AN ABSTRACT APPROACH TO ALGEBRAS OF BRAIDS AND TIES

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ABSTRACT. Generalizing work of Marin (*Artin groups and Yokonuma-Hecke algebras.* Int. Math. Res. Not. IMRN 2018, no. 13, 4022–4062), we construct in a unified way all the "braids and ties" algebras available in literature and new ones.

1. INTRODUCTION

In the paper [1] Aicardi and Juyumaya studied an algebra $\mathcal{E}_n$ defined by generators and relations; the relations were devised by an abstracting procedure of a non-standard presentation for the Yokonuma-Hecke algebra [10], finalized to build a Markov trace on it [11]. They also provided a diagramatic interpretation of this algebra in terms of braid diagrams on n strands, by adding "ties" which can freely move along the strands.

In subsequent papers, the algebra $\mathcal{E}_n$, named the *braids and ties algebra*, has been extensively studied, e.g. in connection with link invariants [1, 3] or from a representation theoretic point of view [17, 8]. From its algebraic presentation the algebra $\mathcal{E}_n$ can be viewed as a "type A" object, and indeed (inequivalent) generalizations to type B have been introduced in [9], [13]. The paper [13] has been another source of inspiration for the present paper. Marin introduced an extension $\mathcal{C}_W$ of the Iwahori-Hecke algebra of a Coxeter system (W,S) and built up a family of generically surjective morphisms of $k[B_W]\to\mathcal{C}_W$ (here B_W is the Artin braid group attached to W). When W is of type A_{n-1}, it turns out that $\mathcal{C}_W\cong\mathcal{E}_n$.

In the present paper we proceed along the lines of [13] at a more abstract level. The main outcome is that the construction we outline in the next paragraphs produces in a unified way all the "braids and ties" algebras available in literature and new ones.

We start introducing the category of Marin data, whose objects are quadruples $((W,S),A,\rho,a)$ consisting of a Coxeter system (W,S), of an action of W on the commutative k-algebra A, and of a W-equivariant map $a\colon S\to A^\times$. Starting from such an object one can construct a ring $\mathcal{E}^{(W,S)}(A,\rho,a)$ as the quotient of $A\rtimes k[B_W]$ by the ideal generated by Hecke type relations on g_s, $s\in S$ (here $w\mapsto g_w$ is the standard set-theoretical section of the projection $B_W\to W$).

The main result, stated in Corollary 2.16, generalizes [13, Theorem 3.4]: if A is free over k, then the algebra $\mathcal{E}^{(W,S)}(A,\rho,a)$ is freely generated as an A-module by the g_w, $w\in W$ if and only if $(1-a(s))(1-\rho(s))=0$ in $\mathrm{End}_k(A)$.

To apply the previous constructions to instances nearer to diagram algebras and at the same time to recover Marin's algebra $\mathcal{C}_W$, inspired by his notion of

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meaningful quotients [13, §3.4], we introduce the notion of Marin-Iwahori-Hecke monoid, as a triple (E, ρ, e) consisting of a commutative monoid E, a W-action ρ on the monoid E, and a W-equivariant map $e\colon S \to \mathrm{Idempotents}(E)$ satisfying relation $e(s)(1-\rho(s)) = 0\,\forall\, s \in S$, which guarantees that $\mathcal{E}^{(W,S)}(R[E], \rho, \hat{e})$ is free over the monoid algebra $R[E]$. Specializing to the monoid of root subsystem $\Sigma(\Phi)$ of a root system Φ, acted by W by conjugation, one recovers Marin's algebras $\mathcal{C}_W$ as well as a surjective algebra map $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi)], \rho, \hat{e}) \to H_R(W,S)(u_s)$ to the Iwahori-Hecke algebra of (W,S).

Let now (W,S), (U,T) be Coxeter systems. A Juyumaya map (cf. Definition 2.21), is a pair (ϕ, e) consisting of a group homomorphism $\phi : W \to U$ and a W-equivariant map $e\colon S \to \Sigma(\Phi_U)$ satisfying condition (2.16). We can then prove the following result, which appears as Theorem 2.23 in the body of the paper and to which we refer for undefined notation.

Theorem 1.1. *Let (W,S) and (U,T) be Coxeter systems and let (ϕ, e) be a Juyumaya map between them. Let ρ_ϕ denote the W-action on $\Sigma(\Phi_U)$ induced by $\phi\colon W \to U$ and by the canonical action of U on Φ_U. Then $(\Sigma(\Phi_U)_{[e]}, \rho_\phi, e)$ is a Marin-Iwahori-Hecke monoid.*
It follows that $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ is a free $R[\Sigma(\Phi_U)_{[e]}]$-module with basis $\{g_w | w \in W\}$. Moreover, the augmentation map $\epsilon\colon R[\Sigma(\Phi_U)_{[e]}] \to R$ induces a surjective algebra homomorphism

$$\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e}) \to H_R(W,S)(u_s).$$

We then proceed, through Lemmas 2.30-2.32, to provide tools yielding examples of Juyumaya maps. The examples are displayed in Section 3; although we proceed in algebraic terms, our treatment has been guided by the geometric point of view of [6]. In the final Section 4 we show how to produce from our general construction several braids and ties algebras; in general if one takes the Juyumaya map ϕ to be the identity of a Coxeter system (W,S) one recovers Marin's $\mathcal{C}_W$ algebras, and in particular the braids and ties algebra of type A [1] and Marin's braids and ties algebra of type B. Non trivial choiches of ϕ give rise to Flores' braids and ties algebra of type B [9], to a braids and ties algebra of type D (possibly new) and more examples arising from affine Weyl groups. A few directions for future research are discussed in the final section.

Notational conventions. Sometimes, for better readability of formulas, we shall denote the image of an element x through a map f by f_x rather than by $f(x)$. For action ρ of a group G on a set X we shall write $g.x$ to mean $\rho_g(x)$.
We denote by k a commutative unital ring that we shall use as a ground ring.

2. Marin Algebras

In this section we will consider a fixed Coxeter system (W,S) and a commutative k-algebra A, where k is a commutative ring. We denote by $\rho_{\mathrm{geom}}\colon W \to O(V_W)$ the standard geometric representation of (W,S) (see e.g. [15, Section 5.3]). When convenient, we use the same symbol s to denote the basis element of V_W corresponding to the generator s of (W,S). We also denote by $\sigma_v\colon V_W \to V_W$ the reflection associated with a unit vector v, so that $\rho_{\mathrm{geom}}(s) = \sigma_s$. Finally, we denote by Φ_W (or simply by Φ when no confusion is possible) the root system of (W,S), by Φ^+ the subset of positive roots with respect to the basis S of V_W, and by $\langle\alpha\rangle$ the root subsystem $\{\alpha, -\alpha\}$ of Φ (cf. [15, §5.3]). (According to our notational convention, if α is a simple root attached to $s \in S$, we write interchangeably $\langle\alpha\rangle$ or $\langle s\rangle$).

Definition 2.1. *Let G be a group and let X and Y be G-sets, with left G-action denoted by $(g, x) \mapsto g.x$. Let $X_0 \subseteq X$ and $Y_0 \subseteq Y$ be subsets of X and Y, not necessarily G-subsets, and let $f\colon X_0 \to Y_0$ be a map. We will say that f is G-equivariant if for any triple (x_1, x_2, g) in $X_0 \times X_0 \times G$ with $g.x_1 = x_2$ we have $g.f(x_1) = f(x_2)$.*

Example 2.2. Given a Coxeter system (W, S), both the group W, with the W-action given by conjugation, and the set $\Sigma(\Phi_W)$ of root subsystems of Φ_W with the W-action induced by the standard geometric representation of (W, S) are W-sets. The map $S \to \Sigma(\Phi_W)$, $s \mapsto \langle s\rangle$,, where $\langle s\rangle$ is the root subsystem generated by s, i.e. $\{s, -s\}$, is W-equivariant. Indeed, if a triple of elemens s_1, s_2, w in $S \times S \times W$ satisfies $s_2 = ws_1w^{-1}$, then we have $\sigma_{s_2} = \rho_{\mathrm{geom}}(w)\sigma_{s_1}\rho_{\mathrm{geom}}(w)^{-1} = \sigma_{w.s_1}$, and so $\langle s_2\rangle = \langle w.s_1\rangle = w.\langle s_1\rangle$. Moreover, since standard geometric representation is faithful, this shows that $s_2 = ws_1w^{-1}$ if and only if $\langle s_2\rangle = w.\langle s_1\rangle$. In particular, for any $s \in S$ the stabilizer of $\langle s\rangle$ for the W-action on root subsystems of Φ coincides with the stabilizer of s for the W-action on itself by conjugation, i.e., with the centralizer $C_W(s)$ of s in W.

Definition 2.3. *A Marin datum is a quadruple $((W, S), A, \rho, a)$ consisting of a Coxeter system (W, S), of an action of W on the commutative k-algebra A, $\rho\colon W \to \mathrm{Aut}_{k\text{-alg}}(A)$, and of a W-equivariant map $a\colon S \to A^\times$, $a(s) = a_s$ where $A^\times$ is the group of units of A.*

Marin data naturally form a category: a morphism $((W, S), A, \rho, a) \to ((U, T), B, \eta, b)$ is a pair (ϕ, ϵ) consisting of a morphism of Coxeter systems $\phi\colon (W, S) \to (U, T)$ and of a homomorphism of k-algebras $f\colon A \to B$ such that ϵ is W-equivariant with respect to the W-action ρ on A and the W-action $\eta \circ \phi$ on B, and such that $\epsilon \circ a = b \circ \phi$, where we use that ϕ maps S to T by definition of morphism of Coxeter systems.

Remark 2.3.1. Since every element s of S is a fixed point for the conjugacy action of s on W, the equivariancy of a implies $s.a_s = a_s$ for any $s \in S$.

Let B_W be the Artin braid group attached to the Coxeter system (W, S). Recall that there is a canonical (set-theoretical) section $w \mapsto g_w$ for the projection π : $B_W \to W$. Let $k[B_W]$ be the group ring of B_W over k. The k-algebra A is a module for $k[B_W]$ via π, so we can form the semidirect product $A \rtimes k[B_W]$, i.e., the algebra generated by the elements c of A and g_w of B_W with the relations

$$g_w c g_w^{-1} = w.c,$$

i.e., equivalently, $g_w c = (w.c)g_w$.

Definition 2.4. *The Marin ring $\mathcal{E}^{(W,S)}(A, \rho, a)$ is the quotient of $A \rtimes k[B_W]$ by the ideal generated by the elements*

$$g_s^2 - a_s - (1 - a_s)g_s \quad \textit{for all } s \in S. \tag{2.1}$$

Remark 2.4.1. Even without assuming the invertibility of g_s from the group structure of B_W, the equation $g_s^2 = a_s + (1 - a_s)g_s$ together with the invertibility of a_s in A implies the invertibility of g_s in $\mathcal{E}^{(W,S)}(A, \rho, a)$. Indeed, the element ${a_s}^{-1}g_s - a_s^{-1}(1 - a_s)$ is the inverse of g_s in $\mathcal{E}^{(W,S)}(A, \rho, a)$.

Remark 2.4.2. Mimicking the construction of generic Hecke algebras given e.g. in [15, §7.1], one can define a more general Marin ring *Marin ring* $\mathcal{E}^{(W,S)}(A, \rho, b, c)$, where $b, c : S \to A^\times$ are W-equivariant maps and relation (2.1) is replaced by

$$g_s^2 - b_s - (1 - c_s)g_s \quad \text{for all } s \in S. \tag{2.2}$$

As for generic Hecke algebras, one reduces (2.2) to (2.1) a suitable rescaling $g_s \rightsquigarrow \lambda_s g_s$. See Remark 4.4.1 for an example of this rescaling.

Remark 2.4.3. When A is a free k-module, recalling the relations defining the Artin ring B_W one obtains a presentation of $\mathcal{E}^{(W,S)}(A, \rho, a)$ as follows: $\mathcal{E}^{(W,S)}(A, \rho, a)$ is the quotient of the free free k-algebra generated by A and by $\{g_s\}_{s\in S}$ by the relations defining A as a k-algebra and

$$\begin{aligned} g_s c &= (s.c)g_s, \quad c \in A,\ s \in S \\ \underbrace{g_{s_i} g_{s_j} g_{s_i} \cdots}_{m_{i,j} \text{ times}} &= \underbrace{g_{s_j} g_{s_i} g_{s_j} \cdots}_{m_{i,j} \text{ times}}, \\ g_s^2 &= a_s + (1 - a_s)g_s, \quad s \in S. \end{aligned}$$

It follows that the datum of a k-algebra homomorphism $\mu\colon \mathcal{E}^{(W,S)}(A, \rho, a) \to R$ is equivalently the datum of a k-algebra homomorphism $\mu_A\colon A \to R$ together with a choice of elements r_s in R such that

$$\begin{aligned} r_s \mu_A(c) &= \mu_A(s.c) r_s, \qquad c \in A, \\ \underbrace{r_{s_i} r_{s_j} r_{s_i} \cdots}_{m_{i,j} \text{ times}} &= \underbrace{r_{s_j} r_{s_i} r_{s_j} \cdots}_{m_{i,j} \text{ times}}, \\ {r_s}^2 &= \mu_A(a_s) + \mu_A(1 - a_s) r_s. \end{aligned}$$

Remark 2.4.4. Let $\{\lambda_s\}_s \in S$ be a collection of invertible elements in A such that $\lambda_{s_i} = \lambda_{s_j}$ whenever $m_{i,j}$ is odd. All of the relations defining $\mathcal{E}^{(W,S)}(A, \rho, a)$ are homogeneous in the g_s's, with the exception of the relation $g_s^2 = a_s + (1-a_s)g_s$, the relation $g_s c = (s.c)g_s$ is homogeneous in the single variable g_s, and the braid-type relations with an even $m_{i,j}$ are separately homogeneous in g_{s_i} and g_{s_j}. Therefore, we can operate the rescaling $g_s \rightsquigarrow -\lambda_s g_s$ to obtain a presentation of $\mathcal{E}^{(W,S)}(A, \rho, a)$ with relations

$$\begin{aligned} g_s c &= (s.c)g_s, \\ \underbrace{g_{s_i} g_{s_j} g_{s_i} \cdots}_{m_{i,j} \text{ times}} &= \underbrace{g_{s_j} g_{s_i} g_{s_j} \cdots}_{m_{i,j} \text{ times}}, \\ g_s^2 &= {\lambda_s}^{-2} a_s - {\lambda_s}^{-1}(1 - a_s)g_s. \end{aligned}$$

Remark 2.4.5. Since B_W acts on A via its projection on W, we have that the elements g_s^2 with $s \in S$ commute with the elements of A in $A \rtimes k[B_W]$ and so in $\mathcal{E}^{(W,S)}(A, \rho, a)$. Indeed, for any $c \in A$,

$$\begin{aligned} g_s^2 c &= (g_s^2 c g_s^{-2}) g_s^2 = (g_s (g_s c g_s^{-1}) g_s^{-1}) g_s^2 \\ &= (g_s (s.c) g_s^{-1}) g_s^2 = (s.(s.c)) g_s^2 = (s^2.c) g_s^2 = c g_s^2. \end{aligned}$$

As an immediate consequence we have

$$g_s c g_s^{-1} = g_s^{-1} g_s^2 c g_s^{-1} = g_s^{-1} c g_s^2 g_s^{-1} = g_s^{-1} c g_s,$$

for any $s \in \Phi$ and any c in A.

