Nexus Network Journal
https://doi.org/10.1007/s00004-022-00636-6

NEXUS NETWORK JOURNAL
Architecture and Mathematics

RESEARCH

Desargues's Perspective Theory: A Critical Interpretation of the Fundamental Theorem

Leonardo Baglioni[1] · **Riccardo Migliari**[1]

Accepted: 20 August 2022

Abstract
This short essay gets to the heart of the debate that arose about the upper part of Table 151 of the *Manière universelle*, written by Abraham Bosse and Girard Desargues, to illustrate how the interpretation of the figure from a perspective point of view (see Jean Pierre Le Goff) not only is legitimate, but it can be also be extended as far as to acknowledge in the figure the complete representation of lines, planes and measurement operations. The *Proposition fondamentale pour la pratique de la Perspective* proves therefore to be the foundation of a general theory of perspective.

Keywords Desargues · Perspective theorem · History of perspective · History of projective geometry

Introduction and Synthesis

Desargues's contributions to the history of perspective are disjointed within a short space of time and in a few surviving publications, which should be considered at the same time both a contribution of an absolutely practical nature—the drafting of which, articulated in many examples, was entrusted to Abraham Bosse and published in the *Manière universelle de Mr. Desargues, pour pratiquer la perspective par petit-pied, comme le géométral* (Bosse 1648)—and theoretical, consisting of the *Exemple de l'une des manières universelles du S. G. D. L. touchant la pratique de la perspective sans employer aucun tiers point, de distance ny d'autre nature, qui soit hors du champ de l'ouvrage* (Desargues 1636), integrated with some postulates of the *Brouillon Project* (Desargues 1639) and with the four theorems contained in the pages from 335 to 342 of *Manière universelle*, the first of which is the *Proposition fondamentale* (Fig. 1).

✉ Leonardo Baglioni
leonardo.baglioni@uniroma1.it

[1] Department of History, Representation and Restoration of Architecture, Sapienza University of Rome, Rome, Italy

Published online: 15 September 2022

Fig. 1 Personal re-elaboration of a late nineteenth-century lithography attributed to J. Ventura. In the original, Desargues shows the fortifications of *La Rochelle* to Cardinal Richelieu, here, instead, he illustrates his fundamental Theorem of Perspective. Image: author

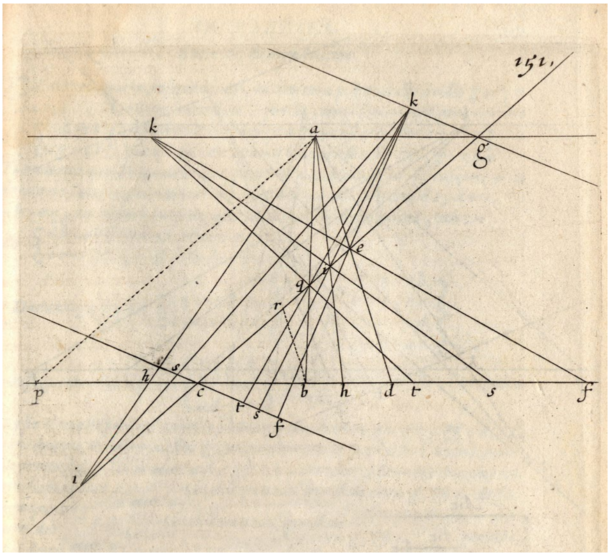


Fig. 2 Detail of Plate 151 of *Manière universelle*, which illustrates the fundamental Theorem of Perspective, according to Desargues. Image: Bosse (1648: 151)

This present essay is dedicated to the image at the top of that publication's Plate 151 (Fig. 2), and illustrates the *Proposition fondamentale de la pratique de la Perspective*. At a first glance, this drawing may appear to be an abstract geometric drawing, similar in all respects to those that illustrate the better known theorem of perspective triangles, but careful observation shows a perspective that represents:

- two planes, one horizontal and the other inclined;
- their intersection line *cg*;
- three measuring operations, distinct but equivalent, that define on the line *cg* the perspectives of three points *q*, *i* and *e*;
- two of these operations are performed on the horizontal plane and use the vanishing points *k* and *a*; a third operation is carried out, instead, on the inclined plane and uses the vanishing point *k*, which is different from the previous one and will hereafter be referred to as uppercase *K*, to distinguish the two.

This perspective brings together, in an extreme synthesis, Girard Desargues's ideas of perspective which were as innovative for the time as they are still relevant today.

The arguments in support of our thesis are the following. First we will see in detail how the above-mentioned contributions can be recomposed today, and how the critics have at times overlooked some of these fragments, almost certainly because they did not recognize in them the dignity of mathematical contribution. We will next look briefly at the state of the art of perspective at the time of Desargues, trying to highlight how much and why his contribution was disruptive, in comparison with the tradition. We will also give an account of the *Exemple de l'une des manières*, highlighting its innovative character, and, in particular, the use of a "pseudo-point of distance" (to quote Jean-Pierre Le Goff (1994: 195), which most likely is also the cause of the refusal of Desargues's idea by many of his contemporaries, and, certainly, the reason that induced him to generalize the procedure with the *Proposition fondamentale* added in the edition of 1648. Finally, we will examine in detail the above-mentioned theorem to justify our interpretation.

Finite and Infinite

First of all, however, it is necessary to explain why, in our opinion, the contribution of Desargues is so important that to him could be attributed the paternity of modern perspective, if there were a need to seek a father.[1] It is not because of the method,

[1] Since the idea that Girard Desargues is the founder of projective geometry, and not of modern perspective, is now well accepted a clarification is necessary. In our opinion, the history of perspective can legitimately be considered in two stages: the phase preceding Desargues and the phase following him. The first phase has a predominantly experimental character. Observations on the perspective image made with the aid of the mirror (Migliari and Baglioni 2018) and other instruments lead first to constructions of a practical nature not justified from a theoretical point of view and then to Guidubaldo del Monte's theorem XXVIII, which justifies the vanishing point even if without relating it to infinity (Guidobaldo Del Monte 1600: 35). Desargues, by introducing the concepts illustrated here, unveils the geometrical and philosophical significance of the vanishing point and therefore opens a second phase, clearly distinct from the first one thanks also to the theorem that is discussed in these pages. He paves the way for Brook Taylor and the many others who helped in developing the present theory. Since we cannot mention them all we would like to at least mention Annibale Angelini (1861: XXII) for being the first to use the term *punto di fuga* now shared in the literature in Italian and Wilhelm Fiedler (1874: 5–66) for

which at the time was the only one that really allowed the artist to work within the physical limits of a painting. The graphical scale procedure, in fact, led to nothing, although the graphical scale procedure would have deserved more fortune.[2]

The most remarkable contribution of Desargues to perspective consists in the definition of points and lines to infinity, found in the incipit of the *Brouillon Project*, because this concept is not only the fundament of projective geometry, but is also an essential moment in the evolution of theory and practice of perspective, since it gives a general meaning to the *punctum concursus* (vanishing point) that Guidobaldo del Monte had described merely as the result of a geometric construction. This broadening of the conception of perspective space also had the effect of generalizing the procedures related to the measurement of a line represented in perspective, a generalization that, on the one hand, is applied in the procedures of 1636 and 1648, and on the other, is the central theme of the fundamental theorem, to which we have more than once alluded. In other words, Desargues understood infinity and encloses it in a solid frame. Perspective became an instrument capable of treating the infinite in finite terms.

In fact, Desargues conceived the revolutionary idea that a sheaf of straight lines can have a vertex at both an accessible point and a point at an indeterminate distance and that, consequently, lines incident in a point and lines that are parallel form analogous and interchangeable figures.[3]

Perspective transforms these sheaves into flat pencils, without limiting this analogy, therefore a flat pencil of lines converging in a point could be the image of a sheaf of parallel lines, so as a flat pencil of parallel lines in perspective may be the image of a sheaf that has its vertex in a point of the front plane.[4]

Desargues moved from the construction of Guidobaldo's *punctum concursus*, extending it to infinite space. He defined the *ligne de l'oeil* as the line, parallel to the objective lines, that passes through the centre of projection *O* and meets the picture plane in the aforesaid *punctum concursus*. We use the participle "projecting"

Footnote 1 (continued)
recognizing the generality of the method. Subsequent developments in the twentieth century have been succinctly illustrated by Jessica Romor (2013: 101–118). That being said, it seems to us correct to point to Desargues as the father of modern perspective, namely, that perspective which is based on a solid theoretical framework and is capable of simulating human perception of space without resorting to the aid of other methods of representation. All this is not to say that Desargues should not also be credited with having enabled through his studies the birth and development of projective geometry, as written by René Taton (1951), Harold S.M.D. Coxeter (2003: 3) and Judith V. Field (1997: 205).

[2] Also because the methods proposed by Desargues's contemporaries completely lacked the property of working always and in whatever way within a picture plane since, using the distance point, they would have needed an aperture of the visual cone of at least 45°, as we will see.

