On decorated representation spaces associated to spherical surfaces

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Abstract

We analyse local features of the spaces of representations of the fundamental group of a punctured surface in SU_2 equipped with a decoration, namely a choice of a logarithm of the representation at peripheral loops. Such decorated representations naturally arise as monodromies of spherical surfaces with conical points. Among other things, in this paper we determine the smooth locus of such absolute and relative decorated representation spaces: in particular, in the relative case (with few special exceptions) such smooth locus is dense, connected, and exactly consists of non-coaxial representations. The present study sheds some light on the local structure of the moduli space of spherical surfaces with conical points, which is locally modelled on the above-mentioned decorated representation spaces.

Contents

1 Introduction

A spherical surface $(S, \boldsymbol{x}, h)$ is a compact, connected, oriented surface S endowed with a metric h of constant curvature 1 with conical singularities at $\boldsymbol{x} = (x_1, \dots, x_n)$ of angles $2\pi\boldsymbol{\vartheta} = (2\pi\vartheta_1, \dots, 2\pi\vartheta_n)$. In local polar coordinates (r, η) such metric takes the form $dr^2 + \sin^2(r)d\eta^2$ near every point of $\dot{S} = S \setminus \boldsymbol{x}$, and the form $dr^2 + \vartheta_i^2 \sin^2(r)d\eta^2$ near x_i for every i.

Associated to the spherical metric h on the punctured surface $\dot{S}$, we have a monodromy homomorphism $\rho_h : \pi_1(\dot{S}) \to \mathrm{SO}_3(\mathbb{R})$. Note that ρ_h is well-defined only up to conjugation by elements of $\mathrm{SO}_3(\mathbb{R})$ but its conjugacy class $[\rho_h]$ is uniquely determined.

We remark that, though the monodromy homomorphism of a spherical metric naturally takes values in $\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{PSU}_2$, it always admits liftings to SU_2. As a representation in SU_2 retains more information, in this paper we will focus on this case.

When no ϑ_i is an integer, the deformation space of $(S, \boldsymbol{x}, h)$ is modelled on the space of conjugacy classes of homomorphisms ρ from $\pi_1(\dot{S})$ to SU_2. This is achieved by showing that the monodromy map, that associates to a metric h its monodromy representation $[\rho_h]$ (see [38, Corollary 1(a)]), is a local homeomorphism.

If ϑ_i is an integer, the monodromy $[\rho_h]$ sends a peripheral loop of $\dot{S}$ that simply winds about the puncture x_i to $\pm I \in \mathrm{SU}_2$. Thus, ρ_h fails to detect the exact position of the conical point x_i: in more precise terms, there exist local surgeries around x_i that move the conical point x_i but keep the monodromy fixed. Hence, passing from the deformation space of $(S, \boldsymbol{x}, h)$ to the representation space results in a loss of information. For this reason we introduce decorations.

Let $\mathcal{B}$ be the subset of $\pi_1(\dot{S})$ consisting of all classes of peripheral loops, namely simple loops that winds about a puncture. A *decoration* of a homomorphism ρ is a map $A : \mathcal{B} \to \mathfrak{su}_2$ to the Lie algebra $\mathfrak{su}_2$, which is equivariant under the action of $\pi_1(\dot{S})$ by conjugation on $\mathcal{B}$ and via Ad_ρ on $\mathfrak{su}_2$, and such that $\rho(\beta) = -\exp(2\pi A(\beta))$ for all $\beta \in \mathcal{B}$ (see Definition 1.13). A *decorated representation* is a conjugacy class of pairs (ρ, A).

There is a very natural way to associate a decorated representation to a spherical metric, in such a way that the norm of $A(\beta_i)$ is ϑ_i for all peripheral loops β_i that wind about x_i.

In a forthcoming paper [43] it will be shown that, in all cases, the deformation space of spherical surfaces with conical points is locally modelled on a space of decorated representations via the decorated monodromy map. Similarly, the locus of surfaces with conical points of assigned angles is locally modelled on the subspace of representations with decoration A of assigned norm.

The aim of this paper is to study the local properties of the above-mentioned decorated representation spaces, and in particular of the subsets of decorated representation with finite stabilizers; in fact, decorated monodromies of spherical metrics will have such properties, as shown in Theorem $\widehat{\text{VII}}$.

We will state our model results for the spaces of undecorated representations (namely, Lemma I, Theorems II-III-IV-V and Proposition VI), and then we will present our results for the spaces of decorated representations (Lemma $\hat{\text{I}}$, Theorems $\widehat{\text{II}}$-$\widehat{\text{III}}$-$\widehat{\text{IV}}$-$\widehat{\text{V}}$ and Corollary $\widehat{\text{VI}}$).

Sometimes we will have to separately treat certain *special* cases, which in fact only arise in genus 0 and 1 when all stabilizers (of the undecorated homomorphisms) are infinite.

In Lemma I we show that real representation spaces are semi-algebraic sets and complex representation spaces are complex algebraic sets. Correspondingly, in Lemma $\hat{\text{I}}$ we will show that decorated representation spaces are semi-analytic sets.

In Theorem II and Theorem $\widehat{\text{II}}$ we study the smooth locus of absolute representation spaces and absolute decorated representation spaces respectively. The relative representation spaces are treated in Theorem III in the undecorated case, and in Theorem $\widehat{\text{III}}$ in the decorated case. Density and connectedness of the non-coaxial locus in non-special cases are proven in Theorem IV and Theorem $\widehat{\text{IV}}$; the special cases are discussed in Theorem V and Theorem $\widehat{\text{V}}$.

In Proposition VI we recall that, if no ϑ_i is an integer, relative representation spaces support a symplectic structure, namely Goldman symplectic structure. Such form on the SU_2-representation space is the restriction of Goldman's complex symplectic form on the $\mathrm{SL}_2(\mathbb{C})$-representation space. Since spherical metrics yield an underlying $\mathbb{CP}^1$-structure, and since $\mathrm{PSL}_2(\mathbb{C})$-representations that are monodromies of $\mathbb{CP}^1$-structures always admit a lifting to $\mathrm{SL}_2(\mathbb{C})$, we will focus our discussion on the $(\mathrm{SU}_2, \mathrm{SL}_2(\mathbb{C}))$-case. Finally, in Corollary $\widehat{\text{VI}}$ we show how such symplectic structure is induced on decorated representation spaces.

The above-mentioned main results of the present paper are stated in Section 1.2.

In Appendix A Daniil Mamaev gives an elementary proof of a known existence criterion for closed polygons in $\mathbb{S}^3$. Such criterion is relevant for the non-emptiness of our SU_2 representations spaces in genus 0 (see Section 3.6.1).

1.1 Setting

Surface. In what follows S will be a compact, connected, oriented surface of genus g and $\boldsymbol{x} = (x_1, \dots, x_n)$ an n-tuple of distinct points of S. Unless differently specified, we will assume that the punctured surface $\dot{S} = S \setminus \boldsymbol{x}$ satisfies $\chi(\dot{S}) < 0$ and $n \geq 1$.

Fundamental group. We fix a basepoint $*$ in $\dot{S}$ and we denote by $\Pi_{g,n}$ the fundamental group $\pi_1(\dot{S}, *)$. We also pick a standard basis $\{\mu_1, \nu_1, \mu_2, \dots, \nu_g, \beta_1, \dots, \beta_n\}$ of $\Pi_{g,n}$, which satisfies the unique relation $[\mu_1, \nu_1] \cdots [\mu_g, \nu_g] \cdot \beta_1 \cdots \beta_n = e$. Note that, as $n \geq 1$, such group is isomorphic to a free group with $2g + n - 1$ generators. We will denote by $\mathcal{B}_i$ the conjugacy class of β_i and by $\mathcal{B}$ the union $\bigcup_i \mathcal{B}_i$. Note that $\mathcal{B}$ is the subset of $\pi_1(\dot{S}, *)$ consisting of all classes of peripheral loops.

Spherical structure with conical points. A metric h on S is a *spherical metric* with angles $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_n)$ at $\boldsymbol{x}$ if h has constant curvature 1 on $\dot{S}$ and it has a conical singularity at x_i of angle $2\pi\vartheta_i$ for $i = 1, \dots, n$. A *spherical surface* is a triple $(S, \boldsymbol{x}, h)$ (see also [14]).

Killing form. We will endow the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ with the Killing form $\mathcal{K}(X, Y) := -2\mathrm{tr}(XY)$, so that its restriction to $\mathfrak{su}_2$ is positive-definite and it induces a norm $\|X\| := \sqrt{\mathcal{K}(X, X)}$.

Adjoint action. We denote by $\mathbb{S}^2$ the unit 2-sphere. In this paper we will often identify it to the unit sphere in $\mathfrak{su}_2$, and will implicitly assume this identification without reminding. The group SU_2 isometrically acts on $\mathbb{S}^2$ via the adjoint action. Such action factorizes through PSU_2 and identifies such group to $\mathrm{SO}(\mathfrak{su}_2) \cong \mathrm{SO}_3(\mathbb{R})$. Upon identifying $\mathbb{CP}^1$ with $\mathbb{S}^2$, the restriction of the $\mathrm{PSL}_2(\mathbb{C})$-action on $\mathbb{CP}^1$ agrees with the $\mathrm{SO}_3(\mathbb{R})$-action on $\mathbb{S}^2$ above mentioned.

The map e. We will often use the map $\mathrm{e} : \mathfrak{su}_2 \longrightarrow \mathrm{SU}_2$, defined as $\mathrm{e}(X) := -\exp(2\pi X)$. Note that $\mathrm{e}(X)$ is conjugate to the diagonal matrix with eigenvalues $\exp(i\pi(1 \pm \|X\|))$. The reason behind the above definition is that we want $\mathrm{e}(X)$ to be conjugate to the SU_2-monodromy of a spherical disk with one conical point of angle $2\pi\|X\|$.

Conjugacy classes in SU_2**.** Let $\bar{\delta} \in [0, 1]$. The elements of SU_2 at distance $2\pi \cdot \bar{\delta}$ from I form a closed algebraic subset and a conjugacy class, which will be denoted by $\mathcal{C}_{\bar{\delta}}$, and consist of all matrices in SU_2 with eigenvalues $e^{\pm i\pi\cdot\bar{\delta}}$. In particular, $\mathcal{C}_0 = \{I\}$, $\mathcal{C}_1 = \{-I\}$ and $\mathcal{C}_{\bar{\delta}}$ is diffeomorphic to a 2-sphere for all $\bar{\delta} \in (0,1)$. Given $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_n)$, we will denote by $\bar{\delta}_i$ the distance $d(\vartheta_i, \mathbb{Z}_o)$ of ϑ_i from the subset $\mathbb{Z}_o \subset \mathbb{R}$ of odd integers.

Algebraic and analytic sets. In what follows, we will use the term real/complex *algebraic set* (resp. *analytic set*) to denote a subset Z of some $\mathbb{K}^n$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ defined by finitely-many equations $f_i = 0$ and inequalities $g_j \neq 0$, with f_i and g_j polynomials (resp. analytic functions). The tangent space T_pZ to $Z \subset \mathbb{K}^n$ at a point $p \in Z$ is the intersection of the hyperplanes in $\mathbb{K}^n$ of equations $(df_i)_p = 0$. The point p of Z is *smooth* if p belongs to a unique irreducible component Z' of Z, and $T_pZ' = T_pZ$ has the same dimension as Z'.

A *semi-algebraic set* (resp. *semi-analytic set*) is a subset of some $\mathbb{R}^n$ locally obtained as a finite union of loci defined by finitely-many equations $f_i = 0$ and inequalities $g_j > 0$, with f_i, g_j polynomials (resp. real-analytic functions). See, for instance, [5].

A map between semi-algebraic sets (resp. semi-analytic sets) is *algebraic* (resp. *analytic*) if locally it is the restriction of an algebraic (resp. analytic) map between open subsets of Euclidean spaces.

A point p of a semi-analytic set $Z \subset \mathbb{R}^n$ is smooth if Z agrees with an analytic variety $W = \{f_i = 0 \,|\, i \in I\}$ in a neighbourhood of p, and p is a smooth point of W. The *smooth locus* of Z is the subset of all smooth points in Z.

An irreducible (semi-)algebraic or (semi-)analytic set has *pure dimension* if all of its irreducible components have the same dimension.

A *closed algebraic subset* of a semi-algebraic set is a subset defined by finitely-many polynomial equations. A *closed analytic subset* of a semi-analytic set is a subset locally defined by finitely-many analytic equations. All such sets will be regarded with the *classical topology*, namely with the topology inherited by the ambient real/complex affine space. Thus a subset is *dense* if it is such in the classical topology: for example, a proper closed analytic subset of an irreducible semi-analytic set has dense complement.

1.2 Main results

1.2.1 Homomorphism spaces and representation spaces. Here we introduce the representation spaces we are interested in. Throughout the whole paper, we constantly identify PSU_2 with $\mathrm{SO}_3(\mathbb{R})$ and $\mathbb{CP}^1$ with $\mathbb{S}^2$.

Definition 1.1 (Homomorphism spaces)**.** The *(absolute) homomorphism space* $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ is the space of homomorphisms $\Pi_{g,n} \to \mathrm{SU}_2$. Given an *angle vector* $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_n) \in \mathbb{R}^n_{>0}$, the *relative homomorphism space* $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is the subspace of $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ that consists of homomorphisms that send $\mathcal{B}_i$ to $\mathcal{C}_{\bar{\delta}_i}$ for every $i = 1, \dots, n$.

Absolute and relative homomorphism spaces will be realized as real algebraic subsets of a Euclidean space, from which they will inherit a classical topology and a Zariski topology (see Section 3).

Consider now the action of the group PSU_2 by conjugation on $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$. The *stabilizer* of a homomorphism ρ under such action is the subgroup $\mathrm{stab}(\rho) = \{g \in \mathrm{PSU}_2 \,|\, g\rho g^{-1} = \rho\}$ of PSU_2.

Definition 1.2 (Representation spaces)**.** The *(absolute) representation space* $\mathcal{R}ep(\Pi_{g,n}, \mathrm{SU}_2)$ is the topological quotient ot $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ by PSU_2. The *relative representation space* $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is the locus in it corresponding to $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$.

The same definition of homomorphism and representation space can be given replacing SU_2 by $\mathrm{SO}_3(\mathbb{R})$.

Remark 1.3 (Other names for $\mathcal{H}om$ and $\mathcal{R}ep$ spaces)**.** The homomorphism space $\mathcal{H}om$ was also called "space of representations" [11], or "representation variety" [37] [50], or "Betti representation space" [49]. In [11] a point $[\rho]$ of the representation space was identified to its "character" $\chi_\rho : \Pi_{g,n} \to \mathbb{C}$, defined as $\chi_\rho(\gamma) := \mathrm{tr}(\rho(\gamma))$, and $\mathcal{R}ep$ itself was then called "space of characters". Such $\mathcal{R}ep$ space was also called "scheme of representations" [37], "Betti moduli space“ [49], "character variety" [22].

Remark 1.4 (Representations and parabolic bundles)**.** Fix a complex structure J on S, and let $w_i \in [0,1)$ be the fractional part of $\frac{1}{2}(\vartheta_i - 1)$. Mehta-Seshadri [**MS**] showed that there is a real-analytic homeomorphism between $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ and the moduli space of (equivalence classes of) semi-stable holomorphic bundles E of rank 2 on S with parabolic structure at $\boldsymbol{x}$ of type $\boldsymbol{\alpha}$ and trivial determinant.

The type $\boldsymbol{\alpha}$ is determined as follows: if $w_i = 0, \frac{1}{2}$, then the filtration of the bundle E at x_i is trivial ($E_{x_i} = E^1_{x_i} \supset \{0\}$) of weight $\alpha^1(x_i) := w_i$; if $w_i \neq 0, \frac{1}{2}$, then the filtration of the bundle E at x_i is non-trivial ($E_{x_i} = E^1_{x_i} \supsetneq E^2_{x_i} \supset \{0\}$) with weights $\alpha^1(x_i) := \min\{w_i, 1 - w_i\}$ and $\alpha^2(x_i) := \max\{w_i, 1 - w_i\}$ (see also [42, Example 5]). Under such homeomorphism, non-coaxial representations correspond to stable parabolic bundles.

The decision of factoring homomorphism spaces of SU_2-homomorphisms by the conjugacy action of PSU_2 can sound a bit non-standard, but we adopt it for the following reason.

Remark 1.5 (Avoiding ubiquitous stabilizers)**.** Since $\pm I$ are in the centre of SU_2, they belong to the stabilizer of every point of $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$. In the context of monodromies of spherical metrics, such $\pm I$ do not correspond to any piece of geometric information; rather, they are a side-effect of lifting the monodromy from $\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{PSU}_2$ to SU_2. Moreover, taking the quotient of $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$ or $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ by SU_2 or by PSU_2 makes no difference from the topological point of view. For the above reasons, it will be more comfortable to avoid such ubiquitous stabilizers, and so taking the quotient by PSU_2 rather than by SU_2.

Definition 1.6 (Central and coaxial homomorphisms)**.** A homomorphism ρ in SU_2 is *central* if its image is contained in the center $Z(\mathrm{SU}_2) = \{\pm I\}$. The central (resp. non-central) locus in $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$ is the locus of central (resp. non-central) homomorphisms. A homomorphism is *coaxial* if its image is contained in a 1-parameter subgroup of SU_2. The coaxial (resp. non-coaxial) locus in $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$ is the locus of coaxial (resp. non-coaxial) homomorphisms. We similarly call the corresponding locus in the represention space, and in their relative versions.

Notation. The non-coaxial locus in $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$ is denoted by $\mathcal{Hom}^{nc}(\Pi_{g,n}, \mathrm{SU}_2)$. Similarly for the represenation space and the relative versions.

We stress that a homomorphism ρ in SU_2 is non-coaxial if and only if its PSU_2-stabilizer is finite and, in this case, such stabilizer is trivial (see Lemma 2.4). All of above definitions still make sense and the above statement still holds true, replacing SU_2 by $\mathrm{SL}_2(\mathbb{C})$, and taking the quotient by the conjugacy action of $\mathrm{PSL}_2(\mathbb{C})$, with the following caveat.

The first issue concerns the definition of coaxiality. Definition 1.6 must be modified for $\mathrm{SL}_2(\mathbb{C})$. In fact, a homomorphism in $\mathrm{SL}_2(\mathbb{C})$ may have image contained inside the subgroup

$$N := \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \;:\; t \in \mathbb{C} \right\},$$

but not be contained inside a 1-parameter subgroup of $\mathrm{SL}_2(\mathbb{C})$. In this case, the stabilizer of such homomorphism would not be discrete, since it would contain the image of N inside $\mathrm{PSL}_2(\mathbb{C})$. Hence, we call a homomorphism in $\mathrm{SL}_2(\mathbb{C})$ *coaxial* if the composition with the projection $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ has image contained in a 1-parameter subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

The second issue concerns relative homomorphism spaces. We will say that $\rho \in \mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ if B_i is conjugate to an element of $\mathcal{C}_{\bar{\delta}_i}$.

The third issue concerns quotients. Since SU_2 is compact, all orbits in $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$ are closed and the topological quotient $\mathcal{Rep}(\Pi_{g,n}, \mathrm{SU}_2)$ is Hausdorff. On the other hand, $\mathrm{SL}_2(\mathbb{C})$ is not compact and its orbits in $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ are not necessarily closed and so its topological quotient will not necessarily be Hausdorff. Thus, the symbol $\mathcal{Rep}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ will be used for the largest Hausdorff quotient: in concrete terms, this is the space obtained from $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ by identifying two homomorphisms if and only if they belong to the closure of the same orbit. Such space is also called *categorical quotient* (as in [10]), or *affine quotient* (as in [45]) as it will naturally be complex algebraic affine. Its points are in bijective correspondence with the closed $\mathrm{SL}_2(\mathbb{C})$-orbits.

In fact, it turns out that $\mathcal{Rep}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ is complex algebraic (see Theorem 1.1 of [46]). As explained in Remark 1.7, on the other hand, $\mathcal{Rep}(\Pi_{g,n}, \mathrm{SU}_2)$ is semi-algebraic and $\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ is a closed algebraic subset of $\mathcal{Rep}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$.

Remark 1.7 (Complex quotients and real quotients)**.** Let V be a vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ with an algebraic action of an algebraic reductive group G over $\mathbb{K}$. Let us remind the standard technique for realizing the quotient of V by G. First consider the finitely generated algebra $\mathbb{K}[V]^G$ of G-invariant polynomial functions on V. Then pick a finite set of generators $p_1, \dots, p_\ell$ for the $\mathbb{K}$-algebra $\mathbb{K}[V]^G$ and consider the algebraic map $p = (p_1, \dots, p_\ell) : V \to \mathbb{K}^\ell$. The image of p is contained inside the zero locus

$V(I)$ of the kernel ideal I of the surjection $\mathbb{K}[y_1,\dots,y_\ell]\to\mathbb{K}[V]^G$ that sends y_j to p_j. Moreover, the largest Hausdorff quotient $V/\!/G$ is homeomorphic to $p(V)$. For $\mathbb{K}=\mathbb{C}$ we have $p(V)=V(I)$ and so $V/\!/G$ can be realized as a complex algebraic subset of $\mathbb{C}^\ell$ (see Theorem 1.24 of [10], or Section 5.1 of [45]). For $\mathbb{K}=\mathbb{R}$ the locus $p(V)$ is a semi-algebraic subset contained inside $V(I)$ but need not be the whole $V(I)$; if G is compact, then the topological quotient V/G is already Hausdorff and so p realizes a homeomorphism between V/G and the semi-algebraic set $p(V)$ of $\mathbb{R}^\ell$ (see, for example, [48]).

Suppose now that X is a G-invariant algebraic subset of V. The largest Hausdorff quotient $X/\!/G$ can be identified to $p(X)$: this is a complex algebraic subset of $\mathbb{C}^\ell$ for $\mathbb{K}=\mathbb{C}$, or a semi-algebraic subset of $\mathbb{R}^\ell$ for $\mathbb{K}=\mathbb{R}$. If $\mathbb{K}=\mathbb{R}$ and G is compact, then the topological quotient X/G is Hausdorff and p induces a homeomorphism between X/G and the semi-algebraic subset $p(X)$.

Example 1.8 (Simplest quotients by compact real groups)**.** Consider the rotation action of $\mathrm{SO}_2(\mathbb{R})$ on $\mathbb{R}^2$. If s,t are the standard coordinates on $\mathbb{R}^2$, then the algebra of invariant polynomials is generated by $p(s,t)=s^2+t^2$ and the image of $p:\mathbb{R}^2\to\mathbb{R}$ consists of the positive half-line $\mathbb{R}_{\geq 0}$, which is a semi-algebraic subset of $\mathbb{R}$ and is identified to the quotient $\mathbb{R}^2/\mathrm{SO}_2(\mathbb{R})$. A similar example is obtained by acting on $\mathbb{R}$ via the multiplication by $\{\pm 1\}$: if t is the standard coordinate in $\mathbb{R}$, then the algebra of invariant polynomials is generated by $p(t)=t^2$ and $p:\mathbb{R}\to\mathbb{R}$ has the positive half-line as image. Hence again $\mathbb{R}/\{\pm 1\}=\mathbb{R}_{\geq 0}$.

We will make use of the following two observations.

Remark 1.9 (Quotient of a manifold by a compact group)**.** The quotient of a manifold by a compact group acting with trivial stabilizers is a manifold Moreover, if the manifold is oriented and the group is connected, then the quotient manifold is oriented too.

Remark 1.10 (Quotient of a variety by a compact group)**.** Consider the quotient of a real algebraic (or real analytic) variety X by a compact group G that acts with connected stabilizers. Since all orbits are closed, the tangent space $T_{[x]}(X/G)$ can be identied to the normal space $N_{G\cdot x/X}$ to the orbit $G\cdot x$ at the point x (see [30] for more details). Since $G\cdot x$ is locally isomorphic to $G/\mathrm{stab}(x)$, we obtain

$$\dim(T_{[x]}(X/G))=\dim(T_xX)-\dim(G)+\dim(\mathrm{stab}(x)).$$

Since X is reduced and irreducible, so is X/G. Moreover both spaces have a Zariski-open dense subset of smooth points. It follows that the smooth locus of X/G consists of points $[x]$ at which $\dim(T_xX)+\dim(\mathrm{stab}(x))$ achieves it minimum.

We will see that homomorphism spaces $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)$ and $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ are real algebraic, since they are defined by certain real algebraic equations on SU_2^{2g+n}. Thus their dimension as real algebraic sets is well-defined. Moreover it is possible to compute their Zariski tangent space, namely the intersection of the kernels of the differentials of the above-mentioned equations. As a consequence, one can detect their smoothness through the implicit function theorem (see, for instance, [36, Theorem 4.12]). Furthermore, the associated representation spaces $\mathcal{Rep}(\Pi_{g,n},\mathrm{SU}_2)$ and $\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ are semi-algebraic and the conjugacy action of SU_2 on non-coaxial homomorphisms has trivial stabilizer (see Lemma 2.4).

Since certain representation spaces $\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ entirely consist of coaxial representations (in other words, classes of homomorphisms with infinite stabilizer), such spaces have special features. This phenomenon only occurs for the following class of triples $(g,n,\boldsymbol{\vartheta})$.

Definition 1.11 (Special triples)**.** Consider triples $(g,n,\boldsymbol{\vartheta})$ with $g\in\mathbb{Z}_{\geq 0}$, $n\in\mathbb{Z}_{>0}$ and $\boldsymbol{\vartheta}\in\mathbb{R}^n_{>0}$. A triple $(g,n,\boldsymbol{\vartheta})$ is *special* if

(i) $g=1$ and $\boldsymbol{\vartheta}\in\mathbb{Z}^n$ with $\sum_{i=1}^n(\vartheta_i-1)\in 2\mathbb{Z}$,

(ii) $g=0$ and $d_1(\boldsymbol{\vartheta}-\mathbf{1},\mathbb{Z}^n_o)\leq 1$,

where $\mathbf{1}=(1,1,\dots,1)$ and d_1 is the distance in $\mathbb{R}^n$ induced by the norm $\|\boldsymbol{p}\|_1:=\sum_{i=1}^n|p_i|$, and $\mathbb{Z}^n_o$ is the subset of $\boldsymbol{m}\in\mathbb{Z}^n$ such that $\|\boldsymbol{m}\|_1$ is odd.

Now we are ready to state a few results about representation spaces that will guide our investigation of decorated representation spaces, which will be defined in Section 1.2.2 where we will also present our main results.

In this first lemma we investigate the semi-algebraic structure of representation spaces.

Lemma I (Semi-algebraicity of representation spaces)**.** *The following properties hold.*

(o) $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ *is a real algebraic set and* $\mathcal{H}om(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ *is a complex algebraic set. Moreover,* $\mathcal{R}ep(\Pi_{g,n}, \mathrm{SU}_2)$ *is a semi-algebraic set and* $\mathcal{R}ep(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ *is a complex algebraic set.*

(i) The maps $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2) \to \mathcal{H}om(\Pi_{g,n}, \mathrm{SO}_3(\mathbb{R}))$ *and* $\mathcal{R}ep(\Pi_{g,n}, \mathrm{SU}_2) \to \mathcal{R}ep(\Pi_{g,n}, \mathrm{SO}_3(\mathbb{R}))$ *are real algebraic, local homeomorphisms.*

(ii) The coaxial loci in $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ *and* $\mathcal{H}om(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ *are closed algebraic subsets.*

(iii) $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \subset \mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ *and* $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C})) \subset \mathcal{H}om(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ *are closed algebraic subsets.*

(iv) The coaxial locus is closed algebraic inside $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *and inside* $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$*.*

Statements (ii-iii-iv) also hold if $\mathcal{H}om$ *spaces are replaced by the corresponding* $\mathcal{R}ep$ *spaces.*

We remark that the above claims are also true if we replace $(\mathrm{SU}_2, \mathrm{SL}_2(\mathbb{C}))$ by $(\mathrm{SO}_3(\mathbb{R}), \mathrm{PSL}_2(\mathbb{C}))$.

We stress that most of Lemma I is already well-known. In particular, the semi-algebraic nature of real representation spaces is already mentioned in [33, page 67] and extensively discussed in [28]. An explicit example of such non-algebraic (though semi-algebraic) SU_2-representation space is described in Example 3.4.

In the following theorem we investigate absolute representation spaces.

Theorem II (Absolute representation spaces)**.** *Fix* $g \geq 0$ *and* $n > 0$ *with* $2g - 2 + n > 0$*.*

(i) Non-coaxial homomorphisms in SU_2 *have trivial stabilizer.*

(ii) The space $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ *is isomorphic to* SU_2^{2g+n-1}*. Inside it,*

- *(ii-a) the coaxial locus is irreducible, of dimension* $2g + n + 1$*;*
- *(ii-b) the non-coaxial locus is an oriented manifold of dimension* $6g - 3 + 3n$*.*

As a consequence, inside $\mathcal{R}ep(\Pi_{g,n}, \mathrm{SU}_2)$

- *(ii-c) the coaxial locus is irreducible, of dimension* $2g + n - 1$*;*
- *(ii-d) the non-coaxial locus is an oriented manifold of* real *dimension* $6g - 6 + 3n$*.*

(iii) The coaxial locus in $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ *is connected, the non-coaxial locus is dense and connected. The same holds in* $\mathcal{R}ep(\Pi_{g,n}, \mathrm{SU}_2)$*.*

In the $\mathrm{SL}_2(\mathbb{C})$ *case, claims (i-ii-iii) hold replacing* SU_2 *by* $\mathrm{SL}_2(\mathbb{C})$ *and "real dimension" by "complex dimension".*

In the following result we consider properties of relative representation spaces.

Theorem III (Relative representation spaces)**.** *Fix* $g \geq 0$ *and* $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_n)$ *and let* k *be the number of integral entries of* $\boldsymbol{\vartheta}$*.*

Then inside $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$

(i) the coaxial locus has pure dimension $2g + 2$*, and it is non-empty if and only if* $\sum_i (\pm(\vartheta_i - 1)) \in 2\mathbb{Z}$ *for some choice of the signs;*

(ii) the non-coaxial locus is an oriented manifold of real *dimension* $6g - 3 + 2(n - k)$*.*

Hence, inside $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$

(iii) the coaxial locus has pure dimension $2g$*;*

(iv) the non-coaxial locus is an oriented manifold of real *dimension* $6g - 6 + 2(n - k)$*.*

The above claims hold replacing SU_2 *by* $\mathrm{SL}_2(\mathbb{C})$ *and "real dimension" by "complex dimension".*

Theorem III(ii) was already obtained by [29] and [47] for $n = 0$, and in [39] for $n > 0$. For general $\boldsymbol{\vartheta}$ smoothness of the whole $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ was also obtained in [4, Proposition 3.3.1] and in [7, Corollary 9.9]. In this paper the proof of Theorem III(ii) relies on a computation, whose case $n = 0$ was already performed in [26], and whose case $n = 1$ was explicitly treated in [31].

Now we separately treat non-special and special cases.

Theorem IV (Non-special relative representation spaces)**.** *Assume that* $(g, n, \boldsymbol{\vartheta})$ *is non-special and let* k *be the number of integral entries of* $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_n)$*. Then* *(i)* $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *has pure dimension* $6g-3+2(n-k)$ *and* $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *has pure dimension* $6g-6+2(n-k)$.

Moreover, inside $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *and* $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$

(ii) their non-coaxial locus coincides with the smooth locus;

(iii) their non-coaxial locus is non-empty and dense;

(iv) their non-coaxial locus is connected.

Claims (i) and (iii) still hold replacing SU_2 *by* $\mathrm{SL}_2(\mathbb{C})$ *and "real dimension" by "complex dimension".*

The situation in special cases is very explicit.

Theorem V (Special relative representation spaces)**.** *Assume that* $(g, n, \boldsymbol{\vartheta})$ *is special and let* k *be the number of integral entries of* $\boldsymbol{\vartheta}$*. Then all representations are coaxial and the following hold.*

(i) Let $g = 1$*. Then* $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$ *is homeomorphic to the* 2*-sphere: four points on such sphere correspond to central representations, all the other points correspond to coaxial non-central representations.*

(ii) Let $g = 0$*. If* $k = n-1$ *or* $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) < 1$*, then* $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ *is empty.*

(iii) Assume $g = 0$*,* $k \neq n-1$ *and* $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) = 1$*.*

(iii-a) If $k = n$*, then* $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ *is isomorphic to a point.*

(iii-b) If $k \leq n-2$*, then* $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ *is isomorphic to* $\mathbb{S}^2$*.*

(iii-c) $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ *consists of one conjugacy class, and the natural structure of algebraic scheme on* $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ *is reduced if and only if* $k = n$ *or* $k = n-2$*.*

Now we turn our attention to symplectic structures. Non-coaxial loci of representation spaces in SU_2 or in $\mathrm{SL}_2(\mathbb{C})$ support Goldman's Poisson structure, whose definition will be recalled in Section 4.6.

Proposition VI (Symplectic structure on relative representation spaces)**.** *Let* $(g, n, \boldsymbol{\vartheta})$ *be non-special.*

(i) The natural map $\mathcal{R}ep^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \to \mathcal{R}ep^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ *is an embedding of real-analytic manifolds.*

(ii) Goldman's Poisson structures on $\mathcal{R}ep^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *and on* $\mathcal{R}ep^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ *are respectively real symplectic and complex symplectic. Moreover, such symplectic structures are compatible with the embedding in (i).*

The above Proposition VI is not original: see [3] [26] [34] for the case of closed surfaces, [53] and [31] for the case $n = 1$, and [18] [27] [1] and [41] for the case $n \geq 1$.

1.2.2 Decorated representation spaces.

View SU_2 as acting on $\mathbb{CP}^1 \cong \mathbb{S}^2$. If no ϑ_i is integer, then a representation $[\rho] \in \mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ sends every $\beta \in \mathcal{B}$ to a non-central element, which thus determines a unique axis of rotation in $\mathbb{S}^2$. On the other hand, if some ϑ_i is integer, then $[\rho]$ sends every $\beta'_i \in \mathcal{B}_i$ to $\pm I$ and so $\rho(\beta'_i)$ does not determine an axis of rotation. In order to add the extra piece of information consisting of axes of rotation of $\rho(\beta)$ for all $\beta \in \mathcal{B}$, we introduce decorations.