Example 2.5 (The Iwahori-Hecke algebra)**.** Let $k = \mathbb{Z}$. Consider the ring of Laurent polynomials $\mathbb{Z}[x_{\langle s\rangle}, x_{\langle s\rangle}^{-1}]$ in variables $x_{\langle s\rangle}$ indexed by the root subsystems $\langle s\rangle$ with $s \in S$. The W-action on root subsystems of Φ induces an action on $\mathbb{Z}[x_{\langle s\rangle}, x_{\langle s\rangle}^{-1}]$. Let $A_{IH}(W,S)$ be the ring of coinvariants. By definition of coinvariants, the W-action on $A_{IH}(W,S)$ is trivial. The map

$$\begin{aligned} \hat{u} \colon S &\mapsto A_{IH}(W,S)^{\times} \\ s &\mapsto [x_{\langle s\rangle}], \end{aligned}$$

where $[x_{\langle s\rangle}]$ is the image of $x_{\langle s\rangle}$ in $A_{IH}(W,S)$, is W-equivariant: if $s_2 = ws_1w^{-1}$ with $w \in W$, then we have $\langle s_2\rangle = w.\langle s_1\rangle$ and so

$$\hat{u}_{s_2} = [x_{\langle s_2\rangle}] = [x_{w.\langle s_1\rangle}] = [w.x_{\langle s_1\rangle}] = [x_{\langle s_1\rangle}] = \hat{u}_{s_1}.$$

The Marin algebra $\mathcal{E}^{(W,S)}(A_{IH}(W,S), \rho_{\mathrm{triv}}, \hat{u})$ is then the Iwahori–Hecke algebra $H(W,S)(\hat{u}_s)$. If R is any ring with trivial W-action and $u \colon S \to R^{\times}$ is a W-equivariant map, then we have a specialization map $\epsilon \colon A_{IH}(W,S) \to R$ mapping $\hat{u}_s$ to u_s; the Marin algebra is $\mathcal{E}^{(W,S)}(R, \rho_{\mathrm{triv}}, u)$ is the Iwahori–Hecke algebra $H_R(W,S)(u_s)$ and the specialization map induces a morphism of algebras $H(W,S)(\hat{u}_s) \to H_R(W,S)(u_s)$.

The specialization map $\epsilon \colon A_{IH}(W,S) \to R$ in the previous example is clearly W-equivariant and satisfies $\epsilon \circ \hat{u} = u$. More generally we have the following.

Lemma 2.6. *Let B a k-algebra with a W-action $\eta \colon W \to \mathrm{Aut}_k(B)$ and a W-equivariant map $b \colon \Phi \to B^{\times}$, and let $\epsilon \colon A \to B$ be a W-equivariant algebra homomorphism such that $b = \epsilon \circ a$. Then one has an induced algebra homomorphism*

$$\epsilon^{\mathcal{E}} \colon \mathcal{E}^{(W,S)}(A, \rho, a) \to \mathcal{E}^{(W,S)}(B, \eta, b)$$

that is the identity on $k[B_W]$ and coincides with ϵ on A. In particular, if ϵ is surjective, then also $\epsilon^{\mathcal{E}}$ is surjective.

Even more generally, we have the functoriality of the construction of the Marin algebra with respect to morphism of Marin data, of which Lemma 2.6 is the particular case corresponding to a morphism of Marin data with the same Coxeter system and with the identity map from this system to itself. In the general situation we have the following.

Lemma 2.7. *Let $(\phi, \epsilon) \colon ((W,S), \rho, a) \to ((U,T), \eta, b)$ be a morphism of Marin data. Then one has an induced algebra homomorphism*

$$(\phi, \epsilon)^{\mathcal{E}} \colon \mathcal{E}^{(W,S)}(A, \rho, a) \to \mathcal{E}^{(U,T)}(B, \eta, b),$$

that is the morphism induced by ϕ on $k[B_W]$ and coincides with ϵ on A. In particular, if $\phi \colon S \to T$ and ϵ are surjective, then also $(\phi, \epsilon)^{\mathcal{E}}$ is surjective.

Example 2.8. Let R be a k-algebra with trivial W-action and $u \colon S \to R^{\times}$ a W-equivariant map. Let (A, ρ, a) be data defining a Marin algebra for (W,S). Then a W-equivariant homomorphism $\epsilon \colon A \to R$ such that $\epsilon \circ a = u$ induces an algebra homomorphism $\epsilon^{\mathcal{E}} \colon \mathcal{E}^{(W,S)}(A, \rho, a) \to H_R(W,S)(u_s)$. If moreover A is an R-algebra and ϵ is an augmentation, i.e., $R \to A \xrightarrow{\epsilon} R$ is the identity of R, then $\mathcal{E}^{(W,S)}(A, \rho, a)$ is an R-algebra and $\epsilon^{\mathcal{E}}$ is a surjective morphism of R-algebras. In particular, this means that $H_R(W,S)(u_s)$ is realized as a quotient of $\mathcal{E}^{(W,S)}(A, \rho, a)$.

The above example suggests the following.

Definition 2.9. *A Marin lift of the Iwahori–Hecke algebra $H_R(W,S)(u_s)$ is a surjective morphism $\epsilon^{\mathcal{E}}\colon \mathcal{E}^{(W,S)}(A,\rho,a) \to H_R(W,S)(u_s)$ as in example 2.8.*

Example 2.10. Let (R,u) be as in the definition the Iwahori–Hecke algebra $H_R(W,S)(u_s)$, and let E be a commutative monoid endowed with a W-action ρ and a W-equivariant map $e\colon S \to \text{Idempotents}(E)$. The monoid R-algebra $R[E]$ inherits a W-action from the W-action on E. The map

$$\hat{e} = 1 + (u-1)e\colon S \to R[E] \tag{2.3}$$

is clearly W-equivariant. Moreover it takes values in $R[E]^\times$, since

$$(1 + (u_s\ -1)e_s)(1 + u_s^{-1}(1-u_s)e_s) = 1.$$

So we can form the Marin algebra $\mathcal{E}^{(W,S)}(E,\rho,\hat{\epsilon})$. The algebra $R[E]$ has a canonical augmentation $\epsilon\colon R[E] \to R$ sending all the elements of E to 1. This augmentation is manifestly W-equivariant; moreover $\epsilon \circ \hat{e} = u$, so we have the surjective morphism $\epsilon^{\mathcal{E}}: \mathcal{E}^{(W,S)}(R[E],\rho,\hat{e}) \to H_R(W,S)(u_s)$ realizing a lift of the Iwahori–Hecke algebra.

The following fact is well-known.

Lemma 2.11 (Matsumoto)**.** *Let w be an element in W and let $w = s_1 \dots s_k$, with $s_i \in S$, be a reduced expression for w. The element $g_{s_1} \dots g_{s_k} \in \mathcal{E}^{(W,S)}(A,\rho,a)$ does not depend on the particular reduced expression for w. Therefore one can define the element g_w in $\mathcal{E}^{(W,S)}(A,\rho,a)$ as $g_w = g_{s_1} \dots g_{s_k}$ for some (hence any) reduced expression for w.*

Lemma 2.12. *The set $\{g_w\}_{w\in W} \subseteq \mathcal{E}^{(W,S)}(A,\rho,a)$ spans $\mathcal{E}^{(W,S)}(A,\rho,a)$ as a left A-module.*

Proof. Let $M \subseteq \mathcal{E}^{(W,S)}(A,\rho,a)$ be the left A-submodule generated by the elements $\{g_w\}_{w\in W}$. Equivalently, M is the k-submodule generated by the elements $\{cg_w\}_{c\in A, w\in W}$. We start by showing that M is a 2-sided ideal of $\mathcal{E}^{(W,S)}(A,\rho,a)$. Since as a ring $\mathcal{E}^{(W,S)}(A,\rho,a)$ is generated by the elements g_s of $k[B_W]$ and by the elements x of A, we have to show that:

1. $x(cg_w) \in M$;
2. $(cg_w)x \in M$;
3. $g_s(cg_w) \in M$;
4. $(cg_w)g_s \in M$;

for any $s \in S$ and any $x \in A$. Relation (1) is immediate: $x(cg_w) = (xc)g_w$.
For relation (2) we compute

$$(cg_w)x = (c(g_w x g_w^{-1}))g_w = (c(w.x))g_w,$$

where we used that B_W acts on A via the projection $B_W \to W$, and that the image of g_w via this projection is precisely w. To prove (3), we distinguish two cases. We can have either $\ell(sw) = \ell(w)+1$, so that sw is a reduced expression for sw, or $\ell(sw) = \ell(w) - 1$ so that $w = sw'$ with w' minimally written. Notice that in the second case we have $w' = sw$ in W. Now, in the first case we compute

$$g_s(cg_w) = (g_s c g_s^{-1})g_s g_w = (g_s c g_s^{-1})g_{sw} = (s.c)g_{sw},$$

while in the second case we compute

$$\begin{aligned} g_s(cg_w) &= (g_s c g_s^{-1}) g_s g_w = (g_s c g_s^{-1}) g_s g_s g_{w'} = (g_s c g_s^{-1}) g_s^2 g_{w'} \\ &= (g_s c g_s^{-1})(a_s + (1-a_s)g_s) g_{w'} \\ &= (s.c) a_s g_{w'} + (s.c)(1-a_s) g_w. \end{aligned}$$

Relation (4) is checked similarly. Now the conclusion is immediate: since $g_1 = 1$, we have $1 \in M$, and so $M = \mathcal{E}^{(W,S)}(A, \rho, a)$. □

For any $s \in S$ we have two k-linear operators on A: the multiplication by a_s and the automorphism ρ_s.

Lemma 2.13. *If $\mathcal{E}^{(W,S)}(A, \rho, a)$ is free as an A-module with basis $\{g_w\}_{w \in W}$, then, for any $s \in S$*

$$(1 - a_s)(1 - \rho_s) = 0 \text{ in } \operatorname{End}_k(A). \tag{2.4}$$

Proof. Let $c \in A$ and $s \in S$. By Remark 2.4.5, we have $g_s^2 c = c g_s^2$ and so

$$(a_s + (1-a_s)g_s)c = c(a_s + (1-a_s)g_s).$$

Since A is commutative, this reduces to the identity

$$(1-a_s)g_s c = (1-a_s) c g_s.$$

Writing $g_s c = (g_s c g_s^{-1}) g_s = (s.c) g_s$, we find $(1-a_s)\rho_s(c) g_s = c(1-a_s)g_s$. Since $\{g_w\}_{w \in W}$ is a spanning set of $\mathcal{E}^{(W,S)}(A, \rho, a)$, this gives $(1-a_s)(\rho_s(c) - c) = 0$. □

When A is a free k-module, condition (2.4) is also sufficient in order to have that $\mathcal{E}^{(W,S)}(A, \rho, a)$ is a free A-module over the basis $\{g_w\}_{w \in W}$. The proof of sufficiency is however more involved and we need a few auxiliary arguments, that we verbatim adapt from [13]. We start by fixing a k-linear basis $\{c_i\}_{i \in I}$ for A, and by noticing that the equations derived in the proof of Lemma 2.12 force the effect of left and right multiplication by elements x of A and by elements g_s with $s \in S$ on the elements $c_i g_w$ of $\mathcal{E}^{(W,S)}(A, \rho, a)$ as follows:

$$x \cdot - \colon c_i g_w \mapsto x c_i g_w.$$

$$- \cdot x \colon c_i g_w \mapsto c_i (w.x) g_w.$$

$$g_s \cdot - \colon c_i g_w \mapsto \begin{cases} (s.c_i) g_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\ \\ (s.c_i) a_s g_{sw} + (s.c_i)(1-a_s) g_w & \text{if } \ell(sw) = \ell(w) - 1 \end{cases}$$

$$- \cdot g_s \colon c_i g_w \mapsto \begin{cases} c_i g_{ws} & \text{if } \ell(sw) = \ell(w) + 1 \\ \\ c_i((ws).a_s) g_{ws} + c_i(1 - (ws).a_s) g_w & \text{if } \ell(sw) = \ell(w) - 1 \end{cases}$$

Let now $\{\gamma_w\}$ be a collection of elements indexed by $w \in W$ and let V be the free left E-module with basis $\{\gamma_w\}_{w \in W}$. It has a k-linear basis given by the elements $\{c_i \gamma_w\}_{i \in I, w \in W}$. We can then define k-linear endomorphisms of V corresponding to the left and right multiplication by elements x and by elements g_s on the elements of the form $c_i g_w$ of $\mathcal{E}^{(W,S)}$. That is we define k-linear operators $C_x^l, C_x^r \colon V \to V$ by

$$C_x^l \colon c_i \gamma_w \mapsto x c_i \gamma_w,$$

$$C_x^r \colon c_i g_w \mapsto c_i (w.x) \gamma_w,$$

and k-linear operators $G^l_s, G^r_s\colon V \to V$ by

$$G^l_s\colon c_i\gamma_w \mapsto \begin{cases} (s.c_i)\gamma_{sw} & \text{if } \ell(sw)=\ell(w)+1 \\ \\ (s.c_i)a_s\gamma_{sw} + (s.c_i)(1-a_s)\gamma_w & \text{if } \ell(sw)=\ell(w)-1 \end{cases}$$

$$G^r_s\colon c_i\gamma_w \mapsto \begin{cases} c_i\gamma_{ws} & \text{if } \ell(sw)=\ell(w)+1 \\ \\ c_i((ws).a_s)\gamma_{ws} + c_i(1-(ws).a_s)\gamma_w & \text{if } \ell(sw)=\ell(w)-1 \end{cases}$$

The key result is the following statement

Proposition 2.14. *If condition* (2.4) *is satisfied, then the map*

$$\mu(c) = C^l_c, \quad \mu(g_s) = G^l_s$$

defines a k-algebra homomorphism $\mu\colon \mathcal{E}^{(W,S)}(A,\rho,a) \to \mathrm{End}_k(V)$.