[3] From here on we will use the term "sheaf" to denote a figure made up of straight lines that belong to the tridimensional space and all of which pass through a given point (the vertex of the sheaf) or are parallel to each other; while we will use the term "flat pencil" to denote a figure composed of straight lines lying all in the same plane and radiating from a given point (the centre or vertex of the pencil) or parallel to each other. About the use of these terms and other related definitions see Luigi Cremona (1893: 22).

[4] This is how we nowadays define the plane parallel to the picture plane that passes through the eye, or projection centre.

Table 1 Correspondences between a sheaf of objective lines and the pencil of their perspectives

Position of the *but* vertex of sheaves and pencils	Objective lines	Perspective lines
but at infinity parallel to the picture plane	Parallel	Parallel
but at infinity incident with the picture plane	Parallel	Converging
but accessible on the front plane	Converging	Parallel
but accessible in space and distinct from the front plane	Converging	Converging

to indicate it. Now, if we imagine a sheaf of lines all incident in an accessible point P, we understand that the flat pencil of lines that represents it in the perspective generated from a centre of projection O must have the vertex in point P', the image or perspective of P. The projecting line OP may be incident to the picture plane or parallel to the picture plane, in which case point P', the image of P, is at infinity and therefore transforms itself into the direction of the flat pencil of parallel lines that represents the subject sheaf.

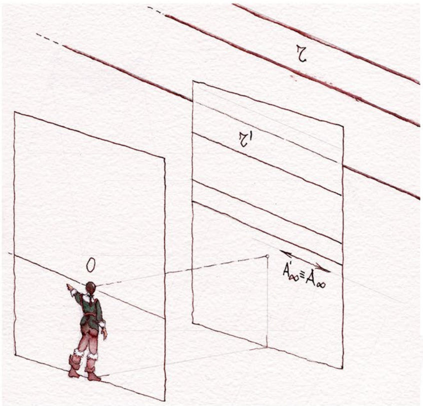
The cases examined by Desargues (1639) are, hence, in the following order (summarized in Table 1):

1. the case where the objective lines r are parallel to each other and the projecting line is parallel to the picture plane (which means that the objective lines are also parallel to the picture plane): in this case the perspectives r' are lines parallel to each other;
2. the case in which the objective lines r are parallel to each other but the projecting line is not parallel to the picture plane, but incident in I'_r, consequently, the perspectives r′ of the lines r converge in the vanishing point I'_r;
3. the case where the objective lines s meet in a point A and the projecting line OA is parallel to the picture plane, because point A is a point on the front plane: in this case the perspectives s' of the lines s are parallel to each other;
4. the case where the objective lines s converge in a point P and the projecting line OP is not parallel to the picture plane: in this case the perspectives s' of the lines s also converge in point P', the perspective of P.

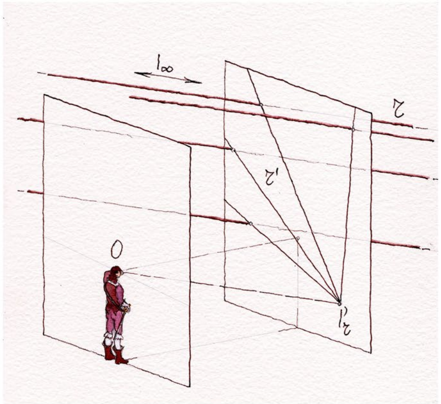
In Fig. 3 we describe the four cases considered by Desargues to show precisely how a sheaf of parallel lines can be transformed into a flat pencil of converging lines and, vice versa, how a sheaf of converging lines can be turned into a flat pencil of lines that are parallel to each other.

This bright idea is so clearly etched in Desargues's mind that he, in discussing the fundamental theorem of perspective, which we will talk about shortly, does not even distinguish the *but*, as centre of the flat pencil, which is the vanishing point of the image of parallel lines, from the *but* that instead is the vertex of a sheaf of objective lines.[5] When he enunciates the theorem which now is called

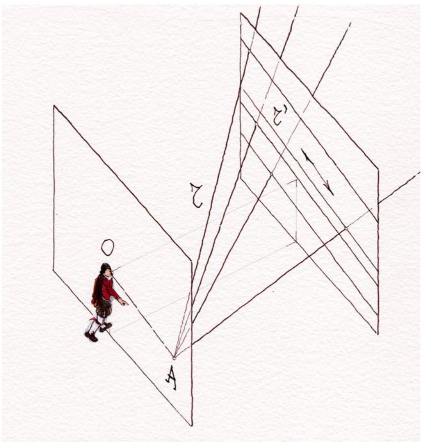
[5] In the *Brouillon Project* the term *but*, which literally means "aim, purpose" indicates the centre or vertex of a sheaf or a flat pencil of lines. Thus, *but* can be both a vanishing point, as the centre of the



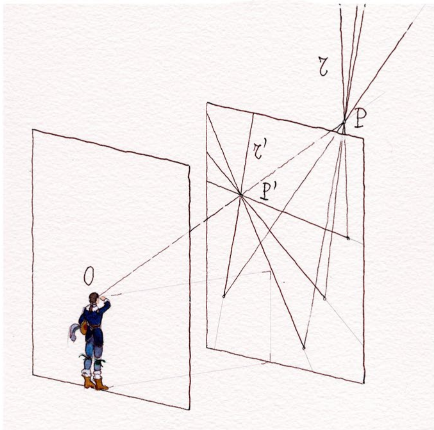
a) First case: in space, the lines of sheaf *r* are parallel to direction A∞ of the picture plane; in perspective, the lines of pencil *r'* are parallel to the direction of picture plane A'∞ ≡ A∞



b) Second case: in space, the lines of sheaf *r* are parallel to the generic direction I∞ incident to the picture plane: in perspective, the lines of pencil *r'* converge in point I'r. This is the recurrent situation in practical perspective.



c) Third case: in space, the lines of sheaf *r* converge in a point A of the front plane; in perspective, the lines of pencil *r'* are parallel to direction OA.



d) Fourth case: in space, the lines of sheaf *r* converge in a point P in space; in perspective, the lines of pencil *r'* converge in a point P' of the picture plane.

Fig. 3 The four cases that occur in perspective if we assume that incident lines and parallel lines form sheaves of the same kind (*ordonnance*), namely sheaves which have vertex in a *but*, both accessible and at infinity

of the theorem of homological or perspective triangles, Desargues emphasizes its generality because of the configuration that it presents graphically—*Quand des droites comme HDa, HEb, cED, lga, lfb, HlK, DgK, EfK, soit en divers plans, soit en un mesme, s'entrerencontrent …* (When the straight lines such as *HDa,*

Footnote 5 (continued)
flat pencil of lines that, in the picture plane, are perspectives of parallel lines in space and the centre of a sheaf of lines, positioned in three-dimensional space.

HEb, cED, lga, lfb, HlK, DgK, EfK are either in distinct planes, or are in the same plane) (Bosse 1648: 340)—namely, objective lines, placed in space, or perspectives of those, therefore belonging to the picture plane. All this leads us to confirm, with Taton, that Desargues's thinking is very difficult indeed: *La figure original de Desargues est assez confuse et ses démostrations très concises …* (The original figure of Desargues is quite confusing and his demonstrations very concise) (Taton 1951: 202). The figure can be read more easily if placed in the perspective space in which it is created, and in that space properly illustrated.

Here we come to the happy intuition of Jean-Pierre Le Goff (1994: 194, 195), whom we mentioned earlier. He intuitively understood, first of all, that there is a remarkable analogy between the Fig. 8 that illustrates the 1636 edition of *Perspective* and plate 151, which, inserted in the pages 336 and 337 of the *Manière universelle* by Bosse, illustrates the *Proposition fondamentale de la pratique de la perspective* and demonstrates this analogy point by point. Further, Le Goff also explained that point *F*, which belongs to the horizon in figure of 1636, is a *pseudopoint de distance*, that is, in our opinion, a measurement point, if examined in the light of the *Proposition fondamentale* and the related figure. This analogy is reinforced by the fact that the criticisms raised against Desargues by his detractors concerned the correctness of the method: not the representation of the lines perpendicular to the picture plane, which is as commonplace today as it was then, but their measurement, namely, the scanning of the depths of space.

From here to construct a simulation of the three-dimensional space that lies concealed behind the abstract drawing of Desargues, is but a short step, indeed very short. In so doing, one also realizes how useful drawing is to illuminate mathematical thought. Perhaps some of the fences that imprison the disciplines, almost as if to avoid their becoming corrupted by coming into contact with drawing and as if perspective were nothing more than a series of empirical rules "for the artists", as they once used to say, should be demolished, when this is possible, and not strengthened, as René Taton did.

