Methodology and Computing in Applied Probability (2024) 26:37
https://doi.org/10.1007/s11009-024-10099-6

RESEARCH

# Pricing and Hedging Contingent Claims by Entropy Segmentation and Fenchel Duality

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Received: 1 November 2023 / Revised: 24 July 2024 / Accepted: 29 July 2024


**Abstract**
We present a new approach to the problem of characterizing and choosing equivalent martingale pricing measures for a contingent claim, in a finite-state incomplete market. This is the *entropy segmentation* method achieved by means of convex programming, thanks to which we divide the claim no-arbitrage prices interval into two halves, the buyer's and the seller's prices at successive entropy levels. Classical buyer's and seller's prices arise when the entropy level approaches 0. Next, we apply Fenchel duality to these primal programs to characterize the hedging positions, unifying in the same expression the cases of super (resp. sub) replication (arising when the entropy approaches 0) and partial replication (when entropy tends to its maximal value). We finally apply linear programming to our hedging problem to find in a price slice of the dual feasible set an optimal partial replicating portfolio with minimal CVaR. We apply our methodology to a cliquet style guarantee, using Heston's dynamic with parameters calibrated on EUROSTOXX50 index quoted prices of European calls. This way prices and hedging positions take into account the volatility risk.

**Keywords** Convex programming · Fenchel duality · Entropy · Finance · Cliquet guarantee

**Mathematics Subject Classification (2010)** 91-10 · 97M30

## 1 Introduction

### 1.1 Scope of the Paper and Motivation

In an incomplete market with a finite number of states and one time period $[0, T]$, we consider a contingent claim with payoff $X$ at $T$. The Fundamental Asset Pricing Theorem (see for instance Avellaneda and Laurence (1999)) states the equivalence between the *no-arbitrage hypothesis* and the existence of an Equivalent Martingale Pricing Measure (EMPM) such that

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Published online: 17 September 2024

the price is the claim payoff expectation discounted at the risk-free rate. In a complete market, this pricing probability measure is unique, while in an incomplete one, there is an infinite family of EMPM giving rise to an infinity of prices. The search for key ideas to characterize and calculate any of those EMPM is of the utmost importance in Financial Mathematics literature. And equally important is the research on static hedging methodologies designed to minimize the risk supported by the claim issuer once the price has been fixed. This paper is firstly devoted to the explanation of a new methodology for characterizing and calculating each of these EMPM for $X$, and secondly, to deduce from the statement of that pricing step an expression for the hedging positions associated with each no-arbitrage price. The first (pricing) step is fulfilled thanks to the methodology of *entropy segmentation*, while the second (hedging) step is developed by applying *Fenchel duality* to the former.

### 1.2 Literature Review

Given a contingent claim $X$ and some benchmark assets traded in the market, the upper and lower extremes of the interval of no-arbitrage prices of $X$ are called *seller's* and *buyer's prices*. In a one-time period and finite state market, these prices can be calculated by means of two linear programs (see for instance Musiela and Rutkowski (2005), p. 95). The solutions to their respective dual linear programs are the *super* and *sub-replicating portfolios* made of those benchmarks (see also Luenberger (2002)). A great deal of interest has been devoted to these positions, especially the super hedging one. In this line of research Bertsimas and Popescu (2002) and Kahale (2017) find methods based on convex programming in discrete time to calculate the price/hedging positions in terms of other derivatives, and King et al. (2005) studies the hedging problem by means of stochastic programming and conjugate duality. This problem of finding a *super(sub)-hedging position* has also been studied for particular options. Pennanen (2011) finds buyer's and seller's prices for American options applying convex optimization, and d'Aspremont and El Ghaoui (2006) finds model-independent upper/lower bounds for European basket options. Also, Antonelli et al. (2013) use linear programming to calibrate buyer's and seller's prices for American options. Unfortunately, the seller's price/super-replicating portfolio position is generally too much expensive and remains of no practical use.

There exist also criteria to choose a specific EMPM driving to a price in the interior of the no-arbitrage interval, for instance, the minimal variance (Follmer and Schweizer 1991), and the minimal martingale measure (Frittelli 2000). There is also the interpretation of no-arbitrage prices in terms of Expected Utility Theory, which tells that in the case of a one-period finite market model, no-arbitrage prices are given by different EMPM pointed out by different concave utility functions of investors solving portfolio optimization problems, see Carr and Zhu (2018). This is a result of Convex Duality Theory applied to Financial Mathematics. For any material on Convex Analysis, Convex Optimization, and Fenchel Duality Theory we refer to Rockafellar (1970) and Mordukhovich and Nam (2013).

*Entropy Pricing Theory* (EPT) (Gulko 1999) is founded on the idea that the asset price should fulfill the *efficient market* hypothesis to its best. The core of EPT is the calculation of the maximum efficient price using a set of benchmarks the prices of which calibrate an EMPM. This is done through the maximization of the entropy as in Bose and Murray (2014) and Neri and Schneider (2012). Alternatively, the *Kullback-Leibler relative entropy* of the EMPM relative to some initial probability can be minimized subject to the benchmark prices, as in Frittelli (2000). When the market states are finite, one particular use of Kullback-Leibler relative entropy minimization is to set the uniform probability as the initial one (Avellaneda et al. 2001), which is known as *the weighted Monte Carlo* method. More generally, any *divergence* (Amari 2016, chapter 3) measuring the informational discrepancy to uniform probability can be used as an objective to minimize, see for instance Sheraz and Preda (2015), Vilar-Zanon and Peraita-Ezcurra (2018) and Vilar-Zanon and Peraita-Ezcurra (2019).

### 1.3 Contributions

We deliver a new methodology to select the EMPM, named *entropy segmentation*, that outputs two EMPM for each entropy level $H \in [0, H^*]$, where the maximum entropy $H^*$ is calibrated on a benchmarks set. This shows that no arbitrage prices can be clustered by pairs associated with entropy levels. It is a generalization to convex programming of the already mentioned Linear Programming (LP) formulation for calculating the buyer's and seller's prices. In the null entropy case $H = 0$, we obtain the extremes of the interval (buyer's and seller's price) while in the maximum case $H = H^*$ we obtain the most efficient price $\pi^*$ as calculated by EPT (see for example Frittelli (2000)). An important advantage is that *entropy segmentation* is an inverse method (see for instance Breeden (1978), Jackwerth and Rubinstein (1996), and Borwein (2012)), so its use is not limited by any parametric hypothesis on the underlying stochastic models.

Applying Fenchel duality we also deliver the dual of the primal-pricing step. By strong duality, the optimal values of the primal and dual programs are equal to the no-arbitrage price calculated at the primal-pricing stage. Depending on the entropy value we use, the dual program changes its appearance. When $H = 0$ it adopts the form of the LPs we find in Musiela and Rutkowski (2005), the output will be the super-replicating (resp. sub-replicating) portfolio. When $H \neq 0$ this dual feasible set consists of all the partial replicating portfolios. We show how we can cut this feasible dual set by any no-arbitrage price to obtain a slice where we can minimize the remaining risk, looking for the optimal hedging position at that fixed price.

This is a way of defining another program to complete the claim pricing and hedging process, and we show how this can be achieved through LP, minimizing the CVaR of that remaining risk following the work of Rockafellar and Uryasev (2000).

To exemplify this process we choose a cliquet-style guarantee that is widely sold in financial markets although its risk hedging is considered of the utmost difficulty. These kinds of guarantees are important because they are widely sold by insurance companies issuing participating life insurance policies.

### 1.4 Organization

The rest of the paper is organized as follows. In Section 2 we set the notation used throughout the paper. In Section 3 we set up the entropy segmentation method by explaining the primal pricing programs and discussing their properties. In Section 4 we apply the Fenchel duality, discuss partial and super (resp. sub) replication, and write down the linear program for CVaR minimization. Technical details about the calculations of the Fenchel conjugates, support, and perspective functions are reported in Appendix A. In Section 5 we introduce the cliquet style guarantee and exemplify our methodology. Section 6 reports our conclusions.

## 2 Notation

We denote random variables and processes with capital letters. Random variable samples are vectors written as lowercase bold letters whose components (realizations) are indexed by the time $t$ and the trajectory $j$. A bold number stands for a vector with the same value in all its coordinates. We account for losses with a positive sign.

We work in one time period $[0, T]$. As in the example we will use forward start options as benchmarks, we shall divide it by yearly periods $t = 1, \ldots, T$ to be able to sample their payoffs. We consider the continuous stochastic process $S = \{S_t\}_{t\in[0,T]}$ modeling an underlying portfolio over several yearly periods whose initial value is $s_0$. The yearly values of the underlying portfolio will be needed to sample the claim and benchmark payoffs, and they are given by the random variables $S_1, S_2, \ldots, S_T$. Using the natural probability $\mathcal{P}$ of the stochastic model we simulate $m$ trajectories of the process $S$ and obtain sample vectors $\boldsymbol{s}_t = (s_t^j)_{j=1}^m$ with $t = 1, 2, \ldots T$ years. The random payoff $X$ at expiry $T$ depends on the yearly evolution of the underlying $S$. By the yearly samples $\boldsymbol{s}_t$ we calculate the payoff sample $\boldsymbol{x} = (x^j)_{j=1}^m \in \mathbb{R}_+^m$ (we omit the time index as the payoff is paid at the expiry $T$). Looking at the time interval $[0, T]$ as one time period, we thus reduce the continuous setting to a discrete one (both in time and states), $\Omega_m$ being the set of the scenarios (market states) at the expiry $T$ and $P(\Omega_m)$ the $\sigma - algebra$ power set of $\Omega_m$. The probability measure $\mathcal{Q}^m$ is *an EMPM* on $(\Omega_m, P(\Omega_m))$ and its mass function is noted by $\boldsymbol{q} = \left(q^j\right)_{j=1}^m \in \mathcal{S}^m$, where $\mathcal{S}^m$ stands for the *m-simplex* and $q_j > 0$ for at least one $j$.