The proof follows closely that of Theorem 3.4 of [13]. We provide a few details, to emphasize the role of condition (2.4).
First note that, by k-linearity, for any $c \in A$ we have

$$C^l_x\colon c\gamma_w \mapsto xc\gamma_w, \quad C^r_x\colon cg_w \mapsto c(w.x)\gamma_w,$$

$$G^l_s\colon c\gamma_w \mapsto \begin{cases} (s.c)\gamma_{sw} & \text{if } \ell(sw)=\ell(w)+1 \\ \\ (s.c)a_s\gamma_{sw} + (s.c)(1-a_s)\gamma_w & \text{if } \ell(sw)=\ell(w)-1 \end{cases}$$

$$G^r_s\colon c\gamma_w \mapsto \begin{cases} c\gamma_{ws} & \text{if } \ell(sw)=\ell(w)+1 \\ \\ c((ws).a_s)\gamma_{ws} + c(1-(ws).a_s)\gamma_w & \text{if } \ell(sw)=\ell(w)-1 \end{cases}$$

Lemma 2.15. *If condition* (2.4) *is satisfied, then $G^l_{s_1}G^r_{s_2} = G^r_{s_2}G^l_{s_1}$ for any $s_1, s_2 \in S$.*

Proof. We have to check that $G^l_{s_1}G^r_{s_2}(c\gamma_w) = G^r_{s_2}G^l_{s_1}(c\gamma_w)$ for any $c \in A$ and any $w \in W$. There are several cases to be examined and we will proceed case by case.
• If $l(s_1w) = l(w)+1$, $l(ws_2) = l(w)+1$ and $l(s_1ws_2) = l(w)+2$, then we have:

$$\begin{aligned} G^l_{s_1}G^r_{s_2}(c\gamma_w) &= G^l_{s_1}(c\gamma_{ws_2}) = (s_1.c)\gamma_{s_1ws_2} \\ G^r_{s_2}G^l_{s_1}(c\gamma_w) &= G^r_{s_2}((s_1.c)\gamma_{s_1w}) = (s_1.c)\gamma_{s_1ws_2}. \end{aligned}$$

If $l(s_1w) = l(w)+1$, $l(ws_2) = l(w)+1$ and $l(s_1ws_2) = l(w)$, then by e.g. [13, Lemma 3.2] we have $s_1ws_2 = w$, so that $ws_2w^{-1} = s_1$ and we have:

$$\begin{aligned} G^l_{s_1}G^r_{s_2}(c\gamma_w) &= G^l_{s_1}(c\gamma_{ws_2}) = (s_1.c)a_{s_1}\gamma_{s_1ws_2} + (s_1.c)(1-a_{s_1})\gamma_{ws_2} \\ &= (s_1.c)a_{s_1}\gamma_w + (s_1.c)(1-a_{s_1})\gamma_{s_1w} \\ G^r_{s_2}G^l_{s_1}(c\gamma_w) &= G^r_{s_2}((s_1.c)\gamma_{s_1w}) \\ &= (s_1.c)((s_1ws_2).a_{s_2})\gamma_{s_1ws_2} + (s_1.c)(1-(s_1ws_2).a_{s_2})\gamma_{s_1w} \\ &= (s_1.c)(w.a_{s_2})\gamma_w + (s_1.c)(1-w.a_{s_2})\gamma_{s_1w} \\ &= (s_1.c)(a_{ws_2w^{-1}})\gamma_w + (s_1.c)(1-a_{ws_2w^{-1}})\gamma_{s_1w} \\ &= (s_1.c)a_{s_1}\gamma_w + (s_1.c)(1-a_{s_1})\gamma_{s_1w}. \end{aligned}$$

• If $l(s_1w) = l(w) - 1$, $l(ws_2) = l(w) - 1$ and $l(s_1ws_2) = l(w)$ we have that in this case $s_1w = ws_2$. This gives $s_1 = ws_2w^{-1}$ and so $w.a_{s_2} = a_{s_1}$. Recalling that by condition (2.4), we have $(1 - a_s)(s.c) = (1 - a_s)c$, and that $s.a_s = a_s$, we get

$$
\begin{aligned}
&G^l_{s_1}G^r_{s_2}(c\gamma_w) = G^l_{s_1}(c((ws_2).a_{s_2})\gamma_{ws_2} + c(1 - (ws_2).a_{s_2})\gamma_w)\\
&= (s_1.(c((ws_2).a_{s_2}))\gamma_{s_1ws_2} + (s_1.(c(1 - (ws_2).a_{s_2})))a_{s_1}\gamma_{s_1w}\\
&\qquad\qquad + (s_1.(c(1 - (ws_2).a_{s_2})))(1 - a_{s_1})\gamma_w\\
&= (s_1.c)((s_1ws_2).a_{s_2})\gamma_{s_1ws_2} + (s_1.c)(1 - (s_1ws_2).a_{s_2})a_{s_1}\gamma_{s_1w}\\
&\qquad\qquad + (s_1.c)(1 - (s_1ws_2).a_{s_2})(1 - a_{s_1})\gamma_w\\
&= (s_1.c)(w.a_{s_2})\gamma_w + (s_1.c)(1 - w.a_{s_2})a_{s_1}\gamma_{s_1w}\\
&\qquad\qquad + (s_1.c)(1 - w.a_{s_2})(1 - a_{s_1})\gamma_w\\
&= (s_1.c)a_{s_1}\gamma_w + (s_1.c)(1 - a_{s_1})a_{s_1}\gamma_{s_1w} + (s_1.c)(1 - a_{s_1})(1 - a_{s_1})\gamma_w\\
&= (s_1.c)a_{s_1}\gamma_w + c(1 - a_{s_1})a_{s_1}\gamma_{s_1w} + c(1 - a_{s_1})^2\gamma_w,
\end{aligned}
$$

$$
\begin{aligned}
&G^r_{s_2}G^l_{s_1}(c\gamma_w) = G^r_{s_2}((s_1.c)a_{s_1}\gamma_{s_1w} + (s_1.c)(1 - a_{s_1})\gamma_w) =\\
&= G^r_{s_2}((s_1.c)a_{s_1}\gamma_{s_1w} + c(1 - a_{s_1})\gamma_w)\\
&= (s_1.c)a_{s_1}\gamma_{s_1ws_2} + c(1 - a_{s_1})(((ws_2).a_{s_2})\gamma_{ws_2} + (1 - (ws_2).a_{s_2})\gamma_w)\\
&= (s_1.c)a_{s_1}\gamma_w + c(1 - a_{s_1})((w.a_{s_2})\gamma_{ws_2} + (1 - w.a_{s_2})\gamma_w)\\
&= (s_1.c)a_{s_1}\gamma_w + c(1 - a_{s_1})(a_{s_1}\gamma_{s_1w} + (1 - a_{s_1})\gamma_w)\\
&= (s_1.c)a_{s_1}\gamma_w + c(1 - a_{s_1})a_{s_1}\gamma_{s_1w} + c(1 - a_{s_1})^2\gamma_w.
\end{aligned}
$$

The cases $l(s_1w) = l(w) + 1$, $l(ws_2) = l(w) - 1$ and $l(s_1ws_2) = l(w)$, $l(s_1w) = l(w)-1$, $l(ws_2) = l(w)+1$ and $l(s_1ws_2) = l(w)$, $l(s_1w) = l(w)-1$, $l(ws_2) = l(w)-1$ and $l(s_1ws_2) = l(w) - 2$ are treated along the same lines.

□

Proof of Proposition 2.14. Let $\mu_A\colon A \to \mathrm{End}_k(V)$ the map defined by $\mu_A(c) = C^l_c$ for any $c \in A$. By Remark 2.4.3, showing that μ defines a k-algebra homomorphism is equivalent to showing that μ_A is a k-algebra homomorphism, i.e., since μ_A is manifestly k-linear, that $C^l_xC^l_y = C^l_{xy}$, $x, y \in A$, and that the following identities hold:

$$G^l_sC^l_x = C^l_{s.x}G^l_s, \tag{2.5}$$

$$(G^l_s)^2 = C^l_{a_s} + C^l_{(1-a_s)}G^l_s, \tag{2.6}$$

$$\underbrace{G^l_{s_i}G^l_{s_j}G^l_{s_i}\cdots}_{m_{i,j}\text{ times}} = \underbrace{G^l_{s_j}G^l_{s_i}G^l_{s_j}\cdots}_{m_{i,j}\text{ times}}. \tag{2.7}$$

For (2.5) we compute

$$(C^l_xC^l_y)(c\gamma_w) = C^l_x(C^l_y(c\gamma_w)) = C^l_x((yc)\gamma_w)) = xyc\gamma_w = C^l_{xy}(c\gamma_w).$$

For (2.6), we find

$$(G_s^l C_x^l)(c\gamma_w) = \begin{cases} (s.(xc))\gamma_{sw} & \text{if } \ell(sw) = \ell(w)+1, \\ \\ (s.(xc))a_s\gamma_{sw} + (s.(xc))(1-a_s)\gamma_w & \text{if } \ell(sw) = \ell(w)-1, \end{cases}$$

and

$$(C_{s.c}^l G_s^l)(c\gamma_w) = \begin{cases} C_{s.c}^l((s.c)\gamma_{sw}) & \text{if } \ell(sw) = \ell(w)+1 \\ \\ C_{s.c}^l((s.c)a_s\gamma_{sw} + (s.c)(1-a_s)\gamma_w) & \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

$$= \begin{cases} (s.(xc))\gamma_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ \\ (s.(xc)a_s\gamma_{sw} + (s.(xc))(1-a_s)\gamma_w & \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

For (2.7) we compute

$$(G_s^l)^2(c\gamma_w) = \begin{cases} G_s^l((s.c)\gamma_{sw}) \quad \text{if } \ell(sw) = \ell(w)+1 \\ \\ G_s^l((s.c)a_s\gamma_{sw} + (s.c)(1-a_s)\gamma_w) \qquad \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

$$= \begin{cases} ((s.((s.c)a_s))\gamma_w + (s.(s.c))(1-a_s)\gamma_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ \\ (s.((s.c)a_s))\gamma_w + ((s.(s.c))a_s\gamma_{sw} + (s.(s.c))(1-a_s)\gamma_w & \\ -((s.((s.c)a_s))a_s\gamma_{sw} + (s.((s.c)a_s))(1-a_s)\gamma_w) & \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

$$= \begin{cases} a_s c\gamma_w + (1-a_s)c\gamma_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ \\ ca_s\gamma_w + ca_s\gamma_{sw} + c(1-a_s)\gamma_w - c(a_s)^2\gamma_{sw} - ca_s(1-a_s)\gamma_w & \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

$$= \begin{cases} a_s c\gamma_w + (1-a_s)c\gamma_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ \\ a_s(1-a_s)c\gamma_{sw} + (1-a_s(1-a_s))c\gamma_w & \text{if } \ell(sw) = \ell(w)-1. \end{cases}$$

By condition (2.4), we have $(1-a_s)(s.c) = (1-a_s)c$, so we find

$$\begin{aligned} &(C_{a_s}^l + C_{(1-a_s)}^l G_s^l)(c\gamma_w) = \\ &= C_{a_s}^l(c\gamma_w) + C_{(1-a_s)}^l(G_s^l(c\gamma_w)) \\ &= \begin{cases} C_{a_s}^l(c\gamma_w) + C_{(1-a_s)}^l((s.c)\gamma_{sw}) \text{if } \ell(sw) = \ell(w)+1 \\ \\ C_{a_s}^l(c\gamma_w) + C_{(1-a_s)}^l((s.c)a_s\gamma_{sw} + (s.c)(1-a_s)\gamma_w) \quad \text{if } \ell(sw) = \ell(w)-1 \end{cases} \\ &= \begin{cases} a_s c\gamma_w + (1-a_s)(s.c)\gamma_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ \\ a_s c\gamma_w + (1-a_s)(s.c)a_s\gamma_{sw} + (1-a_s)(s.c)(1-a_s)\gamma_w & \text{if } \ell(sw) = \ell(w)-1 \end{cases} \\ &= \begin{cases} a_s c\gamma_w + (1-a_s)c\gamma_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ \\ a_s(1-a_s)c\gamma_{sw} + (1-a_s(1-a_s))c\gamma_w & \text{if } \ell(sw) = \ell(w)-1 \end{cases} \end{aligned}$$

Finally, we need to show that the G_s^l satisfy the braid relations. Here we can proceed exactly as in [13]. □

Remark 2.15.1. Notice that the left A-action on V induced by μ via the inclusion $A \hookrightarrow \mathcal{E}^{(W,S)}(A,\rho,a)$ coincides with the left A-action on V given by its A-module structure.

