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ON THE 3-COLORABLE SUBGROUP $\mathcal{F}$ AND MAXIMAL SUBGROUPS OF THOMPSON'S GROUP F

by Valeriano AIELLO & Tatiana NAGNIBEDA (*)

Dedicated to the memory of Vaughan F. R. Jones

Abstract. — In his work on representations of Thompson's group F, Vaughan Jones defined and studied the 3-*colorable subgroup* $\mathcal{F}$ of F. Later, Ren showed that it is isomorphic to the Brown–Thompson group F_4. In this paper we continue with the study of the 3-colorable subgroup and prove that the quasi-regular representation of F associated with the 3-colorable subgroup is irreducible. We show in addition that the preimage of $\mathcal{F}$ under a certain injective endomorphism of F is contained in three (explicit) maximal subgroups of F of infinite index. These subgroups are different from the previously known infinite index maximal subgroups of F, namely the parabolic subgroups that fix a point in $(0,1)$, (up to isomorphism) the Jones' oriented subgroup $\vec{F}$, and the explicit examples found by Golan.

Résumé. — Vaughan Jones a introduit et étudié un sous-groupe $\mathcal{F}$ du groupe de Thompson F dit *le sous-groupe 3-colorable*, apparu naturellement dans son travail sur les représentations unitaires de F. Ren a montré que ce sous-groupe est isomorphe au groupe F_4 de Brown–Thompson. Ici, nous poursuivons l'étude du sous-groupe 3-colorable et démontrons que la représentation quasi-régulière de F qui lui est associée est irréductible. Nous démontrons de plus que la préimage de $\mathcal{F}$ par un certain endomorphisme injectif de F est contenue dans trois sous-groupes maximaux de F que nous construisons explicitement. Ces sous-groupes maximaux sont d'indice infini et sont des nouveaux exemples dans la liste des sous-groupes maximaux d'indice infini connus dans F, tels les sous-groupes paraboliques fixant un point de l'intervalle $(0,1)$, le sous-groupe orienté $\vec{F}$ introduit par Jones, et les exemples construits par Golan.

Keywords: Thompson groups, Brown–Thompson groups, irreducible unitary representations, Jones representation, infinite index maximal subgroups, stabilizer subgroups, chromatic polynomial, closed subgroups.
2020 *Mathematics Subject Classification:* 20F65, 20E28, 22D10.
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Introduction

In [23], Vaughan Jones initiated a research program revolving around Thompson's groups and their unitary representations. In particular, he introduced two big families of such representations, one arising from planar algebras [22] and one arising from the Pythagorean C*-algebra [12]. The stabilizers of a canonical vector (the so-called vacuum vector) in these representations turn out to be interesting subgroups of Thompson's groups. For example, for Thompson's group F, the parabolic subgroups $\mathrm{Stab}(t) \leqslant F$ (with $t \in (0,1)$) arise in this way from the Pythagorean representations [12]. Similarly, a certain representation related to the Temperley–Lieb planar algebra gives rise to the so-called oriented subgroup $\vec{F} \leqslant F$, which corresponds to the oriented links in Jones's encoding of knots and links by elements of F, see [1, 23].

Another interesting example of similar origin is the so-called 3–colorable subgroup $\mathcal{F} \leqslant F$, [24, 31]. It is the main object of this paper and it is worth recalling its story. Roughly speaking, one of the original aims of Jones's project was to obtain representations of $\mathrm{Diff}^+(S^1)$ as limits of representations of Thompson's group T, seen as a group of homeomorphisms of S^1. However, in [24] Jones discovered that, in general, this is not possible. In fact, he defined two families of unitary representations for Thompson's groups by means of the planar algebras of quantum $SO(3)$, one for F and one for T (see [28] for more information on these planar algebras). Then, he computed the weak limit of the rotations by angle 2^{-n} in the representations of T, when n tends to infinity, and saw that contrary to what one would hope, the limit is not the identity. The 3-colorable subgroup $\mathcal{F}$ arises as the stabiliser of the vacuum vector in one of these representations of F, and we will now explain its construction.

Denote by $\mathcal{T}_k$ the set of rooted planar finite k-regular trees. It is well known that the elements of Thompson's group F can be viewed as equivalence classes of pairs of finite planar binary trees. Brown [13] introduced a family of groups that share many properties with F. These groups, sometimes called Brown–Thompson groups F_k, $k \geqslant 2$, are groups of piecewise linear homeomorphisms of the interval $[0,1]$ with slopes powers of k and points of non-continuity of the derivative k-adic rationals. Their elements can be described as equivalence classes of pairs of trees from $\mathcal{T}_k$, $k \geqslant 2$. Besides the Thompson's groups F and T, a third group was also introduced by Richard Thompson, namely V. This group can be described both as a group of homeomorphisms of the Cantor set and in terms of binary trees.

The groups F and T are proper subgroups of V. After the work of Thompson and before that of Brown, Higman [21] introduced a family of groups generalising V. These are the groups $V_{n,r}$ and they form an infinite family of finitely presented, infinite, simple groups, with the additional parameter r for the number of roots of the n-ary trees.

In [31] Ren introduced the following construction of subgroups of F isomorphic with F_k. Let $T \in \mathcal{T}_2$ be a rooted planar binary tree with k leaves. Define an injective map $\alpha_T : F_k \to F$: given a pair of rooted k-regular trees $(T_+, T_-) \in F_k$, replace any vertex of degree $k+1$ with the tree T.

For a tree $T \in \mathcal{T}_2$, denote by $\ell_T(0)$ the number of left edges in the path from the left-most leaf to the root and by $\ell_T(k-1)$ the number of right edges in the path from the right-most leaf to the root. Recall that the abelianization map $\pi : F \to F/[F,F] = \mathbb{Z} \oplus \mathbb{Z}$ can be described as $\pi(f) = (\log_2 f'(0), \log_2 f'(1))$, see [14]. If $f \in F$ is represented by a pair of trees (T_+, T_-), then $\log_2 f'(0)$ is equal to $\ell_{T_+}(0) - \ell_{T_-}(0)$. Similarly, $\log_2 f'(1)$ is equal to $\ell_{T_+}(k-1) - \ell_{T_-}(k-1)$.

Bleak and Wassink considered in [11], for any a, $b \in \mathbb{N}$, the rectangular subgroups of F defined as

$$K_{(a,b)} \coloneqq \{f \in F \mid \log_2 f'(0) \in a\mathbb{Z}, \log_2 f'(1) \in b\mathbb{Z}\}$$

All these subgroups are finite index subgroups of F isomorphic with it. It is not difficult to see that the subgroup $\alpha_T(F_k)$ sits inside $K_{(\ell_T(0), \ell_T(k-1))}$.

The first example of this construction is Jones's oriented subgroup $\vec{F}$. It corresponds to the case $k = 3$ where there is essentially the unique map $\alpha_T : F_3 \to F$ depicted on Figure 1 and its image is precisely the oriented subgroup $\vec{F}$. In fact, Golan and Sapir were the first to prove that $\vec{F}$ is isomorphic with F_3, see [18].

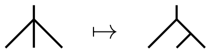

Figure 1. Ren's map for $\vec{F}$.

The oriented subgroup $\vec{F}$ was originally defined as the set of pairs of trees (T_+, T_-) such that the value of the chromatic polynomial $\mathrm{Chr}_{\Gamma(T_+,T_-)}(2)$ is non-zero, where $\Gamma(T_+, T_-)$ is a certain graph associated with (T_+, T_-), see [23]. Since $\Gamma(T_+, T_-)$ is always connected, this specialisation of the chromatic polynomial takes only two values: 0 and 2. In passing, we briefly describe how Ren proved that $\vec{F}$ can be realised as the image of α_T described in Figure 1, [31]. This is done with an inductive argument on the number of the leaves. First, by investigating the 2-colorings of $\Gamma(T_+, T_-)$, he showed that every element of $\vec{F}$ is generated by the *positive*[1] elements in $\vec{F}$. Then he proved that each positive element in $\vec{F}$ contains a binary tree of the form depicted in Figure 1, and so by multiplying this element by $(x_i x_{i+1})^{-1}$, for a suitable i, it was possible to cancel this subtree (thus reducing the number of leaves and ending the inductive argument).

More generally, by using the Temperley–Lieb planar algebra, for every $t \in \{4\cos^2(\pi/n) \mid n \geqslant 4\} \cup [4,\infty)$ one can define a family of representations $(\pi_t)_t$ of F such that

$$\langle \pi_t(T_+,T_-)\Omega,\Omega\rangle = \frac{\mathrm{Chr}_{\Gamma(T_+,T_-)}(t)}{t(t-1)^{n-1}}$$

where n is the number of leaves of T_+ and Ω is a canonical unit vector which is usually called the vacuum vector. The stabiliser of Ω consists of the elements (T_+,T_-) in F such that

$$\mathrm{Chr}_{\Gamma(T_+,T_-)}(t) = t(t-1)^{n-1}$$

For non-integer values of t, the computation of the chromatic polynomial is often more demanding. However, it was shown in [3] that for $t > 4.63$ these subgroups are all trivial. In passing, we mention that several other knot and graph invariants can also be used to define unitary representations of Thompson's groups F and $\vec{F}$, see e.g. [23] and [3, 4, 5, 6].

Similarly, the 3-colorable subgroup $\mathcal{F}$ is the image of the map $\alpha_T : F_4 \to F$ depicted in Figure 2. It can also be defined in terms of a specialisation of the chromatic polynomial of the following graph. Take an element (T_+,T_-) in F, join the two roots by an edge and draw its dual graph $\tilde{\Gamma}(T_+,T_-)$. The elements of $\mathcal{F}$ are exactly those for which the chromatic polynomial of $\tilde{\Gamma}(T_+,T_-)$ evaluated at 3 is non-zero (and in this case the only possible value is 6). See Section 1 for more details.

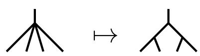

Figure 2. The map for $\mathcal{F}$.

It turns out that the subgroups arising as stabilizers of the vacuum vector in these various representations are also interesting from the viewpoint of understanding the maximal subgroups in Thompson's groups. Recall that parabolic subgroups are natural examples of maximal subgroups of infinite index in F, [32, 33]. The oriented subgroup provided, up to an isomorphism, the first explicit example of a maximal subgroup of infinite index

[1] See Section 1 for a definition of positive elements of F.

of F without fixed points in the open unit interval $(0,1)$, as proven by Golan and Sapir in [19]. Moreover, in the same paper a method for potentially producing more examples of maximal subgroups of infinite index in F was introduced (see Section 4). Later, three further explicit examples of maximal infinite index subgroups without fixed points appeared in [17, Section 10.3]. One might wonder whether all maximal subgroups of infinite index arise as stabilizers of suitable subsets. However, this is not the case, at least when we restrict to the subset of dyadic rationals. Indeed, Golan showed in [17] that one of the aforementioned examples of maximal subgroups acts transitively on this set. A more subtle problem is to determine the isomorphism classes of infinite index maximal subgroups. Despite being all distinct, the parabolic subgroups actually reduce to only three isomorphism classes as shown in [20] (see also the recent paper [16] for a similar classification in the wide context of groups with micro-supported actions). The general problem of describing and classifying the maximal subgroups in F remains very much open.

Maximal subgroups are also of interest in the study of unitary representations by means of quasi-regular representations. By a classical result of Mackey [27], the quasi-regular representations associated with a subgroup is irreducible if the subgroup coincides with its commensurator. For a maximal subgroup it is then enough to exclude its commensurator being equal to the whole group to conclude that the quasi-regular representations is irreducible. It is an important problem to determine whether the unitary representations of Thompson's group defined by Jones are irreducible. Positive evidence for that includes Golan and Sapir's work [18], where they showed that $\vec{F}$ coincides with its commensurator, as well as the recent paper of Jones [26].

Let us now say a few words on the results of our paper. The first main result is that, similarly to the case of $\vec{F}$, the 3-colorable subgroup can be described as the stabilizer of a certain subset of dyadic rationals, namely to any of

$$S_i := \{t \in (0,1) \cap \mathbb{Z}[1/2] \mid \omega(t) = i,\ |t| \in 2\mathbb{N}\} \qquad i = 0,1,2$$

where $|t|$ is the length of t in its binary expansion, i.e., $t = .a_1 \dots a_n$ (with $a_n = 1$) and $|t| = n$, and $\omega(t)$ is some natural function on binary words with values in $\{0,1,2\}$ (see Section 2 for a precise definition). In fact, in Theorem 2.9 we prove that $\mathcal{F}$ is equal to $\mathrm{Stab}(S_i)$, for $i = 0,1,2$. This allows us to show in Theorem 2.10 that $\mathcal{F}$ coincides with its commensurator and, in turn, this implies that the quasi-regular representation of F associated with $\mathcal{F}$ is irreducibile (see Corollary 2.11). The second main result of this paper is contained in Theorem 4.6, where we exhibit three subgroups M_0, M_1, M_2 between $\mathcal{F}$ and the rectangular subgroup $K_{(2,2)}$. These turn out to be maximal subgroups of infinite index in $K_{(2,2)}$. By means of an isomorphism between F and $K_{(2,2)}$, we obtain three infinite index maximal subgroups of F, which are shown to be distinct from all previously known examples of maximal infinite index subgroups of F: the parabolic subgroups, the oriented subgroup, as well as the examples in [18, Section 3.2] and [17, Section 10.3.B]. We also provide partial evidence that the only subgroups between $\mathcal{F}$ and $K_{(2,2)}$ are M_0, M_1, M_2.

We end this introduction with the structure of the paper. In Section 1 we recall the definitions of Thompson's group F and of the 3-colorable subgroup, along with some of their main properties. In Section 2 we provide a description of $\mathcal{F}$ as the homeomorphisms preserving certain subsets of dyadic rationals. We exploit this description to prove the irreducibility of the quasi-regular representation of F associated with $\mathcal{F}$. In Section 3 we exhibit an explicit isomorphism θ between F and the rectangular subgroup $K_{(2,2)}$ consisting of the homeomorphisms whose derivatives at 0 and 1 are in $2^{2\mathbb{Z}}$. In Section 4 we first show that $\mathcal{F}$ is contained in $K_{(2,2)}$, then we exhibit three infinite index maximal subgroups of $K_{(2,2)}$. In Section 5 we compare $\theta^{-1}(M_0)$, $\theta^{-1}(M_1)$, and $\theta^{-1}(M_2)$ to other infinite index maximal subgroups of F that have been identified before.

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1. Preliminaries and notation

In this section we recall the definitions and basic properties of Thompson's group F and of Jones's 3-colorable subgroup. The interested reader is referred to [14] and [9] for a general introduction on Thompson's groups and its basic properties, to [31] for further information on the 3-colorable subgroup.

Thompson's group F is the group of all piecewise linear homeomorphisms of the unit interval $[0,1]$ that are differentiable everywhere except at finitely many dyadic rationals numbers and such that on the intervals of differentiability the derivatives are powers of 2. We adopt the standard notation: $f \cdot g(t) = g(f(t))$.

Thompson's group has the following infinite presentation

$$F = \langle x_0, x_1, \dots \mid x_n x_k = x_k x_{n+1} \quad \forall\, k < n \rangle \,.$$

The monoid generated by $x_0, x_1, \dots$ is denoted by F_+. Its elements are said to be positive. Note that x_0 and x_1 are enough to generate F (see Figure 1.1 for their description in terms of pairs of binary trees).

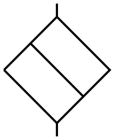
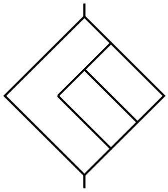
$x_0 =$  $x_1 =$ 

Figure 1.1. The generators of $F = F_2$.

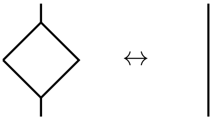
Similarly, for any $k \geqslant 2$, the Brown–Thompson group F_k admits the following presentation

$$\langle y_0, y_1, \dots \mid y_n y_l = y_l y_{n+k-1} \quad \forall\, l < n \rangle \,.$$

The elements $y_0, y_1, \dots, y_{k-1}$ generate F_k (see Figure 1.2 for the generators of F_4).

