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RESEARCH ARTICLE

Heat conduction with "aging" memory

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ABSTRACT

The term material with memory is generally used to indicate materials whose mechanical and/or thermodynamical behavior depends not only on the process at the present time but also on the history of the process itself. Crucial in heat conductors with memory is the heat relaxation function which models the thermal response of the material. The present study is concerned about a thermodynamical problem with memory "aging"; that is, we analyze the temperature evolution within a rigid heat conductor with memory whose relaxation function considers the aging of the material. In particular, we account for variations of the relaxation function due to a possible deterioration of the thermal response of the material related to its age.

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1. Introduction

Mechanical and/or thermodynamical behavior of a material with memory depends not only on the process at the present time but also on the history of the process itself. In particular, rigid heat conduction with memory refers to the study of materials that exhibit rigid behavior (resistant to deformation) while also demonstrating memory effects in their thermodynamic properties. These materials respond not just to their current thermal state, but also to their past history of temperature and temperature gradient. Examples of material with memory include shape-memory alloys, certain polymers, and glasses, which require advanced thermodynamic models to describe their nonequilibrium, path-dependent behaviors. Understanding these materials is crucial for developing applications in actuators, energy storage, and smart materials. Even with the limitation of neglecting the effects of deformation, the interest in rigid material with thermal memory is testified by a wide literature focussing on the mathematical properties of the model [1–3]. Or concerned about how to model particular problem as, for instance, in Ref. [4].

However, we may stress that the present investigation can be regarded as preliminary to describe and investigate more complicated systems in which the thermal effects are combined with the other ones such as when the body is not assumed to be rigid, but also the formation is considered. Thermal effects with memory are important in many fields, such as mechanical engineering, materials science, and structural analysis, as temperature changes can induce stresses and strains in materials, which in turn can affect their mechanical behavior. A wide variety of results concerns models in which the combined effect of different physical phenomena needs to be considered. This is the case of so the called *smart materials* or materials of biological origin.

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In Ref. [5], for instance, the two models of the thermo-elasticity and thermo-viscoelasticity are compared when the body is not considered to be rigid, as in our case, but an elastic or a visco-elastic response is considered. Even more generally further phenomena can be incorporated in the model. In particular of great importance are magnetic media. A notable example of magnetic sensible media is represented by a gel whose magnetic sensibility is due to the injection of magnetic, sensible, micro, or nanoparticles [6].

A first step to approach more complex systems is to consider other time-dependent phenomena that can modify the characteristics of rigid heat conductors, such as thermal conductivity and specific heat. Aging, for example, is a time-dependent process; the macroscopic manifestations of aging, whatever the cause, may be modeled by a variation of the constitutive properties as time passes. In materials with memory, the response of a non-aging substance to an external action changes with the passage of time independently of the moment in which the experiment is started. Instead, the properties of aging materials with memory change with the passage of time even in the absence of external agents. Hence two-time scales are required to unambiguously describe the constitutive properties of an aging material. One-time scale is needed to keep track of the time since the manufacturing of the material and the other time scale is needed to keep track of the time the material is under an external action. Multiscaling models are very much up-to-date: they turn out to represent a useful tool to consider different type of effects. Aim is to develop models capable two capture the memory effects at both the microscopic as well as at the macroscopic level [7].

The problem here addressed to is within the framework of rigid heat conduction with integral form memory. This theory originates from the works of Coleman [8] and Gurtin and Pipkin [9], based on the previous celebrated article by Cattaneo [10]. Further investigations on the physical model concerning thermodynamics of materials with memory are due to Coleman and Dill [11], Giorgi and Gentili [12], Amendola, Fabrizio and Golden [13]. Specifically, on the basis of the Gurtin-Pipkin model [9], the thermodynamic theory developed by by Fabrizio, Gentili and Reynolds [14] is adopted. We provide some hints of this theory below. However, the occurrence of aging requires a specific thermodynamic theory, in which the thermodynamic restrictions on the memory kernels consider the double time scale. This original contribution is developed here in Subection 2.3 and applied to the well-posedness problem of the temperature evolution equation in Section 3.

Many other nonintegral form models have been proposed for rigid heat conductors. For example, we recall the Green-Naghdi [15] and Quintanilla [16] models using the thermal displacement concept, the Tsou dual-phase lag model [17,18], the Burgers-type model [19], and others based on constitutive equations of the rate type [3]. All these models were originally formulated in a framework involving only time derivatives, but we prove that, like the Cattaneo model, they can be described in the form of convolution integrals. When aging is considered, both Quintanilla and Burgers-like models have been generalized here with two timescale kernels.

1.1. Aims of the article

The general strategy of our approach to the problem is introduced in Section 2. The aging effects are modeled by assuming that the time dependence of the heat flux relaxation function on t (the current value) and τ (the past value) does not occur only through their difference $t-\tau$ (the so-called elapsed time), but involves t and τ separately. In particular, we propose here some generalizations of Cattaneo-Maxwell and Quintanilla models with aging memory and show that both lead to the same temperature evolution equation, where the difference is limited to the different expression of the memory kernel. The evolution equation associated with the Burgers-type model is also devised and proved to be slightly more complex than that obtained for the Maxwell and Quintanilla models. Indeed, an additional weak damping proportional to the rate of the relative temperature is present.