Remark 1.12. Decorated representations can be seen as holonomies of flat connections with regular singularities at the punctures, the decoration corresponding to the residues of the connection. Spaces of flat connections with regular singularities were considered in [31], [27], [4], but also in [17]. Flat connections with irregular singularities were considered already in [32] and [8].

Definition 1.13 (Decorated homomorphisms)**.** Let $\rho : \Pi_{g,n} \to \mathrm{SU}_2$ be a homomorphism. A *decoration* for ρ is a map $A : \mathcal{B} \to \mathfrak{su}_2 \setminus \{0\}$ such that

(a) $A(\gamma\beta\gamma^{-1}) = \rho(\gamma)A(\beta)\rho(\gamma)^{-1}$ for all $\beta \in \mathcal{B}$ and $\gamma \in \Pi_{g,n}$

(b) $\rho(\beta) = \mathrm{e}(A(\beta))$ for all $\beta \in \mathcal{B}$.

We call such pair (ρ, A) a *decorated homomorphism.*

Note that, by condition (a), a decoration is determined by its values at $\beta_1, \dots, \beta_n$.

If (ρ, A) is a decorated homomorphism, then the associated *normalized decoration* is $\hat{A}(\beta) := \frac{A(\beta)}{\|A(\beta)\|}$ can be thought of as a map $\hat{A} : \mathcal{B} \to \mathbb{S}^2$. Note that $\hat{A}(\beta)$ is a point of $\mathbb{S}^2$ fixed by $\rho(\beta)$. Thus, using the decoration A, we can think of $\rho(\beta)$ as a rotation of angle $2\pi\|A(\beta)\|$ with centre $\hat{A}(\beta)$.

We will see in Section 1.2.3 that the monodromy of a spherical surface $(S, \boldsymbol{x}, h)$ can be naturally endowed with a decoration, which can be essentially identified to the restriction of the developing map to the boundary points of the completion of the universal cover of $\dot{S}$. Condition (a) for decorated monodromies is then a consequence of the equivariance of the developing map.

Now we introduce the spaces of decorated representations that we want to study, whose topology will be described in Section 4.

Definition 1.14 (Space of decorated representations)**.** The space of *decorated homomorphisms* $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ is the space of couples (ρ, A), where $\rho : \Pi_{g,n} \to \mathrm{SU}_2$ is a homomorphism and A is a decoration for ρ. The subset of (ρ, A) in $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ such that $\|A(\beta_i)\| = \vartheta_i$ for all $i = 1, \dots, n$ is denoted by $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$. The *decorated representation space* $\widehat{\mathcal{Rep}}(\Pi_{g,n}, \mathrm{SU}_2)$ is the space of PSU_2-conjugacy classes of decorated homomorphisms. We denote by $\widehat{\mathcal{Rep}}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ the locus of pairs $[\rho, A]$ such that $\|A(\beta_i)\| = \vartheta_i$ for all $i = 1, \dots, n$.

We say that a decorated homomorphism (ρ, A) is *non-coaxial* if ρ is. The corresponding loci in the spaces of decorated homomorphisms (resp. representations) are denoted by $\widehat{\mathcal{Hom}}^{nc}(\Pi_{g,n}, \mathrm{SU}_2)$ and $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}^{nc}(\Pi_{g,n}, \mathrm{SU}_2)$. (resp. by $\widehat{\mathcal{Rep}}^{nc}(\Pi_{g,n}, \mathrm{SU}_2)$ and $\widehat{\mathcal{Rep}}_{\boldsymbol{\vartheta}}^{nc}(\Pi_{g,n}, \mathrm{SU}_2)$).

Remark 1.15 (Decorated representations and parabolic bundles)**.** In the decorated case, there exists a correspondence similar to the one described in Remark 1.4: namely, $\widehat{\mathcal{Rep}}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is (real-analytically) homeomorphic to a moduli space of (equivalence classes of) semi-stable holomorphic bundles of rank 2 on S with "decorated" parabolic structure at $\boldsymbol{x}$ of type $\boldsymbol{\alpha}$ and trivial determinant. In this case, a "decorated" parabolic bundle E ot type $\boldsymbol{\alpha}$ is just a vector bundle with a non-trivial filtration $(E_{x_i} = E^1_{x_i} \supsetneq E^2_{x_i} \supset \{0\})$ of *each* E_{x_i} and weights $0 \le \alpha^1(x_i) \le \alpha^2(x_i) < 1$ determined as follows: if $w_i = 0$, then $\alpha^1(x_i) = \alpha^2(x_i) = 0$; if $w_i > 0$, then $\alpha^1(x_i) := \min\{w_i, 1 - w_i\}$ and $\alpha^2(x_i) := \max\{w_i, 1 - w_i\}$.

For decorated representations the notion of being "non-elementary" will be also important for us: monodromies of spherical metrics will always have such property, as shown in Theorem $\widehat{\mathrm{VII}}$(i) below.

Definition 1.16 (Elementary decorated homomorphisms)**.** A decorated homomorphism (ρ, A) is *elementary* if there exists a 1-dimensional subalgebra $\mathfrak{h}$ of $\mathfrak{su}_2$ that contains the image of A and such that the 1-parameter subgroup $\exp(\mathfrak{h}) \subset \mathrm{SU}_2$ contains the image of ρ. The elementary (resp. non-elementary) locus in $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ is the locus of elementary (resp. non-elementary) homomorphisms. We similarly call the corresponding locus in the decorated represention space, and in their relative versions.

Notation. The non-elementary locus in $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ is denoted by $\widehat{\mathcal{Hom}}^{ne}(\Pi_{g,n}, \mathrm{SU}_2)$. Similarly for the decorated representation space and the relative versions.

Note that an elementary decorated homomorphism $(\rho, A) \in \widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is coaxial. The converse is not true in general, though it is true if no ϑ_i is integral, as shown in Lemma $\hat{\mathrm{I}}$(a) below.

In Theorem $\widehat{\mathrm{II}}$(ii-b) we will show that the singular locus of $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ consists of the subset $\widetilde{\Sigma}$ that we here introduce. We will also see in Theorem $\widehat{\mathrm{VII}}$(iii) that spherical surfaces with decorated monodromy in $\widetilde{\Sigma}$ are of a very simple type (see Definition 1.22).

Definition 1.17 (The Σ locus)**.** The subset $\widetilde{\Sigma}$ of $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ consists of classes (ρ, A) of decorated homomorphisms such that

(a) ρ is coaxial

(b) $\|A(\beta)\| \in \mathbb{Z}$ for all $\beta \in \mathcal{B}$

(c) some two-dimensional subspace of $\mathfrak{su}_2$ preserved by $\mathrm{Im}(\rho)$ contains the image of A.

We denote by Σ the image of such locus inside $\widehat{\mathcal{Rep}}(\Pi_{g,n}, \mathrm{SU}_2)$.

Let us comment on the above definition. Condition (b) implies that $\widetilde{\Sigma}$ is contained inside the union of all $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ with $\boldsymbol{\vartheta}\in\mathbb{Z}^n$. To decipher condition (c), we note that decorated homomorphisms occurring in $\widetilde{\Sigma}$ can be central or non-central. This permits us to view $\widetilde{\Sigma}$ as the union of two more understandable disjoint subloci $\widetilde{\Sigma}_0$ and $\widetilde{\Sigma}_1$, defined as follows:

- $\widetilde{\Sigma}_0$ is the locus of (ρ, A) with *central* ρ such that the image of A does not span $\mathfrak{su}_2$, and
- $\widetilde{\Sigma}_1$ is the locus of (ρ, A) with *non-central* ρ such that the homomorphism ρ takes values in a 1-parameter subgroup of H with Lie algebra $\mathfrak{h}$ and the decoration A satisfies $A(\beta)\in\mathfrak{h}^\perp$ and $\|A(\beta)\|\in\mathbb{Z}$ for all $\beta\in\mathcal{B}$.

We denote by Σ_0 and Σ_1 the corresponding loci in $\widehat{\mathcal{R}ep}(\Pi_{g,n},\mathrm{SU}_2)$.

The inclusions between the above defined loci inside the space of decorated representations are summarized in the following diagram.

$$\begin{array}{ccccccc} & & \Sigma_0 & \hookrightarrow & \{\text{central}\} & & \\ & & \downarrow & & \downarrow & & \\ \Sigma_1 & \hookrightarrow & \Sigma & \hookrightarrow & \{\text{coaxial}\} & \hookleftarrow & \{\text{special}\} \\ & & & & \uparrow & & \\ & & & & \{\text{elementary}\} & & \end{array}$$

Remark 1.18. The above definitions of decorated homomorphism and representation still make sense when replacing SU_2 by $\mathrm{SO}_3(\mathbb{R})$.

In Lemma $\hat{\mathrm{I}}$, Theorem $\widehat{\mathrm{II}}$, Theorem $\widehat{\mathrm{IV}}$ and Theorem $\hat{\mathrm{V}}$ below we describe the local structure of the decorated representation spaces in SU_2. We show that the spaces of decorated homomorphisms are real-analytic and decorated representation spaces are semi-analytic. Since the conjugacy action of PSU_2 on non-elementary decorated homomorphisms is proper (Corollary 2.7) with trivial stabilizer (Lemma 2.6), we determine the locus of decorated representation spaces that admits a natural manifold structure, separately treating the non-special and the special case.

Lemma $\hat{\mathbf{I}}$ (Semi-analyticity of decorated representation spaces)**.** *Let $g\ge 0$ and $n>0$.*

(o) $\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2)$ *is a real-analytic set and* $\widehat{\mathcal{R}ep}(\Pi_{g,n},\mathrm{SU}_2)$ *is a semi-analytic set.*

(i) The natural map $\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2)\to\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SO}_3(\mathbb{R}))$ *is a real-analytic local homeomorphism.*

(ii) Inside $\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2)$ *the elementary, coaxial and $\widetilde{\Sigma}$ loci are closed analytic. If $g=0$, then $\widetilde{\Sigma}_1$ is empty; if $g\ge 1$, then $\widetilde{\Sigma}_1$ is open and dense inside $\widetilde{\Sigma}$.*

(iii) The subset $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ *of* $\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2)$ *is closed analytic.*

(iv) The coaxial and the elementary loci in $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ *are closed analytic.*

Moreover, if no ϑ_i is integral, then

(a) a decorated representation is elementary if and only if it is coaxial;

(b) $\widehat{\mathcal{H}om}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)\to\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ *is a real-analytic isomorphism.*

Statements analogous to (i-ii-iii-iv) and (b) hold for the corresponding decorated representation spaces.

Note that $\widehat{\mathcal{R}ep}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ and $\widehat{\mathcal{R}ep}(\Pi_{g,n},\mathrm{SU}_2)$ can indeed be not real analytic, but only semi-analytic (see Example 4.2).

The following result mirrors the undecorated case treated in Theorem II.

Theorem $\widehat{\mathbf{II}}$ (Absolute decorated representation spaces)**.** *Fix g and n such that $2g-2+n>0$.*

(i) Non-elementary decorated homomorphisms in SU_2 *have trivial stabilizers.*

(ii) Inside $\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2)$

(ii-a) *the elementary locus has pure dimension* $2g+n+1$*;*

(ii-b) *the singular locus coincides with* $\widetilde{\Sigma}$ *and has pure dimension* $2g+2n+2$*;*

(ii-c) *the components of the coaxial locus have dimensions in* $[n+1, 2n]$ *if* $g=0$*, or in* $[2g+n+1, 2g+2n-1] \cup \{2g+2n+2\}$ *if* $g \geq 1$*;*

(ii-d) $\widehat{\mathcal{H}om}^{ne}(\Pi_{g,n}, \mathrm{SU}_2) \setminus \widetilde{\Sigma}$ *is an oriented manifold of dimension* $6g-3+3n$*.*

As a consequence, inside $\widehat{\mathcal{R}ep}(\Pi_{g,n}, \mathrm{SU}_2)$

(ii-e) *the elementary locus has pure dimension* $2g+n-1$*;*

(ii-f) *the non-elementary singular locus* Σ^{ne} *has pure dimension* $2g+2n-1$*;*

(ii-g) $\widehat{\mathcal{R}ep}^{ne}(\Pi_{g,n}, \mathrm{SU}_2) \setminus \Sigma$ *is an oriented manifold of dimension* $6g-6+3n$*.*

(iii) *Inside* $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$*,*

(iii-a) *the non-elementary and the non-coaxial loci are dense, and so* $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$ *has pure dimension* $6g-3+3n$*;*

(iii-b) *the non-elementary locus is connected, and non-coaxial locus is connected if and only if* $(g,n) \neq (0,3), (1,1)$*.*

Claims analogous to (iii-a) and (iii-b) hold for $\widehat{\mathcal{R}ep}(\Pi_{g,n}, \mathrm{SU}_2)$*, which has pure dimension* $6g-6+3n$*.*

Now we consider relative decorated spaces. As in Theorem III, a major role in the below result will be played by the computation of Zariski tangent spaces.

Theorem $\widehat{\textbf{III}}$ (Relative decorated representation spaces)**.** *Let* $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_n)$*, and* k *be the number of integer entries of* $\boldsymbol{\vartheta}$*. Inside* $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$

(i) *the elementary and the coaxial loci are non-empty if and only if* $\sum_i (\pm(\vartheta_i - 1)) \in 2\mathbb{Z}$ *for some choice of the signs: in this case, the elementary locus is connected, irreducible, of dimension* $2g+2$*, and the coaxial locus is connected, irreducible, of dimension* $2n$ *if* $(g,n) = (0,k)$*, and* $2g+2k+2$ *otherwise;*

(ii) *the non-coaxial locus is an oriented manifold of dimension* $6g-3+2n$*.*

Inside $\widehat{\mathcal{R}ep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$

(iii) *the elementary locus is connected, irreducible, of dimension* $2g$*;*

(iv) *the coaxial locus is connected, irreducible, of dimension* $2n-3$ *if* $(g,n) = (0,k)$*,* $2g$ *if* $k=0$*, and* $2g+2k-1$ *otherwise;*

(v) *the non-coaxial locus is an oriented manifold of dimension* $6g-6+2n$*.*

Moreover, inside $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *and* $\widehat{\mathcal{R}ep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$

(vi) *the non-elementary locus is dense and connected.*

Note that the above Theorem $\widehat{\mathrm{III}}$(ii) deals with the singularities of the relative homomorphisms space, whereas Theorem $\widehat{\mathrm{II}}$(ii) deals with the absolute homomorphisms space. Thus there might be singular points in $\widehat{\mathcal{H}om}^{ne}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ that are smooth in $\widehat{\mathcal{H}om}^{ne}(\Pi_{g,n}, \mathrm{SU}_2)$. When this is the case, the same happens for the corresponding points in the representation spaces.

In the following result we deal with the relative spaces in the non-special case.

Theorem $\widehat{\textbf{IV}}$ (Non-special relative decorated representation spaces)**.** *Let* $(g, n, \boldsymbol{\vartheta})$ *be non-special and let* k *be the number of integer entries of* $\boldsymbol{\vartheta}$*. Then*

(i) $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *has pure dimension* $6g-3+2n$ *and* $\widehat{\mathcal{R}ep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *has pure dimension* $6g-6+2n$*.*

Moreover, inside $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *and* $\widehat{\mathcal{R}ep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$

(ii) *their non-coaxial locus coincides with the smooth locus;*

(iii) *their non-coaxial locus is non-empty and dense;*

(iv) their non-coaxial locus is connected.

In special decorated SU_2-representation spaces non-coaxial loci are empty. Nevertheless, it is still possible to define natural manifold structures on certain open subsets.

Notation (Coaxial subsets). For every integer $m \geq 1$ denote by $\mathrm{Coax}((\mathbb{S}^2)^m)$ the subset of $(\mathbb{S}^2)^m$ consisting of m-tuple of points that sit on the same line.

Theorem $\widehat{\mathbf{V}}$ (Special relative decorated representation spaces). *Assume that $(g, n, \boldsymbol{\vartheta})$ is special and let k be the number of integral entries of $\boldsymbol{\vartheta}$. Then all decorated homomorphisms are coaxial, and the following hold.*

- *(i) Assume $g = 1$ and $k = n$ with $\sum_i(\vartheta_i - 1)$ even. Inside $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$*
  - *(i-a) the central locus is isomorphic to $\{\pm I\}^2 \times (\mathbb{S}^2)^n$ and the central elementary locus corresponds to $\{\pm I\}^2 \times \mathrm{Coax}((\mathbb{S}^2)^n)$;*
  - *(i-b) the non-central locus is isomorphic to an $(\mathbb{S}^2)^{n+1}$-bundle over $S^2 \setminus \{4 \text{ points}\}$ and the non-central elementary locus is a subbundle with fiber isomorphic to $\mathrm{Coax}((\mathbb{S}^2)^{n+1})$.*

  Hence the non-elementary non-central locus in $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$ is a connected oriented manifold of dimension $2n+4$, and the corresponding non-elementary non-central locus in $\widehat{\mathcal{Rep}}_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$ is a connected oriented manifold of dimension $2n+1$.
- *(ii) Assume $g = 0$, and $k = n-1$ or $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) < 1$. Then $\widehat{\mathcal{Rep}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is empty.*
- *(iii) Let $g = 0$ and $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) = 1$.*
  - *(iii-a) If $k = 0$, then $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is isomorphic to $\mathbb{S}^2$ and consist of one class of elementary decorated representations.*
  - *(iii-b) If $1 \leq k \leq n-2$, then $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is isomorphic to $(\mathbb{S}^2)^{1+k}$; the elementary locus corresponds to $\mathrm{Coax}((\mathbb{S}^2)^{1+k})$ and so $\widehat{\mathcal{Rep}}^{ne}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is a connected oriented manifold of dimension $2k-1$.*
  - *(iii-c) If $k = n \geq 3$, then all homomorphisms are central and $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is isomorphic to $(\mathbb{S}^2)^n$; the elementary locus corresponds to $\mathrm{Coax}((\mathbb{S}^2)^n)$, and so $\widehat{\mathcal{Rep}}^{ne}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is a connected, oriented manifold of dimension $2n-3$.*
  - *(iii-d) $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ consists of a single conjugacy class and its natural structure of analytic space is reduced if and only if $k = n$ or $k = n-2$.*

Remark 1.19 (About reduced analytic structures). The space $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is naturally described as the zero locus of finitely many analytic equations inside $\mathrm{SU}_2^{2g} \times \mathfrak{su}_2^{\oplus n}$. For $k < n-2$, Theorem $\widehat{\mathrm{V}}$(iii-d) is saying that at each point of $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$, the kernels of the differentials of such equations intersect in a vector subspace of dimension larger than $2k-1$.

An example of decorated representations as in Theorem $\widehat{\mathrm{V}}$(i-b) is given by decorated monodromies of spherical tori with one conical point of angle $(2m+1)2\pi$ (with $m \in \mathbb{N}$), which are non-elementary non-central (see [15]).

The costraints coming from SU_2-representations in genus 0 as in Theorem $\widehat{\mathrm{V}}$(ii)-(iii) were studied in [44] and [13]. In [54] Zhu has produced a 1-dimensional moduli space of spherical surfaces of genus 0 with $n = 4$ conical points: the decorated monodromies of such spherical surfaces belong to the space treated in Theorem $\widehat{\mathrm{V}}$(iii-b) with $k = 1$.

The existence of a Goldman symplectic form on decorated representation spaces is a direct consequence of Lemma $\hat{\mathrm{I}}$(b) and Proposition VI.

Corollary $\widehat{\mathbf{VI}}$ (Symplectic structure on decorated relative representation spaces). *Consider the map $\widehat{\mathcal{Rep}}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \to \mathcal{Rep}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ that forgets the decoration. If no ϑ_i is integral, then the Goldman symplectic form pulls back to a real symplectic form on $\widehat{\mathcal{Rep}}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$.*

We finally mention that the mapping class group $\mathrm{MCG}_{g,n}$, namely the group of orientation-preserving self-homeomorphisms of $(S, \boldsymbol{x})$ up to isotopy, naturally acts on all representation spaces introduced above (with or without decoration). Moreover, such action preserves the non-coaxial locus, the non-elementary locus, the Σ locus, and the symplectic structures.

1.2.3 Decorated monodromy of a spherical metric. We recall that a metric of constant curvature 1 is locally isometric to a portion of the standard $\mathbb{S}^2$, which can be identified to the unit sphere in $\mathfrak{su}_2$. Thus, simply-connected spherical surfaces admit a global developing map to $\mathbb{S}^2$, which is a local isometry.

Monodromy representation, universal cover and completion. Consider now a spherical surface $(S, \boldsymbol{x}, h)$ with angles $2\pi\boldsymbol{\vartheta}$ at $\boldsymbol{x}$, and lift the spherical metric to the universal cover $\widetilde{\dot{S}}$ of $\dot{S}$. Fix a basepoint $* \in \dot{S}$ and a preimage $\tilde{*} \in \widetilde{\dot{S}}$, so that $\pi_1(\dot{S}, *)$ is identified to the group of deck transformations of $\widetilde{\dot{S}} \to \dot{S}$. Since $\widetilde{\dot{S}}$ is simply-connected, the above observation ensures that there exists a locally isometric developing map $\iota : \widetilde{\dot{S}} \to \mathbb{S}^2$. Moreover such map is equivariant with respect to $\pi_1(\dot{S}, *)$, which acts on $\widetilde{\dot{S}}$ via deck transformations and on $\mathbb{S}^2$ via a *monodromy homomorphism* $\overline{\rho} : \pi_1(\dot{S}, *) \to \mathrm{SO}_3(\mathbb{R})$. We incidentally remark that the overline notation $\overline{\rho}$ is used for homomorphisms to $\mathrm{SO}_3(\mathbb{R})$, to differentiate them from their lifts that take values in SU_2.

Note that the metric completion $\widehat{S}$ of $(\widetilde{\dot{S}}, \tilde{h})$ is independent of the chosen h and in fact it could be defined in purely topological terms: here we will just say that the added points $\partial\widehat{S} := \widehat{S} \setminus \widetilde{\dot{S}}$ correspond to loops in $\pi_1(\dot{S}, *)$ that simply wind about some puncture. It is easy to check that the developing map uniquely extends to $\hat{\iota} : \widehat{S} \to \mathbb{S}^2$.

We recall that we have fixed an isomorphism $\pi_1(\dot{S}, *) \cong \Pi_{g,n}$: hence, $\partial\widehat{S}$ can be identified to $\mathcal{B}$.

Definition 1.20 (Decorated monodromy)**.** Let h be a spherical metric on $(S, \boldsymbol{x})$, whose monodromy homomorphism can thus be viewed as a $\overline{\rho} : \Pi_{g,n} \to \mathrm{SO}_3(\mathbb{R})$. The *decorated monodromy homomorphism* of $(S, \boldsymbol{x}, h)$ is the couple $(\overline{\rho}, A)$, where A is defined by $A(\beta_i) := \vartheta_i \cdot \hat{\iota}(\beta_i)$ for all i. Its $\mathrm{SO}_3(\mathbb{R})$-conjugacy class $[\overline{\rho}, A]$ is uniquely defined and is the *decorated monodromy representation* associated to $(S, \boldsymbol{x}, h)$.

Given a homomorphism ρ to $\mathrm{PSL}_2(\mathbb{C})$, it is clear what is meant by an $\mathrm{SL}_2(\mathbb{C})$-lifting of ρ. It is well-known that all monodromies of $\mathbb{CP}^1$-structures admit a lift to $\mathrm{SL}_2(\mathbb{C})$ (see, for example, [20, Lemma 1.3.1]). We will now introduce the notion of lift for decorated homomorphisms.

Definition 1.21 (SU_2-lifting of decorated homomorphism)**.** An SU_2-*lifting* of a decorated homomorphism $(\overline{\rho}, A) \in \widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SO}_3(\mathbb{R}))$ is a $(\rho, A) \in \widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$, where ρ is an SU_2-lifting of $\overline{\rho}$ and $\mathrm{Ad}_{\rho(\beta_j)}(A(\beta_j)) = e^{\pi i(\vartheta_j - 1)} A(\beta_j)$ for all j. We will also say that $[\rho, A]$ is an SU_2-lifting of $[\overline{\rho}, A]$.

Finally, we will see that there is a special class of spherical surfaces whose decorated monodromy lands in the singular locus of the representation space.

Definition 1.22 (Purely hemispherical surfaces)**.** A *standard hemisphere* is the closed portion of $\mathbb{S}^2$ bounded by a maximal circle. A *purely hemispherical surface* is a spherical surface with conical points obtained from a collection of closed standard hemispheres by identifications along their boundaries.

We observe that purely hemispherical surfaces are a special case of hemispherical surfaces studied in [51] and [21].

The following result collects some properties of decorated monodromy representations of a spherical surface and provides further motivation for the investigation carried on in the present paper.

Theorem $\widehat{\textbf{VII}}$ (Decorated monodromy of a spherical surface)**.** *Let $(\overline{\rho}, A)$ be a decorated monodromy homomorphism of a spherical metric on $(S, \boldsymbol{x})$ and assume $\chi(\dot{S}) < 0$. Then*

(o) the couple $(\overline{\rho}, A)$ admits 2^{2g} SU_2-liftings. More precisely, there exists a bijective corrspondence between SU_2-liftings of $(\overline{\rho}, A)$ and spin structures on S.

Fix now one SU_2-lifting (ρ, A). Then

(i) the couple (ρ, A) is non-elementary and $\hat{A}$ achieves at least three values;

(ii) if no ϑ_i is integer, then ρ is non-coaxial;

(iii) if (ρ, A) belongs to the locus $\widetilde{\Sigma}$, then the spherical surface is purely hemispherical;

(iv) if (ρ, A) belongs to $\widetilde{\Sigma}_0$, then the spherical surface is a cover of $\mathbb{S}^2$ with branch points on a great circle of $\mathbb{S}^2$.

It would be interesting to know exactly which decorated representations arise as decorated monodromies of spherical surfaces with conical points of assigned angles.

Question 1.23 (Geometrization of decorated representations)**.** Given $\boldsymbol{\vartheta}$, which non-elementary decorated representation in $\widehat{\mathcal{H}om}^{ne}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is an SU_2-lifting of the decorated monodromy of some spherical metric on $(S, \boldsymbol{x})$?

We recall that Goldman-Xia [25] proved that the mapping class group ergodically acts on the space of relative (non-decorated) SU_2-representation. Consider now the space of spherical metrics on a fixed surface with conical points of angles $2\pi\boldsymbol{\vartheta}$ and suppose that it is non-empty. The image of the monodromy map associated to such metrics is a non-empty, mapping-class group invariant, open subset of the relative representation space (see [43]). Hence, the locus of spherical monodromy representations is a dense open subset of full measure inside the space of relative representations.

We mention that, in [16, Corollary D], Faraco-Gupta determine which $\mathrm{SO}_3(\mathbb{R})$-representation arises as monodromy of some spherical surface with conical points. However such result only partially answers Question 1.23 for two reasons. First, Corollary D of [16] only deals with non-decorated representations. Moreover, the angles of the conical points of the surface whose monodromy realizes the given representation are only determined up to adding multiples of 2π.

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2 Conventions and centralizers

In this short section we collect notations and conventions employed throughout the paper.

2.1 Conventions

Möbius transformations. We let $\mathrm{SL}_2(\mathbb{C})$ act on $\mathbb{CP}^1$ via projective transformation and we identify $\mathbb{C}$ with the open subset $\{[X_0 : X_1] \in \mathbb{CP}^1 | X_1 \neq 0\}$ via the map $\mathbb{C} \ni z \to [z : 1] \in \mathbb{CP}^1$. This way $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{C}$ via Möbius transformations as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Exponential and volume form. For every $0 \neq X \in \mathfrak{su}_2$, the action of $\mathrm{e}(X)$ on $\mathbb{S}^2$ fixes $\hat{X} := X/\|X\|$ and acts as a rotation of angle $2\pi\|X\|$ on $T_{\hat{X}}\mathbb{S}^2$. In fact, we record the following easy observation without proof.

Lemma 2.1. *For every $\vartheta > 0$ non-integer, the restriction of $\mathrm{e} : \mathfrak{su}_2 \to \mathrm{SU}_2$ to the sphere of radius ϑ is an isomorphism onto its image.*

The volume 3-form on $\mathfrak{su}_2$ associated to $\mathcal{K}$ is $\mathrm{dvol}_{\mathcal{K}} := Y^* \wedge Z^* \wedge [Y, Z]^*$, where Y, Z are orthogonal vectors of $\mathfrak{su}_2$ of norm 1 and $X^* = \mathcal{K}(X, \cdot)$. If v is the outgoing radial vector field on $\mathfrak{su}_2$ whose value at $X \in \mathfrak{su}_2$ is exactly X, we set $\omega_{\mathcal{K}} := \iota_v(\mathrm{dvol}_{\mathcal{K}})$, which is 2-homogeneous and restricts to the standard area 2-form on the unit sphere. We will say that $0 \neq X \in \mathfrak{su}_2$ is *integral* if $\mathrm{e}(X) = \pm I$, namely if $\|X\| \in \mathbb{Z}$.

Tangent space. Given $r \in \mathrm{SU}_2$, we will identify $T_r\mathrm{SU}_2$ with $T_I\mathrm{SU}_2 \cong \mathfrak{su}_2$ via the differential $T_I\mathrm{SU}_2 \to T_r\mathrm{SU}_2$ of the right-multiplication $\mathrm{SU}_2 \ni h \mapsto h \cdot r \in \mathrm{SU}_2$ by r. More explicitly,

$$\begin{array}{ccc} \mathfrak{su}_2 & \longrightarrow & T_r\mathrm{SU}_2 \\ \dot{V} & \longmapsto & [\exp(t\dot{V})r] \end{array}$$

where $[\exp(t\dot{V})r]$ is the tangent vector determined at $t = 0$ by the path $t \mapsto \exp(t\dot{V})r = (I + t\dot{V} + o(t))r$. An analogous identification can be employed for $\mathrm{SO}_3(\mathbb{R})$, $\mathrm{PSL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{C})$.

Distance from the identity. Consider the map $D_I : \mathrm{SU}_2 \longrightarrow [0, 1]$ defined as $D_I(B) := \frac{1}{\pi}\arccos(\mathrm{tr}(B)/2)$.

Lemma 2.2 (Distance from the identity)**.** *The map D_I is continuous. Moreover,*

(i) *D_I is analytic and submersive away from $\pm I$;*

(ii) *$2\pi \cdot D_I(B)$ is the distance between I and B with respect to the metric induced on SU_2 by $\mathcal{K}$;*

(iii) *$D_I(\mathrm{e}(X)) = d(\|X\|, \mathbb{Z}_o)$ for all $X \in \mathfrak{su}_2$, where $d(\cdot, \mathbb{Z}_o)$ is the distance in $\mathbb{R}$ from the subset of odd integer points.*

Proof. Continuity is obvious, since $\arccos : [0,1] \to [0,\pi]$ is continuous.

(i) follows from the fact that the restriction of arccos to $(0,1)$ is analytic and has nonvanishing derivative.

(ii) Let $X_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathbb{S}^2$ and consider the geodesic path $B(t) := \exp(2\pi t X_0)$ on SU_2 for $t \in [0,1]$ Then $B(0) = I$, $B(1) = -I$ and $D_I(B(t)) = t$. On the other hand, $\dot{B}(t) = 2\pi X_0 \cdot B(t)$, and so $\|\dot{B}(t)\| = 2\pi$, which implies that the distance induced by $\mathcal{K}$ on SU_2 between I and $B(t)$ is $2\pi t$. The conclusion follows, since all geodesic paths on SU_2 originating from I with speed 2π are conjugate to $B(t)$.

(iii) Again, by conjugacy invariance it is enough to prove it for $X = tX_0$ with $t \in \mathbb{R}$. Since both hand sides are invariant under $t \mapsto t+2$, we can assume that $t \in [0,2]$. Since $\mathrm{e}(X) = \exp(\pi(1+t)X_0)$, we have $D_I(\mathrm{e}(X)) = |1-t|$. Finally, note that $|1-t| = |1 - \|X\||$ is exactly the distance of $t \in [0,2]$ from the closest odd number. □

2.2 Centralizers

The following notion of centralizer will be useful when considering the conjugacy action of PSU_2 on a product of copies of SU_2 and $\mathfrak{su}_2$.

Definition 2.3 (Centralizer and infinitesimal centralizer)**.** Given a collection $\mathcal{C}$ of elements of SU_2, we denote by $Z(\mathcal{C}) \subset \mathrm{SU}_2$ the *centralizer* of $\mathcal{C}$, namely the subset of elements of SU_2 that commute with every element of $\mathcal{C}$, and by $\mathfrak{z}(\mathcal{C}) \subset \mathfrak{su}_2$ the *infinitesimal centralizer* of $\mathcal{C}$, namely the Lie algebra of $Z(\mathcal{C})$.

Note that $Z(\mathcal{C})$ always contains $Z(\mathrm{SU}_2) = \{\pm I\}$. If ρ is a homomorphism that takes values in SU_2, then the centralizer and the infinitesimal centralizer of ρ, denoted by $Z(\rho)$ and by $\mathfrak{z}(\rho)$ respectively, are just the centralizer and the infinitesimal centralizer of the image of ρ.

The above definition of (infinitesimal) centralizer can be given for subsets of $\mathrm{SL}_2(\mathbb{C})$ or homomorphisms in $\mathrm{SL}_2(\mathbb{C})$.

If (ρ, A) is a decorated homomorphism in SU_2, then $Z(\rho, A) \subseteq Z(\rho)$ denotes the subgroup of elements $g \in Z(\rho)$ such that Ad_g fixes $\mathrm{Im}(A)$, and $\mathfrak{z}(\rho, A)$ is the Lie algebra of $Z(\rho, A)$.