We will work with $n$ benchmarks with random payoffs $(G_i)_{i=1}^n$ (with maturities $t = 1, \ldots, T$ in our application), paid at the expiry $T$ depending on the evolution of the process $S$. They are also sampled by $n$ payoff vectors $\boldsymbol{g}_i = (g_i^j)_{j=1}^m \in \mathbb{R}^m$ with $i = 1, \ldots, n$. The benchmark prices are noted with vector $\boldsymbol{c} = (c_i) \in \mathbb{R}_*^n$, and a benchmarks portfolio with $\boldsymbol{\theta} = (\theta_i) \in \mathbb{R}^n$. Short and long positions are noted with negative and positive coordinates, respectively. A portfolio payoff at maturity $T$ is given by $G = \sum_{i=1}^n \theta_i G_i$, thus the sampled portfolio payoff is $\boldsymbol{g} = \sum_{i=1}^n \theta_i \boldsymbol{g}_i$ with price $\boldsymbol{\theta}'.\boldsymbol{c} = \sum_{i=1}^n \theta_i c_i$.

We suppose a constant yearly risk-free interest rate $R$, thus $r = \log(1 + R)$ is the instantaneous forward rate and the discounting factor is $e^{-rT}$. The risk free zero coupon bond that costs $b_0 = 1$ at $t = 0$ has a sampled payoff $\left(e^{rT}, ..., e^{rT}\right)' \in \mathbb{R}^m$ at expiry $T$. The notation $\boldsymbol{x} \succeq \boldsymbol{0}$ means that the inequality $\geq$ is satisfied by all the vector components.

Finally, *Shanon's entropy* will be noted by:

$$\mathrm{H}(\boldsymbol{q}) = -\sum_{j=1}^{m} q^j \log\left(q^j\right). \tag{1}$$

We consider H continuously prolonged by setting $0 \log(0) = 0$.

## 3 Pricing by Entropy Segmentation

We can calibrate an EMPM by maximum entropy $H^*$ to the benchmark prices $\boldsymbol{c}$ by solving the following mathematical program, that is related to those we can find in Avellaneda et al. (2001), Frittelli (2000) and Borwein (2012):

$$\begin{cases} \max\limits_{\boldsymbol{q}\in I\!R_+^m} \mathrm{H}(\boldsymbol{q}) \\ \text{s.t.: } e^{-rT}\boldsymbol{q}' \cdot \boldsymbol{g}_i - c_i = 0,\ i = 1, 2, \ldots, n \text{ (Benchmark price constraints)} \\ \qquad e^{-rT}\boldsymbol{q}' \cdot \boldsymbol{s}_T - \boldsymbol{s}_0 = 0 \text{ (Martingale constraint)} \\ \qquad \boldsymbol{q}' \cdot \boldsymbol{1} = \boldsymbol{1} \text{ (Probability constraint).} \end{cases} \tag{2}$$

Program (2) reaches its global maximal value $H^* = H(\boldsymbol{q}^*)$ at a unique optimum $\boldsymbol{q}^*$ which is the mass function of a pricing measure $\mathcal{Q}^{m*}$ that correctly prices the benchmarks (benchmark price constraints) and the underlying (martingale constraint). The claim $\boldsymbol{x}$ is then priced using the Fundamental Asset Pricing Theorem:

$$\pi^* = e^{-rT}\mathbb{E}^{\mathcal{Q}^{m*}}[\boldsymbol{x}] = e^{-rT}\boldsymbol{q}^{*\prime}\cdot\boldsymbol{x}. \tag{3}$$

This is the *maximum efficient price* belonging to the interval of no-arbitrage prices, as calculated by EPT.

The method of *entropy segmentation* consists of two primal programs sharing the same feasible set. Firstly, we set the pricing formula as the objective function:

$$\pi(\boldsymbol{q}) = e^{-rT}\mathbb{E}^{\mathcal{Q}^{m}}[\boldsymbol{x}] = e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{x}. \tag{4}$$

Secondly, to define the feasible set we need to replace each equality constraint of Eq. 2 with an equivalent couple of inequality constraints. This is because Fenchel duality has to be applied to a standard program with inequalities. Therefore, we substitute the benchmark price constraints with two inequalities:

$$\begin{aligned} f_k(\boldsymbol{q}) &= e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{g}_k - c_k \le 0, \qquad k = 1, 2, \ldots, n\\ f_k(\boldsymbol{q}) &= c_{k-n} - e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{g}_{k-n} \le 0, \quad k = n+1, \ldots, 2n. \end{aligned} \tag{5}$$

A benchmark price $c_k, c_{k-n}$ in Eq. 5 is *mark-to-market* if it is quoted by the market in which case the decision maker will choose among bid/ask prices. Or it is *mark-to-model* when the benchmark is not traded and it is calculated using a model calibrated on market data.

The martingale and the probability constraints are also rewritten as two inequalities:

$$f_{2n+1}(\boldsymbol{q}) = e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{s}_T - s_0 \le 0, \quad f_{2n+2}(\boldsymbol{q}) = s_0 - e^{-rT}\boldsymbol{q}'\cdot\boldsymbol{s}_T \le 0. \tag{6}$$

$$f_{2n+3}(\boldsymbol{q}) = \boldsymbol{q}'\cdot\boldsymbol{1} - \boldsymbol{1} \le \boldsymbol{0}, \quad f_{2n+4}(\boldsymbol{q}) = \boldsymbol{1} - \boldsymbol{q}'\cdot\boldsymbol{1} \le \boldsymbol{0}. \tag{7}$$

Let us note $\mathfrak{Q}$ for the set of mass distributions satisfying the constraints $f_1, \ldots, f_{2n+4}$. We can now express (2) as $\boldsymbol{q}^* = \arg\max_{\mathfrak{Q}} \mathrm{H}(\boldsymbol{q})$.

We include an inequality constraining the feasible solutions to have an entropy level $\mathrm{H}(\boldsymbol{q})$ greater than some fixed $H \le H^*$,

$$f_{2n+5}(\boldsymbol{q}) = -\mathrm{H}(\boldsymbol{q}) + H \le 0, \tag{8}$$

and finally, make explicit the nonnegativity constraints:

$$f_{2n+6}(\boldsymbol{q}) = \boldsymbol{q} \succeq \boldsymbol{0}. \tag{9}$$

We define our primal feasible set $\mathfrak{Q}_H = \{\boldsymbol{q}\in\mathfrak{Q} : f_{2n+5}(\boldsymbol{q}) \le 0, f_{2n+6}(\boldsymbol{q}) \succeq \boldsymbol{0}\} \subset \mathbb{R}^m$. $\mathfrak{Q}_H$ is convex ((5), (6) and (7) define closed half-spaces of $\mathbb{R}^m$, and $f_{2n+5}$ is a strictly convex function), and compact (it is defined through the intersection with the $m$-simplex). Therefore, as the objective function (4) is linear, the two programs

$$\min\nolimits_{\boldsymbol{q}\in\mathfrak{Q}_H}\pi(\boldsymbol{q}) \tag{10}$$

$$\max\nolimits_{\boldsymbol{q}\in\mathfrak{Q}_H}\pi(\boldsymbol{q}) \tag{11}$$

reach their optimums $\boldsymbol{q}_H^{\min} = \arg\min_{\boldsymbol{q}\in\mathfrak{Q}_H}, \boldsymbol{q}_H^{\max} = \arg\max_{\boldsymbol{q}\in\mathfrak{Q}_H}$, with global minimal and maximal values satisfying:

$$\pi_H^{\min} \le \pi^* \le \pi_H^{\max}, \quad \forall H \in [0, H^*]. \tag{12}$$

We name the prices $\pi_H^{\min}, \pi_H^{\max}$ as the *buyer's price* and *seller's price at entropy level* $H \le H^*$. In the case $H = H^*$, we get $\Omega_H = \{\boldsymbol{q}^*\}$ and the equality in Eq. 12 is reached:

$$\pi_{H^*}^{\min} = \pi^* = \pi_{H^*}^{\max}. \tag{13}$$

Therefore, while programs (10) and (11) satisfy the *Slater condition* for any $H < H^*$, where strong duality applies, this is not true for $H = H^*$.

## 4 Fenchel Duality and Hedging

We will only consider the seller's primal program (11) as the buyer's case (10) can be treated in a similar way.

