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A dynamic game approach for optimal consumption, investment and life insurance problem

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Abstract

In this paper, we consider a multi-agent portfolio optimization model with life insurance for two players with random lifetime under a dynamic game approach. Each player is a price-taker and invests in the market to maximize her own utility for consumption and bequest. The market is complete and consists of n different assets, of which $n-1$ are risky with prices driven by Geometric Brownian motion, while one is risk-free. We analyze both the non-cooperative and cooperative scenarios, and by considering the family of CRRA utility functions, we determine the closed-form expressions of the optimal consumption, investment, and life insurance for both players. A sensitivity analysis is provided both to illustrate the impact of the biometric and risk aversion parameters on the optimal controls and to compare the non-cooperative strategies with the cooperative ones. As a result, we suggest that cooperation favors the consumption optimality, while non-cooperation promotes the coverage of the risk of death.

Keywords Dynamic games · Non-cooperative vs cooperative games · Portfolio choice · Lifetime uncertainty · Life insurance

JEL Classification G11 · C61 · C70 · G22

1 Introduction

The theory of portfolio management has received, during the last century, a great deal of research attention with several contributions and developments. The classical Markowitz

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portfolio model (Markowitz, 1952) is the most important in this vein, despite its limited way of describing the uncertainty related to the portfolio decision and its static formulation, which does not permit portfolio rebalancing within the investment horizon. Different extended portfolio models have been proposed to deal with the presence of multiple criteria as well as the notion of uncertainty and incomplete information, see, e.g., Ben Abdelaziz and Masri (2005); La Torre and Mendivil (2018). Meanwhile, there are several papers considering dynamic portfolio allocation models, and among them, the most famous is surely the one introduced by Merton (1969) (see also Merton (1971)). The portfolio model described in Merton (1969) is a continuous-time problem in which the investor with known lifetime has to decide the allocation of her wealth among several investable risky assets and a risk-free one to maximize her intertemporal utility. One of the generalizations of Merton's model was proposed by Richard (1975) by including optimal consumption, investment, and life insurance decisions for an investor with an arbitrary but known distribution of lifetime. Such a work served as a bridge between Merton's model and the insurance literature, starting with Yaari's work (Yaari, 1965) and the vast number of life-cycle + mortality papers published over the last four decades, as, e.g., Fischer (1973), Davies (1981), Butler (2001) and Lachance (2012). Coming back to Richard (1975), some assumptions have been introduced to simplify the model and make it more tractable from a mathematical perspective. In particular, the market is assumed to be perfect and frictionless, with assets traded continuously. All the assets have a limited liability, and the price of all the risky ones follows a Geometric Brownian Motion process. Finally, there is no default risk for the issuer of the financial assets. Under these assumptions, the author uses stochastic dynamic programming to derive the optimal controls for investment, consumption, and life insurance for the investor, which maximizes her expected utility for consumption and bequest. Moreover, he finds that the uncertain lifetime (that constitutes the uncertain time horizon) and life insurance in no way affect the investment strategies, and this is in part due to the individual's expending (receiving) funds to buy (sell) life insurance.

Several versions of Richard's model have incorporated different features into the models, such as the introduction of transaction costs (Liu & Loewenstein, 2002), the interpretation of the fixed planning horizon as the wage earner's retirement date (see, e.g., Stanley and Jinchun (2007)) or the assumption that the conditional distribution function of the time horizon is stochastic and correlated to stock returns, with subsequent impact on portfolio choices, as in Blanchet-Scalliet et al. (2008). The latter (Blanchet-Scalliet et al., 2008) proved that if the probability of leaving the market increases (respectively, decreases) with the risky asset return, then the share of wealth invested in the risky asset is lower than (respectively, greater than) in the case of a certain time horizon. Other works consider the correlation between the dynamics of human and financial capital (Huang et al., 2008) or include a different way to price the risky assets (Pirvu & Zhang, 2012). In Huang et al. (2008), the authors consider the interaction between the constant relative risk aversion (CRRA) utility-maximizing demand for life insurance and the optimal consumption-investment of a family, as opposed to separating consumption and bequest. They show that the optimal amount of life insurance a family should have depends on the volatility of the income process as well as its correlation to investment returns. In Pirvu and Zhang (2012), allowing the stock price to have a mean reverting drift, the impact of the stochastic market price of risk (MPR) on the optimal investment strategies is analyzed. It is shown that the stock optimal investment strategy is significantly affected by MPR and increases in the MPR. Other extensions can be found, for instance, in Chang (2004), Hamacher et al. (2007) and references therein.

There is also a stream of literature which consider the optimal consumption-investment and life-insurance problem from the viewpoint of a household comprising multiple wage earners, see, e.g., Kenneth Bruhn and Steffensen (2011), Wei et al. (2020) and Wang et al. (2021). In Wei et al. (2020), a couple of wage earners is considered, wherein both individuals independently purchase a life insurance with their partner nominated as the beneficiary. Moreover, the couple has correlated lifetimes, which are modelled using copula and common-shock models. Wang et al. (2021), instead, allows the income of a household consisting of two consecutive generations, say, parents and children, to increase in a random and unobservable way and allow for market ambiguity. Finally, a recent contribution in this area is Moagi and Doctor (2022), which proposes a zero-sum differential game between the market, consisting of a risk asset and a risk-free one, and the investor, which is subject to consumption, purchasing life insurance and stochastic income with inflation risk. In particular, the investor wants to minimize the risk of her terminal wealth to maximize the monetary returns, while the market minimizes the chances of the investor maximizing the investment. The optimal strategies are determined for two different life stages of the investor, pre-death and post-death, and by using both the CRRA utility function and the constant absolute risk aversion (CARA) one.

In this paper, we propose an extension of Richard's model to the case of two players, representing members of two competing households, which do not receive any income but acting on the same financial market and sharing the possibility to invest their wealth in a fixed and common number of possible risky and risk-free assets. Each player aims to maximize her own utility for consumption to the detriment of the other player and does so by deciding the optimal share of wealth to allocate to each asset and the overall level of consumption. Moreover, she selects the optimal amount of life insurance to be purchased to maximize the death benefit paid to the beneficiary or to be sold to maximize her wealth. By considering a CRRA utility function, we determine, for both players, the optimal consumption, the optimal investments across different assets, and the optimal life insurance to buy/sell in both cooperative and non-cooperative scenarios. Then, we analyze the impact of the constant relative risk aversion and mortality intensity on the optimal strategies. We compare such strategies, illustrating that, from one side, the cooperative case is preferable to the non-cooperative one in terms of optimality regarding the consumption activity. On the other side, we highlight that the non-cooperative case promotes the coverage of the risk of death in old age.

The outline of the paper is as follows. Section 2 summarizes Richard's model for an agent in continuous time. In Sect. 3, we present our Richard-type multi-agent model after showing how to pass from a stochastic problem to a deterministic one. Section 4 characterizes the cooperative and non-cooperative equilibrium outcomes, deriving explicitly the players' optimal strategies.

In Sect. 5, we perform an empirical implementation through a sensitivity analysis aiming to compare the non-cooperative and the cooperative strategies. Finally, Sect. 6 presents concluding remarks and highlights directions for future research. All mathematical technicalities are presented in Appendix A.

2 The standard Richard's model

Richard's model is a continuous time model for optimal consumption, investment, and life insurance rules for an investor with arbitrary but known distribution of lifetime. It is defined as a stochastic optimal control problem, and it is solved by means of the Hamilton-Jacobi-Bellman equation. As anticipated in the Introduction, the mathematical tractability of the model is due to the following assumptions: the market is assumed to be perfect and frictionless with assets traded continuously; all the assets have limited liability, and the price of all the risky ones follows a geometric Brownian motion process; there is no default risk for the issuer of the financial assets. Moreover, the model formulation considered in this paper is based on the following conditions:

- the investor does not receive any income and chooses his portfolio by selecting from $n-1$ risky assets which have known normal distribution of returns, and a risk-free asset;
- one can consume any fractional amount of wealth at any time;
- the investor aims at maximizing lifetime utility for consumption and bequest;
- consumption utility $U : \mathbb{R}^+ \to \mathbb{R}$ is a strictly concave C^2 (twice differentiable with continuous derivatives) function, while the bequest function $\mathcal{F}(\widetilde{T}, Z(\widetilde{T}))$ is strictly concave in the contingent bequest $Z(\widetilde{T}) \in \mathbb{R}^+$, being $\widetilde{T} \in [0, T]$ a random variable denoting the age of death of the investor, with T the maximal age beyond which nobody will live.

