|)^{k_0}\min\{1,(2^\ell\theta r_{B_k})^{2M}\} \\
&\qquad \lesssim \sum_{\ell\in\mathbb{Z}}\lambda V(B_k)\max\{1,(2^\ell r_{B_k})^{n\mathfrak{s}_p}\}\sigma^{n/2-k_0}(2^\ell r_{B_k})^{-k_0}(1+|\tau|)^{k_0}\min\{1,(2^\ell\theta r_{B_k})^{2M}\} \\
&\qquad \lesssim \sum_{\ell<0}\lambda V(B_k)\max\{1,(2^\ell r_{B_k})^{n\mathfrak{s}_p}\}\sigma^{n/2-k_0}(2^\ell r_{B_k})^{-k_0}(1+|\tau|)^{k_0}\min\{1,(2^\ell\theta r_{B_k})^{2M}\}.
\end{aligned} \tag{36}
$$

This, together with the fact that

$$
\max\{1,(2^\ell r_{B_k})^{n\mathfrak{s}_p}\}\le\max\{1,(2^\ell\theta r_{B_k})^{n\mathfrak{s}_p}\}\theta^{-n\mathfrak{s}_p},
$$

implies

$$
\begin{aligned}
&\sum_{\ell<0}\sum_{j\ge 2}\lambda(2^j\sigma)^{n/2}V(B_k)^{1/2+1/p}\|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k^*))} \\
&\qquad \lesssim \sum_{\ell}\lambda V(B_k)\max\{1,(2^\ell\theta r_{B_k})^{n\mathfrak{s}_p}\}(2^\ell\theta r_{B_k})^{-k_0}\min\{1,(2^\ell\theta r_{B_k})^{2M}\}\sigma^{n/2-k_0}
\end{aligned}
$$

$$\times (1+|\tau|)^{k_0}\theta^{k_0-n\mathfrak{s}_p}$$
$$\lesssim \lambda V(B_k)\sigma^{n/2-k_0}(1+|\tau|)^{k_0}\theta^{k_0-n\mathfrak{s}_p},$$

as long as $M > k_0/2$.

Recalling that $\theta = (1+|\tau|)^{-p/2}$ and $\sigma = (1+|\tau|)^{1-p/2}$ and by a simple calculation, we come up with

$$\begin{aligned}&\sum_{\ell<0}\sum_{j\geq 2}\lambda(2^j\sigma)^{n/2}V(B_k)^{1/2+1/p}\|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k^*))}\\&\quad\lesssim \lambda V(B_k)\sigma^{n/2}\theta^{-n\mathfrak{s}_p}\\&\quad\lesssim (1+|\tau|)^{pn\mathfrak{s}_p}\lambda V(B_k).\end{aligned}\tag{37}$$

Therefore,

$$\begin{aligned}E_{221}&\lesssim \sum_k (1+|\tau|)^{pn\mathfrak{s}_p}\lambda V(B_k)\\&\lesssim (1+|\tau|)^{pn\mathfrak{s}_p}\frac{\|f\|_p^p}{\lambda^p}.\end{aligned}$$

3.1.2. Estimate of the term E_{222}

This term can be done similarly to the term E_{221} with some modifications. Indeed, similarly to the term E_{221}, we also obtain

$$E_{222}\lesssim \sum_k \sum_{\substack{\ell\geq 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta\geq 1}}\sum_{j\geq 2}(2^j\sigma)^{n/2}V(B_k)^{1/2+1/p}\|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k^*))}.$$

Since $\ell\geq 0$ and $2^{\ell(1-\gamma)}r_{B_k}\theta\geq 1$, we haven $(1+2^{\gamma\ell(k_0-n\mathfrak{s}_p)})\simeq 2^{\gamma\ell(k_0-n\mathfrak{s}_p)}$ and $2^\ell r_{B_k}\geq 2^\ell\theta r_{B_k}\geq 2^{\ell(1-\gamma)}r_{B_k}\theta\geq 1$. Hence, similarly to (36) we obtain that

$$\begin{aligned}&\sum_{\substack{\ell\geq 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta\geq 1}}\sum_{j\geq 2}\lambda(2^j\sigma)^{n/2}V(B_k)^{1/2+1/p}\|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k^*))}\\&\lesssim \sum_{\substack{\ell\geq 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta\geq 1}}\lambda V(B_k)(2^\ell\theta r_{B_k})^{-(k_0-n\mathfrak{s}_p)}2^{\ell\gamma(k_0-n\mathfrak{s}_p)}\sigma^{n/2-k_0}(1+|\tau|)^{k_0}\theta^{k_0-n\mathfrak{s}_p}\\&\lesssim \lambda V(B_k)\sigma^{n/2-k_0}(1+|\tau|)^{k_0}\theta^{k_0-n\mathfrak{s}_p}\sum_{\substack{\ell\geq 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta\geq 1}}(2^{\ell(1-\gamma)}\theta r_{B_k})^{-(k_0-n\mathfrak{s}_p)}\\&\lesssim \lambda V(B_k)\sigma^{n/2-k_0}(1+|\tau|)^{k_0}\theta^{k_0-n\mathfrak{s}_p}\\&\simeq (1+|\tau|)^{pn\mathfrak{s}_p}\lambda V(B_k).\end{aligned}\tag{38}$$

Hence,

$$
\begin{aligned}
E_{221} &\lesssim \sum_k (1+|\tau|)^{pn\mathfrak{s}_p} \lambda V(B_k) \\
&\lesssim (1+|\tau|)^{pn\mathfrak{s}_p} \frac{\|f\|_p^p}{\lambda^p}.
\end{aligned}
$$

3.1.3. Estimate of the term E_{223}

This term is quite complicated and can be estimated by the duality argument.

