Passivity techniques and Hamiltonian structures in discrete time

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Abstract. The object of this paper is to show the impact of representing discrete-time dynamics as two coupled difference/differential equations in establishing passivity properties and describing port-Hamiltonian structures with the corresponding energy-based control strategies.

Keywords: —-

1 Introduction

Energy-based modelling and control are fundamental evergreen concepts that have been extensively investigated with a sustained impact towards the newest methodologies and technologies (see the textbooks or survey-oriented contributions [52, 1, 45, 39, 43, 46, 3]). The underlying idea consists in deducing a representation explicitly catching the energetic features of the dynamics, with the corresponding dissipation and conservative components, so to directly allow analysis and control design. In this framework, the class of port-Hamiltonian systems is paradigmatic [44, 41, 9]. From a theoretical point of view, most of this material is essentially devoted to the continuous-time setting in spite of a pervasive interest in computer-oriented applications. The general gaps in the digital framework are well-known and mostly related to computational burdens and the loss of a geometric structure underlying the evolutions [29]. In addition to these, the definition of dissipation itself provides a further source of difficulty when considering both pure discrete-time dynamics or digital systems issued from sampling [5, 4, 38, 30]. All of this said, a shared definition of discrete-time port-Hamiltonian structures has been missing. To this end, a variety of techniques proposing different discretization schemes can be found in the literature all of them with the objective of preserving energy properties and/or power-balance exchanges. To cite a few [12, 10, 49, 14, 15, 51, 50, 50, 47, 48, 7, 54, 13, 2, 8].

The present work aims at presenting recent advances in the definition of dissipative properties and the corresponding structures for discrete-time systems when represented as a couple of difference and differential equations (DDR). The idea behind the DDR consists in separating the one-step ahead discrete-time dynamics into two components: a difference equation describing the control-free evolution; a differential equation modeling the corresponding variation with respect to the control variable. This structure is propelled to cope with the intrinsic nonlinearity (that is more and more complex through iterative composition along successive time steps) in both the state and input variables of the map defining the usual state-space difference representation. Splitting the free evolution from the controlled part leads to an exponential representation of the discrete-time flow that is useful for further representing the explicit evolution along successive time steps, through the composition of exponential forms. In addition, when considering input-to-output evolutions, the DDR structure recovers an exponential form representation of the Volterra series expansions. Section 2 recalls these developments as useful prolegomena to the paper while details can be found in [24, 25, 27, 32].

The aforementioned representation is then used to characterize passivity. This notion, and more in general the one of dissipativity, relies upon energy-exchanges and the way the system interacts with the environment. Roughly speaking, a system is passive when the internal stored energy does not exceed the one that has been externally supplied [52, 53]. This property is caught by characterizing the variation of a particular function along the system's trajectories [43, 46, 39]. Such a function, referred to as storage function, generally represents the energy and is strictly linked to Lyapunov and/or Hamiltonian functions. With this in mind and for a fixed storage, the DDR representation we propose immediately allows to characterize average passivity [30]. This is done by isolating the control dependent part and thus defining in a very natural way the corresponding average passive output that is, in turn, a conjugate quantity whose product with the control variable is a power unit. The notion of average dissipation is then introduced and exploited for control purposes by extending the usual concept of negative output damping to cope with stabilization at the origin. This is then generalized to describe Passivity-Based Control (PBC) at large for nonlinear discrete-time systems while opening toward the so-called second generation of PBCs, aimed at managing the system energy to cope with the given control specification. Those endow Energy Balancing (EB) and Interconnection and Damping Assignment (IDA) and allow to deal with more general systems and control problems including networked or cascade dynamics. The foundations of average passivity-based techniques are recalled in Section 3 while more detailed studies are in [33, 31].

As mentioned above, port-Hamiltonian dynamics are of extreme interest due to the mathematical structure and the corresponding foundations in physics [16, 17, 18, 40, 44, 9, 42]. Hereinafter, a novel state-space representation of discrete-time port-Hamiltonian dynamics is naturally defined when referring to the DDR form. The proposed pH forms are endowed with the corresponding average passivity properties that are highlighted by the corresponding conjugate output. Thus, they represent a breakthrough in the literature as suitably shaped to verify the fundamental characteristics of port-Hamiltonian structures at large. In particular, closeness under power-preserving interconnection is validated [36]. In addition, their impact in PBC for complex discrete-time systems is shown [37, 20, 21]. These Hamiltonian structures as well as their control properties are discussed in Section 4 while more specific studies are in [35, 37].

1.1 Notations

Throughout the paper all the functions and vector fields defining the dynamics are assumed smooth and complete over the respective definition spaces. The sets $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers including 0 respectively. For any vector $v \in \mathbb{R}^n$, $|v|$ and $v^\top$ define the norm and transpose of v respectively. I_d denotes the identity function on the definition space while I denotes the identity operator and the identity matrix when related to a linear operator. $\circ$ denotes the composition of two functions or operators, depending on the context. Given a real-valued function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ assumed differentiable, ∇V represents the gradient column-vector with $\nabla = col\{\frac{\partial}{\partial x_i}\}_{i=1,n}$. Given a vector-valued function $F(x) = col(F_1(x), \ldots, F_n(x))$, $J_x[F](x) = \{\frac{\partial F_i}{\partial x_j}(x)\}_{i,j=1,n}$ denotes the Jacobian of the function F evaluated at x.

The symbols ">0" and "<0" denote positive and negative definite functions (or matrices), respectively. Given a smooth vector field over $\mathbb{R}^n$, e^{L_f} (indifferently e^f) denotes the exponential Lie operator $e^{\mathrm{L}_f} := I + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$ in the Lie operator $\mathrm{L}_f = \sum_{i=1}^n f_i(x)\frac{\partial}{\partial x_i}$ and L_f^i indicates the power i operator with respect to the usual composition of vector fields (for a linear vector field, the exponential Lie operator recovers the exponential of the matrix representing the operator). For any smooth function $h : \mathbb{R}^n \to \mathbb{R}$ then one verifies $e^f h(x) = h(e^f(x)) = e^f h|_x$ where $|_x$ denotes the evaluation of the function at x. Given two vector fields f, g the Lie bracket is denoted by $ad_f g = [f, g] = \mathrm{L}_f\mathrm{L}_g - \mathrm{L}_g\mathrm{L}_f$ and, iteratively for $i \geq 1$, $ad_f^i g = ad_f^{i-1} \circ ad_f g$ with $ad_f^1 g = ad_f g$. A function $R(x, u) = O(u^p)$ is said in $O(u^p)$ with u^p for all $p \geq 1$ if whenever it is defined, it can be written as $R(x, u) = u^{p-1}\tilde{R}(x, u)$ and there exist a function $\theta \in \mathcal{K}_\infty$ and $u^\star > 0$ such that $\forall u \leq u^\star$, $|\tilde{R}(x, u)| \leq \theta(u)$.

2 Basic on discrete-time dynamics

2.1 Differential/Difference Representation

A single-input nonlinear discrete-time dynamics over $\mathbb{R}^n$ is usually represented by a map $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, smooth in both variables,

$$x_{k+1} = F(x_k, u_k) = x_k + F_0(x_k) + g(x_k, u_k)u_k. \tag{1}$$

For any pair of state and input variables (x_k, u_k) fixed at time instant k, x_{k+1} denotes the state reached at time $k + 1$ from x_k, under the action of the control

u_k. For convenience that will be clear later on and without loss of generality, the free evolution is written as $x + F_0(x) = F(x, 0)$ while $g(x, u)u$ represents the controlled part of the dynamics that is not assumed linear in u.

In [27], we proposed to represent a discrete dynamics via two coupled difference and differential equations. More in detail, for all time steps $k \in \mathbb{N}$, (1) can be represented under mild conditions (e.g., submersivity of $F(x, u)$) by two coupled difference and differential equations (DDR) of the form

$$x^+ = x + F_0(x) \tag{2a}$$

$$\frac{dx^+(u)}{du} = G(x^+(u), u) \quad \text{with} \quad x^+(0) = x^+ \tag{2b}$$

when denoting $x = x_k \in \mathbb{R}^n$, $u = u_k \in \mathbb{R}$, $x^+(u) = x^+(u_k) = x_{k+1}$ and $x^+ = x^+(0)$, that is the one step ahead state in free evolution. We underline that $x^+(u)$ represents a curve in $\mathbb{R}^n$, parameterized by the control variable $u \in \mathbb{R}$ and $G(\cdot, u)$, a vector field on $\mathbb{R}^n$, parameterized by u, satisfying $G(F(x, u), u) = \frac{\partial F}{\partial u}(x, u)$. Accordingly, the difference equation (2a) describes the jump under the drift while the differential equation (2b) models the rate of change of the state dynamics with respect to control variation.