The projection of F onto its abelianisation is denoted by $\pi : F \to F/[F,F] = \mathbb{Z} \oplus \mathbb{Z}$ and it admits a nice interpretation when F is seen as group of homeomorphisms: $\pi(f) = (\log_2 f'(0), \log_2 f'(1))$.

There is still another description of F which is relevant to this paper: the elements of F can be seen as pairs (T_+, T_-) of planar binary rooted trees (with the same number of leaves). We draw one tree upside down on top of the other; T_+ is the top tree, while T_- is the bottom tree. Any pair of trees (T_+, T_-) represented in this way is called a tree diagram. Two pairs of trees are said to be equivalent if they differ by pairs of opposing carets, namely



Every equivalence class of pairs of trees (i.e. an element of F) gives rise to exactly one tree diagram which is reduced, in the sense that the number of its vertices is minimal, [9].

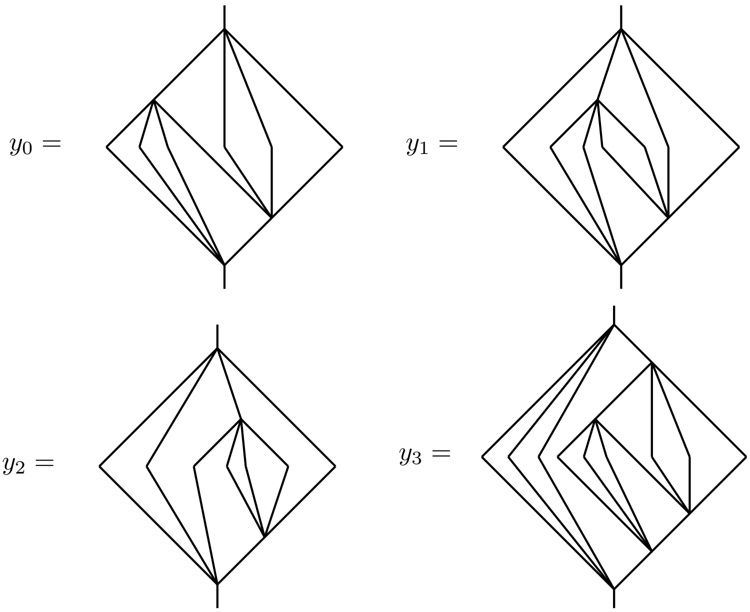


Figure 1.2. The generators of F_4.

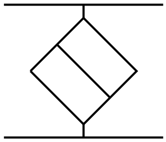
Convention 1.1. — *We make a convention about how we draw trees on the plane. The roots of our planar binary trees are drawn as vertices of degree* 3 *and hence each tree diagram has the uppermost and lowermost vertices of degree* 1*, which lie respectively on the lines $y = 1$ and $y = -1$. The leaves of the trees sit on the x-axis, precisely on the non-negative integers.*

Any tree diagram partitions the strip bounded by the lines $y = 1$ and $y = -1$ in regions. This strip may or may not be 3-colorable, i.e., it may or may not be possible to assign the colors $\mathbb{Z}_3 = \{0, 1, 2\}$ to the regions of the strip in such a way that if two regions share an edge, they have different colors. By convention, we assign the following colours to the regions near the roots

(1.1)


Once we make this convention, if the strip is 3-colourable, there exists a unique colouring. The 3-*colorable subgroup* $\mathcal{F}$ consists of the elements of

F for which the corresponding strip is 3-colorable. For example, this is the strip corresponding to x_0 (which is not 3-colorable)



The 3-colorable subgroup subgroup was introduced by Jones and studied by Ren, who proved the following result.

Theorem 1.2 ([31]). — *The map α_T with T depicted in Figure 1 is an isomorphism between F_4 and the 3-colorable subgroup. In particular, the 3-colorable subgroup is generated by the following elements $w_0 \coloneqq x_0^2x_1x_2^{-1}$, $w_1 \coloneqq x_0x_1^2x_0^{-1}$, $w_2 \coloneqq x_1^2x_3x_2^{-1}$, $w_3 \coloneqq x_2^2x_3x_4^{-1}$ (see Figure 1.3), which are the images of the generators of F_4.*

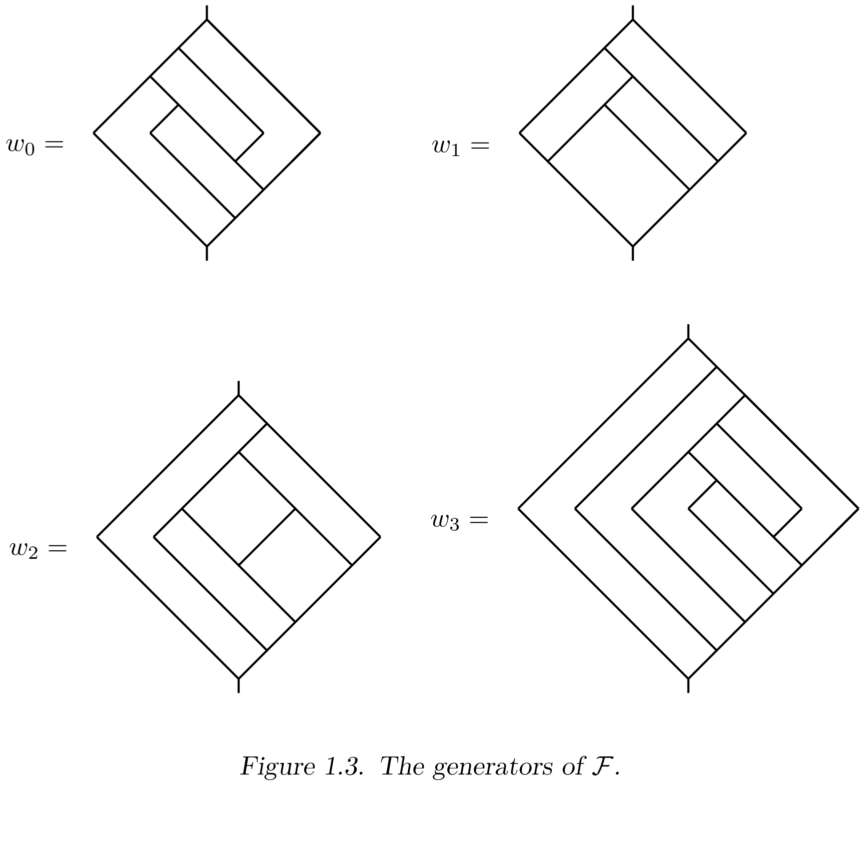


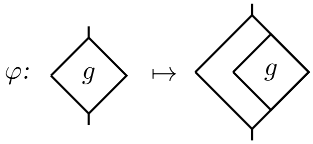
Figure 1.3. The generators of $\mathcal{F}$.

The next couple of easy lemmas will come in handy in the other sections of the paper.

Lemma 1.3. — *Let $\sigma : F \to F$ be the order 2 automorphism obtained by reflecting tree diagrams about a vertical line. For any $g \in F$, we have $g \in \mathcal{F}$ if and only if $\sigma(g) \in \mathcal{F}$.*

Proof. — Clearly the strip associated with g is 3-colorable if and only if the strip associated with $\sigma(g)$ is 3-colorable (the second strip is obtained by the first after a reflection about a vertical line). □

Lemma 1.4. — *Consider the shift homomorphism $\varphi : F \to F$ defined graphically as*



Then $g \in \mathcal{F}$ if and only if $\varphi(g) \in \mathcal{F}$.

Proof. — The claim is clear after drawing the strips corresponding to g and $\varphi(g)$. □

In terms of homeomorphisms of $[0,1]$, the shift homomorphism φ takes elements of F and squeezes them onto $[1/2,1]$. In particular, φ maps x_i to x_{i+1} for all $i \geqslant 0$. We observe that the range of φ is the subgroup of elements of F that act trivially on $[0,1/2]$.

2. The 3-colorable subgroup as a stabiliser subgroup

The goal of this section is to provide a description of $\mathcal{F}$ as a stabiliser of a subset of the dyadic rationals. This result is analogous to the one obtained by Golan and Sapir for $\vec{F}$, where it was realised as the stabiliser of a subset of dyadic rationals, namely that consisting of elements with an even number of digits equal to 1 in the binary expansion. Our approach is similar, but the proof is more involved as our subset of dyadic rationals is more complicated and in order to describe it we introduce a suitable *weight* function ω on dyadic rationals viewed as paths from the root of the tree of standard dyadic intervals, which is intimately related to the coloring of the regions in the strips associated with the elements $\mathcal{F}$.

Consider a rooted binary planar tree T. We draw it in the upper-half plane with its leaves on the x-axis and the highest vertex (of degree 1) on the line $y = 1$. Given a vertex of a tree, there exists a unique minimal path from the root of the tree to the vertex. This path is made by a collection of left, right edges, and may be represented by a word in the letters $\{0, 1\}$ (0 stands for a left edge, 1 for a right edge). An easy inductive argument on the number of vertices shows that the strip bounded by the line $y = 1$ and the x-axis is always 3-colourable. We adopt the same convention as in (1.1), that is the region to the left of the root is colored with 0, the region to the right with 1 and the region below with 2. After this choice, there is a unique coloring of the strip.

Given a vertex v of T, we denote by $\omega(v)$ the color of the region to the left of v. We call $\omega(\cdot)$ the weight associated with T. The weight ω can be actually defined on the infinite binary rooted planar tree (where every vertex has two descendants). In fact, given a rooted subtree S of T, the restriction of the weight of T to the vertices of S is the weight of S. This allows one to consider the weight of the infinite rooted regular binary tree. If we denote by W_2 the set of finite binary words, this yields a function $\omega : W_2 \to \mathbb{Z}_3$. When we consider a tree diagram (T_+, T_-), we denote by $\omega_+(\cdot)$ and $\omega_-(\cdot)$ the weights associated with the top tree and the bottom tree (after a reflection about the x-axis), respectively.

The following result follows at once from the definitions.

Proposition 2.1. — *It holds*

$$\mathcal{F} = \{(T_+, T_-) \in F \mid \omega_+(i) = \omega_-(i)\ \forall i \geqslant 0\}$$

where $\omega_+(i)$ and $\omega_-(i)$ stand for the colors associated with the i-th leaf of the trees T_+ and T_-, respectively.

The next couple of lemmas are instrumental for obtaining a description of $\mathcal{F}$ as a stabiliser subgroup.

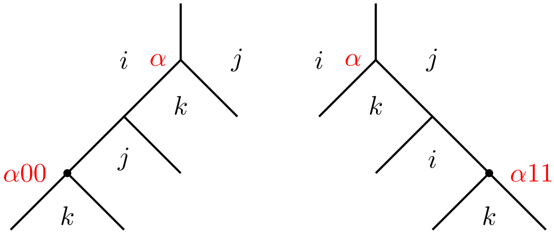
Lemma 2.2. — *For all α, $\beta \in W_2$, it holds*

$$\omega(\alpha 00\beta) = \omega(\alpha\beta) \tag{2.1}$$

$$\omega(\alpha 11\beta) = \omega(\alpha\beta) \tag{2.2}$$

$$\omega(\alpha 0) = \omega(\alpha) \tag{2.3}$$

Proof. — As (2.1) and (2.2), the figure below shows that after two consecutive left and right edges, the colours labelling the regions surrounding a vertex are the same (the black letters denote the colours of the regions, whereas the red ones describe the path from the root to the vertex).



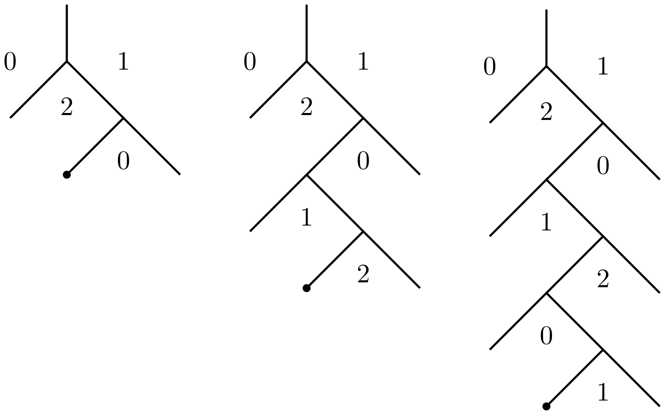
Formula (2.3) is obvious. □

Lemma 2.3. — *For any $n \in \mathbb{N}$, it holds*

$$\omega((10)^n) = \begin{cases} 0 & \text{if } n \equiv_3 0, \\ 2 & \text{if } n \equiv_3 1, \\ 1 & \text{if } n \equiv_3 2. \end{cases} \qquad \omega((01)^n) = \begin{cases} 0 & \text{if } n \equiv_3 0, \\ 1 & \text{if } n \equiv_3 1, \\ 2 & \text{if } n \equiv_3 2. \end{cases} \tag{2.4}$$

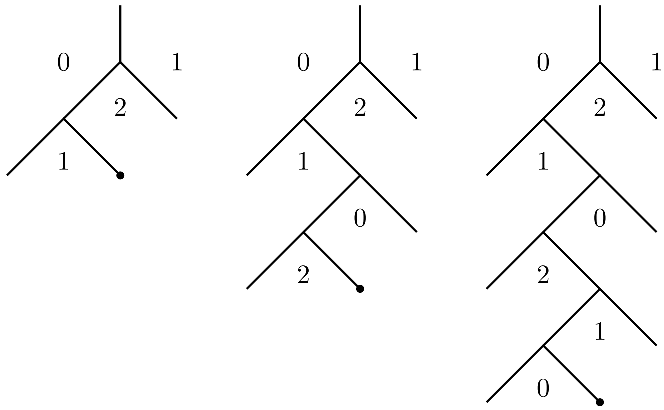
where, for a finite word w in 0 and 1, w^n denotes the word obtained by concatenating n copies of w, and the symbol $\equiv_3$ denotes the equivalence modulo 3.

Proof. — We begin with the formula for $\omega((10)^n)$. The next figure shows that the region to the left of the marked vertices are 2, 1, 0 for the cases $n = 1, 2, 3$, respectively.



Since the regions of the regions surrounding the vertex 101010 have the same colours as those near the root, the former cases suffice to prove the formula.

We now take care of the formula for $\omega((01)^n)$. The next figure displays the cases $n = 1, 2, 3$ which are enough to prove our claim.



□

Remark 2.4. — The properties (2.1), (2.2), (2.3), (2.4) completely determine the function $\omega : W_2 \to \mathbb{Z}_3$.

The next result provides a simple formula for the weight ω.

Proposition 2.5. — *For a binary word $a_1 a_2 \ldots a_n$ it holds*

$$\omega(a_1 a_2 \ldots a_n) \equiv_3 \sum_{i=1}^{n} (-1)^i a_i \,.$$

Proof. — We define the function $f : W_2 \to \mathbb{Z}_3$ by the formula

$$f(a_1 \ldots n_n) \coloneqq \sum_{i=1}^{n} (-1)^i a_i .$$

It is easy to check that f satisfies the properties of Lemmas 2.2 and 2.3. Moreover, $f(1) = -1 \equiv_3 2 = \omega(1)$, $f(101) = -2 \equiv_3 1 = \omega(101)$, $f(10101) = -3 \equiv_3 0 = \omega(10101)$, $f(01) = 1 = \omega(01)$, $f(0101) = 2 = \omega(0101)$, $f(010101) = 3 \equiv_3 0 = \omega(010101)$ and so the functions f and ω coincide. □

Note that the symbols 0 and 1 appearing in the formula for $\omega(\cdot)$ have several meanings, namely those in the word $a_1 a_2 \ldots a_n$ are related to left/right edges, while those in the output of ω are interpreted as elements of $\mathbb{Z}_3$ and denote the color of the regions.