In Section 2 heat conduction models with aging memory are introduced and discussed, also providing some examples. In Section 3 the temperature evolution of the models under investigation is connected to a second order integro-differential problem whose kernel involves the current and past time values separately. On use of the free energy functional, some a priori estimate are obtained. On the basis of the proved results, the existence of a unique strong solution is established in Section 4 borrowing the procedure devised in Ref. [20].

2. Heat conduction models with aging memory

Let $\mathbf{x} \in \Omega \subset \mathrm{IR}^3$ denote the current position within the heat conductor and $t \in [0, +\infty)$ the time variable. Consider the relative temperature

$$u(\mathbf{x}, t) := \theta(\mathbf{x}, t) - \theta_0$$

which is the difference between the absolute temperature θ and a fixed uniform reference temperature θ_0, for instance, the temperature of the surrounding environment assumed not to be affected by the thermodynamical status of the conductor (see, e.g. Ref. [11]). According to Ref. [14] the thermodynamic state of the material at each point is determined when the temperature u and its history u^t,

$$u^t(\mathbf{x}, s) := u(\mathbf{x}, t - s), \tag{2.1}$$

are given together with the temperature gradient $\mathbf{g}$ and its history $\mathbf{g}^t$,

$$\mathbf{g}(\mathbf{x}, t) := \nabla u(\mathbf{x}, t), \qquad \mathbf{g}^t(\mathbf{x}, s) := \mathbf{g}(\mathbf{x}, t - s). \tag{2.2}$$

or, alternatively, the integrated history of the temperature gradient $\overline{\mathbf{g}}^t$ defined as

$$\overline{\mathbf{g}}^t(\mathbf{x}, s) := \int_{t-s}^{t} \mathbf{g}(\mathbf{x}, \tau) d\tau. \tag{2.3}$$

In Eqs. (2.1)–(2.3) $t \in \mathrm{IR}$ represents the current time, $\tau \in (-\infty, t]$ denotes an instant in the past, while $s = t - \tau \in [0, \infty)$ is usually referred to as *elapsed time*.

2.1. Linearized Gurtin-Pipkin model

Hereafter we restrict our attention to linear models of heat conduction. In particular, we assume that the heat flux vector $\mathbf{q}$ is given by

$$\mathbf{q}(\mathbf{x}, t) = -\int_{-\infty}^{t} k(t - \tau)\mathbf{g}(\mathbf{x}, \tau) d\tau = -\int_{0}^{\infty} k(s)\mathbf{g}^t(\mathbf{x}, s) ds \tag{2.4}$$

where the memory kernel $k(t)$ is referred to as *heat flux relaxation function* (see Ref. [14]). This constitutive equation was first obtained in the linearized Gurtin-Pipkin theory [9]. Letting $k_0 := k(0)$ take on a finite value, an integration by parts of (2.4) leads to the alternative integral form

$$\mathbf{q}(\mathbf{x}, t) = -k_0 \mathbf{g}(\mathbf{x}, t) + \int_{0}^{\infty} k'(s)\overline{\mathbf{g}}^t(\mathbf{x}, s) ds, \tag{2.5}$$

where the prime denotes differentiation with respect to the (unique) argument. Otherwise, a derivation of (2.4) with respect to time t gives

$$\mathbf{q}_t(\mathbf{x}, t) = -k_0 \mathbf{g}(\mathbf{x}, t) - \int_{-\infty}^{t} k'(t - \tau)\mathbf{g}(\mathbf{x}, \tau) d\tau = -k_0 \mathbf{g}(\mathbf{x}, t) - \int_{0}^{\infty} k'(s)\mathbf{g}^t(\mathbf{x}, s) ds \tag{2.6}$$

2.2. Maxwell-Cattaneo model

It is well known that (2.4) is a generalization of the Maxwell-Cattaneo model,

$$\xi_0 \mathbf{q}_t(\mathbf{x},t) + \mathbf{q}(\mathbf{x},t) = -\kappa_0 \mathbf{g}(\mathbf{x},t), \tag{2.7}$$

(the subscript $_t$ denotes partial differentiation with respect to time). Unlike Fourier's law, this model takes relaxation into account and hence ξ_0 is referred to as *relaxation time*. Letting $\mathbf{q}(\mathbf{x},t_0) = \mathbf{q}_0(\mathbf{x})$ and solving (2.7) for $t > t_0$ in the unknown $\mathbf{q}$ we find

$$\mathbf{q}(\mathbf{x},t) = \mathbf{q}_0(\mathbf{x}) \exp[-(t-t_0)/\xi_0] - \frac{\kappa_0}{\xi_0}\int_{t_0}^{t} \exp\left[-(t-\tau)/\xi_0\right]\mathbf{g}(\mathbf{x},\tau)d\tau,$$

Hence, letting $t_0 \to -\infty$ we obtain

$$\mathbf{q}(\mathbf{x},t) = -\frac{\kappa_0}{\xi_0}\int_{-\infty}^{t} \exp\left[-(t-\tau)/\xi_0\right]\mathbf{g}(\mathbf{x},\tau)d\tau,$$

and the change of variable $t - \tau = s$ yields

$$\mathbf{q}(\mathbf{x},t) = -\frac{\kappa_0}{\xi_0}\int_{0}^{\infty} \exp\left(-s/\xi_0\right)\mathbf{g}^t(\mathbf{x},s)ds,$$

which provides a special case of the linearized Gurtin-Pipkin model (2.4) by setting $k(s) = \frac{\kappa_0}{\xi_0}\exp\left(-s/\xi_0\right)$.