Remark 1. Provided that the drift $F(x, 0)$ has an inverse as function of the state, $G(\cdot, u)$ can be uniquely defined for u sufficiently small as

$$G(x, u) := \left.\frac{\partial F(x, u)}{\partial u}\right|_{x = F^{-1}(x,u)} \tag{3}$$

through a family of vector fields (G_i)'s on $\mathbb{R}^n$, deduced from the series expansion

$$G(x, u) = G_1(x) + \sum_{i \geq 1} \frac{u^i}{i!} G_{i+1}(x). \tag{4}$$

The vector fields (G_i)s describe the structure of the associated dynamics and can be employed to characterize the accessibility, invariance or decoupling properties in a geometric framework as studied in [24, 25, 27, 6, 28].

Some remarks and comments better explain standard manipulations over these representations.

Remark 2. Particular classes of systems can be discussed. For example, when setting $g(x, u) = g(x)$ in (1), one gets an input-affine dynamics in the map form. In that case, the associated control vector field $G(x, u)$ in (4) still non linearly depends on u but satisfies the algebraic equality

$$G(F(x, u), u) = g(x).$$

On the other hand, assuming the differential dynamics (2b) input-affine by setting $G(x,u) = G_1(x)$, the associated dynamics in map form (1) is generally nonlinear in u and verifies the exponential form

$$F(x,u) = e^{uG_1}(x)\Big|_{x+F_0(x)}$$

where $e^{uG_1}(x)$ represents the flow characterizing the solution to the differential equation (2b).

Remark 3. The DDR representation can be generalized to multi-input dynamics through modeling the rate of change of the dynamics under the action of each control separately. Setting $\boldsymbol{u} = (u^1, \dots, u^m)^\top$, one replaces (2b) by the set of partial derivative equations of motion

$$\frac{\partial x^+(\boldsymbol{u})}{\partial u^j} = G^j(x^+(\boldsymbol{u}), \boldsymbol{u}); \quad j = (1, \dots, m \quad \text{with} \quad x^+(0) = x^+ \tag{5}$$

with $G^j(x, \boldsymbol{u})$ satisfying the condition $G^j(F(x,\boldsymbol{u}), \boldsymbol{u}) := \frac{\partial F(x,\boldsymbol{u})}{\partial u^j}$.

Easy manipulations show how these DDR forms are transformed under coordinates change and feedback [28].

Lemma 1. *Let the coordinates change $z = T(x)$ defined by the diffeomorphism with $T : \mathbb{R}^n \to \mathbb{R}^n$, then the DDR dynamics (2) transforms into*

$$\begin{aligned} z^+ &= z + \bar{F}_0(z) \\ \tfrac{dz^+(u)}{du} &= \bar{G}(z^+(u), u) \quad \textit{with} \quad z^+(0) = z^+ \end{aligned} \tag{6}$$

with

$$\begin{aligned} z + \bar{F}_0(z) &= T(x + F_0(x))\Big|_{x=T^{-1}(z)} \\ \bar{G}(z,u) &= Ad_T G(z,u) \end{aligned}$$

where $Ad_T G(\cdot, u)$ indicates the transport of the vector field $G(\cdot, u)$ along T; i.e.

$$Ad_T G(z,u) = \left[\tfrac{\partial T}{\partial x} G(\cdot, u)\right]_{x=T^{-1}(z)} = \mathrm{L}_{G(\cdot,u)} T(x)\big|_{x=T^{-1}(z)}.$$

Lemma 2. *Consider the state feedback $u(x,v) = \alpha(x) + v$ with $\alpha : \mathbb{R}^n \to \mathbb{R}$ smooth. Then the controlled DDR dynamics (2) is given by*

$$\begin{aligned} x^+_\alpha &= x + F_\alpha(x) \\ \tfrac{dx^+_\alpha(v)}{dv} &= G_\alpha(x^+_\alpha(v), v), \qquad x^+_\alpha = x^+(\alpha(x)) \end{aligned} \tag{7}$$

where

$$\begin{aligned} x^+_\alpha(v) &= F_\alpha(x,v) = F(x,\alpha(x)+v)\\ x + F_\alpha(x) &= F(x,\alpha(x))\\ G_\alpha(x,v) &= G(x,\alpha(x)+v) \end{aligned}$$

so that one obtains

$$\frac{dx^+_\alpha(v)}{dv} = \frac{\partial F(x,u)}{\partial u}\Big|_{u=\alpha(x)+v}\frac{\partial(\alpha(x)+v)}{\partial v} = G(x^+_\alpha(v),\alpha(x)+v) = G_\alpha(x^+_\alpha(v),v).$$

2.2 Input-to-state and input-to-output trajectories along time

Given a pair (x_k, u_k) of state and control variables at any generic time-instant $k \in \mathbb{N}$, integrating (2b) from 0 and u_k with initial condition $x^+(0) = x_k + F_0(x_k)$, one recovers a dynamics in the form of a map (1); i.e.

$$\begin{aligned} x^+(u_k) &= x_k + F_0(x_k) + u_k g(x_k,u_k)\\ &= x_k + F_0(x_k) + \int_0^{u_k} G(x^+(v),v)\mathrm{d}v. \end{aligned} \tag{8}$$

The following exponential forms resulting from integration w.r.t. u hold true.

Theorem 1. *Between two successive time steps* $\{k, k+1\}$ *, one gets*

$$\begin{aligned} x^+(u_k) &= e^{u_k\mathcal{G}(\cdot,u_k)} I_d|_{x_k+F_0(x_k)} = F(x_k,u_k)\\ h(x^+(u_k)) &= e^{u_k\mathcal{G}(\cdot,u_k)} h|_{x_k+F_0(x_k)} = h(F(x_k,u_k)) \end{aligned}$$

where the series exponent is a Lie series $\mathcal{G}(\cdot,u) \in Lie\{G_1,\ldots,G_p,\ldots\}$*, described in [24, 32] through its expansions in powers of* u*. For the first term one gets*

$$\mathcal{G}(\cdot,u) = G_1 + \frac{u}{2}G_2 + \frac{u^2}{3!}(G_3 + [G_1,G_2]) + O(u^3).$$

The proof developed in [32] follows from Lie properties endowed by the flow characterizing the solutions to the nonlinear u-dependent ordinary differential equations (2b).

A major property of these exponential representations of discrete-time flows is to be easily expanded with respect to the control variable and its successive powers. The coefficients of the power terms u^i for $i \geq 1$ in this infinite series expansion are so described in terms of the vector fields $G_{j>0}$ defined in (4) and their Lie brackets. Further on, under the action of a sequence of controls $u_0,\ldots,u_k$, the state or output behaviours along several time steps can be represented through the usual composition of exponential operators so getting, at any generic time instant $k+1>0$, the expressions below.

Proposition 1. *For all $x_0 \in \mathbb{R}^n$, input sequences $\{u_0, \ldots, u_k\}$, $k \in \mathbb{N}$*

$$x_{k+1} = x^+(u_k, \ldots, u_0) = e^{u_0 \mathcal{G}^k(\cdot, u_0)} \circ \cdots \circ e^{u_k \mathcal{G}^0(\cdot, u_k)}|_{(I_d+F_0)^{k+1}(x_0)}$$
$$y_{k+1} = y^+(u_k, \ldots, u_0) = e^{u_0 \mathcal{G}^k(\cdot, u_0)} \circ \cdots \circ e^{u_k \mathcal{G}^0(\cdot, u_k)}(h)|_{(I_d+F_0)^{k+1}(x_0)}$$

and, for $j \geq 0$, the series exponents

$$\mathcal{G}^j(\cdot, u) \in Lie\{G_1^j, \ldots, G_p^j, \ldots\}$$

analogously defined with respect to the transported vector fields $G_i^j = Ad^j_{I_d+F_0} G_i$ with $G_i^0 := G_i$.