Lemma 2.6. — *Let (T_+, T_-) be an element in $\mathcal{F}$. Then, for any leaf, the parity of the length of the paths to the roots is the same.*

Proof. — The claim is true for the generators of $\mathcal{F}$, see Figure 1.3. By using the multiplication algorithm described in [9, Chapter 7], one can easily see that this property is preserved by multiplication (it suffices to check that the two reduction moves preserve the property) and we are done. □

There exists a bijection ρ between the finite sequences of 0 and 1 and the dyadic rationals in the open unit interval, namely the map $\rho(a_1 \dots a_n) := \sum_{i=1}^{n} a_i 2^{-i}$, where $.a_1 \dots a_n$ is word in W_2. For $i \in \mathbb{Z}_3$ consider the subsets S_i of the dyadic rationals consisting of the numbers whose corresponding binary word has even length and weight i, namely

$$S_i := \{t \in (0,1) \cap \mathbb{Z}[1/2] \mid \omega(t) = i,\ |t| \in 2\mathbb{N}\} \qquad i \in \mathbb{Z}_3$$

where $|t|$ is the length of t in its binary expansion, i.e., $|.a_1 \dots a_n| = n$ if $a_n \neq 0$.

Lemma 2.7. — *The* 3*-colorable subgroup* $\mathcal{F}$ *coincides with*

$$\bigcap_{i \in \mathbb{Z}_3} \mathrm{Stab}(S_i).$$

Proof. — The inclusion $\mathcal{F} \subset \bigcap_{i \in \mathbb{Z}_3} \mathrm{Stab}(S_i)$ follows easily by checking that the generators of $\mathcal{F}$, namely $w_0 := x_0^2 x_1 x_2^{-1}$, $w_1 := x_0 x_1^2 x_0^{-1}$, $w_2 := x_1^2 x_3 x_2^{-1}$, $w_3 := x_2^2 x_3 x_4^{-1}$ preserve the sets S_i for all $i \in \mathbb{Z}_3$. For the converse inclusion, let $f = (T_+, T_-)$ be an element of $\bigcap_{i \in \mathbb{Z}_3} \mathrm{Stab}(S_i)$. Denote by $\alpha_+(k)$ and $\alpha_-(k)$ the words associated with the k-th leaf of the top and bottom trees, respectively. Since $f \in \bigcap_{i \in \mathbb{Z}_3} \mathrm{Stab}(S_i)$, it follows that $\omega_+(\alpha_+(k)) \equiv_3 \omega_-(\alpha_-(k))$ for all k. By Proposition 2.1 we have that $f \in \mathcal{F}$. □

Remark 2.8. — We observe that the previous lemma implies that $\mathcal{F} \leqslant K_{(1,2)} = \{f \in F \mid \log_2 f'(1) \in 2\mathbb{Z}\}$. Indeed, given $f = (T_+, T_-) \in \mathcal{F}$, let α_+ and α_- be the words corresponding to the right most leaves of T_+ and T_- (with $k_\pm = |\alpha_\pm|$). These words do not contain any occurrences of the letter 0. Therefore, $\omega(\alpha_\pm) \in \{-1, 0\}$ (as integers). More precisely, $\omega(\alpha_\pm) = -1$ if and only if $k_\pm \in 2\mathbb{Z}+1$. Similarly, $\omega(\alpha_\pm) = 0$ if and only if $k_\pm \in 2\mathbb{Z}$. Since $\log_2 f'(1) = k_+ - k_-$, this implies that $\log_2 f'(1) \in 2\mathbb{Z}$.

We are now in a position to prove the main result of this section.

Theorem 2.9. — *The* 3*-colorable subgroup* $\mathcal{F}$ *coincides with* $\mathrm{Stab}(S_i)$ *for all* $i \in \mathbb{Z}_3$.

Proof. — From Lemma 2.7 we know that $\mathcal{F}$ is contained in $\mathrm{Stab}(S_i)$ for all $i \in \mathbb{Z}_3$, so we only have to prove the converse inclusion. Take an element

$f = (T_+, T_-)$ in $\mathrm{Stab}(S_i)$, for some $i \in \mathbb{Z}_3$. Consider a leaf and denote by α_+ and α_- the words corresponding to the paths to the roots of the top and bottom trees, respectively. We want to show that $\omega(\alpha_+) \equiv_3 \omega(\alpha_-)$. There are several cases to deal with. For convenience we give a proof with the aid of some tables. Here is the idea of the proof. The number $.\alpha_+$ may or may not be in S_i. If it is not in S_i, we take a suitable binary word w such that $.\alpha_+ w \in S_i$. Since f is in $\mathrm{Stab}(S_i)$, we have that $f(.\alpha_+ w) = .\alpha_- w$ is in S_i. Finally, we use this to show that $\omega(\alpha_+) \equiv_3 \omega(\alpha_-)$ and, thus, by Lemma 2.1 f is in $\mathcal{F}$. The following table corresponds to the case $i \equiv_3 0$.

$\lvert\alpha_+\rvert$	$\omega(\alpha_+)$	Substitution for $\alpha_+ : \alpha_+ w$	$\lvert\alpha_- w\rvert$	$\lvert\alpha_-\rvert$	$\omega(\alpha_+ w)$	$\omega(\alpha_- w)$	$\omega(\alpha_-)$
EVEN	0	α_+	EVEN	EVEN	0	0	0
ODD	0	α_+011	EVEN	ODD	0	0	0
EVEN	1	α_+0101	EVEN	EVEN	0	0	1
ODD	1	α_+101	EVEN	ODD	0	0	1
EVEN	2	α_+01	EVEN	EVEN	0	0	2
ODD	2	α_+1	EVEN	ODD	0	0	2

Here are the tables for $f \in \mathrm{Stab}(S_1)$

$\lvert\alpha_+\rvert$	$\omega(\alpha_+)$	Substitution for $\alpha_+ : \alpha_+ w$	$\lvert\alpha_- w\rvert$	$\lvert\alpha_-\rvert$	$\omega(\alpha_+ w)$	$\omega(\alpha_- w)$	$\omega(\alpha_-)$
EVEN	1	α_+	EVEN	EVEN	1	1	1
ODD	1	α_+011	EVEN	ODD	1	1	1
EVEN	2	α_+0101	EVEN	EVEN	1	1	2
ODD	2	α_+101	EVEN	ODD	1	1	2
EVEN	0	α_+01	EVEN	EVEN	1	1	0
ODD	0	α_+1	EVEN	ODD	1	1	0

and $f \in \mathrm{Stab}(S_2)$

$\lvert\alpha_+\rvert$	$\omega(\alpha_+)$	Substitution for $\alpha_+ : \alpha_+ w$	$\lvert\alpha_- w\rvert$	$\lvert\alpha_-\rvert$	$\omega(\alpha_+ w)$	$\omega(\alpha_- w)$	$\omega(\alpha_-)$
EVEN	2	α_+	EVEN	EVEN	2	2	2
ODD	2	α_+011	EVEN	ODD	2	2	2
EVEN	0	α_+0101	EVEN	EVEN	2	2	0
ODD	0	α_+101	EVEN	ODD	2	2	0
EVEN	1	α_+01	EVEN	EVEN	2	2	1
ODD	1	α_+1	EVEN	ODD	2	2	1

$\square$

As a corollary, we obtain the following theorem.

THEOREM 2.10. — *The 3-colorable subgroup coincides with its commensurator.*

Corollary 2.11. — *The quasi-regular representation of F associated with $\mathcal{F}$ is irreducible.*

The irreducibility of the quasi-regular representation associated with $\mathcal{F}$ follows from [27]. The proof of the Theorem is similar to that for the oriented subgroup $\vec{F}$ done by Golan and Sapir [18, Theorem 4.15].

We now recall the natural action of an element $f \in F$ on the numbers in $[0,1]$ expressed in binary expansion. A number t enters into the top of the tree diagram, follows a path towards the root of the bottom tree according to the rules portrayed in Figure 2.1. What emerges at the bottom is the image of t under the homeomorphism represented by the tree diagram, [10]. Note that there is a change of direction only when the number comes across a vertex of degree 3 (i.e., the number is unchanged when it comes across a leaf).

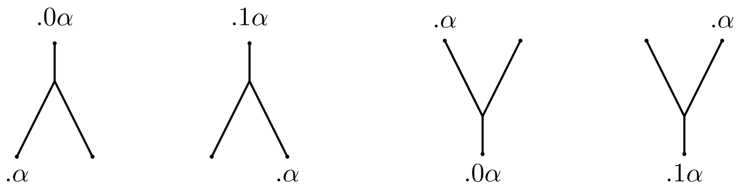


Figure 2.1. The local rules for computing the action of F on numbers expressed in binary expansion.

The next result is analogous to [18, Lemma 4.14].

Lemma 2.12. — *Let $g \in F$. Then there exists $m \in \mathbb{N}$ such that for any $t \in (0, 1/2^m) \cap S_0$, $\omega(t) \equiv_3 \omega(g(t))$.*

Proof. — By definition, $g(t) = 2^l t$ for all $t \in I = [0, 2^{-r}]$, where $r \in \mathbb{N}$, $l \in \mathbb{Z}$. If $l \leqslant 0$ $g(t)$ simply adds l zeros to the beginning of the binary form of t. Therefore, we have two cases depending on whether $\log_2 g'(0)$ is in $2\mathbb{Z}$ or $2\mathbb{Z}+1$. In the first case $\omega(t) = \omega(g(t))$, while in the second $\omega(t) \equiv_3 -\omega(g(t))$. In both cases, it suffices to take $m = r$.

If $l > 0$, take $m = \max\{r, l\}$. Since $m \geqslant r$, the binary word for t begins with at least l zeroes and g erases l of them. We have two cases as before: $\log_2 g'(0)$ is in $2\mathbb{Z}$ or $2\mathbb{Z}+1$. In the first case $\omega(t) = \omega(g(t))$, while in the second $\omega(t) \equiv_3 -\omega(g(t))$. $\square$

Proof of Theorem 2.10. — Let $h \in F \setminus \mathcal{F}$ and set $I := |\mathcal{F} : \mathcal{F} \cap h\mathcal{F}h^{-1}|$. If $I < \infty$, then there is an $r \in \mathbb{N}$ such that $w_0^{-r} \in h\mathcal{F}h^{-1}$, or equivalently $h^{-1}w_0^{-r}h \in \mathcal{F}$ (here w_0 is one of the generators of $\mathcal{F}$ depicted in Figure 1.3).

We will show that for all n big enough, $h^{-1}w_0^{-n}h \notin \mathcal{F}$ (and thus reach a contradiction).

By Lemma 2.12, there is an m such $\omega(h(t)) \equiv_3 \omega(t)$ for all $t \in [0, 2^{-m}] \cap S_0$. Since $h \notin \mathcal{F}$, there exists $t \in S_0$ such that $t_1 := h^{-1}(t) \notin S_0$. We observe that for all $l \in \mathbb{N}$, we have $w_0^{-l}(t_1) \notin S_0$. There exists an $n \in \mathbb{N}$ such that $w_0^{-n}(t_1) < 2^{-m}$. Indeed,

$$w_0^{-1}(t) = \begin{cases} .000\alpha & \text{if } t = .0\alpha, \\ .0010\alpha & \text{if } t = .10\alpha, \\ .0011\alpha & \text{if } t = .1100\alpha, \\ .01\alpha & \text{if } t = .1101\alpha, \\ .1\alpha & \text{if } t = .111\alpha. \end{cases}$$

We observe that $h(w_0^{-n}(t_1)) \notin S_0$. Then, $h^{-1}w_0^{-n}h(t) = h(w_0^{-n}(t_1)) \notin S_0$ and so $h^{-1}w_0^{-n}h \notin \mathrm{Stab}(S_0)$. $\square$

In general, Jones's representations are not well understood. The previous result joins a series of investigations [3, 7, 8, 18, 26], where suitable families of representations were studied.

3. The rectangular subgroup $K_{(2,2)}$

The rectangular subgroups of F were introduced in [11] as

$$K_{(a,b)} := \{f \in F \mid \log_2 f'(0) \in a\mathbb{Z}, \log_2 f'(1) \in b\mathbb{Z}\} \qquad a, b \in \mathbb{N}$$

These are the only finite index subgroups isomorphic with F [11, Theorem 1.1].

As mentioned in the Introduction, the 3-colorable subgroup $\mathcal{F}$ sits inside the rectangular subgroup $K_{(2,2)}$. In this section we are going to define an explicit isomorphism between F and $K_{(2,2)}$ that will on one hand, provide us with a pair of elements generating $K_{(2,2)}$, and on the other hand, will later help to identify new maximal subgroup in F by finding the maximal subgroups in $K_{(2,2)}$ that contain $\mathcal{F}$. First we recall from [19] that the subgroup $K_{(1,2)}$ is generated by x_0x_2 and x_1x_2 (see Figure 3.1). Moreover, the following map is an isomorphism

$$\begin{aligned} \beta :\ F &\to K_{(1,2)} \\ x_0 &\mapsto x_0x_2 \\ x_1 &\mapsto x_1x_2 \end{aligned}$$

Recall that there is an order 2 automorphism $\sigma : F \to F$ obtained by reflecting tree diagrams about a vertical line. Clearly, $\sigma(K_{(1,2)}) = K_{(2,1)}$ and, therefore, the subgroup $K_{(2,1)}$ is generated by $\sigma(x_0x_2) = x_0x_1x_0^{-3}$ and $\sigma(x_1x_2) = x_0x_1^2x_0^{-3}$ (see Figure 3.2). We define the following isomorphism

$$\begin{aligned} \alpha :\ F &\to K_{(2,1)} \\ x_0 &\mapsto x_0x_1x_0^{-3} \\ x_1 &\mapsto x_0x_1^2x_0^{-3} \end{aligned}$$

The isomorphism β first appeared in [19], though we use a different notation from Golan and Sapir: their map Ψ coincides with our map β.

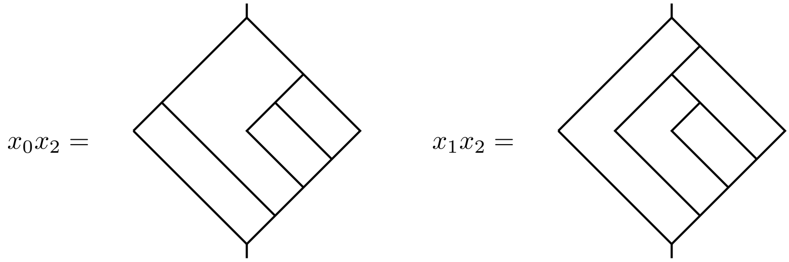


Figure 3.1. The generators of $K_{(1,2)}$.

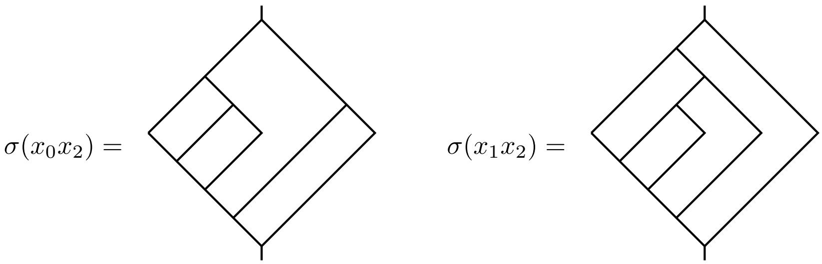


Figure 3.2. The generators of $K_{(2,1)}$.

In the following result we give an explicit isomorphism between F and $K_{(2,2)}$ which will allow us, in Section 4, to obtain maximal subgroups of F from those of $K_{(2,2)}$.

