Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign [1]

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Abstract

We study the global approximate controllability properties of a one dimensional semilinear reaction-diffusion equation governed via the coefficient of the reaction term. It is assumed that both the initial and target states admit no more than finitely many changes of sign. Our goal is to show that any target state $u^* \in H_0^1(0,1)$, with as many changes of sign in the same order as the given initial data $u_0 \in H_0^1(0,1)$, can be approximately reached in the $L^2(0,1)$-norm at some time $T > 0$. Our method employs shifting the points of sign change by making use of a finite sequence of initial-value pure diffusion problems.

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1 Introduction

Our main goal in this paper is to study the global approximate controllability properties of the following semilinear Dirichlet boundary value problem

$$\begin{cases} u_t \;=\; u_{xx} \;+\; v(x,t)u \;+\; f(u) & \text{in} \quad Q_T = (0,1)\times(0,T)\,, \quad T>0, \\ u(0,t) = u(1,t) = 0, & \qquad\qquad t \in (0,T), \\ u\;|_{t=0} = u_0 \in H_0^1(0,1). \end{cases} \tag{1}$$

Here $v \in L^\infty(Q_T)$ is a control function, which affects the reaction rate of the process described by (1). The nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is assumed to be a Lipschitz function satisfying $f(0) = 0$.

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Let us recall that, in general terms, an evolution system is called globally approximately controllable in a given space H at time $T > 0$, if any initial state in H can be steered into any neighborhood of any desirable target state at time T, by selecting a suitable control.

Historically, the concept of controllability emerged in the context of linear ordinary diferenetial equations and was motivated by numerous engineering applications. Then it was extended to various linear partial differential equations governed by *additive* locally distributed (i.e., supported on a bounded subdomain of the space domain), lumped (acting at a point), and boundary controls (see Fattorini in [15], Fattorini and Russell in [16], and Zabczyk in [33]). Methodologically, these studies were typically based on the *linear* duality pairing technique between the control-to-state mapping at hand and its dual observation map. When this mapping is nonlinear, as it happens in (1), the aforementioned approach does not apply and the above-stated concept of controllability becomes, in general, unachievable. For example, the above control system cannot be steered to any nonzero target starting from the origin $u_0 = 0$ (see more about that in Remark 2.1 below).

It is well-known that (1) can be linked to various applied reaction-diffusion models such as chemical reactions, nuclear chain reactions, and biomedical models (see [24], and the references therein). More generally, reaction-diffusion equations or systems describe how the concentration of one or more substances changes under the influence of some processes such as local chemical reactions, where substances are transformed into each other, and diffusion which causes substances to spread out in space.

Unfortunately, additive controls (see, e.g., [1] and [10]) are unfit to treat such problems because, for example, they would require inputs with high energy levels or they are not available due to the physical nature of the process at hand. On the other hand, an approach based on multiplicative controls, where the coefficient v in (1) is used to change the main physical characteristics of the system at hand, seems realistic.

In the area of multiplicative controllability for partial differential equations we would like to mention the pioneering work [3] by Ball, Marsden and Slemrod establishing the approximate controllability of the rod and wave equations, based on the implicit nonharmonic Fourier series approach. Further principal contributions to the field of the multiplicative controllability of linear and semilinear parabolic and hyperbolic equations and of a number of swimming models were made by Khapalov, see [22]-[24] and the references therein. Substantial progress has also been made in the study of the controllability properties of the Schrödinger equation by Beauchard, Coron, Boscain et al., Ervedoza and Puel, Nersesyan, see [4], [6], [13], [14] and [28] and the references therein. Let us also mention along these lines the work by Beauchard [5] for the beam model, the works by Lin et al. [26] and [27], and the work by Fernandez and Khapalov [17] on the bilinear controllability of parabolic equations.

In [22], Khapalov studied the global nonnegative approximate controllability of the one dimensional nondegenerate semilinear convection-diffusion-reaction equation governed in a bounded domain via bilinear control. Similar results were obtained for degenerate parabolic equations by Cannarsa and Floridia in [8],[9],[18] and [19]. In [11] Cannarsa and Khapalov established an approximate controllability property for nondegenerate linear equations in suitable classes of functions that change sign.

In this paper, we are interested in the multiplicative controllability of the semilinear reaction-diffusion system (1) when both the initial and target states admit a finite number of points of sign change. This fact introduces substantial differences with respect to the above works. Indeed, on the one hand, we manage to extend all the approximate controllability results of [11] to semilinear equations as well as those of [22] to initial/target states that may change sign. On the other hand, here we introduce a new technique of proof.

In [11], an implicit "*continuation argument*" was employed to justify the fact that one can always continue to move the points of sign change until their target positions have been reached.

Such a technique was mainly qualitative but sufficient to obtain the conclusion due to the linear structure of the equation at hand. In this paper, on the contrary, a more quantitative approach is needed because the equation of interest is nonlinear.

Indeed, we give an explicit construction of the controls required for the steering process. Such controls are, essentially, obtained by splitting $[0,T]$ into finitely many time intervals

$$[0,T] = [0,S_1] \cup [S_1,T_1] \cup \cdots \cup [T_{N-1},S_N] \cup [S_N,T_N] \cup [T_N,T]$$

on which two alternative actions are applied: on $[S_k,T_k]$ we choose suitable initial data, w_k, in pure diffusion problems ($v \equiv 0$) to move the points of sign change to their desired location, whereas on $[T_{k-1},S_k]$ we use piecewise static multiplicative controls v_k to attain such w_k's as intermediate final conditions. More precisely, on $[S_k,T_k]$ we make use of the boundary problems

$$\begin{cases} w_t \ = \ w_{xx} \ + \ f(w), & \text{in} \quad (0,1) \times [S_k,T_k], \\ w(0,t) = w(1,t) = 0, & t \in [S_k,T_k], \\ w \mid_{t=S_k} = w_k(x), & w_k''(x) \mid_{x=0,1} = 0, \end{cases}$$

where the w_k's are viewed as control parameters to be chosen to generate suitable curves of sign change, which have to be continued along all the N time intervals $[S_k,T_k]$ until each point has reached the desired final position. In order to fill the gaps between two successive $[S_k,T_k]$'s, on $[T_{k-1},S_k]$ we construct v_k that steers the solution of

$$\begin{cases} u_t \ = \ u_{xx} \ + \ v_k(x,t)u \ + \ f(u), & \text{in} \quad (0,1) \times [T_{k-1},S_k], \\ u(0,t) = u(1,t) = 0, & t \in [T_{k-1},S_k], \\ u \ \mid_{t=T_{k-1}} = u_{k-1} + r_{k-1}, & \end{cases}$$

from $u_{k-1}+r_{k-1}$ to w_k, where u_{k-1} and w_k have the same points of sign change, and $\|r_{k-1}\|_{L^2(0,1)}$ is small. The fact that such a process can be completed within a finite number of steps is an important point of the proof. It follows from precise estimates that guarantee that the sum of the distances of each branch of the null set of the resulting solution of (1) from its target points of sign change decreases at a linear-in-time rate for curves which are still far away from their corresponding target points, while the error caused by the possible displacement of points already near their targets is negligible.

We believe that the techniques of this paper could be useful to study multidimensional diffusive systems on Riemannian manifolds of low dimension or special structure.

Motivations and future perspectives

In this section we present some applications. The following nuclear model is a typical example of the applications we plan to study, that is, a reaction-diffusion model in a fissionable material (see Section 2.7 of [31]). By shooting neutrons into a uranium nucleus it may happen that the nucleus breaks into two parts, releasing other neutrons already present in the nucleus and causing a chain reaction. At a macroscopic level, the free neutrons diffuse like a chemical in a porous medium, where reaction and diffusion are competing. Some macroscopic aspects of this phenomenon can be described by means of a simplified reaction-diffusion model, like in (1), where u is the neutron density and the multiplicative coefficient v is the fission rate.

Our study of the reaction-diffusion models is also motivated by mathematical models of tumor growth (see, e.g., Friedman in [21] and Perthame in [29]). There are three distinct main stages in the growth of a tumor (see [30], [32] and [2]) before it becomes so large that it causes patients to die or reduces permanently their quality of life: avascular (tumors without blood vessels), vascular, and metastatic. From a clinical point of view, vascular and metastatic tumor growth are what cause the patient to die. So modeling and understanding these processes is crucial for cancer therapy. Nevertheless, avascular tumor growth is much simpler to model mathematically, and yet contains many of the phenomena that one needs to address also in a general model of vascular or metastatic tumor growth. In the review [30], Roose, Chapman and Maini describe a continuum mathematical model of avascular tumor growth. This model consist of reaction-diffusion-convection equations and was introduced by Casciari, Sotirchos and Sutherland in [12]. Mathematical models describing continuum tumor cell populations and their development classically consider the interactions between the cell number density and one or more chemical species that provide nutrients or influence the cell cycle events. The model introduced in [12] can reduce to the following simplified system

$$\frac{\partial u_i}{\partial t} = \frac{\partial^2 u_i}{\partial^2 x} + v_i(x,t)u_i + f(u_i), \quad i = 1, \ldots, N \;\; (N \in \mathbb{N}),$$

where u_i are the concentrations of the chemical species and v_i are the net rate of consumption/production of the chemical species both by the tumor cells and due to chemical reactions with other species. In future works we will address controllability issues for systems of reaction-diffusion equations of the type outlined in the survey paper [7] (see Section 7, *"Control problems"*).

Outline of the paper

In Section 2, we give the precise formulation of the problem and state our approximate controllability result for system (1) (Theorem 1) together with some of its consequences.
In Section 3, we explain the structure of the proof and, admitting two essential technical tools (Theorem 2 and Theorem 3), we proceed with the proof of Theorem 1.
Section 4 deals with the proof of Theorem 2: a controllability result for pure diffusion problems.
Finally, Section 5 is devoted to the proof of Theorem 3: a smoothing result intended to attain suitable intermediate data while preserving the already-reached points of sign change.

2 Problem formulation and results

Our main goal in this paper is to study the global approximate controllability properties of the semilinear Dirichlet boundary value problem (1)

$$\begin{cases} u_t = u_{xx} + v(x,t)u + f(u) & \text{in} \quad Q_T = (0,1) \times (0,T), \quad T > 0, \\ u(0,t) = u(1,t) = 0, & \qquad t \in (0,T), \\ u \mid_{t=0} = u_0 \in H_0^1(0,1). & \end{cases}$$

Here $v \in L^\infty(Q_T)$ is a bilinear control. The nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is supposed to be a Lipschitz function with $f(0) = 0$, differentiable at 0 and L will denote a Lipschitz constant for f, that is,

$$\mid f(u') - f(u) \mid \leq L \mid u' - u \mid, \quad \forall u, u' \in \mathbb{R}. \tag{2}$$

Remark 2.1 We note that system (1) cannot be steered anywhere from the origin. Moreover, if $u_0(x) \geq 0$ in $(0,1)$, then the strong maximum principle demands that the respective solution to (1) remains nonnegative at any moment of time, regardless of the choice of v. This means that system (1) cannot be steered from any such u_0 to any target state which is negative on a nonzero measure set in the space domain. We remark that the strong maximum principle for linear parabolic PDEs (see, e.g, Chapter 2 in [20], p. 34) can be extend to semilinear parabolic system (1). Indeed, since $f(0)=0$ and $f(u)$ is differentiable at 0, the term $f(u(x,t))$ in (1) can be represented as $\beta(x,t)u(x,t)$, where $\beta = \dfrac{f(u)}{u} \in L^\infty(Q_T)$.

Let us start with the well-posedness for the system (1).

Functional setting and Well-posedness

Hereafter, we use the standard notation for Sobolev spaces, in particular,

$$\begin{aligned} H^1(0,1) &= \{\phi \in L^2(0,1) \mid \ \phi_x \in L^2(0,1)\} \\ H^1_0(0,1) &= \{\phi \in H^1(0,1) \mid \phi(0)=\phi(1)=0\} \\ H^2(0,1) &= \{\phi \in H^1(0,1) \mid \ \phi_{xx} \in L^2(0,1)\}. \end{aligned}$$

By classical well-posedness results (see, for instance, Theorem 6.1 in [25], pp. 466-467) problem (1) with initial data $u_0 \in L^2(-1,1)$ admits a unique solution

$$u \in L^2(0,T;H^1_0(0,1)) \cap C([0,T];L^2(0,1)).$$

Furthermore, if $u_0 \in H^1_0(0,1)$, then the solution u of problem (1) satisfies

$$u \in H^1(0,T;L^2(0,1)) \cap C([0,T];H^1_0(0,1)) \cap L^2(0,T;H^2(0,1)).$$

Problem formulation

In this paper, we assume that $u_0 \in H^1_0(0,1)$ has *finitely many zeros*, that is, there exist points

$$0 = x^0_0 < x^0_1 < \cdots < x^0_n < x^0_{n+1} = 1$$

such that

$$u_0(x) = 0 \Longleftrightarrow x = x^0_l, \ l = 0,\ldots,n+1.$$

Moreover, we assume that the interior zeros (x^0_l, $l=1,\ldots,n$) are *points of sign change*, that is, for $l=1,\ldots,n$,

$$u_0(x)u_0(y) < 0, \quad \forall x \in \left(x^0_{l-1}, x^0_l\right), \forall y \in \left(x^0_l, x^0_{l+1}\right).$$

We will refer to such functions u_0 as the ones with *finitely many changes of sign*.

Our goal is to show that any target $u^* \in H^1_0(0,1)$, with as many changes of sign *in the same order* as the given u_0, can be approximately reached in the $L^2(0,1)$-norm at some time $T>0$. By the above expression we mean that, denoting by x^*_l, $l=0,\ldots,n+1$, the zeros of u^*, we have

$$u_0(x)u^*(y) > 0, \quad \forall x \in \left(x^0_{l-1}, x^0_l\right), \forall y \in \left(x^*_{l-1}, x^*_l\right), \text{ for } l = 1,\ldots,n+1.$$

Remark 2.2 The matching of the initial and target states seems optimal. Indeed, the strong maximum principle (see Remark 2.1), applied on the subdomain of Q_T delimited by any two adjacent curves of sign change (in the sense of Definition 4.1 and Lemma 4.2.), prevents the appearance of new zeros of $u(x,t)$ within this area.

Let us start with the following definition.

Definition 2.1 *We say that a function $v \in L^\infty(Q_T)$ is piecewise static, if there exist $m \in \mathbb{N}$, $c_k(x) \in L^\infty(0,1)$ and $t_k \in [0,T]$, $t_{k-1} < t_k$, $k = 1, \dots, m$ with $t_0 = 0$ and $t_m = T$, such that*

$$v(x,t) = c_1(x)\mathbb{1}_{[t_0,t_1]}(t) + \sum_{k=2}^{m} c_k(x)\mathbb{1}_{(t_{k-1},t_k]}(t),$$

where $\mathbb{1}_{[t_0,t_1]}$ and $\mathbb{1}_{(t_{k-1},t_k]}$ are the indicator function of $[t_0,t_1]$ and $(t_{k-1},t_k]$, respectively.

Here we present our main result for system (1).

Theorem 1 *Let $u_0 \in H_0^1(0,1)$. Assume that u_0 has finitely many points of sign change. Consider any $u^* \in H_0^1(0,1)$ which has exactly as many points of sign change in the same order as u_0. Then, for any $\eta > 0$ there are a $T = T(\eta, u_0, u^*) > 0$ and a piecewise static multiplicative control $v = v(\eta, u_0, u^*) \in L^\infty(Q_T)$ such that for the respective solution u to (1) the following inequality holds*

$$\|u(\cdot,T) - u^*\|_{L^2(0,1)} \leq \eta.$$

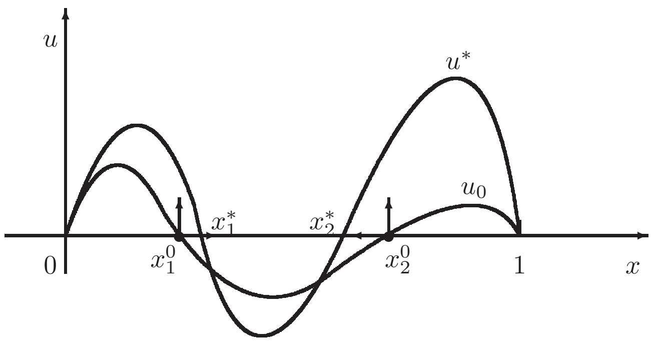

Figure 1. Control of two points of sign change.

2.1 Further results

We present in this section two results that generalize Theorem 1 and are easy consequences of such theorem.

Corollary 2.1 *Let u_0, $u^* \in H_0^1(0,1)$. Assume that u_0 and u^* have finitely many points of sign change and the amount of points of sign change of u^* is less than the one of u_0. Then, for any $\eta > 0$ there are a $T = T(\eta, u_0, u^*) > 0$ and a piecewise static multiplicative control $v = v(\eta, u_0, u^*) \in L^\infty(Q_T)$ such that for the solution u to (1) the following inequality holds*

$$\| u(\cdot,T) - u^* \|_{L^2(0,1)} \leq \eta.$$

In the following Remark 2.3 we clarify the statement of Corollary 2.1.

Remark 2.3 We explain the statement of Corollary 2.1 by the following example. Let us consider an interval $(0, x_1^0)$ of positive values of u_0 followed by an interval (x_1^0, x_2^0) of negative values of $u_0(x)$, which in turn is followed by an interval (x_2^0, x_3^0) of positive values of $u_0(x)$ and so forth. Then the merging of the respective two points of sign change x_1^0 and x_2^0 will result in one single interval $(0, x_3^0)$ of positive values.

In the following picture we describe the situation discussed in Remark 2.3 in the particular case $0 = x_0^0 < x_1^0 < x_2^0 < x_3^0 = 1$.

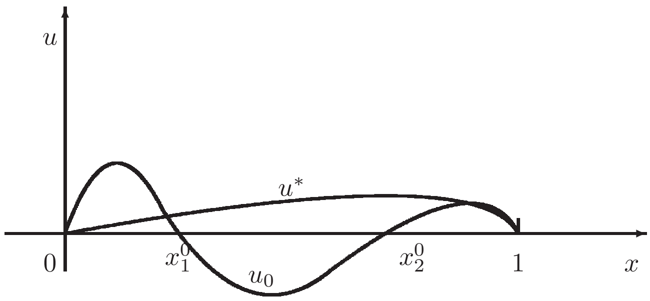

Figure 2. u_0, u^*: merging of the points of change of sign.