2.3. Quintanilla model

An alternative approach starts from the so-called Quintanilla constitutive model. Borrowing from the Green-Naghdi theory, the Quintanilla model involves the temperature displacement $\alpha(\mathbf{x},t)$ such that $\alpha_t(\mathbf{x},t) = \theta(\mathbf{x},t)$, namely

$$\xi_0 \mathbf{q}_t(\mathbf{x},t) + \mathbf{q}(\mathbf{x},t) = -h_0 \nabla\alpha(\mathbf{x},t) - \kappa_0 \mathbf{g}(\mathbf{x},t). \tag{2.8}$$

Taking advantage of our notation of the integrated history (2.3), we can identify the gradient of temperature displacement with the integrated history of $\mathbf{g}$ at time t,

$$\nabla\alpha(\mathbf{x},t) = \int_0^t \mathbf{g}(\mathbf{x},\tau)\mathrm{d}\tau = \bar{\mathbf{g}}^t(\mathbf{x},t).$$

Accordingly, we can write

$$\xi_0 \mathbf{q}_t(\mathbf{x},t) + \mathbf{q}(\mathbf{x},t) = -h_0 \bar{\mathbf{g}}^t(\mathbf{x},t) - \kappa_0 \mathbf{g}(\mathbf{x},t).$$

We can solve it in the unknown $\mathbf{q}$ so obtaining the integral equation

$$\mathbf{q}(\mathbf{x},t) = -\int_{-\infty}^{t} \frac{1}{\xi_0}\exp[-(t-\tau)/\xi_0][h_0 \bar{\mathbf{g}}^\tau(\mathbf{x},\tau) + \kappa_0 \mathbf{g}(\mathbf{x},\tau)]d\tau,$$

from which it follows

$$\mathbf{q}(\mathbf{x},t) = -\frac{\kappa_0}{\xi_0}\bar{\mathbf{g}}^t(\mathbf{x},t) + \int_{-\infty}^{t} \frac{\kappa_0 - \xi_0 h_0}{\xi_0^2}\exp[-(t-\tau)/\xi_0]\bar{\mathbf{g}}^\tau(\mathbf{x},\tau)d\tau.$$

Alternatively, we take advantage of the rate-type form of (2.8) (see Ref. [3]), namely

$$\xi_0 \mathbf{q}_{tt}(\mathbf{x},t) + \mathbf{q}_t(\mathbf{x},t) = -h_0 \mathbf{g}(\mathbf{x},t) - \kappa_0 \mathbf{g}_t(\mathbf{x},t), \tag{2.9}$$

and we solve it in the unknown $\mathbf{q}_t$ so obtaining the integral equation

$$\mathbf{q}_t(\mathbf{x},t) = -\int_{-\infty}^{t} \frac{1}{\xi_0}\exp[-(t-\tau)/\xi_0]\left[h_0 \mathbf{g}(\mathbf{x},\tau) + \kappa_0 \mathbf{g}_\tau(\mathbf{x},\tau)\right]d\tau,$$

from which it follows

$$\mathbf{q}_t(\mathbf{x},t) = -\frac{\kappa_0}{\xi_0}\mathbf{g}(\mathbf{x},t) + \int_{-\infty}^{t} \frac{\kappa_0 - \xi_0 h_0}{\xi_0^2} \exp[-(t-\tau)/\xi_0]\mathbf{g}(\mathbf{x},\tau)d\tau.$$

Assuming $\kappa_0 > \xi_0 h_0$ (see Ref. [19, Proposition 5.3]) and identifying

$$k_0 = \frac{\kappa_0}{\xi_0}, \qquad k(s) = \frac{\kappa_0 - \xi_0 h_0}{\xi_0}\exp[-s/\xi_0]$$

we recover (2.6), the time derivative of the linearized Gurtin-Pipkin model, so proving that both Cattaneo-Maxwell and Quintanilla models lead to the same form of integral constitutive equation. Nevertheless, as to the integral equation obtained from the Quintanilla model, we remark that the value of k_0 does not match with $k(0)$, the initial value of the memory kernel.

2.4. Burgers-type model

According to Ref. [3], a Burgers-type model for heat conductors is characterized by the rate-type equation

$$\xi_0\mathbf{q}_{tt} + \nu_0\mathbf{q}_t + \mathbf{q} = -h_0\mathbf{g} - \nu_0\kappa_0\mathbf{g}_t, \qquad \xi_0, \nu_0 > 0. \tag{2.10}$$

Thermodynamic consistency with the CD inequality is guaranteed by (see Ref. [3, Sect. 3.2.2])

$$h_0 > 0, \qquad \nu_0^2\kappa_0 - \xi_0 h_0 \geq 0. \tag{2.11}$$

Since $\xi_0, \nu_0 > 0$ we can choose $\mu_1, \mu_2 > 0$ such that

$$\begin{cases} \mu_1 + \mu_2 = \nu_0 \\ \mu_1\mu_2 = \xi_0 \end{cases}$$

and Eq. (2.10) can be rewritten as

$$\mu_1(\mu_2\mathbf{q}_t + \mathbf{q})_t + \mu_2\mathbf{q}_t + \mathbf{q} = -h_0\mathbf{g} - \nu_0\kappa_0\mathbf{g}_t.$$

