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Qualitative Properties of Solutions of Degenerate Parabolic Equations via Energy Approaches

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We consider several problems for degenerate parabolic equations exhibiting nonlinearities of various kinds. For example the equations may contain superlinear sources, causing blow up of the solutions, or damping terms; the principal part of the operator is also nonlinear. We mention as unifying features the fact that the spatial domains have non-compact boundary, and the technical approach which is based on energy methods and a priori estimates. The issues investigated include existence under optimal assumptions on the data, asymptotic behavior of solutions, existence or non-existence of global in time solutions.

Contents

1. Introduction

The main subject of this survey are nonlinear degenerate parabolic equations of divergence type. It is well known that many of the main qualitative properties of the Cauchy problem in the whole Euclidean space, such as large time behavior, finite speed of propagation, asymptotic expansion of solutions, are described by means of the so-called Barenblatt–Pattle exponent, which in the case of doubly degenerate parabolic equations of the type of (3.3) is $N(p+m-3)+p$. However, we show that this is not true in the case of Neumann problems in domains with noncompact boundary, with zero flux on the boundary. Some geometrical characteristics are needed to extend the concept of such an exponent, which we express in terms of the volume growth at infinity.

For the Neumann problem in domains with noncompact boundary, for doubly degenerate equations with strongly

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nonlinear sources, we prove a Fujita type result. Here again the corresponding critical Fujita exponent depends strictly on the volume growth. The most difficult situation in this case is the one of domains narrowing at infinity. Here we give delicate sharp estimates of the relevant moments, which may be of independent interest.

Next, we give the asymptotical expansion for large times of solutions of the Porous Media Equation (PME) in paraboloid-like domains. Finally we look at equations with damping terms depending on the gradient of the solution, giving criteria for moment and mass decay in time. Actually here we consider the Cauchy–Dirichlet problem for the PME in cone-like domains and the Cauchy problem for doubly degenerate parabolic equations.

This survey is based on previous work of the authors, quoted as appropriate, presented here in a new and unified way. At the end of each Section we include some Comments on the results and on the literature; the references however are by no means exhaustive.

Let us conclude the Introduction by recalling two open problems which we feel are interesting and may probably be attacked by means of the methods proposed in the following.

1. Prove an asymptotic expansion of the solution to the Neumann problem in paraboloid-like domains for the p-Laplacian

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u). \tag{1.1}$$

2. Give criteria for the moment decay to zero as $t \to +\infty$ for solutions of the Cauchy–Dirichlet problem for (1.1) in cone-like domains.

Our approach is based on new Sobolev–Gagliardo–Nirenberg inequalities depending on the geometry of domain, and on a new streamlined iterative energy approach based on the classical work of DeGiorgi, Ladyzhenskaya and Uraltseva, DiBenedetto.

In the following $\Omega \subset \boldsymbol{R}^N$, $N \geq 2$ denotes an open set with noncompact boundary, connected and with infinite volume; further hypotheses on Ω are assumed as needed. The symbols $\gamma, \gamma_0, \gamma_1, \dots,$ denote positive constants depending on the parameters of the problem (but not on the solution itself, for example), and varying from line to line. We write $\|u\|_{s,G} := \|u\|_{L^s(G)}$ and, even for a positive measure μ, we use the notation $\|\mu\|_{1,G} = \mu(G)$. When $G = \boldsymbol{R}^N$ we sometimes omit it from the notation.

Denote for any measure space (E, μ) with $\mu(E) < +\infty$

$$\fint_E u d\mu = \frac{1}{\mu(E)} \int_E u d\mu,$$

and let $B_\rho \subset \boldsymbol{R}^N$ be the ball of radius ρ centered at $x = 0$, unless noticed otherwise.

All solutions (and thus all initial data) are understood to be nonnegative, though the results of existence are still valid for sign-changing solutions.

2. Sobolev–Gagliardo–Nirenberg Type Inequalities in Domains with Noncompact Boundaries

2.1 Expanding domains

Let $\Omega \subset \boldsymbol{R}^N$, $N \geq 2$ be a domain with noncompact boundary. We are going to describe the geometry of Ω by means of isoperimetric inequalities.

Let for all $v > 0$

$$l(v) = \inf\{|\partial G \cap \Omega|_{N-1} : G \subset \Omega, |G| = v, \partial G \text{ Lipschitz}\}. \tag{2.1}$$

Here we use the symbol $|\cdot|$ for N dimensional Lebesgue measure, while the $N-1$ dimensional Hausdorff measure is denoted by $|\cdot|_{N-1}$. We assume that $l(v) > 0$ for $v > 0$ and that there exists a continuos function g satisfying

$$0 < g(v) \leq l(v), \quad v > 0, \tag{2.2}$$

and

$$\omega(v) := \frac{v^{\frac{N-1}{N}}}{g(v)} \quad \text{is nondecreasing for } v > 0. \tag{2.3}$$

Let us also introduce the volume function V and its inverse R:

$$V(\rho) := |\Omega_\rho|, \quad \rho > 0, \quad \Omega_\rho := \Omega \cap \{|x| < \rho\}, \quad R = V^{(-1)}.$$

Definition 2.1. The set Ω belongs to the class $\mathcal{B}_1(g)$ if all the following requirements are fulfilled: $\Omega \subset \boldsymbol{R}^N$, $N \geq 2$ is an unbounded connected open set, satisfying $|\Omega| = \infty$, with a Lipschitz continuous boundary $\partial\Omega$, such that $0 \in \partial\Omega$. We assume also that a function $g \in C(0, \infty)$ is given as in (2.2), (2.3).

Definition 2.2. We say that Ω belongs to the class $\mathcal{B}_2(g)$ if it belongs to $\mathcal{B}_1(g)$ and also satisfies

$$c_0 \frac{v}{g(v)} \le R(v) \le c_1 \frac{v}{g(v)}, \quad \text{for all } v > 0, \tag{2.4}$$

for suitable constants c_0, $c_1 > 0$.

The domains in $\mathscr{B}_1(g)$, $\mathscr{B}_2(g)$ are often referred to as "expanding" or "non-contracting" domains. The intuition behind this terminology is made clearer by the following example.

Example 2.3 (Paraboloid-like domains). Let $0 \le \alpha \le 1$ be fixed, and define

$$\Omega = \{x \in \boldsymbol{R}^N : |x'| < x_N^\alpha\}, \; x' = (x_1, \dots, x_{N-1}). \tag{2.5}$$

It follows from results of [62, Chapter 4] that $\Omega \in \mathscr{B}_2(g)$, with

$$g(v) = \gamma \min\{v^{\frac{N-1}{N}}, v^\eta\}, \quad v > 0, \quad \eta = \frac{\alpha(N-1)}{\alpha(N-1)+1} \le \frac{N-1}{N}.$$

Theorem 2.4 ([67]). *Let $\Omega \in \mathscr{B}_1(g)$, and set for all $v \in W^{1,p}(\Omega)$*

$$E_q = \int_\Omega |v|^q dx, \quad E_\beta = \int_\Omega |v|^\beta dx,$$

for some $p > 1$, $0 < \beta < q$, $q \ge 1$, $q(N-p) \le Np$. Then we have

$$\|v\|_{q,\Omega} \le \gamma \omega(E) E^{\frac{1}{q} - \frac{1}{p} + \frac{1}{N}} \|\nabla v\|_{p,\Omega}, \tag{2.6}$$

where

$$E := \frac{E_\beta^{\frac{q}{q-\beta}}}{E_q^{\frac{\beta}{q-\beta}}}.$$

2.2 Narrowing domains

Let

$$l(v, \rho) = \inf\{|\partial G \cap \Omega_\rho|_{N-1} : G \subset \Omega_\rho, \; |G| = v, \; \partial G \text{ Lipschitz}\},$$

for all $\rho > 0$, and for all $0 < v < |\Omega_\rho|/2$, where $\Omega_\rho = \Omega \cap \{|x| < \rho\}$ as above and $\Omega_\rho \neq \varnothing$. Let f be a continuous nondecreasing function $f : [1, \infty) \to (0, \infty)$ such that for suitable constants c_0, $c_1 > 0$

$$c_0 \frac{\rho}{f(\rho)} \le V(\rho) = |\Omega_\rho| \le c_1 \frac{\rho}{f(\rho)}, \quad \rho \ge 1. \tag{2.7}$$

Finally, we also require that for all $\delta > 0$

$$\nu_0(\delta) V(\rho) \le V(\delta\rho) \le \nu_1(\delta) V(\rho), \quad \text{for all } \rho \ge \max(1, 1/\delta), \tag{2.8}$$

where ν_0, ν_1 are two given nondecreasing positive functions, such that $\nu_1(\delta) < 1$ for $\delta < 1$. Note that from (2.8) it follows

$$|\Omega| = \infty. \tag{2.9}$$

Moreover, f is required to fulfill for a suitable $c_2 > 0$

$$g(v, \rho) := c_2 \min\left(v^{\frac{N-1}{N}}, \; \frac{1}{f(\rho)} \right) \le l(v, \rho), \quad \rho \ge 1, \quad 0 < v \le V(\rho)/2. \tag{2.10}$$

Then, intuitively, $1/f(\rho)$ takes the geometrical meaning of the area of the section $\Omega \cap \{|x| = \rho\}$.

In the following, for the sake of simplicity, we assume $f(1) = 1$ and extend f by $f(s) = 1$ for $s \in [0, 1)$.

Definition 2.5. We say that an open unbounded connected set $\Omega \subset \boldsymbol{R}^N$, $N \ge 2$, belongs to the class $\mathcal{N}(f)$ if its boundary $\partial\Omega$ is locally Lipschitz continuous and if (2.7)–(2.10) are satisfied.

Domains in the class $\mathcal{N}(f)$ are often called "narrowing" because they can be shaped like infinite cusps, see the following example; note that this is impossible for expanding domains. Clearly, this shape and (2.9) imply that $l(v)$ as defined in (2.1) for expanding domains vanishes identically for domains in the class $\mathcal{N}(f)$; this is the reason that forces us to use local embedding theorems (see Theorem 2.7).

Example 2.6 (Infinite cusp). Let $d > 0$, $0 < \varepsilon < 1/(N-1)$. The domain

$$\Omega^\varepsilon := \{x = (x', x_N) \in \boldsymbol{R}^N : |x'| < x_N^{-\varepsilon}, \; x_N > d\} \subset \boldsymbol{R}^N,$$

belongs to $\mathcal{N}(f)$ when we define $f = \rho^{\beta}$ for $\rho = x_N > 2d$; here $\beta := \varepsilon(N-1) < 1$. In this case $l(v,\rho) \simeq 1/f(\rho)$ if v is large enough.

Theorem 2.7 ([9]). *Let $\Omega \in \mathcal{N}(f)$, $\rho \geq 1$, $v \in W^{1,p}(\Omega_\rho)$ be given, and let*

$$E_q = \int_{\Omega_\rho} |v|^q dx, \quad E_\beta = \int_{\Omega_\rho} |v|^\beta dx,$$

for some $p > 1$, $\beta > 0$, $q \geq 1$, $q > \beta$. Assume moreover that v satisfies

$$E := \frac{E_\beta^{\frac{q}{q-\beta}}}{E_q^{\frac{\beta}{q-\beta}}} \leq \theta_0 V(\rho),$$

where $\theta_0 = \theta_0(q,p,\beta) \in (0,1)$ is a suitable constant. Then for all q such that $q(N-p) \leq Np$, we have

$$\|v\|_{q,\Omega_\rho} \leq \gamma\omega(E,\rho) E^{\frac{1}{q}-\frac{1}{p}+\frac{1}{N}} \|\nabla v\|_{p,\Omega_\rho},$$

where γ depends on q, p, β, and

$$\omega(z,\rho) = \gamma \max(1, z^{\frac{N-1}{N}} f(\rho)).$$

Comment 2.8. Theorem 2.4 was proved in [67], and Theorem 2.7 was proved in [9].
The case $q = p = 2$, $\beta = 1$ in Theorem 2.4 is due to [44] (with a different approach). The class of domains satisfying suitable isoperimetric inequalities was studied in [62] (see also references therein), where precise connections between isoperimetric and Sobolev inequalities were analyzed for various classes of domains. However, in the investigation of qualitative properties of solutions, the precise multiplicative form of the Sobolev inequalities is important.

The embedding Theorems 2.4 and 2.7 are reported here mostly to stress the technical difference between expanding and narrowing domains where, respectively, global or just local embeddings are possible. In both cases, in the application of the embedding one makes often use of the estimate

$$E \leq |\operatorname{supp} v|, \tag{2.11}$$

which follows in turn from Hölder's inequality (here $\operatorname{supp} v$ denotes the support of v). Then assumption $E \leq \theta_0 V(\rho)$ in Theorem 2.7 is handled by means of a suitable cut off technique.

The definition of Ω_ρ given here for narrowing domains clearly understands $r(x) = |x|$ as the coordinate "measuring the distance along Ω." This is done here in order to avoid unnecessary complications, but the whole theory is valid for domains shaped for example like infinite expanding spirals in $\boldsymbol{R}^2$, where such a coordinate $r(x)$ would be different (essentially the polar angular coordinate in this case).

References for this section: [9], [44], [62], [67].

3. Sup Bounds for Solutions $u(t)$ to the Neumann Problem

3.1 Expanding domains

The results of this Subsection were essentially proved in [8]; in this section we demonstrate a simpler approach to the proof of a sharp bound of $\|u(t)\|_\infty$ (see Remark 3.5) using the Faber–Krahn inequality, in the spirit of [13]. We look only at the case of globally integrable initial data.