Proposition 3.1. — *Let $\theta : F \to F$ be the monomorphism $\beta \circ \alpha$. Then, the image of θ is $K_{(2,2)}$.*

Proof. — Easy computations yield the following formulas for $\theta(x_0)$ and $\theta(x_1)$

$$\begin{aligned}
\theta(x_0) &= \beta(x_0x_1x_0^{-3}) = x_0x_2x_1x_2x_2^{-1}x_0^{-1}x_2^{-1}x_0^{-1}x_2^{-1}x_0^{-1} \\
&= x_0x_2x_1x_0^{-1}x_2^{-1}x_0^{-1}x_2^{-1}x_0^{-1} = x_0x_1x_3x_0^{-1}x_2^{-1}x_3^{-1}x_0^{-2} \\
&= x_0x_1x_0^{-1}x_2x_0x_0^{-1}x_2^{-1}x_3^{-1}x_0^{-2} = x_0x_1x_0^{-1}x_3^{-1}x_0^{-2} = x_0x_1x_4^{-1}x_0^{-3} \\
&= x_0x_1x_0^{-3}x_1^{-1} = \alpha(x_0)x_1^{-1}
\end{aligned}$$

$$\begin{aligned}
\theta(x_1) &= \beta(x_0x_1^2x_0^{-3}) = x_0x_2x_1x_2x_1x_2x_2^{-1}x_0^{-1}x_2^{-1}x_0^{-1}x_2^{-1}x_0^{-1} \\
&= x_0x_2x_1x_2x_1x_0^{-1}x_2^{-1}x_0^{-1}x_2^{-1}x_0^{-1} \\
&= x_0x_1x_3x_1x_3x_0^{-1}x_2^{-1}x_0^{-1}x_2^{-1}x_0^{-1} \\
&= x_0x_1^2x_4x_3x_0^{-1}x_2^{-1}x_0^{-1}x_2^{-1}x_0^{-1} \\
&= x_0x_1^2x_4(x_0^{-1}x_2x_0)x_0^{-1}x_2^{-1}x_0^{-1}x_2^{-1}x_0^{-1} \\
&= x_0x_1^2x_4x_0^{-1}x_0^{-1}x_2^{-1}x_0^{-1} \\
&= x_0x_1^2x_0^{-1}x_0^{-1}x_2x_2^{-1}x_0^{-1} \\
&= x_0x_1^2x_0^{-3} = \alpha(x_1)
\end{aligned}$$

In particular, we have $\theta(F) \leqslant K_{(2,2)}$. We now want to show that these two subgroups actually coincide.

First, we observe that $|F : \theta(F)| = 4$. Indeed, $|F : K_{(1,2)}| = |F : K_{(2,1)}| = |K_{(1,2)} : K_{(2,2)}| = 2$. In particular, $F = K_{(1,2)} \cup x_0K_{(1,2)} = K_{(2,1)} \cup x_0^{-1}K_{(2,1)}$, $K_{(1,2)} = K_{(2,2)} \cup gK_{(2,2)}$ for some $g \in K_{(1,2)}$. It holds $F = \alpha(F) \cup x_0^{-1}\alpha(F)$ and $K_{(1,2)} = \beta(F) = \beta(\alpha(F)) \cup \beta(x_0^{-1})\beta(\alpha(F)) = \theta(F) \cup \beta(x_0^{-1})\theta(F)$. Then

$$F = K_{(1,2)} \cup x_0K_{(1,2)} = \theta(F) \cup \beta(x_0^{-1})\theta(F) \cup x_0\theta(F) \cup x_0\beta(x_0^{-1})\theta(F)$$

Since $|F : K_{(2,2)}| = |F : \theta(F)| = 4$ and $\theta(F) \leqslant K_{(2,2)}$, we have that $\theta(F) = K_{(2,2)}$. $\square$

Proposition 3.2. — *The rectangular subgroup $K_{(2,2)}$ is generated by $x_0x_1x_4^{-1}x_0^{-3} = x_0x_1x_0^{-3}x_1^{-1}$ and $x_0x_1^2x_0^{-3}$. Moreover, its elements have normal form of even length and an even number of occurrences of $x_0^{\pm 1}$.*

Remark 3.3. — Since $\sigma(K_{(2,2)}) = K_{(2,2)}$, the elements $x_0^3x_4x_1^{-1}x_0^{-1} = \sigma(x_0x_1x_0^{-3}x_1^{-1})$ and $x_1x_2 = \sigma(x_0x_1^2x_0^{-3})$ generate $K_{(2,2)}$ as well

$$\begin{aligned}\sigma(x_0) &= x_0^{-1}\\ \sigma(x_1) &= x_0x_1x_0^{-2}\\ \sigma(x_0x_1x_0^{-3}x_1^{-1}) &= x_0^{-1}x_0x_1x_0^{-2}x_0^3x_0^2x_1^{-1}x_0^{-1}\\ &= x_1x_0^3x_1^{-1}x_0^{-1} = x_0^3x_4x_1^{-1}x_0^{-1}\\ \sigma(x_0x_1^2x_0^{-3}) &= x_0^{-1}x_0x_1x_0^{-2}x_0x_1x_0^{-2}x_0^3\\ &= x_1x_0^{-1}x_1x_0 = x_1x_2\end{aligned}$$

Therefore

$$\begin{aligned}K_{(2,2)} &= \langle x_0x_1x_0^{-3}x_1^{-1}, x_0x_1^2x_0^{-3}\rangle\\ &= \langle x_0x_1x_0^{-3}x_1^{-1}, x_1x_2\rangle\end{aligned}$$

4. Maximal infinite index subgroups of F containing an isomorphic image of $\mathcal{F}$

We have already mentioned in Section 1 that the 3-colorable subgroup $\mathcal{F}$ is generated by $w_0 = x_0^2x_1x_2^{-1}$, $w_1 = x_0x_1^2x_0^{-1}$, $w_2 = x_1^2x_3x_2^{-1}$, $w_3 = x_2^2x_3x_4^{-1}$, [31]. See Figure 1.3 for the tree diagrams of the generators of $\mathcal{F}$.

Proposition 4.1. — *The group $\mathcal{F}$ sits inside*

$$K_{(2,2)} = \{f \in F \mid \log_2 f'(0), \log_2 f'(1) \in 2\mathbb{Z}\} \cong F\,.$$

Moreover, it holds $\pi(\mathcal{F}) = 2\mathbb{Z} \oplus 2\mathbb{Z}$.

Proof. — In order to prove the claim it suffices to compute the images of the generators of $\mathcal{F}$ under the map π and this can be done as explained in the Introduction. Thanks to Figure 1.3, we see that $\pi(w_0) = (2,-2)$, $\pi(w_1) = (0,-2)$, $\pi(w_2) = (0,-2)$, $\pi(w_3) = (0,-2)$. $\square$

Observe that $\mathcal{F}$ is a proper subgroup of $K_{(2,2)}$ as, for example, $K_{(2,2)}$ is isomorphic with F, whereas $\mathcal{F}$ is isomorphic with F_4. This can also be shown by checking that x_0^2 is in $K_{(2,2)}$, but not in $\mathcal{F}$. In [19, Section 3.2] Golan and Sapir pointed out that every proper subgroup of F that projects onto $F/[F,F]$ is contained in a maximal infinite index subgroup of F. Let us implement this idea on the example of $\mathcal{F}$ and investigate in which infinite index maximal subgroups of F the 3-colorable subgroup is contained, see Corollary 4.18, Corollary 4.25, Theorem 4.27.

Recall that every finite-index subgroup contains a normal subgroup. Since every normal subgroup of F contains contains the commutator subgroup [14, Theorem 4.3], the map $\pi : F \to \mathbb{Z} \oplus \mathbb{Z}$ induces a bijective correspondence between the finite index subgroups of F and the finite index subgroups $\mathbb{Z}\oplus\mathbb{Z}$. As mentioned in Section 3, the only finite index subgroups isomorphic with F are precisely $K_{(a,b)} = \pi^{-1}(a\mathbb{Z} \oplus b\mathbb{Z})$, [11, Theorem 1.1]. Here are some easy consequences of this discussion.

Lemma 4.2. — *Let G be a subgroup of F such that $\pi(G) = a\mathbb{Z} \oplus b\mathbb{Z}$ and $G \neq K_{(a,b)}$. Then, the index of G in F is infinite.*

Corollary 4.3. — *The index of $\mathcal{F}$ in F is infinite.*

Proof. — The claim follows from the fact that, thanks to Proposition 4.1, $\pi(K_{(2,2)}) = \pi(\mathcal{F})$ and $K_{(2,2)} \neq \mathcal{F}$. The two groups are distinct because $\mathcal{F} \cong F_4$, while $K_{(2,2)} \cong F$. □

Remark 4.4. — There is also another way to prove the previous result. All the irreducible finite dimensional representations of F are one dimensional (this follows from [15]). This means that if $\mathcal{F}$ were of finite index in F, then for the quasi-regular representation of F associated with $\mathcal{F}$ we would have $\ell_2(F/\mathcal{F}) = \mathbb{C}$, that is $F = \mathcal{F}$, which is absurd.

Corollary 4.5. — *There exists a maximal infinite index subgroup M of $K_{(2,2)}$ containing $\mathcal{F}$.*

Proof. — Thanks to the Zorn Lemma there exists a maximal subgroup M contained in $K_{(2,2)}$ containing $\mathcal{F}$. Since $\pi(\mathcal{F}) = 2\mathbb{Z} \oplus 2\mathbb{Z} \leqslant \pi(M) \leqslant 2\mathbb{Z} \oplus 2\mathbb{Z}$, we have $\pi(M) = 2\mathbb{Z} \oplus 2\mathbb{Z}$. The subgroup M cannot have finite index by Lemma 4.2. □

We are now ready to exhibit *three infinite index maximal subgroups of $K_{(2,2)}$ containing $\mathcal{F}$*:

$$M_0 := \langle \mathcal{F}, x_0^2 \rangle \qquad M_1 := \langle \mathcal{F}, x_1^2 \rangle \qquad M_2 := \langle \mathcal{F}, \sigma(x_1)^2 \rangle$$

Theorem 4.6. — *For any $g \in K_{(2,2)} \setminus M_i$, the group $\langle M_i, g\rangle$ contains $K_{(2,2)}$, with $i = 0, 1, 2$. In particular, M_0, M_1 and M_2 are maximal infinite index subgroups of $K_{(2,2)}$.*

We will adopt the same strategy as the one deployed in [19] to prove the maximality of the oriented subgroup $\vec{F}$ in $K_{(1,2)}$. We start with the proof of Theorem 4.6 for M_0. Let us start recalling some definitions and present a couple of preliminary lemmas. We will need the notion of closure of a subgroup in F. It first appeared in [19], but we will need the equivalent description from [29]. It will allow us to prove that certain inclusions of subgroups are strict.

Let (T_+, T_-) be an element in F. First direct all the edges in T_+ and T_- away from the roots. Now let us consider the graph obtained by identifying the leaves pairwise. This directed graph with two roots associated with an element of F is called the element's diagram. Such diagram is said to be reduced if the corresponding tree diagram is reduced. The vertices coming from the leaves are the only 2-valent vertices of this graph. Note that for any of these vertices there is exactly one directed path from each root to it.

Definition 4.7. — *The* core *of a finitely generated subgroup* $H = \langle g_1, \dots, g_k \rangle \leqslant F$ *is a vertex-labeled directed graph constructed in the following way. Begin with the reduced diagrams for* $g_1, \dots, g_k$ *and identify all the roots of these diagrams together. Proceed with identifying other vertices according to the following two rules for as long as possible*

1. *If two vertices are identified, identify their left children and left incident edges, along with their right children and right incident edges.*
2. *If two vertices have their left children and their right children identified respectively, then identify the vertices and the edges that connect them to their children.*

As there are only finitely many edges and vertices, this process will eventually end.

See Figure 4.1 for a graphical description of these steps. An example with M_0 will be provided in the proof of Proposition 4.11. This definition can be extended to arbitrary subgroups of F, [19]. An element g of F is said to be *accepted by the core* of a subgroup H if there exists a homomorphism of labeled directed graphs from the core of $\langle g \rangle$ to the core of H. The subset of F accepted by the core of a subgroup H is actually a subgroup of F (see [29, Lemma 18]) and is denoted by $\mathrm{Cl}(H)$. This subgroup is called the *closure* of H. A subgroup is called *closed* if it coincides with its closure.

Remark 4.8. — It was shown in [19, Corollary 5.01] that the stabilizer (in F) of any subset of the unit interval is a closed subgroup. Then, it follows from Theorem 2.9 that $\mathcal{F}$ is closed.

As a preliminary step to proving the maximality of M_0, we show that adding a positive element of length 2 to it either doesn't change it (Lemma 4.12) or gives the whole $K_{(2,2)}$ (Lemma 4.13).

Let us start by understanding better the subgroup M_0. We first show that M_0 is a proper subgroup of $K_{(2,2)}$.

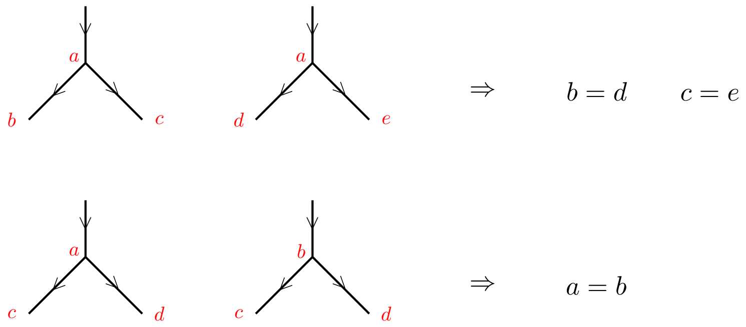
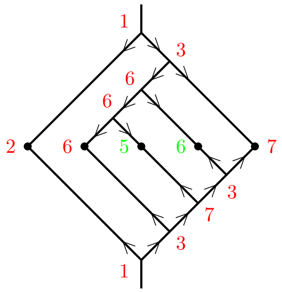


Figure 4.1. Rules for labelling the vertices.

Lemma 4.9. — *The subgroup M_0 is a proper subgroup of $K_{(2,2)}$.*

Proof. — It suffices to show that $x_1^2 \notin M_0$. To this end, we will actually prove an a priori stronger result: x_1^2 does not belong to the closure of M_0. Our main tool is the core of M_0. In order to do this, it suffices to show that x_1^2 does not admit a labelling compatible with that of the generators of M_0, that is x_1^2 does not belong to $Cl(M_0)$ (see Figure 4.2).

$x_1^2 =$ 

□

Remark 4.10. — The previous lemma shows that $\mathcal{F}$ is not a maximal subgroup of $K_{(2,2)}$.

Proposition 4.11. — *The subgroup $M_0 = \langle \mathcal{F}, x_0^2 \rangle$ is generated by x_0^2, x_2^2, x_1x_2.*

Proof. — First we show that the following elements generate M_0

$$\mathcal{S}_{M_0} \coloneqq \{x_{2k}^2, x_{2k+1}x_{2k+2} \mid k = 0, 1, \ldots\} .$$

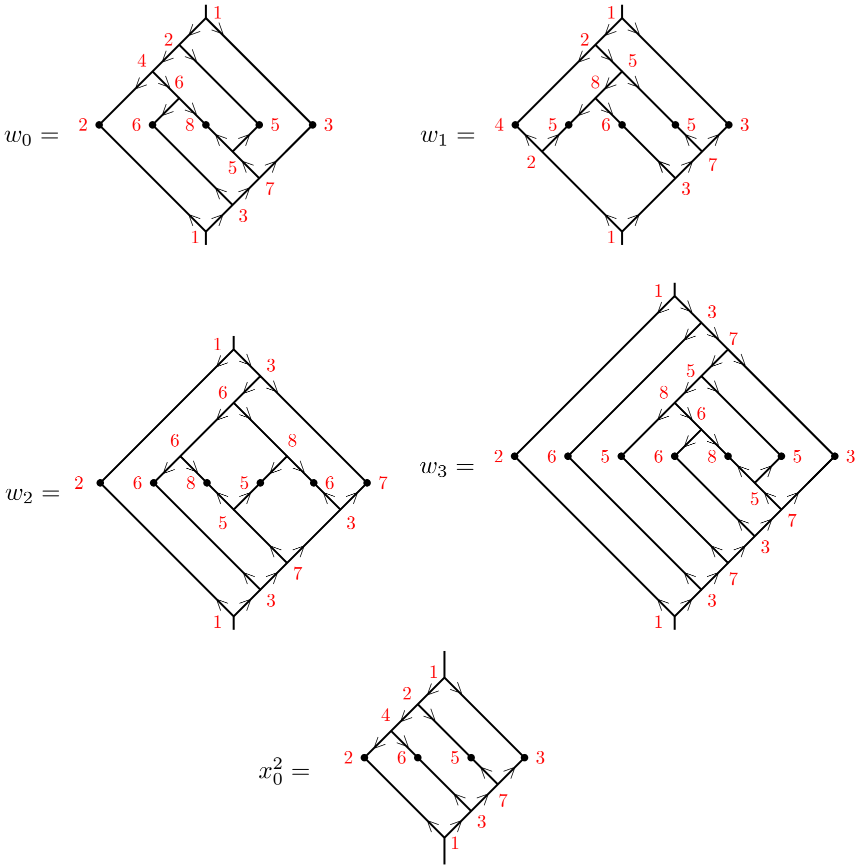


Figure 4.2. The labellings of the generators of M_0.