### 4.1 Fenchel Dual Programs

We follow the general procedure explained in Roos et al. (2020). The feasible set $F$ of the program Eq. 11 is:

$$F = \bigcap_{k=1}^{2n+6} F_k, \tag{14}$$

where

$$F_k = \left\{\boldsymbol{q} \in \mathbb{R}^m : f_k(\boldsymbol{q}) \le 0\right\}, \quad k = 1, \ldots, 2n+5, \tag{15}$$

$$F_{2n+6} = \{\boldsymbol{q} \succeq \boldsymbol{0}\} = \mathbb{R}_+^m. \tag{16}$$

Program (11) can be rewritten as:

$$\mathrm{Sup}\,\{\pi\,(\boldsymbol{q}) : \boldsymbol{q} \in F\} = -\mathrm{Inf}\,\{-\pi(\boldsymbol{q}) : \boldsymbol{q} \in F\}. \tag{17}$$

Let $\delta_F$ be the *indicator function* of the feasible set $F$ (see Mordukhovich and Nam (2013), p. 34):

$$\forall \boldsymbol{q} \in \mathbb{R}^m : \ \delta_F\,(\boldsymbol{q}) = \sum_{k=1}^{2n+6} \delta_{F_k}\,(\boldsymbol{q}) = \begin{cases} 0, & \boldsymbol{q} \in F \\ \infty, & \text{otherwise.} \end{cases} \tag{18}$$

Then calling $g(\boldsymbol{q}) = -\pi(\boldsymbol{q}) + \delta_F(\boldsymbol{q})$, we can write (17) as an unconstrained program:

$$-\,\mathrm{Inf}\,\{-\pi\,(\boldsymbol{q}) : \boldsymbol{q} \in F\} = -\mathrm{Inf}_{\boldsymbol{q}}\,\{g\,(\boldsymbol{q})\}. \tag{19}$$

The Fenchel conjugate $g^*$ of a function $g$ with domain $\mathrm{dom}\, g = \{\boldsymbol{q} \in \mathbb{R}^m : g(\boldsymbol{q}) < \infty\}$ is given by (see Mordukhovich and Nam (2013), p. 77):

$$g^*\,(\boldsymbol{y}) = \sup_{\boldsymbol{q} \in \mathrm{dom}\, g} \left\{\boldsymbol{y}' \cdot \boldsymbol{q} - g\,(\boldsymbol{q})\right\} \quad \forall \boldsymbol{y} \in \mathbb{R}^m,$$

thus we obtain program (19) by substituting $\boldsymbol{y} = \boldsymbol{0}$:

$$g^*\,(\boldsymbol{0}) = \sup_{\boldsymbol{q} \in \mathrm{dom}\, g} \{-g\,(\boldsymbol{q})\} = -\inf_{\boldsymbol{q} \in \mathrm{dom}\, g} \{g\,(\boldsymbol{q})\}. \tag{20}$$

Now, it remains to calculate $g^*\,(\boldsymbol{0})$. As $g$ is the sum of two functions, we can write its Fenchel conjugate as:

$$g^*\,(\boldsymbol{y}) = \min_{\boldsymbol{y}_0, \boldsymbol{y}_1} \left\{(-\pi)^*\,(\boldsymbol{y}_0) + \delta_F^*\,(\boldsymbol{y}_1) : \boldsymbol{y}_0 + \boldsymbol{y}_1 = \boldsymbol{y} \in \mathbb{R}^m\right\} \ \forall \boldsymbol{y}_0, \boldsymbol{y}_1 \in \mathbb{R}^m, \tag{21}$$

thus:

$$g^{*}(\mathbf{0})=\min_{\boldsymbol{y}_0}\left\{(-\pi)^{*}(\boldsymbol{y}_0)+\delta_F^{*}(-\boldsymbol{y}_0)\right\}. \tag{22}$$

Applying the same argument as in Eq. 21, the Fenchel conjugate of the indicator function $\delta_F$ in Eq. 18 is:

$$\delta_F^{*}(-\boldsymbol{y}_0)=\min_{(\boldsymbol{y}_k)_{k=1}^{2n+6}\in\mathbb{R}^m}\left\{\sum_{k=1}^{2n+6}\delta_{F_k}^{*}(\boldsymbol{y}_k):\sum_{k=1}^{2n+6}\boldsymbol{y}_k=-\boldsymbol{y}_0\right\}. \tag{23}$$

As each conjugate $\delta_{F_k}^{*}$ $(k=1,2,\ldots,2n+5)$ is equal to its perspective function $(u_k f_k)^{*}$ (see Eq. A3) and $\delta_{F_{2n+6}}^{*}$ is equal to Eq. A18, we write (23) as:

$$\delta_F^{*}(-\boldsymbol{y}_0)=\min_{(\boldsymbol{y}_k)_{k=1}^{2n+6},\boldsymbol{u}\succeq\mathbf{0}}\left\{\sum_{k=1}^{2n+5}(u_k f_k)^{*}(\boldsymbol{y}_k)\ :\sum_{k=0}^{2n+6}\boldsymbol{y}_k=\mathbf{0},\ \boldsymbol{y}_{2n+6}\preceq\mathbf{0}\right\}. \tag{24}$$

Thus $g^{*}(\mathbf{0})$ in Eq. 21 can be rewritten as:

$$\min_{(\boldsymbol{y}_k)_{k=0}^{2n+6},\boldsymbol{u}\succeq\mathbf{0}}\left\{(-\pi)^{*}(\boldsymbol{y}_0)+\sum_{k=1}^{2n+5}(u_k f_k)^{*}(\boldsymbol{y}_k):\sum_{k=0}^{2n+6}\boldsymbol{y}_k=\mathbf{0},\ \boldsymbol{y}_{2n+6}\preceq\mathbf{0}\right\}, \tag{25}$$

where $(u_k)_{k=1}^{2n+5}$ and $(\boldsymbol{y}_k)_{k=0}^{2n+6}$ are dual variables and vectors, respectively. Expression (22) is the dual program of Eq. 11 containing as particular cases super replication and partial replication, as we are just going to show.

Looking at the perspective function for $k=2n+5$ (see Eqs. A17a and A17b), we must distinguish between two cases consisting of what happens at the interior of the no-arbitrage prices interval or at its upper endpoint (i.e. the seller's price).

When $u_{2n+5}>0$ , we substitute $(u_{2n+5}f_{2n+5})^{*}$ by Eq. A17a, $(-\pi)^{*}$ by Eq. A6, the perspective functions $(u_k f_k)^{*}$ and dual vectors $\boldsymbol{y}_k$ by Eqs. A8, A11, A13 and A15, respectively. We obtain the first particular case of Eq. 22, $\forall u_{2n+5}>0$:

$$\begin{cases}\displaystyle\min_{\boldsymbol{y}_{2n+5},\boldsymbol{y}_{2n+6},\boldsymbol{u}}\sum_{k=1}^{n}(u_k-u_{n+k})\,c_k+(u_{2n+1}-u_{2n+2})\,s_0+(u_{2n+3}-u_{2n+4})+\\ \qquad\qquad+\,u_{2n+5}\left(\displaystyle\sum_{j=1}^{m}e^{\frac{y_{2n+5}^{j}}{u_{2n+5}}-1}-H\right)\\ \text{s.t.:}\quad\displaystyle\sum_{k=1}^{n}(u_k-u_{n+k})\,\boldsymbol{g}_k+(u_{2n+1}-u_{2n+2})\,\boldsymbol{s}_T+e^{rT}(u_{2n+3}-u_{2n+4})\,\mathbf{1}+\\ \qquad\qquad+\,e^{rT}(\boldsymbol{y}_{2n+5}+\boldsymbol{y}_{2n+6})=\boldsymbol{x}\\ \quad\boldsymbol{y}_{2n+5}\succeq\mathbf{0},\quad\boldsymbol{y}_{2n+6}\preceq\mathbf{0},\quad\boldsymbol{u}\succeq\mathbf{0}.\end{cases} \tag{26}$$

When $u_{2n+5} = 0$ we make the same substitutions except for $(u_{2n+5} f_{2n+5})^*$, now replaced by Eq. A17b. This is the second case of Eq. 22, $u_{2n+5} = 0$:

$$\begin{cases} \min\limits_{\boldsymbol{y}_{2n+6},\boldsymbol{u}} \sum\limits_{k=1}^{n} (u_k - u_{n+k})\, c_k + (u_{2n+1} - u_{2n+2})\, s_0 + (u_{2n+3} - u_{2n+4}) \\ \text{s.t.:} \quad \sum\limits_{k=1}^{n} (u_k - u_{n+k})\, \boldsymbol{g}_k + (u_{2n+1} - u_{2n+2})\, \boldsymbol{s}_T + e^{rT}\, (u_{2n+3} - u_{2n+4})\, \mathbf{1} + \\ \qquad\qquad + e^{rT}\, \boldsymbol{y}_{2n+6} = \boldsymbol{x} \\ \boldsymbol{y}_{2n+6} \preceq \mathbf{0}\,, \quad \boldsymbol{u} \succeq \mathbf{0}. \end{cases} \tag{27}$$

The linear program (27) outputs the super-hedging portfolio with a price equal to the upper bound of the no-arbitrage price interval. This portfolio is made of benchmarks, underlying assets, and bonds. The excess vector $e^{rT}\, \boldsymbol{y}_{2n+6}$ in Eq. 27 depicts the issuer's gains over the scenarios at expiry $T$.

Referring now to the program (26), we know by strong duality that its optimal value is equal to $\pi_H^{max}$, the optimal value of the primal (11). Furthermore, the sensitivity of $\pi_H^{max}$ with respect to the entropy level $H$ is given by the optimal value of the dual variable $u_{2n+5} > 0$.