Therefore Richard's model reads as follows:

$$\max_{c, y_1, \dots, y_{n-1}, Q} J(W, c, y_1, \dots, y_{n-1}, Q) := \mathbb{E}\left[\int_0^{\widetilde{T}} e^{-\rho t} U(c(t))\, dt + \mathcal{F}(\widetilde{T}, Z(\widetilde{T}))\right] \quad (1)$$

subject to:

$$\begin{cases} dW(t) = \left[\left(\sum_{i=1}^{n-1} y_i(t)(\mu_i - \mu_n) + \mu_n\right) W(t) - c(t) - \theta Q(t)\right] dt + \sum_{i=1}^{n-1} y_i(t) W(t) \sigma_i dB_i(t), \\ W(0) = W_0 > 0 \ \text{given}, \\ W(t) > 0 \ \forall t \in [0, T], \\ dP_i(t) = P_i(t)(\mu_i\, dt + \sigma_i\, dB_i(t)), \qquad P_i(0) > 0 \ \text{given}, \quad i = 1, \dots, n-1 \\ dP_n(t) = \mu_n P_n(t)\, dt, \qquad P_n(0) > 0 \ \text{given}, \\ \sum_{i=1}^{n} y_i(t) = 1, \\ 0 \le y_i(t) \le \xi_i, \quad i = 1, \dots, n. \end{cases} \quad (2)$$

In (1), ρ is the discount factor derived from factors other than lifetime uncertainty, the state of the investor at age $t \in [0, \widetilde{T}]$ is the wealth $W(t)$, while the control at the same age is described by the triple $(c(t), y_i(t), Q(t))$, where: $c(t) > 0$ is the consumption of any fractional amount of wealth $W(t)$, $y_i(t)$, $i = 1, \dots, n-1$, is the share of wealth invested in the i-th asset and $Q(t)$ is the amount of life insurance of an instantaneous term variety, purchased or sold.
The price of the i-th risky asset, denoted by $P_i(t)$, is generated by a geometric Brownian motion:

$$dP_i(t) = P_i(t)(\mu_i dt + \sigma_i\, dB_i(t)), \quad i = 1, \dots, n-1 \quad (3)$$

where μ_i is the instantaneous conditional expected rate of P_i per unit time, $\sigma_i > 0$ is the volatility per unit time and $dB_i(t)$ is a Wiener process independent from each others for every i. The n-th asset is purely deterministic and its price $P_n(t)$ is given by

$$dP_n(t) = \mu_n(t) P_n(t)\, dt, \quad (4)$$

where μ_n stands for the risk-free interest rate.

Denoting by $N_i(t)$ the units of asset i owned by the investor at age t, it follows that her total wealth can be defined as

$$W(t) = \sum_{i=1}^{n} N_i(t) P_i(t). \quad (5)$$

Since $N_i(t)$ can be expressed in terms of $y_i(t)$ and $P_i(t)$, i.e., $N_i(t) = y_i(t)W(t)/P_i(t)$, then by (5) follows that

$$\sum_{i=1}^{n} y_i(t) = 1. \tag{6}$$

Moreover, differentiating (5) and taking into account the expression of $N_i(t)$, (3), (4) as well as the consumption c and life insurance Q, one derive the budget equation:

$$\begin{aligned} dW(t) &= \left[\left(\sum_{i=1}^{n-1} y_i(t)(\mu_i - \mu_p) + \mu_n \right) W(t) - c(t) - \theta Q(t) \right] dt \\ &\quad + \sum_{i=1}^{n-1} y_i(t)W(t)\sigma_i(t)dB_i(t), \end{aligned} \tag{7}$$

where $\theta > 0$ is the constant insurance premium per dollar of coverage and accordingly $\theta Q(t)$ is the whole insurance premium. Finally, we denote by $\xi_i \in (0, 1]$ the weight in terms of volume of the asset i on the total market volume. Note that this formulation implies that should the investor die at age t, the wealth left behind plus eventual life insurance will go to the beneficiary.

3 A multi-agent model's formulation

In this section, we extend Richard's model with a single player (investor) to the case of two players representing members of two competing households and investing in the same financial market. Our formulation consists of a finite-horizon dynamic game where the terminal date, $\widetilde{T}_j$, is the age of death of each player j, $j = 1, 2$, and hence a random variable. We start by showing that Richard's extended model can be converted into a deterministic model by using the survival function of each player, and, after that, by focusing on the deterministic problem, we determine the non-cooperative and cooperative optimal strategies.

As is common in the literature, we assume that there is a maximal age T beyond which no one will live and call $\pi_j(t)$ the probability that the player j dies at age t, so that $\int_0^T \pi_j(t)\,dt = 1$. Then, we define the survival function at the age t, namely the probability for the player j to live t years or more, as

$$S_j(t) = \int_t^T \pi_j(s)\,ds.$$

By definition, $S_j(0) = 1$, $S_j(T) = 0$ and $S_j(t)' < 0$. The force of mortality is defined as

$$\lambda_j(t) = \frac{\pi_j(t)}{S_j(t)} = -\frac{d}{dt}\log S_j(t), \qquad j = 1, 2$$

therefore, the survival function takes the form

$$S_j(t) = e^{-\int_0^t \lambda_j(s)\,ds} \qquad j = 1, 2.$$

Within this framework, where the force of mortality is deterministic, by Fubini's theorem follows that the expected utility for consumption and bequest that each player wants to maximize can be written as

$$\mathbb{E}\left[\sum_{j=1}^{2}\left(\int_0^{\widetilde{T}_j} e^{-\rho t} U_j(c_1(t), c_2(t))\, dt + \mathcal{F}(\widetilde{T}_j, Z_j(\widetilde{T}_j))\right)\right]$$
$$= \sum_{j=1}^{2}\int_0^T \pi_j(t)\left[\int_0^t e^{-\rho s} U_j(c_1(s), c_2(s))\, ds + \mathcal{F}(t, Z_j(t))\right] dt.$$

Using integration by parts and the equality $S_j(t)' = -\pi_j(t)$, we get

$$\begin{aligned}
&\sum_{j=1}^{2}\int_0^T \pi_j(t)\left[\int_0^t e^{-\rho s} U_j(c_1(s), c_2(s))\, ds + \mathcal{F}(t, Z_j(t))\right] dt \\
&\quad = \sum_{j=1}^{2}\left[-S_j(t)\left(\int_0^t e^{-\rho s} U_j(c_1(s), c_2(s))\, ds\right)\Bigg|_0^T \right. \\
&\qquad \left. + \int_0^T e^{-\rho t} S_j(t)\, U_j(c_1(t), c_2(t)) + \pi_j(t)\mathcal{F}(t, Z_j(t))\, dt\right] \\
&\quad = \int_0^T e^{-\rho t} \sum_{j=1}^{2} S_j(t)\, U_j(c_1(t), c_2(t)) + \pi_j(t)\mathcal{F}(t, Z_j(t))\, dt
\end{aligned} \tag{8}$$

Now, by considering the last expression of the objective function in (8), we formulate the optimization problem that each of the two players, $j = 1, 2$, aims to solve:

$$\begin{aligned}
&\max_{c_j, y_{1,j},\ldots,y_{n-1,j}, Q_j} J(W_j, c_j, y_{1,j}, \ldots, y_{n-1,j}, Q_j) \\
&= \max_{c_j, y_{1,j},\ldots,y_{n-1,j}, Q_j} \int_0^T e^{-\rho t} \sum_{j=1}^{2} S_j(t)\, U_j(c_1(t), c_2(t)) + \pi_j(t)\mathcal{F}(t, Z_j(t))dt
\end{aligned} \tag{9}$$

subject to:

$$\begin{cases}
dW_j(t) = \left[\left(\sum_{i=1}^{n-1} y_{i,j}(t)(\mu_i - \mu_n) + \mu_n\right) W_j(t) - c_j(t) - \theta_j Q_j(t)\right] dt \\
\qquad + \sum_{i=1}^{n-1} y_{i,j}(t) W_j(t)\sigma_i dB_i(t) \\
W_j(0) = W_j > 0 \ \text{ given}, \\
W_j(t) > 0 \ \ \forall t \in [0, T], \\
dP_i(t) = P_i(t)(\mu_i\, dt + \sigma_i\, dB_i(t)), \qquad P_i(0) > 0 \ \text{ given}, \quad i = 1, \ldots, n-1 \\
dP_n(t) = \mu_n P_n(t)\, dt, \qquad P_n(0) > 0 \ \text{ given}, \\
\sum_{i=1}^{n} y_{i,j}(t) = 1, \qquad j = 1, 2 \\
0 \le \sum_{j=1}^{2} y_{i,j}(t) \le \xi_i, \qquad i = 1, \ldots, n.
\end{cases} \tag{10}$$

Most of the parameters and the equations listed in (9)–(10) can be seen as a natural extension of those presented in the previous section for a single investor. However, the following considerations hold: the two players invest in the same financial market consisting of n available assets, $n-1$ of which are risky and one is risk-free, and they share the same number of investments. As a result, it is necessary that each player's utility function also depends on the consumption of the other. Indeed, the utility is defined as $U_j : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $j = 1, 2$, and it is a strictly concave C^2 (twice differentiable with respect to both variables and with continuous derivatives) function, while the bequest function $\mathcal{F}(t, Z_j(t))$ is strictly concave in the contingent bequest Z_j, from which the demand for life insurance is derived, with $Z_j(t)$ such that

$$Z_j(t) = W_j(t) + Q_j(t).$$

Denoting by $y_{i,j}(t)$, $i = 1, \ldots, n-1$, the share of wealth invested in the i-th asset by the player j, follows that

$$\sum_{i=1}^{n} y_{i,j}(t) = 1, \qquad j = 1, 2. \tag{11}$$

Moreover, we can establish the following constraint:

$$0 \leq \sum_{j=1}^{2} y_{i,j}(t) \leq \xi_i, \tag{12}$$

which states that the sum of the shares of wealth of both players invested in the same asset i will at most be equal to the weight, in terms of volume, of the asset i on the total market volume. By (11) and (12) follows the technical condition,

$$\sum_{i=1}^{n} \xi_i \geq 2, \tag{13}$$

because $0 \leq y_{i,j}(\cdot) \leq 1$ for every $i = 1, \ldots, n$ and $j = 1, 2$, and then the sum of the upper bounds ξ_i has to be greater or equal than 2. It expresses the fact that the total availability, in terms of volume, of all the n assets in the market has to be greater or equal than 2.