Lemma 2.4 (Non-coaxial homomorphism has trivial centralizer)**.** *For every ρ in $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ the following hold.*

(i) *The centralizer $Z(\rho) \subseteq \mathrm{SL}_2(\mathbb{C})$ of ρ contains $\pm I$; the stabilizer of ρ is $Z(\rho)/\{\pm I\} \subseteq \mathrm{PSL}_2(\mathbb{C})$.*

(ii) *ρ non-coaxial $\iff$ $Z(\rho)$ finite $\iff$ $Z(\rho) = \{\pm I\}$.*

(iii) *ρ is coaxial non-central $\iff$ $Z(\rho)/\{\pm I\}$ is a 1-parameter subgroup of $\mathrm{PSL}_2(\mathbb{C})$; ρ is central $\iff$ $Z(\rho) = \mathrm{SL}_2(\mathbb{C})$.*

Proof. (i) Clearly $\pm I \in Z(\rho)$. By definition, the stabilizer of ρ under the action of $\mathrm{SL}_2(\mathbb{C})$ by conjugation is $Z(\rho)$. Hence, the stabilizer of ρ under the action of $\mathrm{PSL}_2(\mathbb{C})$ by conjugation is $Z(\rho)/\{\pm I\}$.

(ii) We will show that $Z(\rho) = \{\pm I\} \implies Z(\rho)$ finite $\implies \rho$ non-coaxial $\implies Z(\rho) = \{\pm I\}$.

The first implication is obvious.

Let us prove the second implication, namely $Z(\rho)$ finite $\implies \rho$ non-coaxial, by contradiction. If ρ were coaxial, then $\mathrm{Im}(\rho)$ would be contained in a 1-parameter subgroup H. It would then follow that $H \subseteq Z(\rho)$, and so $Z(\rho)$ would be infinite.

As for the third implication, ρ non-coaxial $\implies Z(\rho) = \{\pm I\}$, let $g \in \mathrm{SL}_2(\mathbb{C})$ such that $g\rho g^{-1} = \rho$. We want to show that $g = \pm I$.

Since ρ is non-coaxial, its image in $\mathrm{PSL}_2(\mathbb{C})$ is not contained in a 1-parameter subgroup. It follows that there are two matrices M, N in the image of ρ that do not commute: in particular, $M, N \neq \pm I$. Since

$gMg^{-1} = M$ and $gNg^{-1} = N$, and every eigenspace of M and N has dimension 1, every eigenvector for M or N must be an eigenvector for g too.

Suppose first that M or N is diagonalizable. Without loss of generality, we can assume that M is diagonalizable and that $\mathbb{C}^2$ decomposes as $\mathbb{C}^2 = V_1 \oplus V_2$ into eigenspaces of dimension 1 for M. Then g must be diagonalizable and V_1, V_2 must be made of eigenvectors for g too. Since N does not commute with M, it follows that g must be $\pm I$.

Suppose now that both M and N are not diagonalizable, and call V_1 the only eigenspace of M and V_2 is the only eigenspace of N. We claim that $\mathbb{C}^2 = V_1 \oplus V_2$, and g must be diagonalizable and V_1, V_2 must consist of eigenvectors for g, from which it follows that $g = \pm I$.

By contradiction, suppose $V_1 = V_2$ and pick a basis of $\mathbb{C}^2$ with the first vector in $V_1 = V_2$. With respect to such a basis, the endomorphisms M, N of $\mathbb{C}^2$ would be represented by two matrices of type $\pm\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, and so M, N would commute.

(iii) Clearly ρ is central, and so takes values $\pm I$, if and only if $Z(\rho) = \mathrm{SL}_2(\mathbb{C})$.

If $Z(\rho)/\{\pm I\}$ is a 1-parameter subgroup of $\mathrm{PSL}_2(\mathbb{C})$, then ρ must be coaxial by (ii) but cannot be central.

Conversely, suppose that ρ is coaxial but not central, and so its image is contained in a 1-parameter subgroup H of $\mathrm{SL}_2(\mathbb{C})$. We want to show that $Z(\rho) = \pm H$.

Let $X \in \mathfrak{sl}_2(\mathbb{C})$ that generate H and such that $\exp(X) \neq \pm I$ is an element in the image of ρ. Clearly, $\pm H$ is contained inside $Z(\rho)$.

If $\exp(X)$ is diagonalizable, then it has distinct eigenvalues; it follows that every element of $Z(\rho)$ different from $\pm I$ must have the same eigenvectors as $\exp(X)$, and so $Z(\rho)$ is contained in $\pm H$.

If $\exp(X)$ is not diagonalizable, then (up to conjugation) we can assume that X is strictly upper-triangular. It can then be checked by hand that $gXg^{-1} = X$ if and only if $g = \pm\exp(tX)$ for some $t \in \mathbb{C}$. The conclusion follows. □

Here is an immediate consequence.

Corollary 2.5. *The group* $\mathrm{PSL}_2(\mathbb{C})$ *acts properly and discontinuously on the non-coaxial locus* $\mathcal{H}om^{nc}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$. *As a consequence,* PSU_2 *acts properly and discontinuously on* $\mathcal{H}om^{nc}(\Pi_{g,n}, \mathrm{SU}_2)$.

Proof. Recall that a homomorphism in $\mathrm{SL}_2(\mathbb{C})$ is non-coaxial if its image in $\mathrm{PSL}_2(\mathbb{C})$ is not contained in a 1-parameter subgroup. Thus, non-coaxial homomorphisms in $\mathrm{SL}_2(\mathbb{C})$ are Zariski-dense. It follows that the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathcal{H}om^{nc}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ is proper by [35, Section 5.3.4] and it is free by Lemma 2.4. The second claim immediately follows. □

A similar conclusion holds for decorated homomorphisms in SU_2.

Lemma 2.6 (Non-elementary decorated homomorphism has trivial centralizer)**.** *For every* (ρ, A) *in* $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$ *the following hold:*

(i) the centralizer $Z(\rho, A) \subseteq \mathrm{SU}_2$ *contains* $\pm I$*; the stabilizer of* (ρ, A) *is* $Z(\rho, A)/\{\pm I\} \subseteq \mathrm{PSU}_2$.

(ii) (ρ, A) *non-elementary* $\iff$ $Z(\rho, A)$ *finite* $\iff$ $Z(\rho, A) = \{\pm I\}$.

Proof. (i) is analogous to Lemma 2.4(i).

(ii) We will show that $Z(\rho, A) = \{\pm I\} \implies Z(\rho, A)$ finite $\implies (\rho, A)$ non-elementary $\implies Z(\rho, A) = \{\pm I\}$.

The first implication again is obvious and the second implication, namely $Z(\rho, A)$ finite $\implies (\rho, A)$ non-elementary, is entirely analogous to the second implication in the proof of Lemma 2.4(ii).

As for the third implication, (ρ, A) non-elementary $\implies Z(\rho, A) = \{\pm I\}$, let $g \in \mathrm{SU}_2$ such that $g(\rho, A)g^{-1} = (\rho, A)$. We want to show that $g = \pm I$.

If ρ is non-coaxial, then the conclusion follows from Lemma 2.4. If ρ is central, then by definition A must achieve two linearly independent values $M, N \in \mathfrak{su}_2$ and g must commute with both of them. As in Lemma 2.4, this implies that $g = \pm I$.

Finally, if ρ is coaxial but non-central with image contained in the 1-parameter subgroup $H = \exp(\mathfrak{h})$, then ρ achieves a value $M \in H$ different from $\pm I$. Thus, $gMg^{-1} = M$ implies that g must be contained in H. Since (ρ, A) is not elementary, A achieves a value $N \in \mathfrak{su}_2$ not in $\mathfrak{h}$. This implies that $g = \pm I$. □

Again, here is an immediate consequence.

Corollary 2.7. *The group* PSU_2 *acts properly and discontinuously on the non-elementary locus* $\mathcal{H}om^{ne}(\Pi_{g,n}, \mathrm{SU}_2)$.

Proof. Properness of the action is immediate because SU_2 is compact. Freeness follows from Lemma 2.6. □

3 Representation spaces

We recall from the introduction that $\Pi_{g,n}$ is the group generated by $\{\mu_1, \nu_1, \mu_2, \dots, \nu_g, \beta_1, \dots, \beta_n\}$ and with the unique relation $[\mu_1, \nu_1] \cdots [\mu_g, \nu_g] \cdot \beta_1 \cdots \beta_n = e$, and that $\mathcal{B}_i$ is the conjugacy class of β_i.
We will use the symbol V to denote $V = \mathcal{M}_{2,2}(\mathbb{C})$. Note that SU_2 is a real algebraic subset and $\mathrm{SL}_2(\mathbb{C})$ is a complex algebraic subset of the vector space V.

3.1 Topology and semi-algebraic structure

The goal of this section is to prove Lemma I. In order to do so, and to endow the absolute SU_2-homomorphism space with an algebraic structure, we follow a classical idea and we embed the absolute SU_2-homomorphism space inside a smooth subset $\mathcal{G}$ of $V^{\oplus(2g+n)}$, so that its image is described by an algebraic equation $R = I$. The action of PSU_2 on the homomorphism space will be induced by a natural action on $V^{\oplus(2g+n)}$ that preserves $\mathcal{G}$ and is compatible with the map R. The relative case and the case of $\mathrm{SL}_2(\mathbb{C})$ are dealt with in an analogous fashion.

3.1.1 The embedding $\boldsymbol{\lambda}$.

Inside $V^{\oplus(2g+n)}$ consider the smooth real algebraic subset $\mathcal{G} := \mathrm{SU}_2^{2g+n}$ of dimension $6g+3n$, whose points $(M_1, N_1, \dots, M_n, N_n, B_1, \dots, B_n)$ will be often denoted by $(\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{B})$ for sake of brevity.
Note that the map

$$\begin{aligned} \lambda : \ \mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2) &\longrightarrow \mathcal{G} \\ \rho &\longmapsto (\rho(\mu_1), \rho(\nu_1), \dots, \rho(\mu_g), \rho(\nu_g), \rho(\beta_1), \dots, \rho(\beta_n)) \end{aligned}$$

is injective. Thus, the set $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ can be identified to the closed algebraic subset of $\mathcal{G}$ consisting of those $(\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{B})$ such that

$$[M_1, N_1] \cdots [M_g, N_g] \cdot B_1 \cdots B_n = I. \tag{1}$$

Remark 3.1. It can be shown that the algebraic structure induced on $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ is independent of the choice of the generators $\{\mu_i, \lambda_i, \beta_j\}$ of $\Pi_{g,n}$.

Fix $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_n) \in \mathbb{R}^n_{>0}$ and let k be the number of integral entries of $\boldsymbol{\vartheta}$. Call $\mathcal{G}_{\boldsymbol{\vartheta}}$ the set of all elements $(\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{B})$ in $\mathcal{G}$ such that $B_i \in \mathcal{C}_{\bar{\delta}_i}$ for $i = 1, \dots, n$. Clearly $\mathcal{G}_{\boldsymbol{\vartheta}}$ is a closed algebraic subset of $\mathcal{G}$, which is smooth of dimension $6g + 2(n-k)$.

3.1.2 The map $\boldsymbol{R}$.

Let now R be the real analytic map defined as

$$\begin{aligned} R : \quad \mathcal{G} &\longrightarrow \mathrm{SU}_2 \\ (\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{B}) &\longmapsto \textstyle\prod_j [M_j, N_j] \prod_i B_i \end{aligned}$$

and denote by $R_{\boldsymbol{\vartheta}} : \mathcal{G}_{\boldsymbol{\vartheta}} \to \mathrm{SU}_2$ the restriction of R to $\mathcal{G}_{\boldsymbol{\vartheta}}$. Via the embedding λ, the space $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ identifies to $R^{-1}(I)$ and $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ identifies to $R_{\boldsymbol{\vartheta}}^{-1}(I)$.

Lemma 3.2 (Number of equations cutting the image of λ)**.** *The space* $\lambda(\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ *inside* $\mathcal{G}$ *is described by* 3 *real algebraic equaitons. If* k *is the number of integer entries in* $\boldsymbol{\vartheta}$*, then the image of* $\lambda(\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *inside* $\mathcal{G}$ *is described by* $n + 2k + 3$ *real algebraic equations.*

Proof. The only equation cutting $\lambda(\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ is $R = I$, which amounts to 3 scalar equations.

Consider now $\lambda(\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$. For each i with non-integer ϑ_i, the condition $B_i \in \mathcal{C}_{\bar{\delta}_i}$ translates into the equation $\mathrm{tr}(B_i) = 2\cos(\pi\bar{\delta}_i)$. For each j with integer ϑ_j, the condition $B_j \in \mathcal{C}_{\bar{\delta}_j}$ translates into $B_j = I$ or $B_j = -I$, which is equivalent to three scalar equations. Moreover $R_{\boldsymbol{\vartheta}} = I$ contributes with another triple of scalar equations. To sum up, we obtain $n - k$ equations for the non-integer ϑ_i, $3k$ equations for the integer ϑ_j, and 3 more equations for $R_{\boldsymbol{\vartheta}} = I$: in total, we obtain $(n - k) + 3k + 3 = n + 2k + 3$. □

The above Lemma 3.2 endows both spaces with a natural structure of algebraic schemes.

3.1.3 The conjugacy action.

The group PSU_2 acts on $V^{\oplus(2g+n)}$ componentwise via conjugation, preserving $\mathcal{G}$ and $\mathcal{G}_{\boldsymbol{\vartheta}}$, and R is PSU_2-equivariant. Hence such action induces the usual conjugacy action on the homomorphism spaces via λ.

We remark that the absolute and relative representation spaces have a natural structure of algebraic schemes too, being described by algebraic equations (induced by $R = I$ and by $R_{\boldsymbol{\vartheta}} = I$ respectively) inside $\mathcal{G}/\mathrm{PSU}_2$ and $\mathcal{G}_{\boldsymbol{\vartheta}}/\mathrm{PSU}_2$ respectively.

3.1.4 The complex case.

As mentioned at the beginning of Section 3.1, in order to endow the absolute and relative $\mathrm{SL}_2(\mathbb{C})$-homomorphism spaces with an algebraic structure, we perform the above constructions just replacing the group SU_2 by $\mathrm{SL}_2(\mathbb{C})$. More explicitly, we proceed as follows.

Denote by $\mathcal{G}_{\mathbb{C}}$ the smooth complex algebraic subset $\mathrm{SL}_2(\mathbb{C})^{2g+n} \subset V^{\oplus(2g+n)}$ of complex dimension $6g+3n$ and let $\lambda_{\mathbb{C}} : \mathcal{H}om(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C})) \hookrightarrow \mathcal{G}_{\mathbb{C}}$ be the injective map defined analogously to λ. Note now that, if $B \in \mathrm{SU}_2$, then its $\mathrm{SL}_2(\mathbb{C})$-conjugacy class is smooth of complex dimension 2 if $\mathrm{tr}(B) \neq \pm 2$, and of dimension 0 if $\mathrm{tr}(B) = \pm 2$. Hence, $\mathcal{G}_{\boldsymbol{\vartheta},\mathbb{C}}$ is a closed complex algebraic subset of $\mathcal{G}_{\mathbb{C}}$, which is smooth of complex dimension $6g + 2(n - k)$. Moreover, via $\lambda_{\mathbb{C}}$, the space $\mathcal{H}om(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ identifies to $R_{\mathbb{C}}^{-1}(I)$ and $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ to $R_{\boldsymbol{\vartheta},\mathbb{C}}^{-1}(I)$, where $R_{\mathbb{C}} : \mathcal{G}_{\mathbb{C}} \to \mathrm{SL}_2(\mathbb{C})$ and $R_{\boldsymbol{\vartheta},\mathbb{C}} : \mathcal{G}_{\boldsymbol{\vartheta},\mathbb{C}} \to \mathrm{SL}_2(\mathbb{C})$ are defined in the obvious way and are $\mathrm{PSL}_2(\mathbb{C})$-equivariant.

As in the SU_2 case with Lemma 3.2, we have the following.

Lemma 3.3 (Number of equations cutting the image of λ)**.** *The space $\lambda_{\mathbb{C}}(\mathcal{H}om(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ inside $\mathcal{G}_{\mathbb{C}}$ is described by 3 complex algebraic equations. If k is the number of integer entries in $\boldsymbol{\vartheta}$, then the image of $\lambda_{\mathbb{C}}(\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ inside $\mathcal{G}_{\mathbb{C}}$ is described by $n + 2k + 3$ complex algebraic equations.*

3.1.5 Algebraic and semi-algebraic structure.

We can now turn to the main result of this section.

Proof of Lemma I. (o) By the constructions performed in Sections 3.1.1-3.1.2-3.1.3 the space $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ is homeomorphic via λ to a real algebraic subset of the vector space $V^{\oplus(2g+n)}$, on which PSU_2 acts algebraically: such λ induces on $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ a real algebraic structure. As discussed in Remark 1.7, if $p_1, \dots, p_\ell$ generate the algebra of PSU_2-invariant polynomial functions on $V^{\oplus(2g+n)}$, then $\mathcal{R}ep(\Pi_{g,n}, \mathrm{SU}_2)$ can be realized as a semi-algebraic subset of $\mathbb{R}^\ell$. As similarly discussed in Section 3.1.4, $\mathcal{H}om(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ embeds via $\lambda_{\mathbb{C}}$ as a complex algebraic subset of $V^{\oplus(2g+n)}$, on which $\mathrm{PSL}_2(\mathbb{C})$ acts algebraically. As in Remark 1.7, if $q_1, \dots, q_s$ generate the algebra of $\mathrm{PSL}_2(\mathbb{C})$-invariant polynomial functions on $V^{\oplus(2g+n)}$, then $\mathcal{R}ep(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ can be realized as a complex algebraic subset of $\mathbb{C}^s$.

(i) Recall that $\Pi_{g,n}$ is a free group and that the natural map $\mathrm{SU}_2 \to \mathrm{SO}_3(\mathbb{R})$ is an algebraic double cover. It follows that the map $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2) \to \mathcal{H}om(\Pi_{g,n}, \mathrm{SO}_3(\mathbb{R}))$ is algebraic, since it is the restriction of the natural map $(\mathrm{SU}_2)^{2g+n} \to \mathrm{SO}_3(\mathbb{R})^{2g+n}$, and that is a finite cover. Hence it is a local homeomorphism. Taking the quotient by $\mathrm{SO}_3(\mathbb{R})$, we obtain a cover between the corresponding representation spaces, which can be easily seen, following the construction in (o), to be an algebraic map.

(ii) Let $\rho \in \mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ with $\lambda(\rho) = (\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{B})$. It is easy to see that ρ is coaxial if and only if the sum of the images of $\mathrm{Ad}_{M_j} - I$, $\mathrm{Ad}_{N_j} - I$, $\mathrm{Ad}_{B_i} - I$ do not span the whole $\mathfrak{su}_2$. Since the latter property can be expressed through algebraic equations, the coaxial locus inside $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ is closed algebraic. The $\mathrm{SL}_2(\mathbb{C})$ case is similar.

(iii) The locus $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ inside $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$ is described by the PSU_2-invariant algebraic equations $\mathrm{tr}(B_i) = -2\cos(\pi\vartheta_i)$ for $i = 1, \dots, n$, and so it is a closed PSU_2-invariant algebraic subset. As a consequence, $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is a closed algebraic subset of $\mathcal{R}ep(\Pi_{g,n}, \mathrm{SU}_2)$. The proof for $G_{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})$ is analogous, with the only caveat that the equation $\mathrm{tr}(B_i) = -2\cos(\pi\vartheta_i)$ must be replaced by $B_i = (-I)^{\vartheta_i - 1}$ if $\vartheta_i \in \mathbb{Z}$.

(iv) The coaxial locus in $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is closed algebraic by (ii) and (iii); hence, so is the coaxial locus in $\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$. □

In the following example we show that SU_2-representation spaces can be non-algebraic.

Example 3.4 (Semi-algebraic nature of SU_2-representation space)**.** Let $g = 1$, $n = 2$ and $\vartheta_1 = \vartheta_2 = t$ for some fixed $t \in (0, 1)$. Via λ, the homomorphism space $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{1,2}, \mathrm{SU}_2)$ is identified to the algebraic subset of $(M_1, N_1, B_1, B_2) \in \mathrm{SU}_2^4$ such that $[M_1, N_1] = B_1B_2$ and $\mathrm{tr}(B_1) = \mathrm{tr}(B_2) = 2\cos\pi(t-1)$. Consider the algebraic subset $\mathcal{X}$ of $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{1,2}, \mathrm{SU}_2)$ defined by $\mathrm{tr}(N_1) = \mathrm{tr}(B_1B_2) = 2$, or equivalently by $N_1 = I$ and $B_1 = (B_2)^{-1}$. We claim that $\mathcal{X}/\mathrm{SU}_2$ is not algebraic. As a consequence, $\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{1,2}, \mathrm{SU}_2)$ itself is not algebraic but only semi-algebraic, according to Lemma I(o). For the same reason, $\mathcal{Rep}(\Pi_{1,2}, \mathrm{SU}_2)$ is not algebraic either.

In order to prove the claim, consider the subset $\mathcal{X}'$ of $\mathcal{X}$ defined by $B_1 = B$ with $B = \begin{pmatrix} e^{i\pi(t-1)} & 0 \\ 0 & e^{-i\pi(t-1)} \end{pmatrix}$. Since the stabilizer of B is the subgroup H of diagonal matrices in SU_2, the natural inclusion $\mathcal{X}' \hookrightarrow \mathcal{X}$ induces an isomorphism $\mathcal{X}'/H \cong \mathcal{X}/\mathrm{SU}_2$. Clearly, $\mathrm{SU}_2 \ni M \mapsto (M, I, B, B^{-1}) \in \mathcal{X}'$ is an isomorphism and so we only have to show that SU_2/H is not algebraic.

A quick way to proceed is to note that SU_2/H is homeomorphic to a closed 2-disk and to recall that an affine, complete, real algebraic variety has a non-zero $\mathbb{Z}/2$-valued fundamental class (see [9, Proposition 11.3.1]). Since this is not the case for the closed 2-disk, SU_2/H cannot be algebraic.

For a direct approach, recall that SU_2-invariant functions on the homomorphism space are polynomials in the trace functions $\rho \mapsto \mathrm{tr}(\rho(\gamma))$ [37, p. 1.31]. Hence, we consider the H-invariant functions $f, g : \mathrm{SU}_2 \to \mathbb{R}$ defined as $f(M) = \mathrm{tr}(M)$ and $g(M) = \mathrm{tr}(MB)$. We assert the f, g generate the algebra of H-invariant functions on SU_2. It follows that $(f, g) : \mathrm{SU}_2 \to \mathbb{R}^2$ induces an isomorphism of SU_2/H onto its image, which is contained inside the box $[-2, 2]^2$. Since SU_2/H has dimension 2, it follows that it cannot be algebraic.

To prove the assertion, let $M = \begin{pmatrix} m_{11} & -\overline{m}_{21} \\ m_{21} & \overline{m}_{11} \end{pmatrix}$ be an element of SU_2. The H-orbit of M consists of matrices of type $\begin{pmatrix} m_{11} & -\overline{m}_{21}e^{-is} \\ m_{21}e^{is} & \overline{m}_{11} \end{pmatrix}$ for all $s \in \mathbb{R}$. Now, the algebra of functions on SU_2 is generated by $\mathrm{Re}(m_{11}), \mathrm{Im}(m_{11}), \mathrm{Re}(m_{21}), \mathrm{Im}(m_{21})$. Since $m_{21}\overline{m}_{21} = 1 - m_{11}\overline{m}_{11}$, it is easy to see that the subalgebra of H-invariant functions is generated by $\mathrm{Re}(m_{11}), \mathrm{Im}(m_{11})$. Since $f(M) = 2\mathrm{Re}(m_{11})$ and $g(M) = 2\mathrm{Re}(m_{11})\cos\pi(t-1) - 2\mathrm{Im}(m_{11})\sin\pi(t-1)$, and since $\sin\pi(t-1) \neq 0$, the algebra of H-invariant functions on SU_2 is generated by f and g.

We can also easily understand some basic properties of the coaxial locus.

Proposition 3.5 (Coaxial locus)**.** *Let* $\boldsymbol{\vartheta} \in \mathbb{R}^n_{>0}$ *and* $2g - 2 + n > 0$.

- *(i) The coaxial locus in* $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$ *is a closed, connected, algebraic subset of pure dimension* $2g + n + 1$. *Hence the coaxial locus in* $\mathcal{Rep}(\Pi_{g,n}, \mathrm{SU}_2)$ *is a closed, connected, irreducible, algebraic subset of dimension* $2g + n - 1$.
- *(ii) The coaxial locus in* $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *is non-empty if and only if there exist* $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$ *such that* $\sum_i \varepsilon_i(\vartheta_i - 1) \in 2\mathbb{Z}$. *In this case, it is a closed, algebraic subset of pure dimension* $2g+2$. *Hence the coaxial locus in* $\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *is a closed, algebraic subset of pure dimension* $2g$.

Claim (ii) still holds if we replace SU_2 *by* $\mathrm{SL}_2(\mathbb{C})$ *and we interpret the dimensions as complex dimensions.*

Proof. (i) Since $\Pi_{g,n}$ is free on $\mu_1, \dots, \beta_{n-1}$, we can identify $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$ to SU_2^{2g+n-1}. Hence, the coaxial locus in SU_2^{2g+n-1} is closed by Lemma I(ii). Upon seeing $\mathbb{S}^2$ as the unit sphere in $\mathfrak{su}_2$, a 1-parameter subgroup of SU_2 can be described as $\exp(\mathbb{R} \cdot X)$ for a certain $X \in \mathbb{S}^2$. As a consequence, the coaxial locus in SU_2^{2g+n-1} is the image of the map

$$\begin{array}{ccc} \mathbb{S}^2 \times \mathbb{R}^{2g+n-1} & \longrightarrow & \mathrm{SU}_2^{2g+n-1} \\ (X, s_1, t_1, \dots, s_g, t_g, \vartheta_1, \dots, \vartheta_{n-1}) & \longmapsto & \mathbf{e}(s_1X, \dots, \vartheta_{n-1}X) \end{array}$$

where $\mathbf{e}$ is meant to operate componentwise. Such map has connected, irreducible domain of dimension $2g+n+1$ and its fibers are discrete over non-central $(2g+n-1)$-tuples and 2-dimensional over $\{\pm I\}^{2g+n-1}$. As a consequence, the coaxial locus in $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)$ is closed algebraic, irreducible, connected, of dimension $2g+n+1$. Moreover the stabilizer of a non-central coaxial $(2g+n-1)$-tuple has dimension 1. Hence the coaxial locus in $\mathcal{Rep}(\Pi_{g,n},\mathrm{SU}_2)$ is closed algebraic, irreducible, connected, of dimension $(2g+n+1)-(3-1)=2g+n-1$.

(ii) Identify $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ to its image via λ. Suppose that an element $(M_1,N_1,\dots,B_1,\dots,B_n)$ belongs to the coaxial locus of $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$. Then there exists a line $\mathfrak{h}\subset\mathfrak{su}_2$ such that $M_1,\dots,B_n$ belong to $H=\exp(\mathfrak{h})$. Moreover, if $X\in\mathbb{S}^2\cap\mathfrak{h}$, then there exist $s_i,t_i\in\mathbb{R}$ and $\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}$ such that $(M_1,\dots,B_n)=\mathbf{e}(s_1X,t_1X,\dots,t_gX,\varepsilon_1\vartheta_1X,\dots,\varepsilon_n\vartheta_nX)$. This implies that $\mathrm{e}(\varepsilon_1\vartheta_1X)\cdots\mathrm{e}(\varepsilon_n\vartheta_nX)=I$, namely that $\sum_j\varepsilon_j(\vartheta_j-1)$ is an even integer. Conversely, suppose that there exist $\varepsilon_1,\dots,\varepsilon_n\in\{\pm1\}$ such that $\sum_j\varepsilon_j(\vartheta_j-1)\in2\mathbb{Z}$. For every such $\boldsymbol{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_n)$, consider the map

$$\begin{array}{ccc}\mathbb{S}^2\times\mathbb{R}^{2g} & \longrightarrow & \mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)\\ (X,s_1,t_1,\dots,s_g,t_g) & \longmapsto & \mathbf{e}(s_1X,\dots,t_gX,\varepsilon_1\vartheta_1X,\dots,\varepsilon_n\vartheta_nX)\end{array}$$

and denote its image by $\mathcal{Hom}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$. Such map has connected, irreducible domain of dimension $2g+2$ and its fibers are discrete over non-central homomorphisms and 2-dimensional over central ones. As a consequence, $\mathcal{Hom}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is connected, irreducible, of dimension $2g+2$. Now, the union of all such $\mathcal{Hom}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is exactly the coaxial locus, which has thus pure dimension $2g+2$. Moreover, it is closed algebraic by Lemma I(ii-iii). The conclusion for $\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ easily follows.

The proof in the $\mathrm{SL}_2(\mathbb{C})$ case is almost identical: it is enough to replace $\mathbb{R}^{2g}$ by $\mathbb{C}^{2g}$ and to note that $\mathfrak{h}$ must be a complex 1-dimensional subspace of $\mathfrak{sl}_2(\mathbb{C})$ that consists of diagonalizable matrices with imaginary eigenvalues, and that $X\in\mathfrak{h}$ must satisfy $\mathcal{K}(X,X)=1$. □

We can now establish the main properties of absolute representation spaces.

Proof of Theorem II. (i) is essentially Lemma 2.4.

Let us first prove (ii-iii) in the SU_2 case.

(ii) Since $n>0$, the group $\Pi_{g,n}$ is free on $2g+n-1\geq2$ generators. It follows that $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)\cong\mathrm{SU}_2^{2g+n-1}$, and so it is a smooth algebraic set of dimension $6g+3n-3$.

(ii-a) The irreducibility of the coaxial locus in $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)$ is proven in Proposition 3.5(i), where its dimension $2g+n+1$ is also computed.

(ii-b) It is enough to note that the non-coaxial locus is open by Lemma I(ii).

(ii-c) It is easy to see that the central locus is clearly 0-dimensional and that the general coaxial homomorphism is non-central, and so has 1-dimensional stabilizer. It follows from (ii-a) that the coaxial locus in $\mathcal{Rep}(\Pi_{g,n},\mathrm{SU}_2)$ has dimension $(2g+n+1)-(3-1)$.

(ii-d) follows from (ii-b), using Lemma 2.4 and Remark 1.9.

(iii) Connectedness of the coaxial locus is proven in Proposition 3.5(i). Connectedness and density of the non-coaxial locus is proven in Proposition 3.17 below.

Finally, the proof of (ii-iii) in the $\mathrm{SL}_2(\mathbb{C})$ case is analogous, provided one notices that the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathcal{Hom}^{nc}(\Pi_{g,n},\mathrm{SL}_2(\mathbb{C}))$ is proper (see, for example, [35, Section 5.3.4]). □

3.1.6 Explicit description of $\mathcal{Hom}$ in simple cases.

Before investigating the properties of relative homomorphism spaces, we wish to analyze two peculiar cases, in which we are able to explicitly describe all homomorphisms up to conjugation. Note that these are exactly the cases of simple triples $(g,n,\boldsymbol{\vartheta})$, namely those for which $k\geq2g-2+n$, where is the number of integer entries of $\boldsymbol{\vartheta}$.

We first treat simple cases of genus 0.

Proposition 3.6 (Surfaces of genus 0 with at most 2 non-integral angles)**.** *Assume $g=0$ and let k be the number of integer entries of $\boldsymbol{\vartheta}\in\mathbb{R}^n_{>0}$.*

- *(o) If $n-k=0$, then $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ is non-empty if and only if $\sum_i(\vartheta_i-1)$ is even: in this case $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ consists of a single point.*
- *(i) If $n-k=1$, then $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ is empty.*

(ii) Suppose $n-k=2$ and $\vartheta_1,\vartheta_2\notin\mathbb{Z}$. Then $\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{0,n},\mathrm{SU}_2)$ is non-empty if and only if $\bar\delta_1=\bar\delta_2$: in this case $\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{0,n},\mathrm{SU}_2)$ can be identified to the conjugacy class $\mathcal{C}_{\bar\delta_1}$.

Proof. Identify $\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{0,n},\mathrm{SU}_2)$ with the locus of $(B_1,\dots,B_n)$ in $\mathcal{C}_{\bar\delta_1}\times\cdots\times\mathcal{C}_{\bar\delta_n}$ such that $B_1\cdots B_n=I$.
(o) This is immediate, since B_i must be equal to $(-I)^{\vartheta_i-1}$.
(i) Suppose $\vartheta_1\notin\mathbb{Z}$. This is again immediate, since $B_2,\dots,B_n\in\{\pm I\}$, but B_1 can be equal to $\pm I$.
(ii) Note that $B_i=(-I)^{\vartheta_i-1}$ for $i\geq 3$ and that $B_1\in\mathcal{C}_{\bar\delta_1}$. It follows that B_2 is uniquely determined by B_1. Hence the map

$$\begin{array}{rcl}\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{0,n},\mathrm{SU}_2) & \longrightarrow & \mathcal{C}_{\bar\delta_1}\\ (B_1,\dots,B_n) & \longmapsto & B_1\end{array}$$

is an isomorphism. □

Now we consider the simple cases in genus 1.

Proposition 3.7 (Surfaces of genus 1 with integral angles)**.** *Suppose that $g=1$ and $\boldsymbol\vartheta\in\mathbb{Z}^n$. Then*

(i) if $\sum_i(\vartheta_i-1)$ is odd, then $(1,n,\boldsymbol\vartheta)$ is not special and every $\rho\in\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ is conjugate to the non-coaxial homomorphism ρ' determined by

$$\lambda(\rho')=\left(\begin{pmatrix}i&0\\0&-i\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix},(-I)^{\vartheta_1-1},\dots,(-I)^{\vartheta_n-1}\right).$$

It follows that $\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ is isomorphic to PSU_2 and that $\mathcal{R}ep_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ consists of a single point.

(ii) if $\sum_i(\vartheta_i-1)$ is even, then $(1,n,\boldsymbol\vartheta)$ is special and every $\rho\in\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ is conjugate to the coaxial homomorphism $\rho'_{s,t}$ determined by

$$\lambda(\rho'_{s,t})=\left(\begin{pmatrix}e^{i(s-1)\pi}&0\\0&e^{-i(s-1)\pi}\end{pmatrix},\begin{pmatrix}e^{i(t-1)\pi}&0\\0&e^{-i(t-1)\pi}\end{pmatrix},(-I)^{\vartheta_1-1},\dots,(-I)^{\vartheta_n-1}\right),$$

for some $t,s\in\mathbb{R}/2\mathbb{Z}$. It follows that $\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ has four points corresponding to central homomorphisms, and the complement is diffeomorphic to an $\mathbb{S}^2$-bundle over $S^2\setminus\{4\text{ points}\}$. As a consequence, $\mathcal{R}ep_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ is homeomorphic to S^2 and four of its points correspond to central representations.