Indeed, these elements are in M_0

$$
\begin{aligned}
&x_0^{-2} w_1 x_0^2 = x_0^{-1} x_1^2 x_0 = x_2^2 \in M_0 \\
&x_0^{-2k} x_2^2 x_0^{2k} = x_{2k+2}^2 \in M_0 \\
&x_0^{-2} w_0 x_2^2 = x_1 x_2 \in M_0 \\
&x_0^{-2k} x_1 x_2 x_0^{2k} = x_{1+2k} x_{2+2k} \in M_0 \, .
\end{aligned}
$$

It is also easy to see that the generators of M_0 can be obtained from elements in $\mathcal{S}_{M_0}$

$$\begin{aligned}
w_0 &= x_0^2(x_1x_2)x_2^{-2}\\
w_1 &= x_0^2x_2^2x_0^{-2}\\
w_2 &= (x_1x_2)(x_1x_2)x_2^{-2}\\
w_3 &= x_2^2(x_3x_4)x_4^{-2}
\end{aligned}$$

□

Lemma 4.12. — *For all $i \geqslant 0$, the subgroups $\langle x_{2i}^2, \mathcal{F}\rangle$ are all equal.*

Proof. — Denote by $R_{2i} := \langle x_{2i}^2, \mathcal{F}\rangle$. Since $\varphi(\mathcal{F}) \subset \mathcal{F}$, it holds $\varphi^{2i}(R_0) = \varphi^{2i}(\langle x_0^2, \mathcal{F}\rangle) = \langle x_{2i}^2, \varphi^{2i}(\mathcal{F})\rangle \subset R_{2i}$. As x_1x_2, $x_2^2 \in R_0$, we have $x_{2i+1}x_{2i+2}$, $x_{2i+2}^2 \in R_{2i}$. In particular, $R_{2i+2} \leqslant R_{2i}$. We want to prove that the converse inclusion holds. First, notice that $w_1x_{2i+1}x_{2i+2}w_1^{-1} = x_{2i-1}x_{2i} \in R_{2i}$ for all $i \geqslant 1$, where $w_1 = x_0x_1^2x_0^{-1}$. Therefore, we have

$$\varphi^{2i-2}(w_0)x_{2i}^2(x_{2i-1}x_{2i})^{-1} = x_{2i-2}^2 \in R_{2i}.$$

This means that $R_{2i-2} \leqslant R_{2i}$ and we are done. □

Lemma 4.13. — *For any $g = x_ix_j \in K_{(2,2)} \setminus M_0$, the subgroup $\langle g, M_0\rangle$ is equal to $K_{(2,2)}$.*

Proof. — We note that since $g = x_ix_j \in K_{(2,2)} \setminus M_0$, then both i and j are different from 0.

For the sake of clarity we divide the proof in seven cases depending on the form of g.

Case 1: $g = x_1^2$. — Since $M_0 \leqslant H_1 := \langle x_0^2, x_1^2, \mathcal{F}\rangle$, by the proof of Proposition 4.11, we already know that $x_{2k}^2, x_{2k+1}x_{2k+2} \in M_0 \leqslant H_1$. We also have

$$\begin{aligned}
&x_0^{-2k}x_1^2x_0^{2k} = x_{1+2k}^2 \in H_1 \qquad k = 0, 1, \dots\\
&x_1^{-2}w_2x_2^2 = x_3x_2 \in H_1\\
&x_2x_3 = x_2^2(x_3x_2)^{-1}x_3^2 \in H_1\\
&x_0^{-2k}x_2x_3x_0^{2k} = x_{2+2k}x_{3+2k} \in H_1 \qquad k = 0, 1, \dots\\
&(x_2x_3)x_3^{-2}(x_3x_4) = x_2x_4 \in H_1\\
&x_0^{-2k}x_2x_4x_0^{2k} = x_{2+2k}x_{4+2k} \in H_1\\
&x_1x_3 = x_2x_1 = w_0^{-1}x_0^2x_1^2 \in H_1\\
&x_0^{-2k}x_1x_3x_0^{2k} = x_{1+2k}x_{3+2k} \in H_1
\end{aligned}$$

Since the elements

$$\{x_{1+2k}x_{3+2k}, x_{2+2k}x_{4+2k}, x^2_{2k}, x^2_{2k+1}, x_{2k+1}x_{2k+2}, x_{2k+2}x_{2k+3}\}_{k\geqslant 0}$$

generate $K_{(2,2)}$ we are done.

Case 2: $g = x^2_{2i+1}$ for any $i \geqslant 1$. — For any $i \geqslant 1$, it holds $x_0^2 x^2_{2i+1} x_0^{-2} = x^2_{2i-1}$. Therefore, the claim follows by Case 1 and an inductive argument.

Case 3: $g = x_{2i}x_{2i+1}$ for any $i \geqslant 1$. — Assume first that $i = 1$. We have

$$x_0^{-2} w_0 x_2 x_3 = x_1 x_3 = x_2 x_1 \in \langle x_2x_3, M_0\rangle$$
$$x_0^{-2} w_0 x_2 x_1 = x_1^2 \in \langle x_2x_3, M_0\rangle$$

where $w_0 = x_0^2 x_1 x_2^{-1}$. In particular, by Case 1 we have $K_{(2,2)} = \langle x_1^2, M_0\rangle \leqslant \langle x_2x_3, M_0\rangle \leqslant K_{(2,2)}$ and we are done.

If $i \geqslant 2$, the claim follows by induction from the equality $x_0^2 x_{2i}x_{2i+1} x_0^{-2} = x_{2(i-1)}x_{2(i-1)+1}$.

Case 4: $g = x_{2i}x_{2j}$ for any $j \geqslant i+1 \geqslant 1$, $i \geqslant 1$. — Suppose first that $j = i+1$. As $x_0^2(x_{2i}x_{2i+2})x_0^{-2} = x_{2(i-1)}x_{2i}$ for $i \geqslant 2$, we may assume also that $i = 1$. We know from Proposition 4.11 that $x_1x_2 \in M_0$ and, thus, $x_1^2 = (x_1x_2)^2(x_2x_4)^{-1} \in \langle x_2x_4, M_0\rangle$. By Case 1 we get $\langle x_{2i}x_{2j}, M_0\rangle = K_{(2,2)}$.

If $j \geqslant i+2$, the claim follows by induction from the equality

$$x^2_{2i}(x_{2i}x_{2j})x^{-2}_{2i} = x_{2i}x_{2(j-1)} \in \langle x_{2i}x_{2j}, M_0\rangle$$

(we recall that $x^2_{2i} \in M_0$ for all $i \geqslant 0$ by Proposition 4.11).

Case 5: $g = x_{2i}x_{2j+1}$ for any $j \geqslant i+1 \geqslant 1$, $i \geqslant 1$. — Since

$$x^{-2}_{2j}(x_{2i}x_{2j+1})x^{-2}_{2i} = x^{-2}_{2j}(x_{2j}x_{2i})x^{-2}_{2i} = x^{-1}_{2j}x^{-1}_{2i} \in \langle x_{2i}x_{2j+1}, M_0\rangle,$$

we have $x_{2i}x_{2j} \in \langle x_{2i}x_{2j+1}, M_0\rangle$ and the claim follows from Case 4.

Case 6: $g = x_{2i+1}x_{2j}$ for any $j > i+1$, $i \geqslant 0$. — Since

$$x^2_{2i+2}(x_{2i+1}x_{2i+2})^{-1}(x_{2i+1}x_{2j}) = x_{2(i+1)}x_{2j}$$

the claim follows from Case 4.

Case 7: $g = x_{2i+1}x_{2j+1}$, for any $j \geqslant i$, $i \geqslant 0$. — When $j = i$ the claim is precisely the content of Case 2.

Suppose that $j \geqslant i+1$. We note that $x^2_{2i+2}(x_{2i+1}x_{2i+2})^{-1}(x_{2i+1}x_{2j+1}) = x_{2i+2}x_{2j+1}$. Now when $j = i+1$ the claim follows from Case 3. When $j \geqslant i+2$ it follows from Case 5. □

The next result is not needed for the proof of the maximality of M_0 in $K_{(2,2)}$, but it will come in handy in the proof of the maximality of the subgroup M_1.

Lemma 4.14. — *For any $j \geqslant 1$, it holds $\langle x_0 x_j, M_0\rangle = K_{(1,2)}$.*

Proof. — As $x_0^{-2} x_0 x_j x_0^{-2} = x_0 x_{j-2}$ for any $j \geqslant 2$, it is enough to consider the cases $j = 1$ and $j = 2$.

Assume $j = 1$. Then, we have

$$x_0^{-2}(x_0x_1)^2 = x_1x_3 = x_2x_1 \in \langle x_0x_1, M_0\rangle$$
$$x_1^2 = (x_1x_2)x_2^{-2}(x_2x_1) \in \langle x_0x_1, M_0\rangle$$

so by Lemma 4.13 we see that $K_{(2,2)} = \langle x_0^2, x_1^2, \mathcal{F}\rangle < \langle x_0x_1, M_0\rangle \leqslant K_{(1,2)}$. Now $|K_{(1,2)} : K_{(2,2)}| = 2$ and, thus, $\langle x_0x_1, M_0\rangle = K_{(1,2)}$.

Similarly, for $j = 2$, first we observe that $x_2x_4 = x_3x_2 = x_0^{-2}(x_0x_2)^2 \in \langle x_0x_2, M_0\rangle$. Recall that $w_2 = x_1^2x_3x_2^{-1}$. Then, $x_1^2 = w_2(x_2x_4)(x_3x_4)^{-1} \in \langle x_0x_1, M_0\rangle$ and, therefore, by Lemma 4.13 we have $K_{(2,2)} = \langle x_0^2, x_1^2, M_0\rangle < \langle x_0x_2, M_0\rangle \leqslant K_{(1,2)}$. Since $\langle x_0x_1, K_{(2,2)}\rangle = K_{(1,2)}$, we are done. $\square$

In the next definition and lemma we collect some notions and results from [19, Section 3].

Definition 4.15. — *Let $w = x_{i_1} \cdots x_{i_n}$ be a positive normal form of F, then*

- *a letter x_i is said to* skip over *w if $x_i w = w x_{i+n}$.*
- *w is a* block *if contains at least two distinct letters and x_{i_1+1} skips over B.*
- *w is a* minimal block *if $w' = x_{i_2} \cdots x_{i_n}$ is not a block.*
- *for any $k \in \mathbb{N}$, the element $w'' = x_{i_1+k} \cdots x_{i_n+k}$ is said to be a* translation *of w.*

Lemma 4.16 ([19, Lemma 3.6, Lemma 3.9, remark in proof of Lemma 3.11]). — *Let $B = x_{i_1} \cdots x_{i_n}$ be a positive normal form. Then*

1. *a letter x_i skips over B if and only if for all $j = 1, \dots, n$ we have $i_j < i + j - 1$;*
2. *if x_i skips over B, then x_k skips over w for all $k > i$.*

Moreover, if $B = x_{i_1} \cdots x_{i_n}$ is a block, then

3. *every translation B' of B is a block;*
4. *for every $j \neq i_1$ we have $x_j^{-1} B = B' x_r^{-1}$, where B' is a translation of B and $r \in \mathbb{N}$;*
5. *let $w = w_1 B w_2$ be a positive normal form, where B is a block and w_1, w_2 are some possibly empty words; then either $x_j^{-1} w$ is shorter than w or $x_j^{-1} w = w_1' B' w_2'$ for some w_1', w_2', B' such that $|w_1'| = |w_1|$, $|w_2'| = |w_2| + 1$, B' is a translation of B;*

(6) *If B is a minimal block, then there exists a $j \in \{2, \dots, n\}$ such that $i_j = i_1 + j - 1$.*

Observe that for a normal form $B = x_{i_1} \cdots x_{i_n}$ being a block amounts to having $x_{i_1} \neq x_{i_n}$ and that x_{i_1+1} skips over B, [19, Remark 3.8].

We will need the following lemma in the proof of Theorem 4.6.

Lemma 4.17. — *For any $g \in K_{(2,2)} \setminus M_0$, the double coset $M_0 g M_0$ contains a positive element w such that for every $w' \in M_0 g M_0$, it holds $|w| \leqslant |w'|$. Moreover, w may be chosen such that it also does not contain any block.*

Proof. — First we show that there is a positive element of minimal length in the double coset $M_0 g M_0$. Take $w \in M_0 g M_0$ of minimal length. If it is positive, we are done. Otherwise, find its normal form:

$$w = x_0^{a_0} \cdots x_n^{a_n} x_n^{-b_n} \cdots x_0^{-b_0}.$$

The last non-zero factor is $x_{i_0}^{-1}$. If $i_0 \in 2\mathbb{N}_0$, take the element $w x_{i_0}^2 \in M_0 g M_0$, otherwise take $w x_{i_0} x_{i_0+1} \in M_0 g M_0$. In both cases, the new element admits a normal form of the same length, but with less negative factors. The claim follows by iteration.

Now we prove that w does not contain blocks. Let w an element in the double coset $M_0 g M_0$ of minimal length. By the previous discussion we may assume that w is in F_+. Suppose that the normal form of w contains a block, that is $w = z_1 B z_2$, with B being a minimal block.

We now show that we can replace w with another element in $M_0 g M_0$ of the same length and such that $w' = B' z_2'$, with B' being a translation of B. If $z_1 = \emptyset$, there is nothing to do. If z_1 is non-empty, then $z_1 = x_j z_1''$. First, we observe that if $j = 0$, then actually $z_1 = x_0^{2k} \tilde{w}$. However, this is in contradiction with the minimality of the length of w because $x_0^{-2} w$ is shorter than w and still in $M_0 g M_0$. Therefore, j is necessarily different from 0. Now if j is odd, consider $z'' := (x_j x_{j+1})^{-1} w = x_{j+1}^{-1} z_1'' B z_2$ (recall that $x_j x_{j+1} \in M_0$). By Lemma 4.16(5) and the minimality of w we get $z'' = z_1' B' z_2'$, where $|z_1'| = |z_1''| = |z_1| - 1$, $|z_2'| = |z_2| + 1$, B' is a translation of B. By iteration we may assume that z_1 is empty. If j is even, consider $x_j^{-2} w = x_j^{-1} z_1'' B z_2$ (note that $x_j^2 \in M_0$) and argue as before.

By the previous discussion we may assume that $w = B z_2$, where $B = x_{i_1} \cdots x_{i_n}$. There are two cases depending on whether i_1 is odd or even. In the first case consider $t_1 = (x_{i_1} x_{i_1+1})^{-1} w = x_{i_1+1}^{-1} x_{i_2} \cdots x_{i_n}$. By Lemma 4.16(6) there exists a j such that $i_j = i_1 + j - 1$. This means that $x_{i_1+1}^{-1}$ cancel the first occurrence of x_{i_j} in t. This is in contradiction with our hypothesis of w being of minimal length and we are done. We want to show that the second case cannot occur. Since w is of minimal length, $i_2 \geqslant i_1+1$ (if $i_2 = i_1$, then the element $x_{i_1}^{-2}w \in M_0gM_0$ is shorter than w). As B is a block $i_2 < i_1+2$ and thus $i_2 = i_1+1$. Similarly $i_3 \geqslant i_2 = i_1+1$ and $i_3 < i_1+3$. Therefore, we have two sub-cases: $i_3 = i_2 = i_1+1$ and $i_3 = i_1+2$. In the first sub-case the element

$$\begin{aligned} x_{i_1}^{-2}(x_{i_1-1}x_{i_1}^{-1})w = x_{i_1}^{-2}(x_{i_1-1}x_{i_1}^{-1})Bz_2 &= x_{i_1}^{-2}x_{i_1-1}x_{i_1+1}^2x_{i_4}\cdots x_{i_n}z_2 \\ &= x_{i_1}^{-2}x_{i_1}^2x_{i_1-1}x_{i_4}\cdots x_{i_n}z_2 = x_{i_1-1}x_{i_4}\cdots x_{i_n}z_2 \in M_0gM_0 \end{aligned}$$

is shorter than w. This is absurd. The second sub-case is impossible as well since we assumed that B is a minimal block, while $\tilde{B} \coloneqq x_{i_2}\cdots x_{i_n}$ is a block. Indeed, set $i'_j \coloneqq i_{j+1}$ for $j = 1,\dots,n-1$ and consider $\tilde{B} = x_{i'_1}\cdots x_{i'_{n-1}}$. We only have to check that $i'_j < i'_1+j$ for all $j = 1,\dots,n-1$. By definition we have $i'_j \coloneqq i_{j+1} < i_1+j+1 = i_2+j = i'_1+j$. □

We are finally in a position to prove Theorem 4.6 for M_0.