Definition 3.1. We say that Ω satisfies the global Faber–Krahn inequality for a given $p > 1$ and a nonincreasing function $\Lambda_p : (0,+\infty) \to (0,+\infty)$ if for any $v > 0$ and precompact domain $G \subset \Omega$ with $|G| = v$ we have

$$\Lambda_p(v) \int_G |\varphi|^p dv \leq \int_G |\nabla\varphi|^p dv, \tag{3.1}$$

for all $\varphi \in W^{1,p}(\Omega)$ such that its support is contained in $\overline{G}$.

First we note that the isoperimetric inequality, or even its consequence (2.6), implies the Faber–Krahn inequality. Indeed from (2.6) written for $v = \varphi$ as in Definition 3.1, when we take into account also (2.11) and take $p = q$, we obtain (3.1) with

$$\Lambda_p(s) = [\gamma\omega(s) s^{\frac{1}{N}}]^{-p}, \quad s > 0. \tag{3.2}$$

We consider in $S = \Omega \times (0,\infty)$ the following Neumann problem

$$\frac{\partial u}{\partial t} - \operatorname{div}(u^{m-1}|\nabla u|^{p-2}\nabla u) = 0, \quad (x,t) \in S, \tag{3.3}$$

$$u^{m-1}|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}=0, \qquad (x,t)\in \partial\Omega\times(0,\infty), \tag{3.4}$$

$$u(x,0)=u_0(x), \quad x\in\Omega; \tag{3.5}$$

in (3.4) ν denotes the outer normal to $\partial\Omega$ and $u_0 \in L^1(\Omega)$. The equation in (3.3) is known in the literature as doubly degenerate parabolic, containing both the nonlinearity of the PME (case $p=2$) and of the p-Laplacian (case $m=1$). In what follows we will consider the slow diffusion case, that is

$$p+m-3>0, \quad p>1. \tag{3.6}$$

The notion of weak solution to this and similar problems is standard, see for example [8]; however for the reader's convenience we reproduce it in the context of the present section. In Sect. 7.1 we present a variant suited for the case considered there.

Definition 3.2. A weak solution to (3.3)–(3.5) is a nonnegative function $u\in L^\infty_{\mathrm{loc}}(\overline{\Omega}\times(0,\infty))$ with $u\in C((0,\infty),L^2_{\mathrm{loc}}(\overline{\Omega}))$, $|\nabla u^\lambda|^p\in L^1_{\mathrm{loc}}(\overline{\Omega}\times(0,\infty))$, where $\lambda=(p+m-2)/(p-1)$, such that

$$\int_0^\infty\int_\Omega\left\{-u\frac{\partial\zeta}{\partial t}+u^{m-1}|\nabla u|^{p-2}\nabla u\nabla\zeta\right\}dxdt=0, \tag{3.7}$$

for all $\zeta\in C^1(\overline{\Omega}\times(0,\infty))$ whose support is a compact set contained in $\overline{\Omega}\times(0,\infty)$. In addition we require $u(t)\to u_0$ as $t\to 0$ in $L^1_{\mathrm{loc}}(\Omega)$.

Theorem 3.3. *Let u be a solution to the problem* (3.3)–(3.5) *in S; let Ω satisfy the global Faber–Krahn inequality in the sense of Definition 3.1. Suppose that $\Lambda_p(v)$ fulfills the condition*

$$\Lambda_p(v)v^a \quad \textit{is a nondecreasing function}, \tag{3.8}$$

for a given $a>0$. Then for any $t>0$ and $r\ge 0$ we have

$$\|u(t)\|_\infty \le \gamma\left(\sup_{t/4<\tau<t}\int_\Omega u(\tau)^{r+1}dx\right)^{\frac{1}{r+1}} \times \psi_r^{(-1)}\left(\frac{1}{t}\left(\sup_{t/4<\tau<t}\int_\Omega u(\tau)^{r+1}dx\right)^{-\frac{p+m-3}{r+1}}\right), \tag{3.9}$$

where $\psi_r^{(-1)}$ is the inverse function of

$$\psi_r(s):=s^{p+m-3}\Lambda_p(s^{-(r+1)}).$$

We have to assume here that $u\in L^\infty_{\mathrm{loc}}(0,+\infty;L^{r+1}(\Omega))$.

We need the following standard result.

Lemma 3.4 (Caccioppoli inequality)**.** *Let $q>0$, and $q>2-m$ if $m<1$, be fixed and define $s=(p+m+q-2)/p$. Fix also $a_1>a_2>0$, $t>\tau_1>\tau_2>0$. Then*

$$\sup_{\tau_1<\tau<t}\int_\Omega(u(t)-a_1)_+^{q+1}dx+\int_{\tau_1}^t\int_\Omega|\nabla(u-a_1)_+^s|^pdxd\tau \le \gamma\frac{H(a_1,a_2)}{\tau_1-\tau_2}\int_{\tau_2}^t\int_\Omega(u-a_2)_+^{q+1}dxd\tau. \tag{3.10}$$

Here $H(a_1,a_2)=[a_1/(a_1-a_2)]^{|m-1|}$.

Proof of Theorem 3.3 (see [13]*).* A) Choose $h_0>h_\infty>0$, $\tau_0>\tau_\infty>0$, and $k_i=h_\infty+(h_0-h_\infty)2^{-i}$, $t_i=\tau_\infty+(\tau_0-\tau_\infty)2^{-i}$, $f_i=(u-k_i)_+^s$, $i\ge 0$. For any given $q>0$ as in Lemma 3.4, let $b=(q+1)/s<p$ and denote $A_k=\{u>k\}$ for all $k>0$; then, on applying in this order Hölder, Faber–Krahn and Young inequalities we have that for a constant $\varepsilon>0$ to be chosen (all integrals and measures calculated at time level τ)

$$\int_\Omega f_{i+1}^b dx \le |A_{k_{i+1}}|^{1-b/p}\Lambda_p(|A_{k_{i+1}}|)^{-b/}\left(\int_\Omega|\nabla f_{i+1}|^pdx\right)^{b/p} \le \frac{b}{p}\varepsilon^{p/b}\int_\Omega|\nabla f_{i+1}|^pdx+\frac{p-b}{p}\varepsilon^{-p/(p-b)}\Lambda_p(|A_{k_{i+1}}|)^{-b/(p-b)}|A_{k_{i+1}}|. \tag{3.11}$$

Integrating in time (3.11) we find for all $t>\tau_0$

$$\int_{t_{i+1}}^{t}\int_{\Omega} f_{i+1}^{b}\,dxd\tau \leq \frac{b}{p}\varepsilon^{p/b}\int_{t_{i+1}}^{t}\int_{\Omega}|\nabla f_{i+1}|^{p}\,dxd\tau$$
$$+\frac{p-b}{p}t\varepsilon^{-p/(p-b)}[\Lambda_p(\sup_{\tau_\infty<\tau<t}|A_{h_\infty}|)]^{-b/(p-b)}\sup_{\tau_\infty<\tau<t}|A_{h_\infty}|. \tag{3.12}$$

Combining Lemma 3.4 and (3.11)–(3.12) with $a_1 = k_i$, $a_2 = k_{i+1}$, $\tau_1 = t_i$, $\tau_2 = t_{i+1}$, we have for all $t > \tau_0$ and for any $\varepsilon_1 > 0$, for a suitable $\varepsilon > 0$

$$\sup_{t_i<\tau<t}\int_{\Omega} f_i^{b}\,dx + \int_{t_i}^{t}\int_{\Omega}|\nabla f_i|^{p}\,dxd\tau \leq \varepsilon_1\int_{t_{i+1}}^{t}\int_{\Omega}|\nabla f_{i+1}|^{p}\,dxd\tau$$
$$+\gamma\gamma_1^{i}\varepsilon_1^{-\frac{b}{p-b}}t(\tau_0-\tau_\infty)^{-\frac{p}{p-b}}\left(\frac{h_0}{h_0-h_\infty}\right)^{\frac{p}{p-b}|m-1|}$$
$$\times[\Lambda_p(\sup_{\tau_\infty<\tau<t}|A_{h_\infty}|)]^{-b/(p-b)}\sup_{\tau_\infty<\tau<t}|A_{h_\infty}|. \tag{3.13}$$

Iterating this inequality on i and choosing ε_1 small enough (see [4, Sect. 2]), we get

$$\sup_{\tau_0<\tau<t}\int_{\Omega}(u(\tau)-h_0)_+^{q+1}\,dx \leq \gamma t(\tau_0-\tau_\infty)^{-\frac{p}{p-b}}\left(\frac{h_0}{h_0-h_\infty}\right)^{\frac{p}{p-b}|m-1|}$$
$$\times[\Lambda_p(\sup_{\tau_\infty<\tau<t}|A_{h_\infty}|)]^{-b/(p-b)}\sup_{\tau_\infty<\tau<t}|A_{h_\infty}|. \tag{3.14}$$

B) We need one more iterative process. Define the sequences

$$l_n = k(1-2^{-n-1}),\quad \bar{l}_n = (l_n + l_{n+1})/2,\quad t_n' = t(1-2^{-n-1}).$$

By invoking (3.14) with $\tau_0 = t_{n+1}'$, $\tau_\infty = t_n'$, $h_0 = \bar{l}_n$, $h_\infty = l_n$ we obtain

$$Y_{n+1} := \sup_{t_{n+1}'<\tau<t}|\{u > l_{n+1}\}|$$
$$\leq \gamma\gamma_1^{n}k^{-(q+1)}\sup_{t_{n+1}'<\tau<t}\int_{\Omega}(u(\tau)-\bar{l}_n)_+^{q+1}\,dx$$
$$\leq \gamma\gamma_1^{n}k^{-(q+1)}t^{-\frac{b}{p-b}}\Lambda_p(Y_n)^{-\frac{b}{p-b}}Y_n.$$

Using now the classical iterative lemma [58, Lemma 5.6, p. 95] we conclude that $Y_n \to 0$ as $n \to \infty$ and therefore $\|u(t)\|_\infty \leq k$, provided

$$k^{-(q+1)}t^{-\frac{b}{p-b}}\Lambda_p(Y_0)^{-\frac{b}{p-b}} \leq \delta, \tag{3.15}$$

for a suitable $\delta = \delta(m, p, q) > 0$, which amounts to

$$k^{-1}t^{-\frac{1}{p+m-3}}\Lambda_p(Y_0)^{-\frac{1}{p+m-3}} \leq \delta^{\frac{1}{q+1}}. \tag{3.16}$$

By Chebyshev inequality we have for any fixed $r \geq 0$, at all time levels

$$|A_{l_0}| \leq \frac{1}{l_0^{r+1}}\int_{\Omega}u^{r+1}\,dx = \frac{2^{r+1}}{k^{r+1}}\int_{\Omega}u^{r+1}\,dx;$$

note that the number r enters the proof only at this stage. Finally, on choosing k from

$$k^{-1}t^{-\frac{1}{p+m-3}}\left[\Lambda_p\left(\frac{2}{k^{r+1}}\sup_{t/4<\tau<t}\int_{\Omega}u^{r+1}\,dx\right)\right]^{-\frac{1}{p+m-3}} = \delta^{\frac{1}{q+1}},$$

taking into account $\|u(t)\|_\infty \leq k$ and monotonicity arguments, we arrive at the desired result. □

Remark 3.5. The integrals on the right hand side of (3.9) must be estimated by taking into account the initial data. For example if $r = 0$ by conservation of mass we conclude from (3.9)

$$\|u(t)\|_\infty \leq \gamma\|u_0\|_{1,\Omega}\psi_0^{(-1)}\left(\frac{1}{t}(\|u_0\|_{1,\Omega})^{-(p+m-3)}\right). \tag{3.17}$$

This amounts to classical estimates when $\Omega = \boldsymbol{R}^N$ so that $\Lambda_p(s) = s^{-p/N}$; see e.g., [70] for the PME, and [35] for the linear case.

3.2 Narrowing domains

Even in the case of narrowing domains one can prove existence of solutions to (3.3)–(3.5) corresponding to initial data growing at infinity; to this end let us introduce the norm which rigorously determines the admissible behavior:

$$[U]_r := \sup_{\rho \geq r} \rho^{-\frac{p}{p+m-3}} \oint_{\overline{\Omega}_\rho} dU,$$

where U is a positive Radon measure and $r > 0$ is a given number.

Theorem 3.6 (Growing initial data ([9])). *Let u_0 be a positive Radon measure in $\overline{\Omega}$, such that $[u]_{\overline{\rho}} < \infty$, $\overline{\rho} \geq 1$. Then problem (3.3)–(3.5) has a solution defined in $\Omega \times (0, T_0)$, where $T_0 = \gamma_0 [u]_{\overline{\rho}}^{3-m-p}$, and*

$$[u(t)]_{\overline{\rho}} \leq \gamma [u_0]_{\overline{\rho}}, \tag{3.18}$$

$$\|u(t)\|_{\infty,\Omega_\rho} \leq \gamma \rho^{\frac{p}{p+m-3}} t^{-\frac{N}{\mathcal{K}}} [u_0]_{\overline{\rho}}^{\frac{p}{\mathcal{K}}}, \tag{3.19}$$

for $0 < t < T_0$, $\rho \geq \overline{\rho}$, where $\mathcal{K} = N(p + m - 3) + p$.

See the Comments at the end of this Section for the general framework of this result. Here we confine ourselves to the remark that formally all the existence results of this kind are very similar, once that the norm $[\cdot]_r$ is defined with the appropriate subdomains Ω_ρ (see e.g., [33]).

Next we deal with solutions uniformly bounded over Ω; here the situation is markedly different. As we saw in the previous Subsection, in expanding domains (e.g., in $\boldsymbol{R}^N$) global integrability of the initial data guarantees that $u(t) \in L^\infty(\Omega)$ for $t > 0$. This is not always the case in narrowing domains, as demonstrated by the explicit counterexample constructed in [9, Sect. 7]. Actually one can prove that $u(t) \to 0$ uniformly as $t \to +\infty$ provided a suitable moment of the initial data is finite. As a first special example of this behavior, interesting on its own, we present the case of initial data with bounded support.