By Chebyshev's inequality and L^2-boundedness of $L^{i\tau L^{\gamma/2}}$,

$$
E_{223} \lesssim \frac{\left\| \sum_k \sum_{\substack{\ell \geq 0 \\ 2^{\ell(1-\gamma)} r_{B_k}\theta < 1}} \psi(2^{-\ell}\sqrt{L}) G_{\theta, r_{B_k}}(\sqrt{L}) b_k \right\|_2^2}{\lambda^2}, \tag{39}
$$

where

$$
G_{\theta, r_{B_k}}(t) = (1+t^2)^{-\gamma n \mathfrak{s}_p/2}(1-\Phi_{\theta, r_{B_k}}(t))^M
$$

By duality,

$$
\begin{aligned}
&\left\| \sum_k \sum_{\substack{\ell \geq 0 \\ 2^{\ell(1-\gamma)} r_{B_k}\theta < 1}} \psi(2^{-\ell}\sqrt{L}) F_{\theta, r_{B_k}}(\sqrt{L}) b_k \right\|_2 \\
&= \sup_{\|u\|_2=1} \int_X \sum_k \sum_{\substack{\ell \geq 0 \\ 2^{\ell(1-\gamma)} r_{B_k}\theta < 1}} \psi(2^{-\ell}\sqrt{L}) G_{\theta, r_{B_k}}(\sqrt{L}) b_k(x) u(x) d\mu(x) \\
&= \sup_{\|u\|_2=1} \sum_k \sum_{\substack{\ell \geq 0 \\ 2^{\ell(1-\gamma)} r_{B_k}\theta < 1}} \int_{B_k} \psi(2^{-\ell}\sqrt{L}) G_{\theta, r_{B_k}}(\sqrt{L}) u(x) b_k(x) d\mu(x) \\
&= \sup_{\|u\|_2=1} \sum_{j=0}^{\infty} \sum_k \sum_{\substack{\ell \geq 0 \\ 2^{\ell(1-\gamma)} r_{B_k}\theta < 1}} \int_{B_k} \psi(2^{-\ell}\sqrt{L}) G_{\theta, r_{B_k}}(\sqrt{L}) u_j(x) b_k(x) d\mu(x),
\end{aligned}
$$

where $u_j = u.1_{S_j(B_k)}$.

By Hölder's inequality, we have

$$
\begin{aligned}
&\int_{B_k} \psi(2^{-\ell}\sqrt{L}) G_{\theta, r_{B_k}}(\sqrt{L}) u_j(x) b_k(x) d\mu(x) \\
&\qquad \leq \|\psi(2^{-\ell}\sqrt{L}) G_{\theta, r_{B_k}}(\sqrt{L}) u_j\|_{L^{p'}(B_k)} \|b_k\|_p \\
&\qquad \leq \|\psi(2^{-\ell}\sqrt{L}) G_{\theta, r_{B_k}}(\sqrt{L})\|_{L^2(S_j(B_k)) \to L^{p'}(B_k)} \|u_j\|_2 \|b_k\|_p
\end{aligned}
$$

$$\simeq \|\psi(2^{-\ell}\sqrt{L})G_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k))}\|u_j\|_2\|b_k\|_p.$$

Therefore,

$$\begin{aligned}&\Big\|\sum_k \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})b_k\Big\|_2\\ &= \sup_{\|u\|_2=1}\sum_{j=0}^{\infty}\sum_k \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \|\psi(2^{-\ell}\sqrt{L})G_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k))}\|u_j\|_2\|b_k\|_p.\end{aligned} \tag{40}$$

We will claim that

$$\begin{aligned}&\sum_{j=0}^{\infty} \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \|\psi(2^{-\ell}\sqrt{L})G_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k))}\|u_j\|_2\|b_k\|_p\\ &\qquad\lesssim \lambda(1+|\tau|)^{pn\mathfrak{s}_p/2}\int_{B_k}\Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2}d\mu(z).\end{aligned} \tag{41}$$

Indeed, for $j=0,1,2$, using Lemma 2.11 we obtain

$$\begin{aligned}&\sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k))}\|b_k\|_p\\ &\qquad\lesssim \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \lambda V(B_k)^{1/p}\sup_{z\in B_k}\frac{1}{V(z,2^{-\ell})^{\mathfrak{s}_p}}2^{-\gamma\ell n\mathfrak{s}_p}\min\{1,(2^{\ell}r_{B_k}\theta)^{2M}\}.\end{aligned}$$

This, in combination with (8), implies that

$$\begin{aligned}&\sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k))}\|b_k\|_p\\ &\lesssim \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \lambda V(B_k)^{1/p}V(B_k)^{-\mathfrak{s}_p}\max\{1,(2^{\ell}r_{B_k})^{n\mathfrak{s}_p}\}2^{-\gamma\ell n\mathfrak{s}_p}\min\{1,(2^{\ell}r_{B_k}\theta)^{2M}\}\\ &\lesssim \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \lambda V(B_k)^{1/2}\max\{1,(2^{\ell}r_{B_k})^{n\mathfrak{s}_p}\}2^{-\gamma\ell n\mathfrak{s}_p}\min\{1,(2^{\ell}r_{B_k}\theta)^{2M}\}\\ &\lesssim \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1,\ 2^{\ell}r_{B_k}\theta<1}}\ldots+\sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1,\ 2^{\ell}r_{B_k}\theta\ge 1}}\ldots\end{aligned}$$

If $2^\ell r_{B_k}\theta < 1$, then

$$\max\{1, (2^\ell r_{B_k})^{n\mathfrak{s}_p}\} \le \theta^{-n\mathfrak{s}_p} \sim (1+|\tau|)^{pn\mathfrak{s}_p/2}.$$