Proof of Theorem 4.6 for M_0. — Let $g \in K_{(2,2)} \setminus M_0$. We need to show that $\langle g, M_0\rangle = K_{(2,2)}$. Note that $\langle g, M_0\rangle = \langle g', M_0\rangle$ for any $g' \in M_0gM_0$. By Lemma 4.17 we may hence assume that g is positive, does not contain any block and is an element of minimal length in M_0gM_0.

We give a proof by induction on the length of the normal form of g (which is even because g is in $K_{(2,2)}$). If the length is 2, then the claim is precisely the content of Lemma 4.13.

Suppose that the length is bigger than 2. We have $g = w'x_j^k$ with $j \in \mathbb{N}$ and $w' = x_{i_1}\cdots x_{i_m}$ is either empty or its last letter is not x_j. If w' is empty (in this case this implies that $k \in 2\mathbb{N}$, $j \in 2\mathbb{N}_0+1$), then $x_j^{-k}(x_jx_{j+1})x_j^k = x_jx_{j+1+k} \in \langle M_0, x_j^k\rangle$ and by Lemma 4.13 we are done.

Suppose that w' is non-empty and let $m = |w'|$. Now we have two cases: (1) j is odd; (2) j is even and $k = 1$. Without loss of generality, we may suppose that $j \geqslant m = |w'|$ (it suffices to replace g by $x_0^{-2l}gx_0^{2l}$ with l big enough). In this case it holds $x_{j-m}w' = w'x_j$ if $j \geqslant m$ ([19, Formula ($\star$) in proof of Theorem 3.12]). For case (1), there are two sub-cases: k and m are even, k and m are odd. If k and m are even take the element

$$\begin{aligned} g^{-1}(x_{j-m}x_{j-m+1})g &= x_j^{-k}w'^{-1}x_{j-m}x_{j-m+1}w'x_j^k \\ &= x_j^{-k}w'^{-1}w'x_jx_{j+1}x_j^k \\ &= x_j^{-k}x_jx_{j+1}x_j^k \\ &= x_jx_{j+1+k} \in \langle g, M_0\rangle \end{aligned}$$

which contains $K_{(2,2)}$ Lemma 4.13.

If k and m are odd take the element

$$\begin{aligned} g^{-1}x_{j-m}^2 g &= x_j^{-k} w'^{-1} x_{j-m}^2 w' x_j^k \\ &= x_j^{-k} w'^{-1} w' x_j^2 x_j^k \\ &= x_j^2 \in \langle g, M_0 \rangle \end{aligned}$$

For some $l \in \mathbb{N}$, we have $x_0^{2l} x_j^2 x_0^{-2l} = x_1^2$. The claim now follows from Lemma 4.13.

For case (2), we observe that m is odd and consider the element

$$\begin{aligned} g^{-1}(x_{j-m}x_{j-m+1})g &= x_j^{-1} w'^{-1}(x_{j-m}x_{j-m+1})w' x_j \\ &= x_j^{-1} w'^{-1} w' x_j x_{j+1} x_j \\ &= x_{j+1}x_j = x_j x_{j+2} \in M_0 g M_0 \end{aligned}$$

Now the claim follows from Lemma 4.13. □

Corollary 4.18. — *The subgroup $\theta^{-1}(M_0)$ is a maximal infinite index subgroup of F.*

Remark 4.19. — In the proof of Lemma 4.9 it was shown that $Cl(M_0)$ is a proper subgroup of $K_{(2,2)}$ (since $x_1^2 \notin Cl(M_0)$). As $M_0 \leqslant Cl(M_0) < K_{(2,2)}$ (the second inclusion follows from [29, Lemma 20]), it follows that the subgroup M_0 is closed, that is $Cl(M_0) = M_0$.

We now consider two other infinite index subgroups of $K_{(2,2)}$ containing $\mathcal{F}$: $M_1 := \langle x_1^2, \mathcal{F} \rangle$ and $M_2 := \langle x_0x_1x_2x_0^{-3}, \mathcal{F} \rangle$. Note that $\sigma(x_1^2) = x_0x_1x_2x_0^{-3}$. Clearly, M_1 is isomorphic with M_2 thanks to $\sigma \restriction_{K_{(2,2)}} \colon K_{(2,2)} \to K_{(2,2)}$, and M_1 is a maximal subgroup if and only if M_2 is a maximal subgroup because $\sigma(M_1) = M_2$.

We begin with a couple of lemmas that allow us to understand better the subgroups M_1 and M_2.

Lemma 4.20. — *The subgroup M_1 contains the subset*

$$\mathcal{S}_{M_1} := \{x_{2k+1}^2, x_{2k+2}x_{2k+3} \mid k = 0, 1, \dots\} .$$

In particular, $\varphi(M_0) = \langle \mathcal{S}_{M_1} \rangle$ is contained in M_1.

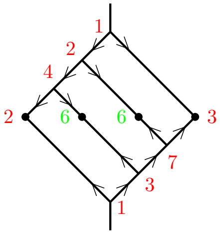
Proof. — The claim follows by easy computations

$$\begin{aligned} &x_1^{-2}\varphi(w_1)x_1^2 = x_3^2 \in M_1 \\ &x_1^{-2k}x_3^2 x_1^{2k} = x_{3+2k}^2 \in M_1 \\ &x_1^{-2}\varphi(w_0)x_3^2 = x_2x_3 \in M_1 \\ &x_1^{-2k}x_2x_3x_1^{2k} = x_{2+2k}x_{3+2k} \in M_1 \end{aligned}$$

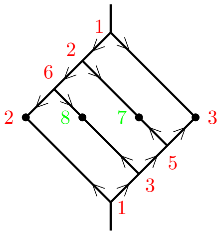
where $w_0 = x_0^2 x_1 x_2^{-1}$, $w_1 = x_0 x_1^2 x_0^{-1}$. □

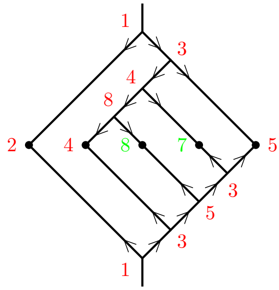
Lemma 4.21. — *The subgroups M_1 and M_2 are proper distinct subgroups of $K_{(2,2)}$.*

Proof. — First we show that $x_0^2 \notin M_1$. In order to do this, it suffices to show that x_0^2 does not admit a labelling compatible with that of the generators of M_1, that is x_0^2 does not belong to $Cl(M_1)$ (see Figure 4.3).

$x_0^2 =$ 

Now we show that x_0^2 and x_1^2 do not belong to M_2 (see Figure 4.4) and, thus, M_2 is different from both M_1 and M_0.

$x_0^2 =$ 

$x_1^2 =$ 

□

As in the case of M_0, we begin proving that adding a positive element of length 2 to M_1 either doesn't change it (Lemma 4.22) or gives the whole $K_{(2,2)}$ (Lemma 4.23).

Lemma 4.22. — *The groups $\langle x_{2k+1}^2, \mathcal{F}\rangle$ are all equal to M_1.*

Proof. — Denote by R_{2i+1} the group $\langle x_{2i+1}^2, \mathcal{F}\rangle$. Since $\varphi(\mathcal{F}) \subset \mathcal{F}$, it holds $\varphi^{2i}(R_1) = \varphi^{2i}(M_1) = \varphi^{2i}(\langle x_1^2, \mathcal{F}\rangle) = \langle x_{2i+1}^2, \varphi^{2i}(\mathcal{F})\rangle \subset R_{2i+1}$. As x_2x_3, $x_3^2 \in R_1 = M_1$, we have $x_{2i+2}x_{2i+3}$, $x_{2i+3}^2 \in R_{2i+1}$. In particular, $R_{2i+3} \leqslant R_{2i+1}$. We want to prove that the converse inclusion holds. First, notice that $w_1 x_{2i+2}x_{2i+3}w_1^{-1} = x_{2i}x_{2i+1} \in R_{2i+1}$ for all $i \geqslant 1$, where $w_1 = x_0x_1^2x_0^{-1}$. Therefore, we have $\varphi^{2i-1}(w_0)x_{2i+1}^2(x_{2i}x_{2i+1})^{-1} = x_{2i-1}^2 \in R_{2i+1}$ where $w_0 = x_0^2x_1x_2^{-1}$. This means that $R_{2i-1} \leqslant R_{2i+1}$ and we are done. □

Lemma 4.23. — *For any $g = x_ix_j \in K_{(2,2)} \setminus M_1$, the subgroup $\langle g, M_1\rangle$ is equal to $K_{(2,2)}$.*

Proof. — We divide the proof into a series of cases.

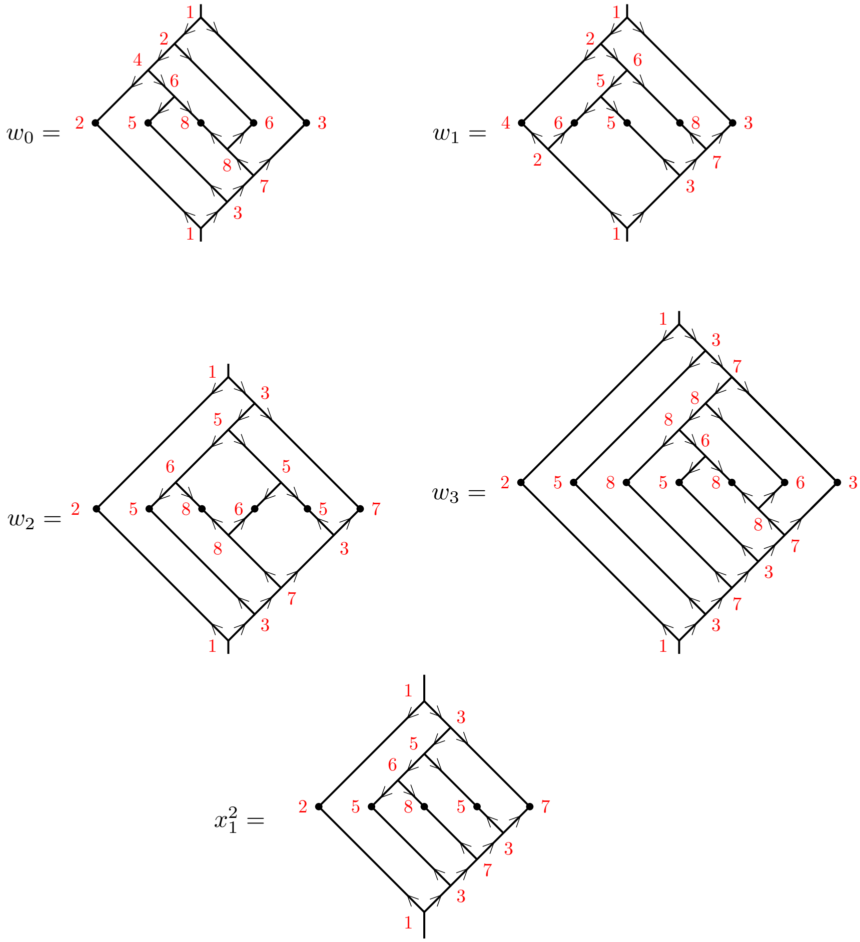


Figure 4.3. The labellings of the generators of M_1.

Case 1: $g = x_{2i}^2$ for $i \geqslant 0$. — In this case the claim follows at once from Lemmas 4.12 and 4.13.

Now we observe that if $K_{(2,2)} \subseteq \langle g, M_0\rangle$, then $K_{(2,2)} \subseteq \langle \varphi(g), M_1\rangle$. Indeed, we have $\varphi(K_{(2,2)}) \subseteq \varphi(\langle g, M_0\rangle) = \langle \varphi(g), \varphi(M_0)\rangle \leqslant \langle \varphi(g), M_1\rangle$ by Lemma 4.20. In particular, $x_2^2 \in \langle \varphi(g), M_1\rangle$ and by Lemmas 4.12 and 4.13 we are done.

Case 2: $g = x_{2i+1}x_{2i+2}$ for any $i \geqslant 0$. — The claim follows from Lemmas 4.13 and 4.14.

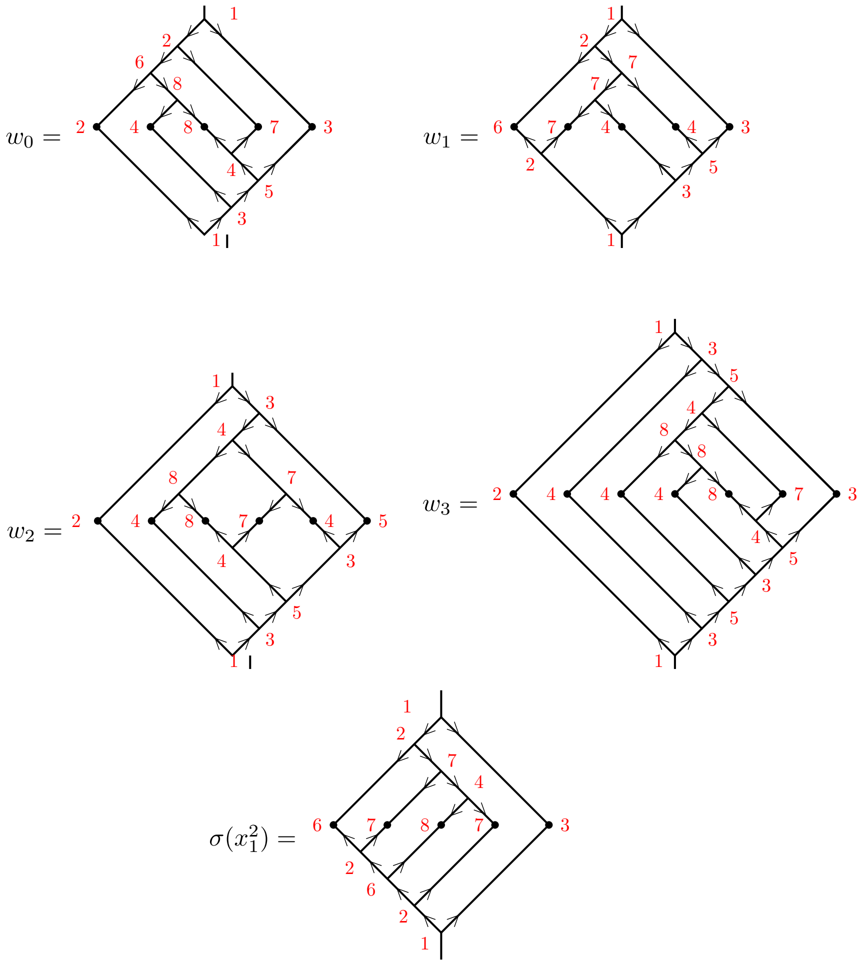


Figure 4.4. The labellings of the generators of M_2.

Case 3: $g = x_{2i+1}x_{2j+2}$ for any $j \geqslant i+1 \geqslant 0$, $i \geqslant 0$. — The claim follows from Lemmas 4.13 and 4.14.

Case 4: $g = x_{2i+1}x_{2j+1}$ for any $j \geqslant i+1$, $i \geqslant 0$. — The claim follows from Lemmas 4.13 and 4.14.