Let

$$Z(t) = \inf\{\rho > 1 \mid u(x,t) = 0,\ x \in \Omega \setminus \Omega_\rho\}, \quad t > 0. \tag{3.20}$$

Clearly, $Z(t)$ gives a measure of the speed of propagation of the support of u, which we expect to be finite in view of the degeneracy of the equation.

Theorem 3.7 (Finite speed of propagation ([9])). *Let u_0 be a positive finite Radon measure in $\overline{\Omega}$ with bounded support:*

$$\operatorname{supp} u_0 \subset \overline{\Omega_{\overline{\rho}}},$$

for a given $\overline{\rho} \in (1, +\infty)$. Then, a suitable $t_0 > 0$ exists such that, for $t > t_0$

$$\gamma_1 P(t) \leq Z(t) \leq \gamma_2 P(t), \tag{3.21}$$

where $\rho = P(t)$ denotes the largest solution of

$$\rho f(\rho)^{-\frac{p+m-3}{2p+m-3}} = \|u_0\|_{1,\Omega}^{\frac{p+m-3}{2p+m-3}} t^{\frac{1}{2p+m-3}}. \tag{3.22}$$

We also have for $t > t_0$ the two-sided estimate

$$\gamma_1 \frac{\|u_0\|_{1,\Omega}}{V(P(t))} \leq \|u(t)\|_{\infty,\Omega} \leq \gamma_2 \frac{\|u_0\|_{1,\Omega}}{V(P(t))}. \tag{3.23}$$

Here t_0 depends on N, p, m, $\|u_0\|_{1,\Omega}$, $\overline{\rho}$.

Note that the left-hand side of (3.22) goes to ∞ as $\rho \to \infty$, because of (2.7) and (2.10), so that $P(t)$ is well defined and $P(t) \to +\infty$ as $t \to +\infty$. Recalling the definition of $P(t)$ the estimates (3.23) amount to bounding $\|u(t)\|_{\infty,\Omega}$ on both sides by multiples of

$$t^{-\frac{1}{2p+m-3}} \|u_0\|_{1,\Omega}^{\frac{p}{2p+m-3}} f(P(t))^{\frac{p}{2p+m-3}}.$$

It can be easily seen that when $f(\rho) =$ constant, i.e., when Ω is cylinder, the results of Theorem 3.7 coincide with those valid for the one-dimensional Cauchy problem; indeed when $N = 1$, $\mathcal{K} = 2p + m - 3$, where $\mathcal{K}$ is the Barenblatt–Pattle exponent defined in Theorem 3.6.

3.2.1 Integrable data of unbounded support

Let us turn to the case of integrable initial data, dropping the assumption of bounded support. As we already remarked, we need an extra assumption to obtain a uniform bound for $u(t)$, ultimately because of the lack of a global Sobolev–Gagliardo–Nirenberg or Faber–Krahn inequality. A natural assumption is limiting the growth of the initial data by requiring the finiteness of a suitable moment; we introduce to this end the moment

$$\mu(t) = \int_\Omega f(|x|)u(x,t)dx, \quad t \geq 0. \tag{3.24}$$

We extend this definition also to measures, in the obvious way. Then we may prove

Theorem 3.8 (Moment bounds ([7])). *Assume that $\Omega \in \mathcal{N}(f)$ and that for some given $c > 0$*

$$\rho \mapsto \frac{f(\rho)}{\rho^c} \quad \textit{is nondecreasing for } \rho \geq 1. \tag{3.25}$$

Let u_0 be a positive finite measure in $\overline{\Omega}$, such that $\mu(0) < +\infty$. Then problem (3.3)–(3.5) *has a solution u defined in $\Omega \times (0,+\infty)$, satisfying for $t > t_0$ the estimates* (3.23) *with $P(t)$ given in* (3.22), *and*

$$\gamma_0 \|u_0\|_{1,\Omega} f(P(t)) \leq \mu(t) \leq \gamma_1 \|u_0\|_{1,\Omega} f(P(t)). \tag{3.26}$$

Here t_0 depends also on $\mu(0)$.

The proof of this result is actually rather complex. We sketch here the main argument, which is essentially an a priori estimate to be then exploited to obtain compactness in a suitable approximation scheme.

First, for all solutions corresponding to an integrable initial data, we obtain

$$\|u(t)\|_{\infty,\Omega_{(1+\sigma)\rho}\setminus\Omega_{(1-\sigma)\rho}} \leq \gamma \max(t^{-\frac{N}{\mathcal{K}}}\Gamma(t)^{\frac{p}{\mathcal{K}}}, t^{-\frac{1}{\mathcal{K}_1}}\Gamma(t)^{\frac{p}{\mathcal{K}_1}} f(\rho)^{\frac{p}{\mathcal{K}_1}}), \tag{3.27}$$

provided $\sigma \in (0,1/2)$, $\rho \geq \gamma P(t)$, with $\mathcal{K}_1 = 2p + m - 3$, and

$$\Gamma(t) = \sup_{0<\tau<t} \|u(\tau)\|_{1,\Omega_{(1+2\sigma)\rho}\setminus\Omega_{(1-2\sigma)\rho}}.$$

Since, as already remarked, $\mathcal{K}_1$ is the one-dimensional version of $\mathcal{K} = N(m+p-3)+p$, it is clear that (3.27) bounds $u(t)$ by combining the (local) N-dimensional bound and the (global but non-uniform) 1-dimensional one. Note that this bound is optimal in general (i.e., if $\mu(0) = +\infty$) as proved by the counterexample in [9].

In (3.27) the integrals in $\Gamma(t)$ are local in space; therefore we may absorb $f(\rho)$ into them; this is the point where the moment appears in the estimate and the dependence on ρ is dropped. We obtain after some calculations

$$\|u(t)\|_{\infty,\Omega} \leq \gamma \max(t^{-\frac{N}{\mathcal{K}}}\|u_0\|_{1,\Omega}^{\frac{p}{\mathcal{K}}}, t^{-\frac{1}{\mathcal{K}_1}}\tilde{\mu}(t)^{\frac{p}{\mathcal{K}_1}},$$
$$t^{-\frac{1}{\mathcal{K}_1}}\|u_0\|_{1,\Omega}^{\frac{p}{\mathcal{K}_1}} f(P(t))^{\frac{p}{\mathcal{K}_1}}), \tag{3.28}$$

where $\tilde{\mu}(t) = \sup_{0<\tau<t}\mu(\tau)$. The third term on the right hand side of (3.28) is necessary to bound $u(t)$ in Ω_ρ with $\rho < P(t)$.

Then we appeal to the equation again, essentially using $f(|x|)\zeta(x)$ as a test function, where ζ is a suitable cut off function. We obtain for $t > 0$

$$\mu(t) \leq \gamma\mu(0) + \gamma f(H(t))\|u_0\|_{1,\Omega}, \tag{3.29}$$

where for all $\lambda \in (0,1/p)$

$$H(t) = t^{\frac{1}{p}-\lambda}\left(\int_0^t \tau^{\frac{\lambda p}{p-1}-1}\|u(\tau)\|_{\infty,\Omega}^{\frac{p+m-3}{p-1}} d\tau\right)^{\frac{p-1}{p}};$$

here γ depends on λ too.

Finally we apply (3.29) in (3.28), obtaining an inequality of (rather non-standard) Gronwall type for $\|u(t)\|_{\infty,\Omega}$. However, this inequality is handled by means of a suitable real analysis lemma, finally yielding the sought after estimate for $\|u(t)\|_{\infty,\Omega}$; then the bound for $\mu(t)$ follows from (3.29).

Comment 3.9. Existence of solutions to the Cauchy problem for (3.3), under optimal assumptions on the initial data, was obtained in [28] in the case $p = 2$ and in [34] in the case $m = 1$. See [17] for an extension of results of this kind to a case where (3.3) contains a space-dependent weight in the form of a capacitary coefficient, in a Riemannian framework.

The Neumann problem for linear uniformly parabolic equations in domains with noncompact boundaries was treated in [44], [45], see also [61], where it was shown that for a large class of domains Ω, whose geometry is described by means of an isoperimetric inequality, the asymptotic behavior of solutions for large times is

$$\|u(t)\|_{\infty,\Omega} \sim \frac{\|u_0\|_{1,\Omega}}{V(\sqrt{t})}.$$

These results were extended to the evolutive p-Laplacian in [66], [68] under the assumption $u_0 \in L_1(\Omega) \cap L_2(\Omega)$. We quote also [41] for an investigation of the PME in unbounded domains.

The approach used in the proof of Theorem 3.3 is taken from the work [13] (see also [16]) and is similar to the one introduced in [11] with the only difference that instead of Gagliardo–Nirenberg inequalities, the Faber–Krahn's inequality is used. The latter is a flexible tool for research on Riemannian manifolds (see [43]). This approach seems to us to essentially simplify the proofs of the corresponding results of [8].

Note that the assumption $m \geq 1$ which appeared in [7], [9] is in fact not needed for the results we present here. The degeneracy of the equation amounts indeed to (3.6), which is the requirement we place on p, m.

Finally, the approach of [11] is developed starting from the classical ideas of DeGiorgi, Ladyzhenskaya–Uraltseva, DiBenedetto. In particular, we don't use here the cylindrical parabolic embedding, which may be restrictive, for example in a Riemannian setting or in some other problems. In addition, this approach allows us to prove the optimal finite speed of propagation for degenerate parabolic equations even for initial data measures (see e.g., the two-sided estimate (3.21)), and it works also in graphs [15].

See also [49] for an approach to the Cauchy problem for doubly nonlinear equations, and [19] for the property of finite speed of propagation. On the behavior of solutions for large times let us also quote [51], [52], [53], [54].

We remark that the proof of Theorem 3.7 employs an iterative sequence of integral estimates on shrinking annuli; the use of bounded domains allows one to use local embedding results. This technique was applied also to higher order parabolic equations in [10].

References for this section: [7], [8], [9], [10], [11], [13], [15], [16], [17], [19], [28], [33], [34], [35], [41], [43], [44], [45], [49], [51], [52], [53], [54], [58], [61], [66], [68], [70].

4. Fujita Type Results for Blow Up Problems

In this section we study the behavior of nonnegative solutions in $S = \Omega \times (0, +\infty)$ of the Neumann problem

$$\frac{\partial u}{\partial t} - \operatorname{div}(u^{m-1}|\nabla u|^{p-2}\nabla u) = u^{\mu}, \qquad (x,t) \in S, \tag{4.1}$$

$$u^{m-1}|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{4.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega; \tag{4.3}$$

here $p \geq 2$, $m \geq 1$, $\mu > 1$.

It is well known that, since $\mu > 1$, global in time solutions to this problem may or may not exist; results in this direction are known in the literature as Fujita type results, after [36]. We are concerned here with existence or nonexistence of such global solutions, and with estimates of the finite speed of propagation of the support of u. We show how these properties are connected with the geometry of Ω, which we assume to be an expanding domain.

Our approach relies on careful integral estimates of solutions which follow in turn from embedding results involving geometrical properties of the domain Ω. Let us define $\Phi = \Psi^{(-1)}$ as the inverse function over $[0,\infty)$ of

$$\Psi(z) = z^{p+m-3+\frac{p}{N}}\omega(z)^p = z^{p+m-3+p}g(z)^{-p}, \quad z \geq 0.$$

Note that under assumption (3.2) we have

$$\Phi(s) = \frac{1}{\psi_0^{(-1)}\left(\frac{1}{s}\right)}, \quad s > 0, \tag{4.4}$$

where ψ_0 has been defined in Theorem 3.3.

Theorem 4.1 (Existence of global solutions [8]). *Let us assume that* $\Omega \in \mathscr{B}_1(g)$ *and that* Φ *satisfies*

$$\int^{+\infty} \Phi(s)^{-\mu+1} ds < +\infty. \tag{4.5}$$

Then (4.1)–(4.3) *has a solution defined for all* $t > 0$, *provided the initial data fulfills*

$$\|u_0\|_{1,\Omega} + \|u_0\|_{r,\Omega} \leq \delta, \tag{4.6}$$

where $r > 1$ *is such that* $N(\mu - m - p + 2) < rp$, *and* $\delta = \delta(N, p, m, \mu, r, g)$ *is chosen suitably small. Moreover,* u *satisfies for large* $t > 0$

$$\|u(t)\|_{\infty,\Omega} \leq \gamma \frac{\|u_0\|_{1,\Omega}}{\Phi(t\|u_0\|_{1,\Omega}^{p+m-3})}. \tag{4.7}$$

We remark that restrictions on the local integrability of the initial data u_0 are in general necessary when superlinear sources are present in the equation; we state here a global requirement for the sake of simplicity; see also the Comments at the end of this Section.

The estimate (4.7) is the same as (3.17), see (4.4). Namely it is the same estimate in force for solutions to the

homogeneous equation.
Finally, (4.5) requires that μ is large enough; in the case $\Omega = \boldsymbol{R}^N$ where $\Phi(s) = s^{N/\mathcal{K}}$ it amounts to the classical threshold

$$\frac{N}{\mathcal{K}}(\mu - 1) > 1. \tag{4.8}$$

Theorem 4.2 (Nonexistence of global solutions [8]). *Assume that $\Omega \in \mathcal{B}_1(g)$, $\mu > p + m - 2$ and that*

$$\int_0 V(\tau^{-\mathcal{E}})^{\frac{p+m-2}{p-1}} d\tau < +\infty, \quad \mathcal{E} := \frac{(p-1)(\mu - p - m + 2)}{p(p+m-2)}. \tag{4.9}$$

Then all non negative solutions $u \not\equiv 0$ to (4.1)–(4.3) *become unbounded in a finite time (in some bounded subset of Ω), provided we assume also that $\rho \mapsto V(\rho)\rho^{-\lambda}$ is nonincreasing for large ρ, for some given $0 < \lambda < p/(\mu - p - m + 2)$.*

Again in the case of $\Omega = \boldsymbol{R}^N$, condition (4.9) coincides with the strict converse of (4.8), and hence it is sharp. The last requirement on V could be weakened somehow, see [8].