Let us focus now on the dual feasible set in Eq. 26. The equality constraint is valued at $T$ and makes the sum of three vectors equal to the claim payoff $\boldsymbol{x}$. These vectors are:

- The payoff of a partial replicating portfolio $\boldsymbol{u} = \{u_1, u_2, \ldots, u_{2n}, u_{2n+1}, \ldots, u_{2n+4}\} \in \mathbb{R}_+^{2n+4}$, made of benchmarks, underlying asset, and bonds, where the dual variables stand for short/long positions. The benchmark portfolio is $\boldsymbol{\theta}$ where $\theta_i = u_i - u_{n+i}$ is the net short/long position on asset $i = 1, 2, \ldots, n$. The quantities $u_S = u_{2n+1} - u_{2n+2}$ and $u_b = u_{2n+3} - u_{2n+4}$ are the net positions on underlying assets and bonds, respectively. Observe that the bond value and weights in the dual come from the probability constraint in the primal (11), and the underlying asset value and weights come from the martingale constraint.
- The vector $e^{rT}\, \boldsymbol{y}_{2n+5} \succeq \mathbf{0}$, coming from the entropy constraint in the primal program, quantifying the part of the claim $\boldsymbol{x}$ not replicated by the portfolio $\boldsymbol{u}$. We call it the *remaining claim* describing the losses over the scenarios.
- The vector $e^{rT}\, \boldsymbol{y}_{2n+6} \preceq \mathbf{0}$, depicting the gains over the scenarios that we call *excess vector*. It comes from the non-negativity constraints in the primal program.

The vector $e^{rT}\, (\boldsymbol{y}_{2n+5} + \boldsymbol{y}_{2n+6})$ quantifies the mismatch over the scenarios between the claim and the partial replicating portfolio payoffs. Observe that both $\boldsymbol{y}_{2n+5}$ and $\boldsymbol{y}_{2n+6}$ are calculated by programs (26) and (27) at $t = 0$.

In summary, the feasible set in Eq. 26 consists of all the portfolios partially replicating the claim $\boldsymbol{x}$, at *any* no-arbitrage price. We also conclude that the primal pricing program (11) contains all the elements to build, by means of Fenchel duality, both the feasible set of all the partial replicating portfolios and a pricing formula.

### 4.2 Optimal Partial Replicating Portfolio

To calculate an optimal portfolio, it is natural to ask for its price to be equal to the claim price. This is done by introducing one additional constraint fixing the price, for any $H \leq H^*$:

$$\begin{cases} \sum_{k=1}^{n} (u_k - u_{n+k})\, c_k + (u_{2n+1} - u_{2n+2})\, s_0 + (u_{2n+3} - u_{2n+4}) = \pi_H^{\max} \\ \sum_{k=1}^{n} (u_k - u_{n+k})\, \boldsymbol{g}_k + (u_{2n+1} - u_{2n+2})\, \boldsymbol{s}_T + e^{rT}\, (u_{2n+3} - u_{2n+4})\, \mathbf{1} + \\ \qquad + e^{rT}\, (\boldsymbol{y}_{2n+5} + \boldsymbol{y}_{2n+6}) = \boldsymbol{x} \\ \boldsymbol{y}_{2n+5} \succeq \mathbf{0}\,, \quad \boldsymbol{y}_{2n+6} \preceq \mathbf{0}\,, \quad \boldsymbol{u} \succeq \mathbf{0}. \end{cases} \tag{28}$$

If we want to calculate an optimal solution in the set (28), sampling all the random variables by means of the natural probability measure $\mathcal{P}$ is very important, because all the risk calculations must be done in the natural world in order to be meaningful, and not in the risk-neutral one.

At this final step, we minimize the CVaR of the remaining claim. This is achieved through a linear program with a fast resolution, applying the methodology we find in Rockafellar and Uryasev (2000). We consider the continuous case where capital letters denote continuous random variables. We focus on the seller's case. Let us consider the decision vector made of the dual variables:

$$\boldsymbol{u} = \{u_1, u_2, \ldots, u_{2n}, u_{2n+1}, \ldots, u_{2n+4}\} \in \mathbb{R}_+^{2n+4},$$

defining a portfolio made of benchmarks, underlying, and bonds, with their short/long position interpretation as given by the signs in the formula 29. We define the loss associated with the *decision* $\boldsymbol{u} \in \mathbb{R}_+^{2n+4}$ (consisting of the chosen portfolio of benchmarks) as the following random variable which is a generalization of the vector constraint in Eq. 28 to the continuous case :

$$\begin{aligned} Y(\boldsymbol{u}, X, S_T) =& e^{-rT}\Big[X - \sum_{k=1}^{n} (u_k - u_{n+k})\, G_k - (u_{2n+1} - u_{2n+2}) S_T - \\ &- e^{rT}\, (u_{2n+3} - u_{2n+4})\Big] = Y^+(\boldsymbol{u}, X, S_T) + Y^-(\boldsymbol{u}, X, S_T). \end{aligned} \tag{29}$$

In Eq. 29 $Y$ is a random variable because it depends on the random variable $S_T$ and the claim payoff $X$ at $t = T$ which is also a random variable depending on the underlying yearly dynamic. By a slight abuse of notation, let us note $Y(\boldsymbol{u}) \equiv Y(\boldsymbol{u}, X, S_T)$, decomposed in Eq. 29 as a sum of two random variables $Y^+(\boldsymbol{u})$ $(\geq 0\,(a.s.))$ and $Y^-(\boldsymbol{u})$ $(\leq 0\,(a.s.))$.

For each decision $\boldsymbol{u} \in \mathbb{R}_+^{2n+4}$, the random variable $Y(\boldsymbol{u})$ has a distribution that we can suppose to be absolutely continuous. Let us note $\varphi$ and $p$ for its cumulative distribution and density functions:

$$P\{Y(\boldsymbol{u}) \leq \alpha\} = \varphi(\alpha, \boldsymbol{u}) = \int_{Y(\boldsymbol{u}) \leq \alpha} p(y, \boldsymbol{u}) dy. \tag{30}$$

The $\mathrm{VaR}_\beta$ and $\mathrm{CVaR}_\beta$ of $Y$ are defined respectively as :

$$\alpha_\beta(\boldsymbol{u}) = \min\{\alpha \in \mathbb{R} : \varphi(\alpha, \boldsymbol{u}) \geq \beta\}, \quad \phi_\beta(\boldsymbol{u}) = (1-\beta)^{-1} \int_{Y \geq \alpha_\beta} y p(y, \boldsymbol{u}) dy. \tag{31}$$

$\alpha_\beta(\boldsymbol{u})$ is the left endpoint of the non-empty interval consisting of the values $\alpha$ such that $\varphi(\alpha, u) = \beta$.

$\phi_\beta(\boldsymbol{u})$ is the conditional expectation of the loss associated with the decision $\boldsymbol{u} \in \mathbb{R}_+^{2n+4}$, relative to the loss being $\alpha_\beta(\boldsymbol{u})$ or greater.

For each decision $\boldsymbol{u}$ the following function is defined:

$$\begin{aligned} F_\beta : \mathbb{R} \times \mathbb{R}^{2n+4} &\longrightarrow \mathbb{R} \\ (\alpha, \boldsymbol{u}) \quad &\longrightarrow \alpha + (1-\beta)^{-1} \int_{y \in \mathbb{R}} [y - \alpha]_+ p(y, \boldsymbol{u}) dy. \end{aligned} \tag{32}$$

In theorem 1 of Rockafellar and Uryasev (2000) p.5, it is shown that Eq. 32 is convex and continuously differentiable, and the $\mathrm{CVaR}_\beta$ of the loss $Y(\boldsymbol{u})$ associated with each $\boldsymbol{u} \in \mathbb{R}_+^{2n+4}$ can be determined by $\varphi_\beta(\boldsymbol{u}) = \min_{\alpha \in \mathbb{R}} F_\beta(\alpha, \boldsymbol{u})$. Calling $A_\beta(\boldsymbol{u}) = \arg\min_{\alpha \in \mathbb{R}} F_\beta(\alpha, \boldsymbol{u})$, the $\mathrm{VaR}_\beta$ of the loss $Y(\boldsymbol{u})$ is given by $\alpha_\beta(\boldsymbol{u})$, the left endpoint of $A_\beta(\boldsymbol{u})$. Moreover, one always has:

$$\alpha_\beta(\boldsymbol{u}) \in \arg\min_{\alpha \in \mathbb{R}} F_\beta(\alpha, \boldsymbol{u}) \quad \text{and} \quad \phi_\beta(\boldsymbol{u}) = F_\beta(\alpha_\beta(\boldsymbol{u}), \boldsymbol{u}). \tag{33}$$

Also in theorem 2 of Rockafellar and Uryasev (2000) p.5, it is shown that for each decision $\boldsymbol{u}$, minimizing the $\mathrm{CVaR}_\beta$ is equivalent to minimizing

$$\min_{\boldsymbol{u} \in \mathbb{R}_+^{2n+4}} \phi_\beta(\boldsymbol{u}) = \min_{(\alpha, \boldsymbol{u}) \in \mathbb{R} \times \mathbb{R}_+^{2n+4}} F_\beta(\alpha, \boldsymbol{u}). \tag{34}$$

Moreover, the pair $(\alpha^*, \boldsymbol{u}^*)$ achieves the second minimum in Eq. 34 if and only if $\boldsymbol{u}^*$ achieves the first minimum and $\boldsymbol{u}^* \in A_\beta(\boldsymbol{u}^*)$. When the interval $A_\beta(\boldsymbol{u}^*)$ reduces to a single point, the minimization of $F(\alpha, \boldsymbol{u})$ produces a pair such that $\boldsymbol{u}^*$ minimizes the $\mathrm{CVaR}_\beta(Y(\boldsymbol{u}^*))$ and $\alpha^*$ gives the corresponding $\mathrm{VaR}_\beta(Y(\boldsymbol{u}^*))$.