Corollary 2.16. *Assume A is a free k-module. Then $\mathcal{E}^{(W,S)}(A,\rho,a)$ is the free A-module over the basis $\{g_w\}_{w\in W}$ if and only if condition* (2.4) *is satisfied.*

Proof. The "only if" part is Lemma 2.13. To prove the "if" part, consider the free A-module V on the basis $\{\gamma_w\}_{w\in W}$ endowed with the left $\mathcal{E}^{(W,S)}(A,\rho,a)$-module structure on the k-module V defined by the algebra homomorphism

$$\mu\colon \mathcal{E}^{(W,S)}(A,\rho,a) \to \mathrm{End}_k(V)$$

from Lemma 2.14. Then we have a k-linear map $\lambda\colon \mathcal{E}^{(W,S)}(A,\rho,a) \to V$, $\lambda(x) = x._\mu\gamma_1$. By Remark 2.15.1, the map λ is a map of left A-modules. Since V is free as an A-module, we can define an A-module map $\eta : V \to \mathcal{E}^{(W,S)}(A,\rho,a)$ by defining it on the A-basis $\{\gamma_w\}_{w\in W}$. We set $\eta(\gamma_w) = g_w$. The morphisms of A-modules λ and η are inverse each other. To see this, it suffices to check $\lambda\circ\eta = \mathrm{id}_V$ and $\eta\circ\lambda = \mathrm{id}_{\mathcal{E}^{(W,S)}(A,\rho,a)}$ on A-generating sets for V and $\mathcal{E}^{(W,S)}(A,\rho,a)$. For V we can take as generating set the A-basis $\{\gamma_w\}_{w\in W}$. We find

$$(\lambda\circ\eta)(\gamma_w) = \lambda(g_w) = g_w\cdot_\mu\gamma_1.$$

If $w = s_1\cdots s_k$ a reduced expression for w, then $g_w = g_{s_1}\cdots g_{s_k}$ and so

$$\begin{aligned} g_w\cdot_\mu\gamma_1 &= g_{s_1}\cdot_\mu(g_{s_1}\cdot_\mu\cdots\cdot_\mu(g_{s_k}\cdot_\mu g_1)\cdots) \\ &= G^l_{s_1}\cdots G^l_{s_k}(\gamma_1) = G^l_{s_1}\cdots G^l_{s_{k-1}}(\gamma_{s_k}) = \ldots \\ &= \gamma_{s_1\cdots s_k} = \gamma_w. \end{aligned} \tag{2.8}$$

Therefore $(\lambda\circ\eta)(\gamma_w) = \gamma_w$, for any $w\in W$. By Lemma 2.12, the set $\{g_w\})w\in W$ is a generating set for $\mathcal{E}^{(W,S)}(A,\rho,a)$ as an A-module. We find, by using (2.8) again,

$$(\eta\circ\lambda)(g_w) = \eta(\lambda(g_w)) = \eta(g_w\cdot_\mu\gamma_1) = \eta(\gamma_w) = g_w,$$

for any $w\in W$. □

Corollary 2.17. *If A is free of finite rank over k and condition* (2.4) *is satisfied, then $\mathcal{E}^{(W,S)}(A,\rho,a)$ is free of finite rank over k and*

$$\mathrm{rk}_k\mathcal{E}^{(W,S)}(A,\rho,a) = |W|\,\mathrm{rk}_k A. \tag{2.9}$$

A k-linear basis of $\mathcal{E}^{(W,S)}(A,\rho,a)$ is $\{c_i g_w\}_{w\in W}$ and $\{c_i\}_{i\in I}$ a k-basis of A.

Remark 2.17.1. In the same assumptions as in Lemma 2.6, we have

$$\epsilon\circ((1-a_s)(1-\rho_s)) = ((1-b_s)(1-\eta_s))\circ\epsilon\colon A\to B$$

In particular, if ϵ is surjective and $(1-a_s)(1-\rho_s) = 0$ then also $(1-b_s)(1-\eta_s)$, so that if ϵ is surjective and $\mathcal{E}^{(W,S)}(A,\rho,a)$ is free as an A-module, then $\mathcal{E}^{(W,S)}(B,\eta,b)$ is free as a B-module.

Remark 2.17.2. One immediately sees form Corollary 2.16 that if the W-action ρ on A is trivial, then $\mathcal{E}^{(W,S)}(A,a)$ is a free A-module with basis $\{g_w|w\in W\}$. In particular, one recovers this way the well known result that $H_R(W,S)(u_s)$ is a free R-module with basis $\{g_w|w\in W\}$.

Remark 2.17.3. In the same notation as in Example 2.10, one has

$$(1-\hat{\epsilon}_s)(1-\rho_s) = (1-u_s)e_s(1-\rho_s).$$

In particular, if

$$e_s(1-\rho_s) = 0 \text{ for all } s \in S \tag{2.10}$$

one has that $\mathcal{E}^{(W,S)}(R[E], \rho, \hat{e})$ is a free $R[E]$-module with basis $\{g_w | w \in W\}$.

Example 2.10 and Remark 2.17.3 motivate the following

Definition 2.18. *A Marin-Iwahori-Hecke monoid is a triple (E, ρ, e) consisting of a commutative monoid E, a W-action ρ on the monoid E, and a W-equivariant map $e\colon S \to \mathrm{Idempotents}(E)$ satisfying equation* (2.10)*.*

Example 2.19 (Marin's algebras)**.** Let $\Sigma(\Phi)$ be the monoid of root subsystems of Φ, with the composition law is given by $\Psi_1 \cdot \Psi_2 = \langle \Psi_1, \Psi_2 \rangle$, where $\langle X \rangle$ denotes the root subsystem generated by X. The W-action on Φ induces a W-action on $\Sigma(\Phi)$. Let us denote this action by ρ. All elements of $\Sigma(\Phi)$ are idempotent, and the map

$$e\colon S \to \Sigma(\Phi), \quad e(s) = \langle s \rangle \tag{2.11}$$

is manifestly W-equivariant. Let R be a commutative k-algebra with trivial W-action, and with a W-equivariant map $u\colon S \to R^\times$. Since $R[\Sigma(\Phi)]$ is generated by the root subsystems $\langle \alpha \rangle$ with $\alpha \in \Phi$, equation (2.10) is equivalent to

$$(e_s(1-\rho_s))(\langle \alpha \rangle) = 0 \text{ for all } s \in S \text{ and all } \alpha \in \Phi.$$

This is the equation $\langle s \rangle \cdot \langle \alpha \rangle = \langle s \rangle \cdot \langle s.\alpha \rangle$, where . is the W-action on Φ, i.e., the equation $\langle s, \alpha \rangle = \langle s, s.\alpha \rangle$. That this relation is satisfied is a general fact in the theory of Coxeter groups. Hence $(\Sigma(\Phi), \rho, e)$ is a Marin-Iwahori-Hecke monoid. It follows that $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi)], \rho, \hat{e})$ is a free $R[\Sigma(\Phi)]$-module with basis $\{g_w | w \in W\}$. Moreover, the augmentation map $\epsilon\colon R[\Sigma(\Phi)] \to R$ induces a surjective algebra homomorphism

$$\mathcal{E}^{(W,S)}(R[\Sigma(\Phi)], \rho, \hat{e}) \to H_R(W,S)(u_s). \tag{2.12}$$

The algebra $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi)], \rho, \hat{e})$ is precisely denoted by $\mathcal{C}_W$ in [13] mentioned in the Introduction.

Lemma 2.20. *Let $\{x_i\}$ be a set of generators for the commutative monoid E. Then condition* (2.10) *is equivalent to*

$$e_s \cdot x_i = e_s \cdot (s.x_i) \quad \text{for any } s \in S \text{ and any generator } x_i. \tag{2.13}$$

Proof. Equation (2.10) is equivalent to

$$e_s \cdot x = e_s \cdot (s.x) \tag{2.14}$$

for any $s \in S$ and any $x \in E$. So we only need to show that if x and y satisfy equation (2.14), then also $x \cdot y$ satisfies it. We have

$$\begin{aligned} e_s \cdot (s.(x \cdot y)) &= e_s \cdot ((s.x) \cdot (s.y)) = (e_s \cdot (s.x)) \cdot (s.y) = (e_s \cdot x) \cdot (s.y) \\ &= (x \cdot e_s) \cdot (s.y) = x \cdot (e_s \cdot (s.y)) = x \cdot (e_s \cdot y) \\ &= e_s \cdot (x \cdot y). \end{aligned}$$

Notice how the commutativity of E played an essential role in the proof. $\square$

Remark 2.20.1. Condition (2.10), which amounts to $e_s \cdot x = e_s \cdot (s.x)$ for any $s \in S$ and any $x \in E$, can be quite restrictive. One can relax it by looking at the W-invariant submonoid $E_{[e]}$ generated by the image of $e\colon S \to \mathrm{Idempotents}(E)$, i.e., the submonoid generated by the elements $w.e_s$ with $w \in W$ and $s \in S$. The W-action on E restricts to a W-action on $E_{[e]}$, and e itself can be seen as a W-equivariant map $e\colon S \to \mathrm{Idempotents}(E_{[e]})$. Then, by Lemma 2.20, one sees that $(E_{[e]}, \rho, e)$ is a Marin-Iwahori-Hecke monoid precisely when

$$e_{s_1} \cdot (w.e_{s_2}) = e_{s_1} \cdot ((s_1 w).e_{s_2}) \quad \text{for any } s_1, s_2 \in S,\, w \in W. \tag{2.15}$$

The above remark suggests an immediate generalization of Example 2.19, where on considers two Coxeter systems instead of a single one. We begin with the following.

Definition 2.21. *Let (W, S) and (U, T) be Coxeter systems. A pair (ϕ, e) where $\phi : W \to U$ is an homomorphisms of groups (not necessarily a morphism of Coxeter systems) and e is a map of sets $e\colon S \to \Sigma(\Phi_U)$, is called a Juyumaya map if e is W-equivariant (via ϕ) and*

$$\langle e_{s_1}, w.e_{s_2}\rangle = \langle e_{s_1}, (s_1 w).e_{s_2}\rangle \text{ for any } s_1, s_2 \in S, w \in W, \tag{2.16}$$

where $\langle X\rangle$ denotes the root subsystem generated by X and . denotes the W-action on Φ_W.

Example 2.22. Let $\phi\colon (W, S) \to (U, T)$ be a morphism of Coxeter systems, and let

$$e_\phi\colon S \to \Sigma(\Phi_U), \quad e_\phi(s) = \langle \phi(s)\rangle.$$

The pair (ϕ, e_ϕ), where the ϕ is the group homomorphism $\phi\colon W \to U$ associated with the morphism of Coxeter systems, is a Juyumaya map. Indeed, since a morphism of Coxeter systems preserves the Coxeter matrix, the map $\phi\colon S \to T$ is W-equivariant, and so also e_ϕ is W-equivariant. Next, as in Example 2.19, the condition $\langle e_{\phi;s_1}, w.e_{\phi;s_2}\rangle = \langle e_{\phi;s_1}, (s_1 w).e_{\phi;s_2}\rangle$ is equivalent to

$$\langle \phi(s_1), \phi(w).\phi(s_2)\rangle = \langle \phi(s_1), \phi(s_1).(\phi(w).\phi(s_2))\rangle.$$

Here we used the W-equivariance of ϕ and the fact that W acts on Φ_U, and so on $\Sigma(\Phi_U)$, via $\phi\colon W \to U$. Since $\phi(s_1) \in T$ and $w.\phi(s_2) \in \Phi_U$, the conclusion follows as in Example 2.19.

Let (ϕ, e) be a Juyumaya map. Recalling the definition of the product in $\Sigma(\Phi_U)$, we see that the second condition in the definition of a Juyumaya map is equivalent to

$$e_{s_1} \cdot (w.e_{s_2}) = e_{s_1} \cdot ((s_1 w).e_{s_2}) \quad \text{for any } s_1, s_2 \in S.$$

But this is precisely condition (2.15). In other words we have proven the following.

Theorem 2.23. *Let (W, S) and (U, T) be Coxeter systems and let (ϕ, e) be a Juyumaya map between them. Let (R, u) be a pair consisting of a commutative ring R with trivial W-action and a W-equivariant map $u\colon S \to R^\times$. Let ρ_ϕ denote the W-action on $\Sigma(\Phi_U)$ induced by $\phi\colon W \to U$ and by the canonical action of U on Φ_U. Then $(\Sigma(\Phi_U)_{[e]}, \rho_\phi, e)$ is a Marin-Iwahori-Hecke monoid. It follows that $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ is a free $R[\Sigma(\Phi_U)_{[e]}]$-module with basis $\{g_w | w \in W\}$ ($\hat{e}$ is as in (2.3)). Moreover, the augmentation map $\epsilon\colon R[\Sigma(\Phi_U)_{[e]}] \to R$ induces a surjective algebra homomorphism $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e}) \to H_R(W, S)(u_s)$.*

The following Lemmas will be instrumental to provide nontrivial examples of Juyumaya maps, which will be used in applying Proposition 2.23.

Lemma 2.24. *Let (W,S) and (U,T) be Coxeter systems, and let $\phi\colon W \to U$ be a group homomorphism (not necessarily a morphism of Coxeter systems). Let $e\colon S \to \Sigma(\Phi_U)$ be a W-equivariant map (via ϕ), and let $\mathcal{S} \subseteq S$ be a set of representatives for the equivalence relation on S induced by the conjugation action of W on itself. Then the following are equivalent.*

(1) *(ϕ, e) is a Juyumaya map;*
(2) *$\langle e_s, w.e_{s_2}\rangle = \langle e_s, (sw).e_{s_2}\rangle$ for each $s \in \mathcal{S}$, $s_2 \in S$ and $w \in W$.*

Proof. One implication is clear. To prove the other, let s_1 be an element in S. Then there exist s in $\mathcal{S}$ and w_1 in W such that $s_1 = w_1 s w_1^{-1}$. Since e is W-equivariant, we have

$$\begin{aligned}\langle e_{s_1}, w.e_{s_2}\rangle &= \langle e_{w_1 s w_1^{-1}}, w.e_{s_2}\rangle = \langle w.e_s, w_1.((w_1^{-1}w).e_{s_2})\rangle \\ &= w_1.\langle e_s, (w_1^{-1}w).e_{s_2}\rangle = w_1.\langle e_s, (s w_1^{-1}w).e_{s_2}\rangle \\ &= \langle w_1.e_s, (w_1 s w_1^{-1}w).e_{s_2}\rangle = \langle e_{w_1 s w_1^{-1}}, (w_1 s w_1^{-1} w).e_{s_2}\rangle \\ &= \langle e_{s_1}, (s_1 w).e_{s_2}\rangle.\end{aligned}$$

□

Definition 2.25. *Let (W,S) and (U,T) be Coxeter systems, and let $\phi\colon W \to U$ be a group homomorphism (not necessarily a morphism of Coxeter systems). Let $\mathcal{S} \subseteq S$ be a set of representatives for the equivalence relation on S induced by the conjugation action of W on itself. For every $s \in \mathcal{S}$, let $K_s \subseteq T$ be the set*

$$K_s = \{t \in T \text{ such that } \phi(C_W(s)) \subseteq C_U(t)\}.$$

We say that the pair $(\phi, \mathcal{S})$ is a Juyumaya pair if K_s is nonepty for every $s \in \mathcal{S}$.