Proof (of Corollary 2.1). Corollary 2.1 follows from Theorem 1. Indeed, all the target states described in Corollary 2.1 can be approximated in $L^2(0,1)$ by those in Theorem 1.

It may be worth noting that the following generalized approximate controllability property can be deduced from Corollary 2.1.

Corollary 2.2 *Let u_0 and u^* be given in $L^2(0,1)$. Then, for any $\eta > 0$ there exists $u_0^\eta \in H_0^1(0,1)$ such that $\| u_0^\eta - u_0 \|_{L^2(0,1)} < \eta$, and there exist $T = T(\eta, u_0, u^*) > 0$ and a piecewise static multiplicative control $v = v(\eta, u_0, u^*) \in L^\infty(Q_T)$ such that the solution u to*

$$\begin{cases} u_t \ = \ u_{xx} \ + \ v(x,t)u \ + \ f(u) & \text{in} \quad Q_T = (0,1) \times (0,T) \\ u(0,t) = u(1,t) = 0 & \qquad t \in (0,T) \\ u(\cdot, 0) = u_0^\eta \in H_0^1(0,1) & \end{cases}$$

satisfy the following inequality

$$\|u(\cdot, T) - u^*\|_{L^2(0,1)} \ \leq \eta.$$

The proof of Corollary 2.2 is similar to one of the Corollary 2 of [11], which deals with linear systems.

3 Control Strategy for the proof of the main result

In this section, we fix the notation and we introduce a control strategy to obtain the complete proof of Theorem 1 in Section 3.3.

Let us fix a number $\vartheta \in (0,1)$ to be used in whole the paper.

Now, we recall some useful functional spaces.

Hölder continuous spaces

We define the Hölder spaces

$$C^\vartheta([0,1]) := \left\{ w \in C([0,1]) \ : \ \sup_{x,y \in [0,1]} \frac{|w(x) - w(y)|}{|x-y|^\vartheta} < +\infty \right\},$$

$$C^{2+\vartheta}([0,1]) := \left\{ w \in C^2([0,1]) \,:\, w'' \in C^{\vartheta}([0,1]) \right\}.$$

Let $Q_T = (0,1) \times (0,T)$. Let us define the following spaces of time dependent functions

$$C^{\vartheta,\frac{\vartheta}{2}}(\overline{Q}_T) := \left\{ u \in C(\overline{Q}_T) \,:\, \sup_{x,y\in[0,1]} \frac{|u(x,t)-u(y,t)|}{|x-y|^{\vartheta}} + \sup_{t,s\in[0,T]} \frac{|u(x,t)-u(x,s)|}{|t-s|^{\frac{\vartheta}{2}}} < +\infty \right\},$$

$$C^{2,1}(\overline{Q}_T) := \left\{ u : \overline{Q}_T \longrightarrow \mathbb{R} \,:\, \exists\, u_{xx}, u_x, u_t \in C(\overline{Q}_T) \right\},$$

$$C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{Q}_T) := \left\{ u \in C^{2,1}(\overline{Q}_T) \,:\, u_{xx}, u_x, u_t \in C^{\vartheta,\frac{\vartheta}{2}}(\overline{Q}_T) \right\}.$$

Notation

Given $N \in \mathbb{N}$, let us set $\mathbb{R}^N_+ = \{(a_1,\dots,a_N) \,|\, a_k \in \mathbb{R},\, a_k > 0,\, k = 1,\dots,N\}$. For every

$$(\tau_1,\dots,\tau_N) = (\tau_k)_1^N \in \mathbb{R}^N_+, \;\; (\sigma_1,\dots,\sigma_N) = (\sigma_k)_1^N \in \mathbb{R}^N_+,$$

we define

$$T_0 := 0, \qquad S_k := T_{k-1} + \sigma_k, \qquad T_k := S_k + \tau_k, \qquad k = 1,\dots,N. \quad ([3]) \tag{3}$$

Noting that $0 = T_0 < S_k < T_k \le T_N,\; k = 1,\dots,N$, we consider the following partition of $[0,T_N]$ in $2N$ intervals:

$$[0,T_N] = [0,S_1] \cup [S_1,T_1] \cup \cdots \cup [T_{N-1},S_N] \cup [S_N,T_N] = \bigcup_{k=1}^{N} (\mathcal{O}_k \cup \mathcal{E}_k), \tag{4}$$

where, for every $k = 1,\dots,N$, we have set $\mathcal{O}_k := [T_{k-1},S_k]$ and $\mathcal{E}_k := [S_k,T_k]$.

Outline and main ideas for the proof of Theorem 1

We obtain the proof of Theorem 1 in Section 3.3 using the partition introduced in (4) and applying two alternative control actions: on $[S_k,T_k]$ we choose suitable initial data, w_k, in pure diffusion problems ($v \equiv 0$) to move the points of sign change to their desired location (Section 3.1), whereas on $[T_{k-1},S_k]$ we give a smoothing result to preserve the reached points of sign change and attain such w_k's as intermediate final conditions, using piecewise static multiplicative controls v_k (Section 3.2).

3.1 Controllability for initial-value pure diffusion problems on disjoint time intervals

In this section, we outline the main idea of the proof of Theorem 1.

Let $N \in \mathbb{N}$. For any fixed $(\sigma_1,\dots,\sigma_N) \in \mathbb{R}^N_+$, let us consider a generic $(\tau_1,\dots,\tau_N) \in \mathbb{R}^N_+$ and, for $k = 1,\dots,N$, recalling (3)-(4), let us introduce the following initial value pure diffusion problems on disjoint time intervals

$$\begin{cases} w_t = w_{xx} + f(w), & \text{in } Q_{\mathcal{E}_k} = (0,1) \times [S_k,T_k], \\ w(0,t) = w(1,t) = 0, & t \in [S_k,T_k], \quad ([4]) \\ w\,|_{t=S_k} = w_k(x), & w_k''(x)\,|_{x=0,1} = 0. \end{cases} \tag{5}$$

[3] We note that $T_0 = 0$, $S_1 = \sigma_1$, $T_1 = \sigma_1 + \tau_1$, and $S_k = \sum_{h=1}^{k-1}(\sigma_h + \tau_h) + \sigma_k$, $T_k = \sum_{h=1}^{k}(\sigma_h + \tau_h)$, $\forall k = 2,\dots,N$.

[4] $w_k''(x)\,|_{x=0,1} = 0$ is a compatibility condition, see [25], pp. 452-453, where it is introduced in a more general parabolic problem.

We will consider the initial data w_k and times $\tau_k,\ k=1,\dots,N$, as control parameters, where w_k's belong to $C^{2+\vartheta}([0,1])$, with $\vartheta \in (0,1)$ fixed at the beginning of this section.

Remark 3.1 For every $k=1,\dots,N$, by a classical well-posedness result (see [25], pp. 452-453), if $w_k \in C^{2+\vartheta}([0,1])$, then any initial-value problem in (5) has a unique classical solution $W_k(x,t)$ on $\overline{Q}_{\mathcal{E}_k}$ and $W_k \in C^{2+\vartheta,1+\vartheta/2}(\overline{Q}_{\mathcal{E}_k})$.

Definition 3.1 *We call solution of* (5) *the function defined in* $(0,1)\times \bigcup_{k=1}^{N}[S_k,T_k]$ *as*

$$w(x,t)=W_k(x,t), \quad \forall (x,t)\in (0,1)\times[S_k,T_k], \ \ k=1,\dots,N,$$

where W_k, for every $k=1,\dots,N$, is the unique solution on $(0,1)\times[S_k,T_k]$ of the k^{th} problem in (5)*, with initial state w_k.*

Remark 3.2 We observe that a solution of (5) is a collection of solutions of finitely many problems which are set on disjoint time intervals. Therefore, it is independent of the choice of $(\sigma_k)_1^N$. We prefer to give Definition 3.2 for a fixed $(\sigma_k)_1^N$, just for technical purposes that will be clarified in the sequel (see Theorem 2 and Section 3.3).

Definition 3.2 *Let $u_0 \in H_0^1(0,1)$ be a function with n points of sign change. For every fixed $N\in\mathbb{N}$ and $(\sigma_k)_1^N \in \mathbb{R}_+^N$, we call a finite "family of Times and Initial Data" of* (5) *associated to u_0, a set of the form $\left\{(\tau_k)_1^N,(w_k)_1^N\right\}$ such that*

- ⋆ $(\tau_k)_1^N \in \mathbb{R}_+^N$;
- ⋆ $w_k \in C^{2+\vartheta}([0,1])$, *for all $k=1,\dots,N$, satisfies:*
  1. $w_k(0)=w_k(1)=0,\ w_k''(0)=w_k''(1)=0$;
  2. *w_1 and u_0 have the same points of sign change, in the same order as the points of sign change of u_0;*
  3. *for $k=2,\dots,N$, $w_k(\cdot)$ and $w(\cdot,T_{k-1})$ have the same points as the points of sign change, in the same order of sign change of u_0, where w is the solution of* (5).

Theorem 2 *Let $u_0\in H_0^1(0,1)$ have n points of sign change at $x_l^0\in(0,1)$, with $0=: x_0^0 < x_l^0 < x_{l+1}^0 \le x_{n+1}^0 := 1$, $l=1,\dots,n$. Let $x_l^*\in(0,1)$, $l=1,\dots,n$, be such that $0:=x_0^* < x_l^* < x_{l+1}^* \le x_{n+1}^* := 1$. Then, for every $\varepsilon>0$ there exist $N_\varepsilon\in\mathbb{N}$ and a finite family of times and initial data $\left\{(\tau_k)_1^{N_\varepsilon},(w_k)_1^{N_\varepsilon}\right\}$ such that, for any $(\sigma_k)_1^{N_\varepsilon}\in\mathbb{R}_+^N$, the solution w^ε of problem* (5) *satisfies*

$$w^\varepsilon(x,T_{N_\varepsilon})=0 \qquad \Longleftrightarrow \qquad x=x_l^\varepsilon,\ \ l=0,\dots,n+1,$$

for some points $x_l^\varepsilon\in(0,1)$, $0:=x_0^\varepsilon < x_l^\varepsilon < x_{l+1}^\varepsilon \le x_{n+1}^\varepsilon := 1$, $l=1,\dots,n$, such that

$$\sum_{l=1}^{n}|x_l^*-x_l^\varepsilon|<\varepsilon.$$

Moreover, $w^\varepsilon(\cdot,T_{N_\varepsilon})$ has the same order of sign change as u_0.

This theorem is obtained in Section 4 by proving a series of preliminar results.

3.2 Smoothing result to preserve the reached points of sign change and to obtain the intermediate data w_k

In this section we introduce a smoothing result to preserve the reached points of sign change and attain regular intermediate final conditions w_k's.

Let $N \in \mathbb{N}$. For any fixed $(\tau_1, \dots, \tau_N) \in \mathbb{R}_+^N$, let us consider a generic $(\sigma_1, \dots, \sigma_N) \in \mathbb{R}_+^N$ and, for $k = 1, \dots, N$, recalling (3)-(4), given u_{k-1}, $r_{k-1} \in H_0^1(0,1)$, $v_k \in L^\infty((0,1) \times [T_{k-1}, S_k])$, let us introduce the following problem

$$\begin{cases} u_t \ = \ u_{xx} \ + \ v_k(x,t)u \ + \ f(u) & \text{in} \quad Q_{\mathcal{O}_k} = (0,1) \times [T_{k-1}, T_{k-1} + \sigma_k], \\ u(0,t) = u(1,t) = 0, & t \in [T_{k-1}, T_{k-1} + \sigma_k], \quad ({}^5) \\ u \ |_{t=T_{k-1}} = u_{k-1} + r_{k-1} \in H_0^1(0,1). \end{cases} \tag{6}$$

Our goal is to show that, given T_{k-1}, there exists a sufficiently small $\sigma_k > 0$ such that, provided $\|r_{k-1}\|_{L^2(0,1)}$ is small, we can steer the system (6) from $u_{k-1} + r_{k-1}$ to a neighborhood of any state w_k, where w_k has the same n changes of sign as u_{k-1}, in the same order of sign change. In Section 3.3 we will apply the following result to obtain regular w_k's, that satisfy suitable properties.

Theorem 3 *Let $u_{k-1}, r_{k-1}, w_k \in H_0^1(0,1)$. Let u_{k-1} and w_k have the same n points of sign change in the same order of sign change. Then, for every $\eta > 0$ there exist a sufficiently small $\sigma_k = \sigma_k(\eta, u_{k-1}, w_k) > 0$ and a piecewise static bilinear control $v_k = v_k(\eta, u_{k-1}, w_k) \in L^\infty((0,1) \times (T_{k-1}, S_k))$ such that*

$$\|U_k(\cdot, S_k) - w_k(\cdot)\|_{L^2(0,1)} \leq \eta + C_k \|r_{k-1}\|_{L^2(0,1)},$$

where U_k is the solution of (6) on $(0,1) \times [T_{k-1}, S_k]$ and $C_k = C(u_{k-1}, w_k)$ is a positive constant.

The above theorem is proved in Section 5.

3.3 Proof of Theorem 1

Let us start this section by the following Lemma.

Lemma 3.1 *Let $u_0 \in H_0^1(0,1)$ be a function with n points of sign change. Let $\{(\tau_k)_1^N, (w_k)_1^N\}$ be a finite family of times and initial data of (5) associated to u_0, and let $w : \big(0,1\big) \times \bigcup_{k=1}^{N} [S_k, T_k] \longrightarrow \mathbb{R}$ be the solution of (5). For every $\delta > 0$, there exists $\sigma_\delta = (\sigma_k)_1^N \in \mathbb{R}_+^N$, $v_\delta \in L^\infty((0,1) \times (0, T_N))$ such that, denoting by $u_\delta : (0,1) \times [0, T_N] \to \mathbb{R}$ the solution of (1) with bilinear control v_δ, we have*

$$\|u_\delta(\cdot, T_k) - w(\cdot, T_k)\|_{L^2(0,1)} \leq \delta, \quad \forall k = 1, \dots, N. \tag{7}$$

Proof. Fix $\{(\tau_k)_1^N, (w_k)_1^N\}$ and $\delta > 0$. Let us consider the partition of $[0, T_N]$ in $2N$ intervals introduced in (4). We will show that the bilinear control v_δ has the following expression

$$v_\delta(x,t) = \begin{cases} v_k^\delta(x,t) & \text{in} \quad Q_{\mathcal{O}_k} = (0,1) \times [T_{k-1}, S_k], \quad k = 1, \dots, N, \\ 0 & \text{in} \quad Q_{\mathcal{E}_k} = (0,1) \times [S_k, T_k], \qquad k = 1, \dots, N. \end{cases}$$

[5] We recall that $S_k = T_{k-1} + \sigma_k$.

Step 1: *A useful energy estimate on* $(0,1)\times\bigcup_{k=1}^{N}[S_k,T_k]$ $(v_\delta\equiv 0)$.

In the following, for every $k=1,\dots,N$, we will consider the following problem on $Q_{\mathcal{E}_k}$

$$\begin{cases} u_t \;=\; u_{xx} \;+\; f(u), & \text{in } \; Q_{\mathcal{E}_k}=(0,1)\times[S_k,T_k],\\ u(0,t)=u(1,t)=0, & t\in[S_k,T_k],\\ u\;|_{t=S_k}=w_k+p_k, \end{cases} \tag{8}$$

where $p_k\in H_0^1(0,1)$ are given functions, and we will represent the solution of (8) as the sum of two functions $w(x,t)$ and $h(x,t)$, which solve the following problems in $Q_{\mathcal{E}_k}$

$$\begin{cases} w_t \;=\; w_{xx} \;+\; f(w) \quad \text{in } \; Q_{\mathcal{E}_k},\\ w(0,t)=w(1,t)=0,\\ w\;|_{t=S_k}=w_k\in C^{2+\vartheta}([0,1]), \end{cases} \qquad \begin{cases} h_t \;=\; h_{xx} \;+\; (f(w+h)-f(w)) \quad \text{in } \; Q_{\mathcal{E}_k},\\ h(0,t)=h(1,t)=0,\\ h\;|_{t=S_k}=p_k. \end{cases} \tag{9}$$

Evaluation of $\|h(\cdot,T_k)\|_{L^2(0,1)},\;k=1,\dots,N.$ Let us fix $k=1,\dots,N$. Multiplying by h each member of the equation in the second problem of (9) and integrating by parts over $Q_{[S_k,t]}=(0,1)\times[S_k,t],\;t\in[S_k,T_k]$, keeping in mind (2), it follows that

$$\begin{aligned}\frac{1}{2}\int_{S_k}^{t}\int_0^1 (h^2)_t dx\,ds &= \int_{S_k}^{t}\int_0^1 h_{xx}h dx\,ds+\int_{S_k}^{t}\int_0^1 (f(w+h)-f(w))h\,dxds\\ &\le -\int_{S_k}^{t}\int_0^1 h_x^2 dx\,ds+L\int_{S_k}^{t}\int_0^1 h^2 dxds, \quad \forall t\in[S_k,T_k].\end{aligned}$$

Then,

$$\int_0^1 h^2(x,t)dx\le\int_0^1 p_k^2(x)dx+2L\int_{S_k}^{t}\int_0^1 h^2 dxds, \quad \forall t\in[S_k,T_k].$$

So, applying Gronwall's inequality we deduce

$$\|\,h(\cdot,T_k)\,\|_{L^2(0,1)}\le\; e^{L\widetilde{T}}\,\|\,p_k\,\|_{L^2(0,1)}, \qquad \text{with} \;\; \widetilde{T}:=\sum_{k=1}^{N}\tau_k. \tag{10}$$

Step 2: *Steering.*

- *Steering the system from* u_0 *to* w_1 *on* $[0,S_1]$. Applying Theorem 3 (with $k=1$ and $r_0=0$ in its statement), for every $\eta_1>0$ there exist $\sigma_1=\sigma_1(\eta_1,u_0,w_1)>0$ and a piecewise static bilinear control $v_1=v_1(\eta_1,u_0,w_1)\in L^\infty((0,1)\times(0,S_1))$ (with $S_1=\sigma_1$) such that

$$\|\overline{U}_1(\cdot,S_1)-w_1(\cdot)\|_{L^2(0,1)}\le\eta_1, \tag{11}$$

  where $\overline{U}_1$ is the solution of (6) on $(0,1)\times[0,S_1]$. Let us set

$$p_1(\cdot):=\overline{U}_1(\cdot,S_1)-w_1(\cdot), \tag{12}$$

  we have $\overline{U}_1(\cdot,S_1)=w_1(\cdot)+p_1(\cdot)$.