Therefore, the problem is how to approximate the integral in Eq. 32. As suggested in the cited reference, we can generate samples of $Y^+(\boldsymbol{u}) \in \mathbb{R}_+^m$, $Y^-(\boldsymbol{u}) \in \mathbb{R}_-^m$, and $Y(\boldsymbol{u}) = Y^+(\boldsymbol{u}) + Y^-(\boldsymbol{u}) \in \mathbb{R}^m$. To connect with the previous sections, let us name them respectively as $\boldsymbol{y}_{2n+5}, \boldsymbol{y}_{2n+6}$. Through the sampling process, we also obtain the sample vectors $\boldsymbol{x}$, $\boldsymbol{s}_T \in \mathbb{R}_+^m$. Then:

$$\widetilde{F}_\beta(\boldsymbol{u}, \alpha) = \alpha + [m(1-\beta)]^{-1} \sum_{j=1}^{m} \left[ y_{2n+5}^j + y_{2n+6}^j - \alpha \right]_+ . \tag{35}$$

We finally have to solve:

$$\begin{cases} \min\limits_{\alpha, \boldsymbol{d}, \boldsymbol{u}, \boldsymbol{y}_{2n+5}, \boldsymbol{y}_{2n+6}} \alpha + [m(1-\beta)]^{-1} \sum\limits_{j=1}^{m} \left[ y_{2n+1}^j + y_{2n+6}^j - \alpha \right]_+ \\ \text{s.t.:} \ \sum\limits_{k=1}^{n} (u_k - u_{n+k})\, c_k + (u_{2n+1} - u_{2n+2})\, s_0 + (u_{2n+3} - u_{2n+4}) = \pi_H^{\max} \\ \qquad \sum\limits_{k=1}^{n} (u_k - u_{n+k})\, \boldsymbol{g}_k + (u_{2n+1} - u_{2n+2})\, \boldsymbol{s}_T + e^{rT}\, (u_{2n+3} - u_{2n+4})\, \mathbf{1} + \\ \qquad\qquad + e^{rT} (\boldsymbol{y}_{2n+5} + \boldsymbol{y}_{2n+6}) = \boldsymbol{x} \\ \qquad \boldsymbol{y}_{2n+5} \succeq \mathbf{0}, \quad \boldsymbol{y}_{2n+6} \preceq \mathbf{0}, \quad \boldsymbol{u} \succeq \mathbf{0}. \end{cases} \tag{36}$$

In terms of the auxiliary variables $d^j$

$$\forall j = 1, \dots, m : y^j_{2n+5} + y^j_{2n+6} - \alpha = d^j \Rightarrow \alpha + d^j - y^j_{2n+5} - y^j_{2n+6} \geq 0,$$

the following LP must be solved:

$$\begin{cases} \min\limits_{\alpha, \boldsymbol{d}, \boldsymbol{u}, \boldsymbol{y}_{2n+5}, \boldsymbol{y}_{2n+6}} \quad \alpha + [m(1-\beta)]^{-1} \sum_{j=1}^{m} d^j \\ \text{s.t.:}\ \alpha + d^j - y^j_{2n+5} - y^j_{2n+6} \geq 0\,, \quad j = 1, \dots, m \\ \sum_{k=1}^{n} (u_k - u_{n+k})\, c_k + (u_{2n+1} - u_{2n+2})\, s_0 + (u_{2n+3} - u_{2n+4}) = \pi_H^{max} \\ \sum_{k=1}^{n} (u_k - u_{n+k})\, \boldsymbol{g}_k + (u_{2n+1} - u_{2n+2})\, \boldsymbol{s}_T + e^{rT} (u_{2n+3} - u_{2n+4})\, \mathbf{1} + \\ \qquad + e^{rT} (\boldsymbol{y}_{2n+5} + \boldsymbol{y}_{2n+6}) = \boldsymbol{x} \\ \boldsymbol{d} \succeq \mathbf{0}, \quad \boldsymbol{y}_{2n+5} \succeq \mathbf{0}\,, \quad \boldsymbol{y}_{2n+6} \preceq \mathbf{0}\,, \quad \boldsymbol{u} \succeq \mathbf{0}\,, \quad \alpha \in \mathbb{R}. \end{cases} \tag{37}$$

This outputs an optimal value, the minimal $\text{CVaR}_\beta$, attained at some $\alpha = \text{VaR}_\beta$ close (if not equal) to the minimal value of $\text{VaR}_\beta$ (see Rockafellar and Uryasev (2000)). The optimum is the optimal partial replicating portfolio $\boldsymbol{u}$ and its corresponding remaining claim $e^{rT}\, \boldsymbol{y_{2n+5}}$ and excess vector $e^{rT}\, \boldsymbol{y_{2n+6}}$. If we fix $H = H^*$, we replace the price $\pi_H^{max}$ by $\pi^*$ (3) and we obtain the optimal partial replicating portfolio $\boldsymbol{u}^* = (\boldsymbol{\theta}^*, u^*_S, u^*_b)$ corresponding to the maximum efficient price. If we decrease the entropy level $H$ towards 0, we obtain the super-hedging case where $\boldsymbol{y_{2n+5}} = \mathbf{0}$ and program (37) outputs the same solution as program (27). For any entropy level $H < H^*$, we could also solve the primal program (10) finding buyer's prices $\pi_H^{\min} < \pi^*$, and using then the analog set to Eq. 28 calculated from the dual program of Eq. 10. Both linear programs, Eq. 37 and the one corresponding to the buyer's side, would give the same solution when substituting the maximum efficient price $\pi^*$ in the price constraint.

## 5 Application

In this section, we exemplify through a cliquet-style guarantee. As our aim is to show how our methodology works, we present a simple application with a maturity of three years, assuming that the total market value of the asset portfolio follows the Heston model. We introduce the benchmark set and finally, we present the numerical results. In order to make our application more realistic, both the Heston parameters and the instantaneous forward rate are estimated on market data.

### 5.1 Cliquet-Style Guarantees

Cliquet-style guarantees are path-dependent contingent claims with a high financial market risk, that are widely used in life insurance in the context of participating life insurance policies (PLI) (see for instance Olivieri and Pitacco (2015)). The payoff $Z$ at time $T$ is:

$$Z = C \prod_{t=1}^{T} \left[1 + \max\{\gamma I_t, i_{\min}\}\right], \tag{38}$$

where $C$ is the capital at time $t = 0$, $\gamma$ is the participation coefficient, $I_t$ is the annual return of the asset portfolio $S_t$ and $i_{\min}$ is the minimum guaranteed interest rate. Without loss of generality, we fix $i_{\min} = 0$ and $\gamma = 1$. Thus, if $I_t$ is positive, at the end of each year the policyholder earns a positive profit kept until the maturity $T$. By substituting $I_t = \frac{S_t}{S_{t-1}} - 1$ in Eq. 38, we write the payoff $Z$ as:

$$Z = C \prod_{t=1}^{T} \left[1 + \max\left\{\frac{S_t}{S_{t-1}} - 1, 0\right\}\right], \tag{39}$$

where we find the payoff of one European call and $(T - 1)$ forward start call options (see Kruse and Nogel 2005). The payoff $Z$ can be easily decomposed into a bond component and a path-dependent derivative component with payoff $X$:

$$Z = C + X. \tag{40}$$

The total payoff $Z$ is valued by means of:

$$\Pi = Ce^{-rT} + \pi, \tag{41}$$

where $\pi$ is the price of the payoff $X$ at time $t = 0$. The problem is how to calculate a price $\pi$ together with its associated static risk hedging position over one period $[0, T]$.

As the payoff (39) consists of the product of one European call and $(T - 1)$ forward start call options, this is our choice for the benchmark assets. Consequently, before the cliquet maturity is fixed at $t = T$, we need to consider a set of yearly maturities $t = 1, \ldots, T$ to be able to sample the benchmark payoffs at their respective maturities.

### 5.2 Stochastic Underlying Model: the Heston Model

For our exemplification, we need to implement a yearly dynamic for the underlying to sample the benchmarks and cliquet payoffs and substitute them into the programs Eqs. 2, 11, 26, 27 and 37. We choose the Heston model Eq. Heston (1993) for the underlying process $S$, because its dynamic depends on a stochastic variance process $V = \{V_t\}_{t\in[0,T]}$:

$$dS_t = \mu_S S_t dt + \sqrt{V_t} S_t dW_t^S, \tag{42}$$

$$dV_t = k_V(\mu_V - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V, \tag{43}$$

where the initial conditions are $s_0$ and $v_0$, and $\mu_S$ is the expected return. The process $V$ is modeled as a mean reverting square root process where $k_V$ is the convergence speed to the long-term mean $\mu_V$, and $\sigma_V$ is the volatility of the variance process. These parameters are assumed to be positive so that the process is well-defined. Notably, if the Feller condition is enforced, i.e., $2k_V\mu_V > \sigma_V^2$, the variance process is positive for any $t$. The standard Brownian motions $W^V = \{W_t^V\}_{t\in[0,T]}$ and $W^S = \{W_t^S\}_{t\in[0,T]}$ are correlated with quadratic variation satisfying $d[W^S, W^V] = \rho dt$ for some constant correlation coefficient $\rho \in [-1, 1]$. We generate claim and benchmark payoff samples $\boldsymbol{x}$ and $\boldsymbol{g}_i$ by simulating yearly values of the processes $V$, $S$ under the real-world probability measure $\mathcal{P}$ given by the natural parameters $(\mu_S, k_V, \mu_V, \sigma_V, \rho)$. This is done by applying the Euler-Maruyama discretization scheme, fixing the problem of a negative value for the variance thanks to the full truncation scheme proposed by Lord et al. (2010).