Proof. Recall that every $\rho\in\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ must satisfy $\rho(\beta_i)=(-I)^{\vartheta_i-1}$.
(i) In this case $\rho(\beta_1)\cdots\rho(\beta_n)=-I$. It was proven in [15, Corollary A.3] that ρ is conjugate to ρ'. Hence, $\mathcal{R}ep_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ consists of a single point. Since ρ' is non-coaxial, by Lemma 2.4 the stabilizer of ρ' is trivial, and so the map $\mathrm{PSU}_2\ni g\mapsto g\rho' g^{-1}\in\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ is an isomorphism.
(ii) In this case $\rho(\beta_1)\cdots\rho(\beta_n)=I$, and so $\rho(\mu_1),\rho(\nu_1)$ commute and can be simultaneously diagonalized. It is immediate that ρ must be conjugate to some $\rho'_{s,t}$.
As for the description of $\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$, note first that the four central homomorphisms are parametrized by the values $\pm I$ taken at μ_1,ν_1. Now, identify $\mathbb{S}^2$ with the unit sphere in $\mathfrak{su}_2$ and let $\mathcal{X}:=\mathbb{S}^2\times(\mathbb{R}^2\setminus\mathbb{Z}^2)/(2\mathbb{Z})^2$. Note that $\{\pm 1\}$ acts on $\mathcal{X}$ by multiplication, namely $(-1)\cdot(\hat X,[s,t]):=(-\hat X,[-s,-t])$ for every element $(\hat X,[s,t])$ of $\mathcal{X}$, and consider the map

$$\begin{array}{rcl}\mathcal{X} & \longrightarrow & \mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)\\ (\hat X,[s,t]) & \longmapsto & \mathbf{e}(s\hat X,t\hat Y)\end{array}$$

Such map factors through the quotient $\mathcal{X}/\{\pm1\}$, and in fact it sends $\mathcal{X}/\{\pm1\}$ isomorphically onto the locus of non-central homomorphisms. Note that $(\mathbb{R}^2\setminus\mathbb{Z}^2)/(2\mathbb{Z})^2$ is diffeomorphic to a 2-torus with four points removed and that its quotient by $\{\pm1\}$ is diffeomorphic to a 2-sphere with four points removed, which we denote by $\dot\Sigma_{0,4}$. It follows that the projection $\mathcal{X}/\{\pm1\}\to\dot\Sigma_{0,4}$ is an $\mathbb{S}^2$-bundle. Finally, fix $\hat X_D:=\mathrm{diag}(i,-i)$ and consider the closed locus in $\mathcal{H}om_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ corresponding to $(\mathrm{e}(s\hat X_D),\mathrm{e}(t\hat X_D))=\rho'_{s,t}$, which is diffeomorphic to $\mathbb{R}^2/(2\mathbb{Z})^2$. Its stabilizer under the conjugacy action of SU_2 is generated by the subgroup of diagonal matrices, and by $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ that acts by multiplication by -1 on $\mathbb{R}^2/(2\mathbb{Z})^2$. Hence, $\mathcal{R}ep_{\boldsymbol\vartheta}(\Pi_{1,n},\mathrm{SU}_2)$ is homeomorphic to $(\mathbb{R}^2/(2\mathbb{Z})^2)/\{\pm1\}$, which is homeomorphic to S^2; moreover the four points $[0,0],[1,0],[0,1],[1,1]$ correspond to central representations. □

3.2 Connectedness and monotone connectedness

In this short section we recall two criteria for the connectedness of the total space and of the fibers of a fibration. We also introduce monotone connected pairs, that will be useful to study relative homomorphism spaces in genus 0.

The following statement is rather standard: a proof is included for completeness.

Lemma 3.8 (Surjective maps and connectedness)**.** *Let $f : X \to Y$ be a continuous, surjective map between locally finite CW-complexes.*

(i) Suppose that f is proper or open, and that f has connected fibers. Then X is connected $\iff$ Y is connected.

(ii) Suppose that X is connected, Y is simply-connected and f is a fiber bundle. Then all fibers of f are connected.

Proof. (i) Since f is surjective, X connected $\implies$ Y connected. Conversely, suppose that Y is connected and let $U, V \subseteq X$ be disjoint, open subsets such that $U \cup V = X$ (which implies that U, V are also closed). We want to show that either U or V is empty.

Since the fibers of f are connected, each fiber of f is completely contained either in U or in V. It follows that $f(U) \cap f(V) = \emptyset$. Since f is surjective, $f(U) \cup f(V) = Y$. We separately consider the following two cases.

Suppose first that f is open. Then $f(U)$ and $f(V)$ are open. Since Y is connected, either $f(U)$ or $f(V)$ is empty. As a consequence, either U or V must be empty, and so X is connected.

Suppose now that f is proper. Then f is closed because Y is locally compact and Hausdorff. It follows that $f(U)$ and $f(V)$ are closed. Since Y is connected, either $f(U)$ or $f(V)$ must be empty. Again, either U or V must be empty and so X is connected.

(ii) Pick $x \in X$, $y = f(x) \in Y$ and let $F = f^{-1}(y)$. Then the exact sequence in homotopy for the fibration $(F, x) \to (X, x) \to (Y, y)$ gives $\pi_1(Y, y) \to \pi_0(F, x) \to \pi_0(X, x)$. Since $\pi_1(Y, y)$ and $\pi_0(X, x)$ are trivial by hypothesis, $\pi_0(F, x)$ is too and so F is connected. □

In order to establish the connectedness of the non-coaxial locus $\mathcal{H}om_{\vartheta}(\Pi_{0,n}, \mathrm{SU}_2)$, we will use the formalism of monotone-connected pairs here presented.

Definition 3.9 (Monotone-connected pair)**.** Consider a CW-complex X with a continuous function $f : X \to \mathbb{R}$. A path $\gamma : [0, 1] \to X$ is called *monotone* if the function $f(\gamma(t))$ is monotonic. We say that the pair (X, f) is *monotone-connected* if any two points $x, y \in X$ can be joined by a monotone path.

Note that, according to this definition, if (X, f) is monotone-connected, then the level sets $\{f = c\}$ are connected. Our aim is to show the following.

Proposition 3.10 (Criterion for monotone-connectedness)**.** *Let M be a connected real-analytic variety and f be an analytic function on M. Then (M, f) is monotone connected if and only if each level set $f^{-1}(c)$ is connected.*

We start with a simpler claim.

Lemma 3.11 (Sufficient conditions for monotone-connectedness)**.** *Suppose X is a compact CW-complex, f is a continuous function on X, and each level set $\{f = c\}$ is connected. Then (X, f) is monotone-connected if one of the following holds.*

(i) The exists a monotone path $\gamma \subset X$ that connects a point where f attains its minimum $\min(f)$ *with a point where f attains its maximums* $\max(f)$*.*

(ii) For any $c \in [\min f, \max f]$ there is $\varepsilon > 0$ and a monotone path $\gamma_{c,\varepsilon} \subset X$ on which f attains all values from the interval $[\min f, \max f] \cap [c - \varepsilon, c + \varepsilon]$*.*

Proof. (i) Since each level set of f is connected, for any $x, y \in X$ we can choose a path that first joins x to a point of γ inside the level set $\{f = f(x)\}$, then follows γ and then connects to y in the level set $\{f = f(y)\}$.

(ii) The interval $[\min f, \max f]$ is covered by sub-intervals for which monotone paths $\gamma_{c,\varepsilon}$ exist. We can choose a finite sub-cover and construct a monotone γ as in (1) that follows the corresponding finite collection of paths and jumps from one to another along level sets of f. □

Now we can prove our criterion.

Proof of Proposition 3.10. "Only if" direction. Assume by contrapositive that a level set $f^{-1}(c)$ is not connected. Then any two points $x, y \in f^{-1}(c)$ that lie in different connected components of $f^{-1}(c)$ cannot be connected by a monotone path in M.

"If" direction. It will be enough to show that condition (ii) of Lemma 3.11 holds. We will prove a half of this statement: namely, that there exists a monotone path $\gamma^-_{c,\varepsilon}$ that attains all values in the interval $[\min f, c] \cap [c-\varepsilon, c]$. The other half is proven identically.

We can assume $c > \min f$. The subset $M_{<c} \subset M$ where $f < c$ is semi-analytic. Since M is connected, there is a point $x \in f^{-1}(c)$ that lies in the closure of $M_{<c}$. By the real-analytic curve selection lemma [40, Paragraph 3], there is a real-analytic map $\gamma : [0,1] \to M$, such that $\gamma(0) = x$, and $f(\gamma(t)) < c$ for $t > 0$. Since $f(\gamma(t))$ is an analytic function, it is monotonic for t small enough. So we choose $\gamma^-_{c,\varepsilon}$ as a sub-path of γ. □

3.3 First-order computations: commutator map and product map

It order to analyze the maps R and $R_{\boldsymbol{\vartheta}}$, we first compute the differentials of the commutator map and of the product map.

The *commutator map* is defined as follows

$$\begin{aligned} c:\ \mathrm{SU}_2 \times \mathrm{SU}_2 &\longrightarrow \mathrm{SU}_2 \\ (M,N) &\longmapsto [M,N] \end{aligned}$$

We also let $\boldsymbol{c} : (\mathrm{SU}_2 \times \mathrm{SU}_2)^g \longrightarrow \mathrm{SU}_2$ be defined as $\boldsymbol{c}(\boldsymbol{M}, \boldsymbol{N}) := c(M_1, N_1) \cdots c(M_g, N_g)$.

The following computation is already contained in Example 3.7 in [26]. We reproduce here for completeness.

Lemma 3.12 (Differential of a product of commutators)**.** *The image of the differential $d\boldsymbol{c}$ at the point $(\boldsymbol{M}, \boldsymbol{N})$ is $\mathfrak{z}(M_1, \dots, N_g)^\perp$, namely the orthogonal in $\mathfrak{su}_2$ to the infinitesimal centralizer of the set $\{M_1, \dots, N_g\}$.*

Proof. Fix $M_1, \dots, N_g \in \mathrm{SU}_2$ and let $\dot V_i, \dot W_i \in \mathfrak{su}_2$ for $i = 1, \dots, g$. Consider the paths

$$M_i(t) = \exp(t\dot V_i) M_i = (I + t\dot V_i + o(t)) M_i, \qquad N_i(t) = \exp(t\dot W_i) N_i = (I + t\dot W_i + o(t)) N_i$$

in SU_2. Under our conventions, the tangent vector $\frac{d}{dt} M_i(t)|_{t=0}$ (resp. $\frac{d}{dt} N_i(t)|_{t=0}$) is identified to the element $\dot V_i$ (resp. $\dot W_i$) of $\mathfrak{su}_2$. Inside the space of complex 2×2 matrices, we now compute

$$\begin{aligned} \frac{d}{dt} c(M_1(t), N_1(t))\big|_{t=0} &= \dot V_1 M_1 N_1 M_1^{-1} N_1^{-1} + M_1 \dot W_1 N_1 M_1^{-1} N_1^{-1} - M_1 N_1 M_1^{-1} \dot V_1 N_1^{-1} - M_1 N_1 M_1^{-1} N_1^{-1} \dot W_1 = \\ &= \Big(\dot V_1 + M_1 \dot W_1 M_1^{-1} - (M_1 N_1 M_1^{-1}) \dot V_1 (M_1 N_1 M_1^{-1})^{-1} \\ &\qquad - (M_1 N_1 M_1^{-1} N_1^{-1}) \dot W_1 (M_1 N_1 M_1^{-1} N_1^{-1})^{-1} \Big) [M_1, N_1] = \\ &= \Big((I - \mathrm{Ad}_{M_1 N_1^{-1} M_1^{-1}}) \dot V_1 + \mathrm{Ad}_{M_1} \circ (I - \mathrm{Ad}_{N_1 M_1 N_1^{-1}}) \dot W_1 \Big) c(M_1, N_1). \end{aligned}$$

Hence, $\mathrm{Im}(dc)_{(M_1,N_1)} = \mathrm{Im}(I - \mathrm{Ad}_{M_1 N_1^{-1} M_1^{-1}}) + \mathrm{Im}(\mathrm{Ad}_{M_1} \circ (I - \mathrm{Ad}_{N_1 M_1 N_1^{-1}}))$. Note that

$$\mathrm{Im}(I - \mathrm{Ad}_{M_1 N_1^{-1} M_1^{-1}}) = \ker(I - \mathrm{Ad}_{M_1 N_1^{-1} M_1^{-1}})^\perp = \mathfrak{z}(M_1 N_1^{-1} M_1^{-1})^\perp = \mathrm{Ad}_{M_1}(\mathfrak{z}(N_1)^\perp).$$

On the other hand,

$$\mathrm{Im}(\mathrm{Ad}_{M_1} \circ (I - \mathrm{Ad}_{N_1 M_1 N_1^{-1}})) = \mathrm{Ad}_{M_1}(\ker(I - \mathrm{Ad}_{N_1 M_1 N_1^{-1}})^\perp) = \mathrm{Ad}_{M_1}(\mathfrak{z}(N_1 M_1 N_1^{-1})^\perp)$$

and so

$$\mathrm{Im}(dc)_{(M_1,N_1)} = \mathrm{Ad}_{M_1}(\mathfrak{Z}(N_1M_1N_1^{-1},N_1)^{\perp}) = \mathrm{Ad}_{M_1}(\mathfrak{Z}(M_1,N_1))^{\perp} = \mathfrak{Z}(M_1,N_1)^{\perp} \tag{2}$$

where the second equality depends on the fact that the subgroup generated by $\{N_1M_1N_1^{-1},\, N_1\}$ agrees with the one generated by $\{M_1,\, N_1\}$, and the third equality depends on the fact that Ad_{M_1} is an isometry of $\mathfrak{su}_2$ and fixes $\mathfrak{Z}(M_1,N_1)$.

Now denote $c(M_i(t),N_i(t))$ by c_i and $\boldsymbol{c}(M_1(t),\dots,N_g(t))$ simply by $\boldsymbol{c}$, and let $\dot{c}_i$ and $\dot{\boldsymbol{c}}$ be their derivative at $t=0$. Observe that

$$\dot{\boldsymbol{c}} = (\dot{c}_1c_1^{-1})\cdot\boldsymbol{c} + c_1(\dot{c}_2c_2^{-1})c_1^{-1}\cdot\boldsymbol{c} + c_1c_2(\dot{c}_3c_3^{-1})c_2^{-1}c_1^{-1}\cdot\boldsymbol{c}+\cdots = \left(\sum_{i=1}^{g}\mathrm{Ad}_{\hat{c}_{i-1}}(\dot{c}_ic_i^{-1})\right)\cdot\boldsymbol{c}$$

where $\hat{c}_0 = I$ and $\hat{c}_j = [M_1,N_1]\cdot[M_2,N_2]\cdots[M_j,N_j]$ for $j=1,\dots,g-1$. Thus,

$$\begin{aligned}\mathrm{Im}(d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})}) &= \sum_{i=1}^{g}\mathrm{Ad}_{\hat{c}_{i-1}}(\mathrm{Im}(dc_i)_{(M_i,N_i)}) = \sum_{i=1}^{g}\mathfrak{Z}(\mathrm{Ad}_{\hat{c}_{i-1}}(M_i),\mathrm{Ad}_{\hat{c}_{i-1}}(N_i))^{\perp} = \\ &= \left(\bigcap_{i=1}^{g}\mathfrak{Z}\big(\mathrm{Ad}_{\hat{c}_{i-1}}(M_i),\mathrm{Ad}_{\hat{c}_{i-1}}(N_i)\big)\right)^{\perp}\end{aligned}$$

where the second equality relies on (2). It follows that $\mathrm{Im}(d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})}) = \mathfrak{Z}(H)^{\perp}$, where $H < \mathrm{SU}_2$ is the subgroup generated by

$$\{M_1, N_1, \mathrm{Ad}_{[M_1,N_1]}M_2, \mathrm{Ad}_{[M_1,N_1]}N_2, \mathrm{Ad}_{[M_1,N_1][M_2,N_2]}M_3,\dots\}$$

Such H agrees with the subgroup generated by $\{\boldsymbol{M},\boldsymbol{N}\}$. Hence $\mathrm{Im}(d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})}) = \mathfrak{Z}(\boldsymbol{M},\boldsymbol{N})^{\perp}$, as desired. □

A first consequence of the computation in Lemma 3.12 is the following.

Corollary 3.13 (Surjectivity and connectedness of the commutator map)**.** *The commutator map c is proper, algebraic, surjective. Moreover the following hold.*

(i) *$c^{-1}(I)$ is connected of dimension 4 and $c^{-1}(I)\setminus(H\times H)$ is connected too for every 1-parameter subgroup H of SU_2.*

(ii) *All the fibers of c different from $c^{-1}(I)$ are smooth, connected of dimension 3, and they are all isomorphic to each other.*

(iii) *For all $d\in[0,1]$ the preimage $c^{-1}(\mathcal{C}_d)$ of the conjugacy class $\mathcal{C}_d$ is connected.*

Proof. Properness and algebraicity are obvious. Consider now $B_0 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $B = \begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix}$, where $|z|=1$. A direct computation gives $[B,B_0]=B^2$.

It follows that every diagonal matrix of SU_2 is in the image of the commutator map. Since such image is invariant under conjugation, it must be the whole SU_2 and so c is surjective.

(i) Consider the map $w:\mathbb{S}^2\times[0,2)^2\to\mathrm{SU}_2\times\mathrm{SU}_2$ that sends (X,t,s) to $(\exp(2\pi tX),\exp(2\pi sX))$. Its image is $c^{-1}(I)$. Moreover, the restriction of w to $\mathbb{S}^2\times((0,1)\cup(1,2))^2$ is injective. Hence, $c^{-1}(I)$ has dimension 4. If $H=\exp(\mathfrak{h})$ for a certain 1-dimensional subspace $\mathfrak{h}\subset\mathfrak{su}_2$, then $c^{-1}(I)\setminus(H\times H)$ is the image of the restriction of w to $(\mathbb{S}^2\setminus\mathfrak{h})\times(0,2)^2$, and so is connected.

(ii) Consider the restriction of the commutator map to $(\mathrm{SU}_2\times\mathrm{SU}_2)\setminus c^{-1}(I)\to\mathrm{SU}_2\setminus\{I\}$. Such restriction is proper, surjective and it is submersive by Lemma 3.12. Hence such restriction is a fiber bundle and so the fibers have dimension $6-3=3$. Since $(\mathrm{SU}_2\times\mathrm{SU}_2)\setminus c^{-1}(I)$ is connected and $\mathrm{SU}_2\setminus\{I\}$ is simply connected, the fibers over an element different from I are connected by Lemma 3.8(ii).

(iii) The case $d=0$ follows from (i). For $d\in(0,1]$ it follows from the discussion in (ii) that $c^{-1}(\mathcal{C}_d)\to\mathcal{C}_d$ is a fiber bundle with connected base and fibers, and so it is connected by Lemma 3.8(i). □

We now consider the *product map*

$$\begin{aligned}P:\quad \mathrm{SU}_2^n &\longrightarrow \mathrm{SU}_2\\ (B_1,\dots,B_n) &\longmapsto B_1B_2\cdots B_n.\end{aligned}$$

Given an angle vector $\boldsymbol{\vartheta}$, we also denote by $P_{\boldsymbol{\vartheta}}$ the restriction of P to $\mathcal{C}_{\boldsymbol{d}} := \mathcal{C}_{\bar{\delta}_1} \times \cdots \times \mathcal{C}_{\bar{\delta}_n}$. Note that $T_{B_i}\mathcal{C}_{\bar{\delta}_i} = \mathfrak{z}(B_i)^{\perp}$.

Lemma 3.14 (Differential of P). *Let $m : \mathrm{SU}_2 \times \mathrm{SU}_2 \to \mathrm{SU}_2$ be the multiplication map $m(Q_1, Q_2) := Q_1 \cdot Q_2$.*

(i) Upon identifying the tangent spaces of SU_2 with $\mathfrak{su}_2$ as in Section 2.1, we have

$$dm_{(Q_1,Q_2)}(\dot{V}_1, \dot{V}_2) = \dot{V}_1 + \mathrm{Ad}_{Q_1}(\dot{V}_2).$$

(ii) The differential of P satisfies

$$dP_{\boldsymbol{B}}(\dot{\boldsymbol{X}}) = \dot{X}_1 + \sum_{i=2}^{n} \mathrm{Ad}_{B_1 \cdots B_{i-1}}(\dot{X}_i).$$

where $\boldsymbol{B} = (B_1, \ldots, B_n)$ and $\dot{\boldsymbol{X}} = (\dot{X}_1, \ldots, \dot{X}_n)$.

(iii) If $\boldsymbol{B} \in \mathcal{C}_{\boldsymbol{d}}$, then $\mathrm{Im}(dP)_{\boldsymbol{B}} = \mathfrak{su}_2$ and $\mathrm{Im}(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}} = \mathfrak{z}(\boldsymbol{B})^{\perp}$.

Proof. (i) Consider the paths $Q_1(t) = \exp(t\dot{V}_1)Q_1$ and $Q_2(t) = \exp(t\dot{V}_2)Q_2$. Then $Q_1(t)Q_2(t) = (I + t\dot{V}_1)Q_1(I + t\dot{V}_2)Q_2 + o(t) = Q_1Q_2 + t(\dot{V}_1Q_1Q_2 + Q_1\dot{V}_2Q_2) + o(t) = Q_1Q_2 + t(\dot{V}_1 + Q_1\dot{V}_2Q_1^{-1})Q_1Q_2 + o(t)$ and the conclusion follows.

(ii) is obtained by iterating (i).

(iii) It immediately follows from (ii) that $dP_{\boldsymbol{B}}$ is surjective. As for the image of $(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}}$, we have three cases.

If all $B_i = \pm I$, then clearly such image is $\{0\}$.

Suppose now that all B_i belong to a 1-parameter subgroup $H = \exp(\mathfrak{h})$ but not all of them are $\pm I$. Since $T_{B_i}\mathcal{C}_{\bar{\delta}_i} \subseteq \mathfrak{h}^{\perp}$, then $\mathrm{Im}(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}} \subseteq \mathfrak{h}^{\perp}$. If k is the smallest index so that $B_k \neq \pm I$, then $(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}}(0, \ldots, 0, \dot{X}_k, 0, \ldots, 0) = \dot{X}_k$ for all $\dot{X}_k \in \mathfrak{h}^{\perp}$ by (ii), and so $\mathrm{Im}(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}} = \mathfrak{h}^{\perp}$.

Finally, if $B_1, \ldots, B_n$ are not contained in a 1-parameter subgroup, then let j be the smallest index so that $B_j \neq \pm I$ and let $k > j$ be the smallest index so that $\{B_j, B_k\}$ are not contained in the same 1-parameter subgroup. Then $(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}}(0, \ldots, 0, \dot{X}_j, 0, \ldots, 0) = \dot{X}_j$ for all $\dot{X}_j \in \mathfrak{z}(B_j)^{\perp}$, and so $\mathrm{Im}(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}} \supseteq \mathfrak{z}(B_j)^{\perp}$. Let now $\dot{X}_k$ be in $\mathfrak{z}(B_k)^{\perp}$ but not in $\mathfrak{z}(B_j)^{\perp}$. Then $(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}}(0, \ldots, 0, \dot{X}_k, 0, \ldots, 0) \notin \mathfrak{z}(B_j)^{\perp}$ by (ii), and so $\mathrm{Im}(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}} \neq \mathfrak{z}(B_j)^{\perp}$, which implies that $(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}}$ is surjective. □

3.4 First-order computations: the maps R and $R_{\boldsymbol{\vartheta}}$

Let us explicitly deal with homomorphisms in SU_2, as the $\mathrm{SL}_2(\mathbb{C})$ case will be analogous. Recall the definitions of the smooth algebraic varieties $\mathcal{G}$ and $\mathcal{G}_{\boldsymbol{\vartheta}}$ and of the algebraic maps R, $R_{\boldsymbol{\vartheta}}$ introduced in Section 3.1.

Since the image of $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ via λ is defined by the equation $R_{\boldsymbol{\vartheta}} = I$ inside $\mathcal{G}_{\boldsymbol{\vartheta}}$, by the implicit function theorem its smooth locus is detected by looking at points at which $dR_{\boldsymbol{\vartheta}}$ has locally constant rank. Since in non-special cases $dR_{\boldsymbol{\vartheta}}$ will have generically fully rank (this will follows from Proposition 3.15(i) and Theorem 3.27), a point $\rho \in \mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ will be smooth if and only if $dR_{\boldsymbol{\vartheta}}$ is surjective at $\lambda(\rho)$. Analogous considerations hold for $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$, whose image via λ is described by the equation $R = I$ inside $\mathcal{G}$.

For the above reason, we begin by analysing the ranks of the differentials of R and $R_{\boldsymbol{\vartheta}}$.

We remind that $\mathfrak{z}(\rho)$ is the infinitesimal centralizer of the image of ρ.

Proposition 3.15 (Differentials of R and $R_{\boldsymbol{\vartheta}}$). *For every $\rho \in \mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$, the images of the differentials of $R : \mathcal{G} \to \mathrm{SU}_2$ and $R_{\boldsymbol{\vartheta}} : \mathcal{G}_{\boldsymbol{\vartheta}} \to \mathrm{SU}_2$ at $\lambda(\rho)$ are*

(i) $\mathrm{Im}(dR_{\boldsymbol{\vartheta}})_{\lambda(\rho)} = \mathfrak{z}(\rho)^{\perp}$;

(ii) dR is sujective at $\lambda(\rho)$.

Observe that, if we identify $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$ with its image inside $\mathcal{G}$ via λ, then

$$R(\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{B}) = \boldsymbol{c}(\boldsymbol{M}, \boldsymbol{N}) \cdot P(\boldsymbol{B}).$$

Such observation will play an important role in the below computation, and motivates why we analyzed the differentials of the maps $\boldsymbol{c}$ and P in Section 3.3.

Proof of Proposition 3.15. First of all, note that by Lemma 3.14(i) the differential dR consists of two summands

$$dR_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})}(\dot{\boldsymbol{V}},\dot{\boldsymbol{W}},\dot{\boldsymbol{X}}) = d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})}(\dot{\boldsymbol{V}},\dot{\boldsymbol{W}}) + \mathrm{Ad}_{\boldsymbol{c}(\boldsymbol{M},\boldsymbol{N})} dP_{\boldsymbol{B}}(\dot{\boldsymbol{X}})$$

and so

$$\mathrm{Im}(dR_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})}) = \mathrm{Im}(d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})}) + \mathrm{Ad}_{\boldsymbol{c}(\boldsymbol{M},\boldsymbol{N})}(\mathrm{Im}(dP_{\boldsymbol{B}})). \tag{3}$$

Analogously for $dR_{\boldsymbol{\vartheta}}$ we have

$$\mathrm{Im}(dR_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})} = \mathrm{Im}(d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})}) + \mathrm{Ad}_{\boldsymbol{c}(\boldsymbol{M},\boldsymbol{N})}(\mathrm{Im}(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}}). \tag{4}$$

Let now $(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B}) = \lambda(\rho)$.

(i) In order to compute the image of $dR_{\boldsymbol{\vartheta}}$ at $(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})$, we separately consider three cases.

If $\dim \mathfrak{Z}(\boldsymbol{M},\boldsymbol{N}) = 0$, then $\mathrm{Im}(d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})}) = \mathfrak{su}_2$ by Lemma 3.12. Thus, $\mathrm{Im}(dR_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})} = \mathfrak{su}_2$ by (4).

Suppose that $\mathfrak{Z}(\boldsymbol{M},\boldsymbol{N}) = \mathfrak{h}$ is 1-dimensional, and let $H = \exp(\mathfrak{h})$. Then $\boldsymbol{c}(\boldsymbol{M},\boldsymbol{N}) = I$ and $\mathrm{Im}(d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})}) = \mathfrak{h}^\perp$. By Lemma 3.14(iii), the image of $(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}}$ is $\mathfrak{Z}(\boldsymbol{B})^\perp$. If $B_1,\dots,B_n \in H$, then $\mathfrak{Z}(\boldsymbol{B}) \supseteq \mathfrak{h}$ and so $\mathrm{Im}(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}} \subseteq \mathfrak{h}^\perp$; then it follows from (4) that $\mathrm{Im}(dR_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})} = \mathfrak{h}^\perp$. If some $B_i \notin H$, then $\mathfrak{Z}(\boldsymbol{B}) \not\supseteq \mathfrak{h}$ and so $\mathrm{Im}(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}} \not\subseteq \mathfrak{h}^\perp$; then it follows from (4) that $\mathrm{Im}(dR_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})} = \mathfrak{su}_2$.

Suppose now that $\dim \mathfrak{Z}(\boldsymbol{M},\boldsymbol{N}) = 3$, and so all $M_i, N_i = \pm I$. Thus the image of $(dR_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})}$ is equal to the image of $(dP_{\boldsymbol{\vartheta}})_{\boldsymbol{B}}$ by (4), and the result follows from Lemma 3.14(iii).

In all cases the image of $dR_{\boldsymbol{\vartheta}}$ at $\lambda(\rho)$ agrees with $\mathfrak{Z}(\rho)^\perp$.

(ii) Since $n > 0$, the group $\Pi_{g,n}$ is a free group on $\mu_1,\nu_1,\dots,\mu_g,\nu_g,\beta_1,\dots,\beta_{n-1}$. It easily follows that dR is surjective. □

Note that the statement of Proposition 3.15 still holds if we replace SU_2 by $\mathrm{SL}_2(\mathbb{C})$: the proof is identical.

3.5 Tangent spaces to homomorphism spaces

Using Proposition 3.15 we can compute the dimensions of the tangent spaces to our homomorphism spaces.

Corollary 3.16 (Tangent spaces to homomorphisms spaces)**.** *Fix $\boldsymbol{\vartheta}$ and let k be the number of integer entries of $\boldsymbol{\vartheta}$.*

(i) The Zariski tangent spaces satisfy

$$\dim T_\rho \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2) = 6g-3+2(n-k)+\dim(Z(\rho))$$
$$\dim T_{\rho_{\mathbb{C}}} \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SL}_2(\mathbb{C})) = 6g-3+2(n-k)+\dim(Z(\rho_{\mathbb{C}})).$$

(ii) The non-coaxial locus $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is an oriented manifold of real dimension $6g-3+2(n-k)$, and $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SL}_2(\mathbb{C}))$ is a complex manifold of complex dimension $6g-3+2(n-k)$.

(iii) The map $\overline{\boldsymbol{\Theta}} : \mathcal{H}om^{nc}(\Pi_{g,n},\mathrm{SU}_2) \to [0,1]^n$ that sends ρ to $(D_I(\rho(\beta_1)),\dots,D_I(\rho(\beta_n)))$ is continuous; moreover, it is real-analytic and submersive (i.e. has surjective differential) over $(0,1)^n$.

Proof. (i) We recall that λ embeds $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ inside the smooth, oriented variety $\mathcal{G}_{\boldsymbol{\vartheta}}$ of dimension $6g+2(n-k)$, and that $\lambda(\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2))$ is defined by $R_{\boldsymbol{\vartheta}} = I$ inside $\mathcal{G}_{\boldsymbol{\vartheta}}$. Thus the tangent space to $\lambda(\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2))$ at $\lambda(\rho) = (\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})$ is the kernel of $(dR_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})}$. Since the image of $(dR_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})}$ has dimension $\dim(\mathfrak{Z}(\rho)^\perp) = 3-\dim(Z(\rho))$ by Proposition 3.15(i), it follows that the tangent space to $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ at ρ has dimension $6g+2(n-k)-3+\dim(Z(\rho))$.

(ii) By Proposition 3.15(i) the map $R_{\boldsymbol{\vartheta}}$ is submersive at non-coaxial representations, and so $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is a real manifold of dimension $6g-3+2(n-k)$. Since SU_2 and so $\mathcal{G}$ are oriented, $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is oriented. The proof for $\mathrm{SL}_2(\mathbb{C})$ is analogous.

(iii) Continuity and analyticity properties of $\overline{\boldsymbol{\Theta}}$ follows from those of D_I (see Lemma 2.2). So we focus on submersiveness.

Let ρ be any homomorphism in $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ with no integral ϑ_i. To show that $\overline{\boldsymbol{\Theta}}$ is submersive at $\lambda(\rho) = (\boldsymbol{M}',\boldsymbol{N}',\boldsymbol{B}')$ it is enough to show that the map $\mathcal{G} \to \mathrm{SU}_2 \times [0,1]^n$ that sends $(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B})$ to

$(R(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B}), D_I(B_1),\dots,D_I(B_n))$ is smooth and submersive at $\lambda(\rho)$. (We are just using that, given linear maps $f_k : W \to W_k$ with $k = 1, 2$ such that $(f_1, f_2) : W \to W_1 \times W_2$ is surjective, the restriction of f_2 to $\ker(f_1)$ is surjective.)

By Lemma 2.2(i) the map $(D_I)^n : \mathrm{SU}_2^n \mapsto [0,1]^n$ at $\boldsymbol{B'}$ is submersive at $\boldsymbol{B'}$, since $B'_i \neq \pm I$ for all i. It follows that $\mathcal{G} \to [0,1]^n$ given by $(\boldsymbol{M},\boldsymbol{N},\boldsymbol{B}) \mapsto (D_I(B_1),\dots,D_I(B_n))$ is submersive at $\lambda(\rho)$. Since $R_{\boldsymbol{\vartheta}} : \mathcal{G}_{\boldsymbol{\vartheta}} \to \mathrm{SU}_2$ is submersive at $\lambda(\rho)$ by Proposition 3.15(i), this implies that the above map $\mathcal{G} \to \mathrm{SU}_2 \times [0,1]^n$ is submersive at $\lambda(\rho)$. □

Now we are ready to complete the investigation of the relative representation spaces.

Proof of Theorem III. (i) follows from Proposition 3.5(ii).

(ii) follows from Corollary 3.16(ii).