Unfortunately, Desargues could not breach these limits, except in his powerful imagination, which was evidently capable of reading in the tangle of Fig. 151 the deep space, or rather, the deep space and the picture plane to which the image belongs. This is clearly evident in his insistence regarding the possibility that the lines used can be situated both in the same plane and in different planes. Moreover, Desargues was bound in the demonstrations to the use of Euclidean geometry and did not have the concepts and logical tools that—thanks to him!—we possess today,

Now, let us come to the theorems that Desargues set out as the foundation of perspective, in the order in which they follow each other in the text:

- the *proposition fondamentale de la pratique de la perspective* (1648: 336);
- an *autre fondement du trait de la perspective, ensemble du fort et faible de ses touches ou couleurs* (1648: 338);
- the *fondement du compas optique* (1648: 339);
- the *proposition geometrique* today known as the theorem of perspective triangles or homologous triangles (1648: 340).

Hereafter, for reasons of space, we will only examine the first, and postpone to another time an in-depth reading of the entire Desarguesian text.

The Treatises of the Early Seventeenth Century

The year 1600, with the publishing of Guidobaldo del Monte's *Perspectivæ libri sex*, marks the clear separation between the empirical perspective of the beginnings, still heavily influenced by its experimental phase and particularly by the use of the mirror (Migliari and Baglioni 2018), and the perspective which instead makes use of geometric speculations. In particular, in sixteenth-century France, the perspective treatises, from those written by Jean Pèlerin (1505) and, later, by Jean Cousin (1560), up until that of Jaques Androuet du Cerceau (1576), repeat the stylish graphical elements and reasoning based on the principal point, understood as reflection of the viewpoint, and on the third points.

Traces of these ideas, transformed from empirical observations to didactic expedients, can still be see in the plates of Abraham Bosse (Fig. 4), in those characters intent on stretching strings between the vertexes of a geometric figure and their eyes, precisely as Filarete[6] suggested in his *Trattato di architettura* (Averlino 1972).

With the publication of Guidobaldo's treatise the confusion between that which is the eye, namely, the viewpoint or centre of projection, and the main point, that is, the foot of the perpendicular drawn from the eye to the picture plane, was finally dissolved in the light of a rational genesis of perspective. The attention of scholars could turn elsewhere and, in particular, towards three practically new problems:

- how to make the construction of perspective faster and easier;
- how to construct the perspective of any object while keeping all graphical constructions within the picture plane;
- how to reproduce the effects of light on the bodies, that is, the *chiaroscuro* (light and shade), in the same functional way.

In this rush towards the pictorial efficacy of the procedure, the theoretical impulse begun by Guidobaldo seems to wane, at least until it is powerfully renewed by Girard Desargues, as we shall see.

Let us focus on the second of the above problems, which arises from the need to work without recourse to reductions or enlargements. Let us consider, for instance, a table that has the shape of a rectangle placed vertically, as in Bosse's plates, and which is observed from a distance equal to the longest side; it is evident that the

[6] Antonio Averlino, better known as il Filarete, writes that the perspectives of the lines that are perpendicular to the picture plane are the images of *razzi visivi* (visual lightning bolts, Averlino 1972: 652), and, again, he explicitly suggests the use of a mirror to study the perspective structure of a space (Averlino 1972: 653) or also two mirrors, positioned opposite each other (Averlino 1972: 677). As we will see shortly, the dual reflection may have suggested to Desargues the scanning of the depth of space, which he translates into the scale of *éloignements*.

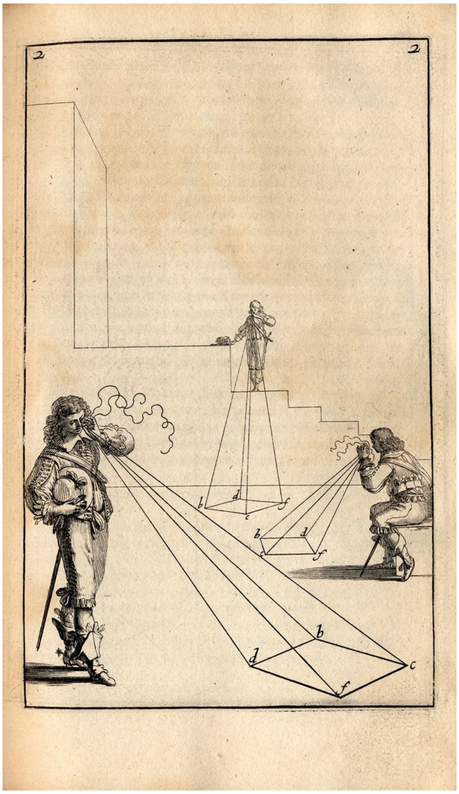


Fig. 4 Legacy of the Renaissance experiment with the mirror, in the second plate of *Manière universelle*. The mirror emphasizes not only the collinearity between a point and its perspective, but also the coplanarity between a line and its perspective; the perspective of lines perpendicular to the picture plane converge into the mirror image of the onlooker's eye

distance point will be situated outside of the perimeter of this rectangle. Therefore, wanting to construct the perspective views with the aid of a distance point, it is necessary to use a drawing support much larger than the picture; alternatively, a smaller sketch can be drawn and then enlarged. The same also happens with horizontal formats. For instance, in the twenty-fifth chapter of the *Perspective* of Salomon De Caus (1612), the wall is 22 feet wide and 14 feet in height and the onlooker, placed at the centre, is about 20 feet from the wall; hence the two distance points are outside of the picture plane, about 9 feet from the edge, left and right.

A first way to solve this problem was suggested by Pietro Accolti (1625) in a treatise dedicated to the "deception of the eyes" which, in its abundance of expedients and problems dealt with, gives a sense of the direction of perspective research in the early seventeenth century. In Fig. 5 is reproduced the original drawing and is accompanied by a scheme that explains the reasoning followed by Accolti.

The triangles BC_1C^* and BEI are similar. Therefore, if you want to draw a perspective of a square whose side EB is situated beyond the picture plane, and where the distance of the onlooker from the picture plane is four times EB, the catheti EI and C_1C^* of the two triangles will stand to each other in the ratio of 1:5, where $C_1C^*=EL$ is the height of the onlooker. Accolti himself, recalling Piero della Francesca, encourages the reader to consider the side AE of the painting as a section

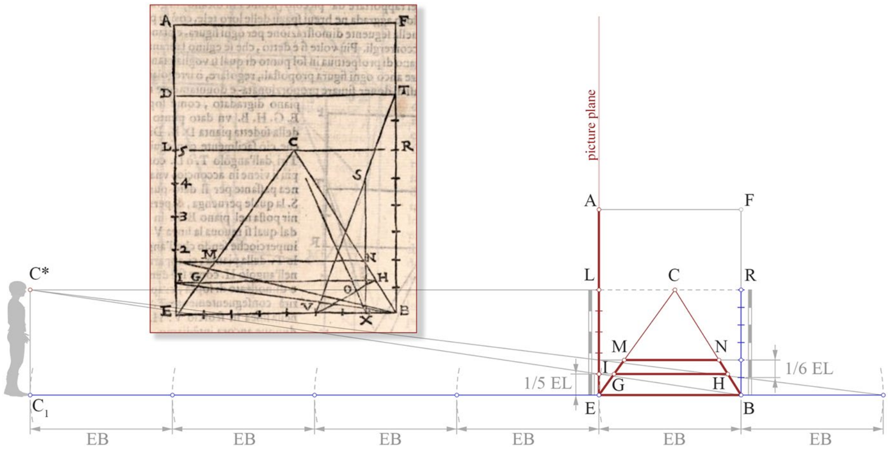


Fig. 5 Pietro Accolti, *Come deva il Pittore, quando non può havere il punto della lontananza sudetta, sonseguire ad ogni modo gli scorci de' piani, & d'ogni altra figura* (What the painter must do, when he cannot have the aforementioned distance point, to achieve in any case the foreshortening of the planes and any other figure). The original figure in page 26, which illustrates the solution of problem in Chapter XIX, is set opposite a scheme which explains the logic of the procedure itself. Image: Accolti (1625: 26)

of the picture plane and the scheme that we have added below as a side view of the perspective square grid. Thus, in practice, if the onlooker is four times *EB* from the picture plane, you divide *EL* into five parts and trace through *I* a parallel to *EB* until it intersects at *G* the perspective *EC* of the side of the square that is perpendicular to the picture plane.

The method used by Accolti began to show signs of replacing the "force of the lines" with arithmetic, to use a *motto* of Piero della Francesca in the incipit of the third book of "De Prospectiva Pingendi", but it is still far from the simplicity of Desargues's *Manière*, as well as from its logic.[7]

We do not know whether Desargues knew Accolti's text, but it is certain that, of all those which precede it and which we will examine more closely later, this is the one that may have "inspired" him. In fact, it exploits the proportional ratios that exists between the onlooker's distance from the picture plane, and the distance from the same picture plane, but on the opposite side, of a line in the ground plane that is parallel to the fundamental line.[8]

As we will see, Desargues starts precisely from this consideration and, in particular, from the line whose distance from the picture plane is equal to that of the onlooker (on the opposite side), a line whose perspective divides into two the stripe

[7] See *Dimostrazione per conseguir l'istesso aritmeticamente con una qualsivoglia immaginata lontananza* (Demonstration to achieve the same arithmetically with whatever distance imagined) (Desargues 1636: 25).