Case 5: $g = x_{2i+2}x_{2j+1}$ for any $j > i+1$, $i \geqslant 0$. — The claim follows from Lemma 4.13.

Case 6: $g = x_{2i+2}x_{2j+2}$ for any $j \geqslant i \geqslant 0$. — The claim follows from Lemma 4.13. □

The following lemma is instrumental in the proof of Theorem 4.6 for M_1 and M_2.

Lemma 4.24. — *For any $g \in K_{(2,2)} \setminus M_1$, the double coset M_1gM_1 contains a positive element w such that for every $w' \in M_1gM_1$, it holds $|w| \leqslant |w'|$. Moreover, w may be chosen such that it does not contain any block and that it lies in $\varphi(F_+)$.*

Proof. — Consider the normal form of $g = x_0^{a_0}x_1^{a_1}\cdots x_1^{-b_1}x_0^{-b_0}$. In the first step of this proof we want to obtain an element in $\varphi(F) \cap M_1gM_1$. There are two cases to deal with: a_0 and b_0 are both even or odd. In the first case take the element $h^{-a_0/2}gh^{b_0/2}$, where $h \coloneqq w_0x_2x_3 = x_0^2x_1x_3 \in M_1$, where $w_0 = x_0^2x_1x_2^{-1}$. This element has the same length as g and is in $\varphi(F)$. In the second case, we observe that $g = x_0^{a_0}\varphi(\tilde{g})x_0^{-b_0}$, where $\tilde{g} \coloneqq x_0^{a_1}x_1^{a_2}\cdots x_1^{-b_1}x_0^{-b_1}$. Now take the element

$$\begin{aligned} h^{-[a_0/2]-1}gh^{[b_0/2]+1} &= x_3^{-1}x_1^{-1}(x_0^{-1}\varphi(\tilde{g})x_0)x_1x_3 \\ &= x_3^{-1}x_1^{-1}\varphi^2(\tilde{g})x_1x_3 \\ &= x_3^{-1}(\varphi(x_0^{-1}\varphi(\tilde{g})x_0))x_3 \\ &= x_3^{-1}\varphi^3(\tilde{g})x_3 \end{aligned}$$

where we used that $x_0^{-1}\varphi(x)x_0 = \varphi^2(x)$ for all $x \in F$, [9, p. 29]. The generators $x_3^{\pm 1}$ may or may not appear in the normal form of $\varphi^3(\tilde{g})$. If they appear, the element $x_3^{-1}\varphi^3(\tilde{g})x_3$ is shorter than g and in $\varphi(F)$. Otherwise, we have $h^{-[a_0/2]-1}gh^{[b_0/2]+1} = \varphi^4(\tilde{g}) \in \varphi(F)$.

Now take $w \in M_1gM_1$ of minimal length. By the previous discussion we may assume that $w \in \varphi(F)$. If it is positive, we are done. Otherwise, we consider its normal form

$$w = x_0^{a_0}\cdots x_n^{a_n}x_n^{-b_n}\cdots x_0^{-b_0}\ .$$

Let $x_{i_0}^{-1}$ be the last non-zero factor. If $i_0 \in 2\mathbb{N}_0 + 1$, the element $wx_{i_0}^2 \in M_1gM_1$ admits a normal form of the same length, but with less negative factors. Similarly, when $i_0 \in 2\mathbb{N}$, one considers the element $wx_{i_0}x_{i_0+1} \in M_1gM_1$. The claim follows by iteration.

Now we want to show that such an element w in $\varphi(F_+)$ does not contain a block. Suppose instead that the normal form of w contains a block, that is $w = z_1Bz_2$, with B being a minimal block.

We now show that we can replace w with another element in $M_1gM_1 \cap \varphi(F)$ of the same length and such that $w' = B'z_2'$, with B' being a translation of B. If $z_1 = \emptyset$, there is nothing to do. If z_1 is non-empty, then $z_1 = x_j z_1''$. If j is even, consider $z'' \coloneqq (x_j x_{j+1})^{-1} w = x_{j+1}^{-1} z_1'' B z_2$ (recall that $x_j x_{j+1} \in M_1$). By Lemma 4.16(5) and the minimality of w we get $z'' = z_1' B' z_2'$, where $|z_1'| = |z_1''| = |z_1| - 1$, $|z_2'| = |z_2| + 1$, B' is a translation of B. If j is odd, consider $x_j^{-2} w = x_j^{-1} z_1'' B z_2$, $x_j^2 \in M_1$ and argue as before. By iteration we may assume that z_1 is empty.

By the previous discussion we may assume that $w = Bz_2$, where $B = x_{i_1} \cdots x_{i_n}$. There are two cases depending on whether i_1 is even or odd.

In the first case consider $t_1 = (x_{i_1} x_{i_1+1})^{-1} w = x_{i_1+1}^{-1} x_{i_2} \cdots x_{i_n}$. By Lemma 4.16(6) there exists a j such that $i_j = i_1 + j - 1$. This means that $x_{i_1+1}^{-1}$ cancel the first occurrence of x_{i_j} in t. This is in contradiction with our hypothesis of w being of minimal length and we are done.

We want to show that the second case (i_1 odd) cannot occur. Since w is of minimal length, $i_2 \geqslant i_1 + 1$ (if $i_2 = i_1$, then the element $x_{i_1}^{-2} w \in M_1gM_1 \cap \varphi(F)$ is shorter than w). We assumed that B is a minimal block, however $\tilde{B} \coloneqq x_{i_2} \cdots x_{i_n}$ is a block. Indeed, set $i_j' \coloneqq i_{j+1}$ for $j = 1, \ldots, n-1$ and consider $\tilde{B} = x_{i_1'} \cdots x_{i_{n-1}'}$. We have to check that $i_j' < i_1' + j$ for all $j = 1, \ldots, n-1$. By definition we have $i_j' \coloneqq i_{j+1} < i_1 + j + 1 = i_2 + j = i_1' + j$. If we show that $\tilde{B}$ contains at least two different letters, then we reached a contradiction. As B is a block $i_3 \geqslant i_2 = i_1 + 1$ and $i_3 < i_1 + 3$. Therefore, we have two sub-cases: $i_3 = i_1 + 2$ and $i_3 = i_2 = i_1 + 1$. In the first we found that there are at least two different letters and, thus, $\tilde{B}$ is a block. In the second sub-case the element

$$\begin{aligned}\varphi^{i_1}(w_1)^{-1} w = x_{i_1} x_{i_1+1}^{-2} x_{i_1}^{-1} B z_2 &= x_{i_1} x_{i_1+1}^{-2} x_{i_1}^{-1} x_{i_1} x_{i_1+1} x_{i_1+1} x_{i_4} \cdots x_{i_n} z_2 \\ &= x_{i_1} x_{i_4} \cdots x_{i_n} w_2 \in M_1 g M_1\end{aligned}$$

is shorter than w (recall that $w_1 = x_0 x_1^2 x_0^{-1}$) and we are done. □

We are now ready to prove that M_1 and M_2 are maximal subgroups of the rectangular subgroup $K_{(2,2)}$.

Proof of Theorem 4.6 for M_1 and M_2. — Let $g \in K_{(2,2)} \setminus M_1$. We need to show that $\langle g, M_1 \rangle = K_{(2,2)}$. Note that $\langle g, M_1 \rangle = \langle g', M_1 \rangle$ for any $g' \in M_1 g M_1$. Therefore, we may replace g with any element in $M_1 g M_1$. By Lemma 4.24 we may suppose that g does not contain any block and is an element of minimal length in $M_1 g M_1 \cap \varphi(F_+)$.

We give a proof by induction on the length of the normal form of g (which is even because g is in $K_{(2,2)}$). If the length is 2, then the claim is exactly the content of Lemma 4.23.

Suppose that the length is bigger than 2. We have $g = w'x_j^k$ with $j \in \mathbb{N}$ and $w' = x_{i_1} \cdots x_{i_m}$ is either empty or its last letter is not x_j. If w' is empty (in this case this implies that $k \in 2\mathbb{N}$, $j \in 2\mathbb{N}_0$), then $x_j^{-k}(x_j x_{j+1})x_j^k = x_j x_{j+1+k} \in \langle M_1, x_j^k \rangle$ and by Lemma 4.23 we are done.

Suppose that w' is non-empty and let $m = |w'|$. Now we have two cases: (1) j is even; (2) j is odd and $k = 1$. Without loss of generality, we may suppose that $j \geqslant m = |w'|$ (it suffices to replace w by $x_1^{-2l} w x_1^{2l}$ with l big enough). In this case it holds $x_{j-m} w' = w' x_j$ if $j \geqslant m$ ([19, Formula ($\star$) in proof of Theorem 3.12]). For case (1), there are two sub-cases: k and m are odd, k and m are even. If k and m are odd (j is even) take the element

$$\begin{aligned} g^{-1} x_{j-m}^2 g &= x_j^{-k} w'^{-1} x_{j-m}^2 w' x_j^k \\ &= x_j^{-k} w'^{-1} w' x_j^2 x_j^k \\ &= x_j^2 \in \langle g, M_1 \rangle \end{aligned}$$

which contains $K_{(2,2)}$ by Lemma 4.23.

If k and m are even (j is even) take the element

$$\begin{aligned} g^{-1}(x_{j-m} x_{j-m+1}) g &= x_j^{-k} w'^{-1} x_{j-m} x_{j-m+1} w' x_j^k \\ &= x_j^{-k} w'^{-1} w' x_j x_{j+1} x_j^k \\ &= x_j^{-k} x_j x_{j+1} x_j^k \\ &= x_j x_{j+1+k} \in \langle g, M_1 \rangle \end{aligned}$$

which contains $K_{(2,2)}$ by Lemma 4.23.

For case (2), that is j is odd and $k = 1$, we observe that m is odd and consider the element

$$\begin{aligned} g^{-1}(x_{j-m} x_{j-m+1}) g &= x_j^{-1} w'^{-1} (x_{j-m} x_{j-m+1}) w' x_j \\ &= x_j^{-1} w'^{-1} w' x_j x_{j+1} x_j \\ &= x_{j+1} x_j = x_j x_{j+2} \in M_1 g M_1 \end{aligned}$$

Now the claim follows from Lemma 4.23. □

Corollary 4.25. — *The subgroups $\theta^{-1}(M_1)$ and $\theta^{-1}(M_2)$ are maximal infinite index subgroups of F.*

Remark 4.26. — Like M_0, the subgroups M_1 and M_2 are closed, that is $Cl(M_1) = M_1$ and $Cl(M_2) = M_2$.

Theorem 4.27. — *The index of $\mathcal{F}$ in M_0, M_1 and M_2 is infinite.*

In order to prove this theorem we need the following lemma.

Lemma 4.28. — *The groups $\langle x_j^{2k}, \mathcal{F}\rangle$ are all equal to M_0 or M_1.*

Proof. — Clearly, $\langle x_j^{2k}, \mathcal{F}\rangle$ is contained in M_0 (if j is even) or M_1 (if j is odd). Since $x_j^{-2k}\varphi^j(w_1)x_j^{2k} = x_{j+2k}^2$, we have that $\langle x_{2k}^2, \mathcal{F}\rangle \leqslant \langle x_j^{2k}, \mathcal{F}\rangle$. By Lemmas 4.12 and 4.22 we are done. □

Proof of Theorem 4.27. — First of all, we observe that it is enough to calculate $|\mathcal{F} : M_0|$ and $|\mathcal{F} : M_1|$. Suppose that $|\mathcal{F} : M_0| = n < \infty$, that is $M_0 = \cup_{k=1}^n g_k\mathcal{F}$ for some $g_1 = 1, \dots, g_n \in M_0$. We claim that this implies that for every $g \in M_0$ there exist infinitely many $m \in \mathbb{N}$ such that $g^m \in \mathcal{F}$. Indeed, there are at least two distinct indices $i, j \in \mathbb{N}$ such that $g^i, g^j \in g_k\mathcal{F}$ for some $k \in \{1, \dots, n\}$. This means that $g^{i-j} \in \mathcal{F}$ and $g^{(i-j)m} \in \mathcal{F}$ for all $m \in \mathbb{N}$. Take $g = x_0^2 \in M_0 \setminus \mathcal{F}$. It follows from Lemma 4.28 that the element g^k does not belong to $\mathcal{F}$ for all $k \in \mathbb{N}$. Therefore, the index of $\mathcal{F}$ in M_0 is infinite.

For M_1 use the same argument with $g = x_1^2$ and Lemma 4.28. □

5. On maximal infinite index subgroups of F

We recall that the *oriented subgroup* $\vec{F}$ is the subgroup of F generated by x_0x_1, x_1x_2, x_2x_3. It can be easily seen that $\vec{F}$ is a subgroup of $K_{(1,2)}$. As mentioned in Section 3, the subgroup $K_{(1,2)}$ is isomorphic to F and an isomorphism is provided by the map $\beta : F \to K_{(1,2)}$, which is defined as $\beta(x_i) \coloneqq x_ix_2$ for $i = 0, 1$. The subgroup $\beta^{-1}(\vec{F})$ is the first example of a maximal subgroup of infinite index in F without fixed points in the open unit interval $(0, 1)$. For further information on $\vec{F}$, we refer to [1, 2, 18, 19, 23, 30, 31].

In this section we compare the subgroups $\theta^{-1}(M_0)$, $\theta^{-1}(M_1)$, $\theta^{-1}(M_2)$ to maximal infinite index subgroups of F that have been identified before: the oriented subgroup $\vec{F}$; the parabolic subgroups $\mathrm{Stab}(t)$ for $t \in (0, 1)$; Golan's examples [17, Examples 10.12 and 10.13, Section 10.3.B] $K_1 \coloneqq \langle H, x_1^2x_2^{-1}\rangle$, $K_2 \coloneqq \langle H, x_1^2x_2x_1^{-3}, x_1^3x_2x_1^{-4}\rangle$, where $H \coloneqq \langle x_0, x_1x_2x_1^{-1}\rangle$, and

$K_3 \coloneqq \langle x_0, x_1x_2x_1^{-3}, x_1x_2x_3x_2^{-3}x_1^{-1}\rangle$. Note that K_3 is the first known example of a maximal infinite index subgroup of F which acts transitively on the set of dyadic rationals.

We mention that K_1 and K_2 are concrete realisations of Golan and Sapir's implicit example of maximal infinite index subgroup containing H described in [19]. In fact, the subgroups M_0, M_1 and M_2 are distinct from all the maximal infinite index subgroups containing H.

