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Traveling waves for a nonlocal KPP equation and mean-field game models of knowledge diffusion

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Abstract. We analyze a mean-field game model proposed by economists Lucas and Moll [J. Political Econ. 122 (2014)] to describe economic systems where production is based on knowledge growth and diffusion. This model reduces to a PDE system where a backward Hamilton–Jacobi–Bellman equation is coupled with a forward KPP-type equation with nonlocal reaction term. We study the existence of traveling waves for this mean-field game system, obtaining the existence of both critical and supercritical waves. In particular, we prove a conjecture raised by economists on the existence of a critical balanced growth path for the described economy, supposed to be the expected stable growth in the long run. We also provide nonexistence results which clarify the role of parameters in the economic model.

In order to prove these results, we build fixed point arguments on the sets of critical waves for the forced speed problem arising from the coupling in the KPP-type equation. To this purpose, we provide a full characterization of the whole family of traveling waves for a new class of KPP-type equations with nonlocal and nonhomogeneous reaction terms. This latter analysis has independent interest since it shows new phenomena induced by the nonlocal effects and a different picture of critical waves, compared to the classical literature on Fisher–KPP equations.

Contents

1. Introduction

There is a huge literature in macroeconomics devoted to the analysis of knowledge-based economic systems, where production and learning play key roles. As a sample reference, we cite here only [4] among the pioneering papers on this topic. The most recent contributions in this field have refreshed the interest in quantitative analysis of this kind of

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model; see e.g. [3, 14, 17, 19]. Along the lines of this research, in 2014 R. E. Lucas and B. Moll introduced a new refined model to describe an economy of knowledge growth and diffusion ([16]). This model, resulting in a system of PDEs, proved to be a source of many interesting mathematical questions, which are the object of this work.

Compared to other previous models in macroeconomics, Lucas and Moll put new emphasis on the interaction between the individual optimization and the evolution of the economic environment which results from individual behaviors. In their model, the agents are characterized by their level of productivity-related knowledge (or technology) and split their time between producing and meeting other people in order to exchange ideas and improve their knowledge. The evolution of this economy is globally described by the productivity distribution function, which is driven by people's choices. Conversely, the individual strategies search for an optimal equilibrium between the time devoted to producing and the time spent on increasing the technological level of production; this choice obviously depends itself on the global status of the economic environment.

This kind of interaction is typical in mean-field game models, which aim at studying the interplay (and the occurrence of Nash equilibria) between individual decisions and collective behavior. So far, mean-field game theory, introduced by Lasry and Lions (see [12, 15]), has been rapidly spreading in many fields of applications and currently leads to new interesting problems in the theory of PDEs. Nowadays, mean-field game theory attracts more and more interest among economists, since it provides support to develop models for heterogeneous agents. We refer the reader to [2] for a discussion of several mean-field game models in macroeconomics and we borrow from this paper the following short presentation of the Lucas–Moll model.

In this model, any single agent has some level of knowledge/productivity z and decides to allocate a fraction $s \in [0, 1]$ of their time (a unit of labor per year) to search for new ideas or technologies (in order to increase the productivity level) by interacting with other people, with $\alpha(s)$ Poisson rate of probability of meeting another agent. As a result of a meeting, the productivity associated to knowledge level becomes the maximum of the productivity of the two agents. Thus, the individual dynamics is described through the stochastic process $x_t := \log(z_t)$ (here z_t is the productivity level), which is governed by the SDE

$$dx_t = \sqrt{2}\kappa\, dB_t + dJ_t,$$

where B_t is a standard 1-dimensional Brownian motion, $\kappa > 0$, and J_t is a Poisson process with intensity $\alpha(s_t)$ that jumps when individuals meet someone with a higher level of productivity during the time s_t. The Brownian motion accounts for fluctuations in the individual productivity. In the language of economists, the Brownian noise may also represent the individual process of experimentation and innovation, whereas the learning process is referred to as imitation.

The agents' goal is to maximize their production, which is of course proportional to the time devoted to producing (that is, $(1 - s)$) and to the current level of productivity z. Then the value function of a single agent, conditionally to the initial condition $x_t = x$, is given by

$$v(t,x) = \sup_{s_\tau \in [0,1]} \mathbb{E}_{t,x} \int_t^{+\infty} e^{-\rho(\tau - t)} (1 - s_\tau) e^{x_\tau} \, d\tau.$$

Here ρ is the discount factor, while the control strategy of the agent is the process $\{s_t\}$ taking values in [0, 1] (recall that s_t also enters into the Poisson process J_t involved in the dynamics of x_t, which has intensity $\alpha(s_t)$).

The Bellman equation of dynamic programming yields the following Hamilton–Jacobi equation for v:

$$-\partial_t v - \kappa^2 \partial_{xx} v + \rho v = \max_{s \in [0,1]} \left\{ (1-s) e^x + \alpha(s) \int_x^{+\infty} (v(y,t) - v(x,t)) f(t,y) \, dy \right\},$$

where $f(t,x)$ is the density of the log-productivity distribution function at time t (i.e. the law of x_t). As derived by Lucas and Moll, the equation for f reads

$$\begin{aligned} \partial_t f - \kappa^2 \partial_{xx} f = \ & f(t,x) \int_{-\infty}^x \alpha(s^*(t,y)) f(t,y) \, dy \\ & - \alpha(s^*(t,x)) f(t,x) \int_x^{+\infty} f(t,y) \, dy, \end{aligned} \tag{1}$$

where $s^*(t,x)$ is the optimal feedback strategy of the agents.

It is not difficult to understand the equation for f as a balance of mass. Indeed, the density $f(t,x)$ changes according not only to the individual noise of the agents, but also to the exchange of knowledge among the population. In this respect, the right-hand side should be understood as a balance (at time t) between new people who upgrade their knowledge to level x by meeting someone with such a level, and people who leave level x because they increase their knowledge by learning from someone with higher technology. In particular, the $L^1(\mathbb{R})$ norm of f is preserved. We point out that the above description applies to an equilibrium configuration, in the spirit of Nash equilibria: indeed, the density f appears a priori as an exogenous datum in the optimization of the agents, and the equilibrium is realized a posteriori by assuming that f is actually driven by the optimal strategy used by the agents.

Summing up, the mean-field game (MFG) system proposed by Lucas and Moll in their knowledge–production model can be stated as follows:

$$\begin{cases} -\partial_t v - \kappa^2 \partial_{xx} v + \rho v = \max\limits_{s \in [0,1]} \left\{ (1-s) e^x + \alpha(s) \int_x^{+\infty} [v(y) - v(x)] f(y) \, dy \right\}, \\ \partial_t f - \kappa^2 \partial_{xx} f = f(x) \int_{-\infty}^x \alpha(s^*) f(y) \, dy - f(x) \alpha(s^*) \int_x^{+\infty} f(y) \, dy, \\ s^* = \operatorname{argmax} \left\{ (1-s) e^x + \alpha(s) \int_x^{+\infty} [v(y) - v(x)] f(y) \, dy \right\}, \\ f(0) = f_0, \end{cases} \tag{2}$$

which is set for $t > 0$, $x \in \mathbb{R}$ and the normalization condition $\int_{\mathbb{R}} f(t,y) \, dy = 1$.

Among the most important questions raised by Lucas and Moll in the analysis of this model, they addressed the problem of existence of traveling wave solutions for system (2). These solutions, which are called *balanced growth paths* in the language of economics, are solutions of the type

$$v(t,x) = e^{ct}\nu(x-ct), \quad f(t,x) = \varphi(x-ct). \tag{3}$$

As explained in [16], this kind of solution (usually rephrased in terms of the productivity variable z) plays a very crucial role in understanding the behavior of the economy in the long run and the existence of sustainable growth strategies. See also Remark 2 where we discuss the interpretation of our results in terms of the original model.

In [16], Lucas and Moll introduced a numerical algorithm to show the existence of balanced growth paths in the case that the agents are not affected by individual noise, which can be called the deterministic case ($\kappa = 0$) for system (2). Further results for the case without diffusion were given in [7, 8].

On one hand, introducing a diffusion term in the form of individual noise for the agents looks very natural for the model, since it allows one to consider fluctuations in the individual productivity and prevents some additional constraint for balance growth paths (like an a priori prescription of a Pareto tail for the initial distribution); see e.g. the discussion in [17].

On the other hand, in the diffusive case the analysis of traveling waves for system (2) looks more challenging and intriguing. In the case of constant learning technology function $\alpha(s) = \alpha_0$ (new ideas arrive without the need to go in search of other people) the cumulative distribution function $F(t,x) = \int_{-\infty}^{x} f(t,y)\,dy$ satisfies the classical Fisher–KPP equation ([13]). This case was extensively discussed in [17].

In the case of variable learning technology function $\alpha(s)$, it was conjectured in [2, 16] that system (2) admits balanced growth paths and, in particular, the limiting profile distribution in the long time should be a solution of the form (3) satisfying

$$c = 2\kappa\sqrt{\int_{\mathbb{R}} \alpha(s^*(y))\varphi(y)\,dy}. \tag{4}$$

This question is very relevant for the economic model because this would identify a critical growth rate in the long run for balanced growth paths.

The purpose of this article is to prove, under fairly general assumptions, the existence of such a critical traveling wave for system (2). From a PDEs viewpoint, this is especially interesting because it involves both a nontrivial extension of the standard analysis of Fisher–KPP equations and the construction of critical equilibria for the mean-field game system, namely a fixed point argument on a family of traveling waves.

Results in this direction were given in [20] for the case of a linear function $\alpha(s) = \alpha s$. By contrast, in the original model suggested by Lucas and Moll, $\alpha(\cdot)$ is supposed to be a strictly concave, increasing function such that $\alpha'(0) = +\infty$ and $\alpha'(1) > 0$. According to [16], this setting of assumptions seems to match real situations on account of experimental data, and power-type functions like $\alpha(s) = s^\eta$, $\eta \in (0,1)$ are typical examples.

In the economic interpretation, assuming $\alpha'(0) = +\infty$ implies that people will never stop searching for new ideas and a possibly small but not trivial fraction of time is devoted to searching for new technology, even at a large productivity level z. This results in the condition that the optimal policy s^* in (2) satisfies

$$s^*(t, x) > 0, \quad x \in \mathbb{R},$$

but of course $s^*(t, x) \to 0$ as $x \to \infty$. The second condition $\alpha'(1) > 0$ also has a clear interpretation in the model, namely that people with a sufficiently low level of knowledge should devote all their time to going in search of new technology; this means that there exists a threshold $z_0 > 0$ such that it is not convenient (or not possible) to start producing if the knowledge level is smaller than z_0 (this typically happens for new producers). In the logarithmic variable $x = \log z$, this implies that there exists $x_0 \in \mathbb{R}$ such that

$$s^*(t, x) \equiv 1 \quad \forall x \leq x_0.$$

Under the above constitutive assumptions on the learning technology function $\alpha(\cdot)$, in this paper we derive the following results:

- If $\rho \geq 2\kappa\sqrt{\alpha(1)}$ and $\alpha(1) > \kappa^2$, there exists a balanced growth path (i.e. a solution of (2) in the form (3)) with a growth rate c satisfying the critical identity (4). Moreover, it holds that $2\kappa^2 < c < 2\kappa\sqrt{\alpha(1)}$.
- For every c such that $2\kappa\sqrt{\alpha(1)} \leq c < \alpha(1) + \kappa^2$ and $c < \rho$, there exist balanced growth paths with growth rate c (which are not critical).
- There are no balanced growth paths with growth rate $c \leq 2\kappa^2$ or $c \geq \alpha(1) + \kappa^2$.

The first item above is our main contribution and proves the conjecture in [2] about the existence of traveling waves with critical growth. We refer to Theorem 2.2 for a precise statement, where we also discuss the optimality of the conditions on ρ, κ, α. Let us mention that the existence of a critical traveling wave for system (2) is also proved independently in the very recent paper [18] under the assumption that the discount factor ρ and the intensity α of the technology function are sufficiently large.

In the second item we show that there is a whole family of other traveling waves with supercritical speed. This proves to be consistent with the typical behavior of KPP-type equations. However, the existence of an upper bound ($\alpha(1) + \kappa^2$) for the velocities, which is optimal owing to the third item, is not an intrinsic feature of KPP equations and it is rather an outcome of the coupling with the value function v through the optimal feedback strategy s^*.

Unfortunately, the picture of all possible waves of system (2) is not yet completely understood, as we will discuss later. However, even if many questions remain open for the system, we believe that our analysis makes a significant advance towards the study of the long time convergence to a stable profile.

As is very typical in mean-field game systems, the construction of equilibria is a consequence of some fixed point argument. In this context, this leads us to a careful analysis of traveling waves for a nonlocal KPP-type equation. Indeed, if $F(t, x) := \int_{-\infty}^{x} f(t, y)\, dy$

is the cumulative distribution function, a direct computation (which we postpone to Section 2) reveals that (1) can be rewritten for $W := 1 - F$ as

$$\partial_t W - \kappa^2 \partial_{xx} W = W \int_{-\infty}^{x} A(t,y)(-\partial_x W)\, dy, \quad \text{with } A := \alpha \circ s^*, \tag{5}$$

together with the limiting conditions

$$W(t,-\infty) = 1, \quad W(t,+\infty) = 0.$$

This is a nonlocal reaction–diffusion equation which, in the case A constant, reduces to the classical Fisher–KPP equation

$$\partial_t W - \kappa^2 \partial_{xx} W = AW(1-W).$$

Traveling waves for system (2) require that $A = A(x - ct)$ (see Section 2.1), hence we are led to consider solutions of the form $W(t,x) = w(x-ct)$, i.e.

$$\begin{cases} -\kappa^2 w'' - cw' = w \displaystyle\int_{-\infty}^{x} A(y)(-w'(y))\, dy, \quad x \in \mathbb{R}, \\ w(-\infty) = 1, \quad w(+\infty) = 0, \quad w' < 0. \end{cases} \tag{6}$$

We point out that $w(x-ct)$ is not just a wave for equation (5) because one additionally assumes that the nonlocal kernel A is also moving with an imposed velocity c. As a matter of fact, (6) has to be understood as a *forced speed* problem. Thus, even though equation (5) can be rewritten as a more standard integro-differential equation by integrating by parts the right-hand side, the results on traveling waves for that class of equations (see e.g. [5,6,11]) do not apply to (6).

A major part of our work consists in the analysis of solutions to (6). This corresponds to the traveling wave problem for the cumulative distribution function with a given imposed policy s (and $A = \alpha \circ s$). Despite the large literature about nonlocal KPP equations, problem (6) presents some peculiar features which have not appeared in previous models. In this respect, we give several new contributions, of independent interest, to the study of forced speed waves for nonlocal KPP equations.

Assuming that $A(\cdot)$ is a nonnegative nonincreasing function satisfying $\bar{A} := A(-\infty) > A(+\infty) =: \underline{A}$, we can summarize as follows our results, to be compared with what is known for the standard local KPP case:

- Problem (6) admits waves for all $c > 2\kappa\sqrt{\underline{A}}$. If $c \geq 2\kappa\sqrt{\bar{A}}$, there exist waves with speed c and arbitrary normalization at any point $x_0 \in \mathbb{R}$. By contrast, if $2\kappa\sqrt{\underline{A}} < c < 2\kappa\sqrt{\bar{A}}$, for any given point x_0 there is a minimal height $\vartheta = \vartheta(x_0, c)$ such that waves with velocity c only exist with $w(x_0) \geq \vartheta$.
- For fixed speed $c \in (2\kappa\sqrt{\underline{A}}, 2\kappa\sqrt{\bar{A}})$, all possible waves are an ordered foliation indexed by the value $\int_{\mathbb{R}} A(y)(-w'(y))\, dy$, whose maximum is given by

$$\frac{c^2}{4} = \kappa^2 \int_{\mathbb{R}} A(y)(-w'(y))\, dy. \tag{7}$$

The unique wave which satisfies (7) is called *critical*; this is the wave of velocity c which, at any point, runs at the lowest possible height.

Let us point out how the analysis of (6) proves to be crucial in the study of system (2). In fact, our approach is built on a fixed point argument which requires an understanding of the full picture of possible waves for the single nonlocal KPP equation. Then imposing condition (7) will lead us to a wave for (2) satisfying the criticality condition (4). In this respect, our construction of the critical wave for the mean-field game system (2) looks completely different from the method employed in [18], where the authors use a topological degree argument and a suitable approximation procedure which automatically provides a wave for (2) satisfying the criticality condition (4), assuming the parameters ρ, α to be sufficiently large. The essential difference between the two approaches even raises the question of whether the obtained critical waves coincide.

The organization of this paper runs as follows. We leave to the next section the derivation of the traveling wave system and a more precise statement of our main results. As we mentioned, they involve both the single nonlocal KPP equation and the mean-field game system. Further comments on the optimality of our results are also given below. Then Section 3 is devoted to a detailed analysis of solutions to (6). In Section 4 we come back to system (2) and we prove the results on the mean-field game model.