[8] The fundamental line is the line generated by the intersection between the ground plane and the picture plane.

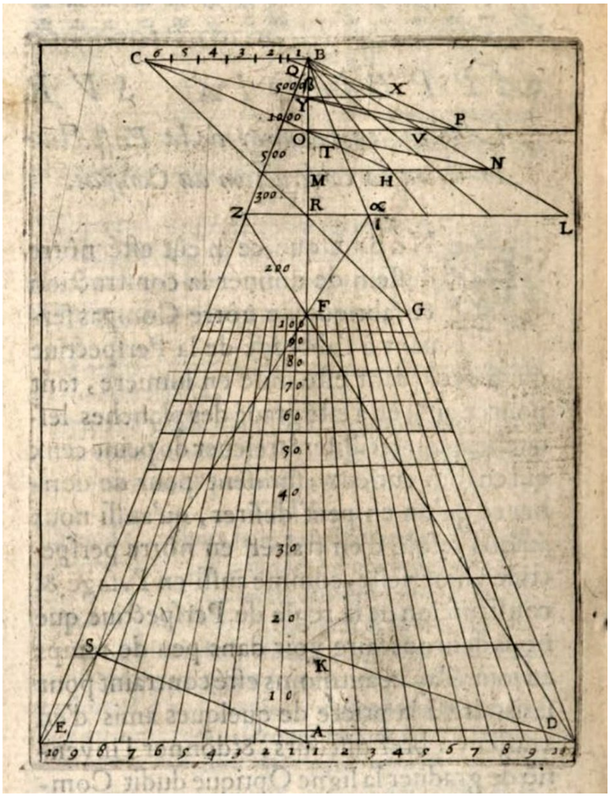


Fig. 6 This illustration from the treatise of Vaulezard, dedicated to the construction of the perspective scales using the proportional compasses, shows the procedure employed for the scanning of the optical line (*ligne optique*) AB. Image: Vaulezard 1631: Appendix, p. 2

of the picture plane which represents the ground plane from the fundamental line to infinity.

A second method is the one followed first by Jean-Louis Vaulezard (1631) and later by Jacques Aleaume (1643), which consists in attributing the ground plane points to two reference axes divided into multiples and submultiples of a unit of measurement, exactly as in the current Cartesian system which, not by chance, was invented in those years by Pierre de Fermat and René Descartes (see Boyer 1980). In perspective (Fig. 6), the axis of the abscissa transforms into the fundamental or any other line of the ground plane that is parallel to the picture plane, while the axis of the ordinate transforms into the perspective of a line that is perpendicular to the picture plane, and whose vanishing point is therefore in the principal point.

It seems evident that the units of measurement defined on the perspective axis of the abscissas are all equal, but those defined on the axis of the ordinates decrease according to the laws of perspective. It is exactly in this "decreasing" and in the manner of constructing it that the various procedures proposed during those years differ. This method is visually analogous to that of Desargues, but very distant as far as the theoretical principles are concerned, because it divides the perspective scale of the ordinates using the distance point. This is what was done by Giacomo Barozzi, called Vignola, whose treatise (Barozzi 1583) was well-known in France, whereas Desargues, like Accolti, used a proportionality that is independent of the distance point.

Finally, it is necessary to mention the contribution of Jacques Aleaume (1643), because, although it was published posthumously in 1643, edited by Estienne Migon, it derives from a manuscript of 1628 (Taton 1951: 50; Laurent 1994) that Desargues might have known[9] and may have "inspired" him, as Kirsti Andersen also writes (2007: 438). Alleaume split the problem into three passages that concern, respectively, the scale of the abscissas, the scale of the ordinates and the scale of the angles, necessary to represent lines that are not perpendicular to the picture plane (Fig. 7). The scale of the abscissas (Migon and Aleaume 1643: 66–73) is shown on the lower edge of the drawing, in multiples and submultiples of the unit of measurement, all equal to each other, since this line is situated on the picture plane. The scale of the ordinates is seen on the lateral edges of the drawing, to the right and to the left, and the perspective decrease is constructed by means of the distance point *K*.

In order to increase the effectiveness of his procedure, Aleaume resorts to a translation of the abovesaid point. As a matter of fact, having determined that *I* is the principal point, *K* would be found outside of the picture plane. Point *K* is instead fixed on the frame and the principal distance shifted from *K* to *N*. The vertical line *MN* now fulfils the same function that, in Accolti's procedure, was carried out by *EA*, namely that of creating a section of the picture plane.

Lastly, the scale of the angles (Migon and Aleaume 1643: 73–76) is indicated on the horizon and each of its divisions corresponds to the vanishing point of horizontal lines that, together with those perpendicular to the picture plane, form a known angle. As a consequence, if the principal point is placed at the centre, the points that the horizon defines on the edges of the picture plane correspond to 45° angles, to the right and to the left, for a total of 90°, which is the aperture of the onlooker's visual cone.

Here we can also notice the discontinuity of the Desarguesian conception of perspective, similar to a sudden leap in time, because the methods that precede it are still embodied in the precepts of Renaissance perspective, while Desargues's conception lives in a space which expands at infinity. For these reasons, which we will better explain shortly, we cannot share the opinion of Kirsti Andersen,who doubts that the idea of geometrical entities at infinity may have been suggested to Desargues by perspective; we agree even less when she attributes to him the will to eliminate the vanishing points, whichever they may be, from his perspective (except

[9] Historical criticism seems anxious to demonstrate the originality of Desargues's method, founded on perspective scales, compared to all the others. In our opinion, however, these efforts are vain because what makes the contribution of Desargues original and incomparable is certainly not the practical procedure, which, moreover, led to nothing, but rather the theoretical content and, particularly, the fundamental theorem on which this present study is focused. This theorem, in fact, embodies the generality of the principles of modern perspective.

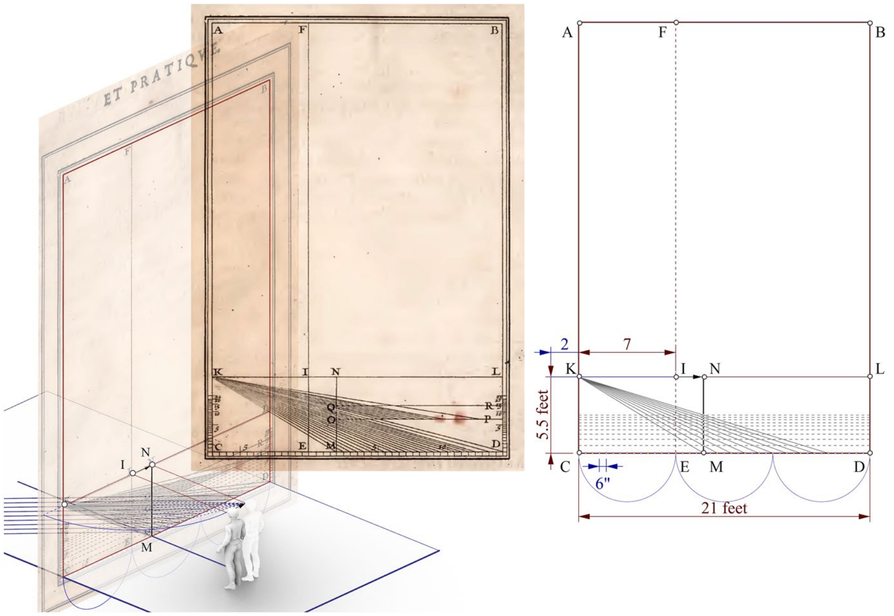


Fig. 7 The construction of the perspective ordinates of Jacques Aleaume. Image: Migon and Aleaume (1643: figure on p. 71), with authors' overlay

for those perpendicular to the picture plane), because the points in which converge the perspectives of lines used for the measurement of space, in practical procedures as well as in the high theory of the fundamental proposition, are vanishing points.[10]

Desargues's Perspective of 1636

As we have already recalled, Desargues became part of the competition for the best practical method of perspective, in 1636, with the booklet entitled *Exemple de l'une des manières universelles* (Desargues 1636). But it should not be forgotten that in the final two pages, after a line of decorative signs that create a strong demarcation, Desargues enunciates the four emblematic cases which we have mentioned above, which are usefully recapitulated here. Given C any sheaf of converging lines, P any star of parallel lines, C' a flat pencil of converging lines and P' a flat pencil of parallel lines, and adopting $\leftrightarrow$ as a symbol of bi-univocal correspondence, then the following cases may arise:

[10] "In pursuing a technique of perspective construction that does not involve any points outside the picture frame, Desargues not only disposed of the vanishing points outside the frame, but also of those inside – apart from the principal vanishing point" (Andersen 1996: 433).

$$P \leftrightarrow P'$$
$$P \leftrightarrow C'$$
$$C \leftrightarrow P'$$
$$C \leftrightarrow C'$$

Note that this analysis only indirectly justifies this practical procedure. It has a much wider value, as Desargues himself observes:

> *En ce reste de place les contemplatifs auront quelques propositions lesquelles peuvent étre enoncées autremente pour les diverses matières, mais elles sont accomodées ici pour la perspective …* (In the space that remains [i.e., on the printed page] thinkers will have some propositions that can be enunciated differently in the various disciplines, but here they are adapted to perspective…) (Desargues 1636: 11).