We underline that, when composing the dynamics along successive time steps, the vector fields $(G_{i>0})$s characterizing the flow are iteratively transported along the free dynamics $x + F_0(x)$ so defining for $j \geq 1$ and according to Lemma 1 the family of vector fields $\{G^j, \text{ with } j > 0\}$ for $j \geq 1$ with

$$G_i^{j+1}(x) = Ad_{I_d+F_0} G_i^j(x) = \mathrm{L}_{G_i^j}(x + F_0(x))\Big|_{(I_d+F_0)^{-1}(x)}$$

and iteratively $G_i^{j+1}(x) = Ad^{j+1}_{I_d+F_0} G_i(x)$.

2.3 Concluding comments

In this section, an alternative description for discrete-time dynamics has been proposed to usual forms of discrete-time dynamics in map form. In doing so, the free evolution has been isolated from the control depend part of the dynamics so exhibiting a family of control vector fields, the (G_i)s that enter in the characterization of the structural and control properties of the dynamics. This representation inspires a modeling approach that would separately identify the free evolution as a map F_0 and the variation with respect to u in terms of these controlled vector fields. Further on, the state dynamics has been rewritten in terms of the exponential operator characterizing the flow associated to the solution to an ordinary differential equation. The whole differential geometry apparatus behind this operator form can be used to describe the behaviours of the input-to-state and input-to-output behaviours along successive time steps. In particular, the composition of nonlinear maps is replaced with the composition of exponential flows parameterized by the control values along time instants that are more tractable in practice. This is well illustrated with the characterization of the Volterra series and the computation of the successive Volterra kernels in terms of the vector fields (G_i^j)s and their Lie brackets. We refer to [23, 26] for further discussion along these lines in relation with realization problems.

3 Passivity techniques

The notion of average passivity has been introduced in [31] to weaken the necessary requirement of direct throughput when adopting the standard notion of passivity in a discrete-time framework. This new definition is in fact an immediate consequence of splitting the state dynamics into free and control parts. Passivity Based Control (PBC) techniques that can be developed according to this new definition are recalled below.

3.1 Average passivity

Denoting by $\Sigma_d(h)$ a generic discrete-time system described by the dynamics (1) (equivalently (2)) with output $h : \mathbb{R}^n \to \mathbb{R}$, the following definition is recalled from [31].

Definition 1 (Average passivity). *$\Sigma_d(h)$ is said average passive if there exists a positive semi-definite function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (the storage function) such that for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$ the following inequality holds*

$$S(x^+(u)) - S(x) \leq \int_0^u h(x^+(s))\mathrm{d}s := uh^{av}(x, u) \tag{9}$$

with the average output defined as

$$h^{av}(x, u) := \frac{1}{u}\int_0^u h(x^+(v))\mathrm{d}v. \tag{10}$$

This definition is inspired by the DDR representation of the dynamics that enables us to rewrite the rate of change of the storage function between two successive states as

$$S(x^+(u)) - S(x) = S(x^+(0)) - S(x) + \int_0^u S(x^+(s))\mathrm{d}s.$$

The so-defined average map $h^{av}(\cdot, u)$ introduces a direct input-output link in such a way that average passivity with respect to h is in fact equivalent to usual passivity with respect to $h^{av}(\cdot, u)$.

More in general average dissipativity can be defined in discrete time making reference to this average output and a supply rate function $s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ so requiring that the dissipation inequality be verified for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$

$$S(x^+(u)) - S(x) \leq s(u, h^{\mathrm{av}}(x, u)).$$

As an immediate consequence of average passivity, setting $u = 0$ in (9), one verifies $S(x^+(0)) - S(x) \leq 0$ so concluding stability of any equilibrium $x_e \in \mathbb{R}^n$ when S qualifies as a Lyapunov function at x_e ($S(x_e) = 0$ and $S(x) > 0$ for $x \neq x_e$) and asymptotic stability provided $S(x^+(0)) - S(x) < 0$.

More generally, the notion of average passivity from some nominal non-zero constant value $\bar{u}$ is recalled from [19, 36]. It is instrumental when discussing the action of feedback law over passivity.

Definition 2 (Average passivity from $\bar{u}$). *$\Sigma_d(h)$ is said* average passive from a given $\bar{u}$ *with $\bar{u} \in U \subseteq \mathbb{R}$, if there exists a positive semi-definite function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (the storage function) such that, for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$*

$$S(x^+_{\bar{u}}(u)) - S(x) \leq u h^{av}_{\bar{u}}(x, u) \tag{11}$$

with u-average output from $\bar{u}$ defined as

$$h^{av}_{\bar{u}}(x, u) := \frac{1}{u} \int_0^u h(x^+(\bar{u} + v))\mathrm{d}v.$$

When $\bar{u} = 0$, one recovers the u-average passivity and $h^{av}_0(x, u) = h^{av}(x, u)$.

3.2 Feedback stabilization and interconnection

Two basic ingredients of control design strategies exploiting passivity are revisited hereafter making reference to the average notion. First, stabilization through negative output feedback and additional damping is specified. Secondly, to enlarge the control action to the interconnection of passive systems, closeness under power preserving input-output interconnection is discussed. The next definition is instrumental.

Definition 3 (Zero-state detectability). *Given $\Sigma_d(h)$ let $x_e \in \mathbb{R}^n$ be an equilibrium and $\mathcal{Z} \in \mathbb{R}^n$ be the largest invariant set contained in the set $\{x \in \mathbb{R}^n$ s.t. $h(x^+(0), 0) = 0\}$. $\Sigma_d(h)$ is said zero-state detectable (ZSD) if $x = x_e$ is an asymptotically stable equilibrium conditionally to $\mathcal{Z}$. tiv*

The following theorem characterizes negative output damping feedback laws.

Theorem 2 (Negative average output feedback). *Given $\Sigma_d(h)$ with equilibrium $x_e \in \mathbb{R}^n$ ($F_0(x_e) = 0$) assumed average passive with storage function $S > 0$, then the feedback $u = \alpha(x)$ solving the algebraic equality*

$$u = -\kappa h^{av}(x, u), \quad \kappa > 0 \tag{12}$$

ensures asymptotic stability provided that $\Sigma_d(h)$ is Zero State Detectability (ZSD). Moreover, setting $u(x, v) = \alpha(x) + v$ with external control $v \in \mathbb{R}$, then the closed loop dynamics is average passive again with respect to h

$$S(x^+_\alpha(v)) - S(x) \leq v h^{av}_\alpha(x, v). \tag{13}$$

To conclude average passivity of the closed-loop dynamics it is sufficient to note that under the feedback $u(x, v) = \alpha(x) + v$, the average output associated to h along the closed-loop dynamics described in Lemma 2 recovers the average output from $\bar{u} = \alpha(x)$ defined in (2). In fact one gets

$$h^{av}_\alpha(x, v) := \frac{1}{v} \int_0^v h(x^+_\alpha(w))\mathrm{d}w.$$

Then by definition of $\alpha(x)$, one gets

$$\begin{aligned}\int_0^{\alpha(x)+v} h(x^+(w))\mathrm{d}w &= \int_0^{\alpha(x)} h(x^+(w))\mathrm{d}w + \int_{\alpha(x)}^{\alpha(x)+v} h(x^+(w))\mathrm{d}w \\ &= \int_0^{\alpha(x)} h(x^+(w))\mathrm{d}w + \int_0^{v} h(x^+(\alpha(x)+w))\mathrm{d}w \\ &= -\kappa(h^{av}(x,\alpha(x)))^2 + \int_0^{v} h(x^+(\alpha(x)+w))\mathrm{d}w \\ &\leq v h^{av}_{\alpha}(x,v)\end{aligned}$$

so concluding that average passivity from $\alpha(x)$ coincides with average passivity under preliminary feedback $\alpha(x)$ with respect to the same output map h.

This negative average output feedback is at the basis of stabilizing strategy through additional damping or average PBC feedback. Its computation requires solving the nonlinear algebraic equality (12) which may be a difficult task but can be performed according to suitable approximation methods as proposed in [19, 20].

The second fundamental property to verify regards the interconnection of two average passive systems through their respective input and output variables setting $u = \phi(h^{av}(x,u))$. Given two average passive systems $\Sigma_d^i(h^i)$ with storage function S^i for $i = 1, 2$, a power-preserving input-output interconnection making reference to the so defined average outputs is naturally defined in [36] as follows.