2. Assumptions and main results

We come back to the mean-field game system (2) in order to make precise the setting of our assumptions. We assume that the learning technology function $\alpha(s)$ satisfies

$$\alpha \in C^0([0,1]) \cap C^2((0,1]) \quad \text{is increasing, strictly concave,} \tag{8}$$

together with

$$\alpha(0) = 0, \tag{9}$$

$$\alpha(1) > \kappa^2, \tag{10}$$

$$\lim_{s\to 0^+} \alpha'(s) = +\infty, \tag{11}$$

$$\alpha'(1) > 0. \tag{12}$$

We already explained in the introduction the interpretation of conditions (11) and (12) in terms of the knowledge diffusion–growth model. Besides, condition (10) will turn out to be necessary in order for balanced growth paths to exist (see Proposition 4.2 and Theorem 2.2). A natural interpretation is that there should be enough probability to meet people and enhance the individual level of knowledge in order for this model of economy to reach a significant balanced growth.

Another necessary assumption involves the discount rate ρ, i.e.

$$\rho > \kappa^2. \tag{13}$$

It will soon appear clear that this is a minimal condition even for the existence of solutions to (2). Further conditions will be needed on the discount rate in order to guarantee the existence of balanced growth paths, which we will discuss after Theorem 2.2.

2.1. Balanced growth paths and traveling waves

Here we derive the system of traveling waves which is associated to balanced growth paths for the Lucas–Moll model. Before giving a proper definition of admissible solutions, we start by making a few heuristic remarks on the solutions of system (2).

First of all, we stress that the Hamiltonian function

$$H(t,x;v) := \max_{s\in[0,1]}\left[(1-s)e^x + \alpha(s)\int_x^{+\infty}[v(t,y)-v(t,x)]f(t,y)\,dy\right] \tag{14}$$

requires the condition $v(t) \in L^1(f(t)\,dx)$ in order to be finite. Since

$$-\partial_t v - \kappa^2\partial_{xx}v + \rho v \geq e^x$$

by comparison (and the condition $\rho > \kappa^2$) we have $v \geq \frac{e^x}{(\rho-\kappa^2)}$, hence we are led to require $f(t)e^x \in L^1(\mathbb{R})$. This is to point out that a natural functional setting for system (2) should require

$$f(1+e^x) \in C^0([0,\infty); L^1(\mathbb{R})), \qquad \frac{v}{1+e^x} \in C^0([0,\infty); L^\infty(\mathbb{R})), \tag{15}$$

plus the natural condition that $f(t)$ be a probability density for all t.

It is also natural to guess that v be monotone with respect to x. This can be observed by differentiating the Hamilton–Jacobi equation. In fact, by standard parabolic regularity, locally bounded solutions (v, f) are at least of class $C^{(1+\vartheta)/2,1+\vartheta}$, in particular v is C^1 in the x variable. Then the strict concavity assumption on α allows us to use some form of the envelope theorem (see e.g. [9, Lemma 1]) which implies that $H(t,x;v)$ is differentiable in x. Differentiating the Bellman equation we deduce that v_x solves the equation

$$-\partial_t v_x - \kappa^2\partial_{xx}v_x + \rho v_x = (1-s^*)e^x - \alpha(s^*)v_x(1-F), \tag{16}$$

where

$$F(t,x) := \int_{-\infty}^x f(t,y)\,dy$$

is the cumulative distribution function, and s^* is given in (2). Heuristically, this equation yields $v_x \geq 0$ and $v_x e^{-x} \in L^\infty((0,\infty)\times\mathbb{R})$.

The monotonicity of v also implies that $\int_x^{+\infty}[v(y)-v(x)]f(y)\,dy \geq 0$ and that the function s^* defined in (2) is a nonincreasing function of x. Indeed, since $\alpha(\cdot)$ is concave, the function

$$g(s;x) := (1-s)e^x + \alpha(s)\int_x^{+\infty}(v(t,y)-v(t,x))f\,dy$$

is also concave with respect to s and so either $g(\cdot; x)$ is decreasing in $[0, 1]$ or $s^*(t, x) := \sup\{\tau \in [0, 1] : g(\cdot; x)$ is increasing in $[0, \tau)\}$. But one can readily check that

$$\begin{aligned}&\{\tau \in [0, 1] : g(\cdot; x_2) \text{ is increasing in } [0, \tau)\}\\ &\qquad \subset \{\tau \in [0, 1] : g(\cdot; x_1) \text{ is increasing in } [0, \tau)\} \quad \forall x_1 < x_2,\end{aligned}$$

hence $s^*(t, x_2) \leq s^*(t, x_1)$.

In fact, it is possible to build a solution (v, f) of (2) satisfying the above properties, provided the discount rate is sufficiently large; however, we postpone to a forthcoming article a more detailed analysis of the existence of solutions to the system, which depends both on conditions on initial data and on the range of the discount factor.

Here, we only concentrate on balanced growth path solutions. The above discussion eventually leads us to the following definition.

Definition 2.1. A *balanced growth path* (BGP) solution of (2) with growth rate $c \in \mathbb{R}$ is a triple (f, v, s^*) such that

$$f = \varphi(x - ct), \quad v = e^{ct}\nu(x - ct), \quad s^* = \sigma(x - ct),$$

and the following properties are satisfied:

- $\varphi, \nu \in C^2(\mathbb{R})$, $\sigma \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$,
- $\varphi(1 + e^x) \in L^1(\mathbb{R})$, $e^x \int_x^{+\infty} \varphi(y)\, dy \in L^1(\mathbb{R})$,
- ν is increasing, nonnegative and $\nu' e^{-x} \in L^\infty(\mathbb{R})$,
- f, v are classical solutions of the MFG system (2) (with $f_0 = \varphi(x)$) and $s^*(t, x) = \operatorname{argmax}\{(1 - s)e^x + \alpha(s) \int_x^{+\infty} [v(t, y) - v(t, x)] f(t, y)\, dy\}$.

Let us anticipate that the growth rate c of a BGP will necessarily be positive (and actually larger than $2\kappa^2$), but we include the case $c \leq 0$ in the above definition in order to derive in particular the nonexistence of stationary solutions to (2).

We now proceed by showing that BGP solutions can be conveniently reformulated in terms of v_x and the CDF function F, and this formulation is well suited for traveling waves. Let (f, v) be a solution to (2). We first observe that, integrating by parts, we can rewrite (omitting the t variable) as

$$\begin{aligned}\int_x^{+\infty} [v(y) - v(x)] f(y)\, dy &= - \lim_{y \to +\infty} (1 - F(y))(v(y) - v(x)) + \int_x^{+\infty} v_x (1 - F)\, dy\\ &= \int_x^{+\infty} v_x (1 - F)\, dy, \end{aligned} \tag{17}$$

where we have used the monotonicity of ν and the integrability of $\varphi\nu$ to see that $(1 - F(y))v(y) \leq \int_y^{+\infty} f(s)v(s)\, ds \to 0$ as $y \to +\infty$. Due to (16), and using (17) in the definition of s^*, we see that the function $\zeta := v_x e^{-x}$ is bounded and satisfies

$$\begin{cases} -\partial_t \zeta - \kappa^2 \partial_{xx}\zeta - 2\kappa^2 \partial_x \zeta + (\rho - \kappa^2)\zeta = (1 - s^*) - \alpha(s^*)\zeta(1 - F), \\ s^*(t, x) = \underset{s \in [0,1]}{\operatorname{argmax}} \left\{ (1 - s)e^x + \alpha(s) \displaystyle\int_x^{+\infty} \zeta(t, y) e^y (1 - F(t, y))\, dy \right\}. \end{cases} \tag{18}$$

The equation for F is also readily found. Integrating the equation for f (and neglecting the terms at infinity), we have

$$\begin{aligned}\partial_t F - \kappa^2 \partial_{xx} F &= \int_{-\infty}^{x} f(\xi) \int_{-\infty}^{\xi} \alpha(s^*) f(y)\, dy - \int_{-\infty}^{x} \alpha(s^*(\xi)) f(\xi) \int_{\xi}^{+\infty} f(y)\, dy \\ &= -\Big[(1 - F(\xi)) \int_{-\infty}^{\xi} \alpha(s^*) f(y)\, dy\Big]_{-\infty}^{x} \\ &= -(1 - F(x)) \int_{-\infty}^{x} \alpha(s^*) f(y)\, dy,\end{aligned}$$

where we just used integration by parts. Therefore, the function F solves the nonlocal KPP equation

$$\partial_t F - \kappa^2 \partial_{xx} F + (1 - F) \int_{-\infty}^{x} \alpha(s^*(y))(\partial_x F(y))\, dy = 0, \tag{19}$$

i.e. $W := 1 - F$ satisfies (5). Now, if W is a traveling wave, i.e. it is of the form $W(t, x) = w(x - ct)$, one can look for ζ and s^* in the form of traveling waves too. This is consistent because if $\zeta(t, x) := v_x(t, x)e^{-x} = z(x - ct)$ for some function z, then

$$\begin{aligned}s^*(t, x) &= \underset{s\in[0,1]}{\operatorname{argmax}} \Big[(1 - s)e^x + \alpha(s) \int_{x}^{+\infty} v_x(t, y)(1 - F(t, y))\, dy\Big] \\ &= \underset{s\in[0,1]}{\operatorname{argmax}}\, e^{ct} \Big[(1 - s)e^{x-ct} + \alpha(s) \int_{x-ct}^{+\infty} z(y)e^y w(y)\, dy\Big],\end{aligned} \tag{20}$$

which implies that s^* is a function of $x - ct$, i.e. it is itself a traveling wave.

Summing up, in the case of BGP solutions, we have that

$$W(t, x) = 1 - \int_{-\infty}^{x} \varphi(y - ct)\, dy, \quad \zeta(t, x) = v_x(t, x)e^{-x} = e^{ct-x} \nu'(x - ct)$$

are traveling wave solutions of (5), (18), i.e. $W = w(x - ct)$, $\zeta = z(x - ct)$ and $s^* = \sigma(x - ct)$ are solutions to

$$\begin{cases} \kappa^2 w'' + cw' + w \displaystyle\int_{-\infty}^{x} A(y)(-w'(y))\, dy = 0, \quad x \in \mathbb{R}, \\ w \geq 0, \quad w(-\infty) = 1, \quad w(+\infty) = 0, \\ -\kappa^2 z'' + (c - 2\kappa^2)z' + (\rho - \kappa^2)z + A(x)wz = 1 - \sigma(x), \quad x \in \mathbb{R}, \\ z \geq 0, \quad z \text{ is bounded}, \quad ze^x w \in L^1(\mathbb{R}), \\ \sigma(x) = \underset{s\in[0,1]}{\operatorname{argmax}} \Big\{(1 - s)e^x + \alpha(s) \displaystyle\int_{x}^{+\infty} z(y)e^y w(y)\, dy\Big\}, \quad A := \alpha \circ \sigma. \end{cases} \tag{21}$$

This will be the framework where traveling waves will be sought. Let us notice that the conditions at infinity for w are induced by mass conservation in the original system, with the normalization condition $\int_{-\infty}^{+\infty} f = 1$. The conditions for z follow from the conditions on v_x discussed above. The condition $e^x w \in L^1(\mathbb{R})$ is necessary to give proper sense to σ in (21); this is why this condition is required in the definition of BGP solutions. We stress that, using elliptic estimates and the Harnack inequality, the first equation in (21) implies $|w'(x)| \leq C w(x)$, so the condition $e^x w \in L^1(\mathbb{R})$ itself implies a similar condition for w' (which is the requirement $\varphi e^x \in L^1(\mathbb{R})$ appearing in Definition 2.1). Let us further recall that σ and A are nonincreasing and, as we will see in Proposition 4.2 below, they are also locally Lipschitz-continuous on $\mathbb{R}$.

The connection between BGP solutions of (2) and traveling wave solutions of (21) will be rigorously analyzed in Proposition 4.8. We only stress here that a one-to-one correspondence is easily given, following the above derivation, between the solutions (z, w) of (21) and the couple (v_x, f). However, an extra condition ($\rho > c$) will be needed in order to build the *balanced growth* value function v. This specifically comes from the requirement that v be positive, and is a natural condition in the described model; see also Remark 2.

2.2. Statement of the main results

We now state the main result of the paper.

Theorem 2.2. *Assume that hypotheses* (8)–(12) *and* (13) *hold true. Then we have the following:*

(i) *If there exists a BGP solution (with growth c) of* (2)*, then necessarily*

$$2\kappa^2 < c < \alpha(1) + \kappa^2 \quad \textit{and} \quad c < \rho$$

(hence (10) *and* (13) *are necessary for a BGP to exist).*

(ii) *If $\rho \geq 2\kappa\sqrt{\alpha(1)}$, there exists a BGP solution of* (2) *with growth $c \in (2\kappa^2, 2\kappa\sqrt{\alpha(1)})$ and such that* (4) *is satisfied.*

(iii) *For every $c \in [2\kappa\sqrt{\alpha(1)}, \alpha(1) + \kappa^2)$ such that $c < \rho$, there exist BGP solutions of* (2) *with growth c (which do not satisfy* (4)*).*

Remark 1. Several comments are in order to describe the above statement.

(a) The lower bound $c > 2\kappa^2$ for the growth rate of BGP solutions not only immediately implies the nonexistence of stationary solutions to (2), but further suggests that solutions to the system emerging from arbitrary initial data should be "driven" rightward with a positive *asymptotic speed*, in both their components.

(b) The condition $\alpha(1) > \kappa^2$ proves to be necessary to leave room for the existence of *some* traveling wave, solution of (21), hence for BGP solutions as well. By contrast, the restriction $\rho > c$ is not needed for the solutions (z, w) of (21) to exist. But this restriction is necessary for BGP solutions. In particular this is needed to build a consistent value function v once the traveling waves $v_x e^{-x}$ and f are proved to exist; we refer the reader to Proposition 4.8 for a better comprehension.

It is interesting to notice that this necessary condition $\rho > c$, linking the discount rate to the possible balanced growth, is very common in the economic literature. Indeed, as pointed out to us by Moll, this condition is usually needed both for neo-classical growth models (see e.g. [1]) and for balanced growth equilibria induced by endogenous growth (see e.g. [21]).

(c) As already mentioned in the previous item, there is a small gap between the pure analysis of system (21) and the BGP solutions of (2). For example, without the requirement $ze^x w \in L^1(\mathbb{R})$, solutions of (21) may be found with $\sigma \equiv 1$ in the (larger) range of parameters

$$\frac{3}{4}\kappa^2 \le \rho \le \kappa^2, \quad \kappa^2 - \rho \le \alpha(1) < \kappa^2,$$

and velocities $c \in [2\kappa\sqrt{\alpha(1)}, 2\kappa^2)$. These solutions however do not correspond to balanced growth paths because the derivation of the equations in (21) from (2) crucially relies on the condition $ze^x w \in L^1(\mathbb{R})$.

(d) The most important output of Theorem 2.2 is the existence of at least one wave with velocity $c \in (2\kappa^2, 2\kappa\sqrt{\alpha(1)})$, which in addition is *critical*, in the sense that it fulfills (4).
Notice that the speed c of this critical wave is not precisely known, and since $c < \rho$ is necessary for a BGP to exist, we have to assume $\rho \ge 2\kappa\sqrt{\alpha(1)}$ in order to guarantee the existence of at least one critical wave.
Unfortunately, not only we do not know whether this is the unique critical wave, but we also do not know yet if there are other traveling waves in this range of velocities (but we conjecture that other noncritical waves exist for c in this range). By contrast, we know much better what happens for $c \ge 2\kappa\sqrt{\alpha(1)}$; indeed, for every $c \in [2\kappa\sqrt{\alpha(1)}, \alpha(1) + \kappa^2)$ there are traveling waves with speed c, and they can have arbitrary normalization at any given point $x_0 \in \mathbb{R}$. This is a whole family of traveling waves with *supercritical speed*, because they cannot satisfy condition (4), since this latter condition implies $c < 2\kappa\sqrt{\alpha(1)}$.

(e) The critical wave found in Theorem 2.2 (ii) also satisfies the expected decay as $x \to \infty$, namely that $\frac{-w'}{w} \to \frac{c}{2\kappa^2}$.

Remark 2. Let us recall, from [16], that the solutions constructed in Theorem 2.2 have a clear interpretation in terms of the productivity variable $z = e^x$. Indeed, for a balanced growth path solution, the cumulative distribution function F, given in terms of z, takes the form

$$F = \Phi(e^{-ct} z)$$

for some increasing function Φ. This implies that all level sets of F (the qth quantiles of the CDF function) grow with the same exponential rate $c > 0$, because

$$\{z : F(t,z) = q\} = \{z = e^{ct}\Phi^{-1}(q)\}.$$

This fact justifies the name of *balanced growth path* solutions, in terms of the economy.

Let us also mention that the decay rate of the critical wave, mentioned in Remark 1 (e), is also significant for the economic model. This is usually interpreted by economists in terms of the Pareto tail of the CDF function; indeed, if $\frac{-w'}{w} \to \frac{c}{2\kappa^2}$, this means that $F(t,z)$ has a tail which decays (in polynomial scale) as $z^{-\frac{c}{2\kappa^2}}$ (the precise behavior for the KPP equation would actually suggest $F = O(z^{-\frac{c}{2\kappa^2}} \log z)$). In the language of economists, the value $\frac{2\kappa^2}{c}$ is called the *tail inequality* associated to the Pareto-like distribution. In this respect, our result also proves the conjecture in [2] that the critical balanced growth path for system (2) should have tail inequality equal to $\kappa(\int_{\mathbb{R}} \alpha(s^*(y))\varphi(y)\,dy)^{-1/2}$.