Note that the constraint (12) will get the following form

$$\sum_{j=1}^{2} y_{i,j}(t) = \xi_i, \tag{14}$$

when the cooperative case is considered, as it represents the external enforcement, which allows the coalition among the players. Every player j aims to find the optimal policy that maximizes her objective function (9). To do that, she will decide the optimal share of wealth to allocate to the asset i by considering each risky asset's volatility and the overall consumption level. Moreover, she will select the optimal amount of life insurance $Q_j(t)$ to be purchased to maximize the death benefit paid to the beneficiary or to be sold to maximize her wealth.

In the following section, we discuss how to obtain the optimal strategies for both players in non-cooperative and cooperative contexts and determine the optimal solution to the HJB equation in the case of power utility and Gompertz force of mortality.

4 Equilibrium outcomes

We now investigate, separately, the scenarios in which the two players do not and do cooperate in their wealth allocation choices. The comparison between the non-cooperative and cooperative strategies will be dealt with in Sect. 5.

4.1 Non-cooperation

In a non-cooperative framework, the player 1 solves problem (9)–(10) by taking the optimal strategies of the player 2 as exogenous data, meaning that each player cares only about her investments, consumption, and life insurance. As a consequence, it is equivalent for the player 1 to solve (9) or another maximization problem in which the utility function U_1 only depends on $c_1(t)$, as $c_2(t)$ can be considered exogenous and, therefore, it will not affect the calculation of the maximum. The player 1 also knows the shares of wealth of player 2 invested in all assets i, as this data is provided exogenously. Then, it is possible to show (the proofs of all the propositions are presented in Appendix A) that the following result for player 1 holds. An equivalent result also holds for the player 2 (see Remark 1).

Proposition 1 *Assume that the force of mortality $\lambda_1(t)$ follows the Gompertz law, i.e,*

$$\lambda_1(t) = \lambda_1(0)\, e^{\gamma_1 t}, \qquad \lambda_1(0) > 0,\ \gamma_1 > 0,$$

and that the utility function takes the form:

$$U_1(c_1(t), c_2(t)) = \frac{1}{1-\alpha_1}(c_1(t))^{1-\alpha_1} + \frac{1}{1-\alpha_2}(c_2(t))^{1-\alpha_2},$$

with constant relative risk aversion $\alpha_j > 0$, $\alpha_j \neq 1$ for $j = 1, 2$. If, in addition,

$$\alpha_1 > \max\left\{\frac{\mu_1-\mu_n}{(\sigma_1)^2(\xi_1 - y_{1,2})}, \frac{\mu_2-\mu_n}{(\sigma_2)^2(\xi_2 - y_{2,2})}, \ldots, \frac{\mu_{n-1}-\mu_n}{(\sigma_{n-1})^2(\xi_{n-1} - y_{n-1,2})}\right\}, \tag{15}$$

and $G_1(t) > 0\ \forall\, t \in [0, T)$ then the non-cooperative optimal consumption, the share of wealth invested on the i-th asset and the optimal amount of life insurance of the player 1 are respectively given by:

$$c_1(t) = \left(\frac{1}{G_1(t)}\right)^{\frac{1}{\alpha_1}} W_1(t), \tag{16}$$

$$y_{i,1}(t) = \frac{\mu_i - \mu_n}{\alpha_1(\sigma_i)^2}, \quad i = 1, \ldots, n-1, \tag{17}$$

$$Q_1(t) = \left[\left(\frac{\lambda_1(0)\, e^{\gamma_1 t}}{\theta_1\, G_1(t)}\right)^{\frac{1}{\alpha_1}} - 1\right] W_1(t), \tag{18}$$

where $G_1(t)$ is given by:

$$\begin{aligned} G_1(t) =& e^{\{\nu_1(t-T) + \frac{\lambda_1(0)}{\gamma_1}(e^{\gamma_1 t} - e^{\gamma_1 T})\}} \\ &\times \left[k^{\frac{1}{\alpha_1}} + \int_t^T \left(1 + (\theta_1)^{\frac{\alpha_1 - 1}{\alpha_1}} (\lambda_1(0))^{\frac{1}{\alpha_1}}\, e^{\frac{\gamma_1 s}{\alpha_1}}\right) e^{\{-\frac{\nu_1}{\alpha_1}(s-T) - \frac{\lambda_1(0)}{\alpha_1\gamma_1}(e^{\gamma_1 s} - e^{\gamma_1 T})\}} ds\right]^{\alpha_1}, \end{aligned} \tag{19}$$

with

$$\nu_1 = -(1-\alpha_1)\left(\frac{1}{2}\sum_{i=1}^{n-1}\frac{(\mu_i - \mu_n)^2}{\alpha_1(\sigma_i)^2} + \mu_n - \frac{\rho}{1-\alpha_1} + \theta_1\right). \tag{20}$$

Provided that some technical conditions hold, Proposition 1 determines, in closed form, the expressions of the optimal consumption, the share of wealth invested in the i−th risky asset and the optimal amount of life insurance. Note that (15) translates the condition $0 \leq y_{i,1} < \xi_i - y_{i,2}, i = 1, \ldots, n-1$, that follows from (12) in the non-cooperative framework. We can observe that the consumption is linear in wealth, and the share of wealth invested in the i−th risky asset is constant as in Merton's model (Merton, 1971). Moreover, from Proposition 1 we also derive the optimal quantity of contingent bequest

$$Z_1(t) = Q_1(t) + W_1(t) = \left(\frac{\lambda_1(0)\, e^{\gamma_1 t}}{\theta_1\, G_1(t)}\right)^{\frac{1}{\alpha_1}} W_1(t),$$

which is proportional to wealth. This fact is in line with the standard result on life insurance 'a la Fischer' (Fischer, 1973). In case that $Z_1(t) < W_1(t)$, as observed in Richard (1975), the consumer becomes the seller and not the buyer of life insurance. Apart from these intuitive observations, we cannot say much about the effects of both biometric and financial parameters on consumption and life insurance, as the expressions in (16) and (18) are particularly cumbersome due to the form of $G_1(t)$ in (19). Therefore, in Sect. 5, we will present a sensitivity analysis to scrutinize the impact of these parameters on optimal controls. Finally, the optimal controls (16), (17) and (18) can be used inside the budget equation (7), which, if solved, gives the optimal wealth of the player 1.

Remark 1 As anticipated, for player 2, an equivalent result to Proposition 1 holds. Indeed, proceeding as in Proposition 1, we get that the optimal controls for the player two are given by:

$$c_2(t) = \left(\frac{1}{G_2(t)}\right)^{\frac{1}{\alpha_2}} W_2(t), \tag{21}$$

$$y_{i,2}(t) = \frac{\mu_i - \mu_n}{\alpha_2(\sigma_i)^2}, \quad i = 1, \ldots, n-1, \tag{22}$$

$$Q_2(t) = \left[\left(\frac{\lambda_2(0)\, e^{\gamma_2 t}}{\theta_2\, G_2(t)}\right)^{\frac{1}{\alpha_2}} - 1\right] W_2(t), \tag{23}$$

where $G_2(t)$ is similar to the one in (19).

4.2 Cooperation

In this section, we focus on the cooperative setup where the two players, sharing market information forced by (14), agree on the investments to be made. Such a binding agreement implies that the two players are active in the financial market in the same years, meaning that the two players must have the same residual lifetime. Therefore, we assume that they have the same survival probability and insurance coverage, i.e., $S_1(t) = S_2(t) = S(t)$ for every $t \in [0, T]$ and $\theta_1 = \theta_2 = \theta > 0$. From the equivalence of the survival probabilities follows the ones of the forces of mortality, $\lambda_1(t) = \lambda_2(t) = \lambda(t)$. Moreover, we assume that the players are characterized by the same constant risk aversion, i.e., $\alpha_1 = \alpha_2 = \alpha$. Then, the following holds.