It follows that

$$\begin{aligned}\sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1,\ 2^\ell r_{B_k}\theta<1}} &\lambda V(B_k)^{1/2}\max\{1, (2^\ell r_{B_k})^{n\mathfrak{s}_p}\}2^{-\gamma\ell n\mathfrak{s}_p}\min\{1, (2^\ell r_{B_k}\theta)^{2M}\}\\ &\lesssim \sum_{\ell\ge 0}\lambda V(B_k)^{1/2}\theta^{-n\mathfrak{s}_p}2^{-\gamma\ell n\mathfrak{s}_p}\\ &\lesssim \lambda V(B_k)^{1/2}(1+|\tau|)^{pn\mathfrak{s}_p/2}.\end{aligned}$$

If $2^\ell r_{B_k}\theta \ge 1$, then

$$\max\{1, (2^\ell r_{B_k})^{n\mathfrak{s}_p}\} = \theta^{-n\mathfrak{s}_p}(2^{\ell(1-\gamma)}\theta r_{B_k})^{n\mathfrak{s}_p}2^{\ell\gamma n\mathfrak{s}_p}.$$

Therefore,

$$\begin{aligned}\sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1,\ 2^\ell r_{B_k}\theta\ge 1}} &\lambda V(B_k)^{1/2}\max\{1, (2^\ell r_{B_k})^{n\mathfrak{s}_p}\}2^{-\gamma\ell n\mathfrak{s}_p}\min\{1, (2^\ell r_{B_k}\theta)^{2M}\}\\ &\lesssim \sum_{2^{\ell(1-\gamma)}r_{B_k}\theta<1}\lambda V(B_k)^{1/2}\theta^{-n\mathfrak{s}_p}(2^{\ell(1-\gamma)}\theta r_{B_k})^{n\mathfrak{s}_p}\\ &\lesssim \lambda V(B_k)^{1/2}(1+|\tau|)^{pn\mathfrak{s}_p/2}.\end{aligned}$$

Consequently,

$$\sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}}\|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k))}\|b_k\|_p \lesssim \lambda V(B_k)^{1/2}(1+|\tau|)^{pn\mathfrak{s}_p/2}.$$

On the other hand, for $j=0,1,2$,

$$\begin{aligned}\|u_j\|_2 &\lesssim V(2^jB_k)^{1/2}\Big(\frac{1}{V(2^jB)}\int_{2^jB}|u|^2\Big)^{1/2}\\ &\lesssim V(B_k)^{1/2}\inf_{z\in B_k}\Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2},\end{aligned}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function defined by

$$\mathcal{M}u(x) = \sup_{\substack{B:\text{balls}\\ B\ni x}}\frac{1}{V(B)}\int_B |u(z)|d\mu(z).$$

Therefore, for $j = 0, 1, 2$,

$$\sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k))}\|b_k\|_p\|u_j\|_2$$
$$\lesssim \lambda V(B_k)(1+|\tau|)^{pn\mathfrak{s}_p/2}\inf_{z\in B_k}\Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2} \tag{42}$$
$$\lesssim \lambda(1+|\tau|)^{pn\mathfrak{s}_p/2}\int_{B_k}\Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2}d\mu(z).$$

For $j \ge 3$, also using Lemma 2.11 and arguing similarly to (35), we have, for $k_0 > n/2$,

$$\sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \|\psi(2^{-\ell}\sqrt{L})F_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k))}$$
$$\lesssim \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \sup_{z\in B_k}\frac{1}{V(z,2^{-\ell})^{\mathfrak{s}_p}}(2^j r_{B_k})^{-k_0}2^{-\ell(k_0+\gamma n\mathfrak{s}_p)}\min\{1,(2^\ell\theta r_{B_k})^{2M}\}$$
$$\lesssim \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} V(B_k)^{-\mathfrak{s}_p}\max\{1,(2^\ell r_{B_k})^{n\mathfrak{s}_p}\}(2^j r_{B_k})^{-k_0}2^{-\ell(k_0+\gamma n\mathfrak{s}_p)}\min\{1,(2^\ell r_{B_k})^{2M}\}$$
$$\lesssim \sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} 2^{-jk_0}V(B_k)^{-\mathfrak{s}_p}\max\{1,(2^\ell r_{B_k})^{n\mathfrak{s}_p}\}(2^\ell r_{B_k})^{-k_0}\min\{1,(2^\ell r_{B_k})^{2M}\}$$
$$\lesssim 2^{-jk_0}V(B_k)^{-\mathfrak{s}_p},$$

as long as $2M > k_0 > n/2 \ge \mathfrak{s}_p$.

It follows that

$$\sum_{j\ge 3}^{\infty}\sum_{\substack{\ell\ge 0\\ 2^{\ell(1-\gamma)}r_{B_k}\theta<1}} \|\psi(2^{-\ell}\sqrt{L})G_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k))}\|u_j\|_2\|b_k\|_p$$
$$\lesssim \sum_{j\ge 3}2^{-jk_0}V(B_k)^{-\mathfrak{s}_p}\lambda V(B_k)^{1/p}\|u_j\|_2$$
$$\lesssim \sum_{j\ge 3}2^{-jk_0}\lambda V(B_k)^{1/2}\|u_j\|_2.$$

In addition,

$$\|u_j\|_2 \lesssim V(2^jB_k)^{1/2}\Big(\frac{1}{V(2^jB)}\int_{2^jB}|u|^2\Big)^{1/2}$$

$$\lesssim V(2^j B_k)^{1/2} \inf_{z\in B_k} \Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2}$$
$$\lesssim 2^{jn/2} V(B_k)^{1/2} \inf_{z\in B_k} \Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2}.$$