As is typical in mean-field game systems, the solutions we find in Theorem 2.2 arise from a fixed point argument. For this purpose, we first develop a deep study of traveling waves for the single nonlocal KPP equation

$$\begin{cases} w'' + cw' + w \displaystyle\int_{-\infty}^{x} A(y)(-w'(y))\,dy = 0, & x \in \mathbb{R}, \\ 0 \le w \le 1, \quad w(-\infty) = 1, \quad w(+\infty) = 0. \end{cases} \tag{22}$$

Here we have set the diffusion coefficient $\kappa = 1$; this is no loss of generality, up to rescaling c and A by $1/\kappa^2$. The main difficulties we have to face, compared with the classical KPP equation, come from the facts that this equation is inhomogeneous and nonlocal in the reaction term, which entail, respectively, that it is not translation invariant and that the comparison principle fails.

Equation (22) is obtained from the mean-field game system with $A := \alpha \circ \sigma$. This motivates the setting of assumptions we are interested in, namely, A is bounded and nonincreasing. We also exclude the case A constant because this reduces to the standard Fisher–KPP equation (for which basically everything is known).

In our analysis of problem (22), we completely characterize the whole family of traveling waves.

Theorem 2.3. *Assume that $A \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ is bounded and nonincreasing and that*

$$\bar{A} := \lim_{s \to -\infty} A(s) > \underline{A} := \lim_{s \to +\infty} A(s) \ge 0.$$

The traveling wave problem (22) admits solution if and only if $c > 2\sqrt{\underline{A}}$. For any $c > 2\sqrt{\underline{A}}$ the family of solutions is given by

$$\mathcal{F} := (w_\vartheta)_{\vartheta \in \Theta},$$

with w_ϑ satisfying $w_\vartheta(0) = \vartheta$ and

$$\Theta = \begin{cases} [\vartheta_c, 1) & \text{if } c \in (2\sqrt{\underline{A}}, 2\sqrt{\bar{A}}), \\ (0,1) & \text{if } c \in [2\sqrt{\bar{A}}, +\infty). \end{cases}$$

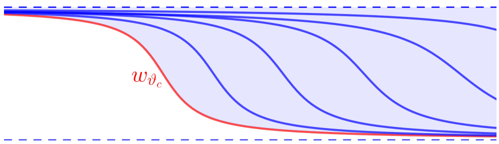
The $(w_\vartheta)_{\vartheta \in \Theta}$ are strictly ordered and $\vartheta \mapsto w_\vartheta$ is a continuous bijection from Θ to $\mathcal{F}$ equipped with the $L^\infty(\mathbb{R})$ norm.

Finally, the "critical" waves w_{ϑ_c} *depend continuously on* $c \in (2\sqrt{\underline{A}}, 2\sqrt{\bar{A}})$ *with respect to the* $L^\infty(\mathbb{R})$ *norm, and the values* $\vartheta_c = w_{\vartheta_c}(0)$ *satisfy*

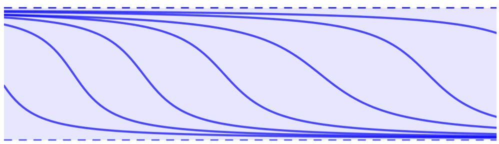
$$\vartheta_c \nearrow 1 \ \text{ as } c \searrow 2\sqrt{\underline{A}}, \quad \vartheta_c \searrow 0 \ \text{ as } c \nearrow 2\sqrt{\bar{A}}. \tag{23}$$

Of course, the choice of the point 0 for parametrizing $\mathscr{F}$ is purely arbitrary.

The fact that the waves with equal speed are ordered seems remarkable, because this property is typically out of reach for nonlocal, inhomogeneous problems, due to the lack of a comparison principle. We stress that the main interest of Theorem 2.3 lies in the range of velocities $(2\sqrt{\underline{A}}, 2\sqrt{\bar{A}})$, which reduces to the empty set when A is constant. So this is the range of traveling waves which come from the genuinely inhomogeneous (and nonlocal) forced speed term A. Outside this range, the picture is similar to the classical KPP equation: for any $c \geq 2\sqrt{\bar{A}}$ the graphs of the family of waves (which in the classical case are simply translations of the same profile) foliate the whole strip $\mathbb{R} \times (0, 1)$. By contrast, for $c \in (2\sqrt{\underline{A}}, 2\sqrt{\bar{A}})$, the foliation does not fill the whole strip, but only the region to the right of the "critical" wave. The situation is depicted in Figure 1.



(a) $c < 2\sqrt{\bar{A}}$



(b) $c \geq 2\sqrt{\bar{A}}$

Figure 1. The two different types of foliation.

The two different scenarios can be heuristically explained as follows: on one hand, if the transition of the wave from 1 to 0 takes place (for its main part) far to the right then it would be affected by values of A close to $\underline{A}$, and for such a value the range of admissible speeds is classically $c \geq 2\sqrt{\underline{A}}$; this is why fronts can be found for any $c > 2\sqrt{\underline{A}}$, and they converge pointwise to 1 as $c \searrow 2\sqrt{\underline{A}}$. Conversely, if the transition occurs far to the left then A would be close to $\bar{A}$ and then necessarily $c \gtrsim 2\sqrt{\bar{A}}$; hence, for a given speed $c < 2\sqrt{\bar{A}}$, the transition cannot occur too much to the left, or, equivalently, there must exist a pointwise lower bound for the wave.

In order to understand what happens for c in the range $(2\sqrt{\underline{A}}, 2\sqrt{\bar{A}})$, the following operator will be of crucial importance:

$$\mathcal{I}(w) := \int_{\mathbb{R}} A(y)(-w'(y))\,dy.$$

From the modeling point of view, $\mathcal{I}(w)$ is related to the total expectation of meetings for the given policy $A := \alpha \circ s$. Because of the condition $w(-\infty) = 1$, it can be equivalently written as

$$\mathcal{I}(w) = \bar{A} + \int_{\mathbb{R}} A'(y)w(y)\,dy.$$

This formulation enlightens the continuity and monotonicity of $\mathcal{I}$. The value of $\mathcal{I}$ on the critical wave w_{ϑ_c} turns out to encode the speed in a very transparent way: $\mathcal{I}(w_{\vartheta_c}) = c^2/4$. This immediately shows that there exist no traveling waves with speed $c \leq 2\sqrt{\underline{A}}$. The relationship between the waves and the functional $\mathcal{I}$ is summarized in the following result.

Theorem 2.4. *The mapping $\vartheta \to \mathcal{I}(w_\vartheta)$ is a decreasing homeomorphism between Θ and J, where*

$$J = \begin{cases} (\underline{A}, c^2/4] & \text{if } 2\sqrt{\underline{A}} < c < 2\sqrt{\bar{A}}, \\ (\underline{A}, \bar{A}) & \text{if } c \geq 2\sqrt{\bar{A}}. \end{cases}$$

Another key feature of the operator $\mathcal{I}$ is that it encodes the exponential rate of decay of the wave; see Proposition 3.12 below. Unfortunately, there is one property that we are still missing: the ordering of critical waves with different speeds. This would be of great help to construct a wave for system (21) through a fixed point argument. Nevertheless, we are able to derive the ordering for large $|x|$, cf. Propositions 3.12 and 3.21, and we use this to cook up a suitable selection principle for the fixed point argument.

3. The nonlocal KPP equation

This section is devoted to the study of the single traveling wave problem (22), which corresponds to an assigned production/research strategy. Namely, throughout this section we assume that $c \in \mathbb{R}$ is assigned and that A is a given function which fulfills the properties derived in Section 2.1, which are

$$A \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) \text{ is nonnegative, nonincreasing,}$$
$$\bar{A} := A(-\infty) > \underline{A} := A(+\infty).$$

In the next section, we start by collecting some tools on the nonlocal equation (22).

3.1. Preliminary toolbox

As a first step, we show basic properties of solutions to (22).

Proposition 3.1. *If* (22) *admits solution for some* $c \in \mathbb{R}$, *then necessarily* $c > 0$ *and* $w' < 0$ *in* $\mathbb{R}$. *In addition, for any* $R > 0$, *there exists a constant* C_R, *only depending on* R, $\bar{A} = A(-\infty)$ *and an upper bound for* c, *such that, for any* $x_0 \in \mathbb{R}$, *it holds that*

$$\max_{[x_0-R,x_0+R]} w \le C_R w(x_0), \qquad \max_{[x_0-R,x_0+R]} (1-w) \le C_R(1-w(x_0)). \tag{24}$$

Proof. We preliminarily observe that $w > 0$ thanks to the elliptic strong maximum principle. We then divide the equation in (22) by w and differentiate to get

$$\frac{w'''}{w} - \frac{w'w''}{w^2} + c\frac{w''}{w} - c\frac{(w')^2}{w^2} - Aw' = 0, \quad x \in \mathbb{R}.$$

Hence, the function $u := w'$ satisfies the equation

$$u'' + \Big(c - \frac{u}{w}\Big)u' - Awu = c\frac{u^2}{w}, \quad x \in \mathbb{R}, \tag{25}$$

with zero-order coefficient $-Aw \le 0$. Moreover, by the boundedness of w, we know that there exist two sequences $(x_n^{\pm})_{n\in\mathbb{N}}$ diverging to $\pm\infty$ respectively, such that $u(x_n^{\pm}) \to 0$ as $n \to \infty$. Applying the weak maximum principle to equation (25) in (x_n^-, x_n^+) and letting $n \to \infty$, we deduce that $u \le 0$ in $\mathbb{R}$ if $c \ge 0$, whereas $u \ge 0$ in $\mathbb{R}$ if $c \le 0$. Then, by the limiting conditions in (22), we necessarily have that $c > 0$ and $u \le 0$. The strict inequality $w' = u < 0$ follows by applying the strong maximum principle to (25).

As for the Harnack inequalities (24), the first one comes from standard elliptic theory, because w solves an equation as $w'' + cw' + Vw = 0$ where the potential V satisfies $0 \le V \le \bar{A}$. As for the second one, we observe that if w solves (22) then $v(x) := 1 - w(x)$ satisfies

$$-v'' - cv' + (1-v)g(x)v = 0,$$

where

$$g(x) := \frac{1}{v}\Big(\bar{A} - A(1-v) + \int_{-\infty}^{x} A'(y)(1-v(y))\,dy\Big).$$

On one hand, using $A'v \le 0$ in the above integral shows that $g \ge 0$. On the other hand, the fact that $A'v$ is decreasing yields

$$g(x) \le \frac{1}{v}(\bar{A} - A(1-v) + (1-v)(A-\bar{A})) = \bar{A}.$$

Therefore, we conclude as before that $v(x) \le C_R u(x_0)$ provided $|x - x_0| \le R$. This gives the second inequality in (24). ∎

It will be handy to reformulate the equation in (22). Namely, under the condition $w(-\infty) = 1$, integrating by parts the nonlocal term leads to

$$w'' + cw' + w\Big(\bar{A} - Aw + \int_{-\infty}^{x} A'(y)w(y)\,dy\Big) = 0, \quad x \in \mathbb{R}. \tag{26}$$

The advantage of this equation, compared with the one in (22), is that the only identically constant solutions are 0 and 1. Now we show that *all other solutions to* (26) *are decreasing waves connecting* 1 *and* 0 and then, in particular, they are solutions of (22).

Lemma 3.2. *Let* $0 \leq w \leq 1$ *be a solution of* (26) *for some* $c \geq 0$. *Then either* $w \equiv 0$, *or* $w \equiv 1$, *or* $c > 0$ *and* w *satisfies*

$$w(-\infty) = 1, \quad w(+\infty) = 0 \quad \text{and} \quad w' < 0 \text{ in } \mathbb{R}.$$

Moreover, there exists a constant L, *only depending on* c, $\bar{A}$, *such that* $\|w'''\|_\infty \leq L$.

Proof. We first observe that globally bounded solutions of (26) are also bounded in C^3. Indeed, using the monotonicity of A and the bounds on w, we notice that w solves a linear equation $w'' + cw' + wV = 0$, where

$$0 \leq V := \left(\bar{A} - Aw + \int_{-\infty}^{x} A'(y)w(y)\,dy \right) \leq \bar{A}.$$

By elliptic estimates (see e.g. [10, Theorem 9.11]), given any point $a \in \mathbb{R}$ we have

$$|w'(a)| \leq L \sup_{x \in [a-1,a+1]} |w(x)| \leq L,$$

where L only depends on c, $\bar{A}$. Then the same conclusion holds true (with a larger L) for $w'' = -cw' - wV$. We then bootstrap by differentiating this equation and observing that $V' = -Aw'$. This shows that $|w'''| \leq L$ for some L depending on c, $\bar{A}$.

Let us now show that bounded solutions are decreasing waves. First we observe that $A' \leq 0$ and $w \leq 1$ imply

$$-w'' - cw' \geq A(x)w(1-w) \geq 0, \quad x \in \mathbb{R}. \tag{27}$$

We treat separately the cases $c = 0$ and $c > 0$.

Case $c = 0$. In this case (27) yields $w'' \leq 0$ in $\mathbb{R}$, hence w is constant. Then, since $A(-\infty) = \bar{A} > 0$, (27) shows that the only possibilities are $w \equiv 0$ or $w \equiv 1$.

Case $c > 0$. Inequality (27) implies that $-(w'e^{cx})' \geq 0$, which, integrated on $(-\infty, x)$ ($w'e^{cx}$ vanishes at $-\infty$ because w' is bounded), yields $w'(x) \leq 0$ for any $x \in \mathbb{R}$. Differentiating (26) we get the following equation for w':

$$(w')'' + c(w')' + w'\left(\bar{A} - 2Aw + \int_{-\infty}^{x} A'(y)w(y)\,dy \right) = 0.$$

We deduce from the elliptic strong maximum principle that either $w' < 0$ in $\mathbb{R}$, or $w' \equiv 0$. In the latter case, as before, we infer from (27) and $\bar{A} > 0$ that $w \equiv 0$ or $w \equiv 1$.

We are left with the case $w' < 0$. In this case $w(\pm\infty)$ exist and satisfy $0 \leq w(-\infty) < w(+\infty) \leq 1$. We integrate the integral in (26) by parts to get

$$-w'' - cw' \geq w\left(\bar{A}(1 - w(-\infty)) + \int_{-\infty}^{x} A(y)(-w'(y))\,dy \right). \tag{28}$$

The two terms on the right-hand side are nonnegative. Suppose by contradiction that $w(-\infty) < 1$. Then there exists $k > 0$ such that the right-hand side is larger than k for $x \leq 0$, i.e.

$$-(w'e^{cx})' \geq ke^{cx}. \tag{29}$$

Integrating on $(-\infty, x)$, for given $x < 0$, we obtain $-w'(x) \geq \frac{k}{c}$, which is impossible. Therefore, $w(-\infty) = 1$. If, on the other hand, $w(+\infty) > 0$, it is the integral term in (28) that is larger than some $k > 0$ for $x \geq x_0$ such that $A(x_0) > 0$, i.e. (29) holds for $x \geq x_0$. Integrating on (x_0, x) yields

$$-w'(x) \geq \frac{k}{c} - e^{-c(x-x_0)}\Big(\frac{k}{c} - w'(x_0)\Big),$$

which is again a contradiction. This concludes the proof. ■

In the next step we study how to build solutions of (26) using shooting and comparison methods for ODEs in truncated domains. A key role will be played by the Cauchy problem in the half-line.

Lemma 3.3. *Let $P, Q \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ and $K \in L^\infty_{\text{loc}}(\mathbb{R})$. Given $c, a, \eta \in \mathbb{R}$, $\vartheta > 0$, the Cauchy problem*

$$\begin{cases} w'' + cw' + w\Big(P(x) + Q(x)w + \displaystyle\int_a^x K(y)w(y)\,dy\Big) = 0, & x > a, \\ w(a) = \vartheta, \\ w'(a) = \eta, \end{cases} \tag{30}$$

admits a unique (classical) positive solution w in some interval $[a, b)$, with either $b = +\infty$, or $b \in (a, +\infty)$ and $w(b^-) = 0$ or $+\infty$. Moreover, such a solution depends continuously on c, a, ϑ, η, as well as on P, Q and K with respect to $W^{1,\infty}_{\text{loc}}(\mathbb{R})$ and $L^\infty_{\text{loc}}(\mathbb{R})$ convergences respectively.

Proof. We formally divide the equation by w and differentiate. We get

$$\frac{w'''w - w'w''}{w^2} + c\frac{w''w - (w')^2}{w^2} + P' + Q'w + Qw' + Kw = 0, \quad x > a,$$

which implies

$$w''' - \frac{w'w''}{w} + cw'' - c\frac{(w')^2}{w} + Qww' + (Q' + K)w^2 + P'w = 0, \quad x > a.$$

We also have that $w''(a) = -c\eta - P(a)\vartheta - Q(a)\vartheta^2$. If $P, Q \in C^1(\mathbb{R})$ and $K \in C^0(\mathbb{R})$, this is a standard Cauchy problem of the third order, as long as w stays bounded away from 0. The existence, uniqueness and continuity with respect to the data then follow from the classical theory. In the general case, the same properties are consequences of Carathéodory's existence theorem. The resulting solution w is such that w'' is absolutely continuous and therefore it is a classical solution of (30). ■

The next tool is a comparison principle and will play a crucial role in our analysis. Here and in the sequel, whatever elliptic equation is given in the form

$$w'' = F(x, w, w'),$$

we say that w is a subsolution (respectively, supersolution) if w satisfies $w'' \geq F(x, w, w')$ (respectively, $w'' \leq F(x, w, w')$).