By solving it in the unknown $z = \mu_2\mathbf{q}_t + \mathbf{q}$ we obtain the integral equation

$$z(t) := \mu_2\mathbf{q}_t(t) + \mathbf{q}(t) = -\int_{-\infty}^{t} \frac{1}{\mu_1}\exp[-(t-\tau)/\mu_1]\left[h_0\mathbf{g}(\tau) + \nu_0\kappa_0\mathbf{g}_\tau(\tau)\right]d\tau,$$

from which it follows

$$\mathbf{q}_t(t) + \frac{1}{\mu_2}\mathbf{q}(t) = -\frac{\nu_0\kappa_0}{\xi_0}\mathbf{g}(t) + \int_{-\infty}^{t} \frac{\nu_0\kappa_0 - \mu_1 h_0}{\mu_1\xi_0}\exp[-(t-\tau)/\mu_1]\mathbf{g}(\tau)d\tau. \tag{2.12}$$

Within the integral term we are allowed to identify

$$k_0 = \frac{\nu_0\kappa_0}{\xi_0}, \qquad k(s) = \frac{\nu_0\kappa_0 - \mu_1 h_0}{\xi_0}\exp[-s/\mu_1] \tag{2.13}$$

provided that $\nu_0\kappa_0 - \mu_1 h_0 > 0$. This condition can be written as

$$\nu_0^2\kappa_0 - \xi_0 h_0 > \mu_1^2 h_0$$

and turns out to be stronger than (2.11). After applying (2.13) and the change of variable $s = t - \tau$ to Eq. (2.12) we get

$$\mathbf{q}_t(t) + \frac{1}{\mu_2}\mathbf{q}(t) = -k_0\mathbf{g}(t) - \int_0^\infty k'(s)\mathbf{g}^t(s)ds. \tag{2.14}$$

This equation corresponds to (2.6) except for the term $\mathbf{q}(t)/\mu_2$.

2.5. Thermodynamics of heat conduction models with memory

The Clausius-Duhem inequality, representing the second law of thermodynamics, can be written as

$$\eta_t \geq \frac{1}{\theta} e_t + \mathbf{q} \cdot \frac{\nabla\theta}{\theta^2} \tag{2.15}$$

wherein, respectively, η denotes the specific entropy and e the specific internal energy. Note that $\nabla\theta = \nabla u = \mathbf{g}$. Within the linear theory of heat conductors with memory, Fabrizio, Gentili and Reynolds [14] introduce the free pseudoenergy

$$\zeta := \theta_0(e - \theta_0\eta),$$

where θ_0 is a given uniform reference temperature and

$$e = \hat{e}\big(\theta(t), \theta^t, \overline{\mathbf{g}}^t\big), \qquad \eta = \hat{\eta}\big(\theta(t), \theta^t, \overline{\mathbf{g}}^t\big), \qquad \zeta = \hat{\zeta}\big(\theta(t), \theta^t, \overline{\mathbf{g}}^t\big).$$

In Ref. [14] it is proved that the approximate canonical free pseudoenergy $\zeta(u(t), u^t, \overline{\mathbf{g}}^t)$, for small values of the relative temperature $u(t)$, enjoys most of the properties that characterize a canonical free energy even if it is not dimensionally homogeneous to such a potential. In particular, the inequality (2.15) reduces to

$$\tilde{\zeta}_t \leq \tilde{e}_t \; u + \mathbf{q} \cdot \mathbf{g}. \tag{2.16}$$

where the approximate internal energy $\tilde{e}$ is assumed to be linear

$$\tilde{e}(\mathbf{x}, t) = \alpha_0 u(\mathbf{x}, t). \tag{2.17}$$

For sake of simplicity, the specific heat α_0 is assumed to be positive and independent on the position within the heat conductor.

Following the arguments set out [12,14], thermodynamic consistency is achieved by assuming that the heat flux relaxation function satisfies the following requirements (often referred to as Graffi's conditions)

$$k(t) > 0, \qquad k'(t) \leq 0, \qquad k''(t) \geq 0, \qquad t \in (0, \infty). \tag{2.18}$$

and

$$k \in L^1(0, T) \cap C^2(0, T) \quad \forall T \in \mathrm{IR}^+. \tag{2.19}$$

Note that inequalities (2.18) together with $\alpha_0 > 0$, guarantee that

$$\tilde{\zeta}_G(u(t), u^t, \overline{\mathbf{g}}^t) = \alpha_0 |u(t)|^2 - \frac{1}{2}\int_0^\infty k'(s)|\overline{\mathbf{g}}^t(s)|^2 \; ds$$

represents a free pseudoenergy which satisfies Clausius-Duhem inequality (2.16). As it is well known, $\tilde{\zeta}_G$ is not unique (up to an additive constant). Indeed, there are many expressions of the free pseudoenergy $\tilde{\zeta}$ since there is a convex set of functionals which satisfies the inequality (2.16) (see, for instance [13,21]).