As we already remarked, due to the degeneracy of the equation one expects solutions with compactly supported initial data to exhibit the property of finite speed of propagation. In our next result we give a sharp estimate of the support of $u(t)$ for large t.

Theorem 4.3 (Finite Speed of Propagation [8]). *Assume $\Omega \in \mathcal{B}_2(g)$, and let the assumptions of Theorem 4.1 be satisfied. Moreover, we require that* $\operatorname{supp} u_0 \subset \Omega_{\rho_0}$ *for a given $\rho_0 > 0$. Then for large t, the bound $Z(t)$ for the support of $u(t)$ (defined in* (3.20)*) satisfies*

$$\gamma_0 R(\Phi(t\|u_0\|_{1,\Omega}^{p+m-3})) \le Z(t) \le \gamma_1 R(\Phi(t\|u_0\|_{1,\Omega}^{p+m-3})), \tag{4.10}$$

and for $R = V^{(-1)}$ we have

$$\|u(t)\|_{\infty,\Omega} \ge \gamma_0 \frac{\|u_0\|_{1,\Omega}}{\Phi(t\|u_0\|_{1,\Omega}^{p+m-3})}. \tag{4.11}$$

Estimate (4.11) shows that (4.7) is sharp.

Example 4.4 (Paraboloid-like Ω). If Ω is one of the domains in Example 2.3, we can check that (4.5) reads

$$\frac{1 + \alpha(N-1)}{(1 + \alpha(N-1))(p + m - 3) + p}(\mu - 1) > 1, \tag{4.12}$$

while (4.9) reduces to the converse strict inequality. Note that for large ρ we have in this case $V(\rho) \sim \rho^{1+\alpha(N-1)}$, so that the numerator in (4.12) accounts for the volume function. Then one should compare (4.12) with the threshold (4.8) valid for the Cauchy problem.

Below we summarily sketch the proofs of the results stated above, which are based on arguments typical of this field. See also [34], [4], [5].

Proof of Theorem 4.1 (sketch). Step 1. If we look at the right hand side of (3.10) we readily see that such an inequality is still valid even for the nonhomogeneous equation (4.1), at least for all times t such that

$$\tau\|u(\tau)\|_{\infty,\Omega}^{\mu-1} \le 1, \quad 0 < \tau < t. \tag{4.13}$$

Clearly this amounts to ignoring the source on the right hand side of (4.1); our goal is exactly to show that this is possible under our assumptions.
Thus, if we assume (4.13), an estimate of the type of (3.17) is in force; recall that Φ and ψ_0 are related by (4.4). Actually, one works with approximating solutions, but we give here the argument in the form of an a priori estimate.

Step 2. We ignore here the time interval $(0, 1)$, where estimates must make use of the extra integrability of the initial data.
Note that estimate (3.17) itself proves that (4.13) is in force, at least for those t such that

$$U(t) := \sup_{0<\tau<t} \|u(\tau)\|_{1,\Omega} \le \delta', \tag{4.14}$$

for a suitable $\delta' > 0$, which is going of course to determine the choice of the δ in assumption (4.6). A more detailed analysis than the one we perform in this sketch, would show that we already use (4.5) and $t > 1$ at this stage.
The point is how to obtain estimate (4.14) for all $t > 1$. Simply by integrating the equation by parts we obtain

$$\int_\Omega u(t)dx \le \int_\Omega u(1)dx + \int_1^t \|u(\tau)\|_{\infty,\Omega}^{\mu-1} \int_\Omega u(\tau)dxd\tau$$
$$\le \gamma\|u_0\|_{1,\Omega} + \gamma \int_1^t \frac{U(\tau)^\mu}{\Phi(t\|u_0\|_{1,\Omega}^{p+m-3})^{\mu-1}} d\tau.$$

As already remarked, we assume here that $U(1)$ has been estimated already by $\gamma\|u_0\|_{1,\Omega}$. Then by comparison arguments, we have that $U(t) \le y(t)$ for $t > 1$, where y is defined by

$$\frac{dy}{dt} = \gamma \frac{y^\mu}{\Phi(t\|u_0\|_{1,\Omega}^{p+m-3})^{\mu-1}}, \quad t > 1; \quad y(1) = \gamma\|u_0\|_{1,\Omega}.$$

But owing to the summability condition (4.5), one can see that $y(t) \le \gamma\delta$ for all $t > 1$, completing the proof. □

Proof of Theorem 4.2 (sketch). The arguments are technically quite involved, but the basic idea is simple. By setting $w = u^a$, $a = (p + m - 2)/(p - 1)$ we rewrite the equation as (here $\beta = 1/a \in (0, 1]$).

$$\frac{\partial w^\beta}{\partial t} - \operatorname{div}(|\nabla w|^{p-2}\nabla w) = w^{\beta\mu}. \tag{4.15}$$

Let $V_*(s) = \max(V(s), 1)$; we use as testing function a suitable cut off of

$$f(w) = \left[\int_0^w V_*(\tau^{-\mathcal{E}})^{\frac{p+m-2}{p-1}} d\tau\right]^{-\varepsilon}, \quad w > 0; \quad f(0) = 0.$$

This definition is possible due to our assumptions.

Essentially, we show that a suitable integral norm of $w(t)$ in a compact set blows up in a finite time, by bounding it from below with the solution to an ordinary differential equation with such a behavior. The technical difficulty is given by absorbing all integrals contributed by the diffusion operator into the nonlinear source, which yields on the other hand the forcing term

$$\int_\Omega w(x,t)^{\beta\mu} f(w(x,t))\zeta(x)dx,$$

where ζ is the cut off function. □

Proof of Theorem 4.3 (sketch). We define a sequence of cut off functions ζ_n, $n \ge 0$, so that

$$\zeta_n(x) = 1, \quad x \in \Omega_{\rho_n} \setminus \Omega_{\overline{\rho}_n}, \quad \zeta_n(x) = 0, \quad x \notin \Omega_{\rho_{n-1}} \setminus \Omega_{\overline{\rho}_{n-1}},$$
$$|\nabla\zeta_n| \le \gamma \frac{2^n}{\sigma\rho},$$

where $0 < \sigma < 1/2$ is given and, for ρ_0 as in the statement,

$$\rho_n = \rho + \sigma 2^{-n}\rho, \quad \overline{\rho}_n = (\rho - \sigma 2^{-n}\rho)/2, \quad n \ge 0, \quad \rho > 4\rho_0.$$

The supports of the ζ_n are shaped like shrinking annuli (intersected with Ω). As $n \to +\infty$ we have $\rho_n \ge \rho_{n+1} \to \rho+$, $\overline{\rho}_n \le \overline{\rho}_{n+1} \to \rho/2-$; note also that the support of ζ_0 is bounded away from the support of u_0. Then choosing as a testing function $\zeta_n^p u^\theta$, where $0 < \theta < p/N$, we have

$$\sup_{0<\tau<t} \int_\Omega u(\tau)^{1+\theta}\zeta_n^p dx + \int_0^t \int_\Omega u^{m+\theta-2}|\nabla u|^p \zeta_n^p dxd\tau$$
$$\le \gamma \frac{2^{np}}{(\sigma\rho)^p} \int_0^t \int_{\Omega_{\rho_{n-1}}\setminus\Omega_{\overline{\rho}_{n-1}}} u^{p+m+\theta-2} dxd\tau + \gamma \int_0^t \int_\Omega u^{\mu+\theta}\zeta_n^p dxd\tau. \tag{4.16}$$

Reasoning as in the proof of Theorem 4.1 and choosing δ small enough, we may bound the last term in (4.16) by

$$\gamma \int_0^t \|u(\tau)\|_\infty^{\mu-1} d\tau \sup_{0<\tau<t} \int_\Omega u(\tau)^{1+\theta}\zeta_n^p dx \le \frac{1}{2} \sup_{0<\tau<t} \int_\Omega u(\tau)^{1+\theta}\zeta_n^p dx.$$

So the contribution of the blow up term can be absorbed into the left hand side and essentially we are reduced to the case of the homogeneous equation. Therefore from this resulting inequality we get for sufficiently large $s > 1$

$$Y_n := \sup_{0<\tau<t} \int_\Omega v_n^\varepsilon dx + \int_0^t \int_\Omega |\nabla v_n|^p dxd\tau \le \gamma_1 \frac{2^{np}}{(\sigma\rho)^p} \int_0^t \int_\Omega v_{n-1}^p dxd\tau, \tag{4.17}$$

where $v_n = u^{(p+m+\theta-2)/p}\zeta_n^s$, $\varepsilon = p(1+\theta)/(p+m+\theta-2)$. Next we need apply the embedding Theorem 2.4, with $\beta = \varepsilon$, $q = p$, $v = v_{n-1}(\tau)$ and replace this in (4.17) with the aim of obtaining an iterative inequality for Y_n. The calculations are made rather intricate by the general functional setting and would be easier in the power nonlinearity setting of

Example 2.3. However, eventually we arrive at

$$Y_n \leq \gamma\gamma_1^n \rho^{-p} t^{\frac{(1+\theta)p}{\mathcal{L}}} Y_{n+1}^{1+\frac{(p+m-3)p}{\mathcal{L}}} f_0(t, \rho, \|u_0\|_{1,\Omega}),$$

where $\mathcal{L} = N(p+m-3) + p(1+\theta)$, f_0 is a suitable positive function connected to the geometry of Ω, and γ, $\gamma_1 > 1$ are constants. Hence by appealing to [58, Lemma 5.6, Chapter II] we prove that $Y_n \to 0$ as $n \to +\infty$ if $\rho \geq \rho(t)$, where $\rho(t)$ is the smallest solution of

$$\rho^{-p} t^{\frac{(1+\theta)p}{\mathcal{L}}} Y_0^{\frac{(p+m-3)p}{\mathcal{L}}} f_0(t, \rho, \|u_0\|_{1,\Omega}) \leq c,$$

for some suitably small $c > 0$. But $Y_n \to 0$ implies that $u(x,t) = 0$ for $|x| \geq \rho(t)$. Next we bound Y_0 by means of an energy inequality like (4.16) and of (4.7). Finally, on tracking the functional structure of f_0 we get the result. □

4.1 Cone-like domains

The methods demonstrated above are in some sense universal and can be applied to other boundary value problems. Consider the Cauchy–Dirichlet problem for the porous media equation with source in unbouded cone-like domains

$$\frac{\partial u}{\partial t} - \Delta u^m = 0, \qquad (x,t) \in \Omega \times (0,T), \tag{4.18}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{4.19}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{4.20}$$

Here $m > 1$, $u_0 \geq 0$ and $\Omega \subset \boldsymbol{R}^N$, $N \geq 2$ satisfies the geometrical assumptions of isoperimetrical type listed below, which involve a suitable harmonic function $H \geq 0$. Before giving the assumptions, let us point out the following examples.

Example 4.5 (Cone-like domains). 1) Standard cones: in this case

$$\Omega = \left\{ x \in \boldsymbol{R}^N \mid \frac{x_N}{|x|} > \cos\theta_0 \right\}, \quad \theta_0 \in (0, \pi];$$

$$H(x) = |x|^\lambda \varphi\left(\arccos \frac{x_N}{|x|} \right),$$

where λ is the positive root of $\lambda(\lambda + N - 2) = \omega$, with ω being the smallest eigenvalue of the Laplace–Beltrami operator on $\Omega \cap \{|x| = 1\}$. The function φ must satisfy a suitable ordinary differential equation.

2) Octants: here for some $1 \leq i < N$

$$\Omega = \{x \in \boldsymbol{R}^N \mid x_1, \dots, x_i > 0\}, \quad H(x) = x_1 \dots x_i.$$

3) Other examples are given by products of the two cases above, by angles in $\boldsymbol{R}^2$, by dihedral angles in $\boldsymbol{R}^3$.

In general, let H be a positive harmonic function in Ω, vanishing on $\partial\Omega$, and let A be any subset of Ω with smooth boundary; define

$$|A|_H = \int_A H(x)dx, \quad |\partial A|_{H,N-1} = \int_{\partial A} H(x) d\sigma_{N-1},$$

where $d\sigma_{N-1}$ denotes the area measure. We also define the perimeter function relative to H as

$$P(V) = \inf\{|\partial A|_{H,N-1} \mid A \subset \Omega, \quad |A|_H = V\}, \quad V > 0.$$

We say that $\Omega \in \mathcal{D}(H, M)$ if there exist constants C, $C_0 > 0$ and $M > 2$ such that

$$\int_{\Omega \cap \{|x| < \rho\}} H(x)dx \leq C\rho^M, \quad P(V) \geq C_0 V^{\frac{M-1}{M}}.$$

The general idea is that M is going to play the role of the dimension (e.g., N in $\boldsymbol{R}^N$); the relevant Barenblatt–Pattle like exponent in this case is

$$\beta := M(m-1) + 2,$$

which indeed coincides with $\mathcal{K}$ when $N = M$, $p = 2$. With reference to the Examples above, we note that in the case of Example 4.5.1 we have $M = N + \lambda$, while in the case of Example 4.5.2 we have $M = N + i$.