What Desargues calls "adaptation" is in reality a justification or geometrical demonstration of the existence of the above-mentioned correspondences, which are, however, based on the postulates relative to the *ordonnances* of lines and planes that Desargues would enunciate only three years later, in the *Brouillon Project*. It seems to us that this observation can settle the question raised by some, namely whether Desargues deduced the idea of points and lines at infinity from perspective or whether, instead, he conceived them in an abstraction of thought, and then acknowledged them in perspective. If this question ever had any relevance, in our opinion there is no doubt that the idea was born in the context of perspective, because only by admitting the idea of point at infinity is it possible to explain why, for example, $C \leftrightarrow P'$, that is, why a sheaf of converging lines can transform into a flat pencil of parallel lines.

It should also be noted that in fact this case is not common in perspective, but occurs, for instance, when representing a circle (or any conic). Moreover, Desargues must have been well aware of this, too, as he concludes his writing with these words:

> *La proposition qui suit ne sè devide pas si brièvement que celles qui precedent. Aiant à pourtraire une coupe de cone plate, y mener deux lignes, don't les aparences soint les essieux de la figure qui la representera"* (Desargues 1636: 12).
>
> (The proposition that follows is not explained so briefly, like the ones that precede. Having to represent the plane section of a cone, draw to this section two lines whose perspectives are the axes of the figure that will represent it.)

Those who know geometry will say, "Of course!" We are sure of that.

Anyone who has a deep knowledge of geometry cannot fail to be astonished at the leap forward taken by these propositions to perspective and more generally to science.

Now it is time to analyse, briefly, the "practice" of perspective in the year 1636. The complete opposite of the treatises of the same period, *L'exemple* is of a disarming simplicity and concision: one single plate, divided into three figures, and the description of the operations, all actually executable, which lead to the

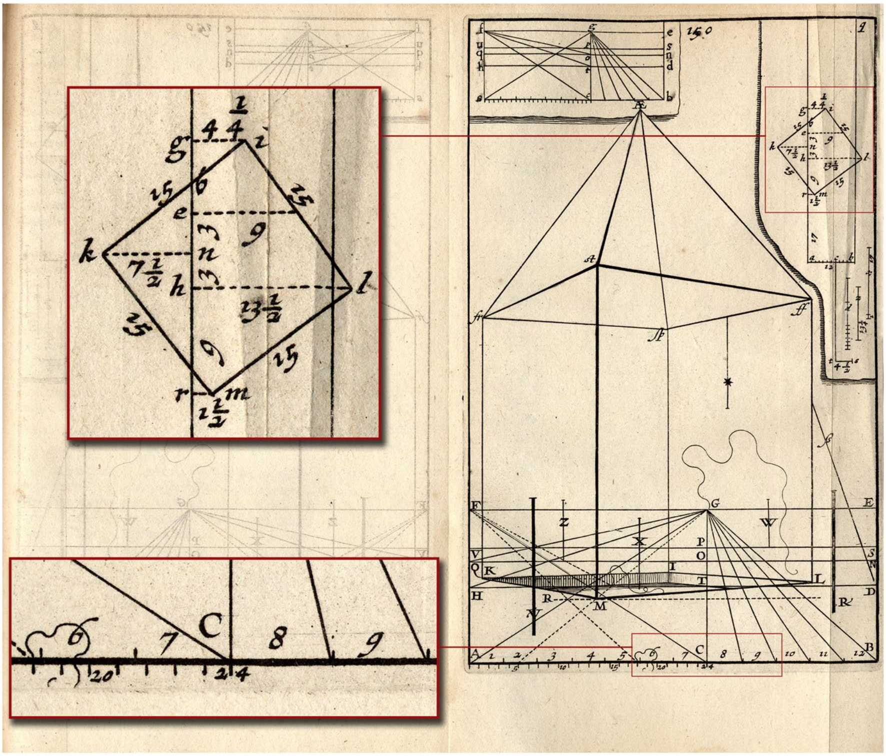


Fig. 8 The only plate that illustrates Desargues's (1636) *Perspective*. On the left, scaled up by a factor of three, the detail of the overlapping of the two superimposed scales, the measurements and depths. Image: Desargues (1636: title page)

result. This is in truth an algorithm that does not contemplate justifications, much less proofs of a mathematical nature. As such, it could be put into practice, blindly, even by someone who had no knowledge of geometry. We refer those who would like to study it in detail to the reading of the original text. Here we just want to highlight some of its noteworthy aspects. First, the reader should be informed about the use of symbols, which employ three different styles, with different references (Fig. 8).

The three figures that appear in the plate represent a plan (top, right), a perspective of the ground plane alone (top, left) and the completed perspective on the remaining part of the sheet. Desargues uses:

- lowercase italic letters such as *a*, *b*, *c* for the plan;
- lowercase Roman letters such as a, b, c for the perspective of the ground plane, reproduced, in a smaller scale, at the top left;
- uppercase Roman letters such as A, B, C …, for the actual perspective.

The same letter indicates the same point in all three representations. In this way there is no need for continual clarifications because it is known, for instance, that A and a are perspectives of point *a*.

The base of the picture AB is two *toises* long (12 feet). The distance of the onlooker from the picture is 4 *toises* (24 feet), and he is 4 1/2 feet tall. The first operation consists in dividing the base of the picture plane AB into twelve parts, each of which, hence, will represent one foot. Desargues call this base the *éschelle des mesures* (scale of measures). Note that the reduction ratio that ensues from this first operation is not given a priori, it is a consequence of the physical dimension of the picture. If a wall were to be painted using this method, there would be no need for any reduction; the base would be divided into as many feet as it is long, at full scale.

Then we would take 4 1/2 parts and draw the vertical segment CG which measures the height of the viewpoint, in proportion to the base of the picture plane. According to the value of the French foot of that time, reported by Agostino Tacchini (1895) as 1 foot = 32.5 cm. This measure corresponds to about 146 cm.

Point G is the principal point, the vanishing point of the lines that are perpendicular to the picture plane: therefore line AG is the perspective of the *ligne indeterminèe* (indeterminate line) ag (Bosse 1648: 324).

At this point, Desargues draws the horizon FE and divides the space between the base AB and the horizon into two equal parts, using the line HD. As the experiment with a mirror teaches (Fig. 9) (but so does a simple geometrical scheme), the lower part of this division represents a depth that is equal to the distance of the viewer from the picture plane, and thus the segment that line HD defines on the perspective AG, on the side of point A, is the perspective of a segment that is 24 feet long, since this is the distance previously established.[11]

In plan Desargues associated this point to the letter *h*, but the corresponding point H, on the perspective, is not situated where it should be, that is, on line AG, but rather on the edge of the picture plane. In order to distinguish it, we will call it H_h.

This observation, as simple as it is brilliant, makes it possible to visually examine the depth of the perspective, without making use of the distance point, which would be situated outside of the picture plane, and instead to draw in complete freedom. The procedure is as follows.

- Define on the horizon any point, for example F, which here is found exactly on the edge of the picture plane, and construct the line FH_h, which intersects the base of the picture plane in point C.[12]

[11] The progression continues following the series 1/2, 2/3, 3/4, 4/5, … $n/(n+1)$ where to each element of the sequence corresponds.
a further step, deeper, equal to the distance of the onlooker from the picture plane (XXX).

[12] This is not a coincidence, because AH_h is equal to H_hG through construction and therefore AFGC is a rectangle. Actually Desargues constructs this rectangle in order to divide into two the area between the fundamental and the horizon, but, to us, it seems simpler to free the procedure from this constraint. If, for instance, point F were not situated on the edge of the picture plane, but moved to the right, on the horizon, line FH_h would have intersected the fundamental, or better, the base of the picture plane, in another point, without invalidating the procedure that follows.