2.6. Temperature evolution

Let $r(\mathbf{x}, t)$ denote the energy supply. The energy balance for a rigid heat conductor is given by

$$e_t(\mathbf{x}, t) = -\nabla \cdot \mathbf{q}(\mathbf{x}, t) + r(\mathbf{x}, t). \tag{2.20}$$

Upon replacing the expressions (2.4) and (2.17) we obtain the equation which models the temperature evolution within the conductor

$$\alpha_0 u_t(\mathbf{x}, t) = \int_0^\infty k(s)\Delta u\ (\mathbf{x}, t-s)ds + r(\mathbf{x}, t).$$

More conveniently, it can be rewritten as

$$\alpha_0 u_t(\mathbf{x}, \tau) = \int_0^t k(t-\tau)\Delta u\ (\mathbf{x}, \tau)d\tau + f(\mathbf{x}, t), \tag{2.21}$$

where the term

$$f(\mathbf{x}, t) = \int_{-\infty}^0 k(t-\tau)\Delta u\ (\mathbf{x}, \tau)d\tau + r(\mathbf{x}, t)$$

denotes an external source term which also includes the history of the material up to $t = 0$.

If in addition to (2.19) we assume the further regularity condition

$$k' \in L^1(0, T), \forall T \in \mathrm{IR}^+, \tag{2.22}$$

then the initial value of the heat-flux relaxation function, $k_0 := k(0)$, also called *initial heat flux relaxation coefficient*, turns out to be finite. This allows us to differentiate (2.21) and obtain

$$\alpha_0 u_{tt}(\mathbf{x}, t) = k_0\Delta u(\mathbf{x}, t) + \int_0^t k'(s)\Delta u(\mathbf{x}, t-s)ds + f_t(\mathbf{x}, t), \tag{2.23}$$

As can be easily verified, this evolution equation also corresponds to the constitutive model (2.6). Therefore, by virtue of the results of the previous section, both Cattaneo-Maxwell and Quintanilla models (in their integral form) lead to the same temperature evolution equation where the difference is limited to the different expression of the memory kernel k.

The evolution equation associated with the integral Burgers-type model (2.14) is slightly more complex than (2.23). Indeed, by adding (2.20) with its time derivative multiplied by μ_2 we have

$$e_t(\mathbf{x}, t) + \mu_2 e_{tt}(\mathbf{x}, t) = -\nabla\cdot\left[\mathbf{q}(\mathbf{x}, t) + \mu_2\mathbf{q}_t(\mathbf{x}, t)\right] + r(\mathbf{x}, t) + \mu_2 r_t(\mathbf{x}, t).$$

Then, applying (2.14) and (2.17) we obtain

$$\alpha_0 u_{tt}(\mathbf{x}, t) + \frac{\alpha_0}{\mu_2}u_t(\mathbf{x}, t) = -k_0\Delta u(\mathbf{x}, t) + \int_0^t k'(t-\tau)\Delta u(\mathbf{x}, \tau)d\tau + F(\mathbf{x}, t). \tag{2.24}$$

where

$$F(\mathbf{x}, t) = \int_{-\infty}^0 k'(t-\tau)\Delta u(\mathbf{x}, \tau)d\tau + r_t(\mathbf{x}, t) + \frac{1}{\mu_2}r(\mathbf{x}, t).$$

Equation (2.24) differs from (2.23) due to the presence of the additional linear damping $\alpha_0 u_t/\mu_2$.

2.7. Aging effects

The most common effect of aging in heat-conducting materials is the change in thermal conductivity over time. For instance, when materials are irradiated with high-energy ions the lattice structures of the materials are largely changed, causing the modification of several physical properties. Since thermal conductivity is dependent on the configuration of atoms in a material's crystal lattice, a more disordered structure leads to lower thermal conductivity, that is,

$$\kappa_0(t) \geq 0, \qquad \kappa_0'(t) \leq 0, \qquad \forall t \in \mathrm{IR}.$$

When aging effects are modeled in heat conductors with memory, it can be assumed that the heat-flux relaxation function k depends on two timescales: the current time t and the past time τ.

Unlike classical memory models [13], where the memory kernel depends only on their difference $t-\tau$ (called elapsed time) as in Eq. (2.4), a memory kernel with aging involves t and τ separately, namely $k(t,\tau)$ (see, for instance, [22, Ch. 8] and [20]). Aging effects are considered by modifying (2.4) as follows (the dependence on $\mathbf{x}$ is understood and not written)

$$\mathbf{q}(t) = -\int_{-\infty}^{t} k(t,\tau)\mathbf{g}(\tau)d\tau. \tag{2.25}$$

As an example, we consider the Maxwell-Cattaneo model (2.7) where the thermal conductivity κ_0 is time dependent,

$$\xi_0 \mathbf{q}_t(t) + \mathbf{q}(t) = -\kappa_0(t)\mathbf{g}(t). \tag{2.26}$$

Accordingly, we obtain the integral form

$$\mathbf{q}(t) = -\frac{1}{\xi_0}\int_{-\infty}^{t} \exp\left[-(t-\tau)/\xi_0\right]\kappa_0(\tau)\mathbf{g}(\tau)d\tau,$$

which provides a special case of (2.25) by setting

$$k(t,\tau) = \frac{\kappa_0(\tau)}{\xi_0}\exp\left(-(t-\tau)/\xi_0\right).$$

The classical (without aging) integral expression is recovered from (2.25) provided that

$$k(t,\tau) \quad \to \quad k(t-\tau), \qquad \tau \le t.$$

In particular, we obtain the following correspondence

$$k_\tau(t,\tau) \quad \to \quad -k'(t-\tau), \qquad k(t,t) \quad \to \quad k_0,$$

wherein the subscript $_\tau$ indicates partial derivative with respect to τ. Hence, to describe memory aging Eq. (2.6) is modified as follows

$$\mathbf{q}_t(t) = -k(t,t)\mathbf{g}(t) + \int_{-\infty}^{t} k_\tau(t,\tau)\mathbf{g}(\tau)d\tau. \tag{2.27}$$