Proposition 2 *Assume that the force of mortality $\lambda(t)$ follows the Gompertz law, i.e,*

$$\lambda(t) = \lambda(0)\, e^{\gamma t}, \qquad \lambda(0) > 0,\ \gamma > 0,$$

and that the utility functions U_1 and U_2 in (9) take the same form:

$$U_1(c_1(t), c_2(t)) = U_2(c_1(t), c_2(t)) = \frac{1}{1-\alpha}\left[(c_1(t))^{1-\alpha} + (c_2(t))^{1-\alpha}\right], \qquad \alpha > 0,\ \alpha \neq 1.$$

If $G(t) > 0\ \forall\ t \in [0, T)$ then the cooperative optimal consumption, the share of wealth invested on the i-th asset and the optimal amount of life insurance of the two players $j = 1, 2$ are respectively given by:

$$c_j(t) = \left(\frac{2}{G(t)}\right)^{\frac{1}{\alpha}} W_j(t), \tag{24}$$

$$y_{i,j}(t) = \frac{\mu_i - \mu_n}{\alpha(\sigma_i)^2}, \quad i = 1, \ldots, n-1, \tag{25}$$

$$Q_j(t) = \left[\left(\frac{\lambda(0)\, e^{\gamma t}}{\theta\, G(t)}\right)^{\frac{1}{\alpha}} - 1\right] W_j(t), \tag{26}$$

where $G(t)$ is given by:

$$G(t) = e^{\{\nu(t-T)+\frac{\lambda(0)}{\gamma}(e^{\gamma t}-e^{\gamma T})\}} \times \left[(\bar{k})^{\frac{1}{\alpha}} + \int_t^T \left(2^{\frac{1}{\alpha}} + \theta^{\frac{\alpha-1}{\alpha}}\lambda(0)^{\frac{1}{\alpha}}\, e^{\frac{\gamma s}{\alpha}}\right) e^{\{-\frac{\nu}{\alpha}(s-T)-\frac{\lambda(0)}{\alpha\gamma}(e^{\gamma s}-e^{\gamma T})\}} ds\right]^{\alpha}, \tag{27}$$

with

$$\nu = -(1-\alpha)\left(\frac{1}{4}\sum_{i=1}^{n-1}\xi_i(\mu_i - \mu_n) + \mu_n - \frac{\rho}{1-\alpha} + \theta\right). \tag{28}$$

Provided that a technical condition holds true, Proposition 2 gives in closed form the expressions of the consumption, the share of wealth invested in the $i-$th risky asset and the optimal amount of life insurance in a cooperative setting. Similar considerations to the ones provided under Proposition 1 for the optimal strategies and the optimal quantity of contingent bequest $Z_j(t)$ hold. The optimal wealth of each player j can be obtained still solving equation (7) but now considering the optimal controls (24), (25) and (26) which, also in this case due their form do not allow the impact of the parameters on them to be observed; hence we refer to the numerical Sect. 5 for that analysis.

5 Empirical implementation

In the present section, we propose an empirical implementation of the theoretical framework exposed in Sects. 3, 4. Such a study is performed through two steps of analysis and distinguishing between the cooperative and non-cooperative cases.

First of all, we proceed to set the value of parameters characterizing the equilibrium outcomes. In particular, given the set of all parameters involved in optimal controls, i.e.,

$$\Gamma = \left\{\alpha_1,\ \alpha_2,\ \alpha,\ \lambda_1(0),\ \lambda(0),\ \gamma_1,\ \gamma,\ \theta_1,\ \theta,\ \mu_i,\ \mu_n,\ \sigma_i,\ \rho,\ k,\ \bar{k}\right\},$$

we proceed to estimate separately the following three sets of parameters, namely:

$$\text{Financial parameters}: \quad \Gamma^{(F)} = \{\mu_i,\ \mu_n,\ \sigma_i\},\ i = 1, \ldots, n-1$$

$$\text{Non-Financial parameters}: \quad \Gamma^{(NF)} = \left\{\alpha_1,\ \alpha_2,\ \alpha,\ \theta_1,\ \theta,\ \rho,\ k,\ \bar{k}\right\}$$

Table 1 Estimates for financial parameters

i	Asset	Parameters	
		μ_i	σ_i
1	S&P500	0.0178	0.1875
2	FTSE100	0.0105	0.1435
3	NIKKEI	0.0178	0.1873
4	DAX	0.0176	0.1865
5	3TB	0.0067	–

All the daily time series employed in the estimation concern the period 01/01/2021–07/06/2023

Table 2 Estimates for biometric and non-financial parameters

$\lambda_1(0) = \lambda(0)$	$\gamma_1 = \gamma$	$\theta_1 = \theta$	$k = \bar{k}$	ρ
0.00128	0.11539	0.2	1	0.01

$$\text{Biometric parameters}: \quad \Gamma^{(B)} = \{\lambda_1(0),\ \lambda(0),\ \gamma_1,\ \gamma\}$$

Concerning the financial parameters, we assume that the players can invest their wealth in a financial market composed of four risky assets and a risk-free one. The former are given by the Standard & Poor's 500 Index (S&P500), the Financial Times Stock Exchange 100 Index (FTSE100), the Japan's Nikkei 225 Stock Average (NIKKEI), and the Deutscher Aktien Index (DAX). On the other side, the risk-free asset is portrayed by the U.S. 3-month Treasury Bill (3TB). To estimate the risky assets parameters, we collect the daily time series of closing prices from Yahoo Finance and, applying the methods of moments to (3), the estimates for μ_i and σ_i, for $i = 1, \ldots, 4$, are the sample mean and the sample standard deviation of the observed log-returns, respectively. The parameter μ_5, instead, is calibrated on the daily time series of the U.S. 3-month Treasury Bill returns gathered by the Federal Reserve Economic Data repository (https://fred.stlouisfed.org/). The estimated values of financial parameters are reported in Table 1.

The biometric parameters are assessed using the mortality data provided by the Human Mortality Database (2018). In particular, we consider period death counts and exposures for the 2019 Italian female population, referring to the age range [50, 100]. Then, our player 1 in the non-cooperative case is an Italian female aged 50, which may reach the age $T = 100$, while player 2 is an agent with a different lifetime (e.g., an Italian male). Note that the age $T = 100$ serves as the maximum attainable age while the age 50 represents the inception age. Conversely, Proposition 2 assumptions imply that, in the cooperative case, we deal with two identical biometrically Italian females. Consequently, we pose $\lambda_1(0) = \lambda$ and $\gamma_1 = \gamma$, and, coherently, the same insurance premium per dollar of coverage, i.e. $\theta_1 = \theta$. To estimate the biometric parameters, $\lambda_1(0)$ and γ_1, we employ the non-linear least square method concerning Gompertz's mortality law parametrized as in Propositions 1 and 2. The obtained values are displayed in Table 2. The latter also contains the values of some of the parameters in $\Gamma^{(NF)}$, which are established by assumption. Regarding the constant relative risk aversion parameters, we set $\alpha_1 = 2$ and $\alpha_2 = 15$ for the non-cooperative case and, with the aim to compare the two scenarios, $\alpha = \alpha_1$ is considered for the cooperative situation.

In addition, the optimal control expressions also incorporate the wealth $W_1(t)$ and the $G_1(t)$ function in the non-cooperative case, while in the cooperative ones, we have $W_j(t)$ and $G(t)$. Concerning the wealth, their expressions are derived by solving the stochastic differential equation in (10) at the equilibrium. In particular, for the non-cooperative scenario, we get:

$$W_1(t) = W_1(0)\exp\Bigg\{\int_0^t \sum_{i=1}^{4} \frac{(\mu_i-\mu_n)^2}{\alpha_1(\sigma_i)^2}\left(1-\frac{1}{2\alpha_1}\right) + \mu_n - \theta_1 - \left(\frac{1}{G_1(s)}\right)^{\frac{1}{\alpha_1}} \times \left(1+\left(\lambda_1(0)e^{\gamma_1 s}\,\theta_1^{\alpha_1-1}\right)^{\frac{1}{\alpha_1}}\right)ds + \sum_{i=1}^{4}\frac{\mu_i-\mu_n}{\alpha_1\sigma_i}\left(B_i(t)-B_i(0)\right)\Bigg\}, \tag{29}$$

while for the cooperative scenario holds the following:

$$W_j(t) = W_j(0)\exp\Bigg\{\int_0^t \sum_{i=1}^{4} \frac{(\mu_i-\mu_n)^2}{\alpha(\sigma_i)^2}\left(1-\frac{1}{2\alpha}\right) + \mu_n - \theta - \left(\frac{1}{G(s)}\right)^{\frac{1}{\alpha}} \times \left(2^{\frac{1}{\alpha}}+\left(\lambda(0)e^{\gamma s}\,\theta^{\alpha-1}\right)^{\frac{1}{\alpha}}\right)ds + \sum_{i=1}^{4}\frac{\mu_i-\mu_n}{\alpha\sigma_i}\left(B_i(t)-B_i(0)\right)\Bigg\}. \tag{30}$$

We notice that integrals in (29) and (30), as well as the functions $G_1(t)$ and $G(t)$, do not have an explicit solution. Then, their value has been obtained by numerical approximation through the trapezoidal rule and by using a constant discretization step with amplitude $\Delta = 0.001$.