We also briefly introduce the risk-neutral dynamics of the underlying asset portfolio $S$ and the stochastic variance $V$. We use it to calculate mark-to-model prices (45) for the benchmarks. The risk-neutral dynamic is obtained by substituting in Eq. 42, $(\mu_S, k_V, \mu_V, \sigma_V, \rho)$ by the risk-neutral parameters $(r, \widetilde{k}_V, \widetilde{\mu}_V, \sigma_V, \rho)$. The prices of equity and volatility risk are:

$$q_S = \frac{\mu_S - r}{\sqrt{V_t}} \quad \text{and} \quad q_V = \frac{\lambda\sqrt{V_t}}{\sigma_V}, \tag{44}$$

with $\lambda \in \mathbb{R}$ a constant parameter. The risk-neutral drifts $\widetilde{f}_S$ and $\widetilde{f}_V$ are given by the relation $\widetilde{f} = f - qg$, where $f$ is the natural drift and $g$ is the diffusion coefficient (see Eqs. 42 and 43). Imposing the Feller condition, process $V$ never reaches zero. Given the value $\mu_V$, the convergence speed is $k_V = \frac{\widetilde{k}_V \widetilde{\mu}_V}{\mu_V}$ and the risk premium is $\lambda = \widetilde{k}_V - k_V$ (see Heston (1993), p. 335).

### 5.3 Numerical Results

We fix the cliquet guarantee maturity at $T = 3$ years and the initial value $s_0 = 100$. The risk-free instantaneous forward rate $r$ is obtained by calibrating the Smith-Wilson method on the quoted swap rates at the date 2020/11/04 [1]. From the estimated risk-free term structure of interest rates, we calculate $R = i(0, T)$. The value of the instantaneous forward rate is $r = -0.00629$ years$^{-1}$.

We need the natural Heston parameters to sample the scenarios and payoffs so that our *risk calculations* give us meaningful results, that is, reserved capital figures calculated with respect to the natural world and not with respect to the risk-neutral one. We need the risk-neutral Heston parameters to calculate the benchmark prices as given by a model calibrated on market data so that our methodology can qualify as *mark-to-model*. Therefore, we calculate the natural Heston parameters in Eqs. 42 and 43 with the estimation of the so-called risk-neutral parameters $(\widetilde{k}_V, \widetilde{\mu}_V, \sigma_V, \rho)$. These are calibrated on the quoted prices of 15 European calls on the same date 2020/11/04 with a maturity of 2023/12/15, whose underlying is the EUROSTOXX50 index. We choose these calls as their maturities are close to 3 years; the strike prices are between 3,000 and 4,350 euros. The observed value of the index on the date 2020/11/04 is 3,161.07. We minimize the sum of the squared errors between the model prices and the observed prices. We use the R function **nmkb** from the package **dfoptim** which implements the Nelder-Mead algorithm for derivative-free optimization (see Nelder and Mead (1965)). The minimization problem is solved by imposing appropriate lower and upper bounds on the parameters, in particular, the parameter $\widetilde{k}_V$ is estimated by enforcing the Feller condition. The solution gives a mean absolute error equal to 0.524. The long-term mean level and the speed of mean-reversion are estimated as $\widetilde{\mu}_V = 0.0246$ and $\widetilde{k}_V = 3.7340$ with a standard deviation $\sigma_V = 0.4252$. The estimated initial value is $v_0 = 0.0184$ and the

[1] The ultimate forward rate is equal to 3.75%. We have estimated the parameter $\alpha = 0.137$ controlling the speed to the convergence point of 60 years. Technical details can be found in European Insurance and Occupational Pensions Authority (2018).

**Table 1** Buyer's and seller's prices calculated by entropy segmentation solving programs (10), (11), simulation number $m = 60,000$, risk premium $\lambda = 0$. Extremes of the no-arbitrage interval in bold characters

| Entropy $H$ | Price | |
|---|---|---|
| | Buyer's | Seller's |
| $\mathbf{H \to 0}$ | **16.873** | **24.029** |
| 8.002 | 16.873 | 24.009 |
| 10.002 | 16.882 | 21.319 |
| 10.502 | 17.012 | 19.924 |
| 10.852 | 17.339 | 18.571 |
| 10.953[1] | 17.792 | 17.792 |

[1] Maximum entropy $H^*$ calibrated to the benchmark set

correlation coefficient is $\rho = -0.3638$. Without loss of generality, we set $\mu_V = 0.0246$ and $\mu_S = 0.02$. Therefore, the risk premium is $\lambda = 0$ (as done by Gatzert (2008)).

Once we have calibrated Heston's risk-neutral parameters on market data, we calculate the forward starting call option prices using the formula in Kruse and Nogel (2005). We adjust the formula of the first two benchmarks to take into account that the payment is made at the maturity $T$. The benchmark prices are:

$$c_1 = 0.055139, \quad c_2 = 0.056772, \quad c_3 = 0.056817. \tag{45}$$

In Table 1 we give the buyer's and seller's prices at successive entropy levels, from 0 to the maximum $H^*$, corresponding to $m = 60,000$ simulations.

In Table 2 we summarize the numerical results. We apply the hedging methodology using $m = 60,000$ simulations [2]in the seller's case $H \in [0, H^*]$. In each case, we write down the maximum efficient price $\pi^*$ (3) corresponding to the maximum entropy level $H^*$ calculated through the program (2). We solve the linear program (37) with $\beta = 99.5\%$, finding the optimal partial replicating portfolio $\boldsymbol{u}^*$ and its components $\boldsymbol{\theta}^*$, $u_S^*$ , and $u_b^*$, corresponding to the price $\pi^*$. We have also reported its $\text{CVaR}_{0.995}$ and $\text{VaR}_{0.995}$. We repeat these calculations for decreasing entropy levels until we obtain the seller's prices $\pi_H^{max}$ as solutions of the program (11). The last row is the super-hedging case (noted in bold), i.e., the solution of the dual program (27) (or Eq. 37 for an enough low entropy level $H$). The corresponding price is the upper bound of the no-arbitrage price interval, as explained in Section 3.

This information is very important for the claim issuer. We find all the available combinations among prices, partial replicating portfolios, and minimal CVaRs hedging the remaining claim risk. The maximum efficient price $\pi^*$ is the *lower bound* for those prices, with a remaining claim CVaR reaching its *highest* value. The *upper bound* of the prices is the seller's price with its associated super-hedging portfolio, its remaining claim being the null vector with CVaR equal to zero (the *lowest* one).

We have also applied our methodology for $m = 60,000$ simulations choosing $\mu_V = 1.20\widetilde{\mu}_V$, so the risk premium is $\lambda = -0.622$, and have reported the results in Table 3. We see that either the seller's prices or the VaR and CVaR have increased compared to the case $\lambda = 0$.

Our solutions to the risk-minimizing program Eq. 37 are optimal in the sense of the CVaR, which is a coherent risk measure, and also with respect to the VaR (at least nearly, as explained by Rockafellar and Uryasev (2000)) which is the risk measure that must be used following the European insurance Solvency II Directive. This is an important advantage of

[2] Results of Table 2 for other simulation numbers to a maximum of $m = 100,000$ are available under request to the authors.

**Table 2** Seller's prices and CVaR minimization at entropy levels $H \leq H^*$

| | Entropy Price[1] | | Portfolio | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $H$ | $\pi_H^{\max}$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $u_S$ | $u_b$ | $\mathrm{VaR}_\beta$ | $\mathrm{CVaR}_\beta$ |
| $H^*$ | 10.953 | 17.792 | 129.693 | 131.044 | 130.519 | 0.000 | −4.214 | 4.214 | 4.866 |
| | 10.852 | 18.571 | 129.693 | 131.044 | 130.519 | 0.000 | −3.435 | 3.435 | 4.087 |
| | 10.502 | 19.924 | 129.693 | 131.044 | 130.519 | 0.000 | −2.082 | 2.082 | 2.734 |
| | 10.002 | 21.319 | 129.693 | 131.044 | 130.519 | 0.000 | −0.688 | 0.688 | 1.340 |
| | 8.002 | 24.009 | 159.297 | 140.352 | 127.730 | 0.000 | 0.000 | 0.000 | 0.001 |
| | **H → 0** | **24.029** | **160.050** | **140.207** | **127.492** | **0.000** | **0.000** | **0.000** | **0.000** |

$\mathrm{VaR}_\beta$ and $\mathrm{CVaR}_\beta$ of the remaining claim, $\beta = 99.5\%$. Super replicating case is reported in bold characters. Simulation number $m = 60,000$, risk premium $\lambda = 0$

[1] $\forall H \leq H^* : \boldsymbol{\theta c} + u_S s_0 + u_b b_0 = \pi_H^{\max}$ and $\pi_{H^*}^{\max} = \pi^*$

our methodology with respect to other risk-minimizing strategies that may be founded on some variation of $L^2$ or $L^1$ distances.

Finally, let us point out that we ran calculations on an i-9, 2.30 GHz, 64GB RAM, intel laptop. We used the Generalized Algebraic Modeling System (GAMS) inside an R language environment. GAMS in turn called CONOPT for *non-linear optimization* and CPLEX for *linear programming*. The execution times in the case of a simulation number $m = 60,000$ were 4.5 min for the program (2), 1.4 min for the program (11), and 0.13 sec for the linear program (37).