Example 2.26. If $\phi\colon (W,S) \to (U,T)$ is a morphism of Coxeter systems, then $(\phi, \mathcal{S})$ is a Juyumaya pair for any set of representatives $\mathcal{S}$. Indeed, the element $\phi(s)$ is an element of T in this case, and $\phi(s) \in K_s$.

Lemma 2.27. *Let (W,S) and (U,T) be Coxeter systems, and let $(\phi,\mathcal{S})$ be a Juyumaya pair. Let $t\colon \mathcal{S} \to T$ be a map such that $t(s) \in K_s$ for every $s \in \mathcal{S}$. Then there exists a unique W-equivariant map $e_t\colon S \to \Sigma(\Phi_U)$ such that $e_{t;s} = \langle t(s)\rangle$ for every $s \in \mathcal{S}$.*

Proof. Let $s \in S$. Then there exist $w \in W$ and $s_0 \in \mathcal{S}$ such that $s = w s_0 w^{-1}$. We set $e_t\colon s \mapsto \phi(w).\langle t(s_0)\rangle$. We need to show this is a well-posed definition. If $s = \tilde{w}\tilde{s}_0\tilde{w}^{-1}$ for another pair $(\tilde{w}, \tilde{s}_0)$, then s_0 and $\tilde{s}_0$ are in the same W-orbit. Since both s and $\tilde{s}$ are in $\mathcal{S}$ this implies $\tilde{s}_0 = s_0$. Then $w^{-1}\tilde{w}$ centralizes s_0 and so $\phi(w^{-1}\tilde{w})$ centralizes $t(s_0)$. Therefore $\phi(\tilde{w}).\langle t(\tilde{s}_0)\rangle = \phi(\tilde{w}).\langle t(s_0)\rangle = \phi(w).\langle t(s_0)\rangle$. The map e is W-equivariant: if $(s_1, s_2, w) \in S \times S \times W$ is a triple such that $w s_1 w^{-1} = s_2$, let $s_0 \in \mathcal{S}$ and $w_0 \in W$ be such that $s_1 = w_0 s_0 w_0^{-1}$. Then $s_2 = (ww_0)s_0(ww_0)^{-1}$ and so

$$\begin{aligned}e_{t;s_2} &= (\phi(ww_0)).\langle t(s_0)\rangle = (\phi(w)\phi(w_0)).\langle t(s_0)\rangle \\ &= \phi(w).(\phi(w_0).\langle t(s_0)\rangle) = \phi(w).e_{t;s_1}.\end{aligned}$$

Uniqueness is clear: if $\tilde{e}$ is another W-equivariant map with $\tilde{e}|_{\mathcal{S}} = \langle t \rangle$, then for any $s \in S$ we have $\tilde{e}_s = \tilde{e}_{ws_0w^{-1}} = \phi(w).\tilde{e}_{s_0} = \phi(w).\langle t(s_0) \rangle = e_{t;s}$, where (w, s) is any pair in $W \times \mathcal{S}$ such that $s = ws_0w^{-1}$. □

Definition 2.28. *In the same notation as in Lemma 2.27, we say that $(\phi, \mathcal{S}, t)$ is a Juyumaya triple if $\phi(s) \in \{1, t(s)\}$ for any $s \in \mathcal{S}$.*

Lemma 2.29. *Let $(\phi, \mathcal{S}, t)$ be a Juyumaya triple. Then (ϕ, e_t) is a Juyumaya map, where $e_t \colon S \to \Sigma(\Phi_U)$ is the map defined in Lemma 2.27.*

Proof. By Lemma 2.24 we only need to check that $\langle e_{t;s}, w.e_{t;s_2} \rangle = \langle e_{t;s}, (sw).e_{t;s_2} \rangle$ for each $s \in \mathcal{S}$, $s_2 \in S$ and $w \in W$. This is equivalent to

$$\langle e_{t;s}, \phi(w).e_{t;s_2} \rangle = \langle e_{t;s}, \phi(s).(\phi(w).e_{t;s_2}) \rangle \tag{2.17}$$

for each $s \in \mathcal{S}$, $s_2 \in S$ and $w \in W$. By assumption we have two possibilities for $s \in \mathcal{S}$: either $\phi(s) = 1$ or $\phi(s) = t(s)$. In the first case, equation (2.17) is trivially satisfied. In the second case (2.17) becomes

$$\langle t(s), \phi(w).e_{t;s_2} \rangle = \langle t(s), t(s).(\phi(w).e_{t;s_2}) \rangle.$$

We have $t(s) = t_0$ for some $t_0 \in T$ and $\phi(w).e_{t;s_2} = \langle \beta \rangle$ for some β in Φ_U. We then use again the general fact from the theory of Coxeter groups that for any t_0 in T and any β in Φ_U one has $\langle t_0, \beta \rangle = \langle t_0, t_0.\beta \rangle$, to deduce $\langle t_0, \langle \beta \rangle \rangle = \langle t_0, t_0.\langle \beta \rangle \rangle$. □

Lemma 2.30. *Let (W, S) and (U, T) be Coxeter systems, and let $\phi \colon W \to U$ be a group homomorphism (not necessarily a morphism of Coxeter systems) such that $\phi(S) \subseteq T \cup \{1\}$. Let $\mathcal{S} \subseteq S$ be a set of representatives for the equivalence relation on S induced by the conjugation action of W on itself, and let $\mathcal{S}_0 = \mathcal{S} \cap \ker(\phi)$. Assume $K_s \neq \emptyset$ for any $s \in \mathcal{S}_0$, and let $t_0 \colon \mathcal{S}_0 \to T$ be a map such that $t(s) \in K_s$ for every $s \in \mathcal{S}_0$. Let $t \colon \mathcal{S} \to T$ be the map defined by*

$$t(s) = \begin{cases} t_0(s) & \text{if } s \in \mathcal{S}_0, \\ \phi(s) & \text{if } s \notin \mathcal{S}_0. \end{cases}$$

Then $(\phi, \mathcal{S}, t)$ is a Juyumaya triple, and so (ϕ, e_t) is a Juyumaya map, where e_t is the map from Lemma 2.27.

Proof. Clearly $\phi(s) \in \{1, t(s)\}$ for any $s \in \mathcal{S}$ so we only need to show that $(\phi, \mathcal{S})$ is a Juyumaya pair and that $t(s) \in K_s$ for any $s \in \mathcal{S}$. If $s \in \mathcal{S}_0$, then $K_s \neq \emptyset$ and $t(s) = t_0(s) \in K_s$ by assumption; if $s \notin \mathcal{S}_0$, then $\phi(s) \in T$ and $t(s) = \phi(s) \in K_s$. □

Remark 2.30.1. If $\phi \colon (W, S) \to (U, T)$ is a morphism of Coxeter systems, then $\phi(S) \subseteq T$ and the assumptions of Lemma 2.30 are trivially satisfied for any $\mathcal{S}$, since $\mathcal{S}_0 = \emptyset$ in this case. It follows that the map t is the restriction of ϕ to $\mathcal{S}$. By uniqueness, $e_t = e_\phi$, where e_ϕ is the map defined in Example 2.22. This way we recover Example 2.22 as a particular case of Lemma 2.30.

Lemma 2.31. *Let (W, S), (U, T) and (Z, R) be Coxeter systems, and let $(\phi, \mathcal{S})$, with $\phi \colon W \to U$ and $\mathcal{S} \subseteq S$, be a pair satisfying the assumptions of Lemma 2.30, and $\psi \colon (U, T) \to (Z, R)$ be a morphism of Coxeter systems. Then the pair $(\psi \circ \phi, \mathcal{S})$ satisfies the assumptions of Lemma 2.30.*

Proof. For any $s \in S$ we have $(\psi \circ \phi)(s) = \psi(1)$ or $(\psi \circ \phi)(s) = \psi(t)$ for some $t \in T$. In either case we have that $(\psi \circ \phi)(s) \in R \cup \{1\}$, since ψ is a morphism of Coxeter systems. Let $s \in \mathcal{S}_0$. Then there exists $t \in T$ such that $\phi(C_W(s)) \subseteq C_U(t)$.

Let $u \in U$ be the element $u = \psi(t)$. Then we have $(\psi \circ \phi)(C_W(s)) \subseteq \psi(C_U(t)) \subseteq C_Z(u)$. $\square$

Lemma 2.32. *Let (W, S), (U, T) and (Z, R) be Coxeter systems, and let $(\phi, \mathcal{S})$, with $\phi\colon W \to U$ and $\mathcal{S} \subseteq S$, be a pair satisfying the assumptions of Lemma 2.30, and $\psi\colon (Z, R) \to (W, S)$ be a morphism of Coxeter systems. Let $\mathcal{R} \subseteq R$ be a set of representatives for the equivalence relation on R induced by the conjugation action of Z on itself. If $\phi(\mathcal{R}) \subseteq \mathcal{S}$, then the pair $(\phi \circ \psi, \mathcal{R})$ satisfies the assumptions of Lemma 2.30.*

Proof. Since the image of $\phi \circ \psi$ is contained in the image of ϕ we have $(\phi \circ \psi)(R) \subseteq T \cup \{1\}$. Let $\mathcal{R}_0 = \mathcal{R} \cap \ker(\phi \circ \psi)$. Since ψ is a morphism of Coxeter systems, if $r \in \mathcal{R}_0$ then $\psi(r) \in \mathcal{S} \cap \ker(\phi) = \mathcal{S}_0$. Then there exists $t \in T$ such that $\phi(C_W(\psi(r)) \subseteq C_U(t)$, and so $(\phi \circ \psi)(C_Z(r)) = \phi(\psi(C_Z(r))) \subseteq \phi(C_W(\psi(r)) \subseteq C_U(t)$. $\square$

3. Examples

We here provide examples of pairs $(\phi, \mathcal{S})$ satisfying the assumptions of Lemma 2.30, and so producing Juyumaya maps.

We denote a root system of finite type X (resp. affine type $\widehat{X}$) with Coxeter diagram of vertices $i_1, \dots, i_n$ by $X_{i_1,\dots,i_n}$ (resp. $\widehat{X}_{i_1,\dots,i_n}$); the corresponding reflection group is denoted by $W(X_{i_1,\dots,i_n})$. When no confusion is possible, we will denote a morphism of Coxeter systems

$$\phi\colon (W(X_{i_1,\dots,i_n}); \{s_{i_1}, \dots, s_{i_n}\}) \to (W(Y_{j_1,\dots,j_m}); \{t_{j_1}, \dots, t_{j_m}\})$$

simply by $\phi\colon X \to Y$. A similar convention applies when one or both of the Coxeter diagrams are of affine type.

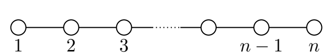
The first series of examples derives from Remark 2.30.1, so they are examples of morphisms of Coxeter systems.

Example 3.1. The Coxeter system A_1 is terminal among Coxeter systems: for any Coxeter system $X_{i_1,\dots,i_n}$ there exists a unique morphism of Coxeter systems

$$\phi\colon (W(X_{i_1,\dots,i_n}); \{s_{i_1}, \dots, s_{i_n}\}) \to (W(A_1); \{t_1\}).$$

Namely, the unique map $\{s_{i_1}, \dots, s_{i_n}\} \to \{t_1\}$ extends to a morphisms of groups $W(X_{i_1,\dots,i_n}) \to W(A_1) \cong S_2$.

Example 3.2. For any Coxeter system X, the identity of X is a morphism of Coxeter systems. In particular, if we consider the Coxeter diagrams

$A_{1\dots n}$: 

and

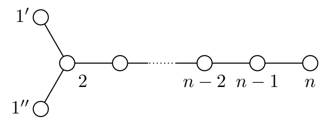
$A_{1\dots n}$: 

then the map

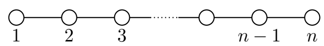
$$\phi(s_i) = t_i$$

defines a morphism of Coxeter systems $\varphi\colon A \to A$. Despite being trivial, this example will play a role in the following section.

Example 3.3. Consider the Coxeter diagrams

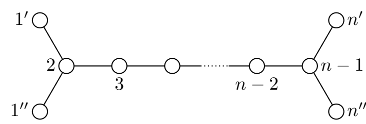
$D_{1',1'',2,\dots n}$: 

and

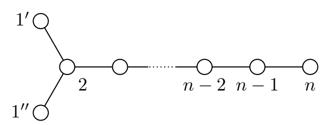
$A_{1\dots n}$: 

The map $\phi(s_i) = \begin{cases} t_1 & \text{if } i \in \{1', 1''\} \\ t_i & \text{if } i \geq 2 \end{cases}$ defines a morphism of Coxeter systems $\varphi\colon D \to A$.

Example 3.4. Consider the Coxeter diagrams

$\widehat{D}_{1',1'',2\dots,n',n''}$: 

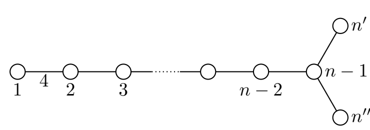
and

$D_{1',1'',2,\dots n}$: 

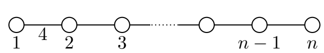
The map $\phi(s_i) = \begin{cases} t_n & \text{if } i \in \{n', n''\} \\ t_i & \text{if } i \leq n-1 \text{ or } i \in \{1', 1''\} \end{cases}$ is a morphism of Coxeter systems $\phi\colon \widehat{D} \to D$.

Example 3.5. By combining Examples 3.4 and 3.3 we get a morphism of Coxeter systems $\hat{D} \to A$.

Example 3.6. Consider the Coxeter diagrams

$\widehat{B}_{1\dots n',n''}$: 

and

$B_{1\dots n}$: 

The map $\phi(s_i) = \begin{cases} t_n & \text{if } i \in \{n', n''\} \\ t_i & \text{if } i \leq n-1 \end{cases}$ is a morphism of Coxeter systems $\phi\colon \widehat{B} \to B$.

To produce more elaborate examples we need the following general result, which describes the centralizer $C_W(r)$ of a simple reflection r in a Coxeter group. Let (W, S) be a Coxeter system. For $s \in S$, let $\Gamma^W(s)$ denote the set of roots orthogonal to α_s, and for $w \in W$, let $N(w)$ denote the inversion set of w, i.e.