- *Steering the system from* w_1+p_1 *to* u_1 *on* $[S_1,T_1]$. We consider the problem (8), written for $k=1$. Due to (9), we have $u(\cdot,T_1)=w(\cdot,T_1)+h(\cdot,T_1)$, where u,w,h, defined on $(0,1)\times\bigcup_{k=1}^{N}[S_k,T_k]$, are the solutions of the problem (8), the first problem in (9) and the second problem in (9), respectively. Thus, let us set

$$u_1(\cdot):=w(\cdot,T_1) \qquad \text{and} \qquad r_1(\cdot):=h(\cdot,T_1). \tag{13}$$

- *Steering the system from $u_{k-1}+r_{k-1}$ to w_k on $[T_{k-1},S_k]$, $k=2,\dots,N$.* Similarly to (13), let us set

$$u_{k-1}(\cdot):=w(\cdot,T_{k-1}) \qquad \text{and} \qquad r_{k-1}(\cdot):=h(\cdot,T_{k-1}).$$

Applying Theorem 3, for every $\eta_k>0$ there exist $\sigma_k=\sigma_k(\eta_k,u_{k-1},w_k)>0$ and a piecewise static bilinear control $v_k=v_k(\eta_k,u_{k-1},w_k)\in L^\infty((0,1)\times(T_{k-1},S_k))$ (with $S_k=T_{k-1}+\sigma_k$), such that

$$\|\overline{U}_k(\cdot,S_k)-w_k(\cdot)\|_{L^2(0,1)}\le \eta_k+C_k\|r_{k-1}\|_{L^2(0,1)}, \tag{14}$$

where $\overline{U}_k$ is the solution of (6) on $(0,1)\times[T_{k-1},S_k]$, $C_k=C(u_{k-1},w_k)\ge 1$ is a constant, as in Theorem 3. Moreover, we note that $\overline{U}_k(\cdot,S_k)=w_k(\cdot)+p_k(\cdot)$, where

$$p_k(\cdot):=\overline{U}_k(\cdot,S_k)-w_k(\cdot). \tag{15}$$

- *Steering the system from w_k+p_k to u_k on $[S_k,T_k]$, $k=2,\dots,N$.* Let us consider the problem (8). Due to (9), we have $u(\cdot,T_k)=w(\cdot,T_k)+h(\cdot,T_k)$, where u,w,h, defined on $(0,1)\times\bigcup_{k=1}^{N}[S_k,T_k]$, are the solutions of the problem (8), the first problem in (9) and the second problem in (9), respectively. Thus, let us set

$$u_k(\cdot):=w(\cdot,T_k) \qquad \text{and} \qquad r_k(\cdot):=h(\cdot,T_k). \tag{16}$$

Step 3: *Conclusions.* Given $\underline{\eta}=(\eta_1,\dots,\eta_N)\in\mathbb{R}^N_+$, let $\overline{U}_1,\dots,\overline{U}_N$ be the solutions of (6) satisfying (11) and (14). Define

$$u_{\underline{\eta}}(x,t)=\begin{cases}\overline{U}_k(x,t) & \text{in } (0,1)\times[T_{k-1},S_k],\ k=1,\dots,N,\\[2mm] w(x,t)+h(x,t) & \text{in } (0,1)\times\bigcup_{k=1}^{N}[S_k,T_k],\end{cases} \quad ({}^6) \tag{17}$$

and observe that $u_{\underline{\eta}}$ is the solution of (1) corresponding to the piecewise static bilinear control

$$v_{\underline{\eta}}(x,t)=\begin{cases}v_k(x,t) & \text{in } (0,1)\times[T_{k-1},S_k],\ k=1,\dots,N,\\[2mm] 0 & \text{in } (0,1)\times\bigcup_{k=1}^{N}[S_k,T_k].\end{cases}$$

Thus, to show (7) it is sufficient to prove by induction that there exists $\underline{\eta}(\delta)=(\eta_1(\delta),\dots,\eta_N(\delta))\in\mathbb{R}^N_+$, such that, for the corresponding $u_\delta:=u_{\underline{\eta}(\delta)}$, as in (17), the following inequality holds

$$\|\overline{U}_k(\cdot,S_k)-w_k(\cdot)\|_{L^2(0,1)}\le\frac{\delta\, C_k}{2^{N-k}e^{(N-k)L\widetilde{T}}\prod_{h=k}^{N}C_h},\quad \forall k=1,\dots,N. \tag{18}$$

Indeed, by (17), (10), (12) and (15), and (18) we obtain (7) and complete the proof.
Step 4: *Proof of* (18).

[6] We note that $\overline{U}_k$, $k=1,\dots,N$, depend on η_k and h depends on $\underline{\eta}$, but w is independent of $\underline{\eta}$.

- *Base case.* Choosing $\eta_1 = \eta_1(\delta) := \dfrac{\delta\, C_1}{2^{N-1} e^{(N-1)L\widetilde{T}} \prod_{h=1}^{N} C_h}$ in (11), we obtain (18) for $k = 1$.

- *Inductive step.* Let $k = 1, \dots, N-1$. By the inductive assumption, let us suppose that the inequality (18) holds for the index k. Thus, we will prove (18) for the index $k+1$. Choosing $\eta_{k+1} = \eta_{k+1}(\delta) := \dfrac{\delta\, C_{k+1}}{2^{N-k} e^{(N-k-1)L\widetilde{T}} \prod_{h=k+1}^{N} C_h}$ in (14), keeping in mind (13) or (16), (10), (12) or (15), and the induction assumption we deduce

$$\begin{aligned}
\|\overline{U}_{k+1}(\cdot, S_{k+1}) - w_{k+1}(\cdot)\|_{L^2(0,1)} &\le \eta_{k+1} + C_{k+1}\|r_k\|_{L^2(0,1)} \\
&\le \eta_{k+1} + C_{k+1} e^{L\widetilde{T}} \|\overline{U}_k(\cdot, S_k) - w_k(\cdot)\|_{L^2(0,1)} \\
&\le \eta_{k+1} + C_{k+1} e^{L\widetilde{T}} \frac{\delta\, C_k}{2^{N-k} e^{(N-k)L\widetilde{T}} \prod_{h=k}^{N} C_h} \\
&= \frac{\delta\, C_{k+1}}{2^{N-k-1} e^{(N-k-1)L\widetilde{T}} \prod_{h=k+1}^{N} C_h},
\end{aligned}$$

from which the inequality (18) is proved. $\diamond$

Now, we can prove Theorem 1.

Proof (of Theorem 1). Let us fix $\eta > 0$. Let $u_0 \in H_0^1(0,1)$ have n points of sign change at $x_l^0 \in (0,1)$, with $0 =: x_0^0 < x_l^0 < x_{l+1}^0 \le x_{n+1}^0 := 1$, $l = 1, \dots, n$. Let $u^* \in H_0^1(0,1)$ have n points of sign change at $x_l^* \in (0,1)$, $l = 1, \dots, n$, such that $0 := x_0^* < x_l^* < x_{l+1}^* \le x_{n+1}^* := 1$.

Step 1: Applying Theorem 2, for every $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and a finite family of times and initial data $\left\{(\tau_k)_1^{N_\varepsilon}, (w_k)_1^{N_\varepsilon}\right\}$ such that, for any $(\sigma_k)_1^{N_\varepsilon} \in \mathbb{R}_+^{N_\varepsilon}$, the solution $w^\varepsilon : (0,1) \times \bigcup_{k=1}^{N_\varepsilon} [S_k, T_k] \longrightarrow \mathbb{R}$ of problem (5) satisfies

$$w^\varepsilon(x, T_{N_\varepsilon}) = 0 \qquad \Longleftrightarrow \qquad x = x_l^\varepsilon, \;\; l = 0, \dots, n+1, \tag{19}$$

for some points $0 := x_0^\varepsilon < x_l^\varepsilon < x_{l+1}^\varepsilon \le x_{n+1}^\varepsilon := 1$, $l = 1, \dots, n$, such that

$$\sum_{l=1}^{n} |x_l^* - x_l^\varepsilon| < \varepsilon. \tag{20}$$

Moreover, $w^\varepsilon(\cdot, T_{N_\varepsilon})$ has the same order of sign change as for u_0.

Step 2: Applying Lemma 3.1, for every $\delta > 0$ there exists $\sigma_{\varepsilon,\delta} = (\sigma_k)_1^{N_\varepsilon} \in \mathbb{R}_+^N$, $v_{\varepsilon,\delta} \in L^\infty((0,1) \times (0, T_{N_\varepsilon}))$ such that, denoted by $u_{\varepsilon,\delta} : (0,1) \times [0, T_{N_\varepsilon}] \longrightarrow \mathbb{R}$ the solution of (1) with bilinear control $v_{\varepsilon,\delta}$, we have

$$u_{\varepsilon,\delta}(\cdot, T_{N_\varepsilon}) = w^\varepsilon(\cdot, T_{N_\varepsilon}) + r_{N_\varepsilon}(\cdot), \tag{21}$$

and

$$\|r_{N_\varepsilon}\|_{L^2(0,1)} \leq \delta\,. \tag{22}$$

Step 3: *Steering the system from $u_{\varepsilon,\delta}(\cdot,T_{N_\varepsilon})$ to u^*.* By (20) it is easy to show that there exist $\varepsilon^* = \varepsilon^*(\eta) > 0$ and $u^*_\varepsilon \in H^1_0(0,1)$ such that, for every $\varepsilon \in (0,\varepsilon^*)$, we have

$$u^*_\varepsilon(x) = 0 \Longleftrightarrow x = x^\varepsilon_l, \;\; l = 0,\ldots,n+1 \qquad \textit{and} \qquad \|u^* - u^*_\varepsilon\|_{L^2(0,1)} \leq \frac{\eta}{3}\,. \tag{23}$$

Since $w^\varepsilon(\cdot,T_{N_\varepsilon})$ and u^*_ε have the same points of sign change (see (19) and (23)), keeping in mind (21) we can steer the system (1) from $u_{\varepsilon,\delta}(\cdot,T_{N_\varepsilon})$ to u^*_ε at some time $T > T_{N_\varepsilon}$. Indeed applying Theorem 3, for every $\overline{\eta} > 0$ there exist $\sigma^* = \sigma^*(\overline{\eta},\varepsilon,u_0,u^*) > 0$ and a piecewise static bilinear control $v^* = v^*(\overline{\eta},\varepsilon,u_0,u^*) \in L^\infty((0,1)\times(T_{N_\varepsilon},T))$, with $T := T_{N_\varepsilon} + \sigma^*$, such that, using also the inequality (22), we obtain

$$\|U^*(\cdot,T) - u^*_\varepsilon(\cdot)\|_{L^2(0,1)} \leq \overline{\eta} + C^*(\varepsilon)\|r_{N_\varepsilon}(\cdot)\|_{L^2(0,1)} \leq \overline{\eta} + C^*(\varepsilon)\delta, \tag{24}$$

where U^* is the solution of (6) on $(0,1)\times[T_{N_\varepsilon},T]$ with initial state $u_{\varepsilon,\delta}(\cdot,T_{N_\varepsilon})$, $C^*(\varepsilon) = C(\varepsilon,u_0,u^*)$.

Conclusions. Let us fix $\varepsilon := \frac{\varepsilon^*(\eta)}{2}$ and $\overline{\eta} := \frac{\eta}{3}$. Then, we consider the constant $C^*(\varepsilon) := C^*\left(\frac{\varepsilon^*(\eta)}{2}\right)$ of (24) and we choose $\delta := \frac{\eta}{3C^*\left(\frac{\varepsilon^*(\eta)}{2}\right)}$. So, by (23) and (24), we obtain the conclusion

$$\|U^*(\cdot,T) - u^*(\cdot)\|_{L^2(0,1)} \leq \|U^*(\cdot,T) - u^*_\varepsilon(\cdot)\|_{L^2(0,1)} + \|u^* - u^*_\varepsilon\|_{L^2(0,1)} \leq \eta. \qquad \diamond$$

4 Proof of Theorem 2

The plan of this section is as follows:

In Section 4.1, by Lemma 4.1 we construct suitable *initial data w_k's* to be used in the proof of Theorem 2. By Lemma 4.2 we construct the n *curves of sign change* associated to the the n initial points of sign change and we prove some useful properties for the construction of suitable control strategies.

In Section 4.2, we construct a suitable particular family of *times and initial data*, that allows to move the n initial points of sign change towards the n target points of sign change. In this section, we also introduce the definitions of *gap and target distance functional*, defined on the set of the order processing steering times and initial data at the beginning of this section).

In Section 4.3, after proving a technical proposition (Proposition 4.1) we show how to steer the points of sign change of the solution arbitrarily close to the target points.

For the notation of this section we refer to Section 3.1.

4.1 Preliminary results

Let us prove the following lemma.

Lemma 4.1 (Construction of suitable initial data w_k's) *Let $x_l \in [0,1]$, $l = 0,\ldots,n+1$, be such that $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$. Let $\alpha = (\alpha_0,\ldots,\alpha_{n+1}) \in \mathbb{R}^{n+2}$, $\beta = (\beta_0,\ldots,\beta_{n+1}) \in \mathbb{R}^{n+2}$ be such that $\alpha_l\,\alpha_{l+1} < 0, \alpha_l \in \{-1,1\}$, $\beta_l \in \{-1,0,1\}$, $l = 0,\ldots,n$, $\beta_0 = \beta_{n+1} = 0$. Let $\widetilde{\rho} = \min_{l=0,\ldots,n}\{x_{l+1} - x_l\}$. Then, there exists $w \in C^\infty([0,1])$ such that*

- ⋆ $w(x) = 0 \iff x = x_l,\ l = 0, \dots, n+1;$
- ⋆ $w'(x_l) = \alpha_l,\ w''(x_l) = \beta_l,\ l = 0, \dots, n+1;$
- ⋆ $\|w\|_{C^k([0,1])} \le C(k, \widetilde{\rho}),\ \forall k \in \mathbb{N}.$

Proof. For every $l = 0, \dots, n+1$, set

$$v_l(x) = \alpha_l(x - x_l) + \frac{\beta_l}{2}(x - x_l)^2,\ \forall x \in \mathbb{R}.$$

Note that each $v_l(x)$ has no critical points in $\left[x_l - \frac{\widetilde{\rho}}{2}, x_l + \frac{\widetilde{\rho}}{2}\right]$. Set $\rho = \frac{1}{2}\widetilde{\rho}$ and define

$$w(x) = \sum_{l=0}^{n+1} \eta_l^\rho(x)\, v_l(x) + \sum_{l=0}^{n} \alpha_l \mathbb{1}_{[x_l, x_{l+1})}(x) \left[1 - (\eta_l^\rho(x) + \eta_{l+1}^\rho(x))\right], \quad x \in [0,1],$$

where $\mathbb{1}_A(x)$ is the characteristic function of a set A, and $\eta_j^\rho \in C^\infty(\mathbb{R}),\ j = 0, \dots, n+1$, are such that $\eta_j^\rho(x) = \eta_0^\rho(x - x_j)$ and η_0^ρ has the following properties:

- $\eta_0^\rho(-x) = \eta_0^\rho(x)$, $0 \le \eta_0^\rho(x) \le 1$, $\forall x \in \mathbb{R}$; $\eta_0^\rho(x) = 1$, $\forall x \in [0, \frac{\rho}{2}]$; $\eta_0^\rho(x) = 0$, $\forall x \in [\rho, +\infty)$;
- $\left|\dfrac{d^h \eta_0^\rho(x)}{dx^h}\right| \le \frac{C_h}{\rho^h}$, $\forall x \in \mathbb{R}$, where C_h is a positive constant and $h \in \mathbb{N}$.

Observe that, for $x \in [x_l, x_{l+1}]$, $l = 0, \dots, n$,

$$w(x) = \begin{cases} v_l(x), & \text{if } x \in [x_l, x_l + \rho/2], \\ \eta_0^\rho(x - x_l)v_l(x) + \alpha_l[1 - \eta_0^\rho(x - x_l)], & \text{if } x \in (x_l + \rho/2, x_l + \rho), \\ \alpha_l, & \text{if } x \in [x_l + \rho, x_{l+1} - \rho], \\ \eta_0^\rho(x - x_{l+1})v_{l+1}(x) + \alpha_l[1 - \eta_0^\rho(x - x_{l+1})], & \text{if } x \in (x_{l+1} - \rho, x_{l+1} - \rho/2), \\ v_{l+1}(x), & \text{if } x \in (x_{l+1} - \rho/2, x_{l+1}]. \end{cases}$$

Notice that w is of class $C^\infty(\mathbb{R})$ by construction. Moreover, our choice of ρ ensures that $w(x)$ has no points of sign change in $(x_l - \rho, x_l)$ or in $(x_l, x_l + \rho)$. This ends the proof of Lemma 4.1. ◇

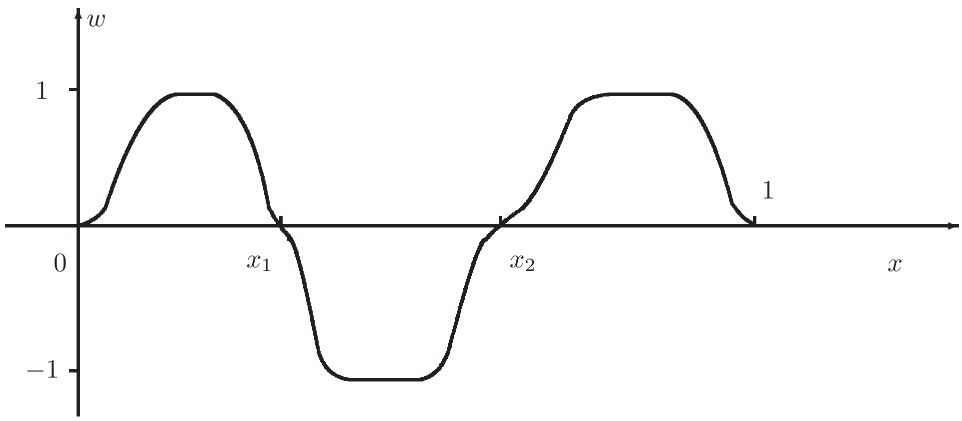

Figure 1: A function w as in Lemma 4.1 with 2 points of change of sign.