Lemma 3.4. *Let $c, a \in \mathbb{R}$ and P, Q, K satisfy the assumptions of Lemma 3.3. In addition, assume that Q and K are nonpositive.*

Let w_1 and w_2 be respectively a positive subsolution and a positive supersolution to the first equation of (30) in an interval $[a, \beta]$, with

$$w_1(a) \geq w_2(a), \quad w_1'(a) \geq w_2'(a), \quad w_2'(a) \leq 0.$$

Then w_1/w_2 is nondecreasing on $[a, \beta]$, and it is increasing if $w_1'(a) > w_2'(a)$.

Proof. Suppose first that $w_1'(a) > w_2'(a)$. Call $\rho := w_1/w_2$. Using all the information at the initial point, this function satisfies

$$\rho(a) \geq 1, \quad \rho'(a) = \frac{w_2(a)w_1'(a) - w_1(a)w_2'(a)}{w_1(a)^2} \geq \frac{w_2(a)}{w_1(a)^2}[w_1'(a) - w_2'(a)] > 0.$$

Let $\tilde{\beta}$ be the largest value in $(a, \beta]$ such that $\rho > 1$ in $(a, \tilde{\beta})$. Assume by contradiction that ρ is not increasing in $[a, \tilde{\beta}]$. This means that there exist $a \leq x_1 < x_2 \leq \tilde{\beta}$ such that $\rho(x_1) \geq \rho(x_2)$. Call

$$h := \max_{[a,x_2]} \rho > 1.$$

Let $\bar{x} \in [a, x_2]$ be such that $\rho(\bar{x}) = h$. We know that $\bar{x} \neq a$. Moreover, if $\rho(x_2) = h$ then necessarily $\rho(x_1) = h$. Hence, in any case, we can take $\bar{x} \in (a, x_2)$. We define $\psi := hw_2 - w_1$. Then $\psi \geq 0$ in (a, x_2) and $\psi(\bar{x}) = 0$. The function ψ satisfies the following differential inequality in $(a, \bar{x})$:

$$-\psi'' - c\psi' \geq hw_2\Big(P + Qw_2 + \int_a^x Kw_2\Big) - w_1\Big(P + Qw_1 + \int_a^x Kw_1\Big),$$

whence, using the fact that $w_1 > w_2 > 0$ in $(a, \tilde{\beta}) \supset (a, \bar{x})$ and that Q and K are nonpositive, we eventually deduce that, in $(a, \bar{x})$,

$$-\psi'' - c\psi' \geq \psi\Big(P + Qw_1 + \int_a^x Kw_1\Big),$$

which means that ψ is a supersolution of some linear elliptic equation. As a consequence, since ψ attains its minimal value 0 at $\bar{x}$, the Hopf lemma yields $\psi'(\bar{x}) < 0$ which implies that $\psi < 0$ is some right neighborhood of $\bar{x}$. This contradicts the definition of h. We have thereby shown that $\rho = w_1/w_2$ is strictly increasing in $[a, \tilde{\beta}]$, whence in particular $\rho(\tilde{\beta}) > 1$. It follows that $\tilde{\beta} = \beta$ and this concludes the proof in the case $w_1'(a) > w_2'(a)$.

Assume now that $w_1'(a) = w_2'(a) =: \hat{\eta}$. Fix a number $\vartheta \in [w_2(a), w_1(a)]$ and, for $\eta \in \mathbb{R}$, let w^η be the solution of (30) provided by Lemma 3.3. Applying the property derived before we deduce, on one hand, that if $\eta > \hat{\eta}$ then w^η/w_2 is increasing on $[a, \beta]$, and on the other that if $\eta < \hat{\eta}$ then w_1/w^η is increasing on some interval $[a, \beta_\eta]$ on which w^η

is positive. It follows from the continuous dependence with respect to the data, ensured by Lemma 3.3, that both $w^{\hat\eta}/w_2$ and $w_1/w^{\hat\eta}$ are nondecreasing on $[a,\beta]$ (and that $w^{\hat\eta}$ is positive there), whence

$$\frac{w_1}{w_2} = \frac{w_1}{w^{\hat\eta}}\frac{w^{\hat\eta}}{w_2}$$

is nondecreasing on $[a,\beta]$. ∎

We now deduce some consequences of the previous comparison principle. An easy one, readily observed, is that if there exists a (positive) constant subsolution of the equation, then any supersolution starting below must be nonincreasing.

Corollary 3.5. *For any $a < b$, $M > 0$, and $\vartheta \geq \gamma > 0$, consider the Dirichlet problem*

$$\begin{cases} w'' + cw' + w\left(M - A(x)w + \displaystyle\int_a^x A'(y)w(y)\,dy\right) = 0, & x \in (a,b),\\ w(a) = \vartheta, \quad w(b) = \gamma, \end{cases} \tag{31}$$

where $A(x) \in W^{1,\infty}(\mathbb{R})$ is nonincreasing.

Assume that $M \geq A(a)\vartheta$ and that there exists a positive solution W of the first equation of (31) such that $W(a) = \vartheta$ and $W'(a) \leq 0$.

Then, for every $\gamma \leq W(b)$ problem (31) admits a unique positive solution, which is nonincreasing and decreasing if $\gamma < W(b)$. Moreover, $\frac{W}{w}$ is increasing unless $w = W$.

Proof. For $\eta \leq 0$, let w_η be the solution to the first equation of (31), with initial condition

$$w^\eta(a) = \vartheta, \quad (w^\eta)'(a) = \eta.$$

Such a solution exists and is positive in a right neighborhood of a thanks to Lemma 3.3. Moreover, since $M \geq A(a)\vartheta$, the constant function $w^0 \equiv \vartheta$ is a subsolution; hence, due to $\eta \leq 0$ and Lemma 3.4, w^η is nonincreasing (and decreasing if $\eta < 0$). Lemma 3.4 also implies that the w^η are increasing with respect to η (in the set where they are positive), and it is readily seen that w^η vanishes before the point b if $-\eta$ is sufficiently large. Finally, if $\eta = W'(a)$, then $w^\eta = W$ by uniqueness. Therefore, for $\eta > W'(a)$, $w^\eta \geq W$ and remains positive up to $x = b$. It then follows from the continuity of the solution with respect to η, that for any $\gamma \leq W(b)$ there exists a unique value $\eta \leq W'(a) \leq 0$ such that $w^\eta(b) = \gamma$. Finally, applying Lemma 3.4 again, with $w_1 = W$, we deduce that W/w^η is nondecreasing and it is increasing if $\eta < W'(a)$. This concludes the proof. ∎

We complete our toolbox with another lemma showing that two waves will be arbitrarily close in the future provided they were sufficiently close in the past.

Lemma 3.6. *Let w be a solution to (22). For every $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $\widetilde{w}$ is another solution satisfying $|w(x) - \widetilde{w}(x)| \leq \delta$ for all $x \in (-\infty, x_0)$, then*

$$|w(x) - \widetilde{w}(x)| \leq \varepsilon \quad \forall x \in \mathbb{R}.$$

Proof. The difference $\psi := \tilde{w} - w$ of two solutions w, $\tilde{w}$ to (22) satisfies

$$\psi'' + c\psi' + \psi(\bar{A} - A(\tilde{w} + w)) = -\tilde{w}\int_{-\infty}^{x} A'\tilde{w} + w\int_{-\infty}^{x} A'w.$$

Assume now that $|w - \tilde{w}| \leq \delta$ in an interval $(-\infty, x_0)$. We can then estimate the right-hand side for $x \leq x_0$ as follows:

$$\left|-\tilde{w}\int_{-\infty}^{x} A'\tilde{w} + w\int_{-\infty}^{x} A'w\right| \leq |w - \tilde{w}|\int_{-\infty}^{x} |A'\tilde{w}| + w\int_{-\infty}^{x} (-A')|\tilde{w} - w| \leq 2\delta\bar{A}.$$

Thus, we can take $a = x_0 - 1$ and deduce from interior elliptic estimates that $|\psi'(a)| \leq C\delta$ for some C only depending on c and $\bar{A}$. We therefore have that at the point a both $\tilde{w} - w$ and $\tilde{w}' - w'$ are of order δ.

The function $\tilde{w}$ satisfies an initial value problem of the type (30), with $P = P_{\tilde{w}}$, Q and K given by

$$P_{\tilde{w}} := \bar{A} + \int_{-\infty}^{a} A'\tilde{w}, \quad Q(x) := -A(x), \quad K(x) := A'(x).$$

The function w satisfies the same type of problem, but with

$$P_w := \bar{A} + \int_{-\infty}^{a} A'w.$$

Using that $|w - \tilde{w}| \leq \delta$ in $(-\infty, a)$ we see that

$$\int_{-\infty}^{a} A'w - \delta(\bar{A} - A(a)) \leq \int_{-\infty}^{a} A'\tilde{w} \leq \int_{-\infty}^{a} A'w + \delta(\bar{A} - A(a)),$$

hence $|P_{\tilde{w}} - P_w| \leq \delta(\bar{A} - A(a))$. Recall that the values of $\tilde{w}$, $\tilde{w}'$ at a are close to those of w, w' up to an order δ. As a consequence, by Lemma 3.3, for any $\varepsilon > 0$ and $x_1 > a$, we can find $\delta < \varepsilon/2$ small enough so that $|\tilde{w} - w| < \varepsilon/2$ in $(-\infty, x_1]$. Choosing x_1 such that $w(x_1) < \varepsilon/2$, and recalling that w and $\tilde{w}$ are decreasing by Proposition 3.1, we conclude that $|w - \tilde{w}| \leq \varepsilon$ in $\mathbb{R}$. ∎

3.2. Construction of the traveling waves

This section is devoted to the construction of waves – i.e. solutions to (22). We will distinguish two cases depending on the range of the velocity c.

Case A: $c \geq 2\sqrt{\bar{A}}$. The construction of solutions of (22) is easier in this case because, for such values of c, we know that there is a traveling wave solution ψ for the classical KPP equation:

$$\begin{cases} \psi'' + c\psi' + \bar{A}\psi(1 - \psi) = 0, & x \in \mathbb{R}, \\ \psi(-\infty) = 1, \quad \psi(+\infty) = 0. \end{cases}$$

Observe that ψ is a supersolution of (22). We further know that $\psi' < 0$. We consider the normalization condition $\psi(0) = \vartheta$, with ϑ arbitrarily fixed in $(0, 1)$.

For $n \in \mathbb{N}$ and $\zeta \in \mathbb{R}$, we introduce the truncated problem

$$\begin{cases} w'' + cw' \\ \quad + w\left(A(-n)\psi(-n-\zeta) - Aw + \displaystyle\int_{-n}^{x} A'(y)w(y)\,dy\right) = 0, & x \in (-n,n), \\ w(-n) = \psi(-n-\zeta), \quad w(n) = \psi(n-\zeta). \end{cases} \tag{32}$$

Lemma 3.7. *Let $c \geq 2\sqrt{\bar{A}}$. For any $n \in \mathbb{N}$ and $\zeta \in \mathbb{R}$, problem (32) admits a unique positive solution $w_{n,\zeta}$. Moreover, $w_{n,\zeta}(x)$ is decreasing in x and satisfies*

$$\forall x \in [-n,n], \quad w_{n,\zeta}(x) \leq \psi(x-\zeta).$$

Finally, the mapping $\zeta \mapsto w_{n,\zeta}$ is continuous with respect to the $L^\infty((-n,n))$ norm.

Proof. The existence and uniqueness of the decreasing solution for (32) is given by Corollary 3.5. To prove the upper bound, we exploit the fact that $w_{n,\zeta}$ is a subsolution of the local equation satisfied by ψ. Indeed, it satisfies in $(-n,n)$,

$$\begin{aligned} -w''_{n,\zeta} - cw'_{n,\zeta} &= w_{n,\zeta} \int_{-n}^{x} A(y)(-w'_{n,\zeta}(y))\,dy \\ &\leq \bar{A}w_{n,\zeta}(\psi(-n-\zeta) - w_{n,\zeta}) \\ &\leq \bar{A}w_{n,\zeta}(1 - w_{n,\zeta}). \end{aligned}$$

We can then use the sliding method to deduce that $w_{n,\zeta} \leq \psi(\cdot - \zeta)$ on $(-n,n)$. Indeed, if this were not the case, calling $\bar{\zeta}$ the value for which

$$\min_{x\in[-n,n]} (\psi(x-\bar{\zeta}) - w_{n,\zeta}(x)) = 0,$$

which exists and is unique by monotonicity, we would have that $\bar{\zeta} > \zeta$. Hence, because $\psi(\cdot - \bar{\zeta}) > \psi(\cdot - \bar{\zeta}) = w_{n,\zeta}$ on the boundary of the interval $[-n,n]$, the minimum would be attained at an interior point but not on the boundary, contradicting the elliptic strong maximum principle.

For the last statement, consider a sequence $(\zeta_j)_{j\in\mathbb{N}}$ converging to some $\tilde{\zeta} \in \mathbb{R}$. Using elliptic estimates up to the boundary, for any subsequence of $(\zeta_j)_{j\in\mathbb{N}}$ we can extract another subsequence $(\zeta_{j_k})_{k\in\mathbb{N}}$ such that the associated $(w_{n,\zeta_{j_k}})_{k\in\mathbb{N}}$ converges in $C^2((-n,n))$ to a solution $\tilde{w}$ of problem (32) with $\zeta = \tilde{\zeta}$. Then, by uniqueness, $\tilde{w} = w_{\tilde{\zeta}}$. This concludes the proof. ∎

The following proposition is the existence result.

Proposition 3.8. *For any $c \geq 2\sqrt{\bar{A}}$ and $\vartheta \in (0,1)$, problem (22) admits a solution w satisfying $w(0) = \vartheta$.*

Proof. Let $(w_{n,\zeta})_{n\in\mathbb{N},\,\zeta\in\mathbb{R}}$ be the family given by Lemma 3.7, associated with the standard traveling wave ψ normalized by $\psi(0)=\vartheta\in(0,1)$. We have

$$w_{n,0}(0)\le\psi(0)=\vartheta=w_{n,n}(n)<w_{n,n}(0).$$

Then, by the continuous dependence with respect to ζ, there exists $\zeta_n\in[0,n)$ such that $w_{n,\zeta_n}(0)=\vartheta$. Using interior elliptic estimates, one sees as in the proof of Lemma 3.2 that the family $(w_{n,\zeta_n})_{n\in\mathbb{N}}$ is equibounded in $C^3(I)$, for any bounded interval I. Hence, as $n\to\infty$, it converges (up to subsequences) in $C^2_{\text{loc}}(\mathbb{R})$ to some function w. We know that w is nonincreasing and satisfies $0\le w\le 1$ and $w(0)=\vartheta$. We can pass to the limit in the equation in (32) using the dominated convergence theorem. Namely, recalling that $\zeta_n\ge 0$ and $\psi(-\infty)=1$, we infer that w is a solution of (26) which satisfies $0\le w\le 1$ and $w(0)=\vartheta$. It then follows from Lemma 3.2 that w is decreasing and that $w(-\infty)=1$, $w(+\infty)=0$. Then, integrating the integral in (26) by parts, we recover a solution of the original problem (22). ∎

Case B: $2\sqrt{\underline{A}}<c<2\sqrt{\bar{A}}$. This range is more interesting since one cannot no longer rely on comparison with the waves of the (usual, local) KPP equation. Indeed, unlike what happens in Case A, now the wave will no longer satisfy any arbitrary normalization at a given point.

The first ingredient is to find a supersolution, which in the previous section was simply given by a wave for a standard KPP equation.

Lemma 3.9. *For any $c>2\sqrt{\underline{A}}$, equation (26) admits a decreasing supersolution ψ satisfying*

$$\psi(-\infty)=1,\quad \psi(+\infty)=0.$$

Proof. Let $\tilde{A}\in(\underline{A},\bar{A})$ be such that $c>2\sqrt{\tilde{A}}$. Then call $s:=1-\tilde{A}/\bar{A}\in(0,1)$ and define

$$h(u):=\begin{cases}\tilde{A}u & \text{if } u\le s,\\ \bar{A}u(1-u) & \text{if } u>s.\end{cases}$$

We know that there is a traveling wave solution ψ for the classical KPP equation with nonlinear term h, i.e. a decreasing solution of

$$\begin{cases}\psi''+c\psi'+h(\psi)=0, & x\in\mathbb{R},\\ \psi(-\infty)=1,\quad \psi(+\infty)=0.\end{cases}$$

We normalize it by $\psi(0)=s$. Let us show that, for ζ sufficiently large, the function $\psi(\cdot-\zeta)$ is a supersolution of (22), or equivalently of (26). For $x<\zeta$, we have that $\psi(x-\zeta)>s$ and thus

$$\psi(x-\zeta)\int_{-\infty}^{x}A(y)(-\psi'(y-\zeta))\,dy\le\bar{A}\psi(x-\zeta)(1-\psi(x-\zeta))=h(\psi(x-\zeta)).$$

This implies that $\psi(\cdot-\zeta)$ is a supersolution of (22) in $(-\infty,\zeta)$, for any choice of ζ. On the other hand, if $\zeta>0$, for $x>\zeta$ we find that

$$\begin{aligned}\int_{-\infty}^{x} A(y)(-\psi'(y-\zeta))\,dy &= \int_{-\infty}^{\zeta/2} A(y)(-\psi'(y-\zeta))\,dy\\ &\quad+\int_{\zeta/2}^{+\infty} A(y)(-\psi'(y-\zeta))\,dy\\ &\leq \bar{A}(1-\psi(-\zeta/2))+A(\zeta/2)\psi(-\zeta/2).\end{aligned}$$

The above right-hand side is independent of x and tends to $\underline{A}$ as $\zeta\to+\infty$. It follows that, for ζ large enough, it holds for $x>\zeta$ that

$$\psi(x-\zeta)\int_{-\infty}^{x} A(y)(-\psi'(y-\zeta))\,dy<\tilde{A}\psi(x-\zeta)=h(\psi(x-\zeta)).$$

Hence, for such values of ζ, the function $\psi(\cdot-\zeta)$ is a supersolution of (22) and thus of (26). ■

The next step is to show that if (22), or equivalently (26), admits a decreasing supersolution then it also admits a solution. We would like to follow the same strategy as in the previous section, going through the approximating problems (32). However, since we cannot no longer use the comparison with the local equation, we will need the following lemma to guarantee that solutions stay bounded away from 1, uniformly in n.