Definition 4 (Power preserving interconnection). *The input-output interconnection between $\Sigma_d(h^1)$ and $\Sigma_d(h^2)$ is said* power preserving *if it satisfies the integral equality*

$$\int_0^{u^1} h^1({x^1}^+(w))\mathrm{d}w + \int_0^{u^2} h^2({x^2}^+(w))\mathrm{d}w = 0 \tag{14}$$

equivalently rewritten

$$u^1 {h^1}^{av}(x^1,u^1) + u^2 {h^2}^{av}(x^2,u^2) = 0. \tag{15}$$

We easily note that the simplest way to solve (15) is to set

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \phi \begin{pmatrix} {h^1}^{av}(x^1,u^1) \\ {h^2}^{av}(x^2,u^2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} {h^1}^{av}(x^1,u^1) \\ {h^2}^{av}(x^2,u^2) \end{pmatrix} \tag{16}$$

so recovering the classical power preserving interconnection expressed with respect to the average outputs. The solution to the implicit equality (15) defines a preliminary power preserving state-feedback that we denote $\alpha(x) = (\alpha^1(x), \alpha^2(x))^\top$ with $x = (x^1, x^2)^\top$. The following Theorem can be stated.

Theorem 3 (Average passivity under power preserving interconnection). *Let, for $i = 1, 2$, the systems $\Sigma_d(h^i)$ be average passive with respective storage functions S^i. Let $\alpha(x)$ be the power-preserving interconnection satisfying (16) and set $u = \alpha(x) + v$ with external control $v = (v^1, v^2)^\top$. Then, the interconnected system*

$$x^{1}{}^{+}_{\alpha^1}(v^1) = F^1_{\alpha^1}(x, v^1) \tag{17a}$$

$$x^2_{\alpha^2}{}^{+}(v^2) = F^2_{\alpha^2}(x, v^2) \tag{17b}$$

with output $h = (h^1, h^2)^\top$ is average passive with storage function $S(x) := S^1(x^1) + S^2(x^2)$. Namely, the dissipation inequality holds; i.e.

$$S^1(x^{1}{}^{+}_{\alpha^1}(v^1)) - S^1(x^1) + S^2(x^2_{\alpha^2}{}^{+}(v^2)) - S^2(x^2) \leq v^\top h^{av}_\alpha(x, v) \tag{18}$$

with average output of the closed loop dynamics defined as

$$h^{av}_\alpha(x, v) = \begin{pmatrix} \frac{1}{v^1}\int_0^{v^1} h^1(x^1_{\alpha^1}{}^{+}(w))\mathrm{d}w \\ \frac{1}{v^2}\int_0^{v^2} h^2(x^2_{\alpha^2}{}^{+}(w))\mathrm{d}w \end{pmatrix}. \tag{19}$$

The proof detailed in [36] works out in two steps. First one concludes that under power-preserving input-output interconnection, the interconnected system is average passive from $\alpha(x)$ according to the Definition 2. Then because of the feedback structure, average passivity from α recovers average passivity of the dynamics under preliminary feedback $\alpha(x)$ so concluding the claim.

3.3 Passivating output map

The previous arguments show that stabilization under feedback can be achieved by exploiting average passivity and through a suitable energy exchange between passive systems. However, stabilization to target equilibria that are local extrema of suitably shaped energy functions can be challenging. For, average passivity with respect to some dummy output function is required. The following Lemma is instrumental for the stabilization of cascade systems. It specifies such dummy output function that preserves average passivity and is zero at the local minima of the storage function. It is reminiscent of the continuous-time context (see [31] for details).

Lemma 3 (Average passivating output). *Let $\Sigma_d(h)$ be average passive with storage function S, then it is average passive with respect to the dummy output function*

$$h(x, u) = \mathrm{L}_{G(\cdot, u)} S(x). \tag{20}$$

$h(\cdot, u)$, defined as the Lie derivative of S along $G(\cdot, u)$, is referred to as an average passivating output because it satisfies the Energy Balance (EB) equality below

$$\underbrace{S(x^+(u)) - S(x)}_{\text{stored energy}} = \underbrace{S(x^+) - S(x)}_{\text{dissipated energy}} + \underbrace{S(x^+(u)) - S(x^+)}_{\text{supplied energy}}$$

so concluding average passivity with respect to the output (20) since

$$S(x^+(u)) - S(x^+(0)) := u\Big(\mathrm{L}_{G(\cdot,u)}S\Big)^{av}(x,u)$$

with $\Big(\mathrm{L}_{G(\cdot,u)}S\Big)^{av}(x,u) := \frac{1}{u}\int_0^u \mathrm{L}_{G(\cdot,w)}S(x^+(w))\mathrm{d}w$ and $S(x^+(0)) - S(x) \leq 0$ from average passivity assumption of $\Sigma_d(h)$.

Remark 4. Lemma 3 generalizes to assuming $\Sigma_d(h)$ average passive from a given $\bar{u}$. In that case, average passivity from $\bar{u}$ with respect to $h(\cdot,u)$ follows too

$$S(x^+(\bar{u}+u)) - S(x) \leq S(x^+(\bar{u}+u)) - S(x^+(\bar{u})) = u\Big(\mathrm{L}_{G_{\bar{u}}(\cdot,u)}S\Big)^{av}(x,u)$$

because by assumption $S(x^+(\bar{u})) - S(x) \leq 0$ and by definition

$$\Big(\mathrm{L}_{G_{\bar{u}}(\cdot,u)}S\Big)^{av}(x,u) := \frac{1}{u}\int_0^u \mathrm{L}_{G(\cdot,\bar{u}+w)}S(x^+(\bar{u}+w))\mathrm{d}w$$

with $G_{\bar{u}}(\cdot,w) = G(\cdot,\bar{u}+w)$.

The stabilizing result in Theorem 2 can be specified on such passivating output so defining by $\alpha(x)$ the feedback law solution to the damping equality

$$u = -\kappa\Big(\mathrm{L}_{G(\cdot,u)}S\Big)^{av}(x) \qquad \kappa > 0.$$

Then, setting $u = \alpha(x) + v$, the closed loop dynamics with output $\mathrm{L}_{G_\alpha(\cdot,v)}S$ remains average passive.

Remark 5. For computational facility, approximations of these stabilizing feedbacks are described so getting for $\alpha(x)$ at the first order around $u = 0$, $\alpha_{ap}(x) = -\kappa\lambda(x)\mathrm{L}_{G_1}S(x + F_0(x))$ with a suitably tuned gain $\lambda(\cdot) > 0$ as discussed in [19].

In the present paper, oriented to characterize Hamiltonian dynamics in discrete time, it is instrumental to rely the passivating output on its discrete gradient.

The definition below is recalled from [22].

Definition 5 (Discrete gradient function). *Given a smooth real-valued function $S:\mathbb{R}^n \to \mathbb{R}$, its* discrete gradient *is a function of two variables $\bar{\nabla}S|_x^z : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying for all $x, z \in \mathbb{R}^n$*

$$(z-x)^\top \bar{\nabla}S|_x^z = S(z) - S(x) \quad \textit{with} \quad \lim_{z\to x} \bar{\nabla}S|_x^z = \nabla S(x). \tag{21}$$

Definition 5 properly states that the discrete gradient function satisfying (21) describes the rate of change of this function between two states and is not uniquely defined as different methods to solve the equality can be worked out; see [10, 22, 11]. Through component-wise integration, one gets the computable expression below

$$\bar{\nabla} S|_x^z = \left[\bar{\nabla}_1 S|_{x_1}^{z_1} \ \dots \ \bar{\nabla}_n S|_{x_n}^{z_n}\right]^\top$$

with

$$\bar{\nabla}_i S|_{x_i}^{z_i} = \frac{1}{z_i - x_i} \int_{x_i}^{z_i} \frac{\partial S(x_1, ..., x_{i-1}, s, z_{i+1}, ..., z_n)}{\partial s} \mathrm{d}s.$$

Remark 6. When $S(x) = \frac{1}{2} x^\top P x$ with $P \in \mathbb{R}^{n\times n}$, the discrete gradient is uniquely expressed as

$$\bar{\nabla} S|_x^z = \frac{1}{2} P(x+z). \tag{22}$$

By definition of the discrete-gradient function, it results that the rate of change of the storage function S in the Definition 1 along the dynamics (1) (equivalently (2)) can be described through its discrete gradient and can be further split into its control free and control dependent parts applying the property below

$$\begin{aligned} S(x^+(u)) - S(x) &= \big(S(x^+) - S(x)\big) + \big(S(x^+(u)) - S(x^+)\big) \\ &= \underbrace{S(x^+) - S(x)}_{=F_0^\top(x)\bar{\nabla} S|_x^{x^+}} + \underbrace{\int_0^u \mathrm{L}_{G(\cdot,v)} S(x^+(v))\mathrm{d}v}_{=ug^\top(x,u)\bar{\nabla} S|_{x^+}^{x^+(u)}} . \end{aligned}$$

The following Lemma describes the passivating output defined in (20) in terms of the discrete gradient function. It is instrumental in the sequel.