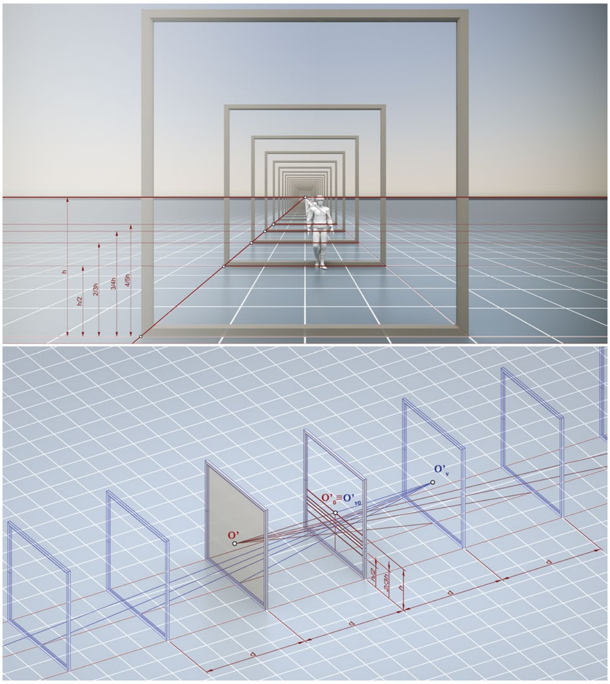


Fig. 9 Antonio Averlino, known as Il Filarete (1464), had already observed the game of opposing mirrors which shows a scan of the perspective space, identical to the one used by Desargues. Image: Authors

- Divide the segment AC into 24 parts and, if necessary, into submultiples. Desargues calls this division the "scale of depths" (*éschelle des éloignements*). In fact, the lines that are drawn from point F and intersect these divisions define, on line AG, the points which divide segment AH_h into a perspective scale that can be used to place the perspectives of the points of the ground plane in their correct position, as may be seen, for instance, in the case of point R.

For our purposes here, there is no need to go further in the examination of the procedure which, at this point, is obvious and repetitive.

The invention of point F, however, is by no means commonplace; it is the root cause which led Desargues to write this brief work and, at the same time, the reason that inspires the fundamental theorem added to the 1648 edition. In fact, point F is not a distance point, because FG is not equal to the distance of the onlooker from the picture plane. The scale of the ordinates (scale of depths), which is associated with this point, is quite distinct from the scale of the abscissas (scale of measures), even if being overlapped. In this case, for instance, to 24 feet of the scale of depths, correspond only 7 feet of the scale of measures.

But what, then, is this point *F*, which Le Goff (1994) defined, intelligently, as a "pseudo-distance" point? Clearly it is the vanishing point of the perspective of a flat pencil of parallel lines that measures off, on the line ag and on the fundamental ab, segments which stand to each other in a known ratio, that of the two aforementioned scales. If we had used the distance point instead of point *F*, this ratio would have been equal to 1 and the scale of depths would have coincided with the scale of measures. Here, instead, the scale of depths consists of smaller units, but so as to obtain the same effect that would have been achieved if the measurement point were used; nowadays, it is customary to say that points like *F* are "reduced measurement points". In particular, and in the case that we are considering, the ratio between a foot in the scale of measures and a foot in the scale of depths is equal to the ratio between the distance of the onlooker from the picture plane and the length of segment FG, namely 7:24, as everyone can read down at the bottom of the picture, in the comparison between the two scales.

All of this justifies, we believe, the addition of the fundamental theorem to the text of *Manière universelle* but at the same time it indicates the aim, which is all in the geometrical demonstration, irreproachable, of the procedure that uses reduced measurement points.

However, as we shall see shortly, there is much more to this theorem, because the validity of the process is demonstrated not only with regards to the flat pencil of lines that belong to the ground plane, but with regard to those that belong to any planes. In so doing, Desargues also shows the correct representation of one of these planes and of their intersection line, anticipating Brook Taylor (1715) and perspective in its present form.

The Fundamental Theorem of Perspective

The first analysis of fundamental theorem can also be merely visual. In fact, if we use, for simplicity, the current language and conventions, we can identify in line *kag* and *chs*, respectively, the vanishing point and trace of a horizontal plane.[13]

Thus, the points *k*, *a* and *g*, which belong to the horizon, are vanishing points. In Fig. 2 can be seen a second plane, which is visibly inclined provided that the first is horizontal: this is the plane that has *Kg* as its vanishing point and *cTS* as the trace. We have written with capital letters the homonymous points that belong to this plane. Desargues makes no distinction, because the relations that he determines and demonstrates are valid both for the first and for the second plane; this highlights the generality of the theorem. Nevertheless, as we shall see, this distinction is useful.

It is clear that line *cg* is the intersection of the two planes and has its vanishing point in *g* which is a point in common to the two vanishing lines, and trace in *c*, which is a common point to the two traces.

It is interesting to note that, in the plate that illustrates the 1636 *Perspective*, the line *cg* is the perspective of a line that is perpendicular to the picture plane; in particular, it is the one that Vaulezard called *ligne optique* (Vaulezard 1631: 38).

[13] From here on we will use the term "trace" to indicate the straight line of intersection between a plane and the picture plane, or also, the point of intersection between a straight line and the picture plane. About this convention, see Luigi Cremona (1893).

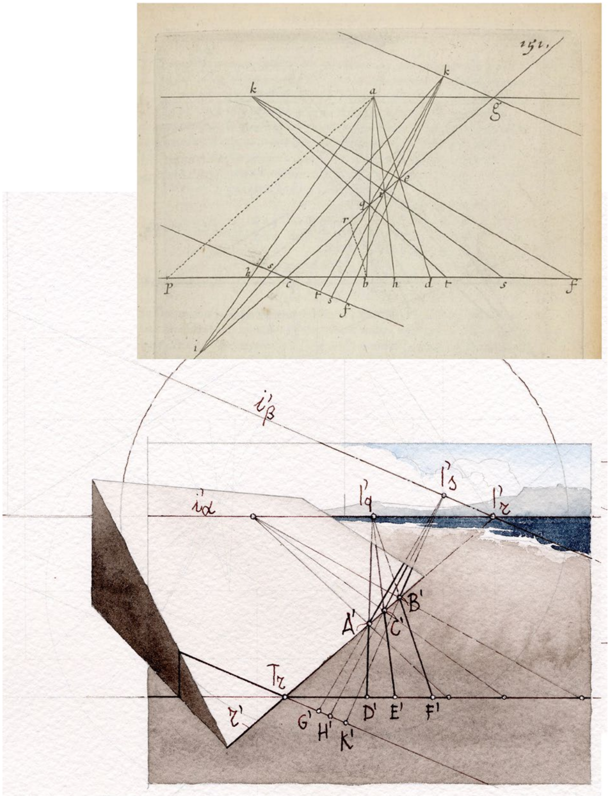


Fig. 10 Desargues's fundamental theorem of perspective in the original drawing (above) and in the one that highlights the perspective characteristic of it (below). This interpretation of the original figure is analogous to that of Fig. 1; here, however, we consider another possibility, namely that line *r* is not perpendicular to the picture plane. Image: Desargues and the authors

In this case there is no reason to consider *cg* as perspective of a perpendicular to the picture plane because the theorem does not require this condition, as shown in Fig. 10.

However, it is useful, for the purpose of our exposition, to give a true dimension to the segments which belong to cg, understood as a line of the real space. To obtain this result we will formulate the following hypothesis (not necessary, but opportune):

- that *g* is the vanishing point of a line of the ground plane perpendicular to the picture plane;
- that *k* is a distance point;
- and, consequently, that segment *kg* measures the distance of the onlooker from the picture plane.

If we accept this hypothesis, the segments *ts*, *sf* measure, at true scale or at the scale of the drawing, the segments *qi* and *ie*, which Desargues identifies on perspective *cg*. This means that the segments *qi* and *ie* in space are as long as *ts* and *sf*, respectively.

Still within the context of this hypothesis, point a represents the vanishing point of lines capable to define on *cg* in space, and on the trace *chs*, segments that stand to the real ones as *kg* stands to *ag*. Since in the figure the ratio *ag*:*kg* is equal to about 12, segments *bh* and *hd* that the lines *aq*, *ai*, *ae* detach on *chs* measure about one-half of the segments *ts* and *sf*, respectively.

Thus, point *a* can be read as a "reduced measuring point". Points of this type are employed, in present-day perspectives, to reduce the graphical measuring constructions within the page of the drawing, which is exactly the use that Desargues makes of it. This is the reason why this fundamental proposition could be called "Theorem of the measurement of perspective space".

We would like to reiterate that the hypothesis we have formulated is not necessary, but it explains in a simple and immediate way, why the ratios that Desargues describes are consistent.