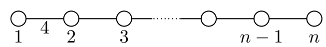
$$N(w) = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in -\Phi^+\}.$$

Proposition 3.7. [8, §2, Theorem] *For every* $r \in S$ *one has*

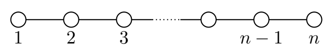
$$C_W(r) = W(\Gamma^W(r) \cup \{\pm\alpha_r\}) \rtimes Y^W_{r,r},$$

where $Y^W_{r,r} = \{u \in W \mid \alpha_r = u(\alpha_r),\ N(u) \cap \Gamma^W(r) = \emptyset\}$*. Moreover* $W(\Gamma^W(r) \cup \{\pm\alpha_r\})$ *is a Coxeter group, and* $Y^W_{r,r}$ *is free of rank* $e(r)-n(r)+1$*, where* $e(r)$ *denotes the number of edges and* $n(r)$ *the number of vertices of the connected component of the odd Coxeter graph of* (W, S)*, i.e, the graph with vertex set* S *and edge set* $\{\{r, s\} \mid m_{r,s}$ *is odd*$\}$*, containing* r*.*

Example 3.8. Consider the Coxeter diagrams

$B_{1\dots n}$: 

and

$A_{1\dots n}$: 

Then

- the map $\phi\colon \{s_1, s_2, \dots, s_n\} \to W(A_{1\dots n})$ defined by $\phi(s_i) = \begin{cases} 1 & \text{if } i = 1 \\ t_i & \text{if } i \geq 2 \end{cases}$ extends to a group homorphism $\phi\colon W(B_{1\dots n}) \to W(A_{1\dots n})$;
- the set $\mathcal{S} = \{s_1, s_2\}$ is a set of representatives for the for the equivalence relation on $\{s_1, s_2, \dots, s_n\}$ induced by the conjugation action of $W(B_{1\dots n})$ on itself;
- the pair $(\phi, \mathcal{S})$ satisfies the assumptions of Lemma 2.30.

The first statement is easily verified by checking that the Coxeter relations are verified, and the second statement is well-known (see e.g. [16]). To prove the third statement, notice that $\mathcal{S}_0 = \{s_1\}$, so we only need to show that the set

$$K_{s_1} = \{t \in \{t_1, \dots, t_n\} \text{ such that } \phi(C_B(s_1)) \subseteq C_A(t)\},$$

is nonempty, where we wrote $C_B(s_1)$ for $C_{W(B_{1\dots n})}(s_1)$ and $C_A(t)$ for $C_{W(A_{1\dots n})}(t)$; we will use similar conventions in the following. We use Proposition 3.7 to determine $C_B(s_1)$. The group $Y^B_{s_1,s_1}$ is free of rank 0, so it is the trivial group. We have

$$\begin{aligned}\Gamma^B(s_1) = &\pm \{\alpha_h + \ldots + \alpha_k, 3 \leq h \leq k \leq n\} \\ &\cup \pm\{\sum_{i=1}^{n} a_i\alpha_i \in \Phi(B_{1,\dots,n}) \mid a_2 = \sqrt{2}a_1 > 0\},\end{aligned}$$

which is a system of type B_{n-1}. A simple system might be $\{\alpha_1 + \sqrt{2}\alpha_2, \alpha_3, \dots, \alpha_n\}$. So $C_B(s_1)$ is of type $A_1 \times B_{n-1}$ and, since $s_{\alpha_1+\sqrt{2}\alpha_2} = s_2s_1s_2$, we have that

$$C_B(s_1) = \langle s_1, s_2s_1s_2, s_3, \dots, s_n\rangle.$$

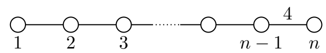
Therefore, $\phi(C_B(s_1)) = \langle\phi(s_1), \phi(s_2s_1s_2), \phi(s_3), \dots, \phi(s_n)\rangle = \langle s_3, \dots, s_n\rangle$. It is clear that $\Gamma^A(s_1) = \{\pm\alpha_3, \dots, \pm\alpha_n\}$, and that $Y^A_{s_1,s_1}$ is the trivial group (being free of rank zero). Hence $C_A(t_1)$ is the group of type $A_1 \times A_{n-2}$ given by $\langle t_1, t_3, \dots, t_n\rangle$. This gives $\phi(C_B(s_1)) \subseteq C_A(t_1)$ and so $t_1 \in K_{s_1}$.

Example 3.9. A set of representatives for the equivalence relation on $\{s_1, s_2, \dots, s_{n'}, s_{n''}\}$ induced by the conjugation action of $W(\widehat{B}_{1,2,\dots,n',n''})$ on itself is given by $\{s_1, s_2\}$. A set of representatives for the equivalence relation on $\{t_1, t_2, \dots, t_n\}$ induced by the conjugation action of $W(B_{1,2,\dots,n})$ on itself is given by $\{t_1, t_2\}$. The morphism of Coxeter systems from Example 3.6 maps $\{s_1, s_2\}$ to $\{t_1, t_2\}$.

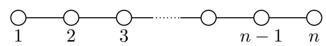
Hence, applying Lemma 2.32 to Examples 3.6 and 3.8 we get a pair $(\phi, \mathcal{S})$ with $\phi\colon W(\widehat{B}_{1,2,\dots,n',n''}) \to W(A_{1,\dots,n})$, satisfying the assumptions of Lemma 2.30.

By flipping Example 3.8 we obtain the following.

Example 3.10. Consider the Coxeter diagrams

$C_{1\dots n}$: 

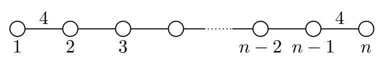
and

$A_{1\dots n}$: 

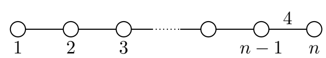
Then

- the map $\phi\colon \{s_1, s_2, \dots, s_n\} \to W(A_{1\dots n})$ defined by $\phi(s_i) = \begin{cases} 1 & \text{if } i = n \\ t_i & \text{if } i \leq n-1 \end{cases}$ extends to a group homorphism $\phi\colon W(C_{1\dots n}) \to W(A_{1\dots n})$;
- the set $\mathcal{S} = \{s_{n-1}, s_n\}$ is a set of representatives for the for the equivalence relation on $\{s_1, s_2, \dots, s_n\}$ induced by the conjugation action of $W(C_{1\dots n})$ on itself;
- the pair $(\varphi, \mathcal{S})$ satisfies the assumptions of Lemma 2.30.

Example 3.11. Consider the Coxeter diagrams

$\widehat{C}_{1\dots n}$: 

and

$C_{1\dots n}$: 

Then

- the map $\phi\colon \{s_1, s_2, \dots, s_n\} \to W(C_{1\dots n})$ defined by $\phi(s_i) = \begin{cases} 1 & \text{if } i = 1 \\ t_i & \text{if } i \geq 2 \end{cases}$ extends to a group homorphism $\varphi\colon W(\widehat{C}_{1\dots n}) \to W(C_{1\dots n})$;
- the set $\mathcal{S} = \{s_1, s_2, s_n\}$ is a set of representatives for the for the equivalence relation on $\{s_1, s_2, \dots, s_n\}$ induced by the conjugation action of $W(\widehat{C}_{1\dots n})$ on itself;
- the pair $(\varphi, \mathcal{S})$ satisfies the assumptions of Lemma 2.30.

The first two statements are checked as in Example 3.8. To prove the third statement, notice that $\mathcal{S}_0 = \{s_1\}$, so we only need to show that the set

$$K_{s_1} = \{t \in \{t_1, \dots, t_n\} \text{ such that } \phi(C_{\widehat{C}}(s_1)) \subseteq C_C(t)\},$$

is nonempty. We use Proposition 3.7 to determine $C_{\widehat{C}}(s_1)$. The group $Y^{\widehat{C}}_{s_1,s_1}$ is free of rank 0, and so it is the trivial group. We have $\Gamma^{\widehat{C}}(s_1) = \Phi(C_{3,\dots,n})$. Next we determine $C_C(t_1)$. The group $Y^{C}_{t_1,t_1}$ is free of rank 0, and so it is the trivial group. Set $\beta = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_{n-1}) + \sqrt{2}\alpha_n$. We have $\Gamma^C(t_1) = \{\pm\beta\} \cup \Phi(C_{3,\dots,n})$, which is a root system of type $A_1 \times C_{n-3}$. A simple system might be $\{\beta, \alpha_3, \dots, \alpha_n\}$. Since $s_\beta = s_2 s_3 \cdots s_{n-1} s_n s_{n-1} \cdots s_3 s_2 s_1 s_2 s_3 \cdots s_{n-1} s_n s_{n-1} \cdots s_2$, we have that

$$\begin{aligned} \phi(s_\beta) &= t_2 t_3 \cdots t_{n-1} t_n t_{n-1} \cdots t_3 t_2 t_2 t_3 \cdots t_{n-1} t_n t_{n-1} \cdots t_2 = 1 \\ \phi(s_i) &= t_i, \quad 3 \leq i \leq n. \end{aligned}$$

This gives $\phi(C_{\widehat{C}}(s_1)) \subseteq C_C(t_1)$ and so $t_1 \in K_{s_1}$.

Example 3.12. Applying lemma 2.32 to Examples 3.11 and 3.10 we get a pair $(\phi, \mathcal{S})$ with $\phi\colon W(\widehat{C}_{1,2,\ldots,n,n'}) \to W(A_{1,\ldots,n})$, satisfying the assumptions of Lemma 2.30.

4. Braids and ties algebras

Given a Juyumaya map (ϕ, e) between the Coxeter systems $X = (W, S)$ and $Y = (U, T)$, we see that the algebra $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ is generated by elements g_s with $s \in S$ and $w.e_s$ with $w \in W$. When X and Y are of A-, B-, C- or D-type, we can interpret the elements g_s as braids and the elements e_s as ties. Consequently the algebra $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ will be called a *braids and ties algebra of type* X-Y. When $Y = A$ we will simply say braids and ties algebra of type X. Somewhat abusing terminology, for the braids and ties algebras arising from the examples in Section 3, with the exception of the terminal morphism from Example 3.1, we will say "the" braids and ties algebra of type X, to mean the distinguished braids and ties algebra of type X arising from those examples.

Example 4.1 (The Hecke algebra "doubled")**.** This is the case of the terminal morphisms from Example 3.1.

If $X = X_{i_1,\ldots,i_n}$, the algebra $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ in this case is generated by the elements g_{s_i} with $i \in \{i_1, \ldots, i_n\}$ and by the single element $e_{s_{i_1}}$ (since we have $e_{s_i} = e_{s_j}$ for any $i, j \in \{i_1, \ldots, i_n\}$). Writing g_i for g_{s_i} and e for e_{s_1}, the relations defining the algebra are

$$\begin{aligned} \underbrace{g_{s_i} g_{s_j} g_{s_i} \cdots}_{m_{i,j} \text{ times}} &= \underbrace{g_{s_j} g_{s_i} g_{s_j} \cdots}_{m_{i,j} \text{ times}} \\ {g_i}^2 &= 1 + (u_i - 1)e(1 - g_i) \\ g_i e &= e g_i \\ e^2 &= e, \end{aligned}$$

where $u\colon \{i_1, \ldots, i_n\} \to R^\times$ is constant over $W(X_{i_1,\ldots,i_n})$-orbits in $\{i_1, \ldots, i_n\}$. For instance, when X is the Coxeter diagram $G_{1,2}$: $\bullet\overset{6}{-}\bullet$ we get the algebra over three generators g_1, g_2, e with the relations

$$\begin{aligned} & g_1 g_2 g_1 g_2 g_1 g_2 = g_2 g_1 g_2 g_1 g_2 g_1 \\ & {g_1}^2 = 1 + (u-1)e(1 - g_1), \quad {g_2}^2 = 1 + (u-1)e(1 - g_2) \\ & g_1 e = e g_1, \quad g_2 e = e g_2 \\ & e^2 = e, \end{aligned}$$

which has manifestly $\{\epsilon\, w \mid w \in W(G_{1,2}),\ \epsilon \in \{1, e\}\}$ as basis over the ground field.

Example 4.2 (Marin's $\mathcal{C}_W$-algebras)**.** We recover here the Example 2.19. It indeed corresponds to the general identity morphism from Example 3.2. If $X = (W, S)$ is a Coxeter system and $e : S \to \Sigma(\Phi_W)$, as in (2.11), is the map $e(s) = \langle s \rangle$, then $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_W)_{[e]}], \rho_{Id}, \hat{e})$ is the algebra $\mathcal{C}_W$. In particular, when $R = k$ then $\mathrm{rk}_k R[\Sigma(\Phi_U)_{[e]}]$ is the number of reflection subgroups of W, which has been called *Bell number* of W in [13]. Hence, formula (2.9) specializes to Marin's rank formula from [13, Theorem 3.4]: $\mathrm{rk}_k \mathcal{C}_W = \mathit{Bell}(W)|W|$.

Example 4.3 (The type A braids and ties algebra)**.** This is a special case of the previous example, when W is of type A_{n-1}. The algebra $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$

in this case is generated by the elements g_{s_i} with $i \in \{1, \ldots, n-1\}$ and by the elements $w.e_{s_i}$ with $w \in W(A_{1,\ldots,n-1}) \simeq S_n$. Let us write g_i for g_{s_i} and e_i for e_{s_i}.