Lemma 4 (Average passivating output in discrete gradient form). *Given the dynamics* (1) *and a real-valued smooth function map* $S : \mathbb{R}^n \to \mathbb{R}$*, the following equalities hold describing the average passivating output in discrete gradient form. One gets*

$$\begin{aligned} S(x^+) - S(x) = S(x + F_0(x)) - S(x) = F_0^\top(x)\bar{\nabla} S|_x^{x^+} \\ S(x^+(u)) - S(x^+) = S(F(x,u)) - S(x + F_0(x)) = ug^\top(x,u)\bar{\nabla} S|_{x^+}^{x^+(u)}. \end{aligned}$$

with the relation

$$ug^\top(x,u)\bar{\nabla} S|_{x^+}^{x^+(u)} = \int_0^u \mathrm{L}_{G(\cdot,v)} S(x^+(v))\mathrm{d}v = u\Big(\mathrm{L}_{G(\cdot,u)} S\Big)^{av}(x,u). \tag{23}$$

Example 1. The discrete integrator

$$x^+(u) = x + u; \quad y = h(x) = x$$

is the simplest storage element. Setting $S(x) = \frac{1}{2}x^2$, as storage function, the system is average passive as

$$S(x^+(u)) - S(x) = xu + \frac{1}{2}u^2 = \int_0^u x^+(v)dv = \int_0^u (x+v)dv = uh^{av}(x,u)$$

with $h^{av}(x,u) = x + \frac{1}{2}u$. Accordingly, one conclude passivity of the input-state-output system

$$x^+(u) = x + u; \quad h^{av}(x,u) = x + \frac{1}{2}u.$$

The associated negative average output feedback satisfies the algebraic equality $u = -\kappa(x + \frac{1}{2}u)$ so computing $u = -\frac{\kappa}{1+\frac{\kappa}{2}}x; \kappa > 0$ that recovers negative output feedback with suitably shaped gain.

3.4 Concluding comments

The introduced notion of average passivity allows the design of discrete-time PBC strategies through damping and interconnection for stabilization purposes. In addition, Lemma 3 introduces a passivating output and its related negative average output feedback so enlarging the control objectives to stabilization through cascade procedures such as backstepping or feedforward strategies [19]. Further on, the second generation of PBCs includes an energy-shaping component aimed at shaping the energy of the system to fulfil the required control specification. Those control strategies include Interconnection and Damping Assignment (IDA), when one also wants to modify the internal pH structure to assign a new equilibrium, or Control by Interconnection (CbI) to manage energy exchanges through an interconnection pattern. Preliminary works in this direction are in [35, 36, 37, 20]. Finally, Lemma 4, which relies the average passivating output to its discrete gradient form, directly inspires the novel port-control Hamiltonian structure we propose in the next section.

4 Port-Hamiltonian structures in discrete time

Port-Hamiltonian structures have a pervasive impact in numerous applied domains enlarging the more traditional mechanical one. These structures are more essentially described in the continuous-time domain while in discrete time a consensus on a specific structure is not reached in spite of a rich literature. In this section, a novel description of port-Hamiltonian structures is proposed exploiting the DDR representation and the introduced average passivating output map.

4.1 Control-free port-Hamiltonian dynamics

From the literature, we first recall the definition of a control-free port-Hamiltonian structure that is quite standard as it extends to the discrete-time framework the analogue continuous-time structure when substituting the gradient function with the discrete gradient function. Let us assume that H, a smooth real-valued function $H : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, denotes the Hamiltonian function.

Definition 6. *A control-free discrete-time port-Hamiltonian dynamics over* $\mathbb{R}^n$ *can be described by the first-order difference equation*

$$x^+ - x = (J(x) - R(x))\bar{\nabla} H|_x^{x^+} \tag{24a}$$

where $J(x) = -J^\top(x)$, $R(x) = R^\top(x) \succeq 0$, *are matrices of functions representing the interconnection and resistive parts respectively.*

By construction, one immediately verifies that:
- any local extremum of $H(x)$ ($\bar{\nabla} H|_{x_e}^{x_e} = \nabla H(x_e) = 0$) is an equilibrium;
- the rate of change of the Hamiltonian along the dynamics satisfies the equality

$$H(x^+) - H(x) = \underbrace{-\bar{\nabla} H^\top|_x^{x^+} J(x) \bar{\nabla} H|_x^{x^+}}_{=0} \underbrace{-\bar{\nabla} H^\top|_x^{x^+} R(x) \bar{\nabla} H|_x^{x^+}}_{\leq 0}.$$

Taking the sum of these increments, energy dissipation from time 0 to time k is described through the equality

$$H(x_k) - H(x) = -\underbrace{\sum_{i=0}^{k-1} \bar{\nabla} H^\top|_{x_i}^{x_i^+} R(x_i) \bar{\nabla} H|_{x_i}^{x_i^+}}_{\text{dissipated energy} \leq 0}.$$

- When $J(x) = 0$, the dynamics is dissipative. The simplest example is the gradient dynamics defined with $R(x) = I$ and $J(x) = 0$ that is

$$x^+ - x = -\bar{\nabla} H|_x^{x^+}$$

satisfying

$$H(x^+) - H(x) = -\bar{\nabla} H^\top|_x^{x^+} \bar{\nabla} H|_x^{x^+} = -||\bar{\nabla} H|_x^{x^+}||_2 \leq 0.$$

- When $R(x) = 0$, the dynamics is conservative

$$H(x^+) = H(x)$$

so concluding that the Hamiltonian is a constant of motion of the so-defined dynamics.

Canonical discrete Hamiltonian dynamics Hamiltonian dynamics are defined over $\mathbb{R}^{2n}$ and specified by setting as skew symmetric interconnection matrix

$$J_c = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.$$

Setting $x = (x^1, x^2)^\top$ with $x^i \in \mathbb{R}^n$ for $i = 1, 2$ and $\bar{\nabla} H|_x^{x^+} = (\bar{\nabla}_1 H|_{x^1}^{x^{1+}}, \bar{\nabla}_2 H|_{x^2}^{x^{2+}})^\top$, one denotes by $\bar{X}_H$, the canonical discrete Hamiltonian vector field associated with a given H satisfying

$$\begin{pmatrix} x^{1+} - x^1 \\ x^{2+} - x^2 \end{pmatrix} = J \bar{\nabla} H|_x^{x^+} = \begin{pmatrix} \bar{\nabla}_2 H|_{x^2}^{x^{2+}} \\ -\bar{\nabla}_1 H|_{x^1}^{x^{1+}} \end{pmatrix} := \bar{X}_H \tag{25}$$

For completeness, we note that for a given Hamiltonian function over R^{2n}, the canonical Hamiltonian dynamics is the solution for all v of the symplectic condition

$$\Omega(x^+ - x, v) = \left\langle \bar{\nabla} H|_x^{x^+}, v \right\rangle$$

where $\Omega(u, v) =< u, J_c v >$ is the usual symplectic form. In fact, easy computations show that this equality rewritten as $< (x^+ - x), J_c v >= \left\langle \bar{\nabla} H(x), v \right\rangle$ is solved by setting $x^+ - x = \bar{X}_H$ defined in (25).