Another example of a constitutive equation like this comes from the Quintanilla model in the rate form (2.9) assuming that the constitutive parameters h_0 and k_0 are functions of time. Namely, let

$$\xi_0 \mathbf{q}_{tt}(t) + \mathbf{q}_t(t) = -h_0(t)\mathbf{g}(t) - \kappa_0(t)\mathbf{g}_t(\mathbf{x},t). \tag{2.28}$$

Solving this equation for $\mathbf{q}_t$ it follows

$$\mathbf{q}_t(\mathbf{x},t) = -\frac{\kappa_0(t)}{\xi_0}\mathbf{g}(\mathbf{x},t) + \int_{-\infty}^{t} \frac{\kappa_0(\tau)+\xi_0[k'(\tau)-h_0(\tau)]}{\xi_0^2}\exp[-(t-\tau)/\xi_0]\mathbf{g}(\mathbf{x},\tau)d\tau.$$

Then assuming $\kappa_0 + \xi_0 k' > \xi_0 h_0$ and identifying

$$k(t,t) = \frac{\kappa_0(t)}{\xi_0}, \qquad k_\tau(t,\tau) = \frac{\kappa_0(\tau)+\xi_0[k'(\tau)-h_0(\tau)]}{\xi_0^2}\exp[-(t-\tau)/\xi_0],$$

we recover (2.27).

In view of further applications, it turn out to be convenient to introduce a different pair of timescales: the current time, t, and the elapsed time, $s = t-\tau$. According to this choice we introduce the scalar function $\mathbb{K}$, defined on $\mathrm{IR}\times\mathrm{IR}^+$ as

$$\mathbb{K}(t,s) = k(t,t-s), \tag{2.29}$$

and hence

$$\mathbb{K}_s(t,s) = k_s(t, t-s) = -k_\tau(t,\tau), \qquad \tau = t-s.$$

Accordingly (2.27) becomes

$$\mathbf{q}_t(t) = -\mathbb{K}_0(t)\mathbf{g}(t) - \int_0^\infty \mathbb{K}_s(t,s)\mathbf{g}^t(s)ds. \tag{2.30}$$

where $\mathbb{K}_0(t) := \mathbb{K}(t,0) = k(t,t)$, and

$$\mathbb{K}(t,s) = -\mathbb{K}_0(t) - \int_0^s \mathbb{K}_\xi(t,\xi)d\xi, \qquad \lim_{s\to+\infty} \mathbb{K}(t,s) = 0.$$

Owing to (2.30), the evolution Eq. (2.23) is rewritten as

$$\alpha_0 u_{tt}(\mathbf{x},t) = \mathbb{K}_0(t)\Delta u\ (\mathbf{x},t) + \int_0^t \mathbb{K}_s(t,s)\Delta u\ (\mathbf{x},t-s)ds + f_t(\mathbf{x},t), \tag{2.31}$$

According to the physics of the model (for viscoelastic materials, see assumptions M1–M4 in Ref. [23]), the relaxation kernel $\mathbb{K}$ and its derivatives must satisfy

$$\mathbb{K}(t,s) > 0, \quad \mathbb{K}_s(t,s) \le 0, \quad \mathbb{K}_{ss}(t,s) \ge 0, \quad (t,s) \in \mathrm{IR}\times\mathrm{IR}^+, \tag{2.32}$$

$$\mathbb{K}_t(t,s) + \mathbb{K}_s(t,s) \le 0, \quad \mathbb{K}_{ts}(t,s) + \mathbb{K}_{ss}(t,s) \ge 0 \quad (t,s) \in \mathrm{IR}\times\mathrm{IR}^+. \tag{2.33}$$

Owing to (2.29), $\mathbb{K}(t,s)$ reduces to $k(s)$ when aging is neglected, in that

$$k(t, t-s) \quad \to \quad k(t-[t-s]) = k(s).$$

It is then apparent that assumptions (2.32) correspond to classical Graffi's conditions (2.18), whereas (2.33) boils down to (2.32).

Examples

A typical example of aging memory kernel is given by (see, e.g., [20])

$$\mathbb{K}(t,s) = K_0 \exp\left(-\frac{s}{\epsilon(t)}\right),$$

where $K_0 > 0$ and $\epsilon \in C^1(\mathrm{IR}, \mathrm{IR}^+)$ satisfies

$$\epsilon(t) \ge 0, \qquad \epsilon'(t) \le 0, \qquad \forall t \in \mathrm{IR}.$$

It is easy to verify that assumptions (2.32), (2.33) are complied.

Another example is obtained by a suitable rescaling of a (nonnegative) non-increasing function k. Given $t_0 > 0$ and $\epsilon \in C^1(\mathrm{IR}, \mathrm{IR}^+)$ satisfying

$$\epsilon'(t) \le 0, \quad \forall t \in \mathrm{IR},$$

we define

$$\mathbb{K}(t,s) = \frac{1}{\epsilon(t)}\ k\left(\frac{s}{\epsilon(t)}\right)$$

In particular, for all $t \in \mathrm{IR}$ we get

$$\int_0^\infty \mathbb{K}(t,s)\mathrm{d}s = \int_0^\infty k(y)\mathrm{d}y = \mu < \infty.$$

Accordingly, if $\epsilon(t) \to 0$ as $t \to \infty$, we obtain the distributional convergence $\lim_{t\to\infty} \mathbb{K}(t,\cdot) = \mu\delta_0$, where δ_0 denotes the Dirac mass at 0^+. For definiteness, we take $k(s) = \mathrm{e}^{-s}$ and $\epsilon(t) = \alpha/t$, $\alpha > 0$, in which case

$$\mathbb{K}(t,s) = \frac{t}{\alpha}\exp\left(-\frac{st}{\alpha}\right)$$

This relaxation kernel complies with assumptions (2.32), (2.33) provided that $t \geq \sqrt{2\alpha}$.