In [1] by Aicardi an Juyumaya introduced the (type A) *algebra of braids and ties*. This is the algebra $\mathcal{E}_n$ generated by the elements $\{g_1, \ldots, g_{n-1}, e_1, \ldots, e_{n-1}\}$ subject to the following relations

$$g_i g_j = g_j g_i \quad \text{if} \quad |i-j| > 1 \tag{4.1}$$

$$g_i g_j g_i = g_j g_i g_j \quad \text{if} \quad |i-j| = 1 \tag{4.2}$$

$$g_i{}^2 = 1 + (u-1)e_i(1-g_i) \tag{4.3}$$

$$g_i e_j = e_j g_i \qquad \text{if} \quad |i-j| \neq 1 \tag{4.4}$$

$$g_j g_i e_j = e_i g_j g_i \qquad \text{if} \quad |i-j| = 1 \tag{4.5}$$

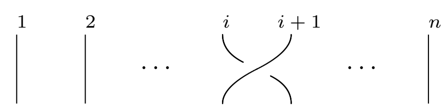
$$e_i e_j = e_j e_i \tag{4.6}$$

$$e_i{}^2 = e_i \tag{4.7}$$

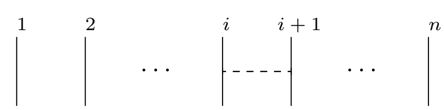
$$e_i e_j g_j = e_i g_j e_i = g_j e_i e_j \qquad \text{if} \quad |i-j| = 1 \tag{4.8}$$

The algebra $\mathcal{E}_n$ has the following useful and important diagrammatic realization, proved in [1]. Consider the algebra generated by concatenation[1] of the following elementary diagrams on n strands: 1 is represented by

g_i is represented by a elementary *braid*



e_i is represented by a elementary *tie*



The resulting diagrams are considered up to planar isotopy; moreover, the endpoints of the ties can freely move along the strand and through the ties themselves. Our general constructions recovers this algebra, as proved in the following

Proposition 4.4. *The algebra $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ coincides with $\mathcal{E}_n$.*

A (sketchy) proof of this statement can be found in [13]. We provide a detailed treatment, which can be taken as a basis for other instances of braids and type algebras admitting a geometric realization.

[1] AB means A on top of B.

Proof. The relations defining $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ are

$$g_i g_j = g_j g_i \quad \text{if} \quad |i-j| > 1 \tag{4.9}$$

$$g_i g_j g_i = g_j g_i g_j \quad \text{if} \quad |i-j| = 1 \tag{4.10}$$

$${g_i}^2 = 1 + (u-1)e_i(1-g_i) \tag{4.11}$$

$$g_i(w.e_j) = (s_i w.e_j)g_i, \qquad \forall w \in W(A_{1,\dots,n-1}) \tag{4.12}$$

$$(w_1.e_i)(w_2.e_j) = (w_2.e_j)(w_1.e_i), \qquad \forall w_1, w_2 \in W(A_{1,\dots,n-1}) \tag{4.13}$$

$$(w.e_i)^2 = w.e_i, \qquad \forall w \in W(A_{1,\dots,n-1}) \tag{4.14}$$

$$e_i(w.e_j) = e_i((s_i w).e_j) \qquad \forall w \in W(A_{1,\dots,n-1}), \tag{4.15}$$

where the fact that all of the s_i's are in the same W-orbit implies that the datum of the W-equivariant map $u\colon \{s_1, \dots, s_n\} \to R^\times$ reduces to that of the choice of an element u in $R^\times$. By Remark 2.4.1 and Example 2.10, equation ${g_i}^2 = 1 + (u_i - 1)e_i(1-g_i)$ implies that g_i is invertible, with inverse given by a polynomial expression in g_i and e_i. So we can rewrite equation $g_i(w.e_j) = (s_i w.e_j)g_i$ as

$$s_i w.e_j = g_i(w.e_j){g_i}^{-1}$$

and argue by induction on the lenght of w that all the elements $w.e_i$ lie in the subalgebra generated by the g_j's and the e_j's. Therefore $\{g_1, \dots, g_{n-1}, e_1, \dots, e_{n-1}\}$ is a set of generators for the algebra $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ in this case. Taking $w = 1$, relation (4.12) gives

$$g_i e_j = (s_i.e_j)g_i. \tag{4.16}$$

Since we have

$$s_i.e_j = \begin{cases} e_j & \text{if} \quad |i-j| \neq 1 \\ s_j.e_i & \text{if} \quad |i-j| = 1, \end{cases} \tag{4.17}$$

equation (4.16) reduces to $g_i e_j = e_j g_i \qquad \text{if} \quad |i-j| \neq 1$, which is relation (4.4).

Moreover, if $|i-j| = 1$, using Remark 2.4.5, we have

$$g_i e_j = (s_i.e_j)g_i = (s_j.e_i)g_i = g_j e_i g_j^{-1} g_i = {g_j}^{-1} e_i g_j g_i,$$

i.e., $g_j g_i e_j = e_i g_j g_i \qquad \text{if} \quad |i-j| = 1$, which is relation (4.5).

Taking $w = w_1 = w_2 = 1$, relation (4.14) becomes ${e_i}^2 = e_i$, which is relation (4.7), and relation (4.13) becomes $e_i e_j = e_j e_i$. which is relation (4.6).

Finally, taking $w = 1$ relation (4.15) gives $e_i e_j = e_i(s_i.e_j)$. This is trivially true if $|i-j| \neq 1$, while for $|i-j| = 1$, by (4.17) and (4.13), it becomes

$$e_i e_j = e_i(s_j.e_i) = (s_j.e_i)e_i,$$

i.e., $e_i e_j = e_i g_j e_i {g_j}^{-1}$ and $e_i e_j = g_j e_i {g_j}^{-1} e_i = {g_j}^{-1} e_i g_j e_i$, i.e., since $e_i e_j = e_j e_i$, $e_j e_i g_j = e_i g_j e_i = g_j e_i e_j$. Hence relation (4.8) holds and since relations (4.9)-(4.11) and (4.1)-(4.3) are identical, $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ satisfies all the defining relations of $\mathcal{E}_n$. Also the converse holds. To get (4.12), notice that it is equivalent to $g_i(w.e_j)g_i^{-1} = (s_i w.e_j)$, which in turn is equivalent to $s_i.(w.e_j) = (s_i w.e_j)$, which clearly holds. Since $w.-$ is an action on a monoid, (4.7) implies $(w.e_i)^2 = w.e_i$ for any w, i.e., (4.14) holds. By the same reason, equation (4.13) is equivalent to

$$e_i(w.e_j) = (w.e_j)e_i, \qquad \forall w \in W(A_{1,\dots,n-1}). \tag{4.18}$$

To prove (4.18) and, finally, (4.15), we resort to the diagrammatic encoding of $\mathcal{E}_n$. Denote by e_{ij} the diagram with a tie joining strands i and j. The key remark is that the W-action on ties can be interpreted as the permutation action of the endpoints of the ties: the diagram of $w.e_i$ is $e_{w(i),w(i+1)}$.

Relations (4.18), (4.15) in $\mathcal{E}_n$ read, respectively

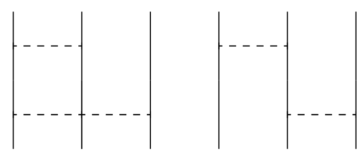
$$e_{i,i+1}e_{w(j),w(j+1)} = e_{w(j),w(j+1)}e_{i,i+1}, \tag{4.19}$$

$$e_{i,i+1}e_{w(j),w(j+1)} = e_{i,i+1}e_{s_iw(j),s_iw(j+1)}. \tag{4.20}$$

The are several cases to analyze. In the following table we display the values of $w(j), w(j+1)$ and the corresponding values of $s_iw(j), s_iw(j+1)$; by $*$ we mean an integer in the range $1, \dots, n$ different from $i, i+1$.

	1	2	3	4	5	6
$w(j)$	i	$i+1$	i	$i+1$	$*$	$*$
$w(j+1)$	$*$	$*$	$i+1$	i	i	$i+1$
$s_iw(j)$	$i+1$	i	$i+1$	i	$*$	$*$
$s_iw(j+1)$	$*$	$*$	i	$i+1$	$i+1$	i

Relation (4.19) in cases 3,4 amounts to (4.6), where in the other cases follows from the fact that the endpoints move freely on the strands (which easily follows from (4.12)). Relation (4.20) is clear in cases 3,4, and in cases 1, 2 (and by the same reason also in cases 5,6) corresponds to the following diagrammatic equality between strands $i, i+1, *$



□

Remark 4.4.1. In [4] the authors constructed a two-parameters braids and ties algebra $\mathcal{E}_n(u, v)$, which specializes to $\mathcal{E}_n$, and which is relevant for applications to knot theory. We can recover this algebra starting from the generalized Marin ring described in Remark 2.4.2 by setting:

$$b_s = 1 + (u-1)e_s, \quad c_s = 1 - (v-1)e_s.$$

With this specialization relation (2.2) (which replaces (4.11)) becomes

$$g_s^2 = 1 + (u-1)e_s + (v-1)e_sg_s, \tag{4.21}$$

which is formula (20) from [4].

To obtain $\mathcal{E}_n$ from $\mathcal{E}_n(u, v)$, one looks for rescalings $g_s \rightsquigarrow \tilde{g}_s = \lambda_s g_s$ which turn (4.21) to (4.11). The equation for λ_s becomes

$$(1 + (u-1)e_s)\lambda_s^2 - (v-1)e_s\lambda_s - 1 = 0.$$

Now we use the idempotency of e_s and write

$$\xi_s = \lambda_s e_s; \qquad z_s = \lambda_s(1 - e_s),$$

so that $\lambda_s = \xi_s + z_s$. By the commutativity of the algebra we have $\xi_s z_s = z_s\xi_s = 0$ and so $\lambda_s^2 = \xi_s^2 + z_s^2$. The quadratic equation for λ_s becomes

$$(1 + (u-1)e_s)(\xi_s^2 + z_s^2) - (v-1)e_s(\xi_s + z_s) - 1 = 0,$$

and so, recalling that $e_s(1 - e_s) = 0$, it becomes the equation

$$(1 + (u-1)e_s)(\xi_s^2 + z_s^2) - (v-1)e_s\xi_s - 1 = 0.$$

By multiplying it by e_s and by $1 - e_s$, this is in turn equivalent to the pair of equations

$$\begin{cases} (u\xi_s^2 - (v-1)\xi_s - 1)e_s = 0 \\ (z_s^2 - 1)(1 - e_s) = 0. \end{cases}$$

Let us consider the system of equations

$$\begin{cases} u\eta_s^2 - (v-1)\eta_s - 1 = 0 \\ \zeta_s^2 - 1 = 0. \end{cases}$$

If (η_s, ζ_s) is a solution, then $\xi_s = \eta_s e_s$, $z_s = \zeta_s(1 - e_s)$ is a solution of the previous one. A solution of $\zeta_s^2 - 1 = 0$ is clearly given by $\zeta_s = 1$. Let then τ be a solution of the quadratic equation $ut^2 - (v-1)t - 1 = 0$ (notice that this equation is independent of s). We find that a solution λ_s to our original problem is given by

$$\lambda_s = (1 - e_s) + \tau e_s = 1 + (\tau - 1)e_s.$$

If we set $\delta = \tau - 1$, then our expression for λ_s takes the form $\lambda_s = 1 + \delta e_s$ so that our rescaling is $\tilde{g}_s = g_s + \delta e_s g_s$. Notice that the quadratic equation satisfied by δ is

$$u(z+1)^2 - (v-1)(z+1) - 1 = 0,$$

i.e., Aicardi-Juyumaya's equation (22) from [4].

The previous example shows how to realize braids and ties algebras of type A, i.e. of type A-A. Similarly, one can produce algebras of type B-B, D-D and so on. There are however also genuine examples of braids and ties algebras of type B and D (i.e., of type B-A, D-A).

Example 4.5 (The type D braids and ties algebra)**.** This is the case when $X = D$ and $Y = A$ coming from Example 3.3. The algebra $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ in this case is generated by the elements g_{s_i} with $i \in \{1', 1'', 2 \ldots, n\}$ and by the elements $w.e_{t_j}$ with $j \in \{1, 2, \ldots, n\}$ and $w \in W(D_{1',1'',\ldots,n})$ (or equivalently, since the morphism $W(D_{1',1'',\ldots,n}) \to W(A_{1,\ldots,n})$ is surjective, $w \in W(A_{1,\ldots,n})$). It is convenient to introduce elements e_{s_i} with $i \in \{1', 1'', 2 \ldots, n\}$ by setting $e_{s_{1'}} = e_{s_{1''}} = e_{t_1}$ and $e_{s_i} = e_{t_i}$ for $i \in \{2, \ldots n\}$. Let us write g_i for g_{s_i} and e_i for e_{s_i}. The relations defining the algebra are then obtained as in Example 4.3:

$$\begin{aligned}
g_i g_j &= g_j g_i \quad \text{if} \quad |i-j| > 1 \\
g_i g_j g_i &= g_j g_i g_j \quad \text{if} \quad |i-j| = 1 \\
{g_i}^2 &= 1 + (u-1)e_i(1 - g_i) \\
g_i e_j &= e_j g_i \qquad \text{if} \quad |i-j| \neq 1 \\
g_j g_i e_j &= e_i g_j g_i \qquad \text{if} \quad |i-j| = 1 \\
e_i e_j &= e_j e_i \\
{e_i}^2 &= e_i \\
e_i e_j g_j &= e_i g_j e_i = g_j e_i e_j \qquad \text{if} \quad |i-j| = 1 \\
e_{1'} &= e_{1''},
\end{aligned}$$

where $|i-j|$ denotes the distance in the Dynkin diagram $D_{1',1'',2,\ldots,n}$, so that, e.g., $|1' - 1''| = 2$. Notice that one has an evident morphism of algebras from the type

D to the type A braids and ties algebra, given by

$$g_i \mapsto \begin{cases} g_1 & \text{if} \quad i \in \{1', 1''\} \\ g_i & \text{if} \quad i \geq 2 \end{cases}, \quad e_i \mapsto \begin{cases} e_1 & \text{if} \quad i \in \{1', 1''\} \\ e_i & \text{if} \quad i \geq 2 \end{cases}.$$