Turning to point *K*, which in the original figure is marked with the lowercase letter, it is again a vanishing point intended to take a measure, and still on line *cg*, but using as a support a different plane and a generic position. That said, the reading of the theorem should be easier. Referring to the original Fig. 151 (Figs. 2 and 10), the theorem states, very briefly, the following:

> Given two pairs of lines parallel one to another, such as *ag* ∥ *cbhd* and *kg* ∥ *ctsf*[14]; and considering the line determined by the two points that these pairs have in common, namely *g* and *c*; and defined on this line two points such as *e* and *q*; and drawn from the points *a* and *k*, respectively, two pairs of lines *aed*, *aqb* and *kef*, *kqt*, then:

1. These two pairs identify, respectively, on line *cbhd* and on line *ctsf*, segments that, if taken in the same order, form, between them, the same ratios, and precisely:

$$\frac{cd}{cb} = \frac{cf}{ct}; \frac{cd}{db} = \frac{cf}{ft}; \frac{cb}{db} = \frac{ct}{ft}.$$

   This means that, if between two segments of the line that has as its image *cg* exists a certain ratio, this same ratio exists among their measures, both if they are true to life, like those that in our hypothesis measure point *k*, and if diminished, like the ones that measure point *a*; pay attention, however, not to confuse the true segments with their image, because the perspective notoriously does not retain the metric properties, except for the cross-ratio, which we will deal with shortly.
2. In each pair, each segment stands to the corresponding segment of the other pair as *ga* stands to *gk*, namely:

[14] Where ∥ means parallelism. Note that Desargues, to identify a line, does not simply use two points, which are sufficient to define it, he often indicates other points that are collinear and whose role will be clarified hereafter.

$$\frac{ga}{gk} = \frac{bh}{ts} = \frac{hd}{sf}$$

$$\frac{ga}{gK} = \frac{bh}{TS} = \frac{hd}{SF}.$$

Now, it is clear that if segment *bh* measures the length of the true segment that has as its image *qi*, reduced by half, for instance, and segment *ts* gives the entire measure if the aforesaid true segment, the ratio between the two will be equal to 1:2. This ratio would not change even if we had to deal with two measures, in their turn bound together with the true one by the same ratio of enlargement or reduction; the aforementioned ratio also exists between the segments *ag* and *gk*, as the theory of measuring points wants. It would be arduous to summarise this here, but it is illustrated in Fig. 11, which shows, on the ground plane and in space, how the flat pencils of parallel lines take these measures. It is important to note that the aforementioned ratios also apply to the pairs that are indicated with capital letters, and to the points that belong to the inclined plane.

3. Again, the pairs *cd*, *cb* and *cf*, *ct* stand to each other as the product of the ratios between the segments *ec*, *eg* e *qg*, *qc*; again, the ratio of the segments *db*, *eq* is equal to the product of the ratios between segments *dc*, *ec* and *ab*, *aq* and, that is:

$$\frac{cd}{cb} = \frac{cf}{ct} = \frac{ec}{eg} \cdot \frac{qg}{qc}$$

$$\frac{db}{eq} = \frac{dc}{ec} \cdot \frac{ab}{aq}.$$

As Maura Boffito (1989) rightly remarked, this is what Desargues calls la *composée des raisons*, namely the anharmonic ratio; in fact, the product of the ratios

$$\frac{ec}{eg} \cdot \frac{qg}{qc}$$

is equal to the cross-ratio

$$\frac{ec}{eg} : \frac{qc}{qg}.$$

Using Oscar Chisini's notation (1967), this cross-ratio can be written in the form *egcq*, which is equivalent to *cqeg*.

4. Lastly, the ratio between the segments *iq* and *ie* is equal to the product of the following ratios

$$\frac{aq}{ab}; \frac{hb}{hc}; \frac{hc}{hd}; \frac{ad}{ae},$$

namely,

$$\frac{iq}{ie} = \frac{aq}{ab} \cdot \frac{hb}{hc} \cdot \frac{hc}{hd} \cdot \frac{ad}{ae}.$$

As a matter of fact, we think that notable among these five ratios are the first two, which concern precisely perspective and the possibility of measuring, with great liberty, the perspective space. It is true that others involve the cross-ratio but they do not directly reveal its invariance in central perspective, a concept which will later be developed by Michel Chasles (1852) (Boffito 1989: 153). Indeed, the first two ratios validate what we have exemplified in our first hypothesis, and that is: if *g* is the principal point, *k* a distance point and *a* a reduced measurement point in the ratio 1:*n*, then segments *ts*, *sf* measure the segments *qi*, *ie* of the true-to-life perspective, whereas segments *bh*, *hd* are of the same dimensions reduced by 1:*n*.

Conclusions

One last crucial question remains: is it possible that Desargues, to explain his fundamental theorem, drew such a perfect perspective, without seeing what it represents? Without realizing that the two pairs of parallel lines represent two planes, and that line *cg* is their intersection?

The doubt is legitimate, since this perspective, read in the manner we said, is far ahead of the ones drawn by the best geometricians of the time, and which, in comparison, look like rough prototypes. We believe that to be able to answer this question, we should consider Desargues himself, in so far as this is possible on the basis of current knowledge. For sure Desargues was an exceptional personality capable of abstract thoughts, and also of an overall view of the design process which have descended into the real and contemporary world. This is shown by the last lines of the *Brouillon Project*,[15] where the idea of projective enlargement of Euclidean space is applied to perspective, gnomonics and stereotomy.

Kirsti Andersen has constructed a well-developed argument about Desargues's creative process, in which she seems to want to claim the priority of the abstraction over that of the application. Thus, regarding the relationship which, in the above-said process, may have existed between the perspective idea of a vanishing point and the idea, totally abstract, of the point at infinity Andersen

[15] *"*30.8* Du contenu dans ce Brouillon il résulte que: Touchant la Perspective. *30.9* Des droictes sujet d'une quelconque même ordonnance, les apparences au tableau plat sont doctes d'une même ordonnance entre elles, e celle de l'ordonnance des sujet qui passe à l'oeil, la quelle est l'essieu de l'ordonnance d'entre les plans de l'oeil e de chacune de ces droictes sujet. *30.10* Touchant les Monstres de l'heure au Soleil. *30.11* En quelconque surface plate, les droictes des heures sont d'une même ordonnance entre elles et l'essieu de l'ordonnance d'entre les plans qui donnent la division de ces heures. *30.12* Touchant la coupe des Pierres de taille. *30.13* En une même face de mur les arestes droictes de pierres de taille sont communicant d'une même ordonnance entre elles et l'essieu de l'ordonnance d'entre les plans des joincts qui passent à ces arestes".* The passage is taken from the paragraphed transcription carried out by Valeria Talarico (2017).

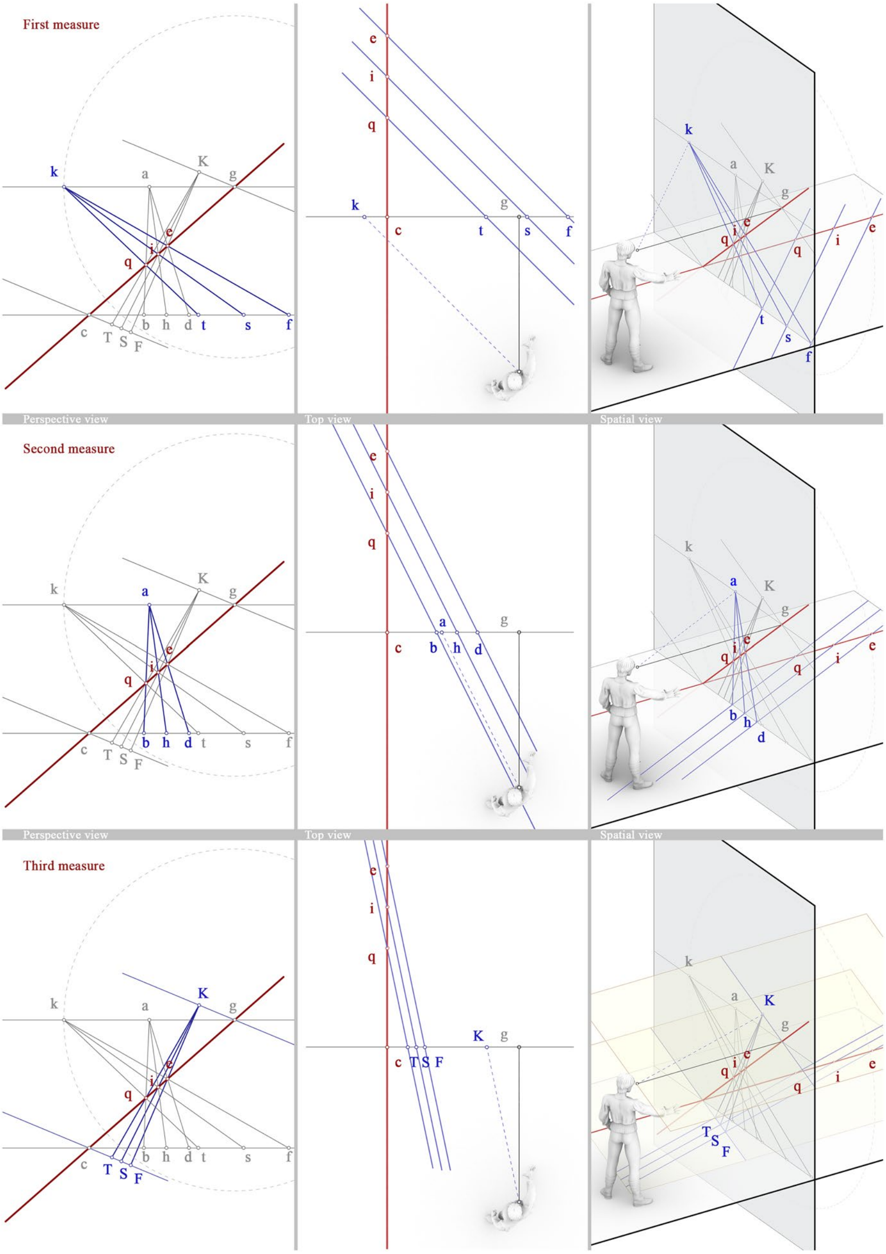


Fig. 11 The three different measurement operations present in plate 151 aim to demonstrate the universality of the fundamental Theorem. Image: authors

concludes: "… he did not use the concept of vanishing points in his arguments, and I therefore do not think that concept inspired him to introduce points at infinity" (Andersen 2007: 444).