Afterwards, we perform a sensitivity analysis showing how parameters affect the optimal allocation of player wealth between consumption, financial assets, and life insurance. To this end, we consider biometric parameters and select the constant relative risk aversion as the more relevant non-financial parameter. For the latter we pose $\alpha_1 \in [2,\ 4]$, so that α varies in the same range. Regarding the biometric parameters, we assume three possible deterministic outcomes: (a) the central scenario, (b) the mortality-increasing scenario, and (c) the mortality-decreasing scenario. The central scenario stems from using biometric parameters mentioned in Table 2 to determine the optimal controls. Instead, scenarios (b) and (c) are obtained by applying, respectively, an instantaneous upward and downward shock to the force of mortality defined in the central scenario. To define such a shock, the following parametrization of Gompertz's mortality law is considered:

$$\lambda_1(t) = \frac{1}{b_1}\exp\left(\frac{t+\tau-M_1}{b_1}\right),$$

where $b_1 = \frac{1}{\gamma_1}$, t is the considered age ($t = 50$ in our application), $t+\tau,\ \tau > 0$, is an attainable age, and $M_1 = t - b\ln\left(b\lambda_1(0)\right)$ is the modal age for the player 1. The mortality-increasing (decreasing) scenario is defined by augmenting (reducing) the biometric parameters of the central scenario. Then, for the parameter M_1, the mortality shocks are derived by shifting the modal age of the central scenario (approximately 89). In the mortality-decreasing scenario, such a shift is equal to the relative variation of the Italian females' modal age over the period 1980–2019 (approximately $+12,41\%$), and it is applied to 2019's modal age. In the mortality-increasing scenario, the shift is employed with the opposite sign. Concerning the parameter $\lambda_1(0)$, the shift is determined as the relative variation of the b_1 parameter over the period 1980–2019. The achieved biometric parameters for scenarios (b) and (c) are collected in Table 3. Such values for the non-cooperative case are also adopted in the cooperative one.

5.1 Non-cooperation versus cooperation

In the following, the sensitivity analysis results are shown comparing the non-cooperative and the cooperative strategies. We stress that the comparison is performed assuming $\alpha = \alpha_1$.

Table 3 Biometric parameters for each mortality scenario

Scenario	Parameters	
	$\lambda_1(0)$	γ_1
Mortality-increasing	0.00187	0.14362
Central	0.00128	0.11539
Mortality-decreasing	0.0009	0.09643

Table 4 Optimal share of wealth invested in financial assets: the non-cooperative versus cooperative case

Asset_i	ξ_i	Non-cooperation		Cooperation
		$y_{i,1}$	$y_{i,2}$	$y_{i,j}$
S&P 500	0.3167	0.1583	0.0211	0.1583
FTSE 100	0.1875	0.0938	0.0125	0.0938
NIKKEI	0.3169	0.1585	0.0211	0.1585
DAX	0.3149	0.1575	0.0210	0.1575
3TB	0.8639	0.4319	0.9243	0.4319

Values are rounded to the fourth decimal. Since $\alpha_1 = \alpha$, we observe that $y_{i,1} = y_{i,j}$, for $i = 1, \ldots, 4$ and $y_{5,1} = y_{5,j}$

Table 4 exhibits the optimal share of wealth allocated in financial assets. Looking at the non-cooperative case, it is worth noting that the constraint (12) is satisfied for all assets except for the risk-free investment. Indeed, by using the optimal strategies (17) and (22) relative to risky investments, the two players have to allocate their residual share of wealth to the risk-free asset to fulfil the condition (6). This fact implies that the sum of optimal risk-free investments exceeds the current asset volume available in the financial markets. Therefore, in the non-cooperative scenario, one of the two players will have to adopt a suboptimal strategy with respect to risk-free investment. On the contrary, due to (14), the cooperative case requires that players are constrained to invest an amount of wealth such that the overall market volume is saturated. On the one hand, this implies that both players optimally allocate their wealth also on the risk-free asset; on the other, both players may have a lower amount of wealth for consumption and life insurance purchasing than in the non-cooperative case.

We notice that, under the assumption $\alpha_1 = \alpha$, both the non-cooperating player 1 and the cooperating players assign the same optimal shares of wealth to financial assets. Then, their allocation of wealth differs in terms of both consumption and life insurance.

Fig. 1 illustrates the effect of an increasing risk aversion on the optimal share of wealth assigned to financial assets. As expected, more wealth is attributed to the risk-free asset when the risk aversion increases. Looking at the risky assets side, both the non-cooperating player 1 and the cooperating players prefer to invest in DAX, S&P500, and NIKKEI indices. As emerged in Table 1, the FTSE100 index is characterized by the lower ratio between return and riskiness, where the latter is represented by σ_i. Hence, the players tend to allocate a lower portion of wealth in the FTSE100, preferring other risky investments, despite an increasing portion of wealth being transferred to the risk-free asset as risk aversion augments.

Figure 2 shows the optimal consumption at three ages, i.e. when the player is 65 years old, 75 years old, and 85 years old, respectively, and how such a consumption varies according to risk aversion increments. For both the non-cooperative and cooperative scenarios, we observe a growing consumption by age and risk aversion, albeit the consumption appears more sensitive to age variations rather than risk aversion changes. Moreover, the cooperative scenario is characterized by a greater consumption activity with respect to the non-cooperative

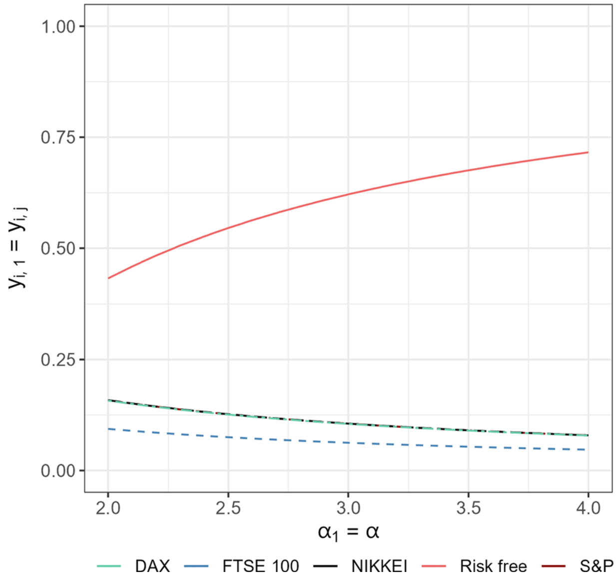

Fig. 1 Optimal share of wealth varying the constant relative risk aversion

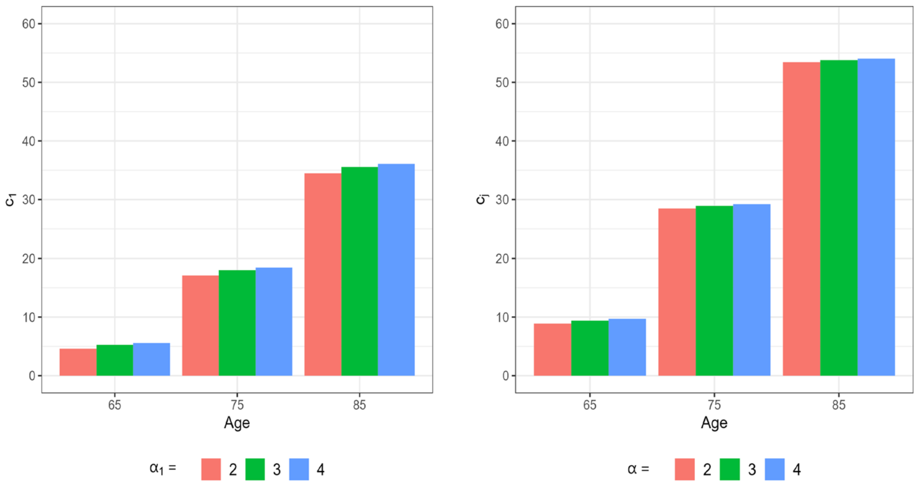

Fig. 2 Optimal consumption varying the constant relative risk aversion: non-cooperative (lhs) versus cooperative (rhs)

ones, by age and risk aversion. Such an interesting finding is justified by the behaviour of both the wealth, $W_1(t)$ and $W_j(t)$, and the functions $G_1(t)$ and $G(t)$.