These two last estimates give us that

$$\sum_{j\geq 3}^{\infty} \sum_{\substack{\ell\geq 0\\ 2^{\ell(1-\gamma)} r_{B_k}\theta<1}} \|\psi(2^{-\ell}\sqrt{L}) G_{\theta,r_{B_k}}(\sqrt{L})\|_{L^p(B_k)\to L^2(S_j(B_k))} \|u_j\|_2 \|b_k\|_p$$
$$\lesssim \sum_{j\geq 3} 2^{-j(k_0-n/2)} \lambda V(B_k) \inf_{z\in B_k} \Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2}$$
$$\lesssim \sum_{j\geq 3} 2^{-j(k_0-n/2)} \lambda \int_{B_k} \Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2} d\mu(z)$$
$$\lesssim \lambda \int_{B_k} \Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2} d\mu(z).$$

This and (42) prove the claim (41). We now insert (41) into (40) and raise both side to the power of 2 to obtain further

$$\Bigg\|\sum_k \sum_{\substack{\ell\geq 0\\ 2^{\ell(1-\gamma)} r_{B_k}\theta<1}} \psi(2^{-\ell}\sqrt{L}) F_{\theta,r_{B_k}}(\sqrt{L}) b_k\Bigg\|_2^2$$
$$\lesssim \lambda^2 (1+|\tau|)^{pn\mathfrak{s}_p} \sup_{\|u\|_2=1} \Big(\sum_k \int_{B_k} \Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2} d\mu(z)\Big)^2$$
$$\lesssim \lambda^2 (1+|\tau|)^{pn\mathfrak{s}_p} \sup_{\|u\|_2=1} \Big(\int_{\cup B_k} \Big[\mathcal{M}(|u|^2)(z)\Big]^{1/2} d\mu(z)\Big)^2.$$

Using Kolmogorov's inequality, the weak type (1,1) of the maximal function $\mathcal{M}$ and (iii),

$$\Bigg\|\sum_k \sum_{\substack{\ell\geq 0\\ 2^{\ell(1-\gamma)} r_{B_k}\theta<1}} \psi(2^{-\ell}\sqrt{L}) F_{\theta,r_{B_k}}(\sqrt{L}) b_k\Bigg\|_2^2$$
$$\lesssim \lambda^2 (1+|\tau|)^{pn\mathfrak{s}_p} \sup_{\|u\|_2=1} \mu(\cup B_k) \|u\|_2^2$$
$$\lesssim \lambda^2 (1+|\tau|)^{pn\mathfrak{s}_p} \frac{\|f\|_p}{\lambda^p}.$$

Plugging this into (39), we obtain

$$E_{223} \lesssim (1+|\tau|)^{pn\mathfrak{s}_p} \frac{\|f\|_p}{\lambda^p}$$

This proves (13) and completes our proof. □

3.2. Sharp estimates for the Schrödinger flows $e^{itL^{\gamma/2}}$ on the Hardy spaces $H^p_L(X)$

This section is dedicated to proving Theorem 1.2.

Proof of Theorem 1.2. Fix $k_0 > n\mathfrak{s}_p$. We also fix $p \in (0,1]$ and an even function $\psi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\psi \subset \{\xi : 1/4 \le |\xi| \le 4\}$ and $\psi = 1$ on $\{\xi : 1/2 < |\xi| < 2\}$. Without any confusion, we still denote

$$S_L f(x) := \Big(\int_0^\infty \int_{d(x,y)<t} |\psi(t\sqrt{L}) f(y)|^2 \frac{d\mu(y)dt}{tV(x,t)}\Big)^{1/2}.$$

Suppose that $a = L^M b$ is a $(p,2,M,L)$-atom associated to a ball $B = B(x_B, r_B)$, where $M > \max\Big\{\frac{n}{2}(\frac{1}{p}-1), k_0/2\Big\}$.

Recall that $\beta = \gamma n\mathfrak{s}_p$ and $F_\tau(\xi) = (1+\xi^2)^{-\beta/2} e^{i\tau|\xi|^{\gamma/2}}$ so that $F_\tau(\sqrt{L}) = (1+L)^{-\beta/2} e^{i\tau L^{\gamma/2}}$. Using the identity

$$I = (I - e^{-r_B^2 L})^M + \sum_{k=1}^M (-1)^{k+1} C_k^M e^{-kr_B^2 L} =: (I - e^{-r_B^2 L})^M + P(r_B^2 L),$$

we write

$$\begin{aligned} S_L\big(F_\tau(\sqrt{L})a\big) &= S_L\big[(I - e^{-r_B^2 L})^M F_\tau(\sqrt{L})a\big] + S_L\big[(r_B^2 L)^M P(r_B^2 L) F_\tau(\sqrt{L}) r_B^{-2M} b\big] \\ &=: E_1 + E_2. \end{aligned}$$

Therefore, by Remark 2.6 it suffices to show that

$$\|E_1\|_p + \|E_2\|_p \lesssim (1+|\tau|)^{n\mathfrak{s}_p}. \tag{43}$$

Since the estimates of $\|E_1\|_p$ and $\|E_2\|_p$ can be considered similarly. We need only to show the contribution of $\|E_1\|_p$. To do this, set $B_\tau = (1+|\tau|)B$, and write

$$\begin{aligned} \|E_1\|^p_{L^p(\mathbb{R}^n)} &= \Big\|S_L\big[(I - e^{-r_B^2 L})^M F_\tau(\sqrt{L})a\big]\Big\|^p_{L^p(4B_\tau)} + \Big\|S_L\big[(I - e^{-r_B^2 L})^M F_\tau(\sqrt{L})a\big]\Big\|^p_{L^p(X\backslash 4B_\tau)} \\ &=: E_{11} + E_{12}. \end{aligned}$$