Further on, for any given real-valued smooth function $C : \mathbb{R}^{2n} \to \mathbb{R}$, its rate of change along the trajectories generated by the Hamiltonian dynamics $\bar{X}_H$ is given by

$$C(x^+) - C(x) = \{C, H\}_D$$

where $\{C, H\}_D$ indicates the discrete Poisson bracket defined with respect to the discrete gradient as follows

$$\{C, H\}_D := \sum_{i=1}^{n} \bar{\nabla}_{1i} C|_{x^{1i}}^{x^{1i+}} \bar{\nabla}_{2i} H|_{x^{2i}}^{x^{2i+}} - \bar{\nabla}_{2i} C|_{x^{2i}}^{x^{2i+}} \bar{\nabla}_{1i} H|_{x^{1i}}^{x^{1i+}}$$

Canonical discrete Hamilton's equations can be alternately written as

$$C(x^+) - C(x) = \{C, H\}_D, \quad \forall C : \mathbb{R}^{2n} \to \mathbb{R}.$$

In our formalism, any function C satisfying $\{C, H\}_D = 0$ is referred to as a discrete integral or constant of the motion with respect to the discrete dynamics generated by $\bar{X}_H$.

4.2 Port-controlled Hamiltonian structures

Along these lines, a novel description of port-controlled Hamiltonian structures exploiting the DDR representation and the previously defined passivating average output map can be proposed. This form is further validated by discussing the Energy Balance equation that is satisfied and its relation with feedback strategies and power-preserving interconnection. From [35], we recall the definition below.

Definition 7 (Port-controlled Hamiltonian system (pH)). *Given a smooth real-valued function $H : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a discrete-time port-Hamiltonian system Σ_d^H over $\mathbb{R}^n$ can be described according to the input/state/output form below*

$$x^+ = x + (J(x) - R(x))\bar{\nabla} H|_x^{x^+} \tag{26a}$$

$$\frac{dx^+(u)}{du} = G(x^+(u), u) \tag{26b}$$

$$h(x,u) = L_{G(\cdot,u)}H(x) \tag{26c}$$

$J(x) = -J^\top(x)$, $R(x) = R^\top(x) \succeq 0$, are matrices of functions representing the interconnection and the resistive parts.

We note that:
• any local extremum of $H(x)$ is an equilibrium;
• the so-defined output is the average passivating output introduced in Lemma 3 when substituting the storage function S with the Hamiltonian function H;
• the rate of change of the Hamiltonian along the dynamics satisfies the power balance equality below

$$H(x^+(u)) - H(x) = \underbrace{-(\bar{\nabla} H|_x^{x^+})^\top R(x) \bar{\nabla} H|_x^{x^+}}_{\leq 0} + \underbrace{\int_0^u \mathrm{L}_{G(\cdot,v)} H(x^+(v))\,\mathrm{d}v}_{=u\left(L_{G(\cdot,u)}H\right)^{av}(x,u)}$$

so concluding average passivity and qualifying the so defined output of conjugate output; the product $u\Big(L_{G(\cdot,u)}H\Big)^{av}(x,u)$ describes the energy brought to the system through the external input and output variables between two successive time steps.
• Taking the sum of these increments from time 0 to time k, one gets the Energy Balance equality in a form that perfectly splits in the total stored energy, the internally dissipated energy from the one supplied by the input/output variables; i.e.

$$\underbrace{H(x_k) - H(x)}_{\text{stored energy}} = \underbrace{-\sum_{i=0}^{k-1} \bar{\nabla} H^\top|_{x_i}^{x_i^+} R(x_i) \bar{\nabla} H|_{x_i}^{x_i^+}}_{\text{dissipated energy}} + \underbrace{\sum_{i=0}^{k-1} u_i^\top \left(L_{G(\cdot,u)}H\right)^{av}(x_i, u_i)}_{\text{supplied energy}}.$$

As an alternative to the DDR representation of Σ_d^H in (26), integration with respect to u transforms the port-controlled Hamiltonian system into its map form. Adopting the discrete gradient form representation of the average output described in Lemma 4 one gets equivalently to (26) the pH structure in map form. The following form specifies the equivalence between these two forms.

Remark 7. The following matrix form representation of the port-Hamiltonian structures we propose is a preamble to defining the associated Dirac description. Easy computations show that the equations (26) satisfy

$$\begin{pmatrix} x^+ - x \\ dx^+(u) \\ h(x^+(u), u) \end{pmatrix} = \begin{pmatrix} J(x) - R(x) & 0 & 0 \\ 0 & 0 & G(x^+(u), u) \\ 0 & G^\top(x^+(u), u) & 0 \end{pmatrix} \begin{pmatrix} \bar{\nabla} H|_x^{x^+} \\ \nabla H(x^+(u)) \\ du \end{pmatrix}.$$

Lemma 5 (pH system in map form). *Σ_d^H satisfying the pH structure* (26) *can be equivalently rewritten in map form as*

$$x^+(u) = x + (J(x) - R(x))\bar{\nabla} H|_x^{x^+} + ug(x, u) \tag{27a}$$

$$y(x, u) = g^\top(x, u)\bar{\nabla} H|_{x^+}^{x^+(u)} \tag{27b}$$

where by definition,

$$\begin{aligned} ug(x, u) &:= \int_0^u G(x^+(v), v)\mathrm{d}v \\ y(x, u) &= h^{av}(x, u) \\ uy(x, u) = uh^{av}(x, u) &= \int_0^u L_{G(\cdot, v)} H(x^+(v))dv = H(x^+(u)) - H(x^+). \end{aligned}$$

Remark 8. The average representation of the conjugate output (26c) is instrumental to describe its series expansion in power of u that gives for the first terms

$$y(\cdot, u) = \left(\mathrm{L}_{G(\cdot, u)} H\right)^{av}(\cdot, u) = \mathrm{L}_{G_1} H|_{x^+} + \frac{u}{2}\left(\mathrm{L}_{G_1}^2 + \mathrm{L}_{G_2}\right) H|_{x^+} + O(u^2) \tag{28}$$

where $O(u^2)$ contains all the remaining terms of higher order in the control variable u. Further details regarding the complete series expansion and its iterative computation are in [31].

4.3 Average PBC strategies for pH systems

Port-controlled Hamiltonian systems represent a largely defunded class of average passive systems over which PBC strategies can be specified. Regarding the negative average output feedback $\alpha(x)$, the algebraic equality to solve (12) specifies as

$$\alpha(x) = -\kappa h^{av}(x, \alpha(x)) = -\frac{\kappa}{\alpha(x)} \int_0^{\alpha(x)} \mathrm{L}_{G(\cdot, w)} H(x^+(w))\mathrm{d}w \tag{29}$$

that can be equivalently rewritten in terms of the discrete gradient function as

$$\alpha(x) = -\kappa g^\top(x, \alpha(x)) \bar{\nabla} H|_{x^+}^{x^+(\alpha(x))}. \tag{30}$$

Accordingly the closed loop port-controlled Hamiltonian state dynamics rewrites

$$x^+(\alpha(x)) = x + (J(x) - R(x)) \bar{\nabla} H|_x^{x^+} - \kappa g(x, \alpha(x)) g^\top(x, \alpha(x)) \bar{\nabla} H|_{x^+}^{x^+(\alpha(x))} \tag{31}$$

that can be represented in matrix form over $\mathbb{R}^{2n}$ as

$$\begin{pmatrix} x^+ - x \\ x^+(\alpha(x)) - x^+ \end{pmatrix} = \begin{pmatrix} J(x) - R(x) & 0 \\ 0 & -\kappa g(x, \alpha(x)) g^\top(x, \alpha(x)) \end{pmatrix} \begin{pmatrix} \bar{\nabla} H|_x^{x^+} \\ \bar{\nabla} H|_{x^+}^{x^+(\alpha(x))} \end{pmatrix}$$

with symmetric and positive-definite matrix expressing the closed-loop dissipation structure of the system modified under feedback into

$$\begin{pmatrix} R(x) & 0 \\ 0 & \kappa g(x, \alpha(x)) g^\top(x, \alpha(x)) \end{pmatrix} \succeq 0.$$

Setting now

$$u(x, v) = \alpha(x) + v$$

the closed loop pH structure is transformed according to Lemma 2 into

$$\begin{aligned} \begin{pmatrix} x^+ - x \\ x^+(\alpha(x)) - x^+ \end{pmatrix} &= \begin{pmatrix} J(x) - R(x) & 0 \\ 0 & -\kappa g(x, \alpha(x)) g^\top(x, \alpha(x)) \end{pmatrix} \begin{pmatrix} \bar{\nabla} H|_x^{x^+} \\ \bar{\nabla} H|_{x^+}^{x^+(\alpha(x))} \end{pmatrix} \\ \frac{dx_\alpha^+(v)}{dv} &= G_\alpha(x_\alpha^+(v), v) \\ h_\alpha(x, v) &= \mathrm{L}_{G_\alpha(\cdot, v)} H(x). \end{aligned}$$

Remark 9. For completeness, we note that the conjugate output computed as the average from $\alpha(x)$ of the output $h(x, u) = \mathrm{L}_{G(\cdot, u)} H(x)$ or in discrete gradient form along the closed loop dynamics satisfies

$${g_\alpha}^\top(x, v) \bar{\nabla} H|_{x_\alpha^+}^{x_\alpha^+(v)} = \frac{1}{v} \int_0^v \mathrm{L}_{G_\alpha(\cdot, w)} H(x_\alpha^+(w)) \mathrm{d}w = h_\alpha^{av}(x, v).$$

with

$$g_\alpha(x, v) = \frac{1}{v} \int_0^v G(x^+(\alpha(x) + w), \alpha(x) + w) dw = \frac{1}{v} \int_0^v G_\alpha(x_\alpha^+(w), w) \mathrm{d}w.$$

Again, the solutions are constructive through easy algebraic manipulations.