## 6 Conclusions

In this article, we have given a new key idea to calculate EMPMs resulting in each no-arbitrage price of a contingent claim. This is the method of entropy segmentation consisting of two convex programs.

**Table 3** Risk premium $\lambda = -0.622$, simulation number $m = 60,000$

| | Entropy Price[1] | | Portfolio | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $H$ | $\pi_H^{\max}$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $u_S$ | $u_b$ | $\mathrm{VaR}_\beta$ | $\mathrm{CVaR}_\beta$ |
| $H^*$ | 10.937 | 17.823 | 131.732 | 132.616 | 136.449 | 0.000 | −4.723 | 4.634 | 5.349 |
| | 10.852 | 18.596 | 131.732 | 132.616 | 136.449 | 0.000 | −3.949 | 3.875 | 4.590 |
| | 10.502 | 20.198 | 131.732 | 132.616 | 136.449 | 0.000 | −2.347 | 2.303 | 2.684 |
| | 10.002 | 21.816 | 131.732 | 132.616 | 136.449 | 0.000 | −0.730 | 0.716 | 1.430 |
| | 8.002 | 24.687 | 157.692 | 149.767 | 131.809 | 0.000 | 0.000 | 0.000 | 0.000 |
| | | **24.695** | **157.987** | **149.708** | **131.728** | **0.000** | **0.000** | **0.000** | **0.000** |

Seller's prices and CVaR minimization at entropy levels $H \leq H^*$. $\mathrm{VaR}_\beta$ and $\mathrm{CVaR}_\beta$, $\beta = 99.5\%$. The super replicating case is reported in bold characters

[1] $\forall H \leq H^* : \boldsymbol{\theta c} + u_S s_0 + u_b b_0 = \pi_H^{\max}$ and $\pi_{H^*}^{\max} = \pi^*$

We have also applied Fenchel duality to get the dual programs interpreting them in terms of risk hedging. The cases of super replication and partial replication are now unified as particular cases of the same formula 25.

We have adapted the LP methodology from Rockafellar and Uryasev (2000) to calculate an optimal partial replicating portfolio at a given no arbitrage claim price whose remaining claim has minimal CVaR. This last amount will have to be reserved by the cliquet guarantee issuer to cover his risk. Then the hedging position of the claim issuer is defined by these three elements: seller's price at an entropy level, partial replicating portfolio, and minimal CVaR.

Finally, we have exemplified considering as a contingent claim a Cliquet-style guarantee.

Our methodology is quite general for it belongs to the family of inverse methods (see for instance Borwein (2012)), which can be applied to any underlying stochastic dynamic. In this regard, in our application, we have chosen Heston's dynamic with parameters calibrated on quoted prices of European calls estimated with real data. This way prices and hedging positions take into account the volatility risk.

## Appendix A: Fenchel Conjugates, Support and Perspective Functions

In this appendix, we write the Fenchel conjugates and the perspective functions necessary to build up the dual programs Eqs. 10 and 11. The Fenchel conjugate of a convex function $f$ with domain $\operatorname{dom} f = \{\boldsymbol{q} \in \mathbb{R}^m : f(\boldsymbol{q}) < \infty\}$, is defined as:

$$\forall \boldsymbol{y} \in \mathbb{R}^m : f^*(\boldsymbol{y}) = \sup_{\boldsymbol{q} \in \operatorname{dom} f} \left\{ \boldsymbol{y}' \cdot \boldsymbol{q} - f(\boldsymbol{q}) \right\}. \tag{A1}$$

Consider the set $F = \{\boldsymbol{q} \in \mathbb{R}^m : f(\boldsymbol{q}) \leq 0\}$. The *indicator function* of $F$ is:

$$\delta_F(\boldsymbol{q}) = \begin{cases} 0, & \boldsymbol{q} \in F \\ \infty, & \text{otherwise.} \end{cases} \tag{A2}$$

The Fenchel conjugate of $\delta_F$ is the *support function* $\delta_F^*$. It is linked to the Fenchel conjugate $f^*$ (see Roos et al. (2020) pp. 3, 27), provided that $F$ and its relative interior are nonempty sets, by the identity:

$$\forall \boldsymbol{y} \in \mathbb{R}^m : \delta_F^*(\boldsymbol{y}) = \sup_{\boldsymbol{q} \in F} \left\{ \boldsymbol{y}' \cdot \boldsymbol{q} \right\} = \min_{u \geq 0} \left\{ (uf)^*(\boldsymbol{y}) \right\}. \tag{A3}$$

The perspective function associated with $F$ is:

$$(uf)^*(\boldsymbol{y}) = \sup_{\boldsymbol{q} \in F} \left\{ \boldsymbol{y}' \cdot \boldsymbol{q} - uf(\boldsymbol{q}) \right\} = uf^*\left(\frac{\boldsymbol{y}}{u}\right). \tag{A4}$$

When working with inequality constraints, perspective functions are key elements to the calculation of support functions, which allow the expression of the Fenchel dual of the primal program in a tractable way.

*Objective function* (4): It is the same for both primal programs (10) and (11). For any $\boldsymbol{y}_0, \boldsymbol{q} \in \mathbb{R}^m$ its Fenchel conjugate $\pi^*$ is:

$$\pi^*(\boldsymbol{y}_0) = \sup_{\boldsymbol{q}} \left\{ \boldsymbol{y}_0' \cdot \boldsymbol{q} - f_0(\boldsymbol{q}) \right\} = \begin{cases} 0, & \boldsymbol{y}_0 = e^{-rT}\boldsymbol{x} \\ \infty, & \text{otherwise.} \end{cases} \tag{A5}$$

The Fenchel conjugate $(-\pi_0)^*$ of the opposite function is:

$$(-\pi)^*(\boldsymbol{y}_0) = \sup_{\boldsymbol{q}} \left\{ \boldsymbol{y}_0{}' \cdot \boldsymbol{q} + f_0(\boldsymbol{q}) \right\} = \begin{cases} 0\,, & \boldsymbol{y}_0 = -e^{-rT}\boldsymbol{x} \\ \infty\,, & \text{otherwise.} \end{cases} \tag{A6}$$

*Benchmark price constraints* (5): As $F_k = \left\{ \boldsymbol{q} \in \mathbb{R}^m : f_k(\boldsymbol{q}) = e^{-rT}\boldsymbol{q}' \cdot \boldsymbol{g}_k - c_k \le 0 \right\}$ for $k = 1, 2, \ldots, n$ and applying Eqs. A1 and A4, the Fenchel conjugate $f_k^*$ and the perspective function $(u_k f_k)^*$ are:

$$\forall \boldsymbol{y}_k \in \mathbb{R}^m: \quad f_k^*(\boldsymbol{y}_k) = \begin{cases} c_k\,, & \boldsymbol{y}_k = e^{-rT}\boldsymbol{g}_k \\ \infty\,, & \text{otherwise}\,. \end{cases} \tag{A7}$$

$$\forall u_k \ge 0: \quad (u_k f_k)^*(\boldsymbol{y}_k) = u_k f_k^*\left(\frac{\boldsymbol{y}_k}{u_k}\right) = \begin{cases} u_k c_k\,, & \boldsymbol{y}_k = e^{-rT}u_k\boldsymbol{g}_k \\ \infty\,, & \text{otherwise.} \end{cases} \tag{A8}$$

The reversed inequalities correspond to

$$F_k = \left\{ \boldsymbol{q} \in \mathbb{R}^m : f_k(\boldsymbol{q}) = c_{k-n} - e^{-rT}\boldsymbol{q}' \cdot \boldsymbol{g}_{k-n} \le 0 \right\},\ k = n+1, \ldots, 2n, \tag{A9}$$

thus:

$$\forall \boldsymbol{y}_k \in \mathbb{R}^m: \quad f_k^*(\boldsymbol{y}_k) = \begin{cases} -c_k\,, & \boldsymbol{y}_k = -e^{-rT}\boldsymbol{g}_k \\ \infty\,, & \text{otherwise} \end{cases} \tag{A10}$$

$$\forall u_k \ge 0: \quad (u_k f_k)^*(\boldsymbol{y}_k) = u_k f_k^*\left(\frac{\boldsymbol{y}_k}{u_k}\right) = \begin{cases} -u_k c_k\,, & \boldsymbol{y}_k = -e^{-rT}u_k\boldsymbol{g}_k \\ \infty\,, & \text{otherwise.} \end{cases} \tag{A11}$$

*Martingale constraints* (6): The two inequalities define the subsets:

$$F_{2n+1} = \left\{ \boldsymbol{q} \in \mathbb{R}^m : f_{2n+1}(\boldsymbol{q}) = e^{-rT}\boldsymbol{s}_T \cdot \boldsymbol{q} - s_0 \le 0 \right\} \tag{A12a}$$

$$F_{2n+2} = \left\{ \boldsymbol{q} \in \mathbb{R}^m : f_{2n+2}(\boldsymbol{q}) = s_0 - e^{-rT}\boldsymbol{s}_T \cdot \boldsymbol{q} \le 0 \right\}. \tag{A12b}$$

Applying Eqs. A1 and A4 we obtain:

$$(u_{2n+1} f_{2n+1})^*(\boldsymbol{y}_{2n+1}) = \begin{cases} s_0 u_{2n+1}\,, & \boldsymbol{y}_{2n+1} = u_{2n+1}e^{-rT}\boldsymbol{s}_T \\ \infty\,, & \text{otherwise} \end{cases} \tag{A13a}$$

$$(u_{2n+2} f_{2n+2})^*(\boldsymbol{y}_{2n+2}) = \begin{cases} -s_0 u_{2n+2}\,, & \boldsymbol{y}_{2n+2} = -u_{2n+2}e^{-rT}\boldsymbol{s}_T \\ \infty\,, & \text{otherwise.} \end{cases} \tag{A13b}$$

for any $\boldsymbol{y}_{2n+1}, \boldsymbol{y}_{2n+2} \in \mathbb{R}^m$ and for any $u_{2n+1}, u_{2n+2} \ge 0$.