We can prove the following theorem

Theorem 4.6 (Existence for growing data ([12])). *Assume that*

$$\sum_{i=1}^N x_i H_{x_i} \geq 0, \tag{4.21}$$

and that $u_0 \in L^1_{\mathrm{loc}}(\Omega)$ *satisfies*

$$[u_0]_H := \sup_{\rho\geq 1} \rho^{-\frac{\beta}{m-1}} \int_{\Omega_\rho} u_0(x)H(x)dx < +\infty,$$

where $\Omega \in \mathcal{D}(H, M)$ *and* H *are as above. Then a weak solution of* (4.18)–(4.20) *exists for* $0 < t < T := \gamma_0[u_0]_H^{-m+1}$. *Moreover*

$$\|u(t)\|_{\infty,\Omega_\rho} \leq \gamma t^{-\frac{M}{\beta}} \rho^{\frac{2}{m-1}} [u_0]_H^{\frac{2}{\beta}}, \quad 0 < t < T, \ \rho > 1. \tag{4.22}$$

The structure of the proof is similar to the one valid in the whole space $\boldsymbol{R}^N$, but the critical exponents strictly depend on the geometry of the domain Ω, which we describe by means of the isoperimetric inequality stated above. We confine ourselves to the statement of the relevant Sobolev–Gagliardo–Nirenberg inequality.

$$\left(\int_\Omega H|v|^q dx\right)^{1/q} \leq \gamma \left(\int_\Omega H|\nabla v|^p dx\right)^{A/p} \left(\int_\Omega H|v|^r dx\right)^{(1-A)/r}, \tag{4.23}$$

where $v \in W^{1,p}(\Omega)$ has bounded support and

$$0 < r < q \leq \frac{Mp}{M-p}, \quad 1 \leq p < M, \quad q > 1,$$
$$A = \frac{p}{q} \frac{M(q-r)}{Mp - (M-p)r}.$$

The inequality (4.23) can be proved according to the approach in [1, Chapter V].

Remark 4.7 (Solutions with finite moment). It follows as a by-product of the proof of Theorem 4.6 that if

$$\mu(0) = \int_\Omega H(x)u_0(x)dx < +\infty \tag{4.24}$$

then

$$\|u(t)\|_{\infty,\Omega} \leq \gamma \mu(0)^{\frac{2}{\beta}} t^{-\frac{M}{\beta}}, \quad t > 0. \tag{4.25}$$

In fact in this case assumption (4.21) is not even needed, since it is necessary only when localizing by means of cut off functions.

Example 4.8 (Sign-changing solutions in the plane). We start by considering the problem (4.18)–(4.20) in the angle

$$\Omega = \{(x_1, x_2) \mid 0 < r < +\infty, \ 0 < \theta < \pi/n\} \subset \boldsymbol{R}^2,$$

$n \in \mathbf{N}$, so that $H(x) = r^n \sin(n\theta)$; here r and θ are the standard polar coordinates in $\boldsymbol{R}^2$.
According to the results in Remark 4.7, we have

$$\|u(t)\|_{\infty,\Omega} \leq \gamma \mu(0)^{\frac{2}{(n+2)(m-1)+2}} t^{-\frac{n+2}{(n+2)(m-1)+2}}, \quad t > 0. \tag{4.26}$$

Actually, u can be extended via $n-1$ odd reflections (about the sides of Ω) to a solution U of the Cauchy problem for the PME in $\boldsymbol{R}^2$. Such a solution changes its sign from a copy of Ω to the adjacent copy. Note that the estimate (4.26) still holds for U; we remark that, by selecting n suitably, we can make the decay rate there as close as we want to the rate valid in bounded domains (with zero Dirichlet data) for nonnegative solutions, i.e., $-1/(m-1)$.

We remark here that, in this setting, Fujita type results might be obtained also for the equation

$$\frac{\partial u}{\partial t} - \Delta u^m = u^q, \quad (x,t) \in \Omega \times (0, +\infty), \tag{4.27}$$

where $q > 1$.

Comment 4.9. The result formulated in this section were published in [8], [12].
The first celebrated result concerning the blow up phenomenon is the classical paper by Fujita [36] for the Cauchy problem for

$$\frac{\partial u}{\partial t} - \Delta u = u^q, \quad \text{in } \boldsymbol{R}^N \times (0, \infty). \tag{4.28}$$

It was proved there that the exponent $q = 1 + 2/N$ is critical in the sense that, if $1 < q < 1 + 2/N$, then (4.28) possesses no global positive solution, while if $q > 1 + 2/N$, then there exist global in time solutions, provided a suitable norm of u_0 is small enough. The critical case $q = 1 + 2/N$ (in which any positive solution blows up in a finite time) was treated later in [46]; see also [65].
As an application of the comparison principle, Fujita type results were established in [38], [37] for the PME and

p-Laplacian equations with sources. By means of a local energy approach, Fujita type results for general PME type equations with blow up term were proved in [5] and [2] (for systems). Some of these results were extended to the p-Laplacian equation with blow up terms in [50]. For the solution of the Neumann problem for doubly degenerate parabolic equations with nonlinear sources in domains with noncompact boundaries Fujita type results were obtained in [7], [8]. It was proven that the critical Fujita exponent depends strictly on the geometry of the domain. Note that these results include the Cauchy problem as well. Later for the same class of equations but in the case of Cauchy problem the blow up phenomena was proved in [63]. Let us quote also [71], [72], [73], [74], and the surveys [60], [31].

In equations with supercritical power sources, we may have existence of solutions only under restrictions on the regularity of the initial data; that is, not for all initial data finite measures, or even L^1 functions, a solution exists; see [5], [2] and for initial data measures [3].

Finally, existence of local in time solutions corresponding to growing initial data (as in Theorem 4.6) has been treated with similar methods, in a Riemannian setting, in [17].

References for this section: [1], [2], [3], [4], [5], [7], [8], [12], [17], [31], [34], [36], [37], [38], [46], [50], [58], [60], [63], [65], [71], [72], [73], [74].

5. Asymptotic Expansion of Solutions of the Filtration Equation in a Paraboloid-like Domain

We consider here, following [14], the domain $\Omega = \Omega(\alpha)$ which has been defined in (2.5) for $0 \le \alpha \le 1$, and the corresponding Neumann problem for the equation of porous-media type

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u^m}{\partial x_i}\right) = 0, \qquad (x,t) \in S = \Omega \times (0,+\infty), \tag{5.1}$$

$$a_{ij}(x,t)\frac{\partial u^m}{\partial x_i}\nu_j = 0, \qquad (x,t) \in \partial\Omega \times (0,+\infty), \tag{5.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{5.3}$$

We understand throughout summation with respect to repeated indexes. Here $N \ge 2$, $m > 1$ and $\nu = (\nu_1,\dots,\nu_N)$ is the unit outer normal to $\partial\Omega$; we assume $u_0 \in L^1(\Omega)$ is a nonnegative function, and the coefficients $a_{ij} \in L^\infty(S)$, $i, j \in \{1,2,\dots,N\}$, satisfy for a constant $c_0 > 1$ and for all $\xi = (\xi_1,\dots,\xi_N) \in \boldsymbol{R}^N$, a.e. in S,

$$c_0^{-1}|\xi|^2 \le a_{ij}(x,t)\xi_i\xi_j \le c_0|\xi|^2. \tag{5.4}$$

The main goal of this section is to get the asymptotic profile as $t \to +\infty$ of solutions to (5.1)–(5.3) in S under additional assumptions on the coefficients.

If $a_{ij}(x,t) = \delta_{ij}$, where δ_{ij} denotes the Kronecker symbol, then (5.1) is the PME. It is well known that if $\Omega = \boldsymbol{R}^N$, then as $t \to +\infty$ we have

$$t^{\frac{N}{\mathcal{K}}}|u(x,t) - E(x,t)| \to 0, \tag{5.5}$$

uniformly in $x \in \boldsymbol{R}^N$, where $\mathcal{K} = N(m-1)+2$ is the Barenblatt–Pattle exponent and $E(x,t)$ is the fundamental solution of the PME with the same total mass as u_0, i.e.,

$$E(x,t) = t^{-\frac{N}{\mathcal{K}}}\left[C - c(m,N)\left(\frac{|x|}{t^{1/\mathcal{K}}}\right)^2\right]_+^{\frac{1}{m-1}}, \tag{5.6}$$

where $c(m,N) = (2\mathcal{K})^{-1}(m-1)$ and C is chosen so that $\|u_0\|_1 = \|E\|_1$ for $t > 0$. That is, E solves the PME and takes the multiple of the Dirac mass $\|u_0\|_1\delta(x)$ as initial data.

Let us go back to our problem set in $\Omega(\alpha)$. Essentially, we prove in [14] that when $\alpha < 1$, the asymptotic profile of the solution to (5.1)–(5.3) is one-dimensional, while when $\alpha = 1$, the asymptotic profile is genuinely N-dimensional; in this instance, our results cover also the Cauchy problem even if $\Omega(1)$ is in fact a cone.

Assume for the moment $\alpha < 1$ and let us introduce the following one-dimensional asymptotic profile:

$$E_\alpha(y,t) = t^{-\frac{n(\alpha)}{b(\alpha)}}\left[C - \frac{m-1}{2mb(\alpha)}\left(\frac{y}{t^{\frac{1}{b(\alpha)}}}\right)^2\right]_+^{\frac{1}{m-1}}, \quad y > 0,\ t > 0. \tag{5.7}$$

Here $n(\alpha) = \alpha(N-1)+1$ and $b(\alpha) = n(\alpha)(m-1)+2$. The power $n(\alpha)$ corresponds to the dimension at infinity of $\Omega(\alpha)$, in the sense that

$$|\Omega(\alpha) \cap \{x_N < \rho\}| = \frac{\omega_{N-1}}{n(\alpha)}\rho^{n(\alpha)}, \quad \rho > 0;$$

here ω_{N-1} is the area of the unit sphere in $\boldsymbol{R}^N$. Note that $|\Omega \cap \{|x| < \rho\}|$ has the same asymptotics for $\rho \to +\infty$. In the following we let $\Omega^\rho = \Omega(\alpha) \cap \{x_N < \rho\}$.
Then for a suitable choice of $C > 0$, E_α solves (in a suitable sense) the problem

$$y^{(N-1)\alpha} \frac{\partial V}{\partial t} - \frac{\partial}{\partial y}\left(y^{(N-1)\alpha} \frac{\partial V^m}{\partial y}\right) = 0, \qquad y > 0,\ t > 0, \tag{5.8}$$

$$y^{(N-1)\alpha} \frac{\partial V^m}{\partial y} = 0, \qquad y = 0,\ t > 0, \tag{5.9}$$

$$y^{(N-1)\alpha} V(y, 0) = M\delta(y), \quad y > 0, \tag{5.10}$$

where δ denotes the Dirac mass, and we let $M = \omega_{N-1}^{-1} \|u_0\|_1$.
Further we need the following assumptions on the asymptotic behavior of the coefficients a_{iN}, $i = 1, \dots, N$:

$$\lim_{\rho\to+\infty} \rho^{-(n(\alpha)+b(\alpha))} \int_0^{\rho^{b(\alpha)}} \int_{\Omega^\rho} |a_{iN}(y,t) - \delta_{iN}|^2 dydt = 0. \tag{5.11}$$

Theorem 5.1 (Case $\alpha < 1$ ([14])). *Let u be a solution of* (5.1)–(5.3) *in S with $0 \le \alpha < 1$, and let E_α be as in* (5.7). *Assume that (a_{iN}) fulfills condition* (5.11). *Then u approaches E_α as $t \to +\infty$ in the following sense, for all $p \in [1, +\infty)$.*

Interior domain: For all $\rho > 0$, $T > \tau > 0$,

$$t^{\frac{n(\alpha)}{b(\alpha)}p} \oint_{t\tau}^{tT} \oint_{\Omega^{t^{1/b(\alpha)}R}} |u(y,s) - E_\alpha(y_N, s)|^p dyds \to 0. \tag{5.12}$$

Outer domain: For a suitable $\Gamma > 0$

$$t^{\frac{n(\alpha)}{b(\alpha)}} \|u(t) - E_\alpha(t)\|_{\infty,\{x_N > \Gamma t^{1/b(\alpha)}\}} \to 0. \tag{5.13}$$

Next, we look at the case $\alpha = 1$, i.e., to cones; in this case the asymptotic profiles is N-dimensional.

Theorem 5.2 (Case $\alpha = 1$ (Cones; [14])). *Let u be a solution of* (5.1)–(5.3) *in S, assume $\alpha = 1$ and let E be as in* (5.6)*, with C such that $\|E(t)\|_{1,\Omega(1)} = \|u_0\|_{1,\Omega(1)}$. Assume that (a_{ij}) fulfills the condition*

$$\lim_{\rho\to+\infty} \rho^{-(N+\mathcal{K})} \int_0^{\rho^{\mathcal{K}}} \int_{\Omega^\rho} |a_{ij}(y,t) - \delta_{ij}|^2 dydt = 0, \quad 0 \le i,j \le N. \tag{5.14}$$

(Note that $n(1) = N$, $b(1) = \mathcal{K}$.) Then

$$t^{\frac{N}{\mathcal{K}}} \|u(t) - E(t)\|_{\infty,\Omega} \to 0, \quad t \to +\infty. \tag{5.15}$$

The same result holds true for the Cauchy problem, that is if $\Omega(1)$ is replaced with $\boldsymbol{R}^N$ above.

We sketch the proof of Theorem 5.1; the one of Theorem 5.2 follows the same lines.