(iii) Since the central locus is 0-dimensional, the general coaxial homomorphism is non-central and so has 1-dimensional stabilizer. By (i), it follows that the coaxial locus in $\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ has pure dimension $(2g+2)-(3-1) = 2g$.

(iv) follows from (ii) by Remark 1.9 and Lemma 2.4(ii). □

We postpone the deeper investigation of the non-coaxial locus of non-special relative homomorphism spaces (namely, the proof of Theorem IV) to the end of Section 3.6. Here we consider the special cases.

Proof of Theorem V. (i) is proven in Proposition 3.7(ii).

(ii) The case $k = n-1$ is clear. The case $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) < 1$ follows from [44, Theorem A], or from Proposition 3.19(o) proven below.

(iii) By Proposition 3.19(i) proven below, the space $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ consists of a single conjugacy class. Hence, $\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ consists of a single point.

(iii-a) For $k = n$ the above-mentioned conjugacy class is the class of a central homomorphism, and so consists of a single point.

(iii-b) For $k \leq n-2$ the conjugacy class is the class of a non-central coaxial homomorphism, and so is isomorphic to $\mathbb{S}^2$.

(iii-c) Since $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ consists of a single conjugacy class, $\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ consists of one point. Moreover, note that the tangent space to $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ at ρ has dimension $t = -3+2(n-k)+\dim(Z(\rho))$ by Corollary 3.16. For $k = n$ we have $t = 0$ and so the structure is reduced; for $k \leq n-2$ we have $t = 2(n-k-1)$. The scheme structure on $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ is reduced if and only if $t = 2$, namely $k = n-2$. □

3.6 Density and connectedness of the non-coaxial locus

In this section we prove the connectedness of absolute and relative homomorphism and representation spaces, and of their non-coaxial loci.

Proposition 3.17 (Connectedness of the absolute non-coaxial locus)**.** *(i) The space $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)$ is smooth, connected, of dimension $6g+3n-3$. The space $\mathcal{Rep}(\Pi_{g,n},\mathrm{SU}_2)$ is connected.*

(ii) The non-coaxial locus in $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)$ and in $\mathcal{Rep}(\Pi_{g,n},\mathrm{SU}_2)$ is dense and connected.

Proof. (i) We have seen in Theorem II(ii) that $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)$ is isomorphic to SU_2^{2g+n-1}, and so it is smooth, connected, of dimension $6g+3n-3$. It follows that the quotient $\mathcal{Rep}(\Pi_{g,n},\mathrm{SU}_2)$ is connected.

(ii) Since $2g+n-1 = 1-\chi(\dot{S}) \geq 2$ and $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)$ is isomorphic to SU_2^{2g+n-1}, it easily follows that any homomorphism can be deformed to a non-coaxial one. This proves the density of the non-coaxial locus in $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)$. As for the connectedness, it is enough to note that $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)$ is smooth, of dimension $6g+3n-3 = 2(2g-2+n)+(2g+n+1)$, and that the coaxial locus has dimension $2g+n+1$ by by Proposition 3.5(i), and so codimension at least 2 in $\mathcal{Hom}(\Pi_{g,n},\mathrm{SU}_2)$. The statement for the representation space immediately follows. □

As for the relative case, a pure codimension count is not enough because the coaxial locus does not sit in the smooth locus of the relative homomorphism space.

Theorem 3.18 (Connectedness of non-special $\mathcal{R}ep^{nc}_{\boldsymbol{\vartheta}}$ space)**.** *Let $(g,n,\boldsymbol{\vartheta})$ be non-special. Then $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ and $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ are connected. Moreover, the non-coaxial locus in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is open, dense, and connected, and so is the non-coaxial locus in $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$.*

We anticipate that the strategies for proving Theorem 3.18 in genus zero and in positive genus are different. In genus zero we will first prove that $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is connected and we will use such result to conclude that the non-coaxial locus is dense and connected. In positive genus we will directly prove that the non-coaxial locus is dense and connected, and obtain that $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is connected as a consequence.

We will first deal with the case of genus 0 and then with the case of positive genus.

3.6.1 Density and connectedness of non-coaxial homomorphisms in genus zero.

In this section we analyse properties of non-emptiness and connectedness for $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ and for its non-coaxial locus. In particular, we first establish when the relative homomorphism space is non-empty and when its non-coaxial locus is nonempty. Then we show that the relative homomorphism space is connected, and we deduce that the non-coaxial locus is dense in nonspecial cases. Finally, using the techniques of Section 3.2 we show that the non-coaxial locus is connected.

Notation. In this section the angle vector $\boldsymbol{\vartheta}=(\vartheta_1,\dots,\vartheta_n)$ will always be an n-tuple of positive real numbers, with $n\geq 2$. We will also use the associated $\bar{\boldsymbol{\delta}}=(\bar{\delta}_1,\dots,\bar{\delta}_n)$ defined by $\bar{\delta}_i:=d(\vartheta_i,\mathbb{Z}_o)\in[0,1]$.

Choosing a basepoint I in $\mathbb{S}^3$ determines an identification $\mathrm{SU}_2\xrightarrow{\sim}\mathbb{S}^3$, which is actually a homothety of factor 1/2 (since the points I and $-I$ on SU_2 lie at distance 2π for the metric induced by $\mathcal{K}$). As discussed in Appendix A.1, there is an isomorphism

$$\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)\longrightarrow \mathcal{P}ol(\bar{\boldsymbol{\delta}})$$

between the relative homomorphism space and the space of closed spherical polygons $(p_1=I,p_2,\dots,p_n)$ in $\mathbb{S}^3$ with edge lengths $\pi\cdot(\bar{\delta}_1,\dots,\bar{\delta}_n)$, obtained by sending ρ to the polygon with $p_i=I\cdot\rho(\beta_1)\cdots\rho(\beta_{i-1})$. Moreover such correspondence preserves coaxiality.

The first result of this section is the following.

Proposition 3.19 (Non-emptiness of $\mathcal{H}om_{\boldsymbol{\vartheta}}$ for $g=0$)**.** *Let $\boldsymbol{\vartheta}=(\vartheta_1,\dots,\vartheta_n)$ with $n\geq 2$.*

(o) If $d_1(\boldsymbol{\vartheta}-\mathbf{1},\mathbb{Z}^n_o)<1$, then $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ is empty.

(i) If $d_1(\boldsymbol{\vartheta}-\mathbf{1},\mathbb{Z}^n_o)=1$, then $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ consists of a single conjugacy class of coaxial homomorphisms.

(ii) If $d_1(\boldsymbol{\vartheta}-\mathbf{1},\mathbb{Z}^n_o)>1$, then the non-coaxial locus in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ is non-empty.

The above result is also essentially proven in [6, Theorem A] and [19], see also [44, Theorem A]. Here we rely on Appendix A for a different and elementary proof, that does not involve (semi)stable holomorphic bundles with parabolic structure.

Proof of Proposition 3.19. By Theorem A.7 the space $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ non-empty if and only if $d_1(\boldsymbol{\vartheta}-\mathbf{1},\mathbb{Z}^n_o)\geq 1$. If $d_1(\boldsymbol{\vartheta}-\mathbf{1},\mathbb{Z}^n_o)=1$, then all homomorphisms in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ belong to a single conjugacy class by Corollary A.8(i). If $d_1(\boldsymbol{\vartheta}-\mathbf{1},\mathbb{Z}^n_o)>1$, then the non-coaxial locus in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ is non-empty by Corollary A.8(ii). □

The second result of this section is the following.

Proposition 3.20 (Connectedness of $\mathcal{H}om_{\boldsymbol{\vartheta}}$ for $g=0$)**.** *Let $n\geq 2$ and assume that $\boldsymbol{\vartheta}=(\vartheta_1,\dots,\vartheta_n)$ satisfies $d_1(\boldsymbol{\vartheta}-\mathbf{1},\mathbb{Z}^n_o)\geq 1$. Then $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n},\mathrm{SU}_2)$ is connected.*

The case $d_1(\boldsymbol{\vartheta}-\mathbf{1},\mathbb{Z}^n_o)=1$ follows from Proposition 3.19(i), thus so does the case $n=2$. Hence we can assume $n\geq 3$. Note that the statement is equivalent to saying that $\mathcal{P}ol(\bar{\boldsymbol{\delta}})$ is connected.

We start by reminding an observation about the length of sides of a convex triangles in $\mathbb{S}^2$.

Lemma 3.21 (Existence and uniqueness of spherical triangles)**.** *Suppose $0\leq a\leq b\leq\pi$. Then there is a continuous family of spherical triangles $A_tB_tC_t$ in $\mathbb{S}^2$ with $|B_tC_t|=a$, $|A_tC_t|=b$, indexed by*

$$t\in I(a,b):=[b-a,\min(a+b,2\pi-a-b)]\,,$$

such that $|B_tC_t| = t$. Moreover, for every t, the triangle $A_tB_tC_t$ is unique up to isometry of $\mathbb{S}^2 cdot$, and it is contained inside a maximal circle if and only if t is an endpoint of $I(a,b)$.

Before proceeding, we introduce certain subspaces of spherical polygons.

Notation. Given U_n in $\mathcal{C}_{\bar{\delta}_n}$, we will denote by $\mathcal{Pol}(\bar{\delta}_1, \ldots, \bar{\delta}_{n-1}, U_n)$ the subset of $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ consisting of polygons $(p_1 = I, p_2, \ldots, p_n)$ such that $p_n = U_n^{-1}$.

Spaces of polygons with one assigned edge will be useful in inductive proof. On the other hand, their connectedness is equivalent to that of ordinary spaces of polygons.

Sublemma 3.22. *The space $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ is connected if and only if $\mathcal{Pol}(\bar{\delta}_1, \bar{\delta}_2, \ldots, \bar{\delta}_{n-1}, U_n)$ is connected.*

Proof. We will show that both connectedness properties are equivalent to the connectedness of $\mathcal{Pol}(\bar{\boldsymbol{\delta}})/\mathrm{SU}_2$, where SU_2 is acting via the isometric conjugacy action on $(\mathrm{SU}_2, I) \cong (\mathbb{S}^3, I)$.
The map $\mathcal{Pol}(\bar{\boldsymbol{\delta}}) \to \mathcal{Pol}(\bar{\boldsymbol{\delta}})/\mathrm{SU}_2$ is proper surjective, and its fibers are quotients of SU_2 and so they are connected. By Lemma 3.8(i) it follows that $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ is connected if and only if $\mathcal{Pol}(\bar{\boldsymbol{\delta}})/\mathrm{SU}_2$ is.
Similary, $\mathcal{Pol}(\bar{\delta}_1, \bar{\delta}_2, \ldots, U_n) \to \mathcal{Pol}(\bar{\boldsymbol{\delta}})/\mathrm{SU}_2$ is proper surjective, and its fibers are quotients of the centralizer of U_n (which is either a 1-parameter subgroup or the whole SU_2) and so they are connected. By Lemma 3.8(i) it follows that $\mathcal{Pol}(\bar{\delta}_1, \bar{\delta}_2, \ldots, U_n)$ is connected if and only if $\mathcal{Pol}(\bar{\boldsymbol{\delta}})/\mathrm{SU}_2$ is. ☐

We can now prove the connectedness of the relative homomorphism spaces.

Proof of Proposition 3.20. We proceed by induction on $n \geq 3$.
If $n = 3$, then $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ consists of a single conjugacy class by Lemma 3.21, and so it is connected. Now we assume $n \geq 4$.
Consider the map

$$\ell_{n-1,n} : \mathcal{Pol}(\bar{\boldsymbol{\delta}}) \longrightarrow [0,1]$$

that sends $(p_1 = I, \ldots, p_n)$ to $\frac{1}{\pi} d_{\mathbb{S}^3}(p_{n-1}, p_1)$, where $d_{\mathbb{S}^3}$ is the usual distance on the unit 3-sphere. We want to show that the image and the fibers of $\ell_{n-1,n}$ are connected, and then conclude that $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ is connected by Lemma 3.8(i).
Connectedness of the image of $\ell_{n-1,n}$.
Let $\mathcal{I}_{n-1,n}$ be the image of the map $\mathcal{C}_{\bar{\delta}_{n-1}} \times \mathcal{C}_{\bar{\delta}_n} \to [0,1]$ that sends (U_{n-1}, U_n) to $D_I(U_{n-1}U_n)$ and let $\mathcal{I}_{1,n-2}$ be the image of $\mathcal{C}_{\bar{\delta}_1} \times \cdots \times \mathcal{C}_{\bar{\delta}_{n-2}} \to [0,1]$ that sends $(U_1, \ldots, U_{n-2})$ to $D_I(U_1 \cdots U_{n-2})$. The interval $\mathcal{I}_{n-1,n}$ consists of all the possible lengths (divided by π) of the third edge of a spherical triangle whose first two edges have assigned lengths $\pi(\bar{\delta}_{n-1}, \bar{\delta}_n)$. Similarly, the interval $\mathcal{I}_{1,n-2}$ consists of all the possible lengths (divided by π) of the $(n-1)$-st edge of a spherical $(n-1)$-gon whose first $n-2$ edges have assigned lengths $\pi(\bar{\delta}_1, \ldots, \bar{\delta}_{n-2})$. Hence, the image of $\ell_{n-1,n}$ is exactly the interval $\mathcal{I}_{1,n-2} \cap \mathcal{I}_{n-1,n}$.
Connectedness of the fibers of $\ell_{n-1,n}$.
Let $d \in \mathcal{I}_{1,n-2} \cap \mathcal{I}_{n-1,n}$ and fix $U \in \mathcal{C}_d$. Consider the subset $\mathcal{Pol}_{n,U}(\bar{\boldsymbol{\delta}})$ of polygons $(p_1 = I, p_2, \ldots, p_n)$ in $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ such that $p_{n-1} \cdot U = p_1$, and note that

$$\mathrm{SU}_2 \cdot \mathcal{Pol}_{n,U}(\bar{\boldsymbol{\delta}}) = \ell_{n-1,n}^{-1}(d)$$

where SU_2 acts by conjugation as usual. As SU_2 is connected, it is enough to show that $\mathcal{Pol}_{n,U}(\bar{\boldsymbol{\delta}})$ is connected. Since the map

$$\begin{array}{ccc} \mathcal{Pol}(\bar{\delta}_1, \ldots, \bar{\delta}_{n-2}, U) \times \mathcal{Pol}(\bar{\delta}_{n-1}, \bar{\delta}_n, U^{-1}) & \longrightarrow & \mathcal{Pol}_{n,U}(\bar{\boldsymbol{\delta}}) \\ ((q_1 = I, q_2, \ldots, q_{n-2}), (r_1 = I, r_2, r_3)) & \longmapsto & (I, q_2, \ldots, q_{n-2}, r_2 \cdot U^{-1}) \end{array}$$

is manifestly an isomorphism, it is enough to show that $\mathcal{Pol}(\bar{\delta}_1, \ldots, \bar{\delta}_{n-2}, U) \times \mathcal{Pol}(\bar{\delta}_{n-1}, \bar{\delta}_n, U^{-1})$ is connected. This follows from Sublemma 3.22 by inductive hypothesis. ☐

Here is a consequence of the above connectedness result.

Corollary 3.23 (Density of non-coaxial locus for non-special $g = 0$)**.** *Assume $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) > 1$. Inside $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$, the coaxial locus consists of finitely many conjugacy classes and the non-coaxial locus is dense.*

Proof. Clearly there are only finitely many polygons in $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ that sit on a given maximal circle of $\mathbb{S}^3$: this implies the first claim.

As for the second claim, it is enough to show that every coaxial conjugacy class of polygons is in the closure of the non-coaxial locus. By Proposition 3.19(ii) such non-coaxial locus is nonempty, and so we can pick a non-coaxial polygon P^{nc} there.

Let now P be any coaxial polygon. By Proposition 3.20, the space $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ is connected and so there exists a path $(P_t)_{t\in[0,1]}$ in $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ such that $P_0 = P$ and $P_1 = P^{nc}$. There exists $t_0 \in [0, 1)$ such that t_0 is the maximum t for which P_t is conjugate to P. It follows that P_{t_0} is in the closure of the non-coaxial locus, and so the whole conjugacy class of P is. □

Given $\bar{\delta}_1, \dots, \bar{\delta}_k \in [0, 1]$, recall that $\mathcal{C}_{\bar{\boldsymbol{\delta}}} = \mathcal{C}_{\bar{\delta}_1} \times \cdots \times \mathcal{C}_{\bar{\delta}_k}$ and let

$$\tau_k : \mathcal{C}_{\bar{\boldsymbol{\delta}}} \longrightarrow [-2, 2]$$

be defined as $\tau_k(U_1, \dots, U_k) := \mathrm{tr}(U_1 \cdots U_k)$.

Another consequence of the connectedness of $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ is the following.

Corollary 3.24 (Monotone-connectedness of $\mathcal{C}_{\bar{\boldsymbol{\delta}}}$)**.** *The pair $(\mathcal{C}_{\bar{\boldsymbol{\delta}}}, \tau_k)$ is monotone-connected.*

Proof. The space $\mathcal{C}_{\bar{\boldsymbol{\delta}}}$ is a smooth, connected, real algebraic manifold, and the function τ_k is real algebraic. The level sets $\{\tau_k = c\}$ are connected, since they are isomorphic to spaces of spherical polygons $\mathcal{Pol}(\bar{\delta}_1, \dots, \bar{\delta}_k, \bar{\delta}_{k+1})$ with $\bar{\delta}_{k+1} := \frac{1}{\pi} \arccos(\frac{c}{2})$. Hence the statement follows from Proposition 3.10. □

We now want to show the following.

Proposition 3.25 (Connectedness of non-coaxial locus for $g = 0$)**.** *Assume that $\boldsymbol{\vartheta}$ satisfies $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) > 1$. Then the non-coaxial locus of $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is connected.*

Before proving the above proposition, we need a technical lemma.

Lemma 3.26 (Misaligning adjacent edges of a non-coaxial polygon)**.** *Let P be a non-coaxial polygon in $\mathcal{Pol}^{nc}(\bar{\boldsymbol{\delta}})$. Suppose that $\bar{\delta}_i, \bar{\delta}_{i+1} \neq 0, 1$ and the edges e_i, e_{i+1} are aligned. Then there is a continuous deformation P_t of $P_0 = P$ such that, for $t > 0$ small enough, the edges $e_i(t), e_{i+1}(t)$ are not aligned.*

Proof. Let m be the number of integral entries in $\bar{\boldsymbol{\delta}}$. Since $\mathcal{Pol}^{nc}(\bar{\boldsymbol{\delta}})$ is identified to $\mathcal{Hom}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$, it is a manifold of dimension $2(n - m) - 3$ by Corollary 3.16(ii).

At the same time, the locus of $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ consisting of polygons for which e_i and e_{i+1} are aligned can be identified to a space of polygons $\mathcal{Pol}(\bar{\boldsymbol{\delta}}')$ with $n - 1$ edges (throwing away the $(i + 1)$-st vertex). Indeed, $\mathcal{Pol}^{nc}(\bar{\boldsymbol{\delta}}')$ can be identified to a submanifold of $\mathcal{Pol}^{nc}(\bar{\boldsymbol{\delta}})$ of codimension 2 or 4. The conclusion clearly follows. □

Proof of Proposition 3.25. We need to show that the locus $\mathcal{Pol}^{nc}(\bar{\boldsymbol{\delta}})$ of non-coaxial polygons inside $\mathcal{Pol}(\bar{\boldsymbol{\delta}})$ is connected.

Let $P_0, P_1 \in \mathcal{Pol}^{nc}(\bar{\boldsymbol{\delta}})$ be two non-coaxial polygons. Suppose first that the edges e_{n-1} and e_n are not aligned neither in P_0 nor in P_1. We will construct a path in $\mathcal{Pol}^{nc}(\bar{\boldsymbol{\delta}})$ that connects P_0 to P_1.

For $i = 0, 1$ let P'_i be the broken geodesic obtained from P_i by removing the edges e_{n-1}, e_n. Using Corollary 3.24 we can find a continuous deformation of broken geodesics $P'_t = (p_1(t) = I, p_2(t), \dots, p_{n-1}(t))$ with fixed edge lengths between P'_0 to P'_1 with the additional property that the distance between $p_{n-1}(t)$ and $p_1(t)$ changes monotonically. Then, using Lemma 3.21 we can insert back edges $e_{n-1}(t)$ and $e_n(t)$ so that the resulting family of polygons $P_t = (p_1(t) = I, p_2(t), \dots, p_n(t))$ varies continuously. It also follows from Lemma 3.21 that edges $e_{n-1}(t)$ and $e_n(t)$ stay non-coaxial during such deformation, which proves the proposition in this case.

In case e_{n-1} and e_n are aligned in P_0 (or P_1), using Lemma 3.26 we first construct a small deformation of P_0 inside $\mathcal{Pol}^{nc}(\bar{\boldsymbol{\delta}})$ so that e_{n-1} and e_n are not aligned any more, and then we proceed as above. □

3.6.2 Connectedness and density of non-coaxial locus in positive genus.

Thoughout this section we focus on non-special cases in positive genus. We begin by proving that the non-coaxial locus is dense.

Theorem 3.27 (Density of non-coaxial locus in non-special cases)**.** *Suppose that $(g, n, \boldsymbol{\vartheta})$ is not special. Then the non-coaxial locus is non-empty and dense in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ and in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$. As a consequence $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ and $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ have pure dimension $6g - 3 + 2(n - k)$ and $2(6g - 3 + 2(n - k))$ respectively.*

Proof. Note that the claimed pure-dimensionality follows from the density of the non-coaxial locus, which is a manifold by Corollary 3.16(ii).

We separately treat non-emptiness and density in the $\mathrm{SL}_2(\mathbb{C})$ and the SU_2-case.

The $\mathrm{SL}_2(\mathbb{C})$ *case.* Since $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ contains $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$, non-emptiness of the non-coaxial locus follows from the SU_2 case treated below.

As for density, let k be the number of integral entries of $\boldsymbol{\vartheta}$. We have seen in Lemma 3.3 that $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ can be realized via the embedding $\lambda_{\mathbb{C}}$ inside $\mathrm{SL}_2(\mathbb{C})^{2g+n}$, and that in $\mathrm{SL}_2(\mathbb{C})^{2g+n}$ it is cut by $n+2k+3$ algebraic equations. Hence, each irreducible component of $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ must have dimension at least $3(2g+n)-(n+2k+3) = 6g+2(n-k)-3$. By Proposition 3.5(ii) the coaxial locus in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SL}_2(\mathbb{C}))$ has dimension $2g+2$. Since $2g+2 < 6g+2(n-k)-3$ in non-special cases, the non-coaxial locus is dense.

The SU_2 *case: non-emptiness.* We want to find $(M_1, N_1, \dots, M_g, N_g, B_1, \dots, B_n)$ in SU_2^{2g+n} that represents a non-coaxial point of $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$. We distinguish three cases.

Case $g \geq 2$. Choose $B_i \in \mathcal{C}_{\bar{\delta}_i}$ for all i. By Corollary 3.13 the commutator map is surjective, and so there exist $M_1, N_1 \in \mathrm{SU}_2$ such that $[M_1, N_1] = (B_1 \cdots B_n)^{-1}$. Note that it is always possible to choose them so that $M_1 \neq \pm I$. Hence, we can pick $M_2 = N_2 \in \mathrm{SU}_2$ that does not belong to the same 1-parameter subgroup as M_1. Finally, we set $M_i = N_i = I$ for $i > 2$. Then $(M_1, N_1, \dots, M_g, N_g, B_1, \dots, B_n)$ works.

Case $g = 1$. Recall that, by non-speciality, there must be some non-integer ϑ_i. As in the above case, pick $B_i \in \mathcal{C}_{\bar{\delta}_i}$ for all i. If $B_1 \cdots B_n \neq I$, then it is enough to pick $M_1, N_1 \in \mathrm{SU}_2$ such that $[M_1, N_1] = (B_1 \cdots B_n)^{-1}$. If $B_1 \cdots B_n = I$, then we pick $M_1 = N_1 \in \mathrm{SU}_2$ not in the same 1-parameter subgroup as B_i.

Case $g = 0$. It follows from Proposition 3.19(ii).

The SU_2 *case: density.* Let ρ be a coaxial homomorphism in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$. By definition, ρ takes values in a 1-parameter subgroup H (if ρ is central, then pick any H). Let $Y \in \mathbb{S}^2$ be an infinitesimal generator for H. Hence,

$$\lambda(\rho) = \mathbf{e}(r_1 Y, s_1 Y, \dots, r_g Y, s_g Y, \epsilon_1 \vartheta_1 Y, \dots, \epsilon_n \vartheta_n Y),$$

where $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$. We want to construct *a non-coaxial deformation of* ρ, namely a path $t \mapsto \rho_t \in \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \subset \mathrm{SU}_2^{2g+n}$ such that $\rho_0 = \rho$ and ρ_t is not coaxial for $t \in (0, \varepsilon)$ for some small $\varepsilon > 0$. In fact what we will construct is the path $t \mapsto (\boldsymbol{M}(t), \boldsymbol{N}(t), \boldsymbol{B}(t))$ inside SU_2^{2g+n} that corresponds to $\lambda(\rho_t)$.

Choose $Z \in \mathfrak{su}_2$ not a multiple of Y.

Case $g \geq 2$. If ρ is central, namely all $\mathrm{e}(r_i Y), \mathrm{e}(s_i Y)$ are $\pm I$ and $\boldsymbol{\vartheta} \in \mathbb{Z}^n$, then a non-coaxial deformation of ρ is obtained by moving the first four entries as follows

$$t \mapsto \lambda(\rho) \cdot \exp(tY, tY, tZ, tZ, 0, \dots, 0).$$

Indeed, for small $t \neq 0$ we have $\pm I \neq \mathrm{e}(r_1 Y) \exp(tY) \in H$ and $\mathrm{e}(r_2 Y) \exp(tZ) \notin H$.

If ρ is not central then, up to renumbering the generators of $\Pi_{g,n}$, we can assume that at least one among $\mathrm{e}(r_1 Y)$, $\mathrm{e}(s_1 Y)$, $\mathrm{e}(\epsilon_1 \vartheta_1 Y)$ is different from $\pm I$. A non-coaxial deformation is then given by

$$t \mapsto \mathbf{e}(r_1 Y, s_1 Y, r_2(Y + tZ), s_2(Y + tZ), r_3 Y, s_3 Y, \dots, \epsilon_n \vartheta_n Y).$$

Indeed, $\mathrm{e}(r_2(Y + tZ))$ does not belong to H for small $t \neq 0$.

Case $g = 1$. We claim that, if $\rho \in \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$ is coaxial, then there must be a non-integral ϑ_i. In fact, coaxiality implies that $[\rho(\mu_1), \rho(\nu_1)] = I$ and then, by Equation (1), that $\sum_i (\pm(\vartheta_i - 1)) \in 2\mathbb{Z}$ for a suitable choice of the signs. Since $(1, n, \boldsymbol{\vartheta})$ is assumed to be non-special, it follows that $\boldsymbol{\vartheta} \notin \mathbb{Z}^n$. Up to renumbering the punctures, we can assume that $\vartheta_1 \notin \mathbb{Z}$. Then

$$t \mapsto \mathbf{e}(r_1(Y + tZ), s_1(Y + tZ), \epsilon_1\vartheta_1 Y, \dots, \epsilon_n\vartheta_n Y)$$

corresponds to a non-coaxial deformation, since $\pm I \neq \mathrm{e}(\epsilon_1\vartheta_1 Y) \in H$ and $\mathrm{e}(r_1(Y + tZ)) \notin H$ for $0 \neq t$ small.

Case $g = 0$. It was proven in Corollary 3.23. □

Now we turn to connectedness of the non-coaxial locus inside the relative homomorphism space. A major role in the argument below will be played by the commutator map $c : \mathrm{SU}_2 \times \mathrm{SU}_2 \to \mathrm{SU}_2$ introduced in Section 3.3.

Proof of Theorem 3.18. The case of genus 0 was treated in Proposition 3.19(ii), Corollary 3.23 and Proposition 3.25. From now on, we assume $g \geq 1$.

The non-coaxial locus in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is open by Lemma I(iv) and is non-empty and dense by Theorem 3.27.

We are thus left to prove the connectedness of $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$. Note that it will imply that the whole relative homomorphism space is connected.

Identify $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ to its image in $\mathrm{SU}_2^{2g} \times \mathcal{C}_{\bar{\delta}_1} \times \cdots \times \mathcal{C}_{\bar{\delta}_n}$ via λ, and consider the map

$$\begin{array}{rrcl} f: & \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) & \longrightarrow & \mathrm{SU}_2^{2g-2} \times \mathcal{C}_{\bar{\boldsymbol{\delta}}} \\ & (M_1, \dots, N_g, B_1, \dots, B_n) & \longmapsto & (M_2, \dots, N_g, B_1, \dots, B_n). \end{array}$$

Let Y be the closed subset of $\mathrm{SU}_2^{2g-2} \times \mathcal{C}_{\bar{\boldsymbol{\delta}}}$ defined as

$$Y := \{(M_2, \dots, B_n) \in \mathrm{SU}_2^{2g-2} \times \mathcal{C}_{\bar{\boldsymbol{\delta}}} \mid [M_2, N_2] \cdots [M_g, N_g] B_1 \cdots B_n = I\}$$

and denote by Y^c the complement of Y inside $\mathrm{SU}_2^{2g-2} \times \mathcal{C}_{\bar{\boldsymbol{\delta}}}$. Every $(M_1, N_1, \dots, B_n)$ in $f^{-1}(Y^c)$ is non-coaxial, because $[M_1, N_1] \neq I$.

Claim: the subset Y *has codimension at least* 2. Observe first that Y can be identified to $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g-1,n}, \mathrm{SU}_2)$. Now we separately analyze two different cases.

Suppose first that $g = 1$ and the triple $(0, n, \boldsymbol{\vartheta})$ is special. It follows that $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) \leq 1$. The conclusion trivially holds if $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) < 1$, since in this case $Y \cong \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is empty by Proposition 3.19(o). Hence, we can assume $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) = 1$. If $n - k = 0$, then $\boldsymbol{\vartheta} \in \mathbb{Z}^n$ and the conclusion follows from Proposition 3.7. If $n - k = 1$, then $Y \cong \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is empty. For $n - k \geq 2$, then $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ has dimension 2 by Proposition 3.19(i), whereas $\mathcal{C}_{\bar{\boldsymbol{\delta}}}$ has dimension $2(n - k) \geq 4$: it follows that Y has codimension at least 2.

Suppose now that $g \geq 2$, or that $g = 1$ and $(0, n, \boldsymbol{\vartheta})$ is not special. The space $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g-1,n}, \mathrm{SU}_2)$ has pure dimension $6(g-1) - 3 + 2(n-k)$ by Theorem 3.27. On the other hand, $\mathrm{SU}_2^{2g-2} \times \mathcal{C}_{\bar{\boldsymbol{\delta}}}$ has dimension $3(2g - 2) + 2(n - k) = 6g - 6 + 2(n - k)$. It follows that Y has codimension at least 3.

We now want to prove that $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is connected by showing that $f^{-1}(Y^c)$ is connected and is dense inside $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$.

Connectedness of $f^{-1}(Y^c)$. Since $\mathrm{SU}_2^{2g-2} \times \mathcal{C}_{\bar{\boldsymbol{\delta}}}$ is smooth and connected and Y has codimension at least 2 inside it, Y^c is connected. Clearly f is proper. We want to show that f is surjective with connected fibers, so that the connectedness of $f^{-1}(Y^c)$ will follow by Lemma 3.8(i). Let

$$\begin{array}{rrcl} p_1: & \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) & \longrightarrow & \mathrm{SU}_2^2 \\ & (M_1, N_1, \dots, B_n) & \longmapsto & (M_1, N_1) \end{array}$$

Observe that p_1 realizes an isomorphism from $f^{-1}(M_2, \dots, B_n)$ to $c^{-1}(C)$ with $C = ([M_2, N_2] \cdots [M_g, N_g] B_1 \cdots B_n)^{-1}$. The conclusion follows, since the map c is surjective and with connected fibers by Corollary 3.13.

Density of $f^{-1}(Y^c)$. Call $f^{nc} : \mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \to \mathrm{SU}_2^{2g-2} \times \mathcal{C}_{\bar{\boldsymbol{\delta}}}$ the restriction of f. We want to show that $(f^{nc})^{-1}(Y)$ has dimension strictly smaller than $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ at every point: it will follow that

$(f^{nc})^{-1}(Y^c) = f^{-1}(Y^c)$ is dense. For every $y \in Y$, the fiber $(f^{nc})^{-1}(y)$ has dimension 4 by Corollary 3.13(i). Moreover, by the above claim Y has dimension at most $6(g-1)+2(n-k)-2$. It follows that $(f^{nc})^{-1}(Y)$ has dimension at most $6(g-1)+2(n-k)+2$. On the other hand, $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is smooth of dimension $6g-3+2(n-k)$ by Corollary 3.16(ii). The conclusion follows. □

An easy consequence of the pure-dimensionality of $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is that the coaxial locus is small in non-special cases.

Corollary 3.28 (Codimension of the coaxial locus in non-special cases)**.** *Let $(g, n, \boldsymbol{\vartheta})$ be non-special. Then the coaxial locus (if non-empty) inside $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ has codimension $4(g-1)+2(n-k)-1 \geq 3$. Hence, the coaxial locus (if non-empty) inside $\mathcal{R}ep_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ has pure codimension $4(g-1)+2(n-k)-2 \geq 2$.*

Proof. Consider first the relative homomorphism space. By Proposition 3.5(ii), the coaxial locus in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ has pure dimension $2g+2$, whereas $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ has pure dimension $3(2g-1)+2(n-k)$ by Theorem 3.27. Denote by $\mathrm{cdim} := (6g-3+2(n-k)) - (2g+2) = 4(g-1)+2(n-k)-1$. The above considerations show that, if non-empty, the the coaxial locus in $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ has pure codimension cdim. We claim that $\mathrm{cdim} \geq 3$.

Case $g = 0$. We must have $n-k \geq 3$ and $d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) > 1$. For $n-k = 3$, Lemma 3.21 implies that the coaxial locus is empty (see also the beginning of the proof of Proposition 3.20). For $n-k \leq 4$, we have $\mathrm{cdim} = 2(n-k)-5 \geq 3$.