Being these pH structures average passive by construction, the results in Theorem 2 apply so providing a variety of stabilizing techniques first and further control strategies based on target structure assignment. Examples are given in [37, 20, 21].

The result in Theorem 3 also applies specifying that pH structures preserved their properties under power preserving interconnection. This property that properly qualifies the proposed pH structure is fundamental to discuss energy management based control schemes of networked dynamics so further enlarging the domain of interest of an Hamiltonian modeling.

Remark 10. For completeness, we report the port-controlled Hamiltonian structure proposed in the literature [14, 2, 54] is as follows

$$x^+(u) = x + (J(x) - R(x))\bar{\nabla} H|_x^{x^+(u)} + u g_{lit}(x,u) \tag{32a}$$

$$h_{lit}(x,u) = g_{lit}^\top(x,u)\bar{\nabla} H|_x^{x^+(u)}. \tag{32b}$$

The discrete gradient of H from x to x^+ is substituted with the discrete gradient of H from x to $x^+(u)$ plus an additive controlled part $u g_{lit}(x,u)$ with $g_{lit}(\cdot,u):\mathbb{R}^n \to \mathbb{R}$ smooth. By construction, the rate of change of H between two successive time instants rewrites as

$$H(x^+(u)) - H(x) = \underbrace{-(\bar{\nabla} H|_x^{x^+(u)})^\top R(x)\bar{\nabla} H|_x^{x^+(u)}}_{\leq 0} + g_{lit}^\top(x,u)\bar{\nabla} H|_x^{x^+(u)} \tag{33}$$

so naturally concluding passivity with respect to the so-defined output map $h_{lit}(x,u)$. However, it comes out that the resistive part depends on the input variable in an unpredictable way through the term $\bar{\nabla} H|_x^{x^+(u)}$. This may represent an obstacle to managing damping or energy exchanges through feedback in particular or to qualify the inner product $u g_{lit}^\top(x,u)\bar{\nabla} H|_x^{x^+(u)}$ of total power supplied to the system because the resistive part contains a control dependent element too.

4.4 Linear Port Hamiltonian structure as an example

Let a quadratic Hamiltonian function $H(x) = \frac{1}{2}x^\top P x$ with symmetric positive matrix P, then Σ_d^H specifies as follows

$$x^+ = x + (J-R)\frac{P}{2}(x + x^+); \quad \frac{dx^+(u)}{du} = B; \quad h(x) = B^\top P x \tag{34}$$

or equivalently in map form as

$$x^+(u) = x + (J-R)\frac{P}{2}(x + x^+) + Bu; \quad y(x,u) = B^\top \frac{P}{2}(x^+ + x^+(u)) \tag{35}$$

with matrices of appropriate dimensions and constant elements. Because the discrete gradient function can be explicitly expressed as a function of x and x^+, the state equations (34) (equivalently (35)) can be rewritten in their explicit form so getting a linear dynamics with an output map admitting a feed-through term as it is required to encompass a passivity property. One gets

$$x^+(u) = A_H x + Bu; \quad y(x,u) = B^\top P A_H x + \frac{B^\top PB}{2} \tag{36}$$

with

$$A_H = \left(I - \tfrac{(J-R)P}{2}\right)^{-1}\left(I + \tfrac{(J-R)P}{2}\right).$$

Easy computations show that the output $y(x,u)$ is exactly the average output associated with $h(x) = B^\top Px$ since by definition

$$\begin{aligned}\left(B^\top Px\right)^{av}(x,u) &= \frac{1}{u}\int_0^u B^\top Px^+(s)ds \\ &= \frac{1}{u}\int_0^u B^\top P(A_H x + Bs)ds = B^\top PA_H x + \frac{B^\top PB}{2}u.\end{aligned}$$

It is also possible to rise the question of when and how a passive system can be written as a port-Hamiltonian system. In this linear case, this question can be answered.

Proposition 2. *Consider the average passive linear system*

$$x^+(u) = Ax + Bu; \qquad y = B^\top Px$$

with positive definite storage $\frac{1}{2}x^\top Px$*, then it can be rewritten in the port-Hamiltonian form (34) (equivalently (35)) with*

$$2(A-I)(I+A)^{-1}P^{-1} = J - R; \quad J = -J^\top; \quad R = R^\top. \tag{37}$$

Proof. Decomposing $2(A-I)(I+A)^{-1}P^{-1}$ into its skew-symmetric and symmetric part as in (37)

$$2(A-I)(I+A)^{-1}P^{-1} = J - R; \quad J = -J^\top; \quad R = R^\top.$$

then $A^\top PA - P \leq 0$ by assumption implies $R \geq 0$ and the system can be rewritten in the port-Hamiltonian form (34) (equivalently (35)).

5 Concluding comments

In the proposed differential algebraic framework, port-Controlled Hamiltonian dynamics have been described, validating the qualifying energy balance properties. On these forms, preliminary energy based control strategies have been developed [37, 20, 21]. Work is progressing to address specific difficulties occurring under energy shaping requirements at large. The property of closeness under power preserving interconnection being validated, perspectives to generalize the techniques of control through interconnection and energy exchanges to the discrete-time context is opened. Work is currently progressing in this direction. All the material reported in this paper regards a purely discrete time setting but it can be specified to the sampled-data context that is unavoidable in practice. How port-Hamiltonian structures are transformed under sampling when assuming both the measurements and control variables available at discrete time-instants is thus a natural and challenging question. In particular we have shown in [27, 32, 34] that the proposed discrete-time structures are recovered under sampling. In that digital framework, the dynamics are necessarily parameterized by the sampling period and the control solutions are described around the continuous-time ones by their infinite series expansions. As a result, the solutions can be computed through an iterative procedure. This renders the approach constructive in a digital environment that is not developed in the present paper but detailed discussions can be found in [34].

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Bibliography

[1] (2022) 50 years of dissipativity, part i [from the editor]. IEEE Control Systems Magazine 42(2):4–5, DOI 10.1109/MCS.2021.3139546

[2] Aoues S, Di Loreto M, Eberard D, Marquis-Favre W (2017) Hamiltonian systems discrete-time approximation: Losslessness, passivity and composability. Systems & Control Letters 110:9–14

[3] Brogliato B, Lozano R, Maschke B, Egeland O (2019) Dissipative Systems Analysis and Control: Theory and Applications. Springer

[4] Byrnes C, Lin W (1994) Losslessness, feedback equivalence, and the global stabilization of discrete-time nonlinear systems. IEEE Transactions on automatic control 39(1):83–98

[5] Byrnes CI, Lin W (1993) Discrete-time lossless systems, feedback equivalence and passivity. In: Proceedings of 32nd IEEE Conference on Decision and Control, IEEE, pp 1775–1781

[6] Califano C, Monaco S, Normand-Cyrot D (2002) Non-linear non-interacting control with stability in discrete-time: a geometric framework. International Journal of Control 75(1):11–22

[7] Castaños F, Michalska H, Gromov D, Hayward V (2015) Discrete-time models for implicit port-Hamiltonian systems. arXiv preprint arXiv:150105097

[8] Celledoni E, Høiseth EH (2017) Energy-preserving and passivity-consistent numerical discretization of port-Hamiltonian systems. arXiv preprint arXiv:170608621