Proof of Theorem 5.1 (sketch). For the choice of $\Omega(\alpha)$ in (5.1)–(5.3), we have from [8] that a solution u satisfies

$$\|u(t)\|_{\infty,\Omega} \le \gamma \max\left\{ \frac{\|u_0\|_{1,\Omega}^{2/\mathcal{K}}}{t^{N/\mathcal{K}}}, \frac{\|u_0\|_{1,\Omega}^{2/b(\alpha)}}{t^{n(\alpha)/b(\alpha)}} \right\}, \quad \text{for all } t > 0. \tag{5.16}$$

The proof uses then Kamin's rescaling arguments, where the scaling is chosen so as to leave Ω fixed. Let $u_k(x,t) = k^{n(\alpha)} u(k^\alpha x', kx_N, k^{b(\alpha)} t)$, $k \ge 1$. Then u_k satisfies essentially the same estimate as above, by direct inspection and for a suitable $\lambda(\alpha) > 0$:

$$\|u_k(t)\|_{\infty,\Omega} \le \gamma \max\left\{ \frac{\|u_0\|_{1,\Omega}^{2/\mathcal{K}}}{k^{\lambda(\alpha)} t^{N/\mathcal{K}}}, \frac{\|u_0\|_{1,\Omega}^{2/b(\alpha)}}{t^{n(\alpha)/b(\alpha)}} \right\}, \quad \text{for all } k > 1,\ t > 0. \tag{5.17}$$

In addition, again by direct inspection one checks that u_k solves

$$\frac{\partial u_k}{\partial t} - \frac{\partial}{\partial x_j}(A_{ij}^k(x,t)(u_k^m)_{x_i}) = 0, \qquad (x,t) \in S, \tag{5.18}$$

$$A_{ij}^k(x,t) \frac{\partial u_k^m}{\partial x_i} \nu_j = 0, \qquad (x,t) \in \partial\Omega \times (0, +\infty), \tag{5.19}$$

$$u_k(x, 0) = u_{k0}(x), \quad x \in \Omega, \tag{5.20}$$

where $u_{k0}(x) = k^{n(\alpha)} u_0(k^\alpha x', kx_N)$ and for $i, j \in \{1, \dots, N-1\}$

$$A^k_{ij} = k^{2(1-\alpha)} a^k_{ij}, \quad A^k_{Nj} = k^{1-\alpha} a^k_{Nj}, \quad A^k_{jN} = k^{1-\alpha} a^k_{jN}, \quad A^k_{NN} = a^k_{NN},$$
$$a^k_{ij}(x,t) = a_{ij}(k^\alpha x', kx_N, k^{b(\alpha)} t).$$

Notice that for $t > 0$ we have the uniform conservation of mass

$$\int_\Omega u_k(x,t)dx = \int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx. \tag{5.21}$$

Moreover from the problem (5.18)–(5.20) we have the energy estimate for all $k \geq 1$, $T > \tau > 0$, $\rho > 0$

$$\int_\tau^T \int_{\Omega^\rho} |(u_k^m)_{x_N}|^2 dxdt + k^{2(1-\alpha)} \int_\tau^T \int_{\Omega^\rho} |\nabla_{x'} u_k^m|^2 dxdt \leq \gamma, \tag{5.22}$$

where γ depends on $\|u_0\|_{1,\Omega}$, τ, T, ρ but not on k. One can prove for the sequence u_k also an equicontinuity result with respect to the time variable; we omit the precise statement here. Then, up to subsequences, we may assume $u_k \to u_\infty$ in $L^2_{\text{loc}}(S)$ and a.e. in S. The estimate (5.22) itself shows that u_∞ does not depend on x', and (5.17) in turn shows

$$\|u_\infty(t)\|_{\infty,\Omega} \leq \gamma \frac{\|u_0\|_{1,\Omega}^{2/b(\alpha)}}{t^{n(\alpha)/b(\alpha)}}, \quad \text{for all } t > 0. \tag{5.23}$$

Let $\varphi(x_N, t)$ be a smooth function in $[0, +\infty) \times [0, +\infty)$ such that

$$\varphi(x_N, t) = 0, \quad x_N \geq R; \quad \varphi(x_N, t) = 0, \quad t \geq T.$$

Then from the weak formulation of the problem for u_k, we obtain on taking the limit $k \to +\infty$ and invoking our assumptions on a_{ij}, as well as the weak convergence $\nabla u_k^m \rightharpoonup \nabla u_\infty^m$ in $L^2_{\text{loc}}(S)$ and all our estimates above,

$$\iint_S (-u_\infty \varphi_t + (u_\infty^m)_{x_N} \varphi_{x_N}) dxdt = \lim_{k\to+\infty} \int_\Omega u_k(x,0)\varphi(x_N,0)dx$$
$$= \varphi(0,0) \int_\Omega u_0 dx = \varphi(0,0)\omega_{N-1} M. \tag{5.24}$$

Recalling that the limit function is independent of x', we infer

$$\int_0^T \int_0^\infty x_N^{\alpha(N-1)} (-u_\infty \varphi_t + (u_\infty^m)_{x_N} \varphi_{x_N}) dx_N dt = M\varphi(0,0). \tag{5.25}$$

Then u_∞ solves (5.8)–(5.10); we can prove uniqueness of such solutions in a class satisfying among other requirements, estimate (5.23); we omit the details, though the proof might have some independent interest. Thus, $u_\infty = E_\alpha$. Therefore, the whole sequence u_k tends to E_α, actually in $L^p_{\text{loc}}(S)$ for all $p \geq 1$, owing to the sup bound (5.16).

We next prove the interior estimate (5.12). Fix $R > 0$, $T > \tau > 0$, $p \geq 1$ and compute, by taking advantage of the self-similar form of E_α and by changing variables,

$$I_k = \fint_\tau^T \fint_{\Omega^R} |u_k(x', x_N, z) - E_\alpha(x_N, z)|^p dxdz$$
$$= k^{n(\alpha)p} \fint_{k^{b(\alpha)}\tau}^{k^{b(\alpha)}T} \fint_{\Omega^{kR}} |u(y,s) - E_\alpha(y_N, s)|^p dyds.$$

Now for any given $t \geq 1$, we select $k = t^{1/b(\alpha)}$ so that by virtue of the already claimed compactness

$$\lim_{t\to+\infty} t^{\frac{n(\alpha)}{b(\alpha)}p} \fint_{t\tau}^{tT} \fint_{\Omega^{Rt^{1/b(\alpha)}}} |u(y,s) - E_\alpha(y_N, s)|^p dxds = \lim_{k\to+\infty} I_k = 0.$$

Concerning the estimate in the outer domain, i.e., (5.13), we note that it reduces to an L^∞ estimate for u_k, since $E_\alpha = 0$ in the outer domain for a suitable choice of Γ. In turn, we may prove a local L^1–L^∞ estimate in the outer domain, that is essentially a bound like (5.16), but where the L^1 norm of the initial data is calculated on a larger outer domain $\Omega \setminus \Omega^{\Gamma t^{1/b(\alpha)}/2}$. Since this norm goes to zero as $t \to +\infty$, the estimate follows. □

Comment 5.3. The asymptotic expansion for $t \to +\infty$ of solutions to the PME in the whole space is well known, see [55], [69]; to the best of our knowledge, for equations with variable coefficients a_{ij} even this case was not treated in the literature. The results of [14], which hold true also in this case, seem therefore interesting in view both of the domain and of the structure of the equation we consider. See [56] for a related interesting investigation in the linear case. An important role is played in all the quoted papers by the uniqueness of the fundamental solution, which we prove by extending to our setting the proof of [64].

References for this section: [8], [14], [55], [56], [64], [69].

6. Universal Bounds Near the Blow-up Time

Following [11] we consider nonnegative solutions to

$$u_t - \operatorname{div}(u^{m-1}|\nabla u|^{p-2}\nabla u) = f(x)u^q, \quad (x,t) \in S_T = \Omega \times (0,T), \tag{6.1}$$

where $0 < T < \infty$. We always assume that

$$q > p + m - 2, \quad q > 1, \quad p > 1.$$

Here f is a radial function in $C(\boldsymbol{R}^N \setminus \{0\})$, with $f(x) > 0$ if $x \neq 0$; we always assume that $f \in L^1_{\rm loc}(\boldsymbol{R}^N)$. The domain Ω is an open subset of $\boldsymbol{R}^N$ such that $0 \in \Omega$. We investigate the behavior of u near the (possible) blow up point $(0,T)$ without any assumption on initial or boundary data. Though the results in [11] are more general, we confine ourselves here to the case of power f, that is

$$f(x) = |x|^{-\alpha}, \quad -\infty < \alpha < \min(N,p), \tag{6.2}$$

setting also

$$\mathcal{H} = \frac{(q-1)p - \alpha(p+m-3)}{q-p-m+2}, \quad \mathcal{B} = \frac{p-\alpha}{(q-1)p - \alpha(p+m-3)}.$$

Theorem 6.1 (Local sup estimates ([11])). *Assume that* $\alpha \geq 0$ *and*

$$0 < p + m - 2 < q < p + m - 2 + \frac{p-\alpha}{N}, \tag{6.3}$$

or that $\alpha < 0$ *and*

$$0 < p + m - 2 < q < p + m - 2 + \frac{p}{N}, \tag{6.4}$$

$$\alpha(p+m-3) < (q-1)p, \tag{6.5}$$

$$|\alpha| \leq \min\{N(q-1),\ N(q-p-m+2)/(p+m-2)\}. \tag{6.6}$$

Then any nonnegative solution to (6.1) *satisfies*

$$u(x,t) \leq \gamma(T-t)^{-\mathcal{B}}, \quad |x| < (T-t)^{\frac{1}{\mathcal{H}}}/2, \tag{6.7}$$

provided $(T-t)^{\frac{1}{\mathcal{H}}} < dist(0,\partial\Omega)$*, and* $T/2 < t < T$.

Let us remark that (6.3) and (6.5) imply that $\mathcal{H} > 0$, $\mathcal{B} > 0$.

The proof of the supremum estimates of Theorem 6.1 relies on two main ingredients: suitable a priori integral estimates, and precise local L^∞ estimates given in terms of such integral norms, in the spirit, e.g., of (3.27). Then we use the first ones in the latter and get the result. Let us first sketch the proof of the following integral estimate.

Proposition 6.2 (A priori integral estimate ([11])). *Let* $N > \alpha \geq 0$, $p + m - 2 > 0$, *and* $0 < \theta < \min\{1, p+m-2\}$ *and let* u *be a nonnegative solution to* (6.1). *Then for* $0 < t < T$, $0 < \rho^{\mathcal{H}} \leq T - t$, $2\rho < dist(0,\partial\Omega)$*, we have*

$$\left(\fint_{B_\rho} u(x,t)^{1-\theta} dx \right)^{1/(1-\theta)} \leq \gamma \rho^{-\mathcal{H}\mathcal{B}}, \tag{6.8}$$

where $\gamma = \gamma(\theta, p, m, q)$. *If* $\alpha < 0$, (6.8) *is still in force, provided* (6.6) *is also satisfied.*

Proof of Proposition 6.2 (sketch). The argument is typical of equations with forcing terms. Fix $0 < \theta < \min(1, p + m - 2)$, and let $\zeta \in C^1(\boldsymbol{R}^N)$ such that $\zeta \geq 0$, $\zeta = 1$ for $x \in B_\rho$, $\zeta = 0$ for $x \notin B_{2\rho}$, and $|\nabla\zeta| \leq 2/\rho$. We choose as a testing function $u^{-\theta}\zeta^s$, $s = p(q-\theta)/(q-p-m+2) > p$, obtaining after applying in a standard way Hölder and Young inequalities,

$$\begin{aligned} \frac{d}{d\tau} \int_{\boldsymbol{R}^N} \zeta^s u(\tau)^{1-\theta} dx &\geq \frac{1-\theta}{2} \int_{\boldsymbol{R}^N} \zeta^s |x|^{-\alpha} u(\tau)^{q-\theta} dx - \gamma \rho^{N - \mathcal{H} - \frac{(p-\alpha)(1-\theta)}{q-p-m+2}} \\ &\geq \gamma_0 \rho^{-N\frac{q-1}{1-\theta} - \alpha} \left(\int_{\boldsymbol{R}^N} \zeta^s u(\tau)^{1-\theta} dx \right)^{\frac{q-\theta}{1-\theta}} - \gamma \rho^{N - \mathcal{H} - \frac{(p-\alpha)(1-\theta)}{q-p-m+2}}. \end{aligned} \tag{6.9}$$

Set then

$$y(\tau) = \frac{1}{\rho^N} \int_{\boldsymbol{R}^N} \zeta^s u(\tau)^{1-\theta} dx;$$

we get from (6.9)

$$y' \geq \gamma_0 \rho^{-\alpha} y^{\frac{q-\theta}{1-\theta}} - \gamma_1 \rho^{-\mathcal{H} - \frac{(p-\alpha)(1-\theta)}{q-p-m+2}}, \quad t < \tau < T, \tag{6.10}$$

where γ_0, γ_1 are now fixed constants for the rest of this proof.

We then consider the alternative

i) $\gamma_0 \rho^{-\alpha} y(t)^{\frac{q-\theta}{1-\theta}} \leq 2\gamma_1 \rho^{-\mathcal{H} - \frac{(p-\alpha)(1-\theta)}{q-p-m+2}}$;

ii) $\gamma_0 \rho^{-\alpha} y(t)^{\frac{q-\theta}{1-\theta}} > 2\gamma_1 \rho^{-\mathcal{H} - \frac{(p-\alpha)(1-\theta)}{q-p-m+2}}$.