*Probability constraints* (7): We have two sets:

$$F_{2n+3} = \left\{ \boldsymbol{q} \in \mathbb{R}^m : f_{2n+3}(\boldsymbol{q}) = \boldsymbol{q}' \cdot \mathbf{1} - 1 \le 0 \right\} \tag{A14a}$$

$$F_{2n+4} = \left\{ \boldsymbol{q} \in \mathbb{R}^m : f_{2n+4}(\boldsymbol{q}) = 1 - \boldsymbol{q}' \cdot \mathbf{1} \le 0 \right\}. \tag{A14b}$$

The perspective functions are:

$$\forall u_{2n+3} \ge 0 : (u_{2n+3} f_{2n+3})^*(\boldsymbol{y}_{2n+4}) = \begin{cases} u_{2n+3}\,, & \boldsymbol{y}_{2n+3} = u_{2n+3}\mathbf{1} \\ \infty\,, & \text{otherwise} \end{cases} \tag{A15a}$$

$$\forall u_{2n+4} \ge 0 : (u_{2n+4} f_{2n+4})^*(\boldsymbol{y}_{2n+4}) = \begin{cases} -u_{2n+4}\,, & \boldsymbol{y}_{2n+4} = u_{2n+4}(-\mathbf{1}) \\ \infty\,, & \text{otherwise.} \end{cases} \tag{A15b}$$

*Entropy constraint* (9): This defines the set:

$$F_{2n+5} = \left\{ \boldsymbol{q} \in \mathbb{R}_+^m : f_{2n+5}(\boldsymbol{q}) = \sum_{j=1}^{m} q^j \log\left(q^j\right) + H \leq 0 \right\}.$$

For any $\boldsymbol{y}_{2n+5}, \boldsymbol{q} \in \mathbb{R}_+^m$ we calculate the Fenchel conjugate of $f_{2n+5}$:

$$\begin{aligned} f_{2n+5}^*(\boldsymbol{y}_{2n+5}) &= \subset \boldsymbol{q} \sup \left\{ \boldsymbol{y}_{2n+5}' \cdot \boldsymbol{q} - f_{2n+5}(\boldsymbol{q}) \right\} \\ &= \subset \boldsymbol{q} \sup \left\{ \sum_{j=1}^{m} y_{2n+5}^j q^j - \sum_{j=1}^{m} q^j \log\left(q^j\right) - H \right\} \\ &= \sum_{j=1}^{m} e^{y_{2n+5}^j - 1} - H. \end{aligned} \tag{A16}$$

For any $\boldsymbol{y}_{2n+5} \in \mathbb{R}_+^m$ the perspective function is (see Eq. A4):

$$u_{2n+5} > 0 : (u_{2n+5} f_{2n+5})^*(\boldsymbol{y}_{2n+5}) = u_{2n+5} \left( \sum_{j=1}^{m} e^{\frac{y_{2n+5}^j}{u_{2n+5}} - 1} - H \right) \tag{A17a}$$

$$u_{2n+5} = 0 : (u_{2n+5} f_{2n+5})^*(\boldsymbol{y}_{2n+5}) = \begin{cases} 0, & \boldsymbol{y}_{2n+5} = \boldsymbol{0} \\ \infty, & \text{otherwise.} \end{cases} \tag{A17b}$$

*Non negativity constraints*: They define the subset $F_{2n+6} = \{\boldsymbol{q} \succeq \boldsymbol{0}\} = \mathbb{R}_+^m$. Its support function is Roos et al. (2020, p. 8):

$$\delta_{F_{2n+6}}^*(\boldsymbol{y}_{2n+6}) = \begin{cases} 0, & \boldsymbol{y}_{2n+6} \in \mathbb{R}_-^m \\ \infty, & \text{otherwise.} \end{cases} \tag{A18}$$

**Author Contributions** José L. Vilar-Zanón is responsible for the entropy segmentation and the application of Rockafellar's methodology in both its theoretical settings and numerical resolutions by G.A.M.S. embedded in R language; also of the dual translation from the pricing primal applying Fenchel duality. Barbara Rogo is responsible for the setting of the Cliquet option, its embedding in a life insurance policy, the valuation of the forward calls, all the simulations, and the calibration of the Heston model. Both authors are responsible of the introduction, discussions along the article, and the conclusions section.

**Funding** Open access funding provided by Università degli Studi di Roma La Sapienza within the CRUI-CARE Agreement. The corresponding author worked under projects $PID2020-115700RB-I00$ and $PID2021-125133NB-I00$.

**Data Availability** All data used in this article are available to the reader under request.

## Declarations

**Competing Interests** The authors declare no competing interests.

## References


Amari SI (2016) Information geometry and its applications (vol 194). Springer

Antonelli F, Mancini C, Pınar M (2013) Calibrated American option pricing by stochastic linear programming. Optimization 62(11):1433–1450

Avellaneda M, Laurence P (1999) Quantitative Modeling of Derivative Securities From Theory to Practice. CRC Press

Avellaneda M, Buff R, Friedman C, Grandechamp N, Kruk L, Newman J (2001) Weighted Monte Carlo: a new technique for calibrating asset-pricing models. Int J Theor Appl Fin 4(1):91–119

Bertsimas D, Popescu I (2002) On the relation between option and stock prices: a convex optimization approach. Oper Res 50(2):358–374

Borwein JM (2012) Maximum entropy and feasibility methods for convex and nonconvex inverse problems. Optimization 61(1):1–33

Bose C, Murray R (2014) Maximum entropy estimates for risk-neutral probability measures with non-strictly-convex data. J Optim Theory Appl 161(1):285–307

Breeden DT (1978) Litzenberger R H Prices of state-contingent claims implicit in option prices. J Bus 51(4):621–651

Carr P, Zhu QJ (2018) Convex Duality and Financial Mathematics. Springer

d’Aspremont A, El Ghaoui L (2006) Static arbitrage bounds on basket option prices. Math Progr Ser A 106:467–489

European Insurance and Occupational Pensions Authority (2018) Technical documentation of the methodology to derive EIOPA’s risk-free interest rate term structure

Föllmer H, Schweizer M (1991) Hedging of contingent claims under incomplete information. Appl Stoch Anal 5(389–414):19–31

Frittelli M (2000) The minimal entropy martingale measure and the valuation problem in incomplete markets. Math Financ 10(1):39–52

Gatzert N (2008) Asset management and surplus distribution strategies in life insurance: an examination with respect to risk pricing and risk measurement. Ins Math Econo 42(2):839–849

Gulko L (1999) The entropy theory of stock option pricing. Int J Theor Appl Fin 2(03):331–355

Heston SL (1993) A closed-form solution for options with stochastic volatility with applications to bond and currency options. Rev Fin Stud 6(2):327–343

Jackwerth JC, Rubinstein M (1996) Recovering probability distributions from option prices. J Financ 51(5):1611–1631

Kahalé N (2017) Superreplication of financial derivatives via convex programming. Manage Sci 63(7):2323–2339

King AJ, Koivu M, Pennanen T (2005) Calibrated option bounds. Int J Theor Appl Fin 8(02):141–159

Kruse S, Nögel U (2005) On the pricing of forward starting options in Heston’s model on stochastic volatility. Fin Stoch 9:233–250

Lord R, Koekkoek R, van Dijk D (2010) A comparison of biased simulation schemes for stochastic volatility models. Quant Fin 10(2):171–194

Luenberger DG (2002) Arbitrage and universal pricing. J Econ Dyn Control 26(9–10):1613–1628

Mordukhovich BS, Nam NM (2013) An easy path to convex analysis and applications. Synth Lect Math Stat 6(2):1–218

Musiela M, Rutkowski M (2005) Martingale Methods in Financial Modelling. Springer-Verlag

Nelder JA, Mead R (1965) A simplex method for function minimization. Comput J 7:308–313

Neri C, Schneider L (2012) Maximum entropy distributions inferred from option portfolios on an asset. Finance Stochast 16(2):293–318

Olivieri A, Pitacco E (2015) Introduction to Insurance Mathematics Technical and Financial Features of Risk Transfers. Springer

Pennanen T (2011) Convex duality in stochastic optimization and mathematical finance. Math Oper Res 36(2):340–362

Rockafellar RT (1970) Convex Analysis (No. 28). Princeton University Press

Rockafellar RT, Uryasev S (2000) Optimization of conditional value-at-risk. J Risk 2:21–42

Roos K, Balvert M, Gorissen BL, den Hertog D (2020) A universal and structured way to derive dual optimization problem formulations. Inf J Optim 2(4):229–255

Sheraz M, Preda V (2015) Entropy Measures in Finance and Risk Neutral Densities
Vilar-Zanón JL, Peraita-Ezcurra O (2018) Pricing Illiquid Assets by Entropy Maximization Through Linear Goal Programming. Mathematical and Statistical Methods for Actuarial Sciences and Finance. Springer, Cham, pp 513–518
Vilar-Zanón JL, Peraita-Ezcurra O (2019) A linear goal programming method to recover risk-neutral probabilities from options prices by maximum entropy. Decisions Econ Finan 42(1):259–276