[9] Duindam V, Macchelli A, Stramigioli S, Bruyninckx H (2009) Modeling and control of complex physical systems: the port-Hamiltonian approach. Springer Science & Business Media

[10] Gonzalez O (1996) Time integration and discrete Hamiltonian systems. Journal of Nonlinear Science 6(5):449

[11] Haier E, Lubich C, Wanner G (2006) Geometric Numerical integration: structure-preserving algorithms for ordinary differential equations. Springer

[12] Itoh T, Abe K (1988) Hamiltonian-conserving discrete canonical equations based on variational difference quotients. Journal of Computational Physics 76(1):85–102

[13] Kotyczka P, Maschke B (2017) Discrete port-Hamiltonian formulation and numerical approximation for systems of two conservation laws. at-Automatisierungstechnik 65(5):308–322

[14] Laila DS, Astolfi A (2005) Discrete-time IDA-PBC design for separable Hamiltonian systems. IFAC Proceedings Volumes 38(1):838–843

[15] Laila DS, Astolfi A (2006) Construction of discrete-time models for port-controlled Hamiltonian systems with applications. Systems & Control Letters 55(8):673–680

[16] Maschke BM, van der Schaft AJ (1992) Port-controlled Hamiltonian systems: modelling origins and system theoretic properties. IFAC Proceedings Volumes 25(13):359–365

[17] Maschke BM, van der Schaft AJ (1993) Port-controlled Hamiltonian systems: modelling origins and systemtheoretic properties. In: Nonlinear Control Systems Design 1992, Elsevier, pp 359–365

[18] Maschke BM, Van Der Schaft AJ, Breedveld PC (1992) An intrinsic Hamiltonian formulation of network dynamics: Non-standard poisson structures and gyrators. Journal of the Franklin institute 329(5):923–966

[19] Mattioni M, Monaco S, Normand-Cyrot D (2019) Forwarding stabilization in discrete time. Automatica 109:108,532

[20] Mattioni M, Moreschini A, Monaco S, Normand-Cyrot D (2022) Discrete-time energy-balance passivity-based control. Automatica 146:110,662

[21] Mattioni M, Moreschini A, Monaco S, Normand-Cyrot D (2022) Quaternion-based attitude stabilization via discrete-time IDA-PBC. IEEE Control Systems Letters 6:2665–2670, DOI 10.1109/LCSYS.2022.3173509

[22] McLachlan RI, Quispel GRW, Robidoux N (1999) Geometric integration using discrete gradients. Philosophical Transactions of the Royal Society of London Series A: Mathematical, Physical and Engineering Sciences 357(1754):1021–1045

[23] Monaco S, Normand-Cyrot D (1984) On the realization of nonlinear discrete-time systems. Systems & Control Letters 5(2):145–152

[24] Monaco S, Normand-Cyrot D (1986) A lie exponential formula for the nonlinear discrete time functional expansion. In: Theory and Applications of Nonlinear Control Systems, Elsevier Sciences North Holland, pp 205–213

[25] Monaco S, Normand-Cyrot D (1986) Nonlinear systems in discrete time. In: Algebraic and geometric methods in nonlinear control theory, Springer, pp 411–430

[26] Monaco S, Normand-Cyrot D (1987) Finite volterra-series realizations and input-output approximations of non-linear discrete-time systems. International Journal of Control 45(5):1771–1787

[27] Monaco S, Normand-Cyrot D (1995) A unified representation for nonlinear discrete-time and sampled dynamics. In: Journal of Mathematical Systems, Estimation and Control, Birkauser-boston

[28] Monaco S, Normand-Cyrot D (2006) Normal forms and approximated feedback linearization in discrete time. Systems & Control Letters 55(1):71–80

[29] Monaco S, Normand-Cyrot D (2007) Advanced tools for nonlinear sampled-data systems' analysis and control. European journal of control 13(2-3):221–241

[30] Monaco S, Normand-Cyrot D (2007) Nonlinear representations and passivity conditions in discrete time. In: Robustness in identification and control, Springer, pp 422–433

[31] Monaco S, Normand-Cyrot D (2011) Nonlinear average passivity and stabilizing controllers in discrete time. Systems & Control Letters 60(6):431–439

[32] Monaco S, Normand-Cyrot D, Califano C (2007) From chronological calculus to exponential representations of continuous and discrete-time dynamics: A lie-algebraic approach. IEEE Transactions on Automatic Control 52(12):2227–2241

[33] Monaco S, Normand-Cyrot D, Tiefensee F (2008) From passivity under sampling to a new discrete-time passivity concept. In: 2008 47th IEEE Conference on Decision and Control, IEEE, pp 3157–3162
[34] Monaco S, Normand-Cyrot D, Mattioni M, Moreschini A (2022) Nonlinear hamiltonian systems under sampling. IEEE Transactions on Automatic Control 67(9):4598–4613, DOI 10.1109/TAC.2022.3164985
[35] Moreschini A, Mattioni M, Monaco S, Normand-Cyrot D (2019) Discrete port-controlled Hamiltonian dynamics and average passivation. In: 2019 IEEE 58th Conference on Decision and Control (CDC), IEEE, pp 1430–1435
[36] Moreschini A, Mattioni M, Monaco S, Normand-Cyrot D (2019) Interconnection through u-average passivity in discrete time. In: 2019 IEEE 58th Conference on Decision and Control (CDC), IEEE, pp 4234–4239
[37] Moreschini A, Mattioni M, Monaco S, Normand-Cyrot D (2020) Stabilization of discrete port-Hamiltonian dynamics via interconnection and damping assignment. IEEE Control Systems Letters 5(1):103–108
[38] Navarro-López EM (2005) Several dissipativity and passivity implications in the linear discrete-time setting. Mathematical Problems in Engineering 2005(6):599–616
[39] Ortega R, van der Schaft AJ, Mareels I, Maschke B (2001) Putting energy back in control. IEEE Control Systems Magazine 21(2):18–33
[40] Ortega R, van der Schaft A, Maschke B, Escobar G (2002) Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. Automatica 38(4):585–596
[41] Ortega R, Romero JG, Borja P, Donaire A (2021) PID Passivity-Based Control of Nonlinear Systems with Applications. John Wiley & Sons
[42] Rashad R, Califano F, van der Schaft AJ, Stramigioli S (2020) Twenty years of distributed port-Hamiltonian systems: a literature review. IMA Journal of Mathematical Control and Information 37(4):1400–1422
[43] van der Schaft (2000) L2-gain and passivity techniques in nonlinear control, vol 2. Springer
[44] van der Schaft A, Jeltsema D (2014) Port-Hamiltonian systems theory: An introductory overview. Foundations and Trends® in Systems and Control 1(2-3):173–378
[45] Sepulchre R (2022) 50 years of dissipativity theory, part ii [about this issue]. IEEE Control Systems Magazine 42(3):5–7, DOI 10.1109/MCS.2022.3156801
[46] Sepulchre R, Jankovic M, Kokotovic PV (2012) Constructive nonlinear control. Springer Science & Business Media
[47] Šešlija M, Scherpen JM, van der Schaft A (2012) Port-Hamiltonian systems on discrete manifolds. IFAC Proceedings Volumes 45(2):774–779
[48] Seslija M, Scherpen JM, van der Schaft A (2014) Explicit simplicial discretization of distributed-parameter port-Hamiltonian systems. Automatica 50(2):369–377
[49] Stramigioli S, Secchi C, van der Schaft AJ, Fantuzzi C (2005) Sampled data systems passivity and discrete port-Hamiltonian systems. IEEE Transactions on Robotics 21(4):574–587

[50] Sümer LG, Yalçın Y (2011) A direct discrete-time IDA-PBC design method for a class of underactuated Hamiltonian systems. In: IFAC World Congress, vol 18, pp 13,456–13,461
[51] Talasila V, Clemente-Gallardo J, van der Schaft A (2006) Discrete port-Hamiltonian systems. Systems & Control Letters 55(6):478–486
[52] Willems JC (1972) Dissipative dynamical systems part i: General theory. Archive for rational mechanics and analysis 45(5):321–351
[53] Willems JC (1972) Dissipative dynamical systems part ii: Linear systems with quadratic supply rates. Archive for rational mechanics and analysis 45(5):352–393
[54] Yalçin Y, Sümer LG, Kurtulan S (2015) Discrete-time modeling of Hamiltonian systems. Turkish Journal of Electrical Engineering & Computer Sciences 23(1):149–170