Recalling the optimal control (16), we stress that the consumption for non-cooperating player 1 is proportional to the wealth $W_1(t)$, where the coefficient of proportionality contains the $G_1(t)$ function. Similarly, by the expression of the optimal control (24), the consumption of cooperating players is proportional to the wealth $W_j(t)$, and the function $G(t)$ shapes the proportionality coefficient. Figure 3 exhibits both the $G_1(t)$ and $G(t)$ functions, as well as the expectation about the wealth $W_1(t)$ and $W_j(t)$. We note that, on average, the cooperation-

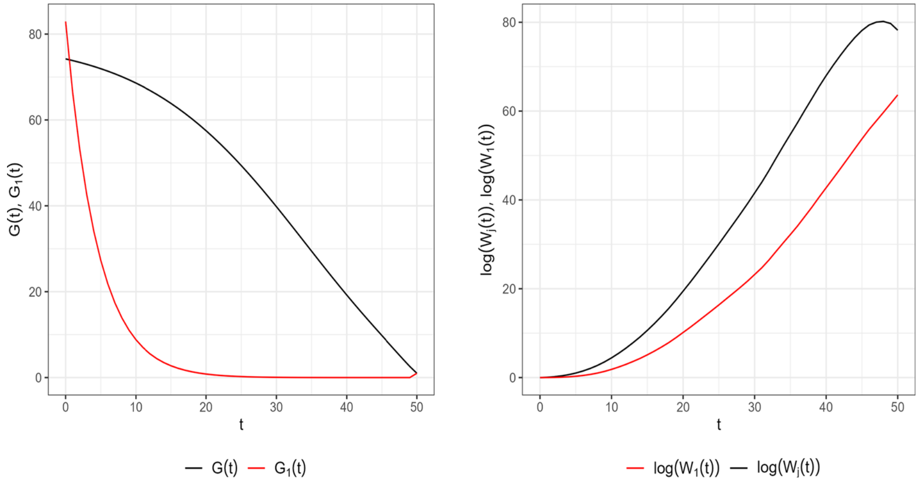

Fig. 3 The left-hand side plot shows $G_1(t)$ and $G(t)$, while the right-hand side plot contains the expectation of the wealth $W_1(t)$ and $W_j(t)$ at the equilibrium. The latter are represented on a log-scale exclusively to facilitate graphic display. For both plots, $\alpha_1 = \alpha = 2$ is considered

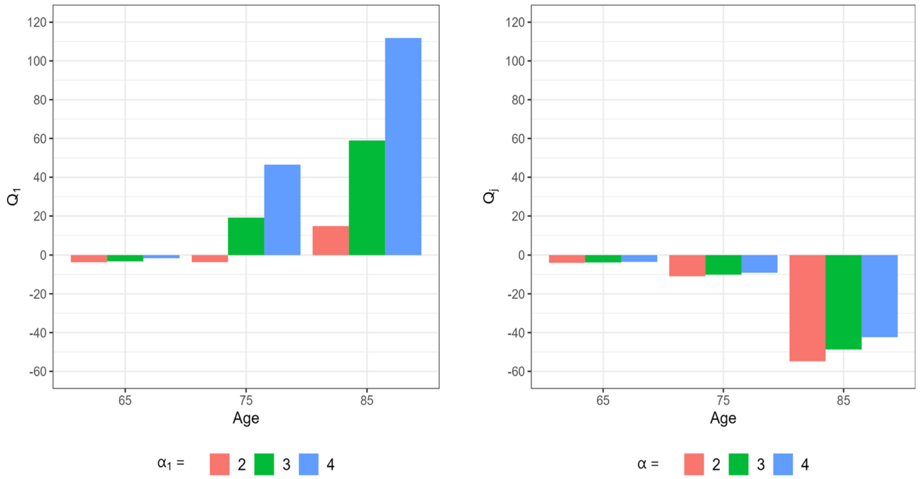

Fig. 4 Optimal life insurance amount varying the constant relative risk aversion: non-cooperative (lhs) versus cooperative (rhs)

based wealth becomes bigger and bigger than the non-cooperation-based one, so that a higher consumption activity for cooperating players is allowed. As (29) and (30) state, the wealth expressions are characterized by the presence of $G_1(t)$ and $G(t)$ functions, respectively, but only the $G(t)$ function embeds the investment constraints ξ_i (see (28)). To some extent, the condition (14) implies, on one side, that cooperating players have to employ wealth to saturate the market volume; on the other, the return on the overall investment activity will be higher than in the non-cooperative scenario, so that cooperating players accrue a greater wealth.

The comparison between the non-cooperative and the cooperative scenarios highlights significant gaps in terms of optimal allocations of life insurance. For instance, Fig. 4 reveals that the non-cooperating player 1 purchases life insurance when older ages are approached,

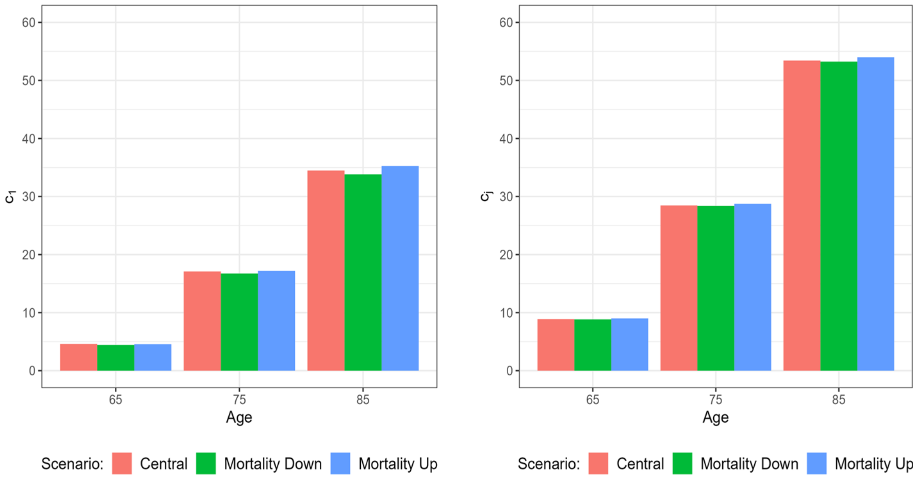

Fig. 5 Optimal consumption for both players varying the biometric parameters: the non-cooperative case(lhs) versus the cooperative case (rhs)

while cooperating players act always as life insurance sellers. As argued in Richard (1975), an agent may either buy or sell life insurance depending on her risk aversion and her preference for consumption or inheritance. Then, in our framework, the cooperating players prefer to sell life insurance aiming to support consumption and, to some extent, they are not, or less, sensitive to the risk of death. On the contrary, the non-cooperating player places more and more importance on the risk of death as she gets older. To understand the feelings driving the players about the optimal life insurance allocations, let us consider:

$$\tilde{\lambda}_1(t) := \left(\frac{\lambda_1(0)}{\theta_1 G_1(t)}\right)^{\frac{1}{\alpha_1}}, \quad \tilde{\gamma}_1 := \frac{\gamma_1}{\alpha_1}, \quad \tilde{\lambda}(t) := \left(\frac{\lambda(0)}{\theta G(t)}\right)^{\frac{1}{\alpha}}, \quad \tilde{\gamma} := \frac{\gamma}{\alpha}.$$

Then, the optimal control expressions (18) and (26) can be rewritten, respectively, as:

$$Q_1(t) = \left(\tilde{\lambda}_1(t)e^{\tilde{\gamma}_1 t} - 1\right) W_1(t), \tag{31}$$

$$Q_j(t) = \left(\tilde{\lambda}(t)e^{\tilde{\gamma} t} - 1\right) W_j(t). \tag{32}$$

The terms $\tilde{\lambda}_1(t)e^{\tilde{\gamma}_1 t}$ and $\tilde{\lambda}(t)e^{\tilde{\gamma} t}$ have the Gompertz mortality law's form with time-dependent baseline mortality, and they express a measure of the exposition to the risk of death for each unit of wealth. Therefore, if $\tilde{\lambda}_1(t)e^{\tilde{\gamma}_1 t} > 1$ then the exposure to the risk of death becomes greater than the available wealth $W_1(t)$, and it is optimal to purchase life insurance. On the contrary, it is preferable to sell life insurance. Since the $G_1(t)$ function characterizing $\tilde{\lambda}_1(t)$ quickly decreases, the exposition to the risk of death grows as the age increases, and the non-cooperating player uses a portion of her wealth to buy life insurance. The opposite situation occurs by considering the cooperative case because the $G(t)$ function reduces slowly.

Finally, Figs. 5 and 6 show how optimal controls vary when different mortality scenarios are addressed (see Table 3). In the non-cooperative case, as well as in the cooperative one, a downward shock of mortality translates into a consumption reduction. The opposite result emerges if an upward shock of mortality occurs. Then, if the non-cooperating player lives longer, she prefers to limit consumption allocating a majority share of wealth to life insurance.

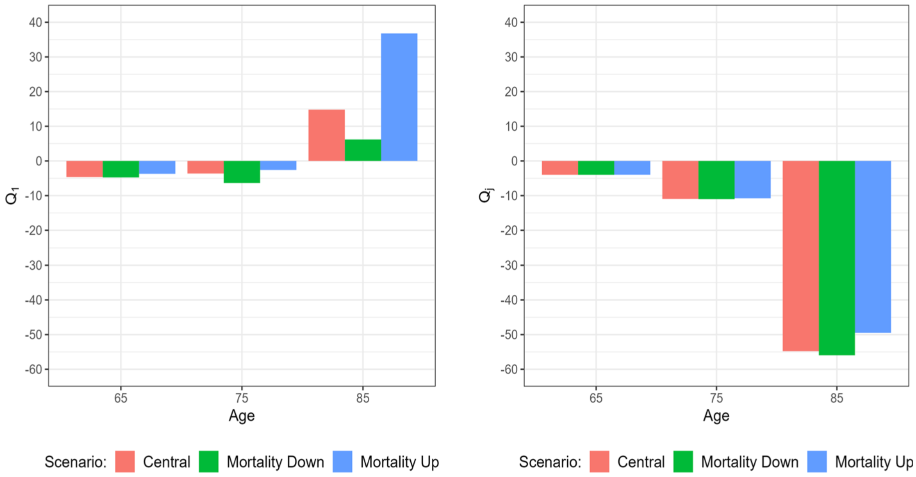

Fig. 6 Optimal life insurance amount for both players varying the biometric parameters: the non-cooperative case (lhs) versus the cooperative case (rhs)

Regarding the cooperating players, Fig. 6 represents the impact of mortality shocks on the cooperative-based allocation of life insurance. Essentially, in line with the central mortality scenario, the cooperative players prefer life insurance selling to support the wealth allocation between financial investments and consumption.