Case $g = 1$. If all ϑ_i are integer, namely if $n-k = 0$, then the sum $\sum(\vartheta_i - 1)$ must be odd, since $(1, n, \boldsymbol{\vartheta})$ is assumed non-special. Hence an element of $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$ identifies to $(M_1, N_1, B_1, \ldots, B_n)$ with $B_i \in \mathcal{C}_{\bar{\delta}_i}$ and $[M_1, N_1] = -I$. This implies that the coaxial locus is empty in this case. If $n-k = 1$, then points of $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$ must satisfy $[M_1, N_1] \neq \pm I$ and so the coaxial locus is empty again. For $n-k \geq 2$, we have $\mathrm{cdim} = 2(n-k)-1 \geq 3$.

Case $g \geq 2$. We have $\mathrm{cdim} = 4(g-1)+2(n-k)-1 \geq 3$.

The statement for the representation space easily follows. Indeed, recalling that the central locus has dimension 0 and that a coaxial non-central homomorphism has 1-dimensional stabilizer. □

As a consequence we have all ingredients to draw our conclusions on the non-coaxial locus of non-special relative homomorphism and representation spaces.

Proof of Theorem IV. Recall that smoothness of the non-coaxial locus is proven in Theorem III(ii-iv) together with the determination of its dimension.

(iii) Non-emptiness and density of the non-coaxial locus is proven in Theorem 3.27.

(iv) Connectedness of the non-coaxial locus is proven in Theorem 3.18.

(i) Since the non-coaxial locus is smooth, pure dimensionality of the relative homomorphism and representation spaces follows from (iii) and (iv).

(ii) Because of (i), Corollary 3.16(i-ii) also shows that the smooth locus of $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ coincides with the non-coaxial locus. It follows that $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is reduced and irreducible.

Concerning the relative representation space, it is reduced and irreducible and its smooth locus consists of $[\rho]$ at which $\dim(T_\rho \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)) + \dim(Z(\rho))$ achieves its minimum by Remark 1.10. Such minimum is achieved at the non-coaxial locus by Corollary 3.16(ii), which is non-empty by (iii). □

4 Decorated representation spaces

We keep the same notation as in the beginning of Section 3. In particular $V = \mathcal{M}_{2,2}(\mathbb{C})$.

4.1 Topology and semi-analytic structure

Analogously to Section 3.1, in order to prove Lemma Î, we endow the absolute decorated homomorphism space with an analytic structure. The purpose is achieved by embedding it inside a smooth algebraic variety $\widehat{\mathcal{G}}$, so that its image is described by the equation $\widehat{R} = I$. Moreover, the conjugacy action by PSU_2 is the restriction of a natural action on $\widehat{\mathcal{G}}$, which is compatible with the map $\widehat{R}$. The relative case is similar.

4.1.1 The embedding $\widehat{\boldsymbol{\lambda}}$.

Let $\widehat{\mathcal{G}}$ be the real algebraic subset $\mathrm{SU}_2^{2g} \times \mathfrak{su}_2^{\oplus n}$ of dimension $6g+3n$ of the real vector space $V^{\oplus 2g} \times \mathfrak{su}_2^{\oplus n}$. and let Nm (for *norm*) be the algebraic map defined as

$$\begin{aligned} \mathrm{Nm}: \quad & \widehat{\mathcal{G}} \longrightarrow \mathbb{R}^n_{\geq 0} \\ & (M_1, N_1, \ldots, X_1, \ldots, X_n) \longmapsto \left(\|X_1\|^2, \ldots, \|X_n\|^2\right) \end{aligned}$$

We call $\widehat{\mathcal{G}}_{\boldsymbol{\vartheta}} := \mathrm{Nm}^{-1}(\vartheta_1^2, \ldots, \vartheta_n^2)$ for any $\boldsymbol{\vartheta} \in \mathbb{R}^n_{>0}$ and $\widehat{\mathcal{G}}_+ := \mathrm{Nm}^{-1}(\mathbb{R}^n_{>0})$.
Clearly, $\widehat{\mathcal{G}}_+$ is a smooth algebraic subset of dimension $6g+3n$. Since Nm is submersive on $\widehat{\mathcal{G}}_+$, it follows from the implicit function theorem that $\widehat{\mathcal{G}}_{\boldsymbol{\vartheta}}$ is a smooth algebraic subset of dimension $6g+2n$.

4.1.2 The map $\widehat{\boldsymbol{R}}$.

Let now $\widehat{R}$ be the real analytic map defined as

$$\begin{aligned} \widehat{R}: \quad & \widehat{\mathcal{G}} \longrightarrow \mathrm{SU}_2 \\ & (M_1, \ldots, X_n) \longmapsto \textstyle\prod_j [M_j, N_j] \prod_i \mathrm{e}(X_i) \end{aligned}$$

and denote by $\widehat{R}_{\boldsymbol{\vartheta}} : \widehat{\mathcal{G}}_{\boldsymbol{\vartheta}} \to \mathrm{SU}_2$ the restriction of $\widehat{R}$.
The injective map

$$\begin{aligned} \widehat{\lambda}: \quad & \widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2) \longrightarrow \widehat{\mathcal{G}} \\ & (\rho, A) \longmapsto (\rho(\mu_1), \rho(\nu_1), \ldots, \rho(\mu_g), \rho(\nu_g), A(\beta_1), \ldots, A(\beta_n)) \end{aligned}$$

identifies $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ to its image $\widehat{R}^{-1}(I) \cap \widehat{\mathcal{G}}_+$, and $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ to the subset $\widehat{R}^{-1}(I) \cap \widehat{\mathcal{G}}_{\boldsymbol{\vartheta}} = \widehat{R}_{\boldsymbol{\vartheta}}^{-1}(I)$.

4.1.3 The map $\boldsymbol{\Theta}$.

Taking the componentwise square root of the map Nm, we obtain a map

$$\boldsymbol{\Theta} : \widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2) \longrightarrow \mathbb{R}^n_+$$

defined as $\boldsymbol{\Theta}(\rho, A) := (\|A(\beta_1)\|, \ldots, \|A(\beta_n)\|)$. In this way, $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ identifies to $\boldsymbol{\Theta}^{-1}(\boldsymbol{\vartheta})$.

4.1.4 The conjugacy action.

Note that PSU_2 acts on $V^{\oplus 2g} \times \mathfrak{su}_2^{\oplus n}$ componentwise via conjugaction and via adjunction. Moreover $\widehat{R}$ and Nm are PSU_2-equivariant and the PSU_2-action on $\widehat{R}^{-1}(I) \cap \widehat{\mathcal{G}}_+$ and $\widehat{R}^{-1}(I) \cap \widehat{\mathcal{G}}_{\boldsymbol{\vartheta}}$ agrees with the wished actions on the decorated homomorphism spaces.

4.1.5 Analytic and semi-analytic structure.

We begin by addressing the first claim of Lemma Î, namely the analyticity of the decorated homomorphism space and the semi-analiticity of the corresponding decorated representation space.

Proof of Lemma Î(o). Analogously to Lemma I(o), the constructions performed in Sections 4.1.1-4.1.2 show that the space $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ is homeomorphic via $\widehat{\lambda}$ to the subset of $\widehat{\mathcal{G}}_+$ described by the analytic equation $\widehat{R} = I$. Hence, $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ is induced a real analytic structure. Taking the quotient of $\widehat{\mathcal{G}}_+$ by the action of PSU_2 described in Section 4.1.4, we obtain that $\widehat{\mathcal{G}}_+/\mathrm{PSU}_2$ is homeomorphic to a semi-algebraic subset of a Euclidean space (the construction is similar to the one described in Remark 1.7). Moreover the locus $\{\widehat{R} = I\}$ descends to a semi-analytic subset of $\widehat{\mathcal{G}}_+/\mathrm{PSU}_2$ homeomorphic to $\widehat{\mathcal{Rep}}(\Pi_{g,n}, \mathrm{SU}_2)$. □

Before completing the proof of Lemma Î, we examine the behaviour of the map

$$\widehat{\mathcal{F}} : \widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2) \longrightarrow \mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$$

that forgets the decoration, defined as $\widehat{\mathcal{F}}(\rho, A) := \rho$, and of its restriction

$$\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}} : \widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \longrightarrow \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$$

to the relative spaces.

Lemma 4.1 (Forgetting the decoration)**.** *The map $\widehat{\mathcal{F}}$ is real-analytic. Moreover, given $\boldsymbol{\vartheta} \in \mathbb{R}^n_{>0}$, the map $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}$ is a trivial bundle with fiber $(\mathbb{S}^2)^k$, where k is the number of integer entries of $\boldsymbol{\vartheta}$.*

Proof. The homomorphism space is real-algebraic by Lemma I(o) and the decorated homomorphism space is real-analytic by Lemma Î(o) proven above. Hence, $\widehat{\mathcal{F}}$ is real-analytic, being the restriction of the real-analytic map $\widehat{\mathcal{G}} \to \mathcal{G}$ that sends $(M_1, \dots, N_g, X_1, \dots, X_n)$ to $(M_1, \dots, N_g, \mathrm{e}(X_1), \dots, \mathrm{e}(X_n))$.
Up to rearranging the indices, we can assume that $\vartheta_i \in \mathbb{Z}$ if and only if $i \leq k$. Consider the map $s : \widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \to \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \times (\mathbb{S}^2)^k$ defined as $s(\rho, A) := (\rho, \hat{A}(\beta_1), \dots, \hat{A}(\beta_k))$. The map s is manifestly a real-analytic isomorphism and the composition of s and of the projection onto $\mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is exactly $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}$: hence, s gives the wished trivialization of the fiber bundle $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}$. □

Now we can complete the proof of our first main statement on decorated representation spaces.

End of the proof of Lemma Î. (i) is very similar to the proof of Lemma I(i).

(ii) The coaxial locus in $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$ is the preimage via the map $\widehat{\mathcal{F}}$ (introduced in Section 4.1.5) of the coaxial locus in $\mathcal{H}om(\Pi_{g,n}, \mathrm{SU}_2)$, and $\widehat{\mathcal{F}}$ is a surjective and analytic by Lemma 4.1. Hence, the coaxial locus is closed analytic by Lemma I(ii). Observe, similarly to the proof of Lemma I(ii), that a decorated homomorphism (ρ, A) is elementary if and only if $\mathrm{Im}(\mathrm{Ad}_{M_j} - I)$, $\mathrm{Im}(\mathrm{Ad}_{N_j} - 1)$, $X_i^{\perp}$ do not span $\mathfrak{su}_2$ for $(\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{X}) = \widehat{\lambda}(\rho, A)$. Such condition can be expressed in terms of analytic equations in $(\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{X})$, and so the elementary locus is closed analytic. Observe moreover that $(\rho, A) \in \widetilde{\Sigma}$ if and only if ρ is coaxial, $\|A(\beta_i)\| \in \mathbb{Z}_+$ for $i = 1, \dots, n$, and $\mathrm{Span}(A(\beta_1), \dots, A(\beta_n)) \neq \mathfrak{su}_2$: all such conditions can be expressed through analytic equations, so $\widetilde{\Sigma}$ is closed analytic.
Note that $\widetilde{\Sigma}_0$ is closed analytic, since the conditions $\rho(\mu_i), \rho(\nu_i), \rho(\beta_j) \in \{\pm I\}$ and $\mathrm{Span}(A(\beta_1), \dots, A(\beta_n)) \leq 2$ can be phrased through analytic equations. Hence $\widetilde{\Sigma}_1$ is open inside $\widetilde{\Sigma}$. If $g = 0$, then $\widetilde{\Sigma}_1$ is empty as (ρ, A) with integer-valued A are necessarily central. Assume now $g \geq 1$ and let $(M_1, \dots, N_g, X_1, \dots, X_n)$ be an element of $\widetilde{\Sigma}_0$. This means that there exists $X \in \mathbb{S}^2$ such that $X_1, \dots, X_n$ are orthogonal to X and $M_i, N_i \in \{\pm I\}$. Define $M_i(t) = M_i e^{tX}$ and $N_i(t) = N_i e^{tX}$, so that $(M_1(t), \dots, N_g(t), X_1, \dots, X_n)_{t \geq 0}$ is a deformation of $(M_1, \dots, N_g, X_1, \dots, X_n)$ that belongs to $\widetilde{\Sigma}_1$ for $t > 0$: this proves that $\widetilde{\Sigma}_1$ is dense inside $\widetilde{\Sigma}$.
(iii) Since $\widehat{\mathcal{G}}_{\boldsymbol{\vartheta}}$ is a closed algebraic subset of $\widehat{\mathcal{G}}_+$, it follows from (i) that $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ is a closed analytic subset of $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$.
(iv) follows from (ii) and (iii).
(a) Clearly, an elementary (ρ, A) is coaxial. Conversely, pick a coaxial decorated homomorphism (ρ, A). By definition, there exists a 1-parameter subgroup H of SU_2 that contains the image of ρ. Since $\vartheta_i \notin \mathbb{Z}$, the element $\mathrm{e}(A(\beta_i)) \in \mathrm{SU}_2$ is different from $\pm I$ and it belongs to H for all i. It follows that all $A(\beta_i)$ belong to the Lie algebra of H and so (ρ, A) is elementary.

(b) By Lemma 4.1 with $k = 0$ we have a real-analytic isomorphism $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \to \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$. Restricting to the non-coaxial locus and taking the quotient by PSU_2, we obtain the wished isomorphism.
Finally, note that the real-analytic maps in (o) and (b) are PSU_2-equivariant and so descend to decorated representation spaces. Similarly, all subets involved in (ii-iii-iv) are PSU_2-invariant and so analogous statements hold for decorated representation spaces. □

Example 4.2 (Semi-analytic nature of decorated SU_2-representation spaces)**.** Let $g = 1$, $n = 2$ and $\vartheta_1 = \vartheta_2 = t$ for some fixed $t \in (0, 1)$. Since no ϑ_i is integer, the map $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}} : \widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{1,2}, \mathrm{SU}_2) \to \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{1,2}, \mathrm{SU}_2)$ that forgets the decoration is an isomorphism of real analytic spaces. Proceeding as in Example 3.4, it is possible to show that $\widehat{\mathcal{R}ep}_{\boldsymbol{\vartheta}}(\Pi_{1,2}, \mathrm{SU}_2)$ is not analytic (but only semi-analytic). As a consequence, so is $\widehat{\mathcal{R}ep}(\Pi_{1,2}, \mathrm{SU}_2)$.

As a further simple application of the forgetful map $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}$, we have the following.

Corollary 4.3 (Space of decorations of a fixed ρ)**.** *Consider the fibration $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}} : \widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \to \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ in Section 4.1.5.*

(i) If ρ is coaxial, then the locus of elementary homomorphisms inside $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}^{-1}(\rho)$ is closed analytic. Moreover, such locus has dimension 2 if ρ is central, and it has dimension 0 if ρ is non-central.

(ii) Inside each fiber $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}^{-1}(\rho)$ the non-elementary locus is open, dense and connected, except if $k = 0$ and ρ is coaxial, or if $n = 1$ and ρ is central.

Proof. (i) The elementary locus is closed analytic by Lemma Î(ii). Identify now $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}^{-1}(\rho)$ to $(\mathbb{S}^2)^k$ via the trivialization s described in the proof of Lemma 4.1. If ρ is central (and so $k = n > 0$), then the elementary locus corresponds to the subset of collinear k-tuples in $(\mathbb{S}^2)^k$, which is a closed analytic subset of dimension 2. If ρ is not central, with image inside $\exp(\mathfrak{h})$ for some line $\mathfrak{h} \subset \mathfrak{su}_2$, then the elementary locus corresponds to $(\mathbb{S}^2 \cap \mathfrak{h})^k$, which consists of 2^k points.
(ii) For $k = 0$ a decorated homomorphism is elementary if and only if it is coaxial by Lemma Î(a), and so the conclusion is immediate. So suppose $k > 0$. If ρ is non-coaxial, then the whole $f^{-1}(\rho)$ is non-elementary. If ρ is coaxial but not central, then $f^{-1}(\rho)$ has dimension $2k \geq 2$ and the elementary locus therein has dimension 0. If ρ is central, there are two cases: if $n = 1$, then $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}^{-1}(\rho)$ consists entirely of elementary homomorphisms; if $n > 1$, then $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}^{-1}(\rho)$ has dimension $2n \geq 4$ and the elementary locus therein has dimension 2. In all cases, the conclusion follows. □

4.2 First-order computations: the maps $\widehat{R}$, $\widehat{R}_{\boldsymbol{\vartheta}}$

As for homomorphism spaces, smoothness of $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$ and $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ are investigated by studying the differentials of the maps $\widehat{R}$ and $\widehat{R}_{\boldsymbol{\vartheta}}$ introduced in Section 4.1. We will denote by $\langle A \rangle \subseteq \mathfrak{su}_2$ the smallest vector subspace that contains the (possibly infinite) image of $A : \mathcal{B} \to \mathfrak{su}_2$.

Proposition 4.4 (Differentials of $\widehat{R}$ and $\widehat{R}_{\boldsymbol{\vartheta}}$)**.** *For every $(\rho, A) \in \widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$, the images of the differentials of $\widehat{R} : \widehat{\mathcal{G}} \to \mathrm{SU}_2$ and $\widehat{R}_{\boldsymbol{\vartheta}} : \widehat{\mathcal{G}}_{\boldsymbol{\vartheta}} \to \mathrm{SU}_2$ at $\widehat{\lambda}(\rho)$ are*

(i) $\mathrm{Im}(d\widehat{R}_{\boldsymbol{\vartheta}})_{\widehat{\lambda}(\rho,A)} = \mathfrak{Z}(\rho)^{\perp}$;

(ii) $\mathrm{Im}(d\widehat{R})_{\widehat{\lambda}(\rho,A)} = \begin{cases} \mathfrak{su}_2 & \text{if some } \vartheta_i \text{ is non-integral} \\ \mathfrak{Z}(\rho)^{\perp} + \langle A \rangle & \text{if all } \vartheta_i \text{ are integral.} \end{cases}$

A preliminary observation concerns the differential of the map $E : (\mathfrak{su}_2 \setminus \{0\})^n \to \mathrm{SU}_2$ defined by

$$E(X_1, \ldots, X_n) := \mathrm{e}(X_1) \cdots \mathrm{e}(X_n).$$

For every $\vartheta > 0$, we denote by $\mathfrak{c}_\vartheta$ the subset of $\mathfrak{su}_2$ consisting of elements of norm ϑ and by e_ϑ the restriction of the map $\mathrm{e}(X)$ to $\mathfrak{c}_\vartheta$. Given an angle vector $\boldsymbol{\vartheta}$, we let $\mathfrak{c}_{\boldsymbol{\vartheta}} := \prod_{i=1}^n \mathfrak{c}_{\vartheta_i}$ and we call $E_{\boldsymbol{\vartheta}}$ the restriction of E to $\mathfrak{c}_{\boldsymbol{\vartheta}}$.

Lemma 4.5 (Differential of E)**.** *The exponential maps defined above satisfy the following properties.*

(i) The differentials of e *and* e_ϑ *at a point $X \in \mathfrak{c}_\vartheta$ have image*

$$\mathrm{Im}(d\mathrm{e}_X) = \begin{cases} \mathfrak{su}_2 & \text{if } \vartheta \notin \mathbb{Z} \\ \mathrm{Span}(X) & \text{if } \vartheta \in \mathbb{Z}, \end{cases} \qquad \mathrm{Im}(d\mathrm{e}_\vartheta)_X = \begin{cases} X^{\perp} & \text{if } \vartheta \notin \mathbb{Z} \\ \{0\} & \text{if } \vartheta \in \mathbb{Z}. \end{cases}$$

(ii) The differential of E satisfies

$$dE_{\boldsymbol{X}}(\dot{\boldsymbol{X}}) := d\mathrm{e}_{X_1}(\dot{X}_1) + \sum_{i=2}^{n} \mathrm{Ad}_{\mathrm{e}(X_1)\cdots\mathrm{e}(X_{i-1})} d\mathrm{e}_{X_i}(\dot{X}_i).$$

(iii) The differentials of E and $E_{\boldsymbol{\vartheta}}$ at a point $\boldsymbol{X} \in \mathfrak{c}_{\boldsymbol{\vartheta}}$ have image

$$\mathrm{Im}(dE)_{\boldsymbol{X}} = \begin{cases} \mathfrak{su}_2 & \text{if some } \vartheta_i \notin \mathbb{Z} \\ \mathrm{Span}(\boldsymbol{X}) & \text{if all } \vartheta_i \in \mathbb{Z}, \end{cases}$$

$$\mathrm{Im}(dE_{\boldsymbol{\vartheta}})_{\boldsymbol{X}} = \begin{cases} \mathfrak{su}_2 & \text{if there exist } i \neq j \text{ with } X_i, X_j \text{ linearly independent and } \vartheta_i, \vartheta_j \notin \mathbb{Z} \\ \mathfrak{h}^\perp & \text{if all } X_i \text{ with } \vartheta_i \notin \mathbb{Z} \text{ belong to a line } \mathfrak{h} \subset \mathfrak{su}_2 \\ \{0\} & \text{if all } \vartheta_i \in \mathbb{Z}. \end{cases}$$

Proof. Part (i) is a straightforward and part (ii) follows from Lemma 3.14(ii). The proof of part (iii) uses (i) and (ii), and is analogous to the proof of Lemma 3.14(iii). □

Similarly to Section 3.4, upon identifying $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ with the image of $\widehat{\lambda}$, we have

$$\widehat{R}(\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{X}) = \boldsymbol{c}(\boldsymbol{M}, \boldsymbol{N}) \cdot E(\boldsymbol{X}).$$

Proof of Proposition 4.4. We follow the same steps as in the proof of Proposition 3.15, with a few variations. Quite as in the proof of Proposition 3.15, Lemma 3.14(i) gives

$$\mathrm{Im}(d\widehat{R}_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})}) = \mathrm{Im}(d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})}) + \mathrm{Ad}_{\boldsymbol{c}(\boldsymbol{M},\boldsymbol{N})}(\mathrm{Im}(dE_{\boldsymbol{X}})). \tag{5}$$

and

$$\mathrm{Im}(d\widehat{R}_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})} = \mathrm{Im}(d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})}) + \mathrm{Ad}_{\boldsymbol{c}(\boldsymbol{M},\boldsymbol{N})}(\mathrm{Im}(dE_{\boldsymbol{\vartheta}})_{\boldsymbol{X}}). \tag{6}$$

Let now $(\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{X}) = \widehat{\lambda}(\rho, A)$.

(i) In order to compute the image of $d\widehat{R}_{\boldsymbol{\vartheta}}$, we separately consider three cases.

If $\dim \mathfrak{z}(\boldsymbol{M}, \boldsymbol{N}) = 0$, then $d\boldsymbol{c}_{(\boldsymbol{M},\boldsymbol{N})} = \mathfrak{su}_2$ by Lemma 3.12 and so $\mathrm{Im}(dR_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})} = \mathfrak{su}_2$.

If $\mathfrak{z}(\boldsymbol{M}, \boldsymbol{N}) = \mathfrak{h}$ is 1-dimensional, then Lemma 4.5(iii) implies that the image $(d\widehat{R}_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})}$ is $\mathfrak{h}^\perp$ if all the non-integral X_i lie in $\mathfrak{h}$, and it is $\mathfrak{su}_2$ otherwise.

Suppose now that $\dim \mathfrak{z}(\boldsymbol{M}, \boldsymbol{N}) = 3$, and so all M_i, N_i are $\pm I$. By Lemma 4.5(iii), the image of $(d\widehat{R}_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})}$ is: $\{0\}$, if all ϑ_i are integer; $\mathfrak{h}^\perp$, if all non-integral X_i span the same line $\mathfrak{h}$ inside $\mathfrak{su}_2$; $\mathfrak{su}_2$, otherwise.

(ii) We now compute the image of $d\widehat{R}$ and we consider Equation (5).

If some ϑ_i is non-integral, then $dE_{\boldsymbol{X}}$ is surjective by Lemma 4.5(iii) and so $d\widehat{R}_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})}$ is too.

Suppose now that all ϑ_i are integral, and so $e(X_i) = \pm I$ for all i. By Lemma 4.5(iii) the image of $dE_{\boldsymbol{X}}$ is equal to $\mathrm{Span}(\boldsymbol{X})$. On the other hand, $\mathrm{Im}(d\boldsymbol{c})_{(\boldsymbol{M},\boldsymbol{N})}$ is exactly $\mathfrak{z}(\rho)^\perp = \mathfrak{z}(\boldsymbol{M}, \boldsymbol{N})^\perp$ by Lemma 3.12.

Thus, if ρ is non-coaxial, then $\mathfrak{z}(\rho) = \{0\}$ and $\mathrm{Im}(dR)_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})} = \mathfrak{su}_2$.

If ρ is coaxial, then $\boldsymbol{c}(\boldsymbol{M}, \boldsymbol{N}) = I$. It follows that the second summand $\mathrm{Ad}_{\boldsymbol{c}(\boldsymbol{M},\boldsymbol{N})}(\mathrm{Im}(dE_{\boldsymbol{X}}))$ of (5) contains $\mathrm{Span}(A(\beta_1), \dots, A(\beta_n))$ and is contained inside $\langle A \rangle$. Thus,

$$\mathfrak{z}(\rho)^\perp + \mathrm{Span}(A(\beta_1), \dots, A(\beta_n)) \subseteq \mathrm{Im}(d\widehat{R})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})} \subseteq \mathfrak{z}(\rho)^\perp + \langle A \rangle.$$

In order to show that the above inclusions are equalities, it is enough to show that the generators of $\langle A \rangle$, namely $A(\beta)$ for all $\beta \in \mathcal{B}$, belong to $\mathfrak{z}(\rho)^\perp + \mathrm{Span}(A(\beta_1), \dots, A(\beta_n))$. Indeed, for every i and every $\beta'_i \in \mathcal{B}_i$, we can write $\beta'_i = \gamma\beta_i\gamma^{-1}$ for some $\gamma \in \Pi_{g,n}$ and so $A(\beta'_i) = \mathrm{Ad}_{\rho(\gamma)}A(\beta_i)$. Then the difference $A(\beta'_i) - A(\beta_i) = (\mathrm{Ad}_{\rho(\gamma)} - I)A(\beta_i)$ belongs to $\mathfrak{z}(\rho(\gamma))^\perp \subseteq \mathfrak{z}(\rho)^\perp$, and so $A(\beta'_i)$ belongs to $\mathfrak{z}(\rho)^\perp + \mathrm{Span}(A(\beta_1), \dots, A(\beta_n))$. □

4.3 Tangent spaces to decorated homomorphism spaces

Using Proposition 4.4 we can compute the dimensions of the tangent spaces to our decorated homomorphism spaces. Recall that in the introduction (Definition 1.17) we defined the locus $\widetilde{\Sigma} = \widetilde{\Sigma}_0 \cup \widetilde{\Sigma}_1$, where

$$\widetilde{\Sigma}_0 := \left\{ (\rho, A) \;\middle|\; \rho \text{ central, and } \langle A \rangle \neq \mathfrak{su}_2 \right\}$$

$$\widetilde{\Sigma}_1 := \left\{ (\rho, A) \;\middle|\; \begin{array}{c} \mathrm{Im}(\rho) \subset \exp(\mathfrak{h}) \text{ non-central, } \rho(\mathcal{B}) \subseteq \{\pm I\}, \ \langle A \rangle \subset \mathfrak{h}^\perp \\ \text{for some 1-dimensional } \mathfrak{h} \subset \mathfrak{su}_2 \end{array} \right\}$$

inside $\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2)$, and we denote by $\Sigma=\Sigma_0\cup\Sigma_1$ the corresponding loci in $\widehat{\mathcal{R}ep}(\Pi_{g,n},\mathrm{SU}_2)$.

Corollary 4.6 (Tangent spaces to decorated homomorphisms spaces)**.** *Fix $\boldsymbol{\vartheta}$ and let $(\rho,A)\in\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$. Then*

(i) The Zariski tangent space to the relative decorated homomorphisms space satisfies

$$\dim T_{(\rho,A)}\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)=6g-3+2n+\dim(Z(\rho)).$$

(ii) The non-coaxial locus $\widehat{\mathcal{H}om}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is an oriented manifold of real dimension $6g-3+2n$.

(iii) The restriction of $\boldsymbol{\Theta}$ to $\widehat{\mathcal{H}om}^{nc}(\Pi_{g,n},\mathrm{SU}_2)$ is real-analytic and submersive.

(iv) The Zariski tangent space to the decorated homomorphisms space satisfies

$$\dim T_{(\rho,A)}\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2)=\begin{cases}6g+3n-\dim\langle A\rangle & \text{if } (\rho,A)\in\widetilde{\Sigma}_0 \quad (a)\\ 6g+3n-2 & \text{if } (\rho,A)\in\widetilde{\Sigma}_1 \quad (b)\\ 6g+3n-3 & \text{otherwise} \quad (c).\end{cases}$$

Away from the locus of decorated homomorphisms in $\widetilde{\Sigma}$, the space $\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2)$ is an oriented manifold of dimension $6g+3n-3$.

Proof. (i) We recall that the image $\widehat{\lambda}(\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2))$ is defined by $\widehat{R}_{\boldsymbol{\vartheta}}=I$ inside the smooth, oriented variety $\widehat{\mathcal{G}}_{\boldsymbol{\vartheta}}$ of dimension $6g+2n$. Thus the tangent space to $\widehat{\lambda}(\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2))$ at $\widehat{\lambda}(\rho,A)=(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})$ is the kernel of $(d\widehat{R}_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})}$. Since the image of $(d\widehat{R}_{\boldsymbol{\vartheta}})_{(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})}$ has dimension $\dim(\mathfrak{Z}(\rho)^\perp)=3-\dim(Z(\rho))$ by Proposition 4.4(i), it follows that the Zariski tangent space to $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ at (ρ,A) has dimension $6g+2n-3+\dim(Z(\rho))$.

(ii) If $(\rho,A)\in\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is non-coaxial, then $Z(\rho)=\{\pm I\}$ by Lemma 2.4 and $\widehat{R}_{\boldsymbol{\vartheta}}$ is submersive at $\widehat{\lambda}(\rho,A)$ by Proposition 4.4(i). Since both $\widehat{\mathcal{G}}_{\boldsymbol{\vartheta}}$ and SU_2 are oriented, the conclusion follows applying the implicit function theorem.

(iii) Let $(\rho,A)\in\widehat{\mathcal{H}om}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},A)$. To show that $\boldsymbol{\Theta}$ is real-analytic and submersive at $\widehat{\lambda}(\rho,A)=(\boldsymbol{M'},\boldsymbol{N'},\boldsymbol{X'})$ it is enough to show that the map $\widehat{\mathcal{G}}_+\to\mathrm{SU}_2\times\mathbb{R}^n_+$ that sends $(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X})$ to $(\widehat{R}(\boldsymbol{M},\boldsymbol{N},\boldsymbol{X}),\|X_1\|,\dots,\|X_n\|)$ is real-analytic submersive at $(\boldsymbol{M'},\boldsymbol{N'},\boldsymbol{X'})$. As in the proof of Corollary 3.16(iii), this follows from the fact that $\widehat{R}_{\boldsymbol{\vartheta}}$ is submersive at $(\boldsymbol{M'},\boldsymbol{N'},\boldsymbol{X'})$ by Proposition 4.4(i), and that the norm map $\|\cdot\|:\mathfrak{su}_2\to\mathbb{R}_+$ is real-analytic and submersive away from 0.

(iv) Since $\widehat{\lambda}(\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2))$ is defined by $\widehat{R}=I$ inside $\widehat{\mathcal{G}}_+$, the tangent space to $\widehat{\lambda}(\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2))$ at $\widehat{\lambda}(\rho,A)$ is the kernel of $d\widehat{R}_{\widehat{\lambda}(\rho,A)}$. Hence, the dimension of $T_{(\rho,A)}\widehat{\mathcal{H}om}(\Pi_{g,n},\mathrm{SU}_2)$ is $6g+2n-\mathrm{rk}(d\widehat{R})_{\widehat{\lambda}(\rho,A)}$.

(a) Suppose that $(\rho,A)\in\widetilde{\Sigma}_0$, namely ρ takes values in $\pm I$ and the image of A does not span $\mathfrak{su}_2$. Then Proposition 4.4(ii) gives $\mathrm{Im}(d\widehat{R})_{\widehat{\lambda}(\rho,A)}=\langle A\rangle$.

(b) Suppose that $(\rho,A)\in\widetilde{\Sigma}_1$, namely ρ is non-central and takes values in the subgroup generated by a 1-dimensional subalgebra $\mathfrak{h}\subset\mathfrak{su}_2$, and A takes values in $\mathfrak{h}^\perp$ of norm in $2\pi\mathbb{Z}$. Then Proposition 4.4(ii) gives $\mathrm{Im}(d\widehat{R})_{\widehat{\lambda}(\rho,A)}=\mathfrak{h}^\perp$, which has dimension 2.

(c) Suppose that $(\rho,A)\notin\widetilde{\Sigma}$. If ρ is non-coaxial or if some $\rho(\beta_i)\neq\pm I$, then Proposition 4.4(ii) implies that $d\widehat{R}_{\widehat{\lambda}(\rho,A)}$ has rank 3. If ρ takes values in $\pm I$, then $\langle A\rangle$ must be the whole $\mathfrak{su}_2$ (otherwise (ρ,A) would belong to $\widetilde{\Sigma}_0$), and so again $d\widehat{R}_{\widehat{\lambda}(\rho,A)}$ has rank 3 by Proposition 4.4(ii). If there exists a 1-dimensional $\mathfrak{h}\subset\mathfrak{su}_2$ such that ρ takes values in $\mathfrak{h}$ but is not central and $\rho(\mathcal{B})\subseteq\{\pm I\}$, then A cannot take values inside $\mathfrak{h}^\perp$ (otherwise (ρ,A) would belong to $\widetilde{\Sigma}_1$) and so $\mathfrak{h}^\perp+\langle A\rangle=\mathfrak{su}_2$, which implies that $d\widehat{R}_{\widehat{\lambda}(\rho,A)}$ has rank 3 by Proposition 4.4(ii).