If i) holds true, we immediately get

$$y(t) \leq (2\gamma_1 \gamma_0^{-1})^{\frac{1-\theta}{q-\theta}} \rho^{-\frac{(p-\alpha)(1-\theta)}{q-p-m+2}}. \tag{6.11}$$

If ii) is in force, it follows that $y' > 0$ over $(0, T)$, thus ii) is valid with $y(t)$ replaced with $y(\tau)$, $t < \tau < T$. Then we may write

$$y' \geq \frac{\gamma_0}{2} \rho^{-\alpha} y^{\frac{q-\theta}{1-\theta}}, \quad t < \tau < T,$$

whence y would blow up before $\tau = T$, unless

$$y(t) \leq \gamma \rho^{\alpha \frac{1-\theta}{q-1}} (T-t)^{-\frac{1-\theta}{q-1}}. \tag{6.12}$$

Then we select $T - t$ so that (6.12) implies (6.11), amounting to our assumption

$$\rho^{\mathcal{H}} \leq T - t.$$

□

We omit the proof of the local sup estimates, confining ourselves to the relevant statement in the case $\alpha > 0$.

Lemma 6.3 (A priori sup estimate ([11])). *Assume here $\alpha > 0$,*

$$q < p + m - 2 + \nu \frac{p - \alpha}{N}, \quad 0 < \nu < \omega + 1.$$

Let $0 < d_1 < d_2$, $0 < T_2 < T_1 < t < T$,

$$M_1 = \frac{p + m - 2 + \nu \frac{p}{N} - (\nu - \omega)}{p + m - 2 + \nu\left(\frac{p}{N} - \frac{1}{s}\right) - q} > 1,$$

$$M_2 = \frac{p + m - 2 + (\omega + q)\frac{p}{N} - q - \frac{1}{s}(\omega + p + m - 2)}{p + m - 2 + \nu\left(\frac{p}{N} - \frac{1}{s}\right) - q} > 1,$$

where $\max(1, N/p) < s < N/\alpha$ is chosen so that the denominator of M_1 is positive, and ω is large enough. Then

$$\|u\|_{\infty, B_{d_1} \times (T_1, t)} \leq d_2^{\frac{\alpha}{q-1}} (T_1 - T_2)^{-\frac{1}{q-1}} + d_2^{\frac{\alpha}{q-p-m-2}} (d_2 - d_1)^{-\frac{p}{q-p-m-2}}$$
$$+ \gamma (t - T_2)^{\frac{1}{\omega+1-\nu}} d_2^{(-\alpha + \frac{N}{s})\frac{M_1}{\omega+1-\nu}} \left(\sup_{T_2 < \tau < t} \int_{B_{d_2}} u(\tau)^\nu dx \right)^{\frac{M_2 - 1}{\omega+1-\nu}}. \tag{6.13}$$

In order to prove the sup estimate we select $\nu = 1 - \theta$ in (6.13), θ as in (6.8). Then for large enough s and ω we choose $d_1 = (T-t)^{1/\mathcal{H}}/2$, $d_2 = (T-t)^{1/\mathcal{H}}$, $T_1 = t - (T-t)/2$, $T_2 = t - (T-t)$, so that all three terms on the right-hand side of (6.13) reduce to constant multiples of $(T-t)^{-\mathcal{B}}$.
In fact in the third term there we use the integral average estimate (6.8) so that the factor $T - t$ appears with the correct power $-\mathcal{B}$. Note that the auxiliary parameters θ, s and ω do not appear in the estimated blow up rate.

6.0.1 Some results in the whole space

The methods of [11] are rather general, and in that paper were also applied to infer other results, like e.g.,

Proposition 6.4 (Nonexistence of local solutions ([11])). *Let $f(x) = (1 + |x|)^{-\alpha}$, with $\alpha < 0$, $p + m - 3 < 0$, $(q - 1)p < \alpha(p + m - 3)$ (so that $\mathcal{H}, \mathcal{B} < 0$). Assume that (6.4) is satisfied. Then the only nonnegative solution in $\boldsymbol{R}^N \times (0, T)$ to (6.1) is the trivial one $u = 0$.*

Theorem 6.5 (Bound below when $f = 1$ ([11])). *Let u be a subsolution to (6.1) defined in $\boldsymbol{R}^N \times (0, T)$. Assume that $f = 1$, and that u blows up at time $\infty > T > 0$. Then*

$$(T-t)^{\frac{1}{q-1}} \|u(t)\|_{\infty, \boldsymbol{R}^N} \geq (q-1)^{-\frac{1}{q-1}}, \quad 0 < t < T. \tag{6.14}$$

Theorem 6.6 (Global boundedness of support ([11])). *Let u be a nonnegative subsolution to the Cauchy problem for*

(6.1). *Assume that* $f = 1$, $p + m - 3 > 0$, *that*

$$u(x,t) \le c(T-t)^{-\frac{1}{q-1}}, \quad x \in \boldsymbol{R}^N, \ 0 < t < T, \tag{6.15}$$

and that $\operatorname{supp} u(x,0)$ *is bounded. Then there exists a finite constant* C *such that*

$$\operatorname{supp} u(x,t) \subset B_C, \quad \textit{for all } 0 < t < T. \tag{6.16}$$

Comment 6.7. Comparison between suitable ordinary differential equations and diffusion equations with sources was applied in [57]. In unbounded domains it was used in [5], [2], [6] and [3] to obtain a priori bounds for solutions to equations and systems with superlinear sources. Indeed, such a comparison yields the integral estimates needed in the independently proved L^1–L^∞ estimates. Clearly in this approach the choice of the correct space-time scaling is of the greatest importance.

References for this section: [2], [3], [5], [6], [11], [57].

7. Decay of Mass and Other Properties of Degenerate Parabolic Equations with Damping

7.1 The Cauchy problem with damping

Consider the following Cauchy problem set in $S_T = \boldsymbol{R}^N \times (0,T)$:

$$u_t - \operatorname{div}(u^{m-1}|\nabla u|^{p-2}\nabla u) = -|\nabla u^\nu|^q \qquad (x,t) \in S_T, \tag{7.1}$$

$$u(x,0) = u_0(x) \ge 0, \quad x \in \boldsymbol{R}^N. \tag{7.2}$$

We assume throughout that $p + m - 3 \ge 0$, $p > 1$, $1 < q < p$, $\nu q > p + m - 2$, $r > 1$, and $u_0 \in L^1(\boldsymbol{R}^N)$, with $u_0 \ge 0$. We use in what follows the notations

$$\mathcal{K} = N(p+m-3) + p, \quad \mathcal{L} = p(\nu q - 1) - q(p+m-3),$$

$$q^* = \frac{\mathcal{K} + N}{N\nu + 1}, \quad \mathcal{A} = \frac{(q^* - q)(N\nu + 1)}{\mathcal{L}}.$$

Note that $\mathcal{L} > 0$ under our assumptions.

Definition 7.1. A weak solution to (7.1)–(7.2) is a function $u \in L^\infty_{\text{loc}}(\boldsymbol{R}^N \times (0,\infty))$ with $u \in C((0,\infty), L^2_{\text{loc}}(\boldsymbol{R}^N))$, $|\nabla u^\lambda|^p$, $|\nabla u^\nu|^q \in L^1_{\text{loc}}(\boldsymbol{R}^N \times (0,\infty))$, where $\lambda = (p+m-2)/(p-1)$, such that

$$\int_0^\infty \int_\Omega \left\{ -u \frac{\partial \zeta}{\partial t} + u^{m-1}|\nabla u|^{p-2}\nabla u \nabla \zeta + |\nabla u^\nu|^q \zeta \right\} dxdt = 0, \tag{7.3}$$

for all $\zeta \in C^1(\boldsymbol{R}^N \times (0,\infty))$ whose support is a compact subset of $\boldsymbol{R}^N \times (0,\infty)$. In addition we require $u(t) \to u_0$ as $t \to 0$ in $L^1_{\text{loc}}(\boldsymbol{R}^N)$.

Let us begin with some estimates for large times.

Theorem 7.2 (Finite speed of propagation ([18])). *Let* $u \ge 0$ *be a solution of* (7.1)–(7.2)*, with* $p + m - 3 > 0$ *and* $\operatorname{supp} u_0 \subset B_{\rho_0}$, $0 < \rho_0 < +\infty$. *Then we have for large enough* t

$$Z(t) := \inf\{\rho > 0 \mid u(x,t) = 0, \quad |x| > \rho\} \le \gamma t^{\frac{\nu q - (p+m-2)}{\mathcal{L}}}. \tag{7.4}$$

Theorem 7.3 (Mass decay ([18])). *Under the same assumptions of Theorem 7.2 we have*

$$\text{if } q < q^*: \quad \|u(t)\|_{1,\boldsymbol{R}^N} \le \gamma t^{-\mathcal{A}}, \tag{7.5}$$

$$\text{if } q = q^*: \quad \|u(t)\|_{1,\boldsymbol{R}^N} \le \gamma |\log t|^{-\frac{1}{\nu q - 1}}. \tag{7.6}$$

Remark 7.4. 1) It is important that the constants γ in Theorems 7.2 and 7.3 do not depend on the initial data.
2) In the case $q > q^*$ on combining (7.5) with the known estimate

$$\|u(t)\|_{\infty,\boldsymbol{R}^N} \le \gamma \sup_{t/4<\tau<t} \|u(t)\|_{1,\boldsymbol{R}^N}^{\frac{p}{\mathcal{K}}} t^{\frac{N}{\mathcal{K}}},$$

we obtain for large t the decay rate predicted by self-similar solutions, i.e.,

$$\|u(t)\|_{\infty,\boldsymbol{R}^N} \le \gamma t^{-\frac{p-q}{\mathcal{L}}}.$$

Even in the case of solutions without bounded support we may prove the following.

Theorem 7.5 (Mass decay ([18])). *Let* u *be a solution to* (7.1)–(7.2)*, with* $q < q^*$. *The for large* t *we have*

$$\|u(t)\|_{1,\boldsymbol{R}^N} \le \gamma \int_{\{|x|>R(t)\}} u_0(x)dx + \gamma t^{-\mathcal{A}}, \quad R(t) = t^{\frac{\nu q-(p+m-2)}{\mathcal{L}}}, \tag{7.7}$$

provided either $\nu = (p+m-2)/(p-1)$ *or* $p = 2$, $N \ge 2$.

On the contrary, for supercritical q we do not have decay of mass.

Theorem 7.6 (Asymptotically positive mass ([18])). *Let u be a solution to* (7.1)–(7.2)*, with $q > q^*$. Then*

$$\|u(t)\|_1 \ge c > 0, \quad t > 0. \tag{7.8}$$

Here c is a constant depending also on u_0.

Proof of Theorem 7.5 (sketch). Let us prove (7.7) when $\nu = (p+m-2)/(p-1)$.
The starting point of the proof of the decay of mass is to represent the total mass as follows

$$\int_{\boldsymbol{R}^N} u(x,t)dx = \int_{\{|x|<2R\}} u(x,t)dx + \int_{\{|x|\ge 2R\}} u(x,t)dx =: E_1 + E_2,$$

for a free parameter R which will be chosen presently. On applying Hölder, Gagliardo–Nirenberg and Young inequalities we have

$$\begin{aligned} E_1 &\le \gamma \Big(\int_{\boldsymbol{R}^N} u(t)dx \Big)^{\frac{1}{\nu q}} R^{\frac{N(\nu q-1)}{\nu q}} \\ &\le \gamma \Big(\int_{\boldsymbol{R}^N} |\nabla u^\nu|^q dx \Big)^{\frac{\alpha}{\nu q}} \|u(t)\|_{1,\boldsymbol{R}^N}^{1-\alpha} R^{\frac{N(\nu q-1)}{\nu q}} \\ &= \gamma \Big(-\frac{d}{dt} \|u(t)\|_{1,\boldsymbol{R}^N} \Big)^{\frac{\alpha}{\nu q}} \|u(t)\|_{1,\boldsymbol{R}^N}^{1-\alpha} R^{\frac{N(\nu q-1)}{\nu q}} \\ &\le \frac{1}{2} \|u(t)\|_{1,\boldsymbol{R}^N} + \gamma \Big(-\frac{d}{dt} \|u(t)\|_{1,\boldsymbol{R}^N} \Big)^{\frac{1}{\nu q}} R^{\frac{N(\nu q-1)+q}{\nu q}}. \end{aligned}$$

Here $\alpha = N(\nu q-1)/[N(\nu q-1)+q]$, and we applied the equation (7.1).
On the other hand, on multiplying the equation by a suitable cut off function and integrating by parts, we obtain that

$$E_2 \le \gamma \int_{\{|x|>R\}} u_0(x)dx + \gamma T R^{N-\frac{q}{q-p+1}} =: E_3;$$

note that in our case the assumptions lead us to $q > p-1$, and that we work in $t \in (0,T)$ for any fixed $T > 0$. Next select

$$R = R(T) = T^{\frac{(q-p+1)(p+m-2)}{(p-1)\mathcal{H}}}.$$

On combining the estimates above for E_1 and E_2, we get for $0 < t < T$

$$F(t) := \|u(t)\|_1 - 2E_3 \le \Big(-\frac{dF}{dt} \Big)^{\frac{1}{\nu q}} R(T)^{\frac{N(\nu q-1)+q}{\nu q}}.$$

The sought after estimate (at $t = T$) follows from an analysis of this nonlinear ordinary differential equation, where we may assume $F(T) > 0$ (otherwise the result follows trivially). □

Proof of Theorem 7.2 (sketch). Let us try to sketch at least how the interplay between diffusion and damping is used in terms of integral estimates.
Define the radii $r_i = 2\rho(1-2^{-i-1})$, $i = 0, 1, \dots$, where $\rho > 2\rho_0$ and the sets $U_i = \{|x| > r_i\}$.