By Hölder's inequality, the L^2 boundedness of S_L and the properties of atoms, we have

$$
\begin{aligned}
E_{11} &\lesssim V\big(4(1+|\tau|)B\big)^{1-p/2}\big\|S_L(I-e^{-r_B^2L})^M F_\tau(\sqrt{L})a\big\|^p_{L^2(4B_\tau)}\\
&\lesssim V\big((1+|\tau|)B\big)^{1-p/2}\|a\|_2^p\\
&\lesssim V\big((1+|\tau|)B\big)^{1-p/2}V(B)^{p/2-1}\\
&\lesssim (1+|\tau|)^{pn\mathfrak{s}_p}.
\end{aligned}
\tag{44}
$$

We now estimate E_{12}. To do this, setting

$$F_{t,\tau,r_B}(\lambda):=\psi(t\lambda)(1-e^{-r_B^2\lambda^2})^M F_\tau(\lambda),$$

we then write

$$
\begin{aligned}
S_L\big[(I-e^{-r_B^2L})^M F_\tau(\sqrt{L})a\big](x) &= \left(\sum_{\ell\in\mathbb{Z}}\int_{2^{-\ell}}^{2^{-\ell+1}}\int_{d(x,y)<t}|F_{t,\tau,r_B}(\sqrt{L})a(y)|^2\frac{d\mu(y)dt}{tV(x,t)}\right)^{1/2}\\
&\le \sum_{\ell\in\mathbb{Z}}\left(\int_{2^{-\ell}}^{2^{-\ell+1}}\int_{d(x,y)<t}|F_{t,\tau,r_B}(\sqrt{L})a(y)|^2\frac{d\mu(y)dt}{tV(x,t)}\right)^{1/2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_{12} &\le \sum_{\ell\in\mathbb{Z}}\left\|\left(\int_{2^{-\ell}}^{2^{-\ell+1}}\int_{d(x,y)<t}|F_{t,\tau,r_B}(\sqrt{L})a(y)|^2\frac{d\mu(y)dt}{tV(x,t)}\right)^{1/2}\right\|^p_{L^p(X\backslash 4B_\tau)}\\
&\le \sum_{(\gamma-1)\ell\ge\ell_0}\ldots+\sum_{(\gamma-1)\ell<\ell_0}\ldots\\
&=: E_{121}+E_{122},
\end{aligned}
$$

where ℓ_0 is the smallest integer such that $2^{\ell_0}\ge r_B$, which implies $2^{\ell_0}\simeq r_B$.

For the first term E_{121}, we can see that

$$
\begin{aligned}
E_{121} \le& \sum_{(\gamma-1)\ell\ge\ell_0}\left\|\left(\int_{2^{-\ell}}^{2^{-\ell+1}}\int_{d(x,y)<t}|F_{t,\tau,r_B}(\sqrt{L})a(y)|^2\frac{d\mu(y)dt}{tV(x,t)}\right)^{1/2}\right\|^p_{L^p(4B(x_B,2^{(\gamma-1)\ell}(1+|\tau|)))}\\
&+\sum_{(\gamma-1)\ell\ge\ell_0}\ \sum_{j\ge(\gamma-1)\ell-\ell_0+2}\left\|\left(\int_{2^{-\ell}}^{2^{-\ell+1}}\int_{d(x,y)<t}|F_{t,\tau,r_B}(\sqrt{L})a(y)|^2\frac{d\mu(y)dt}{tV(x,t)}\right)^{1/2}\right\|^p_{L^p(S_j(B_\tau))}\\
=:& E_{1211}+E_{1212}.
\end{aligned}
$$

By Hölder's inequality and (8),

$$E_{1211} \lesssim \sum_{(\gamma-1)\ell \geq \ell_0} V(x_B, 2^{(\gamma-1)\ell}(1+|\tau|))^{1-p/2} \times \left\| \left(\int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} \right)^{1/2} \right\|^p_{L^2(B(x_B,2^{(\gamma-1)\ell}(1+|\tau|)))}. \tag{45}$$

On the other hand,

$$\begin{aligned} &\left\| \left(\int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} \right)^{1/2} \right\|^2_{L^2(4B(x_B,2^{(\gamma-1)\ell}(1+|\tau|)))} \\ &= \int_{4B(x_B,2^{(\gamma-1)\ell}(1+|\tau|))} \int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} d\mu(x) \\ &\leq \int_X \int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} \frac{d\mu(x)}{V(x,t)} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{dt}{t} d\mu(y). \end{aligned}$$

Since

$$\int_{d(x,y)<t} \frac{d\mu(x)}{V(x,t)} \lesssim 1, \tag{46}$$

we have

$$\begin{aligned} &\left\| \left(\int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} \right)^{1/2} \right\|^2_{L^2(4B(x_B,2^{(\gamma-1)\ell}(1+|\tau|)))} \\ &\lesssim \int_{2^{-\ell}}^{2^{-\ell+1}} \|F_{t,\tau,r_B}(\sqrt{L})a\|_2^2 \frac{dt}{t}, \end{aligned}$$

which, together with the fact that

$$\begin{aligned} \|F_{t,\tau,r_B}(\sqrt{L})a\|_2 &\leq \|F_{t,\tau,r_B}(\sqrt{L})\|_{2\to 2}\|a\|_2 \\ &\leq \|F_{t,\tau,r_B}\|_{2\to 2}\|a\|_2 \\ &\leq \|F_{t,\tau,r_B}\|_\infty \|a\|_2 \end{aligned}$$

$$\lesssim \min\{1, (2^\ell r_{B_k})^{2M}\}(1+2^\ell)^{-\gamma n \mathfrak{s}_p} V(B)^{1/2-1/p},$$

implies that

$$\begin{aligned}
&\left\| \left(\int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} \right)^{1/2} \right\|^2_{L^2(4B(x_B, 2^{(\gamma-1)\ell}(1+|\tau|)))} \\
&\lesssim \min\{1, (2^\ell r_{B_k})^{2M}\}(1+2^\ell)^{-\gamma n \mathfrak{s}_p} V(B)^{1-2/p} \\
&\lesssim V(B)^{1-2/p}.
\end{aligned}$$