Lemma 3.10. *Let $c>2\sqrt{\underline{A}}$ and let ψ be a decreasing supersolution of (26), satisfying $\psi(-\infty)=1$, $\psi(+\infty)=0$. For any $n\in\mathbb{N}$ and $\zeta\geq 0$, problem (32) admits a unique positive solution $w_{n,\zeta}$. Moreover, $w_{n,\zeta}(x)$ is decreasing in x and it holds that*

$$\sup_{n\in\mathbb{N}} w_{n,\zeta}(0)<1.$$

Finally, the mapping $\zeta\mapsto w_{n,\zeta}$ is continuous with respect to the $L^\infty((-n,n))$ norm.

Proof. Firstly, we check that $\psi(\cdot-\zeta)$ is still a supersolution of (26) for any $\zeta\geq 0$. Because of the condition $\psi(-\infty)=1$, it is equivalent to consider equation (22). Using the monotonicity of both A and ψ, we see that for $\zeta\geq 0$ and $x\in\mathbb{R}$,

$$\begin{aligned}-\psi''(x-\zeta)-c\psi'(x-\zeta) &\geq \psi(x-\zeta)\int_{-\infty}^{x-\zeta} A(y)(-\psi'(y))\,dy\\ &=\psi(x-\zeta)\int_{-\infty}^{x} A(y-\zeta)(-\psi'(y-\zeta))\,dy\\ &\geq \psi(x-\zeta)\int_{-\infty}^{x} A(y)(-\psi'(y-\zeta))\,dy,\end{aligned}$$

i.e. $\psi(\cdot-\zeta)$ is a supersolution of (22). We can therefore restrict ourselves to the case $\zeta=0$.

Corollary 3.5 implies the existence, uniqueness and strict monotonicity of the solution to (32) with $\zeta = 0$. We call it w_n. Let us show that $(w_n(0))_{n\in\mathbb{N}}$ stays bounded from above away from 1.

Assume by contradiction that this is not the case. Then, up to extraction of a subsequence, we have that $w_n(0) \to 1$ as $n \to \infty$. We can further assume that, up to another extraction, $w_n(0) > \psi(-1)$ for all $n \in \mathbb{N}$. Let b_n be the smallest intersection point between w_n and ψ on $(0, n]$. Then call

$$k_n := \max_{[-n,b_n]} \frac{w_n}{\psi},$$

and let $x_n \in [-n, b_n]$ be a point where such a maximum is reached. We see that $k_n > \frac{w_n(0)}{\psi(-1)} > 1$, whence x_n lies inside the interval $(-n, b_n)$ because w_n/ψ is equal to 1 on the boundary. We also see that

$$\lim_{n\to\infty} \frac{w_n(0)}{\psi(0)} = \frac{1}{\psi(0)} > \frac{1}{\psi(-1)} \geq \max_{[-n,-1]} \frac{w_n}{\psi}.$$

This implies that $x_n > -1$ for n large enough. The function $g_n := k_n\psi$ touches w_n from above at the point x_n, whence

$$\begin{aligned}
0 &= w_n''(x_n) + c w_n'(x_n) + w_n(x_n) \int_{-n}^{x_n} A(y)(-w_n'(y))\,dy \\
&\leq g_n''(x_n) + c g_n'(x_n) + g_n(x_n) \int_{-n}^{x_n} A(y)(-w_n'(y))\,dy \\
&\leq g_n(x_n)\Big(\int_{-n}^{x_n} A(y)(-w_n'(y))\,dy - \int_{-\infty}^{x_n} A(y)(-\psi'(y))\,dy\Big),
\end{aligned}$$

where, for the last inequality, we have used that ψ is a supersolution of (22). We deduce that

$$\int_{-n}^{x_n} A(y)(\psi'(y) - w_n'(y))\,dy \geq \int_{-\infty}^{-n} A(y)(-\psi'(y))\,dy > 0,$$

and thus, integrating by parts,

$$A(x_n)(\psi(x_n) - w_n(x_n)) + \int_{-n}^{x_n} A'(y)(w_n(y) - \psi(y))\,dy > 0.$$

We recall that $w_n \geq \psi$ in $[-1, 0)$ because $w_n(0) > \psi(-1)$, as well as in $[0, b_n]$ by the definition of b_n. Thus, for n large enough, since $x_n \in (-1, b_n)$, we infer that $w_n \geq \psi$ in $[-1, x_n]$ and therefore the above inequality together with $A' \leq 0$ yields

$$\int_{-n}^{-1} A'(y)(w_n(y) - \psi(y))\,dy > 0.$$

This implies in particular that $A' \not\equiv 0$ in $(-\infty, -1]$. Recall, however, that we are assuming that $(w_n)_{n\in\mathbb{N}}$ converges to 1 at the point 0, hence uniformly in $(-\infty, -1]$. Passing to the limit as $n \to \infty$ in the above integral inequality we then reach a contradiction.

The last statement of the lemma follows from the uniqueness of the solution, exactly as in the proof of Lemma 3.7. ∎

Proposition 3.11. *Problem* (22) *admits a solution for any* $c > 2\sqrt{\underline{A}}$.

Proof. Fix $c > 2\sqrt{\underline{A}}$. Let ψ be the supersolution provided by Lemma 3.9 and let $(w_{n,\zeta})_{n\in\mathbb{N},\,\zeta\geq 0}$ be the family constructed from it in Lemma 3.10. We know from that lemma that there exists ϑ satisfying

$$\sup_{n\in\mathbb{N}} w_{n,0}(0) < \vartheta < 1.$$

For given $n \in \mathbb{N}$, using the fact that $w_{n,\zeta}(0) > \psi(n-\zeta) > \vartheta$ for ζ sufficiently large (depending on n) together with the continuity of $w_{n,\zeta}$ with respect to ζ, we can find $\zeta_n > 0$ such that $w_{n,\zeta_n}(0) = \vartheta$.

By standard elliptic estimates, $(w_{n,\zeta_n})_{n\in\mathbb{N}}$ converges in $C^2_{\mathrm{loc}}(\mathbb{R})$ (up to subsequences) to some function $0 \leq w \leq 1$. Thus, using the dominated convergence theorem, we can pass to the limit in the equation of (32) and deduce that w solves (26). Finally, because $w(0) = \vartheta \in (0,1)$, Lemma 3.2 implies that w is a solution to (22). ∎

3.3. The functional $\mathcal{I}$

We investigate now more deeply the structure of the set of traveling waves. A key role will be played by the following quantity associated to a solution w of (22):

$$\mathcal{I}(w) := \int_{\mathbb{R}} A(y)(-w'(y))\,dy = \bar{A} + \int_{\mathbb{R}} A'(y)w(y)\,dy. \tag{33}$$

Observe that the second formulation of $\mathcal{I}$, obtained after integration by parts, shows that $\mathcal{I}$ is decreasing with respect to w.

We start by collecting some properties of the traveling waves which involve the functional $\mathcal{I}$.

Proposition 3.12. *Let w be a solution to* (22) *and $\mathcal{I}$ be given by* (33)*. Then we have*

(i) $\underline{A} < \mathcal{I}(w) < \bar{A}$;

(ii) $\mathcal{I}(w) \leq \frac{c^2}{4}$;

(iii) *w satisfies*

$$\frac{A(0) - \mathcal{I}(w)}{A(0) - \underline{A}} \leq w(0) \leq \frac{\bar{A} - \mathcal{I}(w)}{\bar{A} - A(0)}, \tag{34}$$

where the inequalities are understood to hold provided the corresponding denominators are not 0;

(iv) *w is strictly* log-*concave (i.e. w'/w is decreasing) and satisfies*

$$\lim_{x\to+\infty} \frac{-w'}{w}(x) = \frac{c}{2} - \sqrt{\frac{c^2}{4} - \mathcal{I}(w)} =: \lambda > 0.$$

In particular, it holds that

$$w(x) = w(0)e^{-\lambda(x)x},$$

where $\lambda(x)$ is an increasing function converging to λ as $x \to +\infty$.

Proof. Since $w' < 0$ from Proposition 3.1, the bounds $\underline{A} < \mathcal{I}(w) < \bar{A}$ immediately follow, recalling that $\underline{A} \leq A \leq \bar{A}$ and that both inequalities are strict somewhere.

Estimates (34) on $w(0)$ easily follow from the definition of $\mathcal{I}$ as well. Indeed, on one hand,

$$\mathcal{I} = \int_{-\infty}^{0} A(y)(-w'(y))\,dy + \int_{0}^{+\infty} A(y)(-w'(y))\,dy \geq A(0)(1 - w(0)) + \underline{A}w(0).$$

On the other hand, an integration by parts shows that

$$\mathcal{I} = \bar{A} + \int_{\mathbb{R}} A'(y)w(y)\,dy \leq \bar{A} + \int_{-\infty}^{0} A'(y)w(y)\,dy \leq \bar{A} + w(0)(A(0) - \bar{A}).$$

Now we investigate the properties of $q := -w'/w$, which is a positive function. Direct computation reveals that

$$q' = q^2 - cq + \int_{-\infty}^{x} A(y)(-w'(y))\,dy, \quad x \in \mathbb{R}. \tag{35}$$

The integral term is positive, nondecreasing in x, and converges to 0 as $x \to -\infty$ and to $\mathcal{I}(w)$ as $x \to +\infty$. We now show that q is bounded and increasing. Recall that $c > 0$ by Proposition 3.1.

First of all we observe that necessarily $q(x) \leq c$, because if $q(x_0) > c$ then (35) would imply that q blows up at some point $x_1 > x_0$. The boundedness of q then implies that $\mathcal{I}(w) \leq c^2/4$, because otherwise by (35) there would exist $\varepsilon > 0$ such that, for large x,

$$q' > q^2 - cq + c^2/4 + \varepsilon \geq \varepsilon,$$

which is impossible because q is bounded. So we also proved that $\mathcal{I}(w) \leq c^2/4$. This allows us to rewrite (35) as

$$q' = (q - \lambda_-(x))(q - \lambda_+(x)), \quad x \in \mathbb{R}, \tag{36}$$

with

$$\lambda_\pm(x) := \frac{c}{2} \pm \sqrt{\frac{c^2}{4} - \int_{-\infty}^{x} A(y)(-w'(y))\,dy}.$$

Observe that $0 < \lambda_-(x) < \lambda_+(x)$ and $\lambda_-'(x) > 0 > \lambda_+'(x)$ for all $x \in \mathbb{R}$, with

$$\lambda_-(-\infty) = 0 < \lambda_-(+\infty) = \frac{c}{2} - \sqrt{\frac{c^2}{4} - \mathcal{I}(w)} =: \lambda, \quad \lambda_+(-\infty) > \lambda_+(+\infty) > 0.$$

We infer that if $q(x_0) \geq \lambda_+(x_0)$ at some x_0 then $q \geq \lambda_+(x_0)$ in $(x_0, +\infty)$, which implies that $q(+\infty) = +\infty$, thus this case is excluded. On the other hand, if $\lambda_-(x_0) \leq q(x_0) \leq \lambda_+(x_0)$ at some x_0 then $\lambda_-(x_0) \leq q \leq \lambda_+(x_0)$ in $(-\infty, x_0)$ and thus $q(-\infty) > 0$, which is impossible because, since $q = -w'/w$, we would have that $w(-\infty) = -\infty$. The unique possibility left is therefore $q < \lambda_-$ in $\mathbb{R}$. We deduce from (36) that $q' > 0$ and that

$$q(+\infty) = \lambda_-(+\infty) = \lambda.$$

So the proof of item (iv) is concluded.

For the last statement of the theorem, we write $w(x) = w(0)e^{-\lambda(x)x}$, with

$$\lambda(x) = -\frac{1}{x}\log\frac{w(x)}{w(0)} = -\frac{1}{x}\int_0^x \frac{w'(y)}{w(y)}\,dy.$$

The convergence of w'/w towards $-\lambda$ implies that $\lambda(x) \to \lambda$ as $x \to +\infty$. Moreover, from the monotonicity of w'/w, we infer that, for $x \neq 0$,

$$\lambda'(x) = -\frac{1}{x}\frac{w'(x)}{w(x)} + \frac{1}{x^2}\int_0^x \frac{w'(y)}{w(y)}\,dy > 0. \qquad \blacksquare$$

We focus now on the case $2\sqrt{\underline{A}} < c < 2\sqrt{\bar{A}}$. We seek a wave for which the bound (ii) in Proposition 3.12 is optimal, i.e. such that

$$\mathcal{I}(w) = \frac{c^2}{4}.$$

This will be called a “critical wave” associated with a given speed. Observe that similar waves can only exist in this range of velocities, since $\mathcal{I}(w) < \bar{A}$ by Proposition 3.12. We are going to show that, for a given velocity c, the critical wave runs at the lowest height.

In order to clarify this fact, we start to investigate the possible heights that are admissible at a given speed c.

Proposition 3.13 (Same speed, different normalization). *Assume that* (22) *admits a solution w. Then, for any $x_0 \in \mathbb{R}$ and $\vartheta \in (w(x_0), 1)$, there exists a solution $\widetilde{w}$ of* (22) *satisfying $\widetilde{w}(x_0) = \vartheta$. Moreover, the function $\widetilde{w}/w$ is nondecreasing on $\mathbb{R}$.*

Proof. Let $n \in \mathbb{N}$. For $\zeta \geq 0$, we consider the initial value problem

$$\begin{cases} \psi'' + c\psi' + \psi\left(\bar{A} - A\psi + \int_{-\infty}^{-n} A'w + \int_{-n}^{x} A'\psi\right) = 0, & x > -n, \\ \psi(-n) = w(-n-\zeta), \\ \psi'(-n) = w'(-n-\zeta). \end{cases} \tag{37}$$

If $\zeta = 0$ then the function w is a solution of this problem. If $\zeta > 0$, we see that the function w_ζ defined by $w_\zeta := w(\cdot - \zeta)$ is a supersolution of this problem. Indeed, calling $A_\zeta := A(\cdot - \zeta)$, for $x \in \mathbb{R}$ we have

$$\begin{aligned} w_\zeta'' + cw_\zeta' = w_\zeta \int_{-\infty}^{x-\zeta} Aw' &= -w_\zeta\left(\bar{A} - Aw_\zeta + (A - A_\zeta)w_\zeta + \int_{-\infty}^{x-\zeta} A'w\right) \\ &= -w_\zeta\left(\bar{A} - Aw_\zeta - w_\zeta\int_x^{x-\zeta} A' + \int_{-\infty}^{-n} A'w + \int_{-n}^{x-\zeta} A'w\right) \\ &\leq -w_\zeta\left(\bar{A} - Aw_\zeta - \int_x^{x-\zeta} A'w_\zeta + \int_{-\infty}^{-n} A'w + \int_{-n}^{x-\zeta} A'w_\zeta\right) \\ &= -w_\zeta\left(\bar{A} - Aw_\zeta + \int_{-\infty}^{-n} A'w + \int_{-n}^{x} A'w_\zeta\right). \end{aligned}$$

On the other hand, the constant $w(-n-\zeta)$ is a subsolution of the same problem. It follows from Lemma 3.4 that (37) admits a unique solution ψ^ζ, which is decreasing and for which the ratio ψ^ζ/w_ζ is nondecreasing in $[-n,+\infty)$, whence in particular $\psi^\zeta \geq w_\zeta$. In the case $\zeta = 0$ we have $\psi^0 \equiv w$. Take $x_0 \in \mathbb{R}$ and $\vartheta \in (w(x_0), 1)$. It holds that

$$\psi^0(x_0) = w(x_0) < \vartheta, \quad \psi^\zeta(x_0) \geq w(x_0 - \zeta) \to 1 \text{ as } \zeta \to +\infty.$$

Thus, the continuous dependence of $\psi^\zeta(x_0)$ with respect to ζ yields the existence of some $\zeta > 0$ such that $\psi^\zeta(x_0) = \vartheta$. Such a function ψ^ζ we will call ψ_n. The ratio ψ^ζ/w_ζ is nondecreasing in $[-n,+\infty)$ and equal to 1 at $-n$. Then, writing

$$\frac{\psi^\zeta}{w} = \frac{\psi^\zeta}{w_\zeta}\frac{w_\zeta}{w},$$

and observing that

$$\left(\frac{w_\zeta}{w}\right)' = \frac{w_\zeta' w - w' w_\zeta}{w^2} = \frac{w_\zeta}{w}\left(\frac{w_\zeta'}{w_\zeta} - \frac{w'}{w}\right) > 0$$

due to Proposition 3.12 (iv), we find that ψ^ζ/w is increasing in $[-n,+\infty)$ and larger than 1.