Nevertheless, the last line of the *Brouillon Project* cited above, sounds not only like an invitation to develop the applications of an innovative theory, but also like a demonstration of the validity and plausibility of that idea. It seems to us, therefore, that the question of priorities allows different answers, all valid, because if it is acceptable that the idea of a point at infinity originated even independently from perspective, it is equally true that in perspective it found its logical and visual evidence.

It is this evidence that makes the fundamental "theorem of measurement, the subject of this study, immediate in its interpretation as it is abstruse in the mathematical proof. Hence our hypothesis is that Desargues imagined, without any uncertainty, the figure that appears here in the incipit and that he did not go further in commenting it simply because the culture of the time was not ready for such a revolutionary development.

Acknowledgements This work was conducted jointly by the two authors who completely shared the methodology and the content of the research. The authors wrote the paragraphs as follows: Riccardo Migliari *Introduction and synthesis*, *Finite and Infinite*, *The treatises of the early seventeenth century*; Leonardo Baglioni *Perspective of 1636*, *The fundamental Theorem of Perspective*, *Conclusions*. The figures have been edited as follows: Figs. 1, 3, 10 watercolor drawings by Riccardo Migliari; Figs. 5, 7, 8, 9 and 11 3D drawings by Leonardo Baglioni.

Funding Open access funding provided by Università degli Studi di Roma La Sapienza within the CRUI-CARE Agreement.

References

Accolti, Pietro. 1625. *Lo inganno de gl'occhi, prospettiva pratica di Petro Accolti gentiluomo fiorentino*. Firenze: Pietro Cecconcelli.

Andersen, Kirsti. 1996. Piero's Place in the History of Descriptive Geometry. In: *Piero della Francesca tra arte e scienza*, ed. M. Dalai Emiliani, 363–374. Venezia: Marsilio.

Andersen, Kirsti. 2007. *The Geometry of an Art. The History of the Mathematical Theory of Perspective from Alberti to Monge*. New York: Springer.

Androuet du Cerceau, Jacques. 1576. *Leçons de Perspective Positive*. Paris: Mamert Patisson.

Angelini, Annibale. 1861. *Trattato teorico e pratico di Prospettiva*. Roma: Enrico Sinimberghi.

Averlino, Antonio [Filarete]. 1972. *Trattato di Architettura* (1460). Milano: Il Polifilo.

Barozzi, Jacopo da Vignola. 1583. *Le due regole della prospettiva pratica di M. Jacomo Barozzi da Vignola. Con i commentari del R.P.M. Egnatio Danti dell'ordine de Predicatori, Matematico dello Studio di Bologna*. Roma: Francesco Zannetti.

Boffito, Maura. 1989. *Dentro la geometria*. Genova: C. D. S.

Bosse, Abraham. 1648. *Manière universelle de Mr. Desargues, pour pratiquer la perspective par petit-pied, comme le géométral*. Paris: De l'imprimerie de Pierre Des-Hayes.

Boyer, Carl B. 1980. *Storia della matematica*. Milano: Arnoldo Mondadori.

Chasles, Michel. 1852. *Traité de Géométrie Supérieure*. Paris: Bachelier.

Chisini, Oscar. 1967. *Lezioni di geometria analitica e proiettiva*. Bologna: Zanichelli.

Cousin, Jean. 1560. *Livre de Perspective de Jean Cousin*. Paris: Iehan le Royer Imprimeur du Roy és Mathematiques.

Coxeter, Harold Scott MacDonald. 2003. *Projective Geometry,* 2nd ed. New York: Springer.

Cremona, Luigi. 1893. *Elements of Projective Geometry*. Oxford: Clarendon Press.

De Caus, Salomon. 1612. *La Perspective avec la raison des ombres et miroirs*. Londres: Jan Norton.

Del Monte, Guidobaldo. 1600. *Guidi Ubaldi e Marchionibus Montis Perspectivæ libri sex*. Pisauri.

Desargues, Girard. 1636. *Exemple de l'une des manières universelles du S. G. D. L. touchant la pratique de la Perspective sans employer aucun tiers point, de distance ny d'autre nature, qui soit hors du champ de l'ouvrage*. Paris: Bidault.

Desargues, Girard. 1639. *Brouillon project d'une atteinte aux evenemens des rencontres du cone avec un plan, par L, S, G, D, L.* Paris.

Fiedler, Wilhelm. 1873. *Trattato di Geometria Descrittiva del dr. Wilhelm Fiedler professore nella Scuola politecnica federale di Zurigo, tradotto dall'ingeg. Antonio Sayno ... e dal dott. Ernesto Padova*. Firenze: 1873.

Field, Judith Veronica. 1997. *The Invention of Infinity: Mathematics and Art in the Renaissance*. New York: Oxford University Press.

Laurent, Roger. 1994. La Perspective et la rupture d'une tradition. In *Desargues en son temps*, ed. Dhombres Jean, Sakarovitch Joël, 231-243. Paris: Albert Blanchard.

Le Goff, Jean-Pierre. 1994. Desargues et la naissance de la "géométrie projective". In *Desargues en son Temps*, ed. Dhombres Jean, Sakarovitch Joël, 157-206. Paris: Albert Blanchard.

Migliari, Riccardo, Baglioni Leonardo. 2018. *Lo specchio alle origini della prospettiva*. Disegnare, idee immagini 56: 42–51.

Migon, Estienne and Jacques Aleaume. 1643. *La Perspective Speculative et Pratique, ou sont demonstrez les fondemens de cet Art et tout qui en esté einsegné jusqu'à present. ... De l'invention du feu Sieur Aleaume, Ingegnieur du Roy*. Paris.

Pèlerin, Jean [Viator]. 1505. *De artificiali perspectiva*. Toul: Petrus Jacobus.

Romor, Jessica. 2013. La prospettiva tra Novecento e albori del terzo millennio, Vol. II. In: *Geometria Descrittiva e Rappresentazione Digitale. Memoria e Innovazione*, ed. Casale Andrea, 101–118. Rome: Edizioni Kappa.

Tacchini, Agostino. 1895. *La metrologia universale ed il codice metrico internazionale*. Milan: Hoepli.

Talarico, Valeria. 2017. Il "Brouillon Projet d'une atteinte aux événements des rencontres du cône avec un plan" e la sua ricaduta sulla storia della prospettiva e sulla geometria descrittiva. Rome: Sapienza University of Rome.

Taton, René. 1951. *L'Œuvre matematique de G. Desargues, Textes publiés et commentés avec une introduction biographique et historique*. Paris: Presses Universitaires de France.

Taylor, Brook. 1715. *Linear Perspective or, a New Method of Representing justly all manner of Objects as they appear to the Eye in all Situations*. London: R. Knaplock.

Vaulezard, Jean Louis de. 1631. *Abrégé ou racourcy de la perspective par l'imitation, dans lequel est traicté du moyen de changer une perspective en une autre semblable [...] ensemble l'invention d'approprier deux ou plusieurs perspectives, ou parties d'icelles en une mesme, et souz une mesme distance d'oeil*. Paris: Chez l'Auteur.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Leonardo Baglioni Architect and PhD is Associate Professor for the scientific discipline ICAR/17 at Sapienza University of Rome, where he teaches in the courses of *Drawing and Fundamentals and Applications of Descriptive Geometry* in the Faculty of Architecture. Since 2017 he has been the scientific coordinator of the Laboratory of Innovation for the Detection, Representation and Analysis of Architecture (LIRALab) in the Department of History, Representation and Restoration of Architecture.

His research is mainly focused on descriptive geometry, its history and its applications, with particular interest in its renewal by means of digital representation methods. In 2006 and 2007 he was part of the interdisciplinary research group for the publication of the National Critical Edition of the treatise "*De prospectiva pingendi*" by Piero della Francesca, promoted in 2010 by the "Piero della Francesca Foundation.

Riccardo Migliari began his academic activity in 1972, after having obtained a degree in Architecture. He has carried out teaching and research activities at various universities in Italy and abroad. In 1990 he obtained the title of full professor for the chair of *Descriptive Geometry* and in 2018, after retirement, the title of professor emeritus. His research concerns perspective and descriptive geometry in theoretical foundations and application developments using information technology. He has published many essays and some volumes on these topics, the last of which is "*Perspective*", in collaboration with Marco Fasolo, published by Hoepli in 2022.