Inserting this into (45) we arrive at

$$\begin{aligned}
E_{1211} &\lesssim \sum_{(\gamma-1)\ell \ge \ell_0} V\big(x_B, 2^{(\gamma-1)\ell}(1+|\tau|)\big)^{1-p/2} V(B)^{p/2-1} \\
&\lesssim \sum_{(\gamma-1)\ell \ge \ell_0} (1+|\tau|)^{pn\mathfrak{s}_p} (2^{(\gamma-1)\ell} r_B^{-1})^{-pn\mathfrak{s}_p} \\
&\lesssim \sum_{(\gamma-1)\ell \ge \ell_0} (1+|\tau|)^{pn\mathfrak{s}_p} (2^{(\gamma-1)\ell-\ell_0})^{-pn\mathfrak{s}_p} \\
&\lesssim (1+|\tau|)^{pn\mathfrak{s}_p},
\end{aligned}$$

where in the second inequality we used (8) and in the third inequality we used the fact $2^{\ell_0} \simeq r_B$.

To estimate E_{1212}, by Hölder's inequality,

$$\begin{aligned}
E_{1212} \lesssim &\sum_{(\gamma-1)\ell \ge \ell_0} \sum_{j \ge (\gamma-1)\ell-\ell_0+2} V(2^j(1+|\tau|)B)^{1-p/2} \\
&\times \left\| \left(\int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} \right)^{1/2} \right\|^p_{L^2(S_j(B_\tau))}. \qquad (47)
\end{aligned}$$

Note that

$$\begin{aligned}
&\left\| \left(\int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} \right)^{1/2} \right\|^2_{L^2(S_j(B_\tau))} \\
&= \int_{S_j(B_\tau)} \int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} d\mu(x)
\end{aligned}$$

$$\leq \int_{2^{-\ell}}^{2^{-\ell+1}} \int_{U_{j,t}(B_\tau)} \int_{d(x,y)<t} \frac{d\mu(x)}{V(x,t)} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{dt}{t} d\mu(y)$$

$$\lesssim \int_{2^{-\ell}}^{2^{-\ell+1}} \int_{U_{j,t}(B_\tau)} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 d\mu(y) \frac{dt}{t},$$

where $U_{j,t}(B_\tau) = \{x \in X : d(x, S_j(B_\tau)) \leq t\}$ and in the last inequality we used (46).

Using the properties of atoms we further obtain

$$\left\| \left(\int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} \right)^{1/2} \right\|^2_{L^2(S_j(B_\tau))}$$

$$\lesssim \int_{2^{-\ell}}^{2^{-\ell+1}} |F_{t,\tau,r_B}(\sqrt{L})\|^2_{L^2(B)\to L^2(U_{j,t}(B_\tau))} \|a\|_2^2 \frac{dt}{t}$$

$$\lesssim V(B)^{1-2/p} \int_{2^{-\ell}}^{2^{-\ell+1}} |F_{t,\tau,r_B}(\sqrt{L})\|^2_{L^2(B)\to L^2(U_{j,t}(B_\tau))} \frac{dt}{t}.$$

By a simple calculation, it can be verified easily that

$$d(B, U_{j,t}(B_\tau)) \simeq 2^j(1+|\tau|)r_B,$$

provided that $2^{-\ell} \leq t \leq 2^{-\ell+1}$, $j \geq (\gamma - 1)\ell - \ell_0 + 2$.

Using (23) in Lemma 2.10 and the fact $t \simeq 2^\ell$ in this situation, we have, for $k_0 > n\mathfrak{s}_p$,

$$\begin{aligned} &\|F_{t,\tau,r_B}(\sqrt{L})\|_{L^2(B)\to L^2(U_{j,t}(B_\tau))} \\ &\quad\lesssim \big(2^j(1+|\tau|)r_B\big)^{-k_0}(1+\tau)^{k_0} 2^{-\ell k_0}(1+2^{\gamma\ell(k_0-n\mathfrak{s}_p)}) \min\{1, (2^\ell r_{B_k})^{2M}\} \\ &\quad\lesssim \big(2^j r_B\big)^{-k_0} 2^{-\ell k_0}(1+2^{\gamma\ell(k_0-n\mathfrak{s}_p)}) \min\{1, (2^\ell r_{B_k})^{2M}\} \\ &\quad\lesssim 2^{-jk_0}(2^\ell r_{B_k})^{-k_0}(1+2^{\gamma\ell(k_0-n\mathfrak{s}_p)}) \min\{1, (2^\ell r_{B_k})^{2M}\}. \end{aligned}$$

Consequently,

$$\left\| \left(\int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} \right)^{1/2} \right\|^2_{L^2(S_j(B_\tau))}$$

$$
\begin{aligned}
&\lesssim V(B)^{1-2/p}(1+2^{2\gamma\ell(k_0-n\mathfrak{s}_p)})\int_{2^{-\ell}}^{2^{-\ell+1}} 2^{-2jk_0}(2^\ell r_{B_k})^{-2k_0}\min\{1,(2^\ell r_{B_k})^{4M}\}\frac{dt}{t}\\
&\lesssim V(B)^{1-2/p}2^{-2jk_0}(2^\ell r_{B_k})^{-2k_0}(1+2^{2\gamma\ell(k_0-n\mathfrak{s}_p)})\min\{1,(2^\ell r_{B_k})^{4M}\}.
\end{aligned}
$$