The sequence $(\psi_n)_{n\in\mathbb{N}}$ converges (up to subsequences) to some function $\widetilde{w}$ in $C^2_{\mathrm{loc}}(\mathbb{R})$. This function is nonincreasing, satisfies $\widetilde{w}(x_0) = \vartheta$ and in addition $\widetilde{w}/w$ is nondecreasing in $\mathbb{R}$ and larger than or equal to 1. We infer that $\widetilde{w}(-\infty) = 1$. For every $x \in \mathbb{R}$, it holds that

$$\widetilde{w}'' + c\widetilde{w}' + \widetilde{w}(\bar{A} - A\widetilde{w}) = -\lim_{n\to\infty} \psi_n \int_{-n}^{x} A'\psi_n = -\int_{-\infty}^{x} A'\widetilde{w},$$

i.e. $\widetilde{w}$ is a solution of (26). It then readily follows that $\widetilde{w}(+\infty) = 0$, and thus that $\widetilde{w}$ satisfies (22). ∎

Corollary 3.14. *Assume that* (22) *admits a solution w. Then, for any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exists a solution $\widetilde{w}$ of* (22) *satisfying*

$$\widetilde{w}(x_0) > w(x_0), \quad w \leq \widetilde{w} < w + \varepsilon.$$

Proof. Let $\delta > 0$ be such that $(1+\delta)w(x_0) < 1$. Applying Proposition 3.13 with $\vartheta = (1+\delta)w(x_0)$ provides us with a solution $\widetilde{w}$ such that $\widetilde{w}(x_0) = (1+\delta)w(x_0)$ and $\widetilde{w}/w$ is nondecreasing. This yields

$$\forall x \leq x_0, \quad w(x) \leq \widetilde{w}(x) \leq (1+\delta)w(x) < w(x) + \delta.$$

By Lemma 3.6, given $\varepsilon > 0$, we can choose δ small enough that $\|w - \widetilde{w}\|_\infty < \varepsilon$. ∎

We have now the ingredients to show the critical role played by the equality $\mathcal{I}(w) = c^2/4$.

Lemma 3.15. *Assume that* (22) *admits a solution* w *for which* $\mathcal{I}(w) < c^2/4$. *Then* (22) *admits another decreasing solution* $\psi < w$.

Before proving this lemma, let us show how it entails the existence of the critical wave.

Proposition 3.16. *For any* $2\sqrt{\underline{A}} < c < 2\sqrt{\bar{A}}$, *there exists a solution* w *to* (22) *for which* $\mathcal{I}(w) = c^2/4$.

Proof. Consider the maximization problem

$$j^* := \sup\{\mathcal{I}(w) : w \text{ is a solution of } (22)\}.$$

We know from Proposition 3.12 that $\underline{A} < j^* \leq c^2/4 < \bar{A}$. Let us show that j^* is attained. Let $(w_n)_{n\in\mathbb{N}}$ be a maximizing sequence for j^*. We use formulation (26) for the equation satisfied by the w_n. Using the C^3 estimate of Lemma 3.2, as well as the dominated convergence theorem, we infer that $(w_n)_{n\in\mathbb{N}}$ converges (up to subsequences) in $C^2_{\mathrm{loc}}(\mathbb{R})$ to a nonincreasing solution w of (26). Moreover, the second formulation in (33) yields

$$j^* = \lim_{n\to\infty} \mathcal{I}(w_n) = \bar{A} + \int_{\mathbb{R}} A'(y)w(y)\,dy.$$

This immediately shows that $w \not\equiv 0, 1$. Therefore, Lemma 3.2 implies that w is a decreasing solution to (22) and in particular that $\mathcal{I}(w) = j^*$.

Assume by way of contradiction that $\mathcal{I}(w) = j^* < c^2/4$. Then by Lemma 3.15 there exists another solution $\widetilde{w} < w$ to (22). The second formulation in (33) yields $\mathcal{I}(\widetilde{w}) > \mathcal{I}(w) = j^*$, contradicting the definition of j^*. ∎

It remains to prove Lemma 3.15.

Proof of Lemma 3.15. We construct the desired wave in two steps.

Step 1. As a first step, we show that, for any $\zeta \in \mathbb{R}$ and $k \in (0,1)$, there exists a solution $\psi_{\zeta,k}$ of (26) for $x < \zeta$ which satisfies

$$\forall x \leq \zeta, \quad \psi'_{\zeta,k}(x) < 0, \quad 1 \geq \frac{\psi_{\zeta,k}}{w}(x) \geq k = \frac{\psi_{\zeta,k}}{w}(\zeta).$$

This is essentially a consequence of Corollary 3.5. Indeed, for $n \in \mathbb{N}$, $n < \zeta$, we consider the problem

$$\begin{cases} \psi'' + c\psi' + \psi\left(\bar{A} - A\psi + \int_{-\infty}^{-n} A'w + \int_{-n}^{x} A'\psi\right) = 0, & x \in (-n,\zeta), \\ \psi(-n) = w(-n), \quad \psi(\zeta) = kw(\zeta). \end{cases} \tag{38}$$

We notice that $\bar{A} + \int_{-\infty}^{-n} A'w = A(-n)w(-n) + \int_{-\infty}^{-n} A(-w') > A(-n)w(-n)$, and we use Corollary 3.5 with $W = w$. Since the target is smaller than $w(\zeta)$, we obtain the existence of a unique positive and decreasing solution ψ^n of (38) with $\frac{\psi^n}{w}$ being decreasing on $[-n,\zeta]$, whence

$$\forall x \in (-n,\zeta), \quad 1 > \frac{\psi^n}{w}(x) > \frac{\psi^n}{w}(\zeta) = k.$$

By elliptic estimates, the ψ^n converge (up to subsequences) as $n \to \infty$, locally uniformly in $(-\infty, \zeta]$, to a solution ψ of (26) for $x < \zeta$. Moreover, ψ satisfies $kw \le \psi \le w$, $\psi(\zeta) = kw(\zeta)$ and ψ/w is nonincreasing on $(-\infty, \zeta]$, whence it holds that

$$0 \ge \psi' w - w'\psi.$$

This is the function $\psi_{\zeta,k}$ we sought.

Step 2. Now the purpose is to extend the function $\psi_{\zeta,k}$ to the whole of $\mathbb{R}$. According to (26), we extend them as the solutions to the problem

$$\begin{cases} \psi'' + c\psi' + \psi\left(\bar{A} - A\psi + \int_{-\infty}^{\zeta} A'\psi_{\zeta,k} + \int_{\zeta}^{x} A'\psi\right) = 0, \quad x > \zeta, \\ \psi(\zeta) = \psi_{\zeta,k}(\zeta), \\ \psi'(\zeta) = \psi'_{\zeta,k}(\zeta). \end{cases} \tag{39}$$

Lemma 3.3 gives the existence and uniqueness of the positive solution in $(\zeta, \xi_{\zeta,k})$, with either $\xi_{\zeta,k} = +\infty$, or $\psi_{\zeta,k}(\xi_{\zeta,k}) = 0$ or $+\infty$. Our aim is to choose $\zeta \in \mathbb{R}$, $k \in (0, 1)$ in such a way that $\xi_{\zeta,k} = +\infty$.

Observing that

$$\bar{A} + \int_{-\infty}^{\zeta} A'\psi_{\zeta,k} = \bar{A}(1 - \psi_{\zeta,k}(-\infty)) + A(\zeta)\psi_{\zeta,k}(\zeta) + \int_{-\infty}^{\zeta} A(-\psi'_{\zeta,k}) > A(\zeta)\psi_{\zeta,k}(\zeta),$$

we deduce that the constant function $\psi_1 \equiv \psi_{\zeta,k}(\zeta)$ is a subsolution of the equation in (39). Hence Lemma 3.4 implies that $\psi_{\zeta,k}$ is decreasing in $[\zeta, \xi_{\zeta,k})$. It satisfies there

$$\psi''_{\zeta,k} + c\psi'_{\zeta,k} + \psi_{\zeta,k}\left(\bar{A} + \int_{-\infty}^{\zeta} A'\psi_{\zeta,k}\right) \ge 0. \tag{40}$$

Since $\psi_{\zeta,k} \ge kw$ on $(-\infty, \zeta]$, we find that

$$\int_{-\infty}^{\zeta} A'\psi_{\zeta,k} \le k \int_{-\infty}^{\zeta} A'w \to k \int_{\mathbb{R}} A'w \quad \text{as } \zeta \to +\infty.$$

Therefore, by definition of $\mathcal{I}$, we have

$$\bar{A} + \int_{-\infty}^{\zeta} A'\psi_{\zeta,k} \xrightarrow{\zeta\to\infty} \bar{A} + k(\mathcal{I} - \bar{A}). \tag{41}$$

On account of (40) and (41), and since $\mathcal{I} < \frac{c^2}{4}$, we can find ζ sufficiently large and k sufficiently close to 1 that $\psi_{\zeta,k}$ satisfies

$$\psi''_{\zeta,k} + c\psi'_{\zeta,k} + \frac{c^2}{4}\psi_{\zeta,k} > 0 \quad \text{in } [\zeta, \xi_{\zeta,k}). \tag{42}$$

Next we apply Proposition 3.12 (iv), which yields

$$\forall x \in \mathbb{R}, \quad \frac{w'}{w}(x) > -\frac{c}{2}.$$

Using the fact that $\psi_{\zeta,k}$ converges to w as $k \nearrow 1$, uniformly in $(-\infty, \zeta]$ and therefore by elliptic estimates in $C^1_{loc}((-\infty, \zeta])$ (up to subsequences), we deduce that for k sufficiently close to 1 (depending on ζ) it holds that

$$\frac{\psi'_{\zeta,k}}{\psi_{\zeta,k}}(\zeta) > -\frac{c}{2}. \tag{43}$$

Summing up, we can pick ζ large enough and then k close enough to 1 in such a way that both (42) and (43) hold. Therefore, the function $q := -\psi'_{\zeta,k}/\psi_{\zeta,k}$ satisfies

$$q' < \left(q - \frac{c}{2}\right)^2 \quad \text{in } [\zeta, \xi_{\zeta,k}), \; q(\zeta) < c/2.$$

It follows that $q(x) < c/2$ for all $x > \zeta$, i.e.

$$\psi_{\zeta,k}(x) > \psi_{\zeta,k}(\zeta)e^{-\frac{c}{2}(x-\zeta)}.$$

This means that $\psi_{\zeta,k}$ remains positive on the whole $\mathbb{R}$. Namely, it is a nontrivial solution of (26) and therefore it solves (22) due to Lemma 3.2. ■

Remark 3. We could have considered two other natural optimization problems. Namely, for given $x_0 \in \mathbb{R}$,

$$\vartheta^* := \inf\{w(x_0) : w \text{ is a solution of (22)}\},$$

or, for given $\vartheta \in (0, 1)$,

$$\zeta^* := \inf\{w^{-1}(\vartheta) : w \text{ is a solution of (22)}\}.$$

Once it has been shown that these infima are actually minima, it follows from Lemma 3.15 that they are both attained by the critical wave (satisfying $\mathcal{I}(w) = c^2/4$). To show that the minima are attained it is sufficient to verify that $\vartheta^* > 0$, $\zeta^* > -\infty$. For this, we consider some corresponding minimizing sequences $(w_n)_{n\in\mathbb{N}}$. By Lemma 3.2, they converge in C^2_{loc} to solutions of (26). On one hand, if $\vartheta^* = 0$, one would have that the limit is identically equal to 0, whence, thanks to Proposition 3.12,

$$c \geq 2 \lim_{n\to\infty} \sqrt{\mathcal{I}(w_n)} = 2\sqrt{A},$$

which is a contradiction. On the other hand, if $\zeta^* = -\infty$ then the limit w of the minimizing sequence would satisfy $w \leq \vartheta$. Being a nonincreasing solution to (26), we would necessarily have that $w \equiv 0$, whence the same contradiction as before.

Let us point out that we do not know whether the optimal waves for the above problems are unique, nor whether an optimal wave for a problem is also critical for the same problem with another choice of the parameter, or for a problem of the other type (except of course that w is optimal for the first problem at a point x_0 if and only if it is optimal for the second problem for the value $\vartheta = w(x_0)$).

We now show the uniqueness of the wave for given $\mathcal{I}$, a crucial step to prove the ordering of waves.

Proposition 3.17. *For given $c, j \in \mathbb{R}$, there exists at most one solution of* (22) *such that*

$$\mathcal{I}(w) = j.$$

Proof. The proof consists in showing that two solutions of (22) on which the operator $\mathcal{I}$ coincides cannot intersect on $\mathbb{R}$, unless they coincide. One then concludes, because two strictly ordered solutions necessarily have distinct values of $\mathcal{I}$, thanks to (33) and the facts that $A' \not\equiv 0$ and that solutions are positive due to Proposition 3.1.

Let w_1, w_2 be two solutions intersecting at some point $x_0 \in \mathbb{R}$ and for which $\mathcal{I}(w_1) = \mathcal{I}(w_2) = j$. Call $q_i := -\frac{w_i'}{w_i}$, $i = 1, 2$. Assume by contradiction that $q_1(x_0) \neq q_2(x_0)$ and, without loss of generality, that $q_1(x_0) > q_2(x_0)$. Notice that

$$q_1 - q_2 = \frac{-w_2 w_1' + w_1 w_2'}{w_1 w_2} = \left(\frac{w_2}{w_1}\right)' \frac{w_1}{w_2}. \tag{44}$$

Hence $q_1(x_0) > q_2(x_0)$ implies that $\frac{w_2}{w_1}$ is increasing near x_0. We claim that $w_2 > w_1$ in $(x_0, +\infty)$. Indeed, set

$$\xi := \inf\{a < x_0 : q_1(x) > q_2(x)\ \forall x \in (a, x_0)\}.$$

Since $\frac{w_2}{w_1} \to 1$ as $x \to -\infty$, by Rolle's theorem there exists some point $a < x_0$ such that $(\frac{w_2}{w_1})'(a) = 0$, i.e. $q_1(a) = q_2(a)$; this means that ξ is finite. Hence we have $q_1(\xi) = q_2(\xi)$ and $q_1'(\xi) \geq q_2'(\xi)$. From equation (35) we deduce that

$$\int_{-\infty}^{\xi} A(-w_1)'\, dy \geq \int_{-\infty}^{\xi} A(-w_2)'\, dy,$$

with equality only if $q_1'(\xi) = q_2'(\xi)$. Next, recalling from (44) that $\frac{w_2}{w_1}$ is increasing as long as $q_1 > q_2$, we find that $q_1 > q_2$ and $w_2 < w_1$ in the interval (ξ, x_0), and therefore, again by (44), that $-w_1' > -w_2'$ in (ξ, x_0), which yields $\int_\xi^x A(-w_1)'\, dy \geq \int_\xi^x A(-w_2)'\, dy$ for all $x \in (\xi, x_0]$, with equality only in the case $A(\xi) = 0$. Summing up, we have seen that $\int_{-\infty}^x A(-w_1)'\, dy \geq \int_{-\infty}^x A(-w_2)'\, dy$ for all $x \in (\xi, x_0]$, and that equality holds if and only if $q_1'(\xi) = q_2'(\xi)$ and $A(\xi) = 0$. However, in such a case, equation (35) yields $(q_1 - q_2)' = (q_1 + q_2)(q_1 - q_2) - c(q_1 - q_2)$ in (ξ, x_0), which, together with $q_1(\xi) = q_2(\xi)$ and $q_1'(\xi) = q_2'(\xi)$, contradicts $q_1(x_0) > q_2(x_0)$. This means that necessarily

$$\int_{-\infty}^{x_0} A(-w_1)'\, dy > \int_{-\infty}^{x_0} A(-w_2)'\, dy. \tag{45}$$

Together with the fact that $w_1(x_0) = w_2(x_0)$, this implies that w_1 is a supersolution of the truncated equation of w_2, namely

$$w_1'' + c w_1' + w_1 \left(\int_{-\infty}^{x_0} A(-w_2)'\, dy + A(x_0) w_2(x_0) - A(x) w_1 + \int_{x_0}^{x} A' w_1\, dy \right) \leq 0.$$

Since $w_1'(x_0) < w_2'(x_0)$, by Lemma 3.4 we deduce that $w_1 < w_2$ in $(x_0, +\infty)$. But this implies, using that $\mathcal{I}(w_1) = \mathcal{I}(w_2)$ in (33),

$$\int_{-\infty}^{x_0} A(w_2 - w_1)'\,dy = -\int_{x_0}^{+\infty} A(w_2 - w_1)'\,dy = \int_{x_0}^{+\infty} A'(w_2 - w_1)\,dy \leq 0,$$

which contradicts (45).