Then, the cooperative players continue to exploit life insurance as an asset financing investments and consumption, independently of mortality shocks and coherently with the central mortality scenario, while the non-cooperative player promotes a greater(lesser) life insurance purchasing(selling) when a mortality-increasing(decreasing) scenario occurs.

6 Conclusion

This work presents a multi-agent portfolio optimization model with life insurance for two players with random lifetimes under a dynamic game approach. In particular, the two players act on the same financial market and maximize their own utility for consumption and bequest. Then, the optimal consumption, investments across different assets, and life insurance are provided in cooperative and non-cooperative scenarios using a CRRA utility function.

We test our theoretical results through a sensitivity analysis, aiming to investigate the effects on optimal controls of changes in risk aversions and biometric parameters. Such analysis suggests that cooperation is preferable to non-cooperation in terms of optimal wealth allocation in consumption. Conversely, non-cooperation promotes the coverage of the risk of death. In future research, we aim to extend our theoretical framework by incorporating stochastic volatility models for risky asset prices. In addition, we will consider age-dependent insurance premiums per dollar of coverage.

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Declarations

Conflict of interest The authors declare no conflict of interest.

A Appendix

A.1 Non-cooperation

By defining $V^1(t, W_1(t)) \in C^{1,2}$ (once differentiable with respect to t and twice with respect to $W_1(\cdot)$, with continuous derivatives) the value function associated with the problem (9)–(10) for the player 1, the quadruple $(W_1(t), c_1(t), y_{i,1}(t), Q_1(t))$ is its solution if V^1 solves the following HJB equation:

$$\begin{cases} D_t V^1 + \max\limits_{\{c_1, y_{i,1}, Q_1\}} \left\{ \left[\left(\sum_{i=1}^{n-1} y_{i,1}(t)(\mu_i - \mu_n) + \mu_n \right) W_1(t) - c_1(t) - \theta_1 Q_1(t) \right] D_{W_1} V^1 \right. \\ \quad + e^{-\rho t} S_1(t) U_1(c_1(t), c_2(t)) + \pi_1(t) \mathcal{F}(t, Z_1(t)) \\ \quad \left. + \frac{1}{2} \sum_{i=1}^{n-1} (y_{i,1}(t))^2 (\sigma_i)^2 (W_1(t))^2 D^2_{W_1} V^1 \right\} = 0, \\ V^1(W_1(T), T) = 0, \end{cases} \tag{33}$$

where $D_t V^1$ is the first derivative of the value function V^1 with respect to t, while $D_{W_1} V^1$ and $D^2_{W_1} V^1$ are, respectively, the first and second derivative of V^1 with respect to $W_1(\cdot)$. The Hamiltonian H_1 associated with the problem (9)–(10) is:

$$\begin{aligned} & H_1(W_1(t), c_1(t), y_{i,1}(t), Q_1(t), D_{W_1} V^1, D^2_{W_1} V^1) \\ & = \left[\left(\sum_{i=1}^{n-1} y_{i,1}(t)(\mu_i - \mu_n) + \mu_n \right) W_1(t) - c_1(t) - \theta_1 Q_1(t) \right] D_{W_1} V^1 \\ & \quad + e^{-\rho t} S_1(t) U_1(c_1(t), c_2(t)) + \pi_1(t) \mathcal{F}(t, Z_1(t)) + \frac{1}{2} \sum_{i=1}^{n-1} (y_{i,1}(t))^2 (\sigma_i)^2 (W_1(t))^2 D^2_{W_1} V^1. \end{aligned}$$

In that framework the constraint $0 \le y_{i,1}(t) < \xi_i - y_{i,2}(t)$ has to be satisfied for every $i = 1, \dots, n-1$, and hence

$$\alpha_1 > \max \left\{ \frac{\mu_1 - \mu_n}{(\sigma_1)^2(\xi_1 - y_{1,2})}, \frac{\mu_2 - \mu_n}{(\sigma_2)^2(\xi_2 - y_{2,2})}, \dots, \frac{\mu_{n-1} - \mu_n}{(\sigma_{n-1})^2(\xi_{n-1} - y_{n-1,2})} \right\}$$

with $\mu_i - \mu_n \ge 0, \forall\, i = 1, \dots, n-1$. Moreover, by assuming

$$\lambda_1(t) = \lambda_1(0)\, e^{\gamma_1 t}, \qquad \lambda_1(0) > 0,\ \gamma_1 > 0, \tag{34}$$

$$U_1(c_1(t), c_2(t)) = \frac{1}{1-\alpha_1}(c_1(t))^{1-\alpha_1} + \frac{1}{1-\alpha_2}(c_2(t))^{1-\alpha_2},\ \alpha_j > 0,\ \alpha_j \ne 1,\quad j = 1, 2, \tag{35}$$

$$V^1(t, W_1(t)) = \frac{e^{-\rho t}}{1-\alpha_1} S_1(t)\, G_1(t)(W_1(t))^{1-\alpha_1},\ \ G_1(t) \in \mathbb{R},\ \alpha_1 > 0,\ \alpha_1 \neq 1, \tag{36}$$

$$\mathcal{F}(t, Z_1(t)) = \frac{e^{-\rho t}}{1-\alpha_1} (Z_1(t))^{1-\alpha_1},\ \ \alpha_1 > 0,\ \alpha_1 \neq 1, \tag{37}$$

we can apply the first order conditions, i.e., $\partial H_1/\partial c_1 = 0$, $\partial H_1/\partial y_{i,1} = 0$ and $H_1/\partial Q_1 = 0$. Therefore, we get

$$c_1(t) = \left(\frac{1}{G_1(t)}\right)^{\frac{1}{\alpha_1}} W_1(t), \tag{38}$$

$$y_{i,1}(t) = \frac{\mu_i - \mu_n}{\alpha_1 (\sigma_i)^2}, \quad i = 1, \ldots, n-1, \tag{39}$$

$$Z_1(t) = \left(\frac{\lambda_1(0)\, e^{\gamma_1 t}}{\theta_1\, G_1(t)}\right)^{\frac{1}{\alpha_1}} W_1(t) \longrightarrow Q_1(t) = \left[\left(\frac{\lambda_1(0)\, e^{\gamma_1 t}}{\theta_1\, G_1(t)}\right)^{\frac{1}{\alpha_1}} - 1\right] W_1(t). \tag{40}$$

Note that since $c_1(t) > 0$ for every $t \in [0, T]$ then $G_1(t) \neq 0$ for every $t \in [0, T]$. Now, by inserting the expressions of (34)–(40) into (33) and recalling that the utility U_1 only depends on the $c_1(t)$ term (due the equivalence of the maximization problems discussed at the begin of the section 4.1) we get that $G_1(t)$ must satisfy the following Bernoulli equation

$$\begin{aligned} \dot{G}_1(t) = & \left[-(1-\alpha_1) \left(\frac{1}{2} \sum_{i=1}^{n-1} \frac{(\mu_i - \mu_n)^2}{\alpha_1 (\sigma_i)^2} + \mu_n - \frac{\rho}{1-\alpha_1} + \theta_1 \right) + \lambda_1(0) e^{\gamma_1 t} \right] G_1(t) \\ & - \alpha_1 \left(1 + (\theta_1)^{\frac{\alpha_1 - 1}{\alpha_1}} (\lambda_1(0))^{\frac{1}{\alpha_1}}\, e^{\frac{\gamma_1 t}{\alpha_1}} \right) (G_1(t))^{\frac{\alpha_1 - 1}{\alpha_1}}, \end{aligned} \tag{41}$$

with terminal condition $G_1(T) = k \in \mathbb{R} \setminus \{0\}$ obtained by evaluating (36) in T and by imposing it equal to 0. Indeed, being $S_1(T) = 0$ by definition, then $G_1(T)$ can assume every real value different from zero. Now, denoting by

$$\nu_1 = -(1-\alpha_1) \left(\frac{1}{2} \sum_{i=1}^{n-1} \frac{(\mu_i - \mu_n)^2}{\alpha_1 (\sigma_i)^2} + \mu_n - \frac{\rho}{1-\alpha_1} + \theta_1 \right),$$

the solutions of (41) dependent on the constant k are the following

$$\begin{aligned} G_1(t) = & \, e^{\{\nu_1 (t-T) + \frac{\lambda_1(0)}{\gamma_1} (e^{\gamma_1 t} - e^{\gamma_1 T})\}} \\ & \times \left[k^{\frac{1}{\alpha_1}} + \int_t^T \left(1 + (\theta_1)^{\frac{\alpha_1 - 1}{\alpha_1}} (\lambda_1(0))^{\frac{1}{\alpha_1}}\, e^{\frac{\gamma_1 s}{\alpha_1}} \right) e^{\{-\frac{\nu_1}{\alpha_1}(s-T) - \frac{\lambda_1(0)}{\alpha_1 \gamma_1} (e^{\gamma_1 s} - e^{\gamma_1 T})\}} ds \right]^{\alpha_1} \end{aligned} \tag{42}$$