Inserting this into (47) and then using (8) and $M > k_0/2$,

$$
\begin{aligned}
E_{1212} \lesssim &\sum_{\substack{(\gamma-1)\ell\geq \ell_0\\ \ell\geq 0}}\ \sum_{j\geq(\gamma-1)\ell-\ell_0+2} V(2^j(1+|\tau|)B)^{1-p/2}\\
&\times V(B)^{p/2-1}2^{-pjk_0}(2^\ell r_{B_k})^{-pk_0}2^{p\gamma\ell(k_0-n\mathfrak{s}_p)}\min\{1,(2^\ell r_{B_k})^{2pM}\}\\
&+\sum_{\substack{(\gamma-1)\ell\geq \ell_0\\ \ell< 0}}\ \sum_{j\geq(\gamma-1)\ell-\ell_0+2} V(2^j(1+|\tau|)B)^{1-p/2}V(B)^{p/2-1}2^{-pjk_0}(2^\ell r_{B_k})^{-pk_0}\\
&\times \min\{1,(2^\ell r_{B_k})^{2pM}\}\\
\lesssim &\sum_{\substack{(\gamma-1)\ell\geq \ell_0\\ \ell\geq 0}}\ \sum_{j\geq(\gamma-1)\ell-\ell_0+2}(1+|\tau|)^{pn\mathfrak{s}_p}2^{-jp(k_0-n\mathfrak{s}_p)}(2^\ell r_{B_k})^{-pk_0}2^{p\gamma\ell(k_0-n\mathfrak{s}_p)}\\
&\times \min\{1,(2^\ell r_{B_k})^{2pM}\}\\
&+\sum_{\substack{(\gamma-1)\ell\geq \ell_0\\ \ell< 0}}\ \sum_{j\geq(\gamma-1)\ell-\ell_0+2}(1+|\tau|)^{pn\mathfrak{s}_p}2^{-jp(k_0-n\mathfrak{s}_p)}(2^\ell r_{B_k})^{-pk_0}\min\{1,(2^\ell r_{B_k})^{2pM}\}\\
\lesssim &(1+|\tau|)^{pn\mathfrak{s}_p}\sum_{\substack{(\gamma-1)\ell\geq \ell_0\\ \ell\geq 0}}(2^\ell r_{B_k})^{-n\mathfrak{s}_p}\{1,(2^\ell r_{B_k})^{2pM}\}\\
&+\sum_{\substack{(\gamma-1)\ell\geq \ell_0\\ \ell< 0}}(1+|\tau|)^{pn\mathfrak{s}_p}(2^{(\gamma-1)\ell-\ell_0})^{-p(k_0-n\mathfrak{s}_p)}(2^\ell r_{B_k})^{-pk_0}\min\{1,(2^\ell r_{B_k})^{2pM}\}\\
\lesssim &(1+|\tau|)^{pn\mathfrak{s}_p}.
\end{aligned}
$$

Therefore,

$$
E_{121}\lesssim (1+|\tau|)^{pn\mathfrak{s}_p}.
$$

It remains to show that

$$
E_{122}\lesssim (1+|\tau|)^{pn\mathfrak{s}_p}.
$$

Applying Hölder's inequality,

$$E_{122} \lesssim \sum_{(\gamma-1)\ell<\ell_0} \sum_{j\geq 2} \times V(2^j(1+|\tau|)B)^{1-p/2} \Bigg\| \Bigg(\int_{2^{-\ell}}^{2^{-\ell+1}} \int_{d(x,y)<t} |F_{t,\tau,r_B}(\sqrt{L})a(y)|^2 \frac{d\mu(y)dt}{tV(x,t)} \Bigg)^{1/2} \Bigg\|^p_{L^2(S_j(B_\tau))}. \tag{48}$$

At this stage, arguing similarly to the estimate of the term E_{121} we come up with

$$E_{122} \lesssim (1+|\tau|)^{pn\mathfrak{s}_p}.$$

For the details we would like to leave to the interested reader.

This completes the proof of (15).

In order to reduce to the sharp estimate (14). We note that by interpolation, the Davies-Gaffney estimates (10) and (11) implies that the operator L satisfies GGE(p) for all $p_0 < p < p_0'$. This, together with Proposition 2.5, implies that

$$H^p_L(X) = L^p(X) \quad \text{for all } p_0 < p < p_0'.$$

At this stage, by using the standard argument (see for example [27]) and Proposition 2.4 the estimate (14) follows immediately from (15).

This completes our proof. □

Data availability

No data was used for the research described in the article.

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References

[1] G. Alexpoulos, Oscillating multiplies on Lie groups and Riemannian manifolds, Tohoku Math. J. 46 (1994) 457–468.
[2] P. Auscher, T. Coulhon, X.T. Duong, S. Hofmann, Riesz transform on manifolds and heat kernel regularity, Ann. Sci. Éc. Norm. Supér. 37 (2004) 911–957.
[3] T.A. Bui, P. D'Ancona, X.T. Duong, J. Li, F.K. Ly, Weighted estimates for powers and smoothing estimates of Schrödinger operators with inverse-square potentials, J. Differ. Equ. 262 (2017) 2771–2807.
[4] T.A. Bui, F.K. Ly, Sharp estimates for Schrödinger groups on Hardy spaces for $0 < p \leq 1$, J. Fourier Anal. Appl. 28 (2023) 70, in press.
[5] N. Burq, F. Planchon, J. Stalker, A.S. Tahvildar-Zadeh, Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential, J. Funct. Anal. 203 (2003) 519–549.
[6] T.A. Bui, P. D'Ancona, X.T. Duong, D. Müller, On the flows associated to selfadjoint operators on metric measure spaces, Math. Ann. 375 (2019) 1393–1426.