Example 4.6 (The type B braids and ties algebra)**.** This is the case when $X = B$ and $Y = A$ coming from Example 3.8. The algebra $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ in this case is generated by the elements g_{s_i} with $i \in \{1, \dots, n\}$ and by the elements $w.e_{s_i}$ with $w \in W(B_{1,\dots,n})$. By Example 3.8 we can choose $t_0 \colon \mathcal{S}_0 = \{s_1\} \to \{t_1, \dots, t_n\}$ to be $t_0(s_1)$. The corresponding map $t \colon \mathcal{S} \to \{t_1, \dots, t_n\}$ from Lemma 2.30 is then given by $t(s_i) = t_i$ for $i = 1, 2$. According to Lemma 2.27, the map

$$e \colon \{s_1, \dots, s_n\} \to \Phi(W(A_{1,\dots,n}, \{t_1, \dots, t_n\})$$

is given by

$$e_{s_i} = \phi(w).\langle t(s_0)\rangle,$$

where $s = ws_0w^{-1}$ with $s_0 \in \{s_1, s_2\}$ and $w \in W(B_{1,\dots,n})$. It then follows that e is explicitly given by

$$e_{s_i} = \langle t_i \rangle$$

for any i. As in the previous example, let us write g_i for g_{s_i} and e_i for e_{s_i}. Following [9], let us also recursively define the elements $f_i \in R[\Sigma_{A_{1,\dots,n},\{t_1,\dots,t_n\}}]_{[e]}$ as $f_1 = e_1$ and $f_i = g_i.f_{i-1}$ for $2 \leq i \leq n$. The relations defining the algebra $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ are then

$$g_i g_j = g_j g_i \quad \text{if} \quad |i - j| \neq 1 \tag{4.22}$$

$$g_1 g_2 g_1 g_2 = g_2 g_1 g_2 g_1 \tag{4.23}$$

$$g_i g_j g_i = g_j g_i g_j \quad \text{if} \quad |i - j| = 1, \{i, j\} \neq \{1, 2\} \tag{4.24}$$

$${g_1}^2 = 1 + (v - 1)e_i(1 - g_i) \tag{4.25}$$

$${g_i}^2 = 1 + (u - 1)e_i(1 - g_i), \qquad i \geq 2 \tag{4.26}$$

$$g_i(w.e_j) = (s_i w.e_j)g_i, \qquad \forall w \in W(B_{1,\dots,n}) \tag{4.27}$$

$$(w_1.e_i)(w_2.e_j) = (w_2.e_j)(w_1.e_i), \qquad \forall w \in W(B_{1,\dots,n}) \tag{4.28}$$

$$(w.e_i)^2 = w.e_i, \qquad \forall w \in W(B_{1,\dots,n}) \tag{4.29}$$

$$e_i(w.e_j) = e_i((s_i w).e_j) \qquad \forall w \in W(B_{1,\dots,n}) \tag{4.30}$$

$$f_1 = e_1 \tag{4.31}$$

$$g_i f_j = f_j g_i \quad \text{if} \quad i = 1 \quad \text{or} \quad i \geq 2, j \neq i - 1, i \tag{4.32}$$

$$g_i f_{i-1} = f_i g_i \quad 2 \leq i \leq n \tag{4.33}$$

$$g_i f_i = f_{i-1} g_i \quad 2 \leq i \leq n \tag{4.34}$$

$$e_{i+1} f_i = e_{i+1} f_{i+1} \quad 1 \leq i < n \tag{4.35}$$

$$e_{i+1} f_i = f_{i+1} f_i \quad 1 \leq i < n, \tag{4.36}$$

where we used the fact that all of the s_i's with the exception of s_1 are in the same W-orbit to reduce the datum of the W-equivariant map $u \colon \{s_1, \dots, s_n\} \to R^\times$ to the pair of elements v and u in $R^\times$. The origin of the relations involving the elements f_i is as follows. The relation $f_1 = e_1$ and the relation $g_i f_{i-1} = f_i g_i$ are the defining relations for the elements f_i. Next, the algebra $R[\Sigma_{A_{1,\dots,n},\{t_1,\dots,t_n\}}]_{[e]}$ is a subalgebra of $R[\Sigma_{A_{1,\dots,n},\{t_1,\dots,t_n\}}]$. In this larger algebra, under the canonical isomorphism $W(A_{1,\dots,n}) \cong S_{n+1}$ mapping the generator t_i to the permutation $(i, i+1)$, the element e_i corresponds to the subgroup of S_{n+1} generated by the permutation $(i, i+1)$, and the element f_i to the subgroup of S_{n+1} generated by the permutation $(1, i+1)$. Then the identity $g_1 f_j = f_j g_1$ for all j is the fact that g_1 is in the kernel of the homomorphism $W(B_{1,\dots,n}) \to W(A_{1,\dots,n})$ and so it acts trivially on $R[\Sigma_{A_{1,\dots,n},\{t_1,\dots,t_n\}}]_{[e]}$. The identity $g_i f_j = f_j g_i$ for $|i-j| \neq 1$ with $i \geq 2$ and $j \neq i-1, i$ is the fact that with these assumptions in S_{n+1} we have the identity $(i, i+1)(1, j+1)(i, i+1) = (1, j+1)$. Therefore in the larger algebra $R[\Sigma_{A_{1,\dots,n},\{t_1,\dots,t_n\}}]$ we have the relation $g_i f_j = f_j g_i$; this is a relation in $R[\Sigma_{A_{1,\dots,n},\{t_1,\dots,t_n\}}]$ among elements in $R[\Sigma_{A_{1,\dots,n},\{t_1,\dots,t_n\}}]_{[e]}$ and so it is a relation in $R[\Sigma_{A_{1,\dots,n},\{t_1,\dots,t_n\}}]_{[e]}$. Similarly, the relation $g_i f_i = f_{i-1} g_i$ is the identity $(i, i+1)(1, i+1)(i, i+1) = (1, i)$ in S_{n+1}. Finally, the relation $e_{i+1} f_i = e_{i+1} f_{i+1}$ expresses the fact that in S_{n+1} the pair of permutations $((i+1, i+2), (1, i+1))$ and $((i+1, i+2), (1, i+2))$ generate the same subgroup; similarly the relation $e_{i+1} f_i = f_{i+1} f_i$ expresses the fact that in S_{n+1} the pair of permutations $((i+1, i+2), (1, i+1))$ and $((1, i+1), (1, i+2))$ generate the same subgroup.

Taking $w = 1$, the relation $g_i(w.e_j) = (s_i w.e_j) g_i$ gives $g_i e_j = (s_i.e_j) g_i$. Since we have the nontrivial relations

$$s_i.e_j = \begin{cases} e_j & \text{if} \quad |i-j| \neq 1 \quad \text{or} \quad (1, j) = (1, 2), \\ s_j.e_i & \text{if} \quad |i-j| = 1, \quad \{i, j\} \neq \{1, 2\}, \end{cases} \tag{4.37}$$

equation $g_i e_j = (s_i.e_j) g_i$ reduces to

$$g_i e_j = e_j g_i \qquad \text{if} \quad |i-j| \neq 1 \quad \text{or} \quad (1, j) = (1, 2)$$

and gives

$$g_i e_j = (s_i.e_j) g_i = (s_j.e_i) g_i = g_j e_i g_j^{-1} g_i = g_j{}^{-1} e_i g_j g_i,$$

where we used Remark 2.4.5, i.e.,

$$g_j g_i e_j = e_i g_j g_i \qquad \text{if} \quad |i-j| = 1, \quad \{i, j\} \neq \{1, 2\}.$$

Taking $w = w_1 = w_2 = 1$, the relation $(w.e_i)^2 = w.e_i$ becomes $e_i{}^2 = e_i$ and the relation $(w_1.e_i)(w_2.e_j) = (w_2.e_j)(w_1.e_i)$ becomes $e_i e_j = e_j e_i$.

Taking $w_1 = 1$, $w_2 = g_j g_{j-1} \cdots g_2$ and $j = 1$, the relation $(w_1.e_i)(w_2.e_j) = (w_2.e_j)(w_1.e_i)$ becomes $e_i f_j = f_j e_i$.

Taking $w_1 = g_i g_{i-1} \cdots g_2$, $w_2 = g_j g_{j-1} \cdots g_2$ and $i = j = 1$, the relation $(w_1.e_i)(w_2.e_j) = (w_2.e_j)(w_1.e_i)$ becomes $f_i f_j = f_j f_i$.

Taking $w = 1$ the relation $e_i(w.e_j) = e_i((s_i w).e_j)$ gives $e_i e_j = e_i(s_i.e_j)$. This is trivial if $|i-j| \neq 1$ or $(i, j) = (1, 2)$, while for $|i-j| = 1$ with $\{i, j\} \neq \{1, 2\}$, by (4.37) and by $(w_1.e_i)(w_2.e_j) = (w_2.e_j)(w_1.e_i)$, it becomes $e_i e_j = e_i(s_j.e_i) = (s_j.e_i)e_i$, i.e.,

$$e_i e_j = e_i g_j e_i g_j{}^{-1}$$

and $e_i e_j = g_j e_i g_j{}^{-1} e_i = g_j{}^{-1} e_i g_j e_i$, i.e., since $e_i e_j = e_j e_i$,

$$e_j e_i g_j = e_i g_j e_i = g_j e_i e_j.$$

For $(i, j) = (2, 1)$ we find the relation $e_2 e_1 = e_2 g_2 e_1 g_2{}^{-1} = g_2 e_1 g_2{}^{-1} e_2$, i.e., since $g_2 e_2 = e_2 g_2$,

$$e_2 e_1 = e_2 g_2 e_1 g_2{}^{-1} = g_2 e_1 e_2 g_2{}^{-1},$$

i.e., since $e_1 e_2 = e_2 e_1$,

$$e_1 e_2 g_2 = e_2 g_2 e_1 = g_2 e_1 e_2.$$

Summing up, we see that relations (4.22)–(4.36) imply relations

$$g_i g_j = g_j g_i \quad \text{if} \quad |i-j| > 1 \tag{4.38}$$

$$g_1 g_2 g_1 g_2 = g_2 g_1 g_2 g_1 \tag{4.39}$$

$$g_i g_j g_i = g_j g_i g_j \quad \text{if} \quad |i-j| = 1, \{i,j\} \neq \{1,2\} \tag{4.40}$$

$${g_i}^2 = 1 + (u_i - 1) e_i (1 - g_i) \tag{4.41}$$

$$g_i e_j = e_j g_i \qquad \text{if} \quad |i-j| \neq 1 \quad \text{or} \quad (i,j) = (1,2) \tag{4.42}$$

$$g_j g_i e_j = e_i g_j g_i \qquad \text{if} \quad |i-j| = 1, \quad \{i,j\} \neq \{1,2\} \tag{4.43}$$

$$e_i e_j = e_j e_i \tag{4.44}$$

$${e_i}^2 = e_i \tag{4.45}$$

$$e_j e_i g_j = e_i g_j e_i = g_j e_i e_j \qquad \text{if} \quad |i-j| = 1, \quad \{i,j\} \neq \{1,2\} \tag{4.46}$$

$$e_1 e_2 g_2 = e_2 g_2 e_1 = g_2 e_1 e_2 \tag{4.47}$$

$$f_1 = e_1 \tag{4.48}$$

$$g_i f_j = f_j g_i \quad \text{if} \quad i = 1 \quad \text{or} \quad i \geq 2, j \neq i-1, i \tag{4.49}$$

$$e_i f_j = f_j e_i \tag{4.50}$$

$$f_i f_j = f_j f_i \tag{4.51}$$

$$g_i f_{i-1} = f_i g_i \quad 2 \leq i \leq n \tag{4.52}$$

$$g_i f_i = f_{i-1} g_i \quad 2 \leq i \leq n \tag{4.53}$$

$$e_{i+1} f_i = e_{i+1} f_{i+1} \quad 1 \leq i < n \tag{4.54}$$

$$e_{i+1} f_i = f_{i+1} f_i \quad 1 \leq i < n. \tag{4.55}$$

The algebra $\mathcal{E}_n^B$ generated by $\{g_i, e_i, f_i\}_{i=1,\ldots,n}$ subject to the relations (4.38)-(4.55) has been considered by Flores in [9], and called the *type B braids and ties algebra*. See Remark 2.4.4 for a direct comparison of the relation ${g_i}^2 = 1 + (u_i - 1)e_i(1 - g_i)$ given above with the relation ${g_i}^2 = 1 + (q_s - {q_s}^{-1}) e_s g_s$ in [9]. The algebra $\mathcal{E}_n^B$ admits a geometric realization, provided in [9], in the spirit of Aicardi-Juyumaya braids and ties algebra. Hence we can repeat the argument given in Example 4.3 to prove that relations (4.38)-(4.55) actually imply (4.22)-(4.36). In particular, $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ is isomorphic to $\mathcal{E}_n^B$.

5. Outlook

The construction of the algebras $\mathcal{C}_W$ has been upgraded to the case of complex reflection groups W in [13, Section 5] and further investigated in [14]. It would be interesting to relate our constructions to this upgraded definition.

We believe that our abstract approach might be useful in handling constructions on "braids and ties"-type monoids and algebras, like the framization and deframization processes treated in [5]. More in detail, let us recall the relationship between the algebra $\mathcal{E}_n$ from Example 4.3 and the Yokonuma-Hecke algebra. In [18] Yokonuma provided a presentation by generators and relations for the Hecke ring $H(G, U)$ where G is a split simple Lie group of adjoint type over $\mathbb{F}_q$, T is a fixed maximal torus and U the unipotent radical of a fixed Borel subgroup containing T. Subsequently, Juyumaya and collaborators ([11, 12]) gave a different presentation which has been successfully related to diagram algebras, as follows. Consider algebra $\mathrm{Y}_{d,n}(q)$ presented with invertible generators $g_1, \ldots, g_{n-1}$, $t_1, \ldots, t_n$ satisfying the following relations:

$$t_i t_j = t_j t_i, \quad t_i^d = 1, \tag{5.1}$$

$$t_j g_i = g_i t_{s_i(j)}, \tag{5.2}$$

$$g_i g_j = g_j g_i \quad \text{for } |i-j| > 1, \quad \text{and} \quad g_i g_j g_i = g_j g_i g_j \quad \text{for } |i-j| = 1, \tag{5.3}$$

$$g_i^2 = 1 + (q - q^{-1})\bar{e}_i g_i, \tag{5.4}$$

where

$$\bar{e}_i := \frac{1}{d}\sum_{k=0}^{d-1} t_i^k t_{i+1}^{-k}. \tag{5.5}$$

It turns out that the algebra $\mathcal{E}_n$ can be constructed by removing the framing generators t_i of the Yokonuma-Hecke algebra and by considering a new abstract generator e_i satisfying the same relations as the $\bar{e}_i$ in the Yokonuma-Hecke algebra. This process has been studied at a more abstract level of generality in [5], and it would be interesting to investigate whether all of the algebras $\mathcal{E}^{(W,S)}(R[\Sigma(\Phi_U)_{[e]}], \rho_\phi, \hat{e})$ can be obtained in general by a similar deframization process.

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