3. The problem

The problem under investigation models the evolution of (relative) temperature in rigid heat conductors with time-dependent memory. For sake of simplicity, the body here considered is one-dimensional. Let $\alpha_0 > 0$, $\Omega = (0,1)$ and $Q := \Omega \times (0,T)$, $T > 0$. After applying (2.31) with $F = f_t$, we consider the following linear integro-differential equation in Q,

$$\alpha_0 u_{tt}(x,t) = \mathbb{K}(t,0)u_{xx}(x,t) + \int_0^t \mathbb{K}_s(t,s)u_{xx}(x,t-s)ds + F(x,t), \tag{3.1}$$

together with the initial and boundary conditions

$$\begin{array}{lll} u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & \text{in} & \Omega \\ u(0,\cdot) = u(1,\cdot) = 0, & & \text{in}(0,T). \end{array} \tag{3.2}$$

In addition, the kernel, $\mathbb{K} : \mathcal{T} \to \mathrm{IR}$, $\mathcal{T} = [0,T]^2$, is supposed to satisfy (2.32) and (2.33).

Borrowing the procedure devised in Ref. [20], problems (3.1)–(3.2) admit a unique strong solution. In particular, the following result holds.

Lemma 3.1. *Denote by u the unique solution admitted to the problems (3.1)–(3.2) with*

$$u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega), \quad F \in L^2(Q). \tag{3.3}$$

Let (2.32)–(2.33) hold. Then for all $t \in [0,T]$ the following estimate is obtained

$$\begin{aligned} &\frac{1}{2}\int_\Omega \mathbb{K}(t,t)|u_x(t)|^2dx + \frac{1}{2}\int_\Omega |u_t(t)|^2dx \\ &\leq \frac{1}{2}\int_\Omega \mathbb{K}(0,0)|u_{0x}|^2dx + \frac{1}{2}\int_\Omega |u_1|^2dx + \int_\Omega\int_0^t F(\tau)u_\tau(\tau)dxd\tau. \end{aligned} \tag{3.4}$$

Proof. First of all, add and subtract to Eq. (3.1) the term

$$\int_0^t \mathbb{K}_s(t,s)u_{xx}(t)ds = [\mathbb{K}(t,t) - \mathbb{K}(t,0)]u_{xx}(t).$$

The result can be written in the equivalent form

$$u_{tt} - \mathbb{K}(t,t)u_{xx} + \int_0^t \mathbb{K}_s(t,s)[u_{xx}(t) - u_{xx}(t-s)]ds = F. \tag{3.5}$$

When Eq. (3.5) is multiplied by u_t, after integration over Ω it follows

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t(t)|^2dx + \int_\Omega \mathbb{K}(t,t)u_x(t)u_{xt}(t)dx + \\ &\quad -\textstyle\int_\Omega u_{xt}(t)\int_0^t \mathbb{K}_s(t,s)[u_x(t) - u_x(t-s)]dsdx = \int_\Omega F u_t(t)dx. \end{aligned} \tag{3.6}$$

Since $\frac{d}{dt}\mathbb{K}(t,t) = [\mathbb{K}_t + \mathbb{K}_s](t,s)|_{s=t}$, it follows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \mathbb{K}(t,t)|u_x|^2\ dx = \frac{1}{2}\int_\Omega [\mathbb{K}_t + \mathbb{K}_s](t,t)|u_x|^2 dx \\ + \int_\Omega \int_0^t \mathbb{K}_s(t,s)u_{xt}(t)[u_x(t) - u_x(t-s)]ds\ dx + \int_\Omega F u_t dx.\end{aligned} \tag{3.7}$$

Now, we observe that

$$\frac{\partial}{\partial t}[u_x(t) - u_x(t-s)] = u_{xt}(t) - \frac{\partial}{\partial s}[u_x(t) - u_x(t-s)], \quad 0 \le s \le t \le T,$$

and then

$$u_{xt}(t)[u_x(t) - u_x(t-s)] = \frac{1}{2}\left[\frac{\partial}{\partial t}|u_x(t) - u_x(t-s)|^2 + \frac{\partial}{\partial s}|u_x(t) - u_x(t-s)|^2\right].$$

Substitution within the double integral in (3.7) gives

$$\begin{aligned}&\int_\Omega \int_0^t \mathbb{K}_s(t,s)u_{xt}[u_x(t) - u_x(t-s)]dxds \\ &= \frac{1}{2}\int_0^t\int_\Omega \mathbb{K}_s(t,s)\frac{\partial}{\partial t}|u_x(t) - u_x(t-s)|^2 dxds \\ &\frac{1}{2}\int_0^t\int_\Omega \mathbb{K}_s(t,s)\frac{\partial}{\partial s}|u_x(t) - u_x(t-s)|^2 dxds \\ &= \frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega \mathbb{K}_s(t,s)|u_x(t) - u_x(t-s)|^2 dxds \\ &- \frac{1}{2}\int_\Omega\int_0^t [\mathbb{K}_{st} + \mathbb{K}_{ss}](t,s)|u_x(t) - u_x(t-s)|^2 dxds.\end{aligned}$$