Remark 4.1 In the above, we can costruct η_0 by the following expression on $\left(\frac{\rho}{2},\rho\right)$

$$\eta_0(x)=\frac{e^{\frac{1}{(x-\rho)(x-\frac{\rho}{2})}}}{e^{\frac{1}{(x-\rho)(x-\frac{\rho}{2})}}+e^{-\frac{1}{(x-\frac{\rho}{2})^2}}}=e^{\frac{1}{1+h_0(x)}},\quad \text{with } h_0(x)=\frac{-(2x-\frac{3}{2}\rho)}{(x-\frac{\rho}{2})^2(x-\rho)}.$$

In the whole section, $\vartheta\in(0,1)$ denotes the number that was fixed at the beginning of Section 3.

Lemma 4.2 (Construction of the curves of sign change) *Let $\alpha=(\alpha_0,\dots,\alpha_{n+1})\in\mathbb{R}^{n+2}$ be such that $\alpha_l\,\alpha_{l+1}<0$, $\alpha_l\in\{-1,1\}$, $\alpha_{n+1}=-\alpha_n$, $l=0,\dots,n$. Let $\widetilde{\rho}>0$ be. Let $x_l\in[0,1]$, $l=0,\dots,n+1$, be such that $0=x_0<x_1<\dots<x_n<x_{n+1}=1$ and $\min\limits_{l=0,\dots,n}\{x_{l+1}-x_l\}=\widetilde{\rho}$. Let $w_k\in C^{2+\vartheta}([0,1])$ be such that $w_k(x)=0$ if and only if $x=x_l$, $w_k'(x_l)=\alpha_l$, $l=0,\dots,n+1$, $w_k''(0)=w_k''(1)=0$ and $\|w_k\|_{C^{2+\vartheta}([0,1])}\le c$, for some positive constant $c=c(\widetilde{\rho})$. Let $T>0$ and let $w\in C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{Q}_T)$ be the solution of the problem*

$$\begin{cases} w_t = w_{xx}+f(w) & \text{in}\quad Q_T=(0,1)\times(0,T)\\ w(0,t)=w(1,t)=0 & t\in(0,T)\qquad ({}^7)\\ w(x,0)=w_k(x) & x\in(0,1)\,. \end{cases}\tag{25}$$

Then, for every $\rho\in(0,\widetilde{\rho}]$ there exist $\widetilde{\tau}=\widetilde{\tau}(\rho)>0$ and $M=M(\rho)>0$ such that, for each $l=1,\dots,n$, there exists a unique solution $\xi_l:[0,\widetilde{\tau}]\longrightarrow\mathbb{R}$ of the initial-value problem

$$\begin{cases}\dot{\xi}_l(t) = -\frac{w_{xx}(\xi_l(t),t)}{w_x(\xi_l(t),t)}, & t\in[0,\widetilde{\tau}],\\ \xi_l(0)=x_l,\end{cases}$$

that satisfies $w(\xi_l(t),t)=0$, $\forall t\in[0,\widetilde{\tau}]$, and

$$\xi_l\in C^{1+\frac{\vartheta}{2}}([0,\widetilde{\tau}]),\qquad \|\xi_l\|_{C^{1+\frac{\vartheta}{2}}([0,\widetilde{\tau}])}\le M,\qquad \|\xi_l(\cdot)-x_l\|_{C([0,\widetilde{\tau}])}<\frac{\rho}{2}.$$

Remark 4.2 In Lemma 4.2, since $\|\xi_l(\cdot)-x_l\|_{C([0,\widetilde{\tau}])}<\dfrac{\rho}{2}$ for each $l=1,\dots,n$, we also have that

$$0:=\xi_0(t)<\xi_l(t)<\xi_{l+1}(t)<\xi_{n+1}(t):=1,\quad \forall t\in[0,\widetilde{\tau}],\ \forall l=1,\dots,n-1.\tag{26}$$

Definition 4.1 *We call the functions $\xi_l:[0,\widetilde{\tau}]\longrightarrow\mathbb{R}$, $l=1,\dots,n$, given by Lemma 4.2, Curves of Sign Change associated to the set of initial points of sign change $X=(x_1,\dots,x_n)$.*

Proof (of Lemma 4.2). Let us fix $\rho\in(0,\widetilde{\rho}]$.
Step 1: *Uniform estimate for w.* Due to Theorem 6.1 of [25] (pp. 452-453), the solution w of (25) belongs to $C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{Q}_T)$ (^{8}) and, for some constant $K=K(\|w_k\|_{C^{2+\vartheta}([0,1])})>0$, depending only on $\|w_k\|_{C^{2+\vartheta}([0,1])}$ (see (6.8)-(6.12) on pp. 451-452 in [25]), we have $\|w\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{Q}_T)}\le K$. Thus, since $\|w_k\|_{C^{2+\vartheta}([0,1])}\le c(\widetilde{\rho})$, we deduce that

$$\|w\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{Q}_T)}\le K(\|w_k\|_{C^{2+\vartheta}([0,1])})\le C,\tag{27}$$

for some positive constant $C=C(\widetilde{\rho})$ depending only on $\widetilde{\rho}$.
Step 2: *Existence and regularity of curves of sign change.* For any fixed $l=1,\dots,n$, since $w_x(x_l,0)=\alpha_l\neq0$ and $w_x(x,t)$ is a continuous function in $(x_l,0)\in\overline{Q}_T$, there exist $\delta_l\in\left(0,\min\left\{\frac{1}{2C},\rho\right\}\right)$ (^{9}) and $T_l>0$ such that $w_x(x,t)\neq0$, $\forall(x,t)\in[x_l-\delta_l,x_l+\delta_l]\times[0,T_l]$.

[7] For the existence, uniqueness and regularity of problem (25) see Remark 3.1 and Theorem 6.1 in [25] (pp. 452-453).
[8] One can note that the initial datum satisfies the compatibility condition $w_k''(0)=w_k''(1)=0$, as in Theorem 6.1 of [25].
[9] C is the constant present in (27).

Let $\delta := \min\limits_{l=1,\dots,n} \delta_l$ be. For every $l = 1,\dots,n$, we consider the Cauchy problems

$$\begin{cases} \dot{\xi}_l(t) \;=\; -\frac{w_t(\xi_l(t),t)}{w_x(\xi_l(t),t)}, \quad t > 0, \\ \xi_l(0) = x_l \;. \end{cases} \tag{28}$$

Observe that $F(x,t) := -\frac{w_t(x,t)}{w_x(x,t)}$ is continuous on $[x_l - \delta, x_l + \delta] \times [0, T_l]$. Therefore, for every $l = 1,\dots,n$, the problem (28) has a solution ξ_l of class C^1 on some interval $[0,\tau_l]$, with $0 < \tau_l \le T_l$. Moreover, since $F \in C^{\frac{\vartheta}{2}}([x_l - \delta, x_l + \delta] \times [0, T_l])$, we conclude that $\xi_l \in C^{1+\frac{\vartheta}{2}}([0,\tau_l])$. Furthermore, $w(\xi_l(t),t) = 0,\ \forall t \in [0,\tau_l]$, because

$$\frac{d}{dt} w(\xi_l(t),t) = w_t(\xi_l(t),t) + w_x(\xi_l(t),t)\dot{\xi}_l(t) = 0, \forall t \in [0,\tau_l], \quad \text{and} \quad w(\xi_l(0),0) = w_k(x_l) = 0.$$

Moreover, since $f(w(\xi_l(t)),t) = 0,\ \forall t \in [0,\tau_l]$, we also have that

$$\dot{\xi}_l(t) = -\frac{w_t(\xi_l(t),t)}{w_x(\xi_l(t),t)} = -\frac{w_{xx}(\xi_l(t),t)}{w_x(\xi_l(t),t)}, \quad \forall t \in [0,\tau_l].$$

Step 3: *Uniform estimates for the curves of sign change.* For any fixed $l = 1,\dots,n$, we consider the number $\delta = \delta(\rho) > 0$, $\delta = \min\limits_{l=1,\dots,n} \delta_l < \min\Big\{\dfrac{1}{2C}, \rho\Big\}$, introduced in Step 2, and the uniform time $\widetilde{\tau} = \widetilde{\tau}(\rho) > 0$,

$$\widetilde{\tau} := \min\Big\{ \Big(\frac{1}{2C} - \delta\Big)^{\frac{2}{\vartheta}}, \frac{\delta^2}{3}, \min_{l=1,\dots,n} \tau_l \Big\}. \ ({}^{10})$$

We remember that the function $t \mapsto w_x(x,t)$ belongs to $C^{\frac{\vartheta}{2}}([0,\widetilde{\tau}])$, and the function $x \mapsto w_x(x,t)$ belongs to $C^{1+\vartheta}([0,\widetilde{\tau}])$. Thus, for every $(x,t) \in (x_l - \delta, x_l + \delta)\backslash\{x_l\} \times (0,\widetilde{\tau})$, by (27) we have

$$\begin{aligned} |w_x(x,t) - \alpha_l| &= |w_x(x,t) - w_x(x_l,0)| \le |w_x(x,t) - w_x(x,0)| + |w_x(x,0) - w_x(x_l,0)| \\ &= \frac{|w_x(x,t) - w_x(x,0)|}{t^{\frac{\vartheta}{2}}}\, t^{\frac{\vartheta}{2}} + \frac{|w_x(x,0) - w_x(x_l,0)|}{|x - x_l|}|x - x_l| \\ &\le \|w\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{Q}_T)}(t^{\frac{\vartheta}{2}} + |x - x_l|) \le\ C(t^{\frac{\vartheta}{2}} + |x - x_l|) \le C(\widetilde{\tau}^{\frac{\vartheta}{2}} + \delta). \end{aligned} \tag{29}$$

Since $\delta < \frac{1}{2C}$ and $\widetilde{\tau} \le \big(\frac{1}{2C} - \delta\big)^{\frac{2}{\vartheta}}$, we have $C(\widetilde{\tau}^{\frac{\vartheta}{2}} + \delta) \le \frac{1}{2}$, so by (29) we deduce

$$\Big||w_x(x,t)| - |\alpha_l|\Big| \le |w_x(x,t) - \alpha_l| \le C(\widetilde{\tau}^{\frac{\vartheta}{2}} + \delta) \le \frac{1}{2}.$$

Therefore, for every $l = 1,\dots,n$, having in mind that $|\alpha_l| = 1$, we obtain

$$|w_x(x,t)| \ge |\alpha_l| - \frac{1}{2} = \frac{1}{2}, \qquad \forall (x,t) \in (x_l - \delta, x_l + \delta) \times (0,\widetilde{\tau})\,. \tag{30}$$

Then, by (27) and (30), keeping in mind that $\widetilde{\tau} \le \frac{\delta^2}{3}$ and $\delta < \min\{\frac{1}{2C}, \rho\}$, we deduce

$$\begin{aligned} |\xi_l(t) - x_l| = \Big|\int_0^t \dot{\xi}_l(s)\,ds\Big| &\le \int_0^{\widetilde{\tau}} \frac{|w_{xx}(\xi_l(s),s)|}{|w_x(\xi_l(s),s)|} ds \ \le\ \frac{\widetilde{\tau}\|w\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{Q}_T)}}{\min\limits_{s\in[0,\widetilde{\tau}]} |w_x(\xi_l(s),s)|} \\ &\le \frac{\widetilde{\tau}C}{\min\limits_{s\in[0,\widetilde{\tau}]} |w_x(\xi_l(s),s)|} \ \le\ 2\widetilde{\tau}C < \frac{\widetilde{\tau}}{\delta} \le \frac{1}{\delta}\frac{\delta^2}{3} < \frac{\rho}{3}, \quad \forall t \in [0,\widetilde{\tau}]. \end{aligned} \tag{31}$$

[10] We note that $\widetilde{\tau} = \widetilde{\tau}(\rho)$ is not dependent on l.

Step 4: *Uniqueness of the curves of sign change.* We note that, although one cannot claim uniqueness for the Cauchy problem (28), a posteriori the ξ_l's turn out to be uniquely determined. Indeed, setting $\xi_0(t) \equiv 0,\ \xi_{n+1}(t) \equiv 0,\ \forall t \in [0, \widetilde{\tau}]$, one can apply the maximum principle for semilinear parabolic equations (see Remark 2.1) on the domains

$$\{(x,t) | x \in [\xi_l(t), \xi_{l+1}(t)]\,,\, t \in [0, \widetilde{\tau})\}\,,$$

for every $l = 0, \ldots, n$. The fact that the initial datum $w_k(x)$ doesn't change sign on (x_l, x_{l+1}) and the boundary conditions in (25) imply that, for every $t^* \in [0, \widetilde{\tau})$,

$$w(x, t^*) = 0 \Longleftrightarrow x = \xi_l(t^*),\ l = 0, \ldots, n+1,$$

this completes the proof of Lemma 4.2. $\diamond$

Remark 4.3 An alternative proof of Lemma 4.2 can be obtained by using the implicit function theorem instead of solving problem (28).

4.2 Construction of Order Processing Steering sets of Times and Initial Data

In this section we define the set of *Order Processing Steering sets of Times and Initial Data* that permit to move the points of sign change towards the desired targets.

Notation

Let us consider the initial state $u_0 \in H_0^1(0,1)$. For simplicity of notation let us set $x_0^0 := 0$ and $x_{n+1}^0 := 1$, and let us consider the set of n points of sign change of u_0, $X^0 = (x_1^0, \ldots, x_n^0)$, where $0 = x_0^0 < x_l^0 < x_{l+1}^0 \le x_{n+1}^0 = 1,\ \ l = 1, \ldots, n$. Let $\rho_0 = \min_{l=0,\ldots,n} \left\{x_{l+1}^0 - x_l^0\right\}$. Let us define

$$\lambda(x_l^0) = \begin{cases} 1, & \text{if } u_0(x) > 0 \text{ on } (x_l^0, x_{l+1}^0), \\ -1, & \text{if } u_0(x) < 0 \text{ on } (x_l^0, x_{l+1}^0), \end{cases} \quad l = 0, \ldots, n \quad \text{and} \quad \lambda(x_{n+1}^0) = -\lambda(x_n^0).\ (^{11}) \qquad (32)$$

For simplicity of notation let us set $x_0^* := 0$ and $x_{n+1}^* := 1$, and let us also consider the set of n target points $X^* = (x_1^*, \ldots, x_n^*)$, where $0 = x_0^* < x_l^* < x_{l+1}^* \le x_{n+1}^* = 1,\ \ l = 1, \ldots, n$.

Order Processing Steering sets of Times and Initial Data

Let us consider the same $\vartheta \in (0,1)$ that was fixed at the beginning of Section 3.
Now, we define the set of the *Order Processing Steering Times and Initial Data* associated to u_0 and X^*

$$\mathcal{W}^*(u_0) := \left\{W^N \mid W^N = \left\{(\tau_k)_1^N, (w_k)_1^N\right\}, N \in \mathbb{N}\right\},$$

where a generic set $W^N = \left\{(\tau_k)_1^N, (w_k)_1^N\right\} \in \mathcal{W}^*(u_0)$ is defined below. Let us fix any $(\sigma_k)_1^N \in \mathbb{R}_+^N$ and, recalling the notation introduced in (3)-(4) in Section 3, for $k = 1, \ldots, N$, we consider the initial value pure diffusion problems on disjoint time intervals (5)

$$\begin{cases} w_t \ = \ w_{xx} \ + \ f(w) \quad \text{in} \quad Q_{\mathcal{E}_k} = (0,1) \times [S_k, T_k] \\ w(0,t) = w(1,t) = 0, \qquad\qquad\qquad t \in [S_k, T_k] \\ w \mid_{t=S_k} = w_k(x), \qquad w_k''(x) \mid_{x=0,1} = 0. \end{cases}$$

[11] Since x_l^0, $l = 1, \ldots, n$, are points of sign change, we note that $\lambda(x_{l+1}^0) = -\lambda(x_l^0)$, $l = 0, \ldots, n$.