We are only left with the possibility that $q_1(x_0) = q_2(x_0)$ (which means $w_1'(x_0) = w_2'(x_0)$). In this case, if (45) holds (up to reversing the roles of the two solutions), we obtain the contradiction as before. Otherwise it holds that $\int_{-\infty}^{x_0} A(-w_1)'\,dy = \int_{-\infty}^{x_0} A(-w_2)'\,dy$, and then $w_1 = w_2$ in $(x_0, +\infty)$ by the uniqueness result of Lemma 3.3. We now rewrite equation (26) satisfied by w_i in terms of the function $\widetilde{w}_i(x) := w_i(-x)$ as follows:

$$\widetilde{w}_i'' - c\widetilde{w}_i' + \widetilde{w}_i\left(\mathcal{I}(w_i) - \int_{x_0}^{+\infty} A' w_i\,dy - A(-x)\widetilde{w}_i - \int_{-x_0}^{x} A'(-y)\widetilde{w}_i(y)\,dy\right) = 0,$$

and we observe that the quantities $\mathcal{I}(w_i) - \int_{x_0}^{+\infty} A' w_i\,dy$ coincide for $i = 1, 2$. It then follows, again by Lemma 3.3, that $w_1 = w_2$ in $(-\infty, x_0)$ as well. ∎

3.4. Decay estimates of the waves

We now derive some estimates about the convergence at $\pm\infty$ in terms of the value of the function at a given point, say, the origin. They will be used in the study of the waves for the mean-field system.

Lemma 3.18. *Let w be a solution of (22). Then we have*

$$1 - w(x) \leq (1 - w(0))e^{\gamma_0 x} \quad \forall x \leq 0, \;\; \textit{where } \gamma_0 := \frac{A(0)w(0)}{\sqrt{\bar{A}} + c}, \tag{46}$$

and

$$w(x) \leq w(0)e^{\tilde{\gamma}_0(\frac{1}{c} - x)} \quad \forall x > 0, \;\; \textit{where } \tilde{\gamma}_0 := \frac{A(0)(1 - w(0))}{c}. \tag{47}$$

In addition, if w is a critical wave, i.e. $\mathcal{I}(w) := \int_{\mathbb{R}} A(-w') = \frac{c^2}{4}$ $(< \bar{A})$, then there exists a constant K, only depending on $\bar{A}$ and positive lower bounds for c and $1 - w(0)$, such that

$$w(x) \leq Kxe^{-\frac{c}{2}x} \quad \forall x \geq 1. \tag{48}$$

Proof. We start with the behavior for $x \to -\infty$. Similarly to the proof of Proposition 3.12, we introduce the function

$$\psi(x) := -\frac{w'}{1 - w}.$$

We know from Lemma 3.2 that ψ is a positive function. It satisfies

$$-\psi' - \psi^2 - c\psi + \frac{w}{1 - w}\int_{-\infty}^{x} A(y)(-w'(y))\,dy = 0. \tag{49}$$

In particular, since $A \leq \bar{A}$ and w satisfies $w < 1$ and $w' < 0$, we have

$$-\psi' - \psi^2 - c\psi + \bar{A} \geq 0.$$

This implies that

$$\psi \leq \sqrt{\bar{A}}, \tag{50}$$

because otherwise we would have $\psi' \leq -c\psi \leq -c\sqrt{\bar{A}}$ in $(-\infty, \bar{x})$ for some $\bar{x}$, which is not possible. Coming back to (49), using that A, w are nonincreasing, we also have

$$-\psi' - \psi^2 - c\psi + A(0)w(0) \leq 0 \quad \forall x < 0.$$

Due to (50), we deduce that

$$\psi' \geq -\psi\big(\sqrt{\bar{A}} + c\big) + A(0)w(0), \quad x < 0.$$

Hence

$$(e^{(\sqrt{\bar{A}}+c)x}\psi)' \geq e^{(\sqrt{\bar{A}}+c)x}A(0)w(0), \quad x < 0.$$

Since ψ is bounded above by (50), integrating in $(-\infty, x)$ we deduce

$$\psi(x) \geq \gamma_0 := \frac{A(0)w(0)}{\sqrt{\bar{A}} + c}.$$

Recalling that $\psi = -\frac{w'}{1-w}$, we readily derive (46).

A similar statement can be obtained as $x \to +\infty$. As in the proof of Proposition 3.12, here we consider the function $q := -w'/w$, which is positive, bounded and satisfies (35). In particular, since A is nonincreasing, for $x > 0$ we deduce

$$q' \geq q^2 - cq + \int_{-\infty}^{0} A(y)(-w'(y))\,dy \geq q^2 - cq + A(0)(1 - w(0)).$$

Hence, always for $x > 0$,

$$q'(x) \geq -cq + c\tilde{\gamma}_0, \quad \text{where } \tilde{\gamma}_0 := \frac{A(0)(1 - w(0))}{c}.$$

This implies

$$q(x) \geq q(0)e^{-cx} + \tilde{\gamma}_0(1 - e^{-cx}) \geq \tilde{\gamma}_0(1 - e^{-cx}).$$

Recalling that $q = -\frac{w'}{w}$ this readily implies (47).

Let us prove the last statement. Suppose that w is a critical wave. In this case, we rewrite (35) as

$$\begin{aligned} q' &= q^2 - cq + \frac{c^2}{4} - \int_{x}^{+\infty} A(y)(-w'(y))\,dy \\ &\geq q^2 - cq + \frac{c^2}{4} - A(x_0)w(x_0), \quad x \geq x_0 \end{aligned} \tag{51}$$

where x_0 is any given point and we used that A, w are nonincreasing.

We take now a number $\beta \in (0, \frac{c}{8})$, and we choose x_0 such that

$$A(x_0)w(x_0) \le c\beta \le \frac{c^2}{4} - c\beta. \tag{52}$$

Notice that, if $A(0) \le c\beta$, then we can take $x_0 = 0$. Otherwise we have $A(0) > c\beta$, and we can use (47) to find a value $x_0 > 0$, only depending on $\bar{A}$, β and a positive lower bound of $1 - w(0)$, such that (52) holds. As a first consequence, from (51) we deduce

$$q' \ge -cq + c\beta, \quad x \ge x_0$$

which leads, as before, to the exponential estimate

$$w(x) \le w(x_0)e^{-\beta(x-x_0-\frac{1}{c})} \quad \forall x \ge x_0.$$

Coming back to (51), now we upgrade it to

$$q' \ge \left(q - \frac{c}{2}\right)^2 - A(x)w(x) \ge \left(q - \frac{c}{2}\right)^2 - \bar{A}e^{-\beta(x-x_0-\frac{1}{c})}, \quad x > x_0. \tag{53}$$

We set

$$\zeta := \frac{c}{2} - q + Be^{-\beta(x-x_0)},$$

where B is sufficiently large, e.g. take $B = \frac{\bar{A}}{\beta}e^{1/8}$, so that $B\beta \ge \bar{A}e^{\frac{\beta}{c}}$. Then we get from (53),

$$-\zeta' = q' + B\beta e^{-\beta(x-x_0)} \ge \left(q - \frac{c}{2}\right)^2 \ge \zeta^2 - 2Be^{-\beta(x-x_0)}\zeta.$$

Notice that ζ is a positive function since $q < \frac{c}{2}$ due to Proposition 3.12. Then ζ satisfies

$$\left(\frac{1}{\zeta}\right)' \ge 1 - 2Be^{-\beta(x-x_0)}\frac{1}{\zeta}$$

and we get, integrating and dropping the term in x_0,

$$\begin{aligned}\frac{1}{\zeta} &\ge \exp\Big(\frac{2B}{\beta}e^{-\beta(x-x_0)}\Big)\int_{x_0}^{x}\exp\Big(-\frac{2B}{\beta}e^{-\beta(y-x_0)}\Big)\,dy\\ &\ge x - M\end{aligned}$$

for some constant M only depending on x_0, β and B (which only depends on $\bar{A}$ and β). Finally, for x sufficiently large (e.g. for $x > M + 1$), we have $\zeta \le \frac{1}{x-M}$ and this implies, by definition of ζ, that

$$\frac{c}{2} - q \le \frac{1}{x-M} \quad \forall x > M + 1.$$

Recalling that $q = -\frac{w'}{w}$, by integration we deduce inequality (48), say for $x > M + 1$, but then of course for any $x \ge 1$ as well. The constant K depends on x_0, β, $\bar{A}$, and then, from the above choices of x_0 and β, the constant depends on $\bar{A}$ and on positive lower bounds of c and $1 - w(0)$. ∎

3.5. The whole family of waves

We are now in a position to characterize the whole family of waves for any given speed $c > 2\sqrt{\underline{A}}$. The key ingredients are Corollary 3.14 and the uniqueness result for any given value of $\mathcal{I}$, Proposition 3.17. We recall that the operator $\mathcal{I}$ is defined on solutions of (22) by the two equivalent formulations in (33).

Lemma 3.19. *Assume that* (22) *admits a solution* w*. Then, for any* $\underline{A} < j < \mathcal{I}(w)$*, there exists a solution* $\widetilde{w} \geq w$ *of* (22) *satisfying* $\mathcal{I}(\widetilde{w}) = j$*.*

Proof. Consider the family of waves

$$\mathcal{F}_{w,j} := \left\{\psi \text{ solution of (22)} : \psi \geq w \text{ and } \mathcal{I}(\psi) \geq j\right\},$$

and call

$$j^* := \inf_{\psi \in \mathcal{F}_{w,j}} \mathcal{I}(\psi).$$

We have that $j^* \geq j$. We now show that j^* is attained. Let $(\psi_n)_{n\in\mathbb{N}}$ be a minimizing sequence for $\mathcal{I}$ on $\mathcal{F}_{w,j}$. This sequence converges (up to subsequences) in $C^2_{\mathrm{loc}}(\mathbb{R})$ to a nonincreasing solution $w \leq \psi^* \leq 1$ of (26). We see from (33) that

$$j^* = \lim_{n\to\infty} \mathcal{I}(\psi_n) = \bar{A} + \int_{\mathbb{R}} A'(y)\psi^*(y)\,dy. \tag{54}$$

Because $j^* \geq j > \underline{A}$, we deduce that $\psi^* \not\equiv 1$ and therefore ψ^* is a solution of (22) thanks to Lemma 3.2. It holds in particular that $\mathcal{I}(\psi^*) = j^*$, i.e. j^* is attained.

Next we assume by contradiction that $j^* > j$. We apply Corollary 3.14 and deduce that, for any $\varepsilon > 0$, there exists a solution $\widetilde{\psi}$ of (22) satisfying

$$\widetilde{\psi}(0) > \psi^*(0), \quad \psi^* \leq \widetilde{\psi} < \psi^* + \varepsilon.$$

It follows from the second formulation in (33) that

$$j^* = \mathcal{I}(\psi^*) > \mathcal{I}(\widetilde{\psi}) > j^* + \varepsilon(\underline{A} - \bar{A}).$$

We can therefore choose ε small enough in such a way that $\mathcal{I}(\widetilde{\psi}) > j$, whence $\widetilde{\psi} \in \mathcal{F}$, and we obtain a contradiction with the definition of j^*. ∎

Proposition 3.20. *Two distinct solutions of* (22) *are strictly ordered.*

Proof. Let w_1, w_2 be two distinct solutions of (22). Proposition 3.17 entails that $\mathcal{I}(w_1) \neq \mathcal{I}(w_2)$. Suppose, to fix ideas, that $\mathcal{I}(w_1) > \mathcal{I}(w_2)$, and assume by contradiction that there exists $x_0 \in \mathbb{R}$ such that $w_1(x_0) \geq w_2(x_0)$. Then, thanks to Corollary 3.14 and the second formulation in (33), we can find a solution $\widetilde{w}_1 \geq w_1$ which still fulfills $\mathcal{I}(\widetilde{w}_1) > \mathcal{I}(w_2)$, but in addition $\widetilde{w}_1(x_0) > w_2(x_0)$. Next, applying Lemma 3.19 one obtains another solution $\widetilde{w} \geq \widetilde{w}_1$ such that $\mathcal{I}(\widetilde{w}) = \mathcal{I}(w_2)$ (note that $\mathcal{I}(w_2) > \underline{A}$ by Proposition 3.12). This contradicts Proposition 3.17, because $\widetilde{w}(x_0) \geq \widetilde{w}_1(x_0) > w_2(x_0)$. ∎

Gathering together all previous results, we can derive the characterization of the family of traveling wave solutions.

Proof of Theorem 2.3. Problem (22) admits solution if and only if $c > 2\sqrt{\underline{A}}$ due to Propositions 3.11 and 3.12. Fix $c > 2\sqrt{\underline{A}}$ and let $\mathcal{F}$ be the family of solutions to (22). We know from Proposition 3.20 that functions in $\mathcal{F}$ are strictly ordered. We can therefore parametrize $\mathcal{F}$ as

$$\mathcal{F} = (w_\vartheta)_{\vartheta\in\Theta},$$

with w_ϑ satisfying $w_\vartheta(0) = \vartheta$, for a suitable set of indices $\Theta \subset (0,1)$. Proposition 3.8 yields $\Theta = (0,1)$ when $c \geq 2\sqrt{\bar{A}}$.

Consider the case $2\sqrt{\underline{A}} < c < 2\sqrt{\bar{A}}$. Let w^c be the critical wave provided by Proposition 3.16, i.e. satisfying $\mathcal{I}(w^c) = c^2/4$. We know from Proposition 3.12 that this realizes the maximum of $\mathcal{I}$ on the family $\mathcal{F}$. As a consequence, since $\mathcal{I}$ is decreasing due to the second formulation in (33) and the functions in $\mathcal{F}$ are strictly ordered, w^c lies below any other function of $\mathcal{F}$. This means that $\min\Theta = w^c(0) \in (0,1)$; let us call this value ϑ_c. Proposition 3.13 eventually shows that $\Theta = [\vartheta_c, 1)$.

Let us show the continuity of the mapping

$$\Theta \ni \vartheta \mapsto w_\vartheta \in \mathcal{F},$$

equipped with the $L^\infty(\mathbb{R})$ norm, for any given $c > 2\sqrt{\underline{A}}$. Consider a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ converging to some $\tilde{\vartheta} \in \Theta$. It follows that $(w_{\vartheta^n})_{n\in\mathbb{N}}$ converges (up to subsequences) locally uniformly to a solution $\widetilde{w}$ of (26) satisfying $\widetilde{w}(0) = \tilde{\vartheta}$. By Lemma 3.2, the function $\widetilde{w}$ satisfies $\widetilde{w}(-\infty) = 1$, $\widetilde{w}(+\infty) = 0$. This means that $\widetilde{w}$ solves (22) and therefore $\widetilde{w} = w_{\tilde{\vartheta}}$. For any $\varepsilon > 0$, consider $x_\varepsilon > 0$ for which

$$w_{\tilde{\vartheta}}(-x_\varepsilon) > 1 - \varepsilon, \quad w_{\tilde{\vartheta}}(x_\varepsilon) < \varepsilon,$$

and, owing to the locally uniform convergence, let n_ε be such that

$$\forall n \geq n_\varepsilon,\ |x| \leq x_\varepsilon, \quad |w_{\vartheta^n}(x) - w_{\tilde{\vartheta}}(x)| < \varepsilon.$$

This means that

$$\forall n \geq n_\varepsilon,\ x > x_\varepsilon, \quad |w_{\vartheta^n}(x) - w_{\tilde{\vartheta}}(x)| < \max\big\{w_{\vartheta^n}(x_\varepsilon), w_{\tilde{\vartheta}}(x_\varepsilon)\big\} < 2\varepsilon$$

and likewise

$$\forall n \geq n_\varepsilon,\ x < -x_\varepsilon, \quad |w_{\vartheta^n}(x) - w_{\tilde{\vartheta}}(x)| < 1 - \min\big\{w_{\vartheta^n}(-x_\varepsilon), w_{\tilde{\vartheta}}(-x_\varepsilon)\big\} < 2\varepsilon.$$

We have thereby shown that $(w_{\vartheta^n})_{n\in\mathbb{N}}$ converges uniformly to $w_{\tilde{\vartheta}}$.

To complete the proof, it remains to analyze the dependence of the critical waves w^c with respect to c. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence converging to some $\tilde{c} \in (2\sqrt{\underline{A}}, 2\sqrt{\bar{A}})$. Then $(w^{c_n})_{n\in\mathbb{N}}$ converges (up to subsequences) locally uniformly to a solution $\widetilde{w}$ of (26) with $c = \tilde{c}$. By dominated convergence, we find that

$$\int_{\mathbb{R}} A'(y)\widetilde{w}(y)\,dy = \lim_{n\to\infty} \int_{\mathbb{R}} A'(y) w^{c_n}(y)\,dy = \lim_{n\to\infty} \mathcal{I}(w^{c_n}) - \bar{A} = \frac{\tilde{c}^2}{4} - \bar{A}. \quad (55)$$

Because $\tilde{c} \in (2\sqrt{\underline{A}}, 2\sqrt{\bar{A}})$, we deduce that $\widetilde{w} \not\equiv 0, 1$ and thus that $\widetilde{w}$ is a solution to (22) due to Lemma 3.2. Hence (55) yields $\mathcal{I}(\widetilde{w}) = \tilde{c}^2/4$, i.e. $\widetilde{w}$ is the critical front $w^{\tilde{c}}$. The same arguments as before show that the convergence of (the subsequence of) $(w^{c_n})_{n\in\mathbb{N}}$ towards $w^{\tilde{c}}$ is uniform in space. This means that the whole sequence $(w^{c_n})_{n\in\mathbb{N}}$ converges uniformly to $w^{\tilde{c}}$.