Considering the sign conditions (2.32)–(2.33), from (3.7) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \mathbb{K}(t,t)|u_x|^2\ dx \le \\ \frac{1}{2}\frac{d}{dt}\int_0^t\int_\Omega \mathbb{K}_s(t,s)|u_x(t) - u_x(t-s)|^2 dxds + \int_\Omega F u_t\ dx.\end{aligned} \tag{3.8}$$

Integration over time, in the range $(0,t)$, $t \in (0,T)$, considering the sign conditions (2.32) implies (3.4) and, hence, completes the proof. □

4. Main results

According to $(2.32)_1$ let

$$g_0 = \min_{t\in[0,T]} \mathbb{K}(t.t) > 0, \qquad g_1 = \max_{t\in[0,T]} \mathbb{K}(t.t) > 0.$$

As a consequence of (3.4)

$$\begin{aligned}g_0\int_\Omega |u_x(t)|^2 dx + \int_\Omega |u_t(t)|^2 dx \le g_1\int_\Omega |u_{0x}|^2 dx + \int_\Omega |u_1|^2 dx \\ + \int_\Omega\int_0^t |F(\tau)|^2 dxd\tau + \int_\Omega\int_0^t |u_\tau(\tau)|^2 dxd\tau,\end{aligned}$$

which, on application of Gronwall's Lemma, implies

$$\|u(t)\|^2_{H^1_0(\Omega)} + \|u_t(t)\|^2_{L^2(\Omega)} \le Ce^T, \tag{4.1}$$

where $C = \hat{C}(\|u_0\|_{H^1_0(\Omega)}, \|u_1\|_{L^2(\Omega)}, \|F\|_{L^2(Q)}) > 0$. The estimate thus obtained is needed to prove the following theorem.

Theorem 4.1. *Let $T > 0$, $\alpha_0 > 0$ and (2.32)–(2.33) hold. There exists a unique solution u to the problems (3.1)–(3.2), subject to conditions (3.3), which satisfies*

$$u \in C^0([0,T]; H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega)).$$

Finally, it is easy to check that the temperature equation associated with the integral Burgers-type model also admits a unique solution. Indeed, the application of Gronwall's Lemma as in (4.1), yields

$$\|u(t)\|^2_{H^1_0(\Omega)} + \|u_t(t)\|^2_{L^2(\Omega)} + \int_0^t \|u_t(s)\|^2_{L^2(\Omega)} ds \le Ce^T. \tag{4.2}$$

Hence, we obtain the following result.

Remark 4.2. *Under the same assumptions of Theorem 3.2, there exists a unique solution u to the problem*

$$\alpha_0 u_{tt}(x,t) + \alpha_1 u_t(x,t) = \mathbb{K}(t,0)u_{xx}(x,t) + \int_0^t \mathbb{K}_s(t,s)u_{xx}(x,t-s)ds + F(x,t), \tag{4.3}$$

subject to $\alpha_0, \alpha_1 > 0$ and conditions (3.2)–(3.3), which satisfies

$$u \in C^0([0,T]; H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega)).$$

These results prove the well posedness of the integrodifferential systems ruling the temperature evolution in heat conductors with linear memory whose kernels are time-dependent due to aging. They cover a variety of models in integral form, such as the Gurtin-Pipkin one, and the integral versions of Quintanilla and Burgers-like models.

5. Conclusions

In this article, we develop some new integral models of heat conductors with memory, whose kernels depend on the current time t and not only on the elapsed time $s = t - \tau$. This means that the heat-flux relaxation function $\mathbb{K}$ depends on two timescale and a linear constitutive equation for the heat flux with memory and aging is given by (2.30). We have shown that integral models of this type are obtained as generalizations of rate-type models in which the material coefficients, instead of being constant, depend on time. For example, the Cattaneo-Maxwell model in which the thermal conductivity is time dependent generates the integral constitutive Eq. (2.27). Similar results are proved starting from Quintanilla and Burgers-like models with time-dependent coefficients.

In this connection, we study the one-dimensional boundary value problem with memory influenced by aging (3.1)–(3.2). In particular, this problem concerns the temperature propagation behavior of Cattaneo-Maxwell conductors and materials obeying the Quintanilla and Burgers models. We show that it admits a unique strong solution by means of suitable estimates based on the physical properties of the phenomena involved. Since the models considered here considers the aging of the material, this result can be applied to a wide variety of real materials. However, we limited our attention to rigid heat conductors with aging memory. This hypothesis narrows the field of applicability of the results as some aging phenomena are closely connected with the deformation history of the material.

A possible further development is to also consider deformations in viscoelastic media. In particular, we are interested to temperature propagation in semiconductor media, for which interesting models based on the two-temperature hyperbolic theory have been recently developed [24,25]. The results obtained in this study represent the needed theoretical background to possible future experimental investigations. Indeed, when an existence and uniqueness result of the solution is established, then the possibility to devise and perform a numerical test is open. The further development we aim to consider is to generalize the results in Ref. [23] following the lines of Ref. [26] to embrace the wider class of materials.

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