Properties of $\{\tau_1, w_1\}$. By Lemma 4.1, there exists $w_1 \in C^{2+\vartheta}([0,1])$, with $\|w_1\|_{C^{2+\vartheta}([0,1])} \leq c_1$, for some positive constant $c_1 = c(\rho_0)$, such that

- $w_1(x) = 0 \Longleftrightarrow x = x_l^0,\ l = 0, \dots, n+1;$
- $w_1'(x_l^0) = \lambda(x_l^0),\ w_1''(x_l^0) = -\lambda(x_l^0)\mu_1(x_l^* - x_l^0),\ l = 0, \dots, n+1,$ where $\mu_1(x_l^* - x_l^0) := sgn(x_l^* - x_l^0) = \begin{cases} 1, & \text{if } x_l^0 < x_l^*, \\ 0, & \text{if } x_l^0 = x_l^*, \\ -1, & \text{if } x_l^0 > x_l^*. \end{cases}$

Let w be the solution to

$$\begin{cases} w_t \ = \ w_{xx} \ + \ f(w) & (x,t) \in (0,1) \times (S_1, +\infty) \\ w(0,t) = w(1,t) = 0 & t \in (S_1, +\infty) \\ w(x, S_1) \ = w_1(x) & x \in (0,1), \end{cases}$$

where $S_1 = \sigma_1$. By Lemma 4.2, for every $\rho \in (0, \rho_0]$ there exist $\widetilde{\tau}_1 = \widetilde{\tau}_1(\rho) > 0$, $M_1 = M_1(\rho) > 0$ and n *small curves of sign change* (associated to the set of n initial points of sign change $X^0 = (x_1^0, \dots, x_n^0)$) $\xi_l^1 \in C^{1+\frac{\vartheta}{2}}([S_1, \widetilde{T}_1])$, $\widetilde{T}_1 = S_1 + \widetilde{\tau}_1$, $l = 1, \dots, n$, such that $w(\xi_l^1(t), t) = 0, \forall t \in [S_1, \widetilde{T}_1]$. Moreover,

$$\begin{cases} \dot{\xi}_l^1(t) \ = \ -\frac{w_{xx}(\xi_l^1(t),t)}{w_x(\xi_l^1(t),t)}, & t \in [S_1, \widetilde{T}_1], \\ \xi_l^1(S_1) = x_l^0\ , \end{cases} \quad \text{and} \quad \|\xi_l^1\|_{C^{1+\frac{\vartheta}{2}}([S_1, \widetilde{T}_1])} \leq M_1. \tag{33}$$

Let us set $\xi_0^1(t) \equiv 0$ and $\xi_{n+1}^1(t) \equiv 1$ on $[S_1, \widetilde{T}_1]$. Furthermore, for every $l = 1, \dots, n-1$, by Remark 4.2 we have

$$0 = \xi_0^1(t) < \xi_l^1(t) < \xi_{l+1}^1(t) < \xi_{n+1}^1(t) = 1,\ \forall t \in [S_1, \widetilde{T}_1]. \tag{34}$$

Let us introduce the *Inactive Set*

$$L_{IS}^0 := \{l \,|\, l \in \{1, \dots, n\},\ x_l^0 = x_l^*\}$$

and let us consider the set of the *stopping times*

$$\Theta_1 := \{s \in (0, \widetilde{\tau}_1] \,|\, \xi_l^1(S_1 + s) = x_l^*, \ \ \text{for some} \ \ l \in \{1, \dots, n\} \backslash L_{IS}^0\}.$$

Let us set

$$\tau_1 = \begin{cases} \widetilde{\tau}_1 & \text{if} \quad \Theta_1 = \varnothing, \\ \min \Theta_1, & \text{otherwise}\ , \end{cases} \tag{35}$$

by (3) we have $T_1 = S_1 + \tau_1$.

Properties of $\{\tau_k, w_k\}$, $k = 2, \dots, N$. By iterate application of Lemma 4.1 and Lemma 4.2, one can set $x_l^{k-1} := \xi_l^{k-1}(T_{k-1})$, $l = 1, \dots, n$ and $X^{k-1} = (x_1^{k-1}, \dots, x_n^{k-1})$, where $\xi_l^{k-1}(t),\ t \in [S_{k-1}, T_{k-1}]$, are the n *curves of sign change* associated to the initial state w_{k-1} and to the set of n points of sign change $X^{k-2} = (x_1^{k-2}, \dots, x_n^{k-2})$. For simplicity of notation, let us set $x_0^{k-1} := 0$ and $x_{n+1}^{k-1} := 1$. Let $\rho_{k-1} = \min\limits_{l=0,\dots,n} \{x_{l+1}^{k-1} - x_l^{k-1}\}$.

Let us introduce the *Inactive Set*

$$L_{IS}^{k-1} := \{l,\ l \in \{1, \dots, n\} | \exists h_l \in \{1, \dots, k-1\} : x_l^{h_l} = x_l^*\},\ ([12])$$

[12] We note that $L_{IS}^{k-1} \subseteq \{1, \dots, n\}$ is a family of sets increasing in k.

that is, the set of the indexes of the points of sign change that have already reached the corresponding target points of the sign change set $X^* = (x_1^*, \dots, x_n^*)$ in some previous time instant. Then, let us set

$$\mu_k(x_l^* - x_l^0) = \begin{cases} 0, & \text{if } l \in L_{IS}^{k-1}, \\ sgn(x_l^* - x_l^0) & \text{if } l \notin L_{IS}^{k-1}. \end{cases} \quad (^{13})$$

Thus, by Lemma 4.1 we can choose $w_k \in C^{2+\vartheta}([0,1])$, with $\|w_k\|_{C^{2+\vartheta}([0,1])} \le c_k$, for some positive constant $c_k = c(\rho_{k-1})$, such that

- $w_k(x) = 0 \Longleftrightarrow x = x_l^{k-1}, \; l = 0, \dots, n+1;$
- $w_k'(x_l^{k-1}) = \lambda(x_l^0), \; w_k''(x_l^{k-1}) = -\lambda(x_l^0)\mu_k(x_l^* - x_l^0), \; l = 0, \dots, n+1.$

Let w be the solution to

$$\begin{cases} w_t \; = \; w_{xx} \; + \; f(w) & (x,t) \in (0,1) \times (S_k, +\infty) \\ w(0,t) = w(1,t) = 0 & t \in (S_k, +\infty) \\ w(x, S_k) \; = w_k(x) & x \in (0,1), \end{cases}$$

where $S_k = \sum_{h=1}^{k-1} (\sigma_h + \tau_h) + \sigma_k$. By Lemma 4.2, for every $\rho \in (0, \rho_{k-1}]$ there exist $\widetilde{\tau}_k = \widetilde{\tau}_k(\rho) > 0$, $M_k = M_k(\rho) > 0$ and n *small curves of sign change* (associated to the set of n intermediate points of sign change X^{k-1}), $\xi_l^k \in C^{1+\frac{\vartheta}{2}}([S_k, T_k])$, $\widetilde{T}_k = S_k + \widetilde{\tau}_k, \; l = 1, \dots, n$, such that $w(\xi_l^k(t), t) = 0$, $\forall t \in [S_k, \widetilde{T}_k]$. Moreover,

$$\begin{cases} \dot{\xi}_l^k(t) \; = \; -\frac{w_{xx}(\xi_l^k(t),t)}{w_x(\xi_l^k(t),t)}, & t \in [S_k, \widetilde{T}_k], \\ \xi_l^k(S_k) = x_l^{k-1}, \end{cases} \quad \text{and} \quad \|\xi_l^k\|_{C^{1+\frac{\vartheta}{2}}([S_k, \widetilde{T}_k])} \le M_k. \quad (36)$$

Let us set $\xi_0^k(t) \equiv 0$ and $\xi_{n+1}^k(t) \equiv 1$ on $[S_k, \widetilde{T}_k]$. Furthermore, for every $l = 1, \dots, n-1$, by Remark 4.2 we have

$$0 = \xi_0^k(t) < \xi_l^k(t) < \xi_{l+1}^k(t) < \xi_{n+1}^k(t) = 1, \; \forall t \in [S_k, \widetilde{T}_k]. \quad (37)$$

Let us introduce the set of the *stopping times*

$$\Theta_k := \{s \in (0, \widetilde{\tau}_k] \,|\, \xi_l^k(S_k + s) = x_l^*, \;\; \text{for some } \; l \in \{1, \dots, n\} \backslash L_{IS}^{k-1}\},$$

and let us set

$$\tau_k = \begin{cases} \widetilde{\tau}_k & \text{if} \quad \Theta_k = \varnothing, \\ \min \Theta_k, & \text{otherwise}. \end{cases} \quad (38)$$

Then, by (3) we have $T_k = S_k + \tau_k$.

Now, we give some remarks about the above introduced set of the order processing steering times and initial data associated to u_0 and X^*.

Remark 4.4 We note that $\tau_k < \widetilde{\tau}_k$ for at most n values of $k \in \{1, \dots, N\}$.

[13] We observe that the definition of μ_k, $k = 2, \dots, N$, is consistent with the one of μ_1.

Remark 4.5 We note that, for each index $l \notin L_{IS}^{k-1}$ $(k = 1, \dots, N)$, by (36) and the choice of the initial data w_k we deduce

$$\dot{\xi}_l^k(S_k) \;=\; -\frac{w_{xx}(\xi_l^k(S_k), S_k)}{w_x(\xi_l^k(S_k), S_k)} = -\frac{-\lambda(x_l^0)\mu_k(x_l^* - x_l^0)}{\lambda(x_l^0)} = \mu_k(x_l^* - x_l^0), \quad \xi_l^k(S_k) = x_l^{k-1}.$$

Then, if $x_l^0 < x_l^*$ we have $\dot{\xi}_l^k(S_k) \;=\mu_k(x_l^* - x_l^0) > 0$, so the initial conditions w_k permits to move the points of sign change x_l^{k-1} on the right toward the desired x_l^*. Otherwise, if $x_l^* < x_l^0$ the initial conditions w_k permits to move the points of sign change on the left.

Remark 4.6 We note that, for each inactive index $l \in L_{IS}^{k-1}$ $(k = 1, \dots, N)$, we choose the initial data such that the second derivative is equal to 0 in the intermediate points of sign change x_l^{k-1}. So, as we will see later, such points will remain forever near the corresponding target points of sign change already reached (we will see it in the next inequality (59) in the proof of Proposition 4.1).

Curves of Sign Change, Gap and Target Distance functional

Given $W^N = \left\{(\tau_k)_1^N, (w_k)_1^N\right\} \in \mathcal{W}^*(u_0)$, we introduce the n *Curves of Sign Change* associated to W^N, as the functions $\xi_l^W : \bigcup_{k=1}^{N} [S_k, T_k] \longrightarrow \mathbb{R}$, $l = 1, \dots, n$, such that

$$\xi_l^W(t) = \xi_l^k(t), \qquad S_k \le t \le T_k, \;\; k = 1, \dots, N,$$

where any curve ξ_l^k, $k = 1, \dots, N$, is given by Lemma 4.2 and is defined on $[S_k, T_k]$. For simplicity of notation, let us set $\xi_0^W(t) \equiv 0$ and $\xi_{n+1}^W(t) \equiv 1$. Moreover, for every $l = 1, \dots, n-1$, by (34) and (37), we deduce that

$$0 = \xi_0^W(t) < \xi_l^W(t) < \xi_{l+1}^W(t) < \xi_{n+1}^W(t) = 1, \;\; \forall t \in \bigcup_{k=1}^{N} [S_k, T_k].$$

Definition 4.2 *We define the* ***gap functional*** $\rho : \mathcal{W}^*(u_0) \to [0,1]$ *in the following way*

$$\rho(W^N) = \min_{l=0,\dots,n} \min_{t \in \mathcal{E}} \left\{\xi_{l+1}^W(t) - \xi_l^W(t)\right\}, \quad \forall\, W^N = \left\{(\tau_k)_1^N, (w_k)_1^N\right\} \in \mathcal{W}^*(u_0),$$

where $\mathcal{E} := \bigcup_{k=1}^{N} [S_k, T_k]$.

Definition 4.3 *We define the* ***target distance functional*** *associated to the set* X^*, $J^* : \mathcal{W}^*(u_0) \to [0,1]$ *such that*

$$J^*(W^N) = \sum_{l=1}^{n} \left|\xi_l^W(T_N) - x_l^*\right|, \quad \forall\, W^N = \left\{(\tau_k)_1^N, (w_k)_1^N\right\} \in \mathcal{W}^*(u_0).$$

4.3 End of the proof of Theorem 2

Notation

Let us consider the initial state $u_0 \in H_0^1(0,1)$. Let us set $x_0^0 := 0$ and $x_{n+1}^0 := 1$, and let us consider the set of n points of sign change of u_0, $X^0 = (x_1^0, \ldots, x_n^0)$ where $0 = x_0^0 < x_l^0 < x_{l+1}^0 \leq x_{n+1}^0 = 1, \;\; l = 1, \ldots, n$. Let $\rho_0 = \min_{l=0,\ldots,n} \left\{x_{l+1}^0 - x_l^0\right\}$ be.

Let us set $x_0^* := 0$ and $x_{n+1}^* := 1$, and let us consider the set of n target points $X^* = (x_1^*, \ldots, x_n^*)$, where $0 = x_0^* < x_l^* < x_{l+1}^* \leq x_{n+1}^* = 1, \;\; l = 1, \ldots, n$.

Let $\rho_0^* = \min_{l=0,\ldots,n} \left\{x_{l+1}^* - x_l^*, \; x_{l+1}^0 - x_l^0\right\}$ be. Let us consider the same $\vartheta \in (0,1)$ that was fixed at the beginning of Section 3. Let $s_\vartheta = \sum_{k=1}^{\infty} \frac{1}{k^{1+\frac{\vartheta}{2}}}$ be. Let $\tau_0^* = \tau(\frac{\rho_0^*}{2}) > 0$ and $M_0^* = M(\frac{\rho_0^*}{2}) > 0$ be the positive time and constant of Lemma 4.2, associated to $\rho = \frac{\rho_0^*}{2} \in (0, \rho_0]$.

To obtain the proof of Theorem 2 it need the following Proposition 4.1.

Proposition 4.1 *There exists $\varepsilon_0^* = \varepsilon_0^*(\frac{\rho_0^*}{2}) \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_0^*]$, $N \in \mathbb{N}$, there exists*

$$W^N = \left\{(\tau_k)_1^N, (w_k)_1^N\right\} \in \mathcal{W}^*(u_0) \;\; with \;\; \tau_k \leq \widetilde{\tau}_k := \left(\frac{\varepsilon \rho_0^*}{4 M_0^* \, s_\vartheta}\right)^{\frac{2}{2+\vartheta}} \frac{1}{k}, \;\; k = 1, \ldots, N \; (^{14}). \quad (39)$$

For every $N \in \mathbb{N}$, $N > n$ and $W^N \in \mathcal{W}^*(u_0)$, as in (39), we have $\rho(W^N) \geq \frac{\rho_0^*}{2}$ and

$$J^*(W^N) \leq \sum_{l=1}^{n} |x_l^0 - x_l^*| + c_1(\varepsilon) \sum_{k=1}^{N} \frac{1}{k^{1+\frac{\vartheta}{2}}} - c_2(\varepsilon) \sum_{k=n+1}^{N} \frac{1}{k}, \quad (40)$$

where $c_1(\varepsilon) = \frac{\varepsilon \rho_0^* n}{4 \, s_\vartheta}$, $c_2(\varepsilon) = \left(\frac{\varepsilon \rho_0^*}{4 M_0^* \, s_\vartheta}\right)^{\frac{2}{2+\vartheta}}$.

Let us give the following definition.

Definition 4.4 *We call Separating order processing steering set of times and initial data a $W^N \in \mathcal{W}^*(u_0)$ such that $\rho(W^N) \geq \frac{\rho_0^*}{2}$. We denote with $\mathcal{W}_{\mathcal{S}}^*(u_0)$ the set of all the Separating order processing steering strategies, that is*

$$\mathcal{W}_{\mathcal{S}}^*(u_0) = \left\{ W^N \in \mathcal{W}^*(u_0) : \; \rho(W^N) \geq \frac{\rho_0^*}{2} \right\}.$$

Before proving Proposition 4.1, we show how thanks this proposition, we can easily obtain the proof of Theorem 2.

Proof (of Theorem 2). To prove Theorem 2 it is sufficient to show that

$$\forall \varepsilon > 0 \;\; \exists \, N_\varepsilon \in \mathbb{N}, \, W^{N_\varepsilon} \in \mathcal{W}_{\mathcal{S}}^*(u_0) : \qquad J^*(W^{N_\varepsilon}) < \varepsilon \; (^{15}) \quad \text{and} \quad L_{IS}^{N_\varepsilon} = \{1, \ldots, n\}. \quad (41)$$

This follows from Proposition 4.1. Indeed, by contradiction,

- ⋆ if $\inf_{W^N \in \mathcal{W}_{\mathcal{S}}^*(u_0)} J^*(W^N) > 0$,

$$\exists \bar{\varepsilon} > 0 : \; \forall N \in \mathbb{N}, \, \forall W^N \in \mathcal{W}_{\mathcal{S}}^*(u_0) \text{ we have } J^*(W^N) > \bar{\varepsilon};$$

[14] $\widetilde{\tau}_k$ and τ_k are defined in (35) and (38), see also Remark 4.4.

[15] $\inf_{W^N \in \mathcal{W}_{\mathcal{S}}^*(u_0)} J^*(W^N) = 0$.

$\star$ if $L_{IS}^N \neq \{1,\ldots,n\}, \forall N \in \mathbb{N}, \quad \exists j \notin L_{IS}^N$, for every $N \in \mathbb{N}$, that is

$$\exists \tilde{\varepsilon} > 0 : \forall N \in \mathbb{N}, \forall W^N \in \mathcal{W}_{\mathcal{S}}^*(u_0) \text{ we have } \tilde{\varepsilon} < |\xi_j^N(T_N) - x_j^*|.$$

Let us consider $\varepsilon^* := \min\{\bar{\varepsilon}, \tilde{\varepsilon}, \varepsilon_0^*\}$, where $\varepsilon_0^* \in (0,1)$ is given by Proposition 4.1. In both the previous cases, for every $N \in \mathbb{N}$, $N > n$, choose $W^N = \{(\tau_k)_1^N, (w_k)_1^N\} \in \mathcal{W}^*(u_0)$ such that $\tau_k \leq \widetilde{\tau}_k = \left(\dfrac{\varepsilon^* \rho_0^*}{4M_0^*\, s_\vartheta}\right)^{\frac{2}{2+\vartheta}} \dfrac{1}{k}$, $\quad k = 1,\ldots,N$. By inequality (40) of Proposition 4.1, we obtain that, for every $N \in \mathbb{N}$, $N > n$, the following inequality holds

$$\varepsilon^* < |\xi_j^N(T_N) - x_j^*| \leq J^*(W^N) \leq \sum_{l=1}^{n} |x_l^0 - x_l^*| + c_1(\varepsilon^*) \sum_{k=1}^{N} \frac{1}{k^{1+\frac{\vartheta}{2}}} - c_2(\varepsilon^*) \sum_{k=n+1}^{N} \frac{1}{k}.$$

Keeping in mind that $\displaystyle\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$, for N enough large, from the previous inequality we have a contradiction, then (41) holds. $\diamond$

Now, we give the proof of Proposition 4.1.