Finally, the limits in (23) are deduced from the bounds (34) in Proposition 3.12. This is immediate if $\underline{A} < A(0) < \bar{A}$. Otherwise, we need to apply the inequalities (34) at a point x_0 where $\underline{A} < A(x_0) < \bar{A}$, which imply that

$$w^c(x_0) \nearrow 1 \ \text{ as } c \searrow 2\sqrt{\underline{A}}, \quad w^c(x_0) \searrow 0 \ \text{ as } c \nearrow 2\sqrt{\bar{A}}.$$

Then we can use Harnack's inequalities (24) to transport these limits at the origin. ∎

Proof of Theorem 2.4. The monotonicity and continuity of the mapping $\vartheta \to \mathcal{I}(w_\vartheta)$ immediately follow from Theorem 2.3 and the second formulation of $\mathcal{I}$ in (33). The image J of the mapping is an interval contained in $(\underline{A}, \bar{A}) \cap (\underline{A}, c^2/4]$ and with lower bound $\underline{A}$, due to Proposition 3.12 and Lemma 3.19. Then by Proposition 3.16, $J = (\underline{A}, c^2/4]$ if $2\sqrt{\underline{A}} < c < 2\sqrt{\bar{A}}$.

In the case $c \geq 2\sqrt{\bar{A}}$, we consider a sequence of waves $(w_{\vartheta^n})_{n\in\mathbb{N}}$ with $(\vartheta^n)_{n\in\mathbb{N}}$ converging to 0, and we deduce from (3.1) that $(w_{\vartheta^n})_{n\in\mathbb{N}}$ converges locally uniformly to 0. It then follows from (33) that $(\mathcal{I}(w_{\vartheta^n}))_{n\in\mathbb{N}}$ converges to $\bar{A}$. This means that $J = (\underline{A}, \bar{A})$ in this case. ∎

A question remains open after Theorem 2.3: Can two distinct critical waves intersect? We are not able to answer this question in general, but only in the region where A is local.

Proposition 3.21. *Assume that $A(x_0) = \bar{A}$. Let w^{c_1}, w^{c_2} be the critical waves associated with $2\sqrt{\underline{A}} < c_1 < c_2 < 2\sqrt{\bar{A}}$; then $w^{c_1}(x_0) > w^{c_2}(x_0)$.*

Proof. Assume by contradiction that $w^{c_1}(x_0) < w^{c_2}(x_0)$. Then by Theorem 2.3 there exists another wave $\widetilde{w}$ for (22) with $c = c_1$ satisfying $\widetilde{w} > w^{c_1}$ on $\mathbb{R}$ and $\widetilde{w}(x_0) = w^{c_2}(x_0)$. Observe that $\widetilde{w}$ is a supersolution of the equation in (22) with $c = c_2$. Then, using the fact that $A \equiv \bar{A}$ on $(-\infty, x_0]$, one checks that necessarily $\widetilde{w} > w^{c_2}$ on $(-\infty, x_0)$, whence $\widetilde{w}'(x_0) \leq w^{c_2}(x_0)$. Therefore, Lemma 3.4 yields $\widetilde{w} \leq w^{c_2}$ on $[x_0, +\infty)$. Then we derive from Theorem 2.4,

$$\frac{c_1^2}{4} \geq \mathcal{I}(\widetilde{w}) = \bar{A} + \int_{x_0}^{+\infty} A'(y)\widetilde{w}(y)\,dy \geq \bar{A} + \int_{x_0}^{+\infty} A'(y)w^{c_2}(y)\,dy = \frac{c_2^2}{4},$$

which is a contradiction. ∎

4. Traveling waves for the mean-field game system

We now prove the main result of the paper. This will be the outcome of a thorough analysis of the system of traveling waves (21). Namely, we are going to provide necessary and sufficient conditions for the existence of traveling waves, as in the following statement.

Theorem 4.1. *Assume that hypotheses* (8)–(13) *hold true. Then we have the following properties:*

(i) *there are no solutions of* (21) *with* $c \le 2\kappa^2$ *or with* $c \ge \alpha(1) + \kappa^2$*;*

(ii) *there exists a solution* (c, w, z) *of* (21) *such that* $c \in (2\kappa^2, 2\kappa\sqrt{\alpha(1)})$ *and*

$$\frac{c^2}{4} = \kappa^2 \int_{\mathbb{R}} \alpha(\sigma(y))(-w'(y))\,dy;$$

(iii) *for every* $c \in [2\kappa\sqrt{\alpha(1)}, \alpha(1) + \kappa^2)$*, there exist solutions of* (21) *(with arbitrary normalization at any given point).*

The three statements of this theorem are separately proved in the next subsections. Then, in Section 4.4, we will eventually show the equivalence between BGP solutions of (2) and traveling wave solutions of (21). The proof of Theorem 2.2 will then be achieved.

4.1. Preliminary properties and necessary conditions

In this section we derive some necessary conditions for the existence of waves. This will enlighten in particular the optimality of the assumptions $\rho > \kappa^2$, $\alpha(1) > \kappa^2$, so those two conditions (hypotheses (10), (13)) will not be assumed to hold a priori here.

First of all, it is convenient to observe that, by the concavity of α, the function σ associated with a solution (z, w) of (21) can be computed as

$$\sigma(x) = \begin{cases} s \in (0,1) & \text{if } \displaystyle\int_x^{+\infty} z(y)e^y w(y)\,dy = \frac{e^x}{\alpha'(s)}, \\ 1 & \text{if } \displaystyle\int_x^{+\infty} z(y)e^y w(y)\,dy \ge \frac{e^x}{\alpha'(1)}. \end{cases} \tag{56}$$

We will see in the next proposition that $\sigma \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$, and it is positive and nonincreasing. Hence, when dealing with the first equation of (21), we will be allowed to make use of the results of Section 3 with $A := \alpha \circ \sigma$ (and the obvious rescaling by $1/\kappa^2$).

Proposition 4.2. *Under assumptions* (8)*,* (9)*,* (11)*,* (12)*, problem* (21) *admits solution only if*

$$\rho > \kappa^2 \quad \textit{and} \quad \alpha(1) > \kappa^2.$$

Moreover, for any solution (c, w, z)*, the following properties hold:*

$$2\kappa^2 < c < \alpha(1) + \kappa^2,$$

$$z(-\infty) = 0, \quad z' > 0 \textit{ in } \mathbb{R}, \quad z(+\infty) = \frac{1}{\rho - \kappa^2},$$

and the associated σ *belongs to* $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$*, is nonincreasing and satisfies*

$$\exists x_0 \in \mathbb{R}, \quad \sigma = 1 \textit{ in } (-\infty, x_0], \quad 0 < \sigma < 1 \textit{ in } (x_0, +\infty), \quad \sigma(+\infty) = 0. \tag{57}$$

In particular, we have that $A := \alpha \circ \sigma \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ *is positive and nonincreasing.*

Proof. Assume that (21) admits a solution (c, w, z). We preliminarily observe that $z > 0$: indeed, with z being a supersolution of a linear elliptic equation, the strong maximum principle implies that either $z > 0$ or $z \equiv 0$. But in the latter case, the equation itself yields $\sigma \equiv 1$, while, from (56), we get $\sigma \equiv 0$. The strong maximum principle also yields $w > 0$. We now derive the properties stated in the proposition separately.

Properties of σ. Owing to the characterization (56) for the function σ, properties $zw > 0$ and $\alpha'(0) = +\infty$ entail that σ is strictly positive. Moreover, (56) also implies that σ is nonincreasing, because $\alpha'' < 0$; that $\sigma(x) = 1$ for $-x$ large enough, because $\alpha'(1) > 0$; and that $\sigma(+\infty) = 0$, because of the condition $ze^x w \in L^1(\mathbb{R})$ in (21). This proves (57). Finally, from (56) we deduce, using the regularity of $\alpha(s)$,

$$\begin{aligned}\sigma'(x) &= \frac{1}{\alpha''(\sigma(x))}\left(\frac{e^x}{\int_x^{+\infty} e^y zw\, dy} + z(x)w(x)\left(\frac{e^x}{\int_x^{+\infty} e^y zw\, dy}\right)^2\right) \\ &= \frac{\alpha'(\sigma(x))}{\alpha''(\sigma(x))}\big(1 + z(x)w(x)\alpha'(\sigma(x))\big) \quad \forall x : \sigma(x) < 1. \end{aligned} \tag{58}$$

Since z, w are locally bounded, and $\alpha''(s) < 0$, we deduce that $\sigma'(x)$ is locally bounded in the interval $(x_0, +\infty)$ where $0 < \sigma(x) < 1$, and it admits a finite limit as $x \to x_0^+$. Hence $\sigma \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$. We notice indeed that σ is piecewise C^1 but it is not differentiable at x_0, because $\lim_{x \to x_0^+} \sigma'(x) < 0$. The regularity of α then yields $A = \alpha \circ \sigma \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$.

The condition $\rho > \kappa^2$. Integrating the equation for z in (21) in an interval (x, y) yields

$$\kappa^2 z'(x) - \kappa^2 z'(y) = (c - 2\kappa^2)(z(x) - z(y)) + \int_x^y [1 - \sigma + (\kappa^2 - \rho - \alpha(\sigma)w)z]. \tag{59}$$

Supposing by contradiction that $\rho \le \kappa^2$, and using that $\sigma(+\infty) = 0$, we find that the term under the integral satisfies

$$\liminf_{x \to +\infty} [1 - \sigma + (\kappa^2 - \rho - \alpha(\sigma)w)z] \ge 1,$$

hence (59) yields $z'(y) \to -\infty$ as $y \to +\infty$, contradicting the boundedness of z.

Properties of z. Now that we know that $\rho > \kappa^2$, we infer from (57) that the term under the integral in (59) satisfies

$$\limsup_{x \to -\infty} [1 - \sigma + (\kappa^2 - \rho - \alpha(\sigma)w)z] \le (\kappa^2 - \rho) \liminf_{x \to -\infty} z(x).$$

Hence, if $z(x)$ does not tend to 0 as $x \to -\infty$, using the fact that z is uniformly continuous (by elliptic estimates) we obtain by (59) the contradiction $z'(x) \to -\infty$ as $x \to -\infty$. This proves that $z(-\infty) = 0$.

Next, the properties of the function σ derived above allow us to apply the results of Section 3 with $A := \alpha \circ \sigma$. In particular, Proposition 3.1 asserts that $c > 0$ and that w is decreasing. Then, differentiating the equation for z in (21), and using that σ, A, w are nonincreasing, we find that z' satisfies

$$-\kappa^2 (z')'' + (c - 2\kappa^2)(z')' + (\rho - \kappa^2)z' + A(x)wz' \ge 0, \quad x \in \mathbb{R}. \tag{60}$$

Moreover, with z being bounded, there exist two sequences $(x_n^\pm)_{n\in\mathbb{N}}$ diverging to $\pm\infty$ respectively, such that $z'(x_n^\pm) \to 0$ as $n \to \infty$. Hence, applying the weak maximum principle in the intervals (x_n^-, x_n^+) and letting $n \to \infty$, we deduce that $z' \geq 0$ in $\mathbb{R}$. Next, the strong maximum principle yields $z' > 0$, because otherwise $z \equiv z(-\infty) = 0$, while we know that $z > 0$.

The monotonicity and boundedness of z imply that $z(x)$ converges to a positive limit $z(+\infty)$ as $x \to +\infty$. Due to elliptic estimates, the convergence holds in C^2_{loc} and thus we deduce that the value $z(+\infty)$ satisfies

$$(\rho - \kappa^2)z(+\infty) = -\lim_{x\to+\infty}(A(x)wz - 1 + \sigma(x)) = 1.$$

The condition $\alpha(1) > \kappa^2$ *and the bounds* $2\kappa^2 < c < \alpha(1) + \kappa^2$. First of all, Proposition 3.1 yields $c > 0$ and Proposition 3.12 (iv) yields

$$w(x) \geq w(0)e^{-\lambda x} \quad \forall x \geq 0,$$

with

$$\lambda = \frac{1}{\kappa^2}\left(\frac{c}{2} - \sqrt{\frac{c^2}{4} - \kappa^2 \mathcal{I}(w)}\right),$$

where the operator $\mathcal{I}$ is defined in (33). If $c \leq 2\kappa^2$, we see that $\lambda \leq 1$. If $c \geq \alpha(1) + \kappa^2$, the same conclusion follows from the inequalities $\mathcal{I}(w) \leq \bar{A} := \alpha(1)$ (and $\mathcal{I}(w) \leq \frac{c^2}{4\kappa^2}$) provided by Proposition 3.12; indeed,

$$\lambda \leq \frac{1}{\kappa^2}\left(\frac{c}{2} - \sqrt{\frac{c^2}{4} - \kappa^2\alpha(1)}\right) \leq \frac{1}{\kappa^2}\left(\frac{c}{2} - \sqrt{\frac{c^2}{4} - c\kappa^2 + \kappa^4}\right) = 1.$$

Hence, in both cases, we find that

$$w(x) \geq w(0)e^{-x} \quad \forall x \geq 0.$$

Since z does not tend to 0 at $+\infty$, because $z' > 0$, the condition $ze^x w \in L^1(\mathbb{R})$ in (21) is violated. As a by-product, we have shown that necessarily $\alpha(1) > \kappa^2$. ■

The first statement of Proposition 4.2 proves Theorem 4.1 (i).

Next we derive a pointwise lower bound for z only using that it satisfies the equation in (21), i.e.

$$-\kappa^2 z'' + (c - 2\kappa^2)z' + (\rho - \kappa^2)z + A(x)wz = 1 - \sigma(x), \tag{61}$$

for some given functions $A, w \in L^\infty(\mathbb{R})$.

Lemma 4.3. *Let* z *be a nonnegative solution of* (61) *in* $[0, 2]$, *with* $c \in \mathbb{R}$, $\rho \geq \kappa^2$ *and* A, w *nonnegative and bounded. Then*

$$z(1) \geq \frac{1 - \sup_{(0,2)}\sigma}{C(1 + \kappa^2 + |c| + \rho + \sup A \sup w)},$$

for some universal constant $C > 0$.

Proof. It is sufficient to find a positive subsolution of (61) on $(0,2)$ vanishing at the boundary. This is simply provided by $\underline{z} := 1-(x-1)^2$. On $(0,2)$ it satisfies

$$-\kappa^2\underline{z}'' + (c-2\kappa^2)\underline{z}' + (\rho-\kappa^2)\underline{z} + A(x)w\underline{z} \le C(1+\kappa^2+|c|+\rho+\sup A \sup w),$$

for some universal constant $C>0$. Hence, calling

$$k := \frac{1-\sup_{(0,2)}\sigma}{C(1+\kappa^2+|c|+\rho+\sup A \sup w)},$$

and supposing that $\sup_{(0,2)}\sigma < 1$ (otherwise the result trivially holds), we have that $k\underline{z}$ is a subsolution of (61) on $(0,2)$. Observe that the zeroth-order coefficient of this equation is nonnegative. Thus, the standard maximum principle yields

$$z(1) \ge k\underline{z}(1) = k. \qquad \blacksquare$$

We conclude this subsection with a stability lemma for problem (22) that will often be used in the sequel. Following the terminology employed for the nonlocal KPP equation, we say that a solution to (22) is *critical* if

$$\mathcal{I}(w) := \int_{\mathbb{R}} A(y)(-w'(y))\,dy = \frac{c^2}{4}. \tag{62}$$

Lemma 4.4. *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers, $(A_j)_{j\in\mathbb{N}}$ be a sequence of equibounded, nonincreasing, nonnegative functions in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ satisfying*

$$A_j(-\infty) > A_j(+\infty) \quad \forall j\in\mathbb{N},$$

and for $j\in\mathbb{N}$, let w_j be a solution to (22) with $c=c_j$ and $A=A_j$. Assume that $(c_j)_{j\in\mathbb{N}}$ converges to some $c>0$, that $(A_j)_{j\in\mathbb{N}}$ converges pointwise to some function A satisfying $A(0)>0$ and that $(w_j(0))_{j\in\mathbb{N}}$ converges to some value in $(0,1)$. Then $(w_j)_{j\in\mathbb{N}}$ converges uniformly towards a decreasing solution w of (22).

In addition, if the w_j are critical (in the sense of (62)) then w is critical too.

Proof. Let $\bar{A}>0$ be such that $A_j\le\bar{A}$ for all $j\in\mathbb{N}$. By Lemma 3.2, $(w_j)_{j\in\mathbb{N}}$ converges in $C^2_{\mathrm{loc}}(\mathbb{R})$, up to subsequences, towards a function w. Moreover, since $A_j(0)\to A(0)>0$, Lemma 3.18 implies that the convergence is also uniform in $\mathbb{R}$, with $1-w_j(x)\le Ke^{\gamma x}$ and $w_j(x)\le Ke^{-\gamma x}$ for some positive K,γ independent of j. Hence, since the w_j are decreasing by Proposition 3.1, we find that

$$\left|\int_{-\infty}^{x} A_j(y)(-w_j'(y))\,dy\right| \le \bar{A}(1-w_j(x)) \le \bar{A}Ke^{\gamma x},$$

which is arbitrarily small up to choosing $-