[7] T.A. Bui, P. D'Ancona, F. Nicola, Sharp L^p estimates for Schrödinger groups on spaces of homogeneous type, Rev. Mat. Iberoam. 36 (2020) 455–484.
[8] T.A. Bui, X.T. Duong, Q. Hong, G. Hu, On Schrödinger groups of fractional powers of Hermite operators, Int. Math. Res. Not. 7 (2023) 6164–6185.
[9] S. Huang, M. Wang, Q. Zheng, Z. Duan, L^p estimates for fractional Schrödinger operators with Kato class potentials, J. Differ. Equ. 265 (2018) 4181–4212.
[10] T. Coulhon, A. Sikora, Gaussian heat kernel upper bounds via Phragmén–Lindelöf theorem, Proc. Lond. Math. Soc. 96 (2008) 507–544.
[11] X.T. Duong, L. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds, J. Am. Math. Soc. 18 (2005) 943–973.
[12] G. Carron, T. Coulhon, E.M. Ouhabaz, Gaussian estimates and L^p-boundedness of Riesz means, J. Evol. Equ. 2 (2002) 299–317.
[13] P. Chen, X.T. Duong, J. Li, L. Yan, Sharp endpoint estimates for Schrödinger groups on Hardy spaces, Available at https://arxiv.org/abs/1902.08875.
[14] P. Chen, X.T. Duong, J. Li, L. Yan, Sharp endpoint L^p estimates for Schrödinger groups, Math. Ann. 378 (2020) 667–702.
[15] R.R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Am. Math. Soc. 83 (1977) 569–645.
[16] P. D'Ancona, F. Nicola, Sharp L^p estimates for Schrödinger groups, Rev. Mat. Iberoam. 32 (3) (2016) 1019–1038.
[17] P. D'Ancona, V. Pierfelice, F. Ricci, On the wave equation associated to the Hermite and the twisted Laplacian, J. Fourier Anal. Appl. 16 (2010) 294–310.
[18] X.T. Duong, A. McIntosh, Singular integral operators with non-smooth kernels on irregular domains, Rev. Mat. Iberoam. 15 (2) (1999) 233–263.
[19] Z. Fan, Weak type (p, p) bounds for Schrödinger groups via generalized Gaussian estimates, J. Math. Anal. Appl. 495 (2) (2021) 124766.
[20] H. Kalf, U.W. Schmincke, J. Walter, R. Wüst, On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials, in: Spectral Theory and Differential Equations, in: Lecture Notes in Math., vol. 448, Springer, Berlin, 1975, pp. 182–226.
[21] P.C. Kunstmann, M. Uhl, Spectral multiplier theorems of Hörmander type on Hardy and Lebesgue spaces, J. Oper. Theory 73 (1) (2015) 27–69.
[22] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea, L. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates, Mem. Am. Math. Soc. 214 (2011).
[23] R. Jiang, D. Yang, Orlicz-Hardy spaces associated with operators satisfying Davies–Gaffney estimates, Commun. Contemp. Math. 13 (2011) 331–373.
[24] N. Lohoué, Estimations des sommes de Riesz d'opérateurs de Schrödinger sur les variétés riemanniennes et les groupes de Lie, C. R. Acad. Sci. Paris, Ser. I 315 (1992) 13–18.
[25] P.D. Milman, Y.A. Semenov, Global heat kernel bounds via desingularizing weights, J. Funct. Anal. 212 (2004) 373–398.
[26] A. Miyachi, On some estimates for the wave equation in L^p and H^p, J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. 27 (1980) 331–354.
[27] A. Miyachi, On some Fourier multipliers for $H^p(\mathbb{R}^n)$, J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. 27 (1) (1980) 157–179.
[28] A. Miyachi, On some singular Fourier multipliers, J. Fac. Sci., Univ. Tokyo, Sect. 1A, Math. 28 (2) (1981) 267–315.
[29] D. Müller, A. Seeger, L^p bounds for the wave equation on groups of Heisenberg type, Anal. PDE 8 (5) (2015) 1051–1100.
[30] D. Müller, E.M. Stein, L^p-estimates for the wave equation on the Heisenberg group, Rev. Mat. Iberoam. 15 (2) (1999) 297–334.
[31] A. Nagel, E.M. Stein, The $\square_b$-heat equation on pseudoconvex manifolds of finite type in $\mathbb{C}_2$, Math. Z. 238 (1) (2001) 37–88.
[32] J.C. Peral, L^p estimates for the wave equation, J. Funct. Anal. 36 (1980) 114–145.
[33] F. Planchon, J. Stalker, A.S. Tahvildar-Zadeh, L^p estimates for the wave equation with the inverse-square potential, Discrete Contin. Dyn. Syst. 9 (2003) 427–442.
[34] A. Seeger, C.D. Sogge, E.M. Stein, Regularity properties of Fourier integral operators, Ann. Math. 134 (1991) 231–251.
[35] A. Sikora, J. Wright, Imaginary powers of Laplace operator, Proc. Am. Math. Soc. 129 (2001) 1745–1754.
[36] S. Sjöstrand, On the Riesz means of the solutions of the Schrödinger equation, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) 24 (1970) 331–348.

[37] E.M. Stein, Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Univ. Press, Princeton, NJ, 1993.

[38] E.C. Titchmarsh, Eigenfunction Expansions Associated with Second-Order Differential Equations, University Press, Oxford, 1946.