Proof (of Proposition 4.1). Let $(\sigma_k)_1^N \in \mathbb{R}_+^N$ be. Given $N \in \mathbb{N}$ and a generic $W^N = \{(\tau_k)_1^N, (w_k)_1^N\} \in \mathcal{W}^*(u_0)$, let us recall that the curves of sign change associated to W^N (see Section 4.2) are defined in the following way

$$\xi_l^W(t) = \xi_l^k(t), \qquad \forall t \in [S_k, T_k],\ \ k = 1,\ldots,N,\ l = 0,\ldots,n+1,$$

where all the curves ξ_l^k, $l = 0,\ldots,n+1$, are defined on $[S_k,T_k]$ and are associated to the initial state w_k. Let us recall that in Section 4.2, for every $k = 2,\ldots,N$, we have defined $x_j^{k-1} := \xi_j^{k-1}(T_{k-1}) = \xi_j^k(S_k)$, $j = 0,\ldots,n+1$ and $\rho_{k-1} = \min\limits_{l=0,\ldots,n} \{x_{l+1}^{k-1} - x_l^{k-1}\}$. We recall that by Lemma 4.2 (see also the properties of $\{\tau_k, w_k\}$, $k = 1,\ldots,N$, in Section 4.2)

$$\forall \rho \in (0, \rho_{k-1}]\ \exists \overline{\tau}_k = \overline{\tau}_k(\rho) > 0,\ M_k = M_k(\rho) > 0,\ \xi_l^k \in C^{1+\frac{\vartheta}{2}}([S_k,T_k]),\ l = 0,...,n+1, \tag{42}$$
$$\text{such that} \qquad \|\xi_l^k\|_{C^{1+\frac{\vartheta}{2}}([S_k,T_k])} \leq M_k.$$

By the properties of $\{\tau_k, w_k\}$, keeping in mind Remark 4.5, we observe that

$$\dot{\xi}_l^k(S_k) = \mu_k(x_l^* - x_j^0), \quad l = 0,\ldots,n+1. \tag{43}$$

Step 1: *Some preliminary evaluations.*
Gap estimate. Let $k = 1,\ldots,N$ and $l = 0,\ldots,n$ be. We note that

$$\begin{aligned}
\xi_{l+1}^k(t) - \xi_l^k(t) =& \xi_{l+1}^k(t) - x_{l+1}^{k-1} + x_{l+1}^{k-1} - x_l^{k-1} + x_l^{k-1} - \xi_l^k(t) \\
=& x_{l+1}^{k-1} - x_l^{k-1} + \int_{S_k}^t \left(\dot{\xi}_{l+1}^k(s) - \dot{\xi}_l^k(s)\right) ds \\
=& x_{l+1}^{k-1} - x_l^{k-1} + \int_{S_k}^t \left(\dot{\xi}_{l+1}^k(s) - \dot{\xi}_{l+1}^k(S_k)\right) ds - \int_{S_k}^t \left(\dot{\xi}_l^k(s) - \dot{\xi}_l^k(S_k)\right) ds \\
&+ \left(\dot{\xi}_{l+1}^k(S_k) - \dot{\xi}_l^k(S_k)\right)(t - S_k),\ \ \forall t \in [S_k, T_k].
\end{aligned} \tag{44}$$

By (42), for every $j = 0, \dots, n+1$, we deduce that

$$\int_{S_k}^{t} \left|\dot{\xi}_j^k(s) - \dot{\xi}_j^k(S_k)\right| ds \le M_k \int_{S_k}^{t} (s - S_k)^{\frac{\vartheta}{2}} \, ds \le M_k \, (t - S_k)^{1+\frac{\vartheta}{2}}\,, \qquad \forall t \in [S_k, T_k]. \tag{45}$$

By (45) we have

$$\left|\int_{S_k}^{t} \left(\dot{\xi}_{l+1}^k(s) - \dot{\xi}_{l+1}^k(S_k)\right) ds - \int_{S_k}^{t} \left(\dot{\xi}_l^k(s) - \dot{\xi}_l^k(S_k)\right) ds\right| \le 2M_k \, (t - S_k)^{1+\frac{\vartheta}{2}}; \forall t \in [S_k, T_k]. \tag{46}$$

By (43), (44) and (46), for every $t \in [S_k, T_k]$, we deduce

$$\xi_{l+1}^k(t) - \xi_l^k(t) \ge x_{l+1}^{k-1} - x_l^{k-1} - 2M_k \tau_k^{1+\frac{\vartheta}{2}} + \left(\mu_k(x_{l+1}^* - x_{l+1}^0) - \mu_k(x_l^* - x_l^0)\right)(t - S_k)\,. \tag{47}$$

Distance evaluation. Let $k = 1, \dots, N$ and $l = 1, \dots, n$ be. By (43) we have

$$\begin{aligned}
\xi_l^k(t) - x_l^* =& \xi_l^k(t) - x_l^{k-1} + x_l^{k-1} - x_l^* = x_l^{k-1} - x_l^* + \int_{S_k}^{t} \dot{\xi}_l^k(s) ds \\
=& x_l^{k-1} - x_l^* + \dot{\xi}_l^k(S_k)\,(t - S_k) + \int_{S_k}^{t} \left(\dot{\xi}_l^k(s) - \dot{\xi}_l^k(S_k)\right) ds \\
=& x_l^{k-1} - x_l^* + \mu_k(x_l^* - x_l^0)\,(t - S_k) + \int_{S_k}^{t} \left(\dot{\xi}_l^k(s) - \dot{\xi}_l^k(S_k)\right) ds, \; \forall t \in [S_k, T_k].
\end{aligned} \tag{48}$$

Step 2: *Uniform time τ_0^* and constant M_0^*.* Let $k = 1, \dots, N$. Without loss of generality, we can suppose, by an induction argument, that we have already proved that

$$\rho_{h-1} = \min_{l=0,\dots,n} \{x_{l+1}^{h-1} - x_l^{h-1}\} \ge \frac{\rho_0^*}{2}, \text{ for every } h = 1, \dots, k. \text{ ([16])} \tag{49}$$

Then, choosing $\rho = \frac{\rho_0^*}{2} \in (0, \rho_{h-1}]$, $h = 1, \dots, k$ in (42), we obtain

$$M_h = M_h\Big(\frac{\rho_0^*}{2}\Big) = M_0^*, \quad \overline{\tau}_h = \overline{\tau}_h\Big(\frac{\rho_0^*}{2}\Big) = \tau_0^*, \; \forall h = 1, \dots, k. \tag{50}$$

Define $\varepsilon_0^* = \min\left\{\frac{4M_0^* s_\vartheta}{\rho_0^*} {\tau_0^*}^{1+\frac{\vartheta}{2}}, 1\right\}$, fix $\varepsilon \in (0, \varepsilon_0^*]$, and let

$$\tau_h \le \widetilde{\tau}_h = \widetilde{\tau}_h(\varepsilon) := \left(\frac{\varepsilon \rho_0^*}{4M_0^* \, s_\vartheta}\right)^{\frac{2}{2+\vartheta}} \frac{1}{h}, \;\; h = 1, \dots, k\,. \text{ ([17])} \tag{51}$$

So, we have

$$\widetilde{\tau}_h(\varepsilon) \le \tau_0^*, \;\; h = 1, \dots, k. \tag{52}$$

In the final Step.5, by (47), (50)-(52), and a technical proof, for every $l = 0, \dots, n$ and $k = 1, \dots, N$, we will prove the following alternative inequalities

$$\begin{aligned}
\xi_{l+1}^k(t) - \xi_l^k(t) \ge& x_{l+1}^{k-1} - x_l^{k-1} - 2M_0^* \tau_k^{1+\frac{\vartheta}{2}}, && \forall t \in [S_k, T_k], \\
&\text{or} \\
\xi_{l+1}^k(t) - \xi_l^k(t) \ge& \rho_0^* - \varepsilon \frac{\rho_0^*}{4} \ge \frac{3}{4}\rho_0^*, && \forall t \in [S_k, T_k].
\end{aligned} \tag{53}$$

[16] Let us recall that by definition $\rho_0 = \min_{l=0,\dots,n} \{x_{l+1}^0 - x_l^0\} \ge \frac{\rho_0^*}{2}$.

[17] $\widetilde{\tau}_h$ and τ_h are defined in (35) and (38), see also Remark 4.4.

Step 3: *Let us prove* $\rho(W^N) \geq \frac{\rho_0^*}{2}$. In this step, we will prove that $\xi_{l+1}^k(t) - \xi_l^k(t) \geq \frac{\rho_0^*}{2}$, $\forall t \in [S_k, T_k]$, $k = 1, \ldots, N$. By (51), (53) for k=1, we have

$$\begin{aligned}\min_{l=0,\ldots,n} \min_{t\in[S_1,T_1]} \left(\xi_{l+1}^1(t) - \xi_l^1(t)\right) &\geq \min\left\{\min_{l=0,\ldots,n}\left(x_{l+1}^0 - x_l^0\right) - 2M_0^*\tau_1^{1+\frac{\vartheta}{2}}, \frac{3}{4}\rho_0^*\right\}\\ &\geq \min\left\{\rho_0^* - 2M_0^*\frac{\varepsilon\rho_0^*}{4M_0^*\, s_\vartheta}, \frac{3}{4}\rho_0^*\right\}\\ &= \min\left\{\rho_0^*\left(1 - \frac{\varepsilon}{2s_\vartheta}\right), \frac{3}{4}\rho_0^*\right\} \geq \frac{\rho_0^*}{2}.\end{aligned}$$

By (51), (53) for $k = 2$ and $k = 1$, we obtain

$$\begin{aligned}\min_{t\in[S_1,T_1]} \min_{l=0,\ldots,n} \left(\xi_{l+1}^2(t) - \xi_l^2(t)\right) &\geq \min\left\{\min_{l=0,\ldots,n}\left(x_{l+1}^1 - x_l^1\right) - 2M_0^*\tau_2^{1+\frac{\vartheta}{2}}, \frac{3}{4}\rho_0^*\right\}\\ &\geq \min\left\{\min_{l=0,\ldots,n}\left(x_{l+1}^0 - x_l^0\right) - 2M_0^*\left(\tau_1^{1+\frac{\vartheta}{2}} + \tau_2^{1+\frac{\vartheta}{2}}\right), \frac{3}{4}\rho_0^*\right\}\\ &\geq \min\left\{\rho_0^* - 2M_0^*\frac{\varepsilon\rho_0^*}{4M_0^*\, s_\vartheta}\left(1 + \frac{1}{2^{1+\frac{\vartheta}{2}}}\right), \frac{3}{4}\rho_0^*\right\}\\ &= \min\left\{\rho_0^*\left(1 - \frac{1 + \frac{1}{2^{1+\frac{\vartheta}{2}}}}{2s_\vartheta}\varepsilon\right), \frac{3}{4}\rho_0^*\right\} \geq \frac{\rho_0^*}{2}.\end{aligned}$$

In general, in the case $N \geq 3$, for every $k = 3, \ldots, N$, by (51) and (53) we deduce

$$\begin{aligned}\min_{l=0,\ldots,n} \min_{t\in[S_k,T_k]} \left(\xi_{l+1}^k(t) - \xi_l^k(t)\right) &\geq \min\left\{\min_{l=0,\ldots,n}\left(x_{l+1}^{k-1} - x_l^{k-1}\right) - 2M_0^*\tau_k^{1+\frac{\vartheta}{2}}, \frac{3}{4}\rho_0^*\right\}\\ \geq \min\left\{\min_{l=0,\ldots,n}\left(x_{l+1}^0 - x_l^0\right) - 2M_0^*\left(\tau_1^{1+\frac{\vartheta}{2}} + \tau_2^{1+\frac{\vartheta}{2}} + \ldots + \tau_k^{1+\frac{\vartheta}{2}}\right), \frac{3}{4}\rho_0^*\right\}&\\ \geq \min\left\{\rho_0^* - 2M_0^*\frac{\varepsilon\rho_0^*}{4M_0^*\, s_\vartheta}\left(1 + \frac{1}{2^{1+\frac{\vartheta}{2}}} + \ldots + \frac{1}{k^{1+\frac{\vartheta}{2}}}\right), \frac{3}{4}\rho_0^*\right\}&\\ = \min\left\{\rho_0^*\left(1 - \frac{1 + \frac{1}{2^{1+\frac{\vartheta}{2}}} + \ldots + \frac{1}{k^{1+\frac{\vartheta}{2}}}}{2s_\vartheta}\varepsilon\right), \frac{3}{4}\rho_0^*\right\} \geq \frac{\rho_0^*}{2}.&\end{aligned}$$

Thus, it follows

$$\rho(W^N) := \min_{l=0,\ldots,n} \min_{t\in\mathcal{E}} \left\{\xi_{l+1}^W(t) - \xi_l^W(t)\right\} = \min_{l=0,\ldots,n} \min_{k=1,\ldots,N} \min_{t\in[S_k,T_k]} \left(\xi_{l+1}^k(t) - \xi_l^k(t)\right) \geq \frac{\rho_0^*}{2}.$$

Step 4: *Proof of* (40). For $N \in \mathbb{N}$, $N > n$, let $W^N = \left\{(\tau_k)_1^N, (w_k)_1^N\right\} \in \mathcal{W}^*(u_0)$, as in (39). Let $k = 1, \ldots, N$ and $l = 1, \ldots, n$. Since in Step. 3 we have proved that $\rho(W^N) \geq \frac{\rho_0^*}{2}$, we note that inequality (45) holds with $M_k = M_k(\frac{\rho_0^*}{2}) = M_0^*$. Then,

- ⋆ if $l \in L_{IS}^{k-1}$, keeping in mind that $\mu_k(x_l^* - x_l^0) = 0$, by (48) and (45) with $M_k = M_0^*$, we obtain

$$|\xi_l^k(t) - x_l^*| \leq |x_l^{k-1} - x_l^*| + M_0^*\,(t - S_k)^{1+\frac{\vartheta}{2}}, \qquad \forall t \in [S_k, T_k]; \qquad (54)$$

⋆ if $l \in \{1,\dots,n\}\backslash L_{IS}^{k-1}$, keeping in mind that $x_l^{k-1} \neq x_l^*$, we have

$$x_l^{k-1} - x_l^* + \mu_k(x_l^* - x_l^0)\ (t - S_k) =(x_l^{k-1} - x_l^*)\left(1 + \frac{\mu_k(x_l^* - x_l^0)}{x_l^{k-1} - x_l^*}\,(t - S_k)\right)$$
$$=(x_l^{k-1} - x_l^*)\left(1 - \frac{t - S_k}{|x_l^{k-1} - x_l^*|}\right),\ \forall t \in [S_k, T_k]. \quad (55)$$

Thus, by (48), (55) and (45) with $M_k = M_0^*$, keeping in mind that by the definition of τ_k (see (35) and (38)) it follows that $\tau_k \leq \min\limits_{l\in\{1,\dots,n\}\backslash L_{IS}^{k-1}} |x_l^{k-1} - x_l^*|$, we deduce

$$|\xi_l^k(t) - x_l^*| \leq |x_l^{k-1} - x_l^*|\left|1 - \frac{t - S_k}{|x_l^{k-1} - x_l^*|}\right| + M_0^*\ (t - S_k)^{1+\frac{\vartheta}{2}}$$
$$= |x_l^{k-1} - x_l^*| - (t - S_k) + M_0^*\ (t - S_k)^{1+\frac{\vartheta}{2}}, \qquad \forall t \in \Big[S_k, T_k\Big]. \quad (56)$$

Let us set $J_k(t) := \sum_{l=1}^{n} |\xi_l^k(t) - x_l^*|$, $t \in [S_k, T_k]$.
By (54) and (56), for every $k = 1,\dots,N$, for every $t \in [S_k, T_k]$, we obtain

$$\begin{aligned}
J_k(t) =& \sum_{l\in\{1,\dots,n\}\backslash L_{IS}^{k-1}} |\xi_l^k(t) - x_l^*| + \sum_{l\in L_{IS}^{k-1}} |\xi_l^k(t) - x_l^*| \\
\leq& \sum_{l\in\{1,\dots,n\}\backslash L_{IS}^{k-1}} |x_l^{k-1} - x_l^*| - \big(n - card(L_{IS}^{k-1})\big)\ (t - S_k) \\
&+ \big(n - card(L_{IS}^{k-1})\big)\ M_0^*\ (t - S_k)^{1+\frac{\vartheta}{2}} + \sum_{l\in L_{IS}^{k-1}} |x_l^{k-1} - x_l^*| + card(L_{IS}^{k-1})\, M_0^*\ (t - S_k)^{1+\frac{\vartheta}{2}} \\
=& J_k(S_k) - \big(n - card(L_{IS}^{k-1})\big)\ (t - S_k) + n\, M_0^*\ (t - S_k)^{1+\frac{\vartheta}{2}} \\
\leq& J_k(S_k) + n\, M_0^*\ \tau_k^{1+\frac{\vartheta}{2}} - (t - S_k), \qquad \forall t \in [S_k, T_k]. \qquad (57)
\end{aligned}$$

Now, keeping in mind that $J_h(S_h) = J_{h-1}(T_{h-1})$, $h = 2,\dots,N$, by (57) we can deduce that

$$\begin{aligned}
J_1(t) =& \sum_{l=1}^{n} |\xi_l^1(t) - x_l^*| \leq J_1(S_1) + n\, M_0^*\ \tau_1^{1+\frac{\vartheta}{2}} - (t - S_1),\ \forall t \in [S_1, T_1], \\
J_2(t) =& \sum_{l=1}^{n} |\xi_l^2(t) - x_l^*| \leq J_2(S_2) + n\, M_0^*\tau_2^{1+\frac{\vartheta}{2}} - (t - S_2) = J_1(T_1) + n\, M_0^*{\tau_2}^{1+\frac{\vartheta}{2}} - (t - S_2) \\
&\leq J_1(S_1) + n\, M_0^*\left(\tau_1^{1+\frac{\vartheta}{2}} + \tau_2^{1+\frac{\vartheta}{2}}\right) - (\tau_1 + t - S_2), \qquad \forall t \in [S_2, T_2], \\
J_k(t) =& \sum_{l=1}^{n} |\xi_l^k(t) - x_l^*| \leq J_k(S_k) + n\, M_0^*\ \tau_k^{1+\frac{\vartheta}{2}} - (t - S_k) \\
&\leq J_1(S_1) + n\, M_0^*\ \left(\tau_1^{1+\frac{\vartheta}{2}} + \dots + \tau_k^{1+\frac{\vartheta}{2}}\right) - (\tau_1 + \dots + \tau_{k-1} + t - S_k), \\
&\qquad \forall t \in [S_k, T_k],\ \forall k = 3,\dots,N. \quad (58)
\end{aligned}$$