Theorem 5.1. — *The subgroup $\theta^{-1}(M_0)$, $\theta^{-1}(M_1)$, $\theta^{-1}(M_2)$ are distinct from the parabolic subgroups, from $\beta^{-1}(\vec{F})$, from K_1, K_2, and K_3.*

Proof. — First of all we show that $\theta^{-1}(M_0)$ does not stabilise any number in $[1/2,1)$. The element $\sigma(x_1) = x_0x_1x_0^{-2} \in \mathrm{Stab}(t)$ for $t \in [1/2,1)$. The following computations show that $\sigma(x_1) \notin \theta^{-1}(M_0)$ and, thus, $\theta^{-1}(M_0) \neq \mathrm{Stab}(t)$ for all $t \in [1/2,1)$

$$\begin{aligned}
\theta(\sigma(x_1)) &= (x_0x_1x_4^{-1}x_0^{-3})(x_0x_1^2x_0^{-3})(x_0^3x_4x_1^{-1}x_0^{-1})(x_0^3x_4x_1^{-1}x_0^{-1})\\
&= x_0x_1x_4^{-1}x_0^{-2}(x_1^2x_4)x_1^{-1}x_0^2x_4x_1^{-1}x_0^{-1}\\
&= x_0x_1x_4^{-1}x_0^{-2}(x_1x_3x_0^2)x_4x_1^{-1}x_0^{-1}\\
&= x_0x_1x_4^{-1}(x_3x_5)x_4x_1^{-1}x_0^{-1}\\
&= x_0(x_1x_3x_4)x_1^{-1}x_0^{-1}\\
&= x_0x_2x_3x_0^{-1} = x_1x_2 \notin M_0
\end{aligned}$$

The element $x_2 = x_0^{-1}x_1x_0 \in \mathrm{Stab}(t)$ for all $t \in (0,3/4]$. Since

$$\begin{aligned}
\theta(x_2) &= (x_0^3x_4x_1^{-1}x_0^{-1})(x_0x_1^2x_0^{-3})(x_0x_1x_4^{-1}x_0^{-3})\\
&= x_0^3x_4x_1(x_0^{-2}x_1)x_4^{-1}x_0^{-3}\\
&= x_0^3x_4x_1x_3x_0^{-2}x_4^{-1}x_0^{-3}\\
&= x_0^3x_4x_1x_3x_6^{-1}x_0^{-5}\\
&= x_0^3x_1x_5x_3x_6^{-1}x_0^{-5}\\
&= x_0^3x_1x_3x_6x_6^{-1}x_0^{-5}\\
&= x_0^3x_1x_3x_0^{-5}
\end{aligned}$$

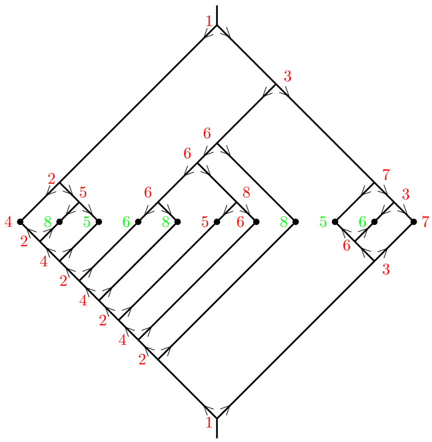
we see that $\theta(x_2) \in M_0$ if and only if $x_0^{-4}\theta(x_2)x_0^6 = x_2x_4 \in M_0$. By Lemma 4.23 we know that $x_2x_4 \notin M_0$ and, thus, $\theta^{-1}(M_0)$ does not stabilise any number in $(0,3/4]$. In particular, $\theta^{-1}(M_0)$ is not a parabolic subgroup.

We now show that $\theta^{-1}(M_0)$ does not coincide with the maximal subgroup $\beta^{-1}(\vec{F})$ exhibited in [19]. Recall that x_0x_1 is one of the generators of $\vec{F}$, [18]. It was shown in the proof of [19, Theorem 3.15] that $\beta^{-1}(x_0x_1) = x_0x_1x_2^{-1} = x_0x_1x_0^{-1}x_1x_0$. We have that

$$
\begin{aligned}
\theta(x_0x_1x_2^{-1}) &= (x_0x_1x_4^{-1}x_0^{-3})(x_0x_1^2x_0^{-3})(x_0^3x_4x_1^{-1}x_0^{-1})(x_0x_1^2x_0^{-3})(x_0x_1x_4^{-1}x_0^{-3})\\
&= x_0x_1x_4^{-1}(x_0^{-2}x_1^2)x_4x_1x_0^{-2}x_1x_4^{-1}x_0^{-3}\\
&= x_0x_1x_4^{-1}x_3^2(x_0^{-2}x_4)x_1x_0^{-2}x_1x_4^{-1}x_0^{-3}\\
&= x_0x_1x_4^{-1}x_3^2x_6(x_0^{-2}x_1)x_0^{-2}x_1x_4^{-1}x_0^{-3}\\
&= x_0x_1x_4^{-1}x_3^2x_6x_3(x_0^{-4}x_1)x_4^{-1}x_0^{-3}\\
&= x_0x_1x_4^{-1}x_3^2x_6x_3x_5(x_0^{-4}x_4^{-1})x_0^{-3}\\
&= x_0x_1x_4^{-1}x_3^2x_6x_3x_5x_8^{-1}x_0^{-7}\\
&= x_0x_1x_4^{-1}x_3^3x_7x_5x_8^{-1}x_0^{-7}\\
&= x_0x_1x_4^{-1}x_3^3x_5x_8x_8^{-1}x_0^{-7}\\
&= x_0x_1x_4^{-1}x_3^3x_5x_0^{-7}\\
&= x_0x_1x_3^3x_5x_8^{-1}x_0^{-7}
\end{aligned}
$$

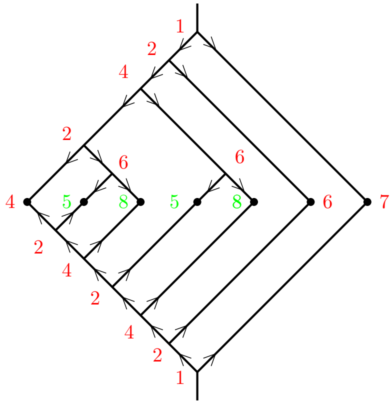
where we used that $x_n^{-1}x_k = x_kx_{n+1}^{-1}$ and $x_k^{-1}x_n = x_{n+1}x_k^{-1}$ for all $k < n$. The following figure shows that $\theta(x_0x_1x_2^{-1})$ does not belong to M_0 and, therefore, $\theta^{-1}(M_0) \neq \beta^{-1}(\vec{F})$

$$\theta(x_0x_1x_2^{-1}) = x_0x_1x_3^3x_5x_8^{-1}x_0^{-7} =$$



Now we take care of $\theta^{-1}(M_1)$. First, we show that it does not stabilise any number in $(0, 3/4]$. The following computations show that $\theta(x_2) =$

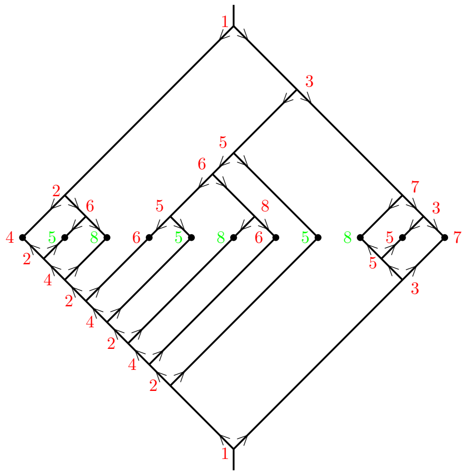
$x_0^3x_1x_3x_0^{-5} \notin M_1$ and, thus, $\theta^{-1}(M_1) \neq \mathrm{Stab}(t)$ for all $t \in (0, 3/4]$

$\theta(x_2) =$  $\notin M_1$

Similarly, $\theta^{-1}(M_1)$ is different from $\mathrm{Stab}(t)$ for all $t \in [1/4, 1)$ because

$$\begin{aligned}
\theta(\sigma(x_2)) &= \theta(\sigma(x_0^{-1}))\theta(\sigma(x_1))\theta(\sigma(x_0)) \\
&= \theta(x_0)x_1x_2\theta(x_0)^{-1} \\
&= (x_0x_1x_4^{-1}x_0^{-3})(x_1x_2)(x_0^3x_4x_1^{-1}x_0^{-1}) \\
&= x_0x_1x_4^{-1}x_4x_5x_4x_1^{-1}x_0^{-1} \\
&= x_0x_1x_5x_4x_1^{-1}x_0^{-1} \\
&= x_0x_4x_3x_1x_1^{-1}x_0^{-1} \\
&= x_0x_4x_3x_0^{-1} = x_3x_2 = x_2x_4 \notin M_1
\end{aligned}$$

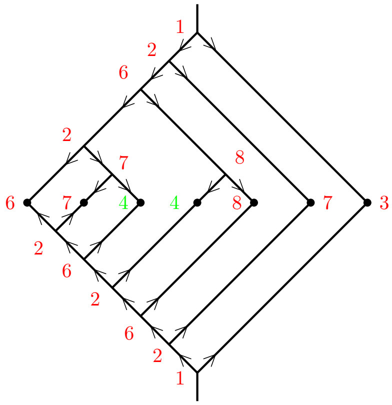
The following figure shows that $\theta(x_0x_1x_2^{-1})$ does not belong to M_1

$\theta(x_0x_1x_2^{-1}) = x_0x_1x_3^3x_5x_8^{-1}x_0^{-7} =$ 

This means that $\theta^{-1}(M_1) \neq \beta^{-1}(\vec{F})$.

Now, we deal with $\theta^{-1}(M_2)$. The element $x_2 \in \mathrm{Stab}(t)$ for $t \in (0, 3/4]$. The following computations show that $\theta^{-1}(M_2) \neq \mathrm{Stab}(t)$ for all $t \in (0, 3/4]$

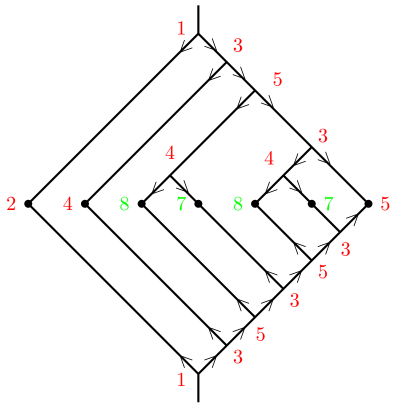
$\theta(x_2) =$



$\notin M_2$

Since

$\theta(\sigma(x_2)) = x_2 x_4 =$

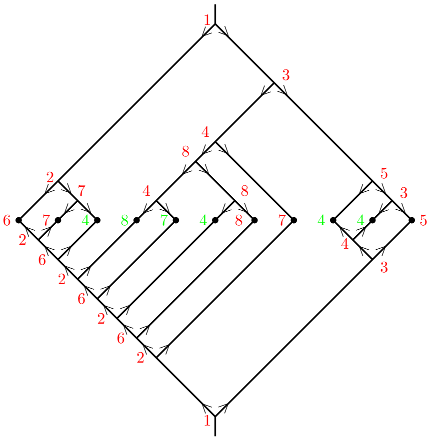


$\notin M_2$

we see that $\theta^{-1}(M_2) \neq \mathrm{Stab}(t)$ for all $t \in (0, 1)$.

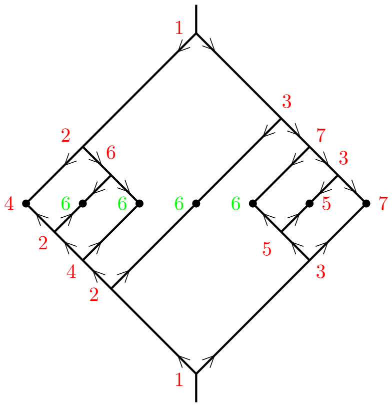
In order to prove that $\theta^{-1}(M_2) \neq \beta^{-1}(\vec{F})$, it suffices to show that $\theta(x_0x_1x_2^{-1})$ does not belong to M_2 (recall that $x_0x_1x_2^{-1} \in \beta^{-1}(\vec{F})$)

$$\theta(x_0x_1x_2^{-1}) = x_0x_1x_3^3x_5x_8^{-1}x_0^{-7} =$$



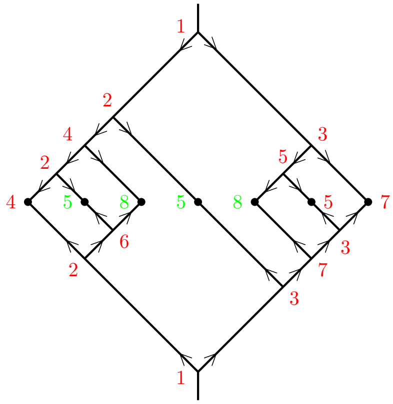
Finally, we deal with K_1, K_2, and K_3. H is also not contained in $\theta^{-1}(M_0)$ because $\theta(x_1) \in M_0$ (if also $\theta(x_0) \in M_0$, then $M_0 = K_{(2,2)}$). For the same reason K_3 is distinct from $\theta^{-1}(M_0)$. As for M_1 and M_2, it suffices to check that $\theta(x_0) = x_0x_1x_4^{-1}x_0^{-3} \notin M_1$ and $\theta(x_0) \notin M_2 = \sigma(M_1)$ (which is equivalent to showing that $\sigma(\theta(x_0)) \notin M_1 = \sigma(M_2)$). This is done in the following figure.

$$\theta(x_0) = x_0x_1x_4^{-1}x_0^{-3} =$$



$$\notin M_1$$

$$\sigma(\theta(x_0)) = x_0^3 x_4 x_1^{-1} x_0^{-1} =$$



$$\notin M_1$$

$\square$

Up to isomorphism, there are at least four maximal infinite index subgroups of F: Stab(1/2), Stab(1/3), Stab($\sqrt{2}/2$), $\beta^{-1}(\vec{F})$

Question 5.2. — *Are M_0, M_1, and M_2 isomorphic to the other known maximal infinite index subgroups of F? What about Golan's subgroups K_1, K_2, K_3?*

So far we have exhibited three subgroups containing $\mathcal{F}$: M_0, M_1 and M_2. It is natural to wonder if $\mathcal{F}$ is of quasi-finite index in $K_{(2,2)}$, that is, if there are only finitely many subgroups of $K_{(2,2)}$ containing $\mathcal{F}$. Here we prove some partial results suggesting that M_0, M_1 and M_2 might be the only subgroups between $\mathcal{F}$ and $K_{(2,2)}$.

Lemma 5.3. — *Let g be a normal form in $F_+ \cap K_{(2,2)}$ containing at least two letters, then there exists a block h in $\langle g, \mathcal{F}\rangle$.*

Proof. — Consider the normal form of $g = x_{i_1}^a x_{i_2} \cdots x_{i_n}$, $a \in \mathbb{N}$. For every j, there exists a $k_j \in \mathbb{N}$ such that $i_j < i_1 + a + k_j + 1$. Let k be $\sum_j k_j$ and consider

$$\begin{aligned} h \coloneqq \varphi^{i_1}(w_1^k)g = \varphi^{i_1}(x_0 x_1^{2k} x_0^{-1})g &= x_{i_1} x_{i_1+1}^{2k} x_{i_1}^{a-1} x_{i_2} \cdots x_{i_n} \\ &= x_{i_1}^a x_{i_1+a}^{2k} x_{i_2} \cdots x_{i_n}\ . \end{aligned}$$

Now h is a block. Indeed,

$$\begin{aligned} &i_1 + a < i_1 + a + 1, \\ &i_2 < i_1 + a + k_2 + 1 < i_1 + a + 2k + 1 \\ &i_3 < i_1 + a + k_3 + 1 < i_1 + a + 2k + 2 \\ &i_j < i_1 + a + k_j + 1 < i_1 + a + 2k + (j-1) \qquad j \geqslant 3 \end{aligned}$$

$\square$

Proposition 5.4. — *Let $g \in F_+ \cap K_{(2,2)}$, then $\langle g, \mathcal{F}\rangle$ is equal to M_0, M_1, or $K_{(2,2)}$.*

Proof. — Thanks to Lemma 4.28 we may assume that the normal form of g contains at least two letters. By Lemma 5.3 there exists a block $h \in \langle g, \mathcal{F}\rangle$. Consider its normal form $h = x_{i_1} \cdots x_{i_n}$, $w_1 = x_0 x_1^2 x_0^{-1}$. Recall that if h is a block, then $x_k^{\pm 1} h = h x_{k+|h|}^{\pm 1}$ for all $k > i_1$ (here $|h|$ is the length of the normal form of h). Then,

$$\begin{aligned} h^{-1}\varphi^{i_1}(w_1)h &= h^{-1}x_{i_1}x_{i_1+1}^2 x_{i_1}^{-1} x_{i_1}x_{i_2}\cdots x_{i_n} \\ &= h^{-1}x_{i_1}x_{i_2}\cdots x_{i_n}x_{i_1+n}^2 = h^{-1}hx_{i_1+n}^2 = x_{i_1+n}^2 \in \langle g, \mathcal{F}\rangle \end{aligned}$$

where we used that x_{i_1+1} skips $x_{i_2}\cdots x_{i_n}$. Now if $i_1 + n \equiv_2 0$, then $M_0 \leqslant \langle g, \mathcal{F}\rangle \leqslant K_{(2,2)}$. Otherwise, we have $M_1 \leqslant \langle g, \mathcal{F}\rangle \leqslant K_{(2,2)}$. In both cases now the claim follows from the maximality of M_0 and M_1 in $K_{(2,2)}$. ☐

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