The last claim is a consequence of the implicit function theorem, since $\widehat{\mathcal{G}}$ and SU_2 are oriented manifolds and $\widehat{R}$ is submersive at $(\rho,A)\notin\widetilde{\Sigma}$. ☐

Lemma 4.7 (Coaxial and elementary decorated homomorphisms). *Let $g \geq 0$ and $n > 0$ so that $2g-2+n > 0$.*

(i) Inside $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ the elementary locus has pure dimension $2g+n+1$, the singular locus agrees with $\widetilde{\Sigma}$ and has pure dimension $2g+2n+2$, the components of the coaxial locus have dimensions in $[2g+n+1, 2g+2n-1] \cup \{2g+2n+2\}$ if $g > 0$, and $[n+1, 2n]$ if $g = 0$.

Let $\boldsymbol{\vartheta} \in \mathbb{R}^n_{>0}$ and let k be the number of integer entries of $\boldsymbol{\vartheta}$.

(ii) Inside $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ the elementary and the coaxial loci are non-empty if and only if $\sum_i(\pm(\vartheta_i - 1)) \in 2\mathbb{Z}$ for some choice of the signs. The elementary locus is connected, irreducible, of dimension $2g+2$, the coaxial locus is connected, irreducible, of dimension $2n$ if $(g,n) = (0,k)$, and $2g+2k+2$ otherwise. Moreover, if $k > 0$, then the coaxial non-elementary locus is dense and connected inside the coaxial locus.

Proof. (i) The elementary locus in $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ is the image of

$$\begin{array}{ccc} \mathbb{S}^2 \times \mathbb{R}^{2g} \times \mathbb{R}^n_0 \times \mathbb{Z} & \longrightarrow & \widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2) \\ (X, s_1, t_1, \dots, t_g, \vartheta_1, \dots, \vartheta_n) & \longmapsto & (\mathbf{e}(s_1 X, t_1 X, \dots, t_g X), \vartheta_1 X, \dots, \vartheta_n X) \end{array}$$

where $\mathbb{R}^n_0 = \{\boldsymbol{\vartheta} \in \mathbb{R}^n \mid \sum_i (\vartheta_i - 1) \in 2\mathbb{Z}, \text{ and } \vartheta_i \neq 0 \text{ for all } i\}$. The fiber of such map is discrete, and so the elementary locus has dimension $2g+n+1$.

Corollary 4.6(iv) implies that the singular locus inside $\widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ is $\widetilde{\Sigma}$.

For $g = 0$ we have $\widetilde{\Sigma} = \widetilde{\Sigma}_0$ by Lemma Î(ii) and $\widetilde{\Sigma}_0$ can be described as the locus of coplanar $(X_1, \dots, X_n)$ in $\mathfrak{su}_2^{\oplus n}$ such that $\boldsymbol{\vartheta} \in \mathbb{Z}_+$, which is analytic of dimension $2n+2$.

Let now $g \geq 1$ and consider the $(2g+2n+2)$-dimensional analytic subset of $\mathbb{S}^2 \times (\mathbb{R}^{2g} \setminus \mathbb{Z}^{2g}) \times \mathfrak{su}_2^{\oplus n}$ defined as

$$\widetilde{\mathcal{S}}_1 := \left\{ (X, s_1, t_1, \dots, t_g, X_1, \dots, X_n) \;\middle|\; \sum_j (\|X_j\| - 1) \in 2\mathbb{Z}, \text{ and } 0 \neq X_j \perp X \text{ for all } j \right\}.$$

The map $\widetilde{\mathcal{S}}_1 \to \widetilde{\Sigma}_1$ that sends $(X, s_1, \dots, X_n)$ to $(\mathbf{e}(s_1 X, \dots, t_g X), X_1, \dots, X_n)$ is real-analytic, surjective, with discrete generic fiber: it follows that $\widetilde{\Sigma}_1$ has pure dimension $2g+2n+2$. By Lemma Î(ii), $\widetilde{\Sigma}_1$ is dense in $\widetilde{\Sigma}$ and so $\widetilde{\Sigma}$ has pure dimension $2g+2n+2$.

The coaxial locus is the preimage of the coaxial locus in $\mathcal{Hom}(\Pi_{g,n}, \mathrm{SU}_2)$ via the forgetful map $\widehat{\mathcal{F}}$. Such locus can be decomposed into subloci indexed by the number k of integer entries of $\boldsymbol{\vartheta}$. Indeed, consider the $(2g+n+k+2)$-dimensional analytic subset $\widetilde{\mathcal{C}}_k$ of $\mathbb{S}^2 \times \mathbb{R}^{2g} \times \mathbb{R}^{n-k}_{>0} \times (\mathbb{S}^2 \times \mathbb{Z}_{\neq 0})^k$ consisting of $(X, s_1, \dots, t_g, \vartheta_1, \dots, \vartheta_{n-k}, X_{n-k+1}, \dots, X_n)$ such that $\sum_{i=1}^{n-k} (\vartheta_i - 1) + \sum_{j=n-k+1}^{n} (\|X_j\| - 1) \in 2\mathbb{Z}$. Then the image of the map $\widetilde{\mathcal{C}}_k \to \widehat{\mathcal{Hom}}(\Pi_{g,n}, \mathrm{SU}_2)$ that sends $(X, s_1, \dots, t_g, \vartheta_1, \dots, \vartheta_{n-k}, X_{n-k+1}, \dots, X_n)$ to $(\mathbf{e}(s_1 X, \dots, t_g X), \vartheta_1 X, \dots, \vartheta_{n-k} X, X_{n-k+1}, \dots, X_n)$ corresponds to the subset of coaxial (ρ, A) such that $\vartheta_1, \dots, \vartheta_{n-k} \notin \mathbb{Z}$ and $\vartheta_{n-k+1}, \dots, \vartheta_n \in \mathbb{Z}$. If $g = 0$ and $k = n$, then such map has $(2n+2)$-dimensional domain and 2-dimensional fiber, and so its image has dimension $2n$. If $g = 0$ and $k \leq n-2$, then such map has $(n+k+1)$-dimensional domain and discrete fiber, and so its image has dimension $n+k+1 \in [n+1, 2n-1]$. If $g > 0$ and $k = n$, then the map has $(2g+2n+2)$-dimensional domain and discrete fiber, and so its image has dimension $2g+2n+2$. If $g > 0$ and $k \leq n-2$, then the map has $(2g+n+k+1)$-dimensional domain and discrete fiber, and so its image has dimension $2g+n+k+1 \in [2g+n+1, 2g+2n-1]$.

(ii) The non-emptiness claim for the coaxial locus follows from Proposition 3.5(ii). Assume now that $\sum_i \varepsilon_i (\vartheta_i - 1) \in 2\mathbb{Z}$ for certain $\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}$. The following analytic map

$$\begin{array}{ccc} \mathbb{S}^2 \times \mathbb{R}^{2g} & \longrightarrow & \widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \\ (X, s_1, t_1, \dots, t_g) & \longmapsto & (\mathbf{e}(s_1 X, \dots, t_g X), \vartheta_1 X, \dots, \vartheta_n X) \end{array}$$

has discrete fibers, and its image is the elementary locus: hence, the elementary locus is connected, irreducible, of dimension $2g+2$. As for the coaxial locus, suppose that $\vartheta_1, \dots, \vartheta_k \in \mathbb{Z}$ and $\vartheta_{k+1}, \dots, \vartheta_n \notin$

$\mathbb{Z}$, and consider the following analytic map

$$\begin{array}{ccc}\mathbb{S}^2 \times \mathbb{R}^{2g} \times (\mathbb{S}^2)^k & \longrightarrow & \widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \\ (X, s_1, t_1, \dots, t_g, \hat{X}_1, \dots, \hat{X}_k) & \longmapsto & (\mathbf{e}(s_1 X, \dots, t_g X), \vartheta_1 \hat{X}_1, \dots, \vartheta_k \hat{X}_k, \vartheta_{k+1} X, \dots, \vartheta_n X)\end{array}$$

Its image is the coaxial locus, which is thus connected and irreducible. It $g \geq 1$ or if $k < n$, then the fibers are discrete and so the coaxial locus has dimension $2g + 2k + 2$. If $g = 0$ and $k = n$, then the fibers are 2-dimensional and so the coaxial locus has dimension $2n$.

Note finally that the elementary locus is the image via the latter map above of the subset $(X, s_1, \dots, t_g, X, \dots, X)$ inside $\mathbb{S}^2 \times \mathbb{R}^{2g} \times (\mathbb{S}^2)^k$: hence, the non-elementary locus is dense and connected inside the coaxial locus if $k > 0$. □

4.4 Density and connectedness of the non-coaxial and non-elementary decorated loci

In this short section we investigate the non-coaxial and the non-elementary loci of decorated homomorphism spaces, first in the relative case and then in the absolute case.

The following statement relies on the properties of the map $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}} : \widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \to \mathcal{H}om_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$, introduced in Section 4.1.5, and on density and connectedness of the non-coaxial locus in undecorated relative homomorphism spaces proven in Theorem 3.18.

Proposition 4.8 (Density and connectedness of non-coaxial and non-elementary relative decorated loci)**.** *Let* $g \geq 0$ *and* $\boldsymbol{\vartheta} \in \mathbb{R}^n_{>0}$.

- *(i) If* $(g, n, \boldsymbol{\vartheta})$ *is non-special, then the non-coaxial locus in* $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *is non-empty, dense and connected; and so* $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *has pure dimension* $6g - 3 + 2n$. *If* $(g, n, \boldsymbol{\vartheta})$ *is special, then the non-coaxial locus is empty.*
- *(ii) The non-elementary locus inside* $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ *is dense and connected.*

Analogous claims hold in $\widehat{\mathcal{R}ep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$.

Proof. Recall that, by Lemma 4.1, the map $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}$ is a $(\mathbb{S}^2)^k$-bundle that preserves the coaxial loci, where k is the number of integer entries of $\boldsymbol{\vartheta}$. Clearly, it will be enough to prove (i) and (ii) for $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$, and the conclusion will hold for $\widehat{\mathcal{R}ep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ too.

(i) Consider first the non-special case. Recall that undecorated non-special $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ are smooth of dimension $6g - 3 + 2(n - k)$, dense and connected by Theorem 3.18. By the above considerations on $\widehat{\mathcal{F}}_{\boldsymbol{\vartheta}}$, the decorated non-special $\widehat{\mathcal{H}om}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ are smooth of dimension $6g - 3 + 2n$, dense and connected; so $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ has pure dimension $6g - 3 + 2n$.

Consider now the special cases. Recall that the undecorated special $\mathcal{H}om^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$ are empty by Theorem V. By the above considerations on $\widehat{\mathcal{F}}$, the same holds in the decorated case.

(ii) In non-special cases, the conclusion follows from (i). In the special cases, $k = n > 0$ and all homomorphisms are coaxial. So it is enough to show that the subset of non-elementary coaxial homomorphisms are dense inside the subset of coaxial homomorphisms: this is proven in Lemma 4.7(ii). □

In order to prove density and connectedness for absolute decorated homomorphism spaces, we want to view $\widehat{\mathcal{H}om}^{nc}(\Pi_{g,n}, \mathrm{SU}_2)$ as the total space of a fibration, whose fibers are the spaces $\widehat{\mathcal{H}om}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2)$, and then exploit Proposition 4.8. Thus, we first need to understand the base space of such fibration.

We denote by $\boldsymbol{\Theta}^{nc}$ the restriction of $\boldsymbol{\Theta} : \widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2) \to \mathbb{R}^n$ the the non-coaxial locus and let $\mathcal{A}_{g,n}$ and $\mathcal{A}^{nc}_{g,n}$ be the images of $\boldsymbol{\Theta}$ and $\boldsymbol{\Theta}^{nc}$ respectively.

Lemma 4.9 (Angles realized by decorated homomorphisms)**.** *For* $g \geq 0$ *and* $n > 0$ *such that* $2g - 2 + n > 0$

$$\mathcal{A}_{g,n} = \begin{cases} \mathbb{R}^n_{>0} & \text{if } g \geq 1 \\ \{\boldsymbol{\vartheta} \in \mathbb{R}^n_{>0} \,|\, d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) \geq 1\} & \text{if } g = 0 \end{cases}$$

and it is always connected; on the other hand,

$$\mathcal{A}^{nc}_{g,n} = \begin{cases} \mathbb{R}^n_{>0} & \text{if } g \geq 2 \\ \mathbb{R}^n_{>0} \setminus \{\boldsymbol{\vartheta} \in \mathbb{Z}^n \mid \sum_1 (\vartheta_i - 1) \in 2\mathbb{Z}\} & \text{if } g = 1 \\ \{\boldsymbol{\vartheta} \in \mathbb{R}^n_{>0} \mid d_1(\boldsymbol{\vartheta} - \mathbf{1}, \mathbb{Z}^n_o) > 1\} & \text{if } g = 0 \end{cases}$$

and it is connected if and only if $(g, n) \neq (0, 3), (1, 1)$.

Proof. The characterization of $\mathcal{A}_{g,n}$ and $\mathcal{A}^{nc}_{g,n}$ follows from Proposition 3.19 for genus 0, from Proposition 3.7(ii) in the special cases of genus 1, and from Theorem 3.27 in non-special cases.
The connectedness of $\mathcal{A}_{g,n}$ and $\mathcal{A}^{nc}_{g,n}$ for genus 0 is analyzed in [44, Lemma B]; the case of genus 1 is straightforward. □

Now we can treat the non-coaxial locus inside absolute decorated homomorphism spaces.

Proposition 4.10 (Density and connectedness of non-coaxial and non-elementary decorated loci)**.** *Let* $g \geq 0$ *and* $n > 0$ *with* $2g - 2 + n > 0$. *Inside* $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$

(i) the non-coaxial locus is dense; moreover it is connected if and only if $(g, n) \neq (0, 3), (1, 1)$*;*

(ii) the non-elementary locus is dense and connected;

(iii) $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$ *has pure dimension* $6g - 3 + 3n$.

The same holds for $\widehat{\mathcal{R}ep}(\Pi_{g,n}, \mathrm{SU}_2)$.

Proof. Again, the conclusion for $\widehat{\mathcal{R}ep}(\Pi_{g,n}, \mathrm{SU}_2)$ will immediately follow once claims (i) and (ii) for decorated homomorphism spaces are proven.

(i) Observe non-special points are dense in $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$. By Proposition 4.8(i) non-coaxial points are dense inside the subset of non-special points. It follows that the non-coaxial locus is dense in $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$. Recall also that the map $\boldsymbol{\Theta}^{nc} : \widehat{\mathcal{H}om}^{nc}(\Pi_{g,n}, \mathrm{SU}_2) \to \mathcal{A}^{nc}_{g,n}$ is surjective and submersive (and so open) by Corollary 4.6(iii), and that its fibers are connected by Proposition 4.8(i). By Lemma 3.8(i) it follows that the connected components of $\widehat{\mathcal{H}om}^{nc}(\Pi_{g,n}, \mathrm{SU}_2)$ are exactly the inverse images via $\boldsymbol{\Theta}^{nc}$ of the connected components of $\mathcal{A}^{nc}_{g,n}$. The conclusion follows by Lemma 4.9.

(ii) Density of the non-elementary locus follows from the density of the non-coaxial locus in (i). As for the connectedness, if $(g, n) \neq (0, 3), (1, 1)$, then again it follows from the connectedness of the non-coaxial locus in (i).

Consider the case $(g, n) = (1, 1)$. The subset $(\boldsymbol{\Theta}^{nc})^{-1}(2m-1, 2m+1)$ is connected for every integer $m \geq 1$. Thus we only need to connect a non-elementary point in $\boldsymbol{\Theta}^{-1}(2m+1)$ to a point in $\boldsymbol{\Theta}^{-1}(2m+1-\varepsilon)$ and to a point in $\boldsymbol{\Theta}^{-1}(2m+1+\varepsilon)$. By Corollary 3.13(i) every non-elementary point in $\boldsymbol{\Theta}^{-1}(2m+1)$ can be connected inside $\widehat{\mathcal{H}om}^{ne}_{2m+1}(\Pi_{1,1}, \mathrm{SU}_2)$ to the point $(M(0), N(0), X(0)) = (I, D, (2m+1)R)$, where $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $R = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So we can for example use the path $(M(t), N(t), X(t)) = (\exp(\pi t R), D, (2m + 1 + t)R)$ with $t \in (-\varepsilon, \varepsilon)$.

Consider finally the case $(g, n) = (0, 3)$. By Lemma 4.9 the set $\mathcal{A}_{0,3}$ is connected. Indeed, if $\mathcal{C} = [a, a + 1] \times [b, b + 1] \times [c, c + 1]$ with $a, b, c \in \mathbb{Z}_{\geq 0}$ and $(a, b, c) \neq (0, 0, 0)$, then $\mathcal{A}_{0,3} \cap \mathcal{C}$ is a closed tetrahedron with vertices in the even vertices of $\mathcal{C}$. Thus, up to reordering the labels, it is enough to show that a non-elementary point $(X_1(0), X_2(0), X_3(0))$ in $\widehat{\mathcal{H}om}_{\boldsymbol{\vartheta}}(\Pi_{0,3}, \mathrm{SU}_2)$ with $\vartheta_3 \in \mathbb{Z}$ and $\vartheta_1, \vartheta_2 \notin \mathbb{Z}$ is part of a path $(X_1(t), X_2(t), X_3(t))$ such that $t \mapsto \|X_3(t)\|$ is strictly increasing. We choose $X_3(t) = (1+t)X_3(0)$ and $X_2(t) = X_2(0)$. Since $\vartheta_1 \notin \mathbb{Z}$, for $t \in (-\varepsilon, \varepsilon)$ there exists a unique continuous path $X_1(t)$ with given $X_1(0)$ and such that $\mathrm{e}(X_1(t))\mathrm{e}(X_2(t))\mathrm{e}(X_3(t)) = I$. The conclusion follows.

(iii) follows from the fact that the non-coaxial locus inside $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$ is dense by (i) and is smooth of dimension $6g - 3 + 3n$ by Corollary 4.6(iv). □

4.5 Absolute, non-special and special decorated homomorphism spaces

In this section we prove our main results on decorated absolute and relative homomorphism and representation spaces.

For absolute decorated spaces, all the work has already been done.

Proof of Theorem $\widehat{II}$. (i) is the content of Lemma 2.6.

(ii-a,b,c) follow from Lemma 4.7(i).

(ii-d) It follows from Corollary 4.6(iv) it is an oriented manifold of dimension $6g - 3 + 3n$.

(ii-e) follows from (ii-a), since elementary decorated homomorphisms have 1-dimensional stabilizer.

(ii-f) follows from (ii-b) by Lemma 2.6.

(ii-g) follows from (ii-c) and Remark 1.9 and Lemma 2.6.

(iii) is proven in Proposition 4.10(iii). □

Also for relative decorated spaces we have already proven almost everything.

Proof of Theorem $\widehat{III}$. (i) follows from Lemma 4.7(ii).

(ii) follows from Corollary 4.6(i-ii).

(iii) follows from (i), since elementary decorated homomorphisms have 1-dimensional stabilizer.

(iv) By Lemma $\hat{\mathrm{I}}$(a), if $k = 0$, then coaxial is equivalent to elementary: in this case, coaxial decorated homomorphisms have 1-dimensional stabilizer. If $k > 0$, then the general coaxial decorated homomorphism is non-elementary by Lemma 4.7(ii): in this case, the general coaxial decorated homomorphism has trivial stabilizer by Lemma 2.6. The conclusion then follows from (i).

(v) follows from Remark 1.9 and Lemma 2.6.

(vi) is proven in Proposition 4.8(ii). □

The wished conclusion for the non-coaxial loci of non-special relative decorated spaces follows.

Proof of Theorem $\widehat{IV}$. Recall that smoothness of the non-coaxial locus is proven in Theorem $\widehat{\mathrm{III}}$(ii-v) together with the determination of its dimension.

(iii-iv) Non-emptiness, density and connectedness of the non-coaxial locus is proven in Proposition 4.8(i).

(i) Pure dimensionality of the relative decorated homomorphism and representation spaces follows from (iii)-(iv).

(ii) Because of pure-dimensionality, Corollary 4.6(i-ii) also shows that the smooth locus of $\widehat{\mathcal{Hom}_{\boldsymbol{\vartheta}}}(\Pi_{g,n}, \mathrm{SU}_2)$ coincides with the non-coaxial locus. It follows that $\widehat{\mathcal{Hom}_{\boldsymbol{\vartheta}}}(\Pi_{g,n}, \mathrm{SU}_2)$ is reduced and irreducible. Concerning the relative decorated representation space, it is reduced and irreducible and its smooth locus consists of $[\rho, A]$ at which $\dim(T_{(\rho,A)}\widehat{\mathcal{Hom}_{\boldsymbol{\vartheta}}}(\Pi_{g,n}, \mathrm{SU}_2)) + \dim(Z(\rho, A))$ is minimum by Remark 1.10. Such minimum is achieved at the non-coaxial locus by Corollary 4.6(i), which is non-empty by (iii). □

The analysis of special decorated spaces is more explicit.

Proof of Theorem $\widehat{V}$. Preliminarily observe that the map

$$\begin{array}{ccc} \mathbb{S}^2 \times \{\pm 1\}^{m-1} & \longrightarrow & \mathrm{Coax}((\mathbb{S}^2)^m) \\ (X, \varepsilon_2, \dots, \varepsilon_m) & \longmapsto & (X, \varepsilon_2 X, \dots, \varepsilon_m X) \end{array}$$

is an isomorphism. For $m = 1$, clearly $\mathrm{Coax}(\mathbb{S}^2) = \mathbb{S}^2$. For $m \geq 2$, the coaxial locus in $(\mathbb{S}^2)^m$ is a compact submanifold of codimension $2m - 2$: hence the non-coaxial locus in $(\mathbb{S}^2)^m$ is open, dense and connected.

(i-a) The central locus in $\widehat{\mathcal{Hom}_{\boldsymbol{\vartheta}}}(\Pi_{1,n}, \mathrm{SU}_2)$ consists of $(M_1, N_1, X_1, \dots, X_n)$ such that $M_1, N_1 \in \{\pm I\}$ and $|X_i\| = \vartheta_i$ for all i. As a consequence, it is isomorphic to $\{\pm I\}^2 \times (\mathbb{S}^2)^n$. Inside it, the elementary locus of $\widehat{\mathcal{Hom}_{\boldsymbol{\vartheta}}}(\Pi_{1,n}, \mathrm{SU}_2)$ corresponds to $\{\pm I\} \times \mathrm{Coax}((\mathbb{S}^2)^n)$. The conclusion follows from (o).

(i-b) The proof of Proposition 3.7(ii) shows that the non-central locus in $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$ is isomorphic to the quotient of $\mathbb{S}^2 \times (\mathbb{R}^2 \setminus \mathbb{Z}^2)/(2\mathbb{Z})^2$ by $\{\pm 1\}$, which is an $\mathbb{S}^2$-bundle over $S^2 \setminus \{4 \text{ points}\}$. By Lemma

4.1, the non-central locus of $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$ is isomorphic to the quotient of $(\mathbb{S}^2)^{1+n} \times (\mathbb{R}^2 \setminus \mathbb{Z}^2)/(2\mathbb{Z})^2$ by $\{\pm 1\}$, which is an $(\mathbb{S}^2)^{1+n}$-bundle over $S^2 \setminus \{4 \text{ points}\}$. The elementary non-central locus corresponds to the $\mathrm{Coax}((\mathbb{S}^2)^{1+n})$-subbundle over $S^2 \setminus \{4 \text{ points}\}$.

(i-c) The claim for $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$ is a consequence of (i-b) and (o). The corresponding conclusion for $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{1,n}, \mathrm{SU}_2)$ follows from Remark 1.9 and Lemma 2.6.

(ii) is a consequence of Theorem V(ii).

(iii) By Theorem V(iii), $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ consists of a single conjugacy class, and it is isomorphic to $\mathbb{S}^2$ if $k \leq n-2$, or to a point if $k = n$.

(iii-a) follows from Lemma Î(b).

(iii-b) Assume $\vartheta_1, \dots, \vartheta_k \in \mathbb{Z}$ and $\vartheta_{k+1}, \dots, \vartheta_n \notin \mathbb{Z}$. By Theorem V(iii) the map $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2) \to \mathbb{S}^2$ that sends $(X_1, \dots, X_n)$ to $\hat{X}_{k+1}$ is an isomorphism. As a consequence,

$$\begin{aligned} \widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2) &\longrightarrow (\mathbb{S}^2)^{k+1} \\ (X_1, \dots, X_n) &\longmapsto (\hat{X}_1, \dots, \hat{X}_{k+1}) \end{aligned}$$

is an isomorphism. Clearly, the elementary locus corresponds to $\mathrm{Coax}((\mathbb{S}^2)^{k+1})$. The conclusion then follows from (o), Remark 1.9 and Lemma 2.6.

(iii-c) By Lemma 4.1 the map

$$\begin{aligned} \widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2) &\longrightarrow (\mathbb{S}^2)^{n} \\ (X_1, \dots, X_n) &\longmapsto (\hat{X}_1, \dots, \hat{X}_{n}) \end{aligned}$$

is an isomorphism. The elementary locus corresponds to $\mathrm{Coax}((\mathbb{S}^2)^n)$. The conclusion for $\widehat{\mathcal{Rep}}_{\boldsymbol{\vartheta}}^{ne}(\Pi_{0,n}, \mathrm{SU}_2)$ follows by Remark 1.9 and Lemma 2.6.

(iii-d) Since $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ consists of a single conjugacy class, $\widehat{\mathcal{Rep}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ consists of one point. By Lemma 4.1 the analytic structure on $\widehat{\mathcal{Hom}}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is reduced if and only if the algebraic structure on $\mathcal{Hom}_{\boldsymbol{\vartheta}}(\Pi_{0,n}, \mathrm{SU}_2)$ is reduced. Hence, the conclusion follows from Theorem V(iii-c). □

In [24, Theorem 1] it was shown that, for $n = 0$, the singularities of the homomorphisms spaces are cut by quadrics in suitable analytic coordinates. The singularities of relative representation spaces and of relative decorated representations spaces, and in particular of their non-elementary locus, are worthwhile further investigations.

4.6 Symplectic structures

We begin by recalling the definition of Goldman's symplectic structure on $\mathcal{Rep}_{\boldsymbol{\vartheta}}^{nc}(\Pi_{g,n}, \mathrm{SU}_2)$, following Guruprasad-Huebschmann-Jeffrey-Weinstein [27]. We refer to [27] for full treatment of the topic (see also [23, Section 1.4]).

The classical Riemann-Hilbert correspondence establishes a bijective correspondence between flat $\mathfrak{su}_2$-vector bundles E on $\dot{S}$ with monodromy in PSU_2 (up to isomorphism) and representations $\bar{\rho} : \pi_1(\dot{S}) \to \mathrm{PSU}_2$, and it works as follows. For every $\bar{\rho}$ we define $E_{\bar{\rho}} := \mathfrak{su}_2 \times \widetilde{\dot{S}}/\pi_1(\dot{S})$, where $\pi_1(\dot{S})$ acts on $\mathfrak{su}_2$ via $\mathrm{Ad} \circ \bar{\rho} : \pi_1(\dot{S}) \to \mathrm{Aut}(\mathfrak{su}_2) \cong \mathrm{PSU}_2$. Conversely, given a flat E we define $\bar{\rho}_E$ to be the monodromy representation of E.

Since PSU_2 is compact, first-order deformations of $[\bar{\rho}]$ are parametrized by the de Rham cohomology group $H^1(\dot{S}, E_{\bar{\rho}})$. Moreover, first-order deformations that do not change the conjugacy class of the image of the β_i's correspond to elements of the *parabolic cohomology* group $H^1_{\mathrm{par}}(\dot{S}, E_{\bar{\rho}})$, namely classes in $H^1(\dot{S}, E_{\bar{\rho}})$ that can be represented by compactly supported 1-forms on $\dot{S}$ (see [52, Section 6]).

Now, recall that $\Pi_{g,n}$ is the fundamental group of $\dot{S}$ with a chosen basepoint. If $\bar{\rho} : \Pi_{g,n} \to \mathrm{PSU}_2$ is induced by some $\rho : \Pi_{g,n} \to \mathrm{SU}_2$, then first-order deformations of ρ bijectively correspond to first-order deformation of $\bar{\rho}$, namely $T_{[\rho]}\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \cong T_{[\bar{\rho}]}\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{PSU}_2)$. Hence,

$$T_{[\rho]}\mathcal{Rep}_{\boldsymbol{\vartheta}}(\Pi_{g,n}, \mathrm{SU}_2) \cong H^1_{\mathrm{par}}(\dot{S}, E_{\bar{\rho}}) \subset H^1(\dot{S}, E_{\bar{\rho}}) \cong T_{[\rho]}\mathcal{Rep}(\Pi_{g,n}, \mathrm{SU}_2).$$

The Goldman 2-vector field Λ on $\mathcal{Rep}(\Pi_{g,n},\mathrm{SU}_2)$ defined as

$$\Lambda_{[\rho]}:T^*_{[\rho]}\mathcal{Rep}(\Pi_{g,n},\mathrm{SU}_2)\cong H^1(\dot{S},E_{\bar{\rho}})\to H^1_c(\dot{S},E_{\bar{\rho}})\cong T_{[\rho]}\mathcal{Rep}(\Pi_{g,n},\mathrm{SU}_2)$$

determines a Poisson structure. Now, the space $\mathcal{Rep}(\Pi_{g,n},\mathrm{SU}_2)$ is in general singular, but its non-coaxial locus is smooth by Theorem III(iv).

In [27, Section 9] it is also shown that $\mathcal{Rep}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ is a symplectic leaf for such Poisson structure. In fact, if we denote by Ω the symplectic form induced by Λ on $\mathcal{Rep}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$, then

$$\Omega_{[\rho]}:T_\rho\mathcal{Rep}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)\otimes T_\rho\mathcal{Rep}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)\longrightarrow\mathbb{R}$$

at a non-coaxial representation $[\rho]$ can be identified to the alternate pairing

$$\begin{array}{ccc} H^1_{\mathrm{par}}(\dot{S},E_{\bar{\rho}})\times H^1_{\mathrm{par}}(\dot{S},E_{\bar{\rho}}) & \longrightarrow & \mathbb{R} \\ ([\phi],[\psi]) & \longmapsto & \int_S \mathcal{K}(\phi\wedge\psi) \end{array}$$

where the representatives ϕ,ψ are chosen to vanish in a neighbourhood of $\boldsymbol{x}$. Note that such alternate pairing is non-degenerate by Poincaré duality and by the non-degeneracy of the Killing form $\mathcal{K}$.

The above considerations still hold when replacing SU_2 by $\mathrm{SL}_2(\mathbb{C})$: in this case, we obtain a holomorphic symplectic form $\Omega_{\mathbb{C}}$ on $\mathcal{Rep}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SL}_2(\mathbb{C}))$.

Proof of Proposition VI. (i) It is enough to observe that the inclusion $\mathcal{Hom}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)\to\mathcal{Hom}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SL}_2(\mathbb{C}))$ is a real-analytic map and that both $\mathcal{Rep}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SU}_2)$ and $\mathcal{Rep}^{nc}_{\boldsymbol{\vartheta}}(\Pi_{g,n},\mathrm{SL}_2(\mathbb{C}))$ are manifolds by Theorem III(ii-iv).

(ii) The compatibility between $\Omega_{\mathbb{C}}$ and Ω is clear, since the Killing form on SU_2 is the restriction of the Killing form on $\mathrm{SL}_2(\mathbb{C})$. Since $\Omega_{\mathbb{C}}$ is holomorphic, its restriction Ω is real-analytic. The non-degeneracy of Ω and $\Omega_{\mathbb{C}}$ discussed above is proven in [27]. □

Functoriality of cohomology implies that Ω and $\Omega_{\mathbb{C}}$ are $\mathrm{MCG}_{g,n}$-invariant. As a consequence, the symplectic form on the decorated representation space mentioned in Corollary $\widehat{\mathrm{VI}}$ is mapping class group invariant too.

4.7 Decorated monodromy of spherical surfaces

In this section we discuss general properties of the decorated monodromy representations of spherical surfaces with conical points and prove Theorem $\widehat{\mathrm{VII}}$. Before doing that, we will show how to restrict a decorated homomorphism to a finite-index subgroup.

4.7.1 Restriction of a decorated homomorphism Given a finite-index subgroup $\Pi'\subset\Pi_{g,n}$, we define the peripheral set of Π' to be the set $\mathcal{B}'$ of elements $\beta'\in\Pi'$ that are indivisible in Π' and such that $\beta'=\beta^k$ for some $\beta\in\mathcal{B}$ and some $k\neq 0$.

Remark 4.11. It is easy to see that, if $\dot{S}$ is a punctured surface with a base point $*$ endowed with an isomorphism $\Pi_{g,n}\cong\pi_1(\dot{S},*)$, then Π' corresponds to a finite unbranched cover $\dot{S}'\to\dot{S}$ and $\mathcal{B}'$ corresponds to the set of peripheral elements in $\pi_1(\dot{S}')$.

Definition 4.12 (Restriction of a decorated homomorphism)**.** Let (ρ,A) be a decorated homomorphism with $\rho:\Pi_{g,n}\to\mathrm{SU}_2$ and $A:\mathcal{B}\to\mathfrak{su}_2\setminus\{0\}$, and let $\Pi'\subseteq\Pi_{g,n}$ be a finite-index subgroup and $\mathcal{B}'$ its peripheral set. The *decorated homomorphism induced by* (ρ,A) *on* Π' is the couple (ρ',A'), where $\rho':\Pi'\to\mathrm{SU}_2$ is simply the restriction of ρ, and $A':\mathcal{B}'\to\mathfrak{su}_2\setminus\{0\}$ is defined by requiring that $A'([\beta']):=k\cdot A(\beta)$, whenever $\beta'=\beta^k$ with $\beta\in\mathcal{B}$.