By (58), keeping in mind that $J_1(S_1) = \sum_{l=1}^{n} |x_l^0 - x_l^*|$ and $L_{IS}^{N-1} \subseteq \{1, \dots, n\}$, by the definition of $\widetilde{\tau}_k$ and τ_k (see (35) and (38)) and Remark 4.4, inequality (40) follows. Indeed,

$$
\begin{aligned}
J^*(W^N) &= \sum_{l=1}^{n} |\xi_l^W(T_N) - x_l^*| = J_N(T_N) \le \sum_{l=1}^{n} |x_l^0 - x_l^*| + nM_0^* \sum_{k=1}^{N} \tau_k^{1+\frac{\vartheta}{2}} - \sum_{k=1}^{N} \tau_k \\
&\le \sum_{l=1}^{n} |x_l^0 - x_l^*| + \frac{\varepsilon \rho_0^* n}{4 s_\vartheta} \sum_{k=1}^{N} \frac{1}{k^{1+\frac{\vartheta}{2}}} - \sum_{\{k \mid \Theta_k \neq \emptyset\}} \tau_k - \left(\frac{\varepsilon \rho_0^*}{4M_0^* \, s_\vartheta} \right)^{\frac{2}{2+\vartheta}} \sum_{\{k \mid \Theta_k = \emptyset\}} \frac{1}{k} \\
&\le \sum_{l=1}^{n} |x_l^0 - x_l^*| + \frac{\varepsilon \rho_0^* n}{4 s_\vartheta} \sum_{k=1}^{N} \frac{1}{k^{1+\frac{\vartheta}{2}}} - \left(\frac{\varepsilon \rho_0^*}{4M_0^* \, s_\vartheta} \right)^{\frac{2}{2+\vartheta}} \sum_{k=n+1}^{N} \frac{1}{k}.
\end{aligned}
$$

Step 5: *Proof of* (53). In this final step, assuming that (49) holds, we prove that inequality (47) implies the two alternative inequalities in (53). Let us set

$$
A_l^k := \mu_k(x_{l+1}^* - x_{l+1}^0) - \mu_k(x_l^* - x_l^0), \quad \forall k = 1, \dots, N, \ \forall l = 0, \dots, n.
$$

Let $k = 1, \dots, N$ and $l = 0, \dots, n$.
We note that if $l, l+1 \in L_{IS}^{k-1}$ it follows $A_l^k = 0$, thus we obtain the first of the two alternatives in (53). Instead, if at least one between l and $l+1$ is not an inactive index (that is $l, l+1 \notin L_{IS}^{k-1}$ or $l \in L_{IS}^{k-1}, l+1 \notin L_{IS}^{k-1}$ or $l \notin L_{IS}^{k-1}, l+1 \in L_{IS}^{k-1}$), since the conditions

$$
x_l^{k-1} < x_{l+1}^{k-1} \quad \text{and} \quad x_l^* < x_{l+1}^*, \quad l = 0, \dots, n,
$$

hold, we have to distinguish six possible configurations to calculate A_l^k, that are presented in the following table:

Table 1

Configurations of the points	i)	ii)
a	$x_l^{k-1} < x_{l+1}^{k-1} \le x_l^* < x_{l+1}^*$	$x_l^* < x_{l+1}^* \le x_l^{k-1} < x_{l+1}^{k-1}$
b	$x_l^{k-1} \le x_l^* \le x_{l+1}^{k-1} \le x_{l+1}^*$	$x_l^* \le x_l^{k-1} \le x_{l+1}^* \le x_{l+1}^{k-1}$
c	$x_l^{k-1} \le x_l^* < x_{l+1}^* \le x_{l+1}^{k-1}$	$x_l^* \le x_l^{k-1} < x_{l+1}^{k-1} \le x_{l+1}^*$

Thus, in the following table we can compute A_l^k in any configuration in the 3 cases:
$l, l+1 \notin L_{IS}^{k-1}$; $l \in L_{IS}^{k-1}$ and $l+1 \notin L_{IS}^{k-1}$; $l \notin L_{IS}^{k-1}$ and $l+1 \in L_{IS}^{k-1}$.

Table 2:

$l, l+1 \notin L_{IS}^{k-1}$

A_l^k	i)	ii)
a	0	0
b	0	0
c	-2 ♣	2

$l \in L_{IS}^{k-1}, l+1 \notin L_{IS}^{k-1}$

A_l^k	i)	ii)
a	1	-1$^\diamond$
b	1	-1$^\diamond$
c	-1$^\diamond$	1

$l \notin L_{IS}^{k-1}, l+1 \in L_{IS}^{k-1}$

A_l^k	i)	ii)
a	-1♠	1
b	-1♠	1
c	-1♠	1

We can observe that when $A_l^k \ge 0$ (in Table 2), that is in the cases and in the configurations without superscript symbols (without ♣ or $\diamond$ or ♠), by (47) we easily obtain the first of the two alternative inequalities in (53).

Before analyzing the cases with superscript symbols, we remark that if $j \in L_{IS}^{k-1}$, there exists h_j, $0 \le h_j \le k-1$ such that $x_j^{h_j} = x_j^*$, then keeping in mind that $\mu_{h_j}(x_j^* - x_j^0) = \ldots = \mu_k(x_j^* - x_j^0) = 0$, thus, by (48), (45), (49)-(52), we obtain

$$\begin{aligned}|\xi_j^k(t) - x_j^*| \le& |x_j^{k-1} - x_j^*| + M_0^* \, \tau_k^{1+\frac{\vartheta}{2}} \le |x_j^{k-2} - x_j^*| + M_0^* \left(\tau_{k-1}^{1+\frac{\vartheta}{2}} + \tau_k^{1+\frac{\vartheta}{2}}\right)\\ \le& |x_j^{h_j} - x_j^*| + M_0^* \left(\tau_{h_j+1}^{1+\frac{\vartheta}{2}} + \ldots + \tau_{k-1}^{1+\frac{\vartheta}{2}} + \tau_k^{1+\frac{\vartheta}{2}}\right)\\ =& M_0^* \sum_{s=h_j+1}^{k} \tau_s^{1+\frac{\vartheta}{2}} \le M_0^* \frac{\varepsilon \rho_0^*}{4 M_0^* \, s_\vartheta} \sum_{s=h_j+1}^{k} \frac{1}{k^{1+\frac{\vartheta}{2}}} \le \varepsilon \frac{\rho_0^*}{4}, \qquad \forall t \in [S_k, T_k], \quad (59)\end{aligned}$$

so, the point of sign change with index $j \in L_{IS}^{k-1}$ remain forever near the corresponding target point of sign change already reached.

Now, we can analyze the following 3 cases:

♣ In the case of $l, l+1 \notin L_{IS}^{k-1}$, by the configuration $i.c$), for every $t \in [S_k, T_k]$ we have

$$x_l^{k-1} \le \xi_l^k(t) \le x_l^* < x_{l+1}^* \le \xi_{l+1}^k(t) \le x_{l+1}^{k-1} \Longrightarrow \xi_{l+1}^k(t) - \xi_l^k(t) \ge x_{l+1}^* - x_l^* \ge \rho_0^*,$$

from which the second option of (53) follows.

♢ In the case of $l \in L_{IS}^{k-1}, l+1 \notin L_{IS}^{k-1}$, we analyze the following 3 configurations:

ii.a) By (59), we deduce $x_l^* < x_{l+1}^* \le x_l^{k-1} \le x_l^* + \frac{\varepsilon\rho}{4}$, then $\rho_0^* \le x_{l+1}^* - x_l^* \le x_l^{k-1} - x_l^* \le \frac{\varepsilon\rho_0^*}{4} \le \frac{\rho_0^*}{4}$, from which a contradiction follows, so this configuration is not admissible.

ii.b) By (59), we deduce

$$\begin{aligned}\xi_l^k(t) \le x_l^* + \frac{\varepsilon\rho_0^*}{4} \le x_{l+1}^* \le \xi_{l+1}^k(t) \le x_{l+1}^{k-1} \Longrightarrow \xi_{l+1}^k(t) - \xi_l^k(t) \ge& x_{l+1}^* - \left(x_l^* + \frac{\varepsilon\rho_0^*}{4}\right)\\ \ge& \rho_0^* - \frac{\varepsilon\rho_0^*}{4} \ge \frac{3}{4}\rho_0^*,\end{aligned}$$

from which the second option of (53) follows.

i.c) It is similar to the configuration $ii.b$).

♠ In the case of $l \notin L_{IS}^{k-1}, l+1 \in L_{IS}^{k-1}$, we analyze the following 3 configurations:

i.a) It is similar to $ii.a$) of the case ♢.

i.b) By (59), we deduce

$$\begin{aligned}x_l^{k-1} \le \xi_l^k(t) \le x_l^* \le x_{l+1}^* - \frac{\varepsilon\rho_0^*}{4} \le \xi_{l+1}^k(t) \Longrightarrow \xi_{l+1}^k(t) - \xi_l^k(t) \ge& \left(x_{l+1}^* - \frac{\varepsilon\rho_0^*}{4}\right) - x_l^*\\ \ge& \rho_0^* - \frac{\varepsilon\rho_0^*}{4} \ge \frac{3}{4}\rho_0^*,\end{aligned}$$

from which the second option of (53) follows.

i.c) It is similar to the configuration $i.b$). ⋄

5 Proof of Theorem 3

In this section, we prove Theorem 3. Without loss of generality, we can reformulate problem (6) with a generic time interval $(0,T)$, in the following way

$$\begin{cases} u_t \ = \ u_{xx} \ + \ v(x,t)u \ + \ f(u) & \text{in} \quad Q_T=(0,1)\times(0,T), \\ u(0,t)=u(1,t)=0, & t\in(0,T), \\ u \ |_{t=0}=u_{in}+r_{in}, & \end{cases} \tag{60}$$

where u_{in}, $r_{in}\in H^1_0(0,1)$, and u_{in} have exactly n points of sign change at $x_l\in(0,1)$, with $0=: x_0<x_l<x_{l+1}\le x_{n+1}:=1,\;\; l=1,\ldots,n$.

Throughout this section, we represent the solution of (60) as the sum of two functions $w(x,t)$ and $h(x,t)$, which solve the following problems in Q_T

$$\begin{cases} w_t \ = \ w_{xx} \ + \ v(x,t)w \ + \ f(w) \\ w(0,t)=w(1,t)=0 \\ w \ |_{t=0}=u_{in}, \end{cases} \qquad \begin{cases} h_t \ = \ h_{xx} \ + \ v(x,t)h \ + \ (f(w+h)-f(w)) \\ h(0,t)=h(1,t)=0 \\ h \ |_{t=0}=r_{in}. \end{cases} \tag{61}$$

In this section, we denote the target state by $\overline{u}\in H^1_0(0,1)$ instead of the specific w_k introduced in the statement of Theorem 3.

Lemma 5.1 *Let $\overline{u}\in H^1_0(0,1)$ have the same points of sign change of u_{in}, in the same order of sign change. Let us suppose that*

$$\exists\nu>0:\ \nu\le\frac{\overline{u}(x)}{u_{in}(x)}<1, \qquad \forall\, x\in(0,1)\setminus\bigcup_{l=1}^{n}\{x_l\}\,. \tag{62}$$

Then, for every $\eta>0$ there exist a sufficiently small time $T=T(\eta,u_{in},\overline{u})>0$ and a piecewise static bilinear control $v=v(\eta,u_{in},\overline{u})\in L^\infty(Q_T)$ such that

$$\|u(\cdot,T)-\overline{u}(\cdot)\|_{L^2(0,1)}\le\eta+\sqrt{2}\|r_{in}\|_{L^2(0,1)}, \tag{63}$$

where u is the solution of (60) *on Q_T.*

Proof. Here, we adapt the proof introduced in Section 2 of [11] from the linear case to the semilinear problem (60). We consider the following function defined on $[0,1]$

$$v_0(x) \ = \ \begin{cases} \ln\left(\frac{\overline{u}(x)}{u_{in}(x)}\right), & \text{for} \ \ x\neq 0,1,x_l,\ l=1\ldots,n \\ 0, & \text{for} \ \ x=0,1,x_l,\ l=1\ldots,n \end{cases}$$

By (62) we note that $v_0\in L^\infty(0,1)$ and $v_0(x)\ \le 0,$ for every $x\in[0,1]$. We select the bilinear control

$$v(x,t) \ := \ \frac{1}{T}v_0(x).$$

Then, let us represent the solution u of (60), associated to the previous choice of the coefficient v, as a sum of two functions $w(x,t)$ and $h(x,t)$, which solve the two problems introduced in (61), respectively.

Step 1: *Representation formula for $w(\cdot,T)$.* For every fixed $\bar{x}\in(0,1)$, let us consider the non-homogeneous first-order ODE $w'(\bar{x},t)=\frac{v_0(\bar{x})}{T}w(\bar{x},t)\ +\big(w_{xx}(\bar{x},t)+f(w(\bar{x},t))\big),\ t\in(0,T),$

associated to the first problem in (61). Then, we easy deduce that the corresponding solution w to the first problem in (61) admits the following representation

$$w(x,t) \ = \ e^{v_0(x)\frac{t}{T}} u_{in}(x) \ + \ \int_0^t e^{v_0(x)\frac{(t-\tau)}{T}} (w_{xx}(x,\tau) + f(w(x,\tau)))d\tau, \quad \forall (x,t) \in Q_T,$$

and, at time $t = T$ we have

$$w(x,T) \ = \ \overline{u}(x) \ + \int_0^T e^{v_0(x)\frac{(T-t)}{T}} (w_{xx}(x,\tau) + f(w(x,\tau)))dt, \quad \forall x \in (0,1). \tag{64}$$

Let us show that the integral in the right-hand side of (64) tends to zero in $L^2(0,1)$ as $T \to 0+$, which would mean that $w(\cdot,T) \to \overline{u}$ in $L^2(0,1)$ at the same time.
Note that, since $v_0(x) \leq 0$, we deduce the following estimate

$$\|w(x,T) - \overline{u}(x)\|^2_{L^2(0,1)} = \int_0^1 \left(\int_0^T e^{v_0(x)\frac{(T-\tau)}{T}} (w_{xx}(x,\tau) + f(w(x,\tau)))d\tau \right)^2 dx$$
$$\leq \ T \parallel w_{xx} + f(w) \parallel^2_{L^2(Q_T)} . \tag{65}$$

Step 2: *Evaluation of* $\parallel w_{xx} + f(w) \parallel^2_{L^2(Q_T)}$. In this step, let us suppose that $v_0 \in C^2([0,1])$; this assumption will be removed in Step. 3.
Multiplying by w_{xx} the equation in the first problem in (61) with $v(x,t) = \frac{1}{T}v_0(x) \leq 0$, integrating over Q_T and applying Hölder's inequality, we have

$$\parallel w_{xx} \parallel^2_{L^2(Q_T)} = \int_0^T \int_0^1 w_t w_{xx} dxdt \ - \ \frac{1}{T}\int_0^T \int_0^1 v_0 w w_{xx} dxdt - \int_0^T \int_0^1 f(w) w_{xx} dxdt$$
$$\leq \int_0^T \int_0^1 w_t w_{xx} dxdt \ - \ \frac{1}{T}\int_0^T \int_0^1 v_0 w w_{xx} dxdt + \frac{1}{2}\int_0^T \int_0^1 f^2(w) dxdt + \frac{1}{2}\int_0^T \int_0^1 w^2_{xx} dxdt \ .$$

Thus, integrating by parts and recalling that $v_0(x) \leq 0$, we obtain

$$\parallel w_{xx} \parallel^2_{L^2(Q_T)} \leq \ 2\int_0^T \int_0^1 w_t w_{xx} dxdt \ - \ \frac{2}{T}\int_0^T \int_0^1 v_0 w w_{xx} dxdt \ + \ \int_0^T \int_0^1 f^2(w) dxdt$$
$$= \ - \int_0^T \int_0^1 (w^2_x)_t dxdt \ + \ \frac{2}{T}\int_0^T \int_0^1 v_0 w^2_x dxdt \ + \ \frac{1}{T}\int_0^T \int_0^1 v_{0x}(w^2)_x dxdt \ + \int_0^T \int_0^1 f^2(w) dxdt$$
$$\leq \ \int_0^1 u^2_{in\,x} dx \ - \int_0^1 w^2_x(T,x) dxdt \ - \ \frac{1}{T}\int_0^T \int_0^1 v_{0xx} w^2 dxdt \ + \ \int_0^T \int_0^1 f^2(w) dxdt$$
$$\leq \int_0^1 u^2_{in\,x} dx + \frac{1}{T} \max_{x \in [0,1]} \mid v_{0xx} \mid \int_0^T \int_0^1 w^2 dxdt + \int_0^T \int_0^1 f^2(w) dxdt. \tag{66}$$