By means of calculations similar to the ones showed in the proof of Theorem 3.3, starting from Caccioppoli inequality we arrive at

$$\begin{aligned} y_{i+1} &:= \sup_{0<\tau<t} \int_{U_{i+1}} u^{1+\theta} dx + \int_0^t \int_{U_{i+1}} u^{m+\theta-2} |\nabla u|^p dx d\tau \\ &\quad + \int_0^t \int_{U_{i+1}} u^\theta |\nabla u^\nu|^q dx d\tau + \frac{2^{ip}}{\rho^p} \int_0^t \int_{U_{i+1}\setminus U_{i+2}} u^{p+m+\theta-2} dx d\tau \\ &\le \gamma \frac{2^{ip}}{\rho^p} t^{\frac{(1+\theta)p}{\mathcal{K}_\theta}} y_i^{1+\frac{p(p+m-3)}{\mathcal{K}_\theta}}. \end{aligned} \tag{7.9}$$

Here $\mathcal{K}_\theta = N(p+m-3) + (\theta+1)p$, for a suitable $\theta > 0$.

Next we use the damping term. Since we know that the support of u is bounded, we have the Poincaré inequality

$$\int_{U_i} w^q dx \le \gamma Z(t)^q \int_{U_i} |\nabla w|^q dx,$$

where $w = u^{(\nu q+\theta)/q}$. Then, by Hölder inequality

$$\int_0^t \int_{U_i\setminus U_{i+1}} w^{\frac{q(p+m+\theta-2)}{\nu q+\theta}} dx d\tau \le \gamma \left(\int_0^t \int_{U_i} w^q dx d\tau \right)^{\frac{p+m+\theta-2}{\nu q+\theta}} (t\rho^N)^{\frac{\nu q-(p+m-2)}{\nu q+\theta}}$$
$$\le \gamma (t\rho^N)^{\frac{\nu q-(p+m-2)}{\nu q+\theta}} Z(t)^{q\frac{p+m+\theta-2}{\nu q+\theta}} y_i^{\frac{p+m+\theta-2}{\nu q+\theta}}. \tag{7.10}$$

Thus, Caccioppoli inequality and (7.10) imply

$$y_{i+1} \le \gamma 2^{ip} t^{\frac{\nu q-(p+m-2)}{\nu q+\theta}} \rho^{N\frac{\nu q-(p+m-2)}{\nu q+\theta}-p} Z(t)^{q\frac{p+m+\theta-2}{\nu q+\theta}} y_i^{\frac{p+m+\theta-2}{\nu q+\theta}}. \tag{7.11}$$

Let

$$a = \frac{\mathcal{K}_\theta}{\mathcal{K}_\theta + p(p+m-3)}, \quad A = [t^{\frac{(1+\theta)p}{\mathcal{K}_\theta}} \rho^{-p}]^a,$$
$$b = \frac{\nu q+\theta}{p+m+\theta-2} > 1, \quad B = [(t\rho^N)^{\frac{\nu q-(p+m-2)}{\nu q+\theta}} \rho^{-p} Z(t)^{q\frac{p+m+\theta-2}{\nu q+\theta}}]^b.$$

Then from (7.9) and (7.11) it follows that

$$\frac{y_{i+1}^a}{A} + \frac{y_{i+1}^b}{B} \le \gamma C^i y_i,$$

for a suitable $C > 1$. On applying Young inequality we obtain

$$\frac{y_{i+1}^{\epsilon_1}}{A^{\epsilon_1} B^{1-\epsilon_1}} \le \gamma C^i y_i, \tag{7.12}$$

where $\epsilon_1 = b/(b+1-a) < 1$. Therefore from the iterative result [58, Lemma 5.6, Chapter 2] we conclude that $y_i \to 0$ if

$$(y_0 B)^{\frac{1-a}{b}} A \le \gamma_0. \tag{7.13}$$

This yields that $u(x,t) = 0$ for $|x| > 2\rho$. Finally we derive a careful bound of y_0 in terms of t, ρ, $Z(t)$, which replaced in (7.13) implies the stated estimate of $Z(t)$; since this step, which makes use again of the damping term, is nontrivial, we omit it here. □

7.2 The Cauchy problem with damping and blow up

We consider next the existence of a priori global in time bounds of solutions to the Cauchy problem for the equation

$$u_t - \operatorname{div}(u^{m-1}|\nabla u|^{p-2}\nabla u) = u^r - \epsilon |\nabla u^\nu|^q, \tag{7.14}$$

for $r > 1$, $\epsilon > 0$ and p, m, q as above. We have to introduce the threshold

$$r^* := \frac{q}{p-q}[\nu q - (p+m-2)].$$

We only state the following results.

Theorem 7.7 (A priori bounds of global solutions ([18])). *Let u be a solution to* (7.14), (7.2), *which can be approximated by bounded subsolutions. Assume $\nu = (p+m-2)/(p-1)$, $r > p+m-2$, $r > r^*$, $q < q^*$, $\epsilon = 1$. Moreover, assume that*

$$\sup_{\rho>1} \rho^{\frac{\mathcal{A}\mathcal{L}}{\nu q-(p+m-2)}} \int_{|x|>\rho} u_0(x) dx < +\infty, \tag{7.15}$$

and $\|u_0\|_{\beta,\mathbf{R}^N} < \gamma_0$ for a suitable γ_0 and $\beta > N(r-p-m+2)/p$. Then the following bound is valid:

$$\|u(t)\|_{\infty,\mathbf{R}^N} \le \gamma t^{-\frac{p-q}{\mathcal{L}}}, \quad t > 1. \tag{7.16}$$

Theorem 7.8 (Blow up ([18])). *Assume that $r = r^*$, $q < q^*$, $0 < \epsilon < \bar{\epsilon}$, where $\bar{\epsilon}(N, \nu, p, m, q) > 0$. Then any nontrivial solution to* (7.14), (7.2) *blows up in a finite time.*

Since equation (7.14) has a positive explicit stationary solution (for a certain $\epsilon > 0$), the requirement $\epsilon < \bar{\epsilon}$ in Theorem 7.8 seems to be not simply technical.

7.3 The Cauchy–Dirichlet problem with damping in cone-like domains

We consider here again domains of the class $\mathcal{D}(H, M)$ introduced in Sect. 4.1. In [12], still in the case of cone-like domains, we study also the Cauchy–Dirichlet problem for the equation with a damping term

$$\frac{\partial u}{\partial t} - \Delta u^m = -a(x)b(t)|\nabla u^\nu|^q, \quad (x,t) \in \Omega \times (0, +\infty), \tag{7.17}$$

for $m \geq 1$, $q \in (1,2)$, $\nu q > m$, while initial and boundary data are still prescribed as in (4.19)–(4.20). In what follows we assume that $a(x) = a(|x|)$ and $b(t)$ are strictly positive functions, satysfing

$$a(s), \quad s^q/a(s), \quad \text{are nondecreasing for } s > 1, \tag{7.18}$$

and, for a given $c \in (0,1)$,

$$b(t), \quad t^c/b(t)^{m/(\nu q - m)}, \quad \text{are nondecreasing for } t > 1. \tag{7.19}$$

Denote for $s > 0$

$$\Phi(s) = [s^{2(\nu q-1)-q(m-1)} a(s)^{m-1}]^{1/(\nu q-m)}, \quad \theta = tb(t)^{-\frac{m-1}{\nu q-m}},$$

$$\widetilde{\mu}(t) = \left[\frac{\Phi^{(-1)}(\theta)^{M(\nu q-1)+q}}{a(\Phi^{(-1)}(\theta))b(t)t}\right]^{1/(\nu q-1)}.$$

Recall that in the setting of cone-like domains, we denote also $\beta = M(m-1) + 2$.

Theorem 7.9 (Moment bound ([12]))**.** *Let $m > 1$, u be a solution of* (7.17), (4.19)–(4.20) *in* $\Omega \times (0, +\infty)$*, and let* $\operatorname{supp} u_0$ *be bounded. Assume also that*

$$|\nabla H(x) \cdot x| \leq \gamma H(x), \quad |x| \geq 1. \tag{7.20}$$

Then a time $t_0 > 0$ depending on u_0 exists such that for $t > t_0$ both

$$\mu(t) := \int_\Omega u(x,t)H(x)dx \leq \gamma \widetilde{\mu}(t), \tag{7.21}$$

and

$$\mu(t) \leq \gamma \left(\int_1^t a(\tau^{\frac{1}{\beta}}) b(\tau) \tau^{-\beta(M(\nu q-1)+q)} d\tau \right)^{-1/(\nu q-1)} \tag{7.22}$$

hold true. Moreover

$$\|u(t)\|_{\infty,\Omega} \leq \gamma \widetilde{\mu}(t)^{2/\beta} t^{-M/\beta}, \quad t > t_0. \tag{7.23}$$

Theorem 7.10 (No moment decay in standard cones ([12]))**.** *Let u be a solution of* (7.17), (4.19)–(4.20) *in* $\Omega \times (0, +\infty)$*, where Ω is a standard cone (see Example 4.5). Let $a(s) = s^\alpha$ for $s > 1$, $\alpha \in [0, q)$, and $b(t) = t^\chi$, $t > 1$; assume also that* $\operatorname{supp} u_0$ *is bounded and that*

$$q > q^* := (M(m + \chi(m-1)) + 2(\chi+1) + \alpha)/(M\nu + 1). \tag{7.24}$$

Then suitable $t_1 > 0$ and $c > 0$ depending on u_0 exist, such that

$$\mu(t) > c > 0, \quad t > t_1. \tag{7.25}$$

More generally we can prove the next theorem.

Theorem 7.11 (Moment bound in standard cones ([12]))**.** *Let u be a solution of* (7.17), (4.19)–(4.20) *in* $\Omega \times (0, +\infty)$*, where Ω is a standard cone. Assume that u can be approximated by solutions with bounded support. Then*

$$\mu(t) \leq \gamma \int_{\Omega \cap \{|x| > \Phi^{(-1)}(t)\}} u_0(x)H(x)dx + \gamma \widetilde{\mu}(t), \quad t > 0. \tag{7.26}$$

Remark 7.12 (Standard cones ([12])). If Ω, $a(s)$ and $b(t)$ are as in Theorem 7.10, then from Theorems 7.9 and 7.10 it follows that $\mu(t) \to 0$ as $t \to +\infty$ if $q \leq q^*$, while $\mu(t) > c$ if $q > q^*$, where q^* is defined in (7.24). Moreover, for large t we have for $h_1 = 2(\nu q-1) + (\alpha - q)(m-1)$,

$$\mu(t) \leq \gamma t^{-(q^*-q)(M\nu+1)/h_1}, \quad q < q^*, \tag{7.27}$$

$$\mu(t) \leq \gamma [\ln t]^{-1/(\nu q-1)}, \quad q = q^*, \tag{7.28}$$

$$\|u(t)\|_{\infty,\Omega} \leq \gamma t^{-(2+2\chi+\alpha-q)/h_1}, \quad q < q^*. \tag{7.29}$$

Remark 7.13 (Octants ([12])). According to the results and methods in [12] we also have the following results in octants-like domains, as in Example 4.5.2, i.e., for $2 \le i \le N$

$$\Omega = \{(x_1, \dots, x_N) \in \boldsymbol{R}^N \mid x_1, x_2, \dots, x_i > 0\}, \quad H(x) = x_1 \dots x_i.$$

Let $m > 1$, $a = 1$, $b = 1$, u_0 with bounded support and

$$q^* = \frac{(N+i)m + 2}{(N+i)\nu + 1}.$$

Then if $q \le q^*$ we have $\mu(t) \to 0$ as $t \to +\infty$; if instead $q > q^*$ we have $\mu(t) \ge c > 0$ for large t. More exactly for large t

$$\mu(t) \le \gamma t^{-(q^*-q)[(N+i)\nu+1]/h_0}, \quad q < q^*; \tag{7.30}$$

$$\mu(t) \le \gamma [\ln t]^{-1/(\nu q - 1)}, \quad q = q^*; \tag{7.31}$$

here $h_0 = 2(\nu q - 1) - q(m-1)$.

Comment 7.14. Behavior of mass plays an important role in the investigation of the asymptotic behavior for large times of solutions of parabolic equations with damping terms. It is well known (see Theorem 3.3) that for solutions or subsolutions to the Cauchy problem for a homogeneous doubly degenerate parabolic equation we have for any $t > 0$

$$\|u(t)\|_{\infty, \boldsymbol{R}^N} \le \gamma t^{-\frac{N}{\mathcal{K}}} \left(\sup_{t/4 < \tau < t} \int_{\boldsymbol{R}^N} u(x,\tau) dx \right)^{\frac{p}{\mathcal{K}}}. \tag{7.32}$$

For the same Cauchy problem with the damping term as in Sect. 7.1 the mass

$$\int_{\boldsymbol{R}^N} u(x,\tau) dx$$

decays in time, but not necessarily to 0. Namely our results answer, in terms of critical exponents, the question of when the mass decays to 0. As a consequence of this result and of (7.32) we have faster decay of $\|u(t)\|_{\infty, \boldsymbol{R}^N}$ as $t \to +\infty$. This kind of estimates also are helpful in discussing whether the fundamental solution does exist. To the best of our knowledge the first result in this direction for the heat equation with gradient damping was proven in [21]; see also [24]. We also quote on the subject of equations with nonlinear source terms depending on the gradient [3], [27], [22], [25], [26], [42], [23], [30], [40], [39], [59], [47], [29], [20], [48], [32].

As to the Cauchy–Dirichlet probem, we investigated in [12] for the first time in this framework, as far as we know, the role played by a suitable moment depending on the geometry of the domain; this role amounts to the one of global mass in the whole space.

References for this section: [3], [12], [18], [20], [21], [22], [23], [24], [25], [26], [27], [29], [30], [32], [39], [40], [42], [47], [48], [59].

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