Observe that (ρ',A') is a decorated homomorphism of Π' and that the image of $\hat{A}'$ coincides with the image of $\hat{A}$. Indeed, if $[\beta']\in\mathcal{B}'$, then $\beta'=\beta^k$ for some $\beta\in\mathcal{B}$ and some $k\neq 0$ and so $\hat{A}'(\beta')=\hat{A}(\beta)$ belongs to the image of $\hat{A}$: this shows that $\mathrm{Im}(\hat{A}')\subseteq\mathrm{Im}(\hat{A})$. Given $[\beta]\in\mathcal{B}$, there exists a $k\neq 0$ such that $[\beta^k]\in\mathcal{B}'$ because Π' has finite index in $\Pi_{g,n}$. It follows that $\hat{A}(\beta)=\hat{A}'(\beta^k)$ belongs to the image of $\hat{A}'$: this shows that $\mathrm{Im}(\hat{A})\subseteq\mathrm{Im}(\hat{A}')$.

We also have the following dichotomy.

Lemma 4.13. *Let $g \geq 0$ and $n > 0$ so that $2g-2+n>0$, and let (ρ, A) be a decorated homomorphism with $\rho : \Pi_{g,n} \to \mathrm{SU}_2$ and $A : \mathcal{B} \to \mathfrak{su}_2 \setminus \{0\}$. Suppose that all decorated homomorphisms induced on finite-index subgroups of $\Pi_{g,n}$ are non-elementary. Then either $\hat{A}$ achieves infinitely many values or ρ has finite image.*

Proof. Since $n > 0$, the map $\hat{A}$ achieves at least one value. Suppose that $\mathrm{Im}(\hat{A})$ is finite. We want to show that $\mathrm{Im}(\rho)$ is finite.

Since $\mathrm{Im}(\hat{A})$ is $\mathrm{Im}(\rho)$-invariant, there exists a finite-index subgroup Π' of $\Pi_{g,n}$ such that $\rho(\Pi')$ fixes $\mathrm{Im}(\hat{A})$ pointwise. Denote by (ρ', A') the decorated homomorphisms induced by $\Pi' \subset \Pi_{g,n}$.

Now, the images of $\hat{A}'$ and of $\hat{A}$ coincide by the above observation and ρ' fixes them pointwise, thus ρ' is coaxial. If ρ' were non-central, then $\mathrm{Im}(\hat{A}')$ would contain either one point or two antipodal points: in either case, (ρ', A') would be elementary, against our hypothesis. Hence, ρ' is central and so ρ has finite image. □

4.7.2 Spin structures Let S be a compact Riemann surface. By [2] a spin structure on S corresponds to a holomorphic line bundle L on S such that $L^{\otimes 2} = K$. Since $H^1(S;\mathbb{Z}/2)$ can be identified to the group of holomorphic line bundles on S of order two, the set of spin structures is simply transitively acted on by $H^1(S;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{2g}$.

For the Riemann surface $\mathbb{CP}^1$, there is a unique spin structure up to isomorphism. Indeed, the square of $\mathbb{L} = \mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathbb{CP}^1$ is isomorphic to $K_{\mathbb{CP}^1}$. Note also that the total space of $\mathbb{L}$ can be identified to $\mathbb{C}^2 \setminus \{0\}$ and that the usual action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{CP}^1$ lifts to the standard action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{L} = \mathbb{C}^2 \setminus \{0\}$.

Note that, upon identifying the center $Z(\mathrm{SU}_2) = \{\pm I\}$ with $\mathbb{Z}/2$, an element of $H^1(S;\mathbb{Z}/2)$ can be viewed as a homomorphism $\sigma : \Pi_{g,n} \cong \pi_1(\dot{S}) \to \{\pm I\}$ that sends every peripheral element to I. Thus, $H^1(S;\mathbb{Z}/2)$ acts on $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SU}_2)$ by sending (ρ, A) to $(\sigma \cdot \rho, A)$, where $\sigma \cdot \rho$ simply denotes the product in SU_2: in particular, $H^1(S;\mathbb{Z}/2)$ simply transitively acts on the set of SU_2-liftings of a given $(\overline{\rho}, A)$ in $\widehat{\mathcal{H}om}(\Pi_{g,n}, \mathrm{SO}_3(\mathbb{R}))$.

Proof of Theorem $\widehat{VII}$(o). Let h be a spherical metric on $(S, \boldsymbol{x})$ and let $\iota : \widetilde{\dot{S}} \to \mathbb{S}^2$ be a developing map for h, with associated decorated monodromy homomorphism $(\overline{\rho}, A)$.

The existence of an SU_2-lifting for $(\overline{\rho}, A)$ was shown in [15, Proposition A.1]. Since such liftings can be put in bijection with $H^1(S;\mathbb{Z}/2)$, there are exactly 2^{2g} of such.

In order to see that an SU_2-lifting of $(\overline{\rho}, A)$ bijectively corresponds to a choice of L satisfying $L^{\otimes 2} \cong K_S$, it is enough to construct an $H^1(S;\mathbb{Z}/2)$-equivariant map

$$\mathcal{L} : \{\mathrm{SU}_2\text{-liftings of } (\rho, A)\} \longrightarrow \{L \mid L^{\otimes 2} \cong K_S\}/\text{isom}.$$

Consider the action of $\Pi_{g,n} \cong \pi_1(\dot{S})$ by deck transformations on $\widetilde{\dot{S}}$. Given an SU_2-lifting ρ of $\overline{\rho}$, we can lift the $\Pi_{g,n}$-action from $\widetilde{\dot{S}}$ to

$$\iota^*\mathbb{L} := \{(\tilde{x}, \iota(\tilde{x}), v) \in \widetilde{\dot{S}} \times \mathbb{CP}^1 \times (\mathbb{C}^2 \setminus \{0\}) \mid [v] = \iota(\tilde{x})\}.$$

The quotient determines a line bundle $\dot{L}$ on $\dot{S}$: the isomorphism $\mathcal{O}_{\mathbb{CP}^1}(-1)^{\otimes 2} \cong K_{\mathbb{CP}^1}$ induces the isomorphism $\dot{L}^{\otimes 2} \cong K_{\dot{S}}$ on $\dot{S}$, since ι is a local biholomorphism.

In order to extend the isomorphism $\dot{L}^{\otimes 2} \cong K_{\dot{S}}$ over the puncture x_i for all i, choose a coordinate z in a disk neighbourhood $V_i \subset S$ of x_i and a local coordinate w on $\mathbb{CP}^1$ in such a way that the developing map on $\dot{V}_i := V_i \setminus \{x_i\}$ can be written as a multi-valued function $\hat{\iota}(z) = z^{\vartheta_i}$. This way the section $\iota^*\sqrt{dw}$ of $\dot{L}\big|_{\dot{V}_i}$ can be multi-valued. Since the pull-back of dw via $z \mapsto z^{\vartheta} = w$ is $\vartheta z^{\vartheta - 1} dz$, the section $\sigma_i = z^{\frac{1-\vartheta_i}{2}}(\iota^*\sqrt{dw})$ of $\dot{L}\big|_{\dot{V}_i}$ is single-valued. So we define $L \to S$ to be the unique extension of $\dot{L}$ whose local sections on V_i are generated by σ_i. It follows that the isomorphism $\dot{L}^{\otimes 2} \cong K_{\dot{S}}$, whose restriction to $\dot{V}_i$ sends $\sigma_i^{\otimes 2}$ to $\vartheta_i dz$, extends over V_i to an isomorphism $L^{\otimes 2} \cong K_S$. Then we set $\mathcal{L}(\rho, A) := L$. It is easy to check that $\mathcal{L}$ is $H^1(S;\mathbb{Z}/2)$-equivariant. □

4.7.3 Properties of decorated monodromy homomorphisms

By Theorem $\widehat{\text{VII}}$(o) proved above, we know that decorated monodromy homomorphisms admit SU_2-liftings. Using this piece of information, we are ready to complete the proof of our last main result.

End of the proof of Theorem $\widehat{VII}$. As in the previous part of the proof, let h be a spherical metric on $(S, \boldsymbol{x})$ and let $\iota : \widetilde{\dot{S}} \to \mathbb{S}^2$ be a developing map for h, with associated decorated monodromy homomorphism $(\overline{\rho}, A)$. Also, fix an SU_2-lifting (ρ, A) of $(\overline{\rho}, A)$.

(i) By contradiction, suppose that (ρ, A) is elementary. Since $n \geq 1$, the image of A is non-trivial and so the infinitesimal centralizer $\mathfrak{h} = \mathfrak{Z}(\rho, A) \subset \mathfrak{su}_2$ is 1-dimensional. Thus the decoration A takes values in $\mathfrak{h}$ and ρ takes values in $H = \exp(\mathfrak{h}) \subset SU_2$. Fix now an element $0 \neq X \in \mathfrak{h}$. It induces a nontrivial Killing vector field on $\mathbb{S}^2$ that vanishes at $\mathbb{S}^2 \cap \mathfrak{h}$ (remember that we are viewing $\mathbb{S}^2$ as the unit sphere inside $\mathfrak{su}_2$). Its pull-back $\widetilde{V}$ on $\widetilde{\dot{S}}$ is $\pi_1(\dot{S})$-invariant holomorphic vector field, which descends to a nontrivial holomorphic vector field V on $\dot{S}$. Note that the developing map extends to $\widehat{S}$ by sending $\partial\widehat{S}$ to $\mathbb{S}^2 \cap \mathfrak{h}$, and so the vector field V vanishes at $\boldsymbol{x}$. Since $\deg(T_S(-\boldsymbol{x})) = 2 - 2g - n < 0$, it follows that V must vanish. This contradiction proves that (ρ, A) must be non-elementary.

As for the image of $\hat{A}$ in $\mathbb{S}^2$, note first that every finite-index subgroup $\Pi' \subset \Pi_{g,n}$ corresponds to a finite unbranched cover $\dot{S}' \to \dot{S}$ and that the decorated homomorphism (ρ', A') induced by (ρ, A) corresponds to the monodromy of the spherical metric h' induced on $\dot{S}'$ by pull-back as in Remark 4.11. At the beginning of the proof of (i) we have already shown that (ρ', A') cannot be elementary. Hence, by Lemma 4.13 there are two possibilities: either $\hat{A}$ achieves infinitely many values, or the image of ρ is finite. In the former case, we have already achieved the wished conclusion. In the latter case, we consider the cover $\dot{S}' \to \dot{S}$ associated to $\Pi' = \ker(\rho)$, which has finite-index in $\Pi_{g,n}$. Since ρ' is trivial, the developing map ι descends to $\dot{S}'$ and extends to a finite cover $\overline{\iota}' : S' \to \mathbb{S}^2$. Such cover $\overline{\iota}'$ ramifies at the conical points of (S', h') and its branching locus is contained inside the image of $\hat{A}'$. Since $\chi(\dot{S}') < 0$, it follows that the branching locus of $\overline{\iota}'$ contains at least three points and so $\hat{A}'$ must achieve at least three values.

(ii) It follows from (i) by Lemma $\hat{\text{I}}$(a). It is also in [12, Theorem 5].

(iii) Since $(\rho, A) \in \widetilde{\Sigma}$, the image of A sits on a ρ-invariant plane P. The pull-back $\iota^{-1}(P \cap \mathbb{S}^2)$ descends to a graph in S that passes through the conical points. Let $\{H_i\}$ be the connected components of the complement of such graph. We want to show that each H_i is a hemisphere.

Let $\widetilde{H}_i$ be a connected component of the preimage of H_i inside $\widehat{S}$. Since each isometry of $\widetilde{H}_i$ fixes a point in $\widetilde{H}_i$, the stabilizer of $\widetilde{H}_i$ inside $\pi_1(\dot{S})$ is trivial and so $\widetilde{H}_i \cong H_i$. Moreover ι properly maps $\widetilde{H}_i$ to $\mathbb{S}^2 \setminus P$ as a local isometry. It follows that ι is an isometry of $\widetilde{H}_i$ onto a connected component of $\mathbb{S}^2 \setminus P$, and so $\widetilde{H}_i \cong H_i$ is a hemisphere.

(iv) Since ρ is central, then the developing map ι descends $\dot{S}$ and extends to a cover $\overline{\iota} : S \to \mathbb{S}^2$, whose branch locus is contained inside the image of $\hat{A}$. Since (ρ, A) belongs to $\widetilde{\Sigma}_0$, the decoration A takes values in a plane $P \subset \mathfrak{su}_2$, and so the branch locus is contained in the maximal circle $P \cap \mathbb{S}^2$. □

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Appendix A. Constraints on edge lengths of spherical polygons (by Daniil Mamaev)

Abstract

We give a short proof of a well-known criterion for the existence of polygons in $\mathbb{S}^3$ with given edge lengths.

A.1 Introduction

We describe a solution to a classical problem on standard collections of unitary matrices, and restate it in the language of spherical polygons.

Definition A.1. A collection of elements $U_1, \dots, U_n \in \mathrm{SU}(2)$ is called *standard* if $U_1 \cdot \dots \cdot U_n = 1$.

Question A.2 (Existence of standard collections)**.** For which $\boldsymbol{l} = (l_1, \dots, l_n) \in [0,1]^n$ there exist a standard collection of matrices $U_1, \dots, U_n \in \mathrm{SU}(2)$ such that the eigenvalues of U_k are $e^{\pm i\pi \cdot l_k}$?

Recall that SU(2) can be naturally identified with the unit sphere $\mathbb{S}^3$, and as a result we can rephrase Question A.2 as follows.

Let $U_1, \dots, U_n$ be a standard collection such that the eigenvalues of U_k are $e^{\pm i\pi \cdot l_k}$ with $l_k \in [0,1]$. Consider the set of partial products $p_i = U_1 \cdot \dots \cdot U_{i-1}$ so that

$$1 = p_1, \quad U_1 = p_2, \dots, \quad U_1 \cdot \dots \cdot U_{n-1} = p_n \in \mathrm{SU}(2).$$

Then for any two consecutive points p_{k-1}, p_k the distance $d_{\mathbb{S}^3}(p_{k-1}, p_k)$ is $\pi \cdot l_{k-1}$. Note that in the case $l_{k-1} \notin \{0,1\}$ there exists a unique geodesic $p_{k-1}p_k$ of length $\pi \cdot l_{k-1}$ that joins p_{k-1} and p_k.

Definition A.3. A *spherical polygon* in $\mathbb{S}^3$ with *edge lengths* $\pi \cdot (l_1, \dots, l_n)$ is a collection of points $p_1, \dots, p_n$ such that $d_{\mathbb{S}^3}(p_{k-1}, p_k) = \pi \cdot l_{k-1}$ and $d_{\mathbb{S}^3}(p_n, p_1) = \pi \cdot l_n$. A spherical polygon is called *coaxial* if its vertices lie on one great circle.

For $\boldsymbol{l} = (l_1, \dots, l_n) \in [0,1]^n$ we denote by $\mathcal{Pol}(\boldsymbol{l}) \subset (\mathbb{S}^3)^n$ the space of all spherical polygons with edge lengths $\pi \cdot \boldsymbol{l}$.

Question A.4 (Existence of spherical polygons)**.** For which $\boldsymbol{l} \in [0,1]^n$ there exists a spherical polygon in $\mathbb{S}^3$ with edge lengths $(\pi l_1, \dots, \pi l_n)$? In other words, for which $\boldsymbol{l}$ is the space $\mathcal{Pol}(\boldsymbol{l})$ non-empty?

In order to state the answer we need one more definition.

Definition A.5. A vertex of the unit n-cube $[0,1]^n$ is *odd* (resp. *even*) if its coordinates add up to an odd (resp. even) integer. The *n-demicube* DC_n is the convex hull of all even vertices of $[0,1]^n$.

We recall that the standard ℓ^1-distance on $\mathbb{R}^n$ is defined as $d_1(\boldsymbol{x}, \boldsymbol{y}) := \sum_{i=1}^n |x_i - y_i|$.

Remark A.6. The n-demicube can be obtained from the unit n-cube by cutting out 2^{n-1} tetrahedra each containing one odd vertex and n adjacent even vertices. Denote the set of odd vertices of $[0,1]^n$ by V_{odd}. Then the n-demicube DC_n coincides with the subset of $[0,1]^n$ consisting of points at ℓ^1-distance at least 1 from V_{odd}.

The main result of the appendix is the following theorem, which is essentially contained in the works of Galitzer [2] and Biswas [1].

Theorem A.7. *For $\boldsymbol{l} = (l_1, \ldots, l_n) \in [0,1]^n$ a spherical polygon with edges of lengths $\pi \cdot l_i$ exists if and only if $\boldsymbol{l} \in DC_n$, namely, if and only if $d_1(\boldsymbol{l}, V_{\text{odd}}) \geq 1$.*

We conclude with a slight refinement of Theorem A.7.

Corollary A.8. *Let $\boldsymbol{l} \in [0,1]^n$.*

(i) If $d_1(\boldsymbol{l}, V_{\text{odd}}) = 1$, then a polygon with edge lengths $\pi \cdot \boldsymbol{l}$ is unique up to an isometry of $\mathbb{S}^3$ and is coaxial.

(ii) If $d_1(\boldsymbol{l}, V_{\text{odd}}) > 1$, then there exists a non-coaxial polygon with edge lengths $\pi \cdot \boldsymbol{l}$.

A.2 Proof of Theorem A.7

The proof of Theorem A.7 is organized as follows. In Lemma A.9 we introduce the group $\text{Iso}(DC_n)$ of self-isometries of the n-demicube DC_n. By Lemma A.11 the spaces of polygons $\mathcal{P}ol(\boldsymbol{l})$ and $\mathcal{P}ol(\boldsymbol{l'})$ are isomorphic whenever $\boldsymbol{l}, \boldsymbol{l'}$ belong to the same $\text{Iso}(DC_n)$-orbit. Hence it is sufficient to determine a fundamental domain $F_n \subset [0,1]^n$ for the action of $\text{Iso}(DC_n)$ on $[0,1]^n$ and prove that, for $\boldsymbol{l} \in F_n$, the space $\mathcal{P}ol(\boldsymbol{l})$ is non-empty if and only if $\boldsymbol{l} \in DC_n \cap F_n$. This last step is achieved in Proposition A.13.

Lemma A.9. *The group $\text{Iso}(DC_n)$ of self-isometries of the n-demicube is generated by the subgroup (isomorphic to S_n) of linear trasformations that permute the n coordinates and by the symmetry τ_{12} defined as*

$$\tau_{12}(l_1, l_2, l_3, \ldots, l_n) := (1 - l_1, 1 - l_2, l_3, \ldots, l_n).$$

Proof. Any isometry of the n-demicube is induced by an isometry of the unit n-cube that sends even vertices to even. Hence, $\text{Iso}(DC_n)$ is an index 2 subgroup in the group of isometries of $[0,1]^n$. The group $\text{Lin}(DC_n)$ of linear transformations that permute the coordinates is clearly isomorphic to S_n and acts by isometries on DC_n. It is easy to see that the subgroup G of isometries generated by $\text{Lin}(DC_n)$ and τ_{12} acts transitively on the set of vertices of DC_n. Now, DC_n has 2^{n-1} even vertices and the stabilizer of $(0, \ldots, 0)$ inside G is exactly $\text{Lin}(DC_n)$, which has order $n!$. It follows that G has order $n! \cdot 2^{n-1}$. We conclude that G is a subgroup of $\text{Iso}([0,1]^n)$ of index 2, contained inside $\text{Iso}(DC_n)$, and so $G = \text{Iso}(DC_n)$. □

From now on we identify S_n with $\text{Lin}(DC_n)$.

Remark A.10. The group $\text{Iso}(DC_n)$ is acting on the unit cube sending its odd vertices to odd vertices. As a result, the action of $\text{Iso}(DC_n)$ preserves the ℓ^1-distance from points of DC_n to V_{odd}. Hence all 2^{n-1} simplicial faces of DC_n that are at ℓ^1-distance 1 from V_{odd} are permuted by the action of $\text{Iso}(DC_n)$.

Lemma A.11. *(i) For any $\boldsymbol{l} \in [0,1]^n$ and any $g \in \text{Iso}(DC_n)$ there is an isomorphism $g_* : \mathcal{P}ol(\boldsymbol{l}) \to \mathcal{P}ol(g \cdot \boldsymbol{l})$. In particular, $\mathcal{P}ol(\boldsymbol{l})$ is empty if and only if $\mathcal{P}ol(g \cdot \boldsymbol{l})$ is.*

(ii) Furthermore, the image of a coaxial polygon in $\mathcal{P}ol(\boldsymbol{l})$ under g_* is coaxial and if two polygons in $\mathcal{P}ol(\boldsymbol{l})$ are congruent, then so are their images in $\mathcal{P}ol(g \cdot \boldsymbol{l})$.

Proof. (i). Since S_n is generated by transpositions $(i, i+1)$, according to Lemma A.9 it is enough to prove the statement for all $g = (i, i+1)$ and $g = \tau_{12}$.

Suppose $g = (1,2)$ (the case $g = (i, i+1)$ is identical). If $l_1 = l_2$, there is nothing to prove. Otherwise, let $P \in \mathcal{P}ol(\boldsymbol{l})$ be a polygon with vertices $p_1, \ldots, p_n$. Since $l_1 \neq l_2$ we have $p_1 \neq p_3$. Hence, there is a unique geodesic sphere $\mathbb{S}^2 \subset \mathbb{S}^3$ equidistant from p_1 and p_3 (this 2-sphere depends algebraically on positions of p_1 and p_3). Let σ be the reflection of $\mathbb{S}^3$ with respect to such $\mathbb{S}^2$ and set $p_2' = \sigma(p_2)$. Then define $g_*(P)$ to be the polygon with vertices $p_1, p_2', \ldots, p_n$ We obtained the desired isomorphism $\mathcal{P}ol(l_1, l_2, \ldots, l_n) \cong \mathcal{P}ol(l_2, l_1, \ldots, l_n)$.

Suppose now $g = \tau_{12}$. Given a polygon $P \in \mathcal{P}ol(\boldsymbol{l})$ with vertices $p_1, \ldots, p_n$, we let $g_*(P)$ be the polygon with vertices $p_1, -p_2, p_3, \ldots, p_n$, where $-p_2$ is the point of $\mathbb{S}^3$ antipodal to p_2.

(ii) It is clear that both $g = (i, i+1)$ and $g = \tau_{12}$ send coaxial polygons to coaxial polygons, and congruent polygons to congruent polygons. □

Lemma A.11 implies that in order to understand the spaces $\mathcal{P}ol(\boldsymbol{l})$ of n-gons in $\mathbb{S}^3$ with arbitrary edge lengths it will be enough to consider $\mathcal{P}ol(\boldsymbol{l})$ with $\boldsymbol{l} \in F_n$, where:

$$F_n := \{(l_1, \ldots, l_n) \in [0,1]^n \mid l_1 + l_2 \leq 1, \text{ and } \quad l_1 \geq l_2 \geq \ldots \geq l_n\}.$$

Indeed, we have the following.

Lemma A.12. *F_n is a fundamental domain for the action of* $\mathrm{Iso}(DC_n)$ *on* $[0,1]^n$.

In order to motivate the above claim, endow $\mathbb{R}^n$ with the standard Euclidean product $(\cdot,\cdot)$ and recall from [3, pag.5] that the elements $\alpha_1,\ldots,\alpha_n \in \mathbb{R}^n$ defined as

$$\alpha_1 = -e_1 - e_2, \qquad \alpha_i = e_{i-1} - e_i \text{ for } i = 2,\ldots,n$$

form a simple root system of type D_n. The reflections $v \mapsto v - \frac{2(v,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$ generate the Weyl group $W(D_n)$ and the chamber

$$\begin{aligned}\Delta_{D_n} &= \{v \in \mathbb{R}^n \mid (v,\alpha_j) \geq 0 \quad \text{for } j=1,\ldots,n\} = \\ &= \{v \in \mathbb{R}^n \mid v_1 + v_2 \leq 0, \text{ and } \quad v_{i-1} \geq v_i \quad \text{for } j = 1,\ldots,n\}\end{aligned}$$

is a fundamental domain for the action of $W(D_n)$ on $\mathbb{R}^n$.
Note that the translation operator $T(v) := v + (\frac{1}{2},\ldots,\frac{1}{2})$ conjugates the actions of $W(D_n)$ and of $\mathrm{Iso}(DC_n)$, namely $T \cdot W(D_n) \cdot T^{-1} = \mathrm{Iso}(DC_n)$.

Proof of Lemma A.12. By the above discussion, $T(\Delta_{D_n}) \cap [0,1]^n$ is a fundamental domain for the action of $\mathrm{Iso}(DC_n)$ on $[0,1]^n$. We conclude by observing that $F_n = T(\Delta_{D_n}) \cap [0,1]^n$. □

Next we determine for which $\boldsymbol{l}$ in a certain subdomain of $[0,1]^n$ the space $\mathcal{Pol}(\boldsymbol{l})$ is non-empty.

Proposition A.13. *Let $n \geq 2$ and $l_1,\ldots,l_n$ be real numbers satisfying $1 \geq l_1 \geq l_2 \geq \ldots \geq l_n \geq 0$ and $1/2 \geq l_2$. Then $\mathcal{Pol}(l_1,\ldots,l_n)$ is non-empty if and only if $l_1 \leq l_2 + \ldots + l_n$.*

Proof. The 'only if' part of this proposition is just the triangle inequality in $\mathbb{S}^3$. We prove the 'if' part by induction on n.
If $n = 2$, then we have $1/2 \geq l_1 = l_2 \geq 0$ and all polygons in $\mathcal{Pol}(\boldsymbol{l})$ consist of two overlapping edges, joining two points at distance $\pi \cdot l_1$ from each other.
Suppose $n = 3$ and (l_1,l_2,l_3) satisfies $1 \geq l_1 \geq l_2 \geq l_3 \geq 0$ and $l_1 \leq l_2 + l_3$. Then there exist two degenerate triangles T_- and T_+ lying on a great circle with two edges of lengths $\pi \cdot (l_2,l_3)$. The third edge of T_- and T_+ has length $\pi \cdot (l_2 - l_3)$ and $\pi \cdot (l_2 + l_3)$. Since we can continuously deform T_- to T_+ so that the length of the first two edges stays equal to l_2 and l_3, the third edge can achieve any length in $\pi \cdot [l_2 - l_3, l_2 + l_3]$, and so in particular it can achieve length $\pi \cdot l_1$. Note that in this case $n = 3$ we are not using the hypothesis $l_2 \leq \frac{1}{2}$.
Now let $n \geq 4$ and $(l_1,\ldots,l_n)$ be an n-tuple satisfying $1 \geq l_1 \geq \ldots \geq l_n \geq 0$, $l_2 \leq 1/2$ and $l_1 \leq l_2 + \ldots + l_n$. There are two cases.

1. $l_1 \leq l_2 + \ldots + l_{n-2} + (l_{n-1} - l_n)$.
   In this case, by induction there exists a polygon with $(n-1)$ vertices and edge lengths $\pi \cdot (l_1,\ldots,l_{n-2},l_{n-1} - l_n)$. Replacing the last edge by two edges of lengths $\pi \cdot l_{n-1}$ and $\pi \cdot l_n$ lying on the same great circle, we obtain the desired polygon with edge lengths $\pi \cdot (l_1,\ldots,l_n)$.
2. $l_1 > l_2 + \ldots + l_{n-2} + (l_{n-1} - l_n)$.
   In this case we construct the desired polygon from scratch. The first edge is a segment of length $\pi \cdot l_1$ on a great circle C starting at a point p_1. The edges from the second to the $(n-2)$-th lie on the great circle C and go in the direction opposite to the first one. Let p_{n-1} be the end of the $(n-2)$-th edge. We have $\frac{1}{\pi} \cdot |p_1,p_{n-1}| = l_1 - l_2 - \ldots - l_{n+2}$ and therefore

   $$0 \leq l_{n-1} - l_n < \frac{1}{\pi} \cdot |p_1,p_{n-1}| \leq (l_{n-1} + l_n) \leq 1.$$

   We have seen in the case $n = 3$ that there exists a triangle with edge lengths $\pi \cdot l_{n-1}, \pi \cdot l_n, |p_1,p_{n-1}|$. Thus there exists a point $p_n \in \mathbb{S}^3$ such that $|p_{n-1}p_n| = \pi \cdot l_{n-1}$ and $|p_n p_1| = \pi \cdot l_n$. Thus the points $p_1,\ldots,p_n$ are the vertices of a polygon in $\mathcal{Pol}(\boldsymbol{l})$.

□

Another advantage of working with F_n instead of $[0,1]^n$ is the following.

Lemma A.14. *Let $\boldsymbol{l} = (l_1, \ldots, l_n) \in F_n$. Then*

$$d_1(\boldsymbol{l}, V_{\text{odd}}) = d_1(\boldsymbol{l}, e_1) = (1 - l_1) + l_2 + \ldots + l_n.$$

Proof. It is clearly enough to show that $d_1(\boldsymbol{l}, V_{\text{odd}})$ is achieved at $e_1 = (1, 0, \ldots, 0)$.
Assume that the distance $d_1(\boldsymbol{l}, V_{\text{odd}})$ is achieved at the point $\boldsymbol{x} = (x_1, \ldots, x_n) \in V_{\text{odd}}$. If $x_1 = 0$ then for some $i \geq 2$ we have $x_i = 1$, but since $l_1 \geq l_i$, replacing (x_1, x_i) by $(1, 0)$ will not increase the distance $d_1(\boldsymbol{l}, \boldsymbol{x})$. So we can assume $x_1 = 1$. Now, if for some $i > j \geq 2$ we have $x_i = x_j = 1$, then replacing x_i and x_j by 0 does not increase the distance $d_1(\boldsymbol{l}, \boldsymbol{x})$, since $l_i, l_j \leq l_2 \leq (l_1 + l_2)/2 \leq 1/2$. We conclude that $e_1 = (1, 0, \ldots, 0)$ is indeed closest to $\boldsymbol{l}$. □

Proposition A.13 motivates the introduction of the following set:

$$\begin{aligned} T_n &= \{\boldsymbol{l} \in F_n \mid l_1 \leq l_2 + \ldots + l_n\} = \\ &= \{\boldsymbol{l} \in [0,1]^n \mid l_1 \geq \ldots \geq l_n,\ l_1 + l_2 \leq 1, \text{ and } l_1 \leq l_2 + \ldots + l_n\}. \end{aligned}$$

Lemma A.15. *$T_n = DC_n \cap F_n$.*

Proof. Recall that $DC_n = \{\boldsymbol{l} \in [0,1]^n \mid d_1(\boldsymbol{l}, V_{\text{odd}}) \geq 1\}$ by Remark A.6. By Lemma A.14 we conclude that

$$\begin{aligned} DC_n \cap F_n &= \{\boldsymbol{l} \in F_n \mid d_1(\boldsymbol{l}, V_{\text{odd}}) \geq 1\} = \\ &= \{\boldsymbol{l} \in F_n \mid (1 - l_1) + l_2 + \ldots + l_n \geq 1\} = T_n. \end{aligned}$$

□

Proof of Theorem A.7. Let E_n be the subset of $\boldsymbol{l} \in [0,1]^n$ for which $\mathcal{Pol}(\boldsymbol{l})$ is not empty. We want to show that $E_n = DC_n$.
Now DC_n is clearly Iso(DC_n)-invariant and E_n is Iso(DC_n)-invariant by Lemma A.11. Since F_n is a fundamental domain (Lemma A.12), it is enough to show that $DC_n \cap F_n = E_n \cap F_n$.
Since every $\boldsymbol{l} \in F_n$ satisfies the hypotheses of Proposition A.13, we obtain $T_n = E_n \cap F_n$. The proof is complete, as $T_n = DC_n \cap F_n$ by Lemma A.15. □

Proof of Corollary A.8. Recall that F_n is the fundamental domain of the action of Iso(DC_n) on the unit cube (Lemma A.12), and that such action preserves the ℓ^1-distance from V_{odd} (Remark A.10). Moreover every element $g \in \text{Iso}(DC_n)$ determines an isomorphism between $\mathcal{Pol}(\boldsymbol{l})$ and $\mathcal{Pol}(g \cdot \boldsymbol{l})$ that sends congruent polygons to congruent polygons and coaxial polygons to coaxial polygons (Lemma A.11). So, it is enough to prove the claims for $\boldsymbol{l} \in F_n$.
Let now $\boldsymbol{l} \in F_n$. By Lemma A.14, we have $d_1(\boldsymbol{l}, V_{\text{odd}}) = d_1(\boldsymbol{l}, e_1) = (1 - l_1) + l_2 + \cdots + l_n$.
(i) Assume $d_1(\boldsymbol{l}, V_{\text{odd}}) = 1$, so that $l_1 = l_2 + \ldots + l_n$. By the triangular inequality in $\mathbb{S}^3$, a polygon $P \in \mathcal{Pol}(\boldsymbol{l})$ must be a segment in $\mathbb{S}^3$ of length $\pi \cdot l_1$ traced twice, so it is coaxial and unique up to isometry of $\mathbb{S}^3$.
(ii) Assume $d_1(\boldsymbol{l}, V_{\text{odd}}) > 1$, so that $l_1 < l_2 + \ldots + l_n$. We first observe that, for $n = 3$, no such polygon in $\mathcal{Pol}(\boldsymbol{l})$ is coaxial. So we can assume $n \geq 4$. Note that, if $l_n = 0$, we can reduce to studying the case of $(n-1)$-gons with edge lengths $(l_1, \ldots, l_{n-1})$. Thus we can assume $l_n > 0$.
The below argument consists just on a closer inspection of the proof of Proposition A.13.
If $l_1 \leq l_2 + \cdots + l_{n-2} + (l_{n-1} - l_n)$, then by induction there exists a non-coaxial polygon with $(n-1)$ vertices and edge lengths $\pi \cdot (l_1, \ldots, l_{n-2}, l_{n-1} - l_n + \eta)$ with $\eta = l_n/2 > 0$. Since $l_{n-1} - l_n + \eta \in (l_{n-1} - l_n, l_{n-1} + l_n)$, the last edge can then be replaced by two edges of lengths $\pi \cdot l_{n-1}$ and $\pi \cdot l_n$, and we obtain the desired non-coaxial polygon with edge lengths $\pi \cdot \boldsymbol{l}$.
If $l_1 > l_2 + \cdots + l_{n-2} + (l_{n-1} - l_n)$, then the last two edges of the polygon constructed in the proof of Proposition A.13 do not lie on the same great circle, because

$$0 \leq l_{n-1} - l_n < \frac{1}{\pi} \cdot |p_1, p_{n-1}| < (l_{n-1} + l_n) \leq 1.$$

It follows that such polygon has edge length $\pi \cdot \boldsymbol{l}$ and is not coaxial. □

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