Now, we have to evaluate $\parallel w \parallel_{C([0,T];L^2(0,1))}$ and $\parallel f(w) \parallel_{C([0,T];L^2(0,1))}$. Since $v_0(x) \leq 0$, multiplying by w the equation in the first problem of (61) and integrating by parts yields

$$\frac{1}{2}\int_0^t \int_0^1 (w^2)_t dxds = \int_0^t \int_0^1 w_t w dxds$$
$$= \int_0^t \int_0^1 w_{xx} w \, dxds + \frac{1}{T}\int_0^t \int_0^1 v_0 w^2 dxds + \int_0^t \int_0^1 f(w) w dxds$$
$$\leq - \int_0^t \int_0^1 w^2_x \, dxds + L\int_0^T \int_0^1 w^2 dxdt \leq L\int_0^T \int_0^1 w^2 dxdt,$$

where L is the Lipschitz constant in (2). Then, for $T \in \left(0, \frac{1}{4L}\right)$ we deduce

$$\int_0^1 w^2(x,t)dx \leq \int_0^1 u_{in}^2(x)dx + 2L\int_0^T \int_0^1 w^2 dxdt$$

$$\leq \int_0^1 u_{in}^2(x)dx + 2LT\|w\|^2_{C([0,T],L^2(0,1))} \leq \| u_{in} \|^2_{L^2(0,1)} + \frac{1}{2}\|w\|^2_{C([0,T],L^2(0,1))}, \quad t \in (0,T), \tag{67}$$

so,

$$\| w \|_{C([0,T];L^2(0,1))} \leq \sqrt{2} \| u_{in} \|_{L^2(0,1)} . \tag{68}$$

From assumption (2) and (68) it follows that $f(w) \in C([0,T]; L^2(0,1))$ and the following estimate holds

$$\| f(w) \|_{C([0,T];L^2(0,1))} \leq L \| w \|_{C([0,T];L^2(0,1))} \leq \sqrt{2}L \| u_{in} \|_{L^2(0,1)} . \tag{69}$$

Due to (66) and (68)-(69), we also have

$$\begin{aligned}
&\| w_{xx} + f(w) \|^2_{L^2(Q_T)} \leq 2 \| w_{xx} \|^2_{L^2(Q_T)} + 2 \| f(w) \|^2_{L^2(Q_T)} \\
&\quad \leq 2\int_0^1 u_{in\,x}^2 dx + \frac{2}{T}\max_{x\in[0,1]} | v_{0xx} | \int_0^T \int_0^1 w^2 dxdt + 3\int_0^T \int_0^1 f^2(w)dxdt \\
&\quad \leq 2\int_0^1 u_{in\,x}^2 dx + \frac{2}{T}\max_{x\in[0,1]} | v_{0xx} | \cdot T \| w \|^2_{C([0,T];L^2(0,1))} + 3T \| f(w) \|^2_{C([0,T];L^2(0,1))} \\
&\quad \leq 2\int_0^1 u_{in\,x}^2 dx + \big(4\max_{x\in[0,1]} | v_{0xx} | + 6TL^2\big)\int_0^1 u_{in}^2 dx \\
&\quad \leq 2\left(1 + 2\max_{x\in[0,1]} | v_{0xx} | + 3TL^2\right) \| u_{in} \|^2_{H_0^1(0,1)} .
\end{aligned} \tag{70}$$

Step 3: *Convergence of* $w(\cdot, T)$ *to* $\overline{u}(\cdot)$. Note that in the previous step we can remove the assumption $v_0 \in C^2([0,1])$. Namely, if $v_0 \notin C^2([0,1])$ we could consider a sequence of uniformly bounded functions $\{v_{0j}\}_{j\in\mathbb{N}}, v_{0j} \in C^2([0,1]), v_{0j}(x) \leq 0, \forall x \in [0,1]$, approximating v_0 in $L^2(0,1)$. Then making use of the following limit relation

$$e^{v_{0j}(x)t/T}u_{in}(x) \mid_{t=T} \to e^{v_0(x)t/T}u_{in}(x) \mid_{t=T} = \overline{u}(x) \text{ in } L^2(0,1) \text{ as } j \to \infty,$$

we conclude that equality (64) still holds. Moreover, by (65) and (70), we deduce that

$$\|w(x,T) - \overline{u}(x)\|_{L^2(0,1)} \leq C(T) \| u_{in} \|^2_{H_0^1(0,1)}, \text{ where } C(T) \to 0 \text{ as } T \to 0^+. \tag{71}$$

Step 4: *Evaluation of* $\| h(\cdot, T) \|_{L^2(0,1)}$. Multiplying by h in the equation of the second problem of (61) and integrating by parts over Q_T, proceeding similarly to Step 2 (see in particular (67)) and keeping in mind (2), for every $T \in (0, \frac{1}{4L})$, yields

$$\begin{aligned}
\int_0^1 h^2(x,t)dx &\leq \int_0^1 r_{in}^2(x)dx + 2\int_0^T \int_0^1 (f(w+h) - f(w))h\,dxdt \\
&\leq \int_0^1 r_{in}^2(x)dx + 2L\int_0^T \int_0^1 h^2 dxdt \leq \|r_{in}\|^2_{L^2(0,1)} + 2LT\|h\|^2_{C([0,T];L^2(0,1))} \\
&\leq \|r_{in}\|^2_{L^2(0,1)} + \frac{1}{2}\|h\|^2_{C([0,T];L^2(0,1))}, \quad t \in (0,T).
\end{aligned}$$

Hence,

$$\| h \|_{C([0,T];L^2(0,1))} \leq \sqrt{2} \| r_{in} \|_{L^2(0,1)} . \tag{72}$$

Conclusions. Thus, recalling (61), (71) and (72), we obtain the conclusion. $\diamond$

Now we need to extend this result to the general case, that is, we have to prove Theorem 3.
Proof (of Theorem 3). Let us fix $\eta > 0$.
Step 0: *Approximating argument.* Without loss of generality we can suppose $u_{in}, \overline{u} \in C^1([0,1])$ and

$$|u'_{in}(x_l)| = 1, \qquad l = 0, \ldots, n+1. \tag{73}$$

Namely, if $u_{in}, \overline{u} \notin C^1([0,1])$ we could consider two sequences in $C^1([0,1])$, approximating in $L^2(0,1)$, u_{in} and $\overline{u}$, respectively, such that any function of the sequence approximating u_{in} satisfies condition (73).
Step 1: *Steering the system from u_{in} to Ku_{in}, with $K > 1$.* We consider the auxiliary function $\psi : [0,1] \longrightarrow \mathbb{R}$, defined in the following way

$$\psi(x) = \begin{cases} \dfrac{\overline{u}(x)}{u_{in}(x)}, & \text{if } x \in (0,1) \setminus \bigcup_{l=1}^{n} \{x_l\} \\ |\overline{u}'(x)|, & \text{if } x = x_l,\ l = 0, \ldots, n+1. \end{cases}$$

By (73) we have that $\lim_{x \to x_l} \dfrac{\overline{u}(x)}{u_{in}(x)} = |\overline{u}'(x_l)|, \ \forall l = 0, \ldots, n+1$, so $\psi \in C([0,1])$. Let us introduce the universal constant

$$K = K(u_{in}, \overline{u}) := \max_{x \in [0,1]} \psi(x) + 1 > 1.$$

For any $0 < \rho < \dfrac{\rho_0}{2} := \dfrac{1}{2} \min_{l=0,\ldots,n} \{x_{l+1} - x_l\}$, consider the following set

$$A_\rho := \bigcup_{l=0}^{n} \Big(x_l + \rho, x_{l+1} - \rho \Big).$$

From the definition of K it follows that

$$K > \max_{x \in \overline{A}_\rho} \left\{ \frac{\overline{u}(x)}{u_{in}(x)} \right\}, \quad \forall \rho \in \left(0, \frac{\rho_0}{2}\right). \tag{74}$$

Let us select

$$v(x,t) = m := \frac{\ln K}{t_1}, \quad (t,x) \in (0,1) \times (0,t_1),$$

for some arbitrarily small $t_1 > 0$. Then, let us apply the auxiliary constant bilinear control $v(x,t) = m > 0, \forall x \in (0,1)$ on the interval $(0,t_1)$. For $t = t_1$ the solution of the first problem in (61) has the following representation in Fourier series

$$\begin{aligned} w(x,t_1) = &\ H(x,t_1) \ + \ e^{mt_1} \sum_{p=1}^{\infty} 2e^{-(p\pi)^2 t_1} \left(\int_0^1 u_{in}(r) \sin p\pi r \, dr \right) \sin p\pi x \\ = &\ H(x,t_1) \ + \ e^{mt_1} \sum_{p=1}^{\infty} 2(e^{-(p\pi)^2 t_1} - 1) \left(\int_0^1 u_{in}(r) \sin p\pi r \, dr \right) \sin p\pi x \ + \ e^{mt_1} u_{in}(x) \\ = &\ H(x,t_1) \ + R(x,t_1) + K u_{in}(x), \end{aligned} \tag{75}$$

where

$$H(x,t_1) := \sum_{p=1}^{\infty} 2 \left[\int_0^{t_1} e^{(m-(p\pi)^2)(t_1-t)} \left(\int_0^1 f(w(r,t)) \sin p\pi r \, dr \right) dt \right] \sin p\pi x,$$

$$R(x,t_1) := K \sum_{p=1}^{\infty} 2(e^{-(p\pi)^2 t_1} - 1) \left(\int_0^1 u_{in}(r) \sin p\pi r \, dr \right) \sin p\pi x \, .$$

By Parseval's equality, we deduce

$$
\begin{aligned}
\|R(\cdot,t_1)\|^2_{L^2(0,1)} &= K^2 \left\| \sum_{p=1}^{\infty} (e^{-p\pi^2 t_1}-1) \left(\int_0^1 u_{in}(r)\sqrt{2}\sin p\pi r\, dr\right) \sqrt{2}\sin p\pi x \right\|^2_{L^2(0,1)} \\
&= K^2 \sum_{p=1}^{\infty} (e^{-p\pi^2 t_1}-1)^2 \left| \int_0^1 u_{in}(r)\sqrt{2}\sin p\pi r\, dr \right|^2 \\
&\le K^2 (1-e^{-\pi^2 t_1})^2 \sum_{p=1}^{\infty} \left| \int_0^1 u_{in}(r)\sqrt{2}\sin p\pi r\, dr \right|^2 \\
&= K^2 (1-e^{-\pi^2 t_1})^2 \|u_{in}\|^2_{L^2(0,1)}. \qquad (76)
\end{aligned}
$$

In the same way, using assumption (2), $f(0)=0$, and Hölder's inequality we obtain

$$
\begin{aligned}
\| H(\cdot,t_1) \|^2_{L^2(0,1)} &= \left\| \sum_{p=1}^{\infty} \left[\int_0^{t_1} e^{(m-(p\pi)^2)(t_1-t)} \left(\int_0^1 f(w(r,t))\sqrt{2}\sin p\pi r\, dr \right) dt \right] \sqrt{2}\sin p\pi x \right\|^2_{L^2(0,1)} \\
&= \sum_{p=1}^{\infty} \left| \int_0^{t_1} e^{(m-(p\pi)^2)(t_1-t)} \left(\int_0^1 f(w(r,t))\sqrt{2}\sin p\pi r\, dr \right) dt \right|^2 \\
&\le \sum_{p=1}^{\infty} \left(\int_0^{t_1} e^{2(m-(p\pi)^2)(t_1-t)} dt \right) \int_0^{t_1} \left| \int_0^1 f(w(r,t))\sqrt{2}\sin p\pi r\, dr \right|^2 dt \\
&\le \sum_{p=1}^{\infty} e^{2mt_1} t_1 \int_0^{t_1} \left| \int_0^1 f(w(r,t))\sqrt{2}\sin p\pi r\, dr \right|^2 dt \\
&= K^2 t_1 \int_0^{t_1} \sum_{p=1}^{\infty} \left| \int_0^1 f(w(r,t))\sqrt{2}\sin p\pi r\, dr \right|^2 dt = K^2 t_1 \int_0^{t_1} \int_0^1 f^2(w(r,t)) dr dt \\
&\le L^2\, K^2 t_1 \int_0^{t_1} \int_0^1 w^2(r,t) dr\, dt \le L^2\, K^2 t_1^2 \,\| w \|^2_{C([0,t_1];L^2(0,1))} . \qquad (77)
\end{aligned}
$$

Now, we have to evaluate $\|w\|_{C([0,t_1];L^2(0,1))}$. Multiplying by w the equation in the first problem of (61), integrating by parts, and arguing as in the proof of (67), for every $t \in (0,t_1)$, we have

$$
\begin{aligned}
\frac{1}{2}\int_0^t \int_0^1 (w^2)_t dx ds &= -\int_0^t \int_0^1 w_x^2\, dx ds + m \int_0^t \int_0^1 w^2 dx ds + \int_0^t \int_0^1 f(w) w dx ds \\
&\le (m+L) \int_0^t \int_0^1 w^2 dx ds,
\end{aligned}
$$

where L is as in (2). Then

$$
\int_0^1 w^2(x,t) dx \le \int_0^1 u_{in}^2(x) dx + 2(m+L) \int_0^t \int_0^1 w^2 dx ds, \quad t \in (0,t_1).
$$

Thus, applying Grönwall's inequality we deduce $\|w(t,\cdot)\|^2_{L^2(0,1)} \le e^{2(m+L)t_1} \|u_{in}\|^2_{L^2(0,1)}$, $t \in (0,t_1)$. So,

$$
\|w\|_{C([0,t_1];L^2(0,1))} \le e^{(m+L)t_1} \|u_{in}\|_{L^2(0,1)} = K e^{L\, t_1} \|u_{in}\|_{L^2(0,1)} \quad . \qquad (78)
$$

Making use of (75)-(78), we have that

$$\begin{aligned}\|w(\cdot,t_1)-Ku_{in}\|_{L^2(0,1)} &= \|H(x,t_1)\ +R(x,t_1)\|_{L^2(0,1)}\\ &\le K\left[(1-e^{-\pi^2 t_1})+t_1 K\,L\,e^{L\,t_1}\right]\|u_{in}\|_{L^2(0,1)}.\end{aligned} \tag{79}$$

Now, we evaluate $\|\ h(\cdot,t_1)\ \|_{L^2(0,1)}$. Multiplying by h both members of the equation in the second problem of (61) and integrating by parts, proceeding similarly to (78) and to the proof of Lemma 5.1, and keeping in mind (2) we obtain

$$\begin{aligned}\int_0^1 h^2(x,t)dx &\le \int_0^1 r_{in}^2(x)dx + 2\,m\int_0^{t_1}\int_0^1 h^2 dxdt + 2\int_0^{t_1}\int_0^1 (f(w+h)-f(w))h\,dxdt\\ &\le \int_0^1 r_{in}^2(x)dx + 2(m+L)\int_0^{t_1}\int_0^1 h^2 dxdt\,.\end{aligned}$$

Hence, using Grönwall's inequality, we have $\|h(t,\cdot)\|^2_{L^2(0,1)} \le e^{2(m+L)t_1}\|r_{in}\|^2_{L^2(0,1)},\ t\in(0,t_1)$, so

$$\|h(t_1,\cdot)\|^2_{L^2(0,1)} \le \|h\|_{C([0,t_1];L^2(0,1))} \le Ke^{L\,t_1}\|r_{in}\|_{L^2(0,1)}. \tag{80}$$

Thus, by (79) and (80), there exists $t_1 = t_1(\eta) > 0$, $t_1 \ll 1$ such that the following inequality holds

$$\|u(\cdot,t_1) - Ku_{in}(\cdot)\|_{L^2(0,1)} \le \frac{\sqrt{2}}{8}\eta + Ke^{L}\|r_{in}\|_{L^2(0,1)}. \tag{81}$$

Step 2: *Steering the system from $Ku_{in} + r_{in}$ to $\overline{u}$.* In this step, let us represent again the solution u of (60) as a sum of two functions $w(x,t)$ and $h(x,t)$, which solve the problems in (61) in $(0,1)\times(t_1,T)$, with the modified initial states Ku_{in} instead of u_{in} and $r_{in}(\cdot) = u(\cdot,t_1)-Ku_{in}(\cdot)$. By (74) it follows that

$$\frac{\overline{u}(x)}{Ku_{in}(x)} < 1, \qquad \forall x\in A_\rho,\ \forall\rho\in\left(0,\frac{\rho_0}{2}\right), \tag{82}$$

moreover,

$$\forall\rho\in\left(0,\frac{\rho_0}{2}\right),\ \exists\,\nu=\nu(\rho)>0\ :\ \ \nu\le\frac{\overline{u}(x)}{Ku_{in}(x)},\ \ \forall x\in A_\rho. \tag{83}$$

Keeping in mind the proof of Lemma 5.1, we consider the following function defined on $[0,1]$

$$v_0(x)\ =\ \begin{cases}\ln\left(\frac{\overline{u}(x)}{Ku_{in}(x)}\right), & x\in A_\rho,\\ 0, & \text{elsewhere in } [0,1]\,.\end{cases} \tag{84}$$

By (83) $v_0 \in L^\infty(0,1)$, and by (82) we have $v_0(x)\ \le 0,\ \forall x\in[0,1]$. Let us select the bilinear control

$$v(x,t)\ :=\ \frac{1}{T}v_0(x)\quad \forall x\in(0,1)\times(t_1,T).$$

We remark that, thanks to (82)-(83), the assumption (62) of Lemma 5.1 holds. Then proceeding similarly to Lemma 5.1, with a proof essentially identical, there exists $T = T(\eta) > t_1$ with $T-t_1$ sufficiently small ($0 < T - t_1 \ll \frac{1}{4L}$) we can obtain the following inequality (similar to (63) of Lemma 5.1)

$$\|u(\cdot,T)-\overline{u}_\rho\|_{L^2(0,1)} \le \frac{\eta}{4} + \sqrt{2}\|u(\cdot,t_1)-Ku_{in}(\cdot)\|_{L^2(0,1)}, \tag{85}$$

with

$$\overline{u}_\rho(x)\ =\ \begin{cases}\overline{u}(x), & x\in A_\rho,\\ 0, & \text{elsewhere in } [0,1].\end{cases}$$

Note that here exists $\overline{\rho} = \overline{\rho}(\eta) > 0$ such that

$$\| \overline{u}_{\overline{\rho}} - \overline{u} \|_{L^2(0,1)} < \frac{\eta}{2}. \tag{86}$$

Then, from (85), (86) and (81) we obtain the conclusion

$$\| u(\cdot, T) - \overline{u}(\cdot) \|_{L^2(0,1)} \leq \| u(\cdot, T) - \overline{u}_{\overline{\rho}}(\cdot) \|_{L^2(0,1)} + \| \overline{u}_{\overline{\rho}} - \overline{u} \|_{L^2(0,1)} \leq \eta + \sqrt{2} K e^L \|r_{in}\|_{L^2(0,1)} . \quad \diamond$$

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