Therefore, for the value function V^1 to be concave for $t \in [0, T)$, as its second derivative with respect to W_1 evaluated in T is equal to zero, is required that $G_1(t) > 0$ for $t \in [0, T)$ and this happens in three cases:

- if $\alpha_1 > 1$ and $k > 0$,
- if $0 < \alpha_1 < 1$ with $\alpha_1 = \frac{1}{m}$ such that $\frac{m}{2} \in \mathbb{Z}$, $m \geq 2$ and $k \in \mathbb{R} \setminus \{0\}$,
- if $0 < \alpha_1 < 1$ with $\alpha_1 = \frac{1}{m}$ such that $\frac{m}{2} \notin \mathbb{Z}$ and $k > 0$.

Finally, by substituting (19) in (38) and (40) we obtain the closed-form expression of the controls given by (16), (17), and (18), respectively.

A.2 Cooperation

Let $V(t, W_1(t), W_2(t)) \in C^{1,2}$ (once differentiable with respect to t and twice both with respect to W_1 and W_2, with continuous derivatives) be the value function associated with the problem (9)–(10) in the cooperative setting. The quadruple $(W_j(t), c_j(t), y_{i,j}(t), Q_j(t))$ for $j = 1, 2$ is the solution of the problem if V solves the following HJB equation:

$$\begin{cases} D_t V + \max_{\{c_1,c_2,y_{i,1},y_{i,2},Q_1,Q_2\}} \Bigg\{ \sum_{j=1}^{2} \Big\langle \Big(\sum_{i=1}^{n-1} y_{i,j}(t)(\mu_i - \mu_n) + \mu_n \Big) W_j(t) - c_j(t) - \theta Q_j(t), D_{W_j} V \Big\rangle \\ \quad + e^{-\rho t} S(t) U_1(c_1(t), c_2(t)) + \pi(t)\mathcal{F}(t, Z_1(t)) \\ \quad + e^{-\rho t} S(t) U_2(c_1(t), c_2(t)) + \pi(t)\mathcal{F}(t, Z_2(t)) \\ \quad + \frac{1}{2} \sum_{j,k=1}^{2} \Big(\sum_{i=1}^{n-1} (y_{i,j}(t))(y_{i,k}(t))(\sigma_i)^2 (W_j(t))(W_k(t)) D^2_{W_j W_k} V \Big) \Bigg\} = 0, \\ V(T, W_1(T), W_2(T)) = 0. \end{cases} \tag{43}$$

where $D_{W_j} V$ is the first derivative of V with respect to W_j, while $D^2_{W_j W_k} V$ is its second partial derivative first with respect to W_j and then with respect to W_k, for $j, k = 1, 2$.

We denote by $H(W, c, y_i, Q, D_W V, D^2_W V)$ the Hamiltonian associated with the problem (9)–(10), where

$$W = [W_1, W_2], \quad c = [c_1, c_2], \quad y_i = [y_{i,1}, y_{i,2}], \quad D_W V = [D_{W_1} V, D_{W_2} V],$$

and

$$D^2_W V = \begin{pmatrix} D^2_{(W_1)^2} V & D^2_{W_1 W_2} V \\ D^2_{W_2 W_1} V & D^2_{(W_2)^2} V \end{pmatrix}.$$

Supposing that

$$\lambda(t) = \lambda(0)\, e^{\gamma t}, \qquad \lambda(0) > 0,\ \gamma > 0, \tag{44}$$

$$U_1(c_1(t), c_2(t)) = U_2(c_1(t), c_2(t)) = \frac{1}{1-\alpha} \left[(c_1(t))^{1-\alpha} + (c_2(t))^{1-\alpha} \right],\ \alpha > 0,\ \alpha \neq 1, \tag{45}$$

$$V(t, W_1(t), W_2(t)) = \frac{e^{-\rho t}}{1-\alpha} S(t)\, G(t) \left[(W_1(t))^{1-\alpha} + (W_2(t))^{1-\alpha} \right],\ G(t) \in \mathbb{R},\ \alpha > 0,\ \alpha \neq 1, \tag{46}$$

$$\mathcal{F}(t, Z_j(t)) = \frac{e^{-\rho t}}{1-\alpha} (Z_j(t))^{1-\alpha},\ \alpha > 0,\ \alpha \neq 1,\ j = 1, 2 \tag{47}$$

we can apply, as in the non-cooperative case, the first order conditions so to get the expressions of the controls:

$$c_j(t) = \left(\frac{2}{G(t)} \right)^{\frac{1}{\alpha}} W_j(t), \quad j = 1, 2 \tag{48}$$

$$y_{i,j}(t) = \frac{\mu_i - \mu_n}{\alpha (\sigma_i)^2}, \quad j = 1, 2, \quad i = 1, \ldots, n-1, \tag{49}$$

$$Z_j(t) = \left(\frac{\lambda(0)\, e^{\gamma t}}{\theta\, G(t)} \right)^{\frac{1}{\alpha}} W_j(t) \longrightarrow Q_j(t) = \left[\left(\frac{\lambda(0)\, e^{\gamma t}}{\theta\, G(t)} \right)^{\frac{1}{\alpha}} - 1 \right] W_j(t), \quad j = 1, 2. \tag{50}$$

Note that by (48) follows $G(t) \neq 0$ for every $t \in [0, T]$. Furthermore, by plugging the expressions (44)–(50) we get that $G(t)$ has to satisfy a Bernoulli equation:

$$\dot{G}(t) = \left[-(1-\alpha)\left(\frac{1}{2}\sum_{i=1}^{n-1}\frac{(\mu_i-\mu_n)^2}{\alpha(\sigma_i)^2} + \mu_n - \frac{\rho}{1-\alpha} + \theta\right) + \lambda(0)e^{\gamma t}\right]G(t) \\ - \alpha\left(2^{\frac{1}{\alpha}} + \theta^{\frac{\alpha-1}{\alpha}}\lambda(0)^{\frac{1}{\alpha}}\, e^{\frac{\gamma t}{\alpha}}\right)G(t)^{\frac{\alpha-1}{\alpha}}, \tag{51}$$

with terminal condition $G(T) = \overline{k} \in \mathbb{R} \setminus \{0\}$ obtained by evaluating (46) in T and by imposing it equal to 0. Since we are in the cooperative scenario, to the solution of (51) has to be applied the constraint (14), $\sum_{j=1}^{2} y_{i,j}(t) = \xi_i$, which by using (49) can be written as

$$\frac{\mu_i - \mu_n}{\alpha(\sigma_i)^2} = \frac{\xi_i}{2}, \qquad i = 1, \cdots, n-1.$$

Finally, denoting by

$$\nu = -(1-\alpha)\left(\frac{1}{4}\sum_{i=1}^{n-1}\xi_i(\mu_i-\mu_n) + \mu_n - \frac{\rho}{1-\alpha} + \theta\right),$$

the solutions of (51) dependent on $\overline{k}$ are

$$G(t) = e^{\{\nu(t-T)+\frac{\lambda(0)}{\gamma}(e^{\gamma t}-e^{\gamma T})\}} \\ \times\left[(\overline{k})^{\frac{1}{\alpha}} + \int_t^T\left(2^{\frac{1}{\alpha}} + \theta^{\frac{\alpha-1}{\alpha}}\lambda(0)^{\frac{1}{\alpha}}\, e^{\frac{\gamma s}{\alpha}}\right)e^{\{-\frac{\nu}{\alpha}(s-T)-\frac{\lambda(0)}{\alpha\gamma}(e^{\gamma s}-e^{\gamma T})\}}ds\right]^{\alpha}. \tag{52}$$

Similarly to the non-cooperative case, it is necessary that $G(t) > 0\,\forall\, t \in [0, T)$ to guarantee the concavity of the value function V. The positive sign of $G(t)$ is provided under the following cases:

- if $\alpha > 1$ and $\overline{k} > 0$,
- if $0 < \alpha < 1$ with $\alpha_1 = \frac{1}{m}$ such that $\frac{m}{2} \in \mathbb{Z}$, $m \geq 2$ and $\overline{k} \in \mathbb{R}\setminus\{0\}$,
- if $0 < \alpha < 1$ with $\alpha = \frac{1}{m}$ such that $\frac{m}{2} \notin \mathbb{Z}$ and $\overline{k} > 0$.

Finally, by inserting (27) in (48) and (50) we get the cooperative closed-form expression of the controls given by (24), (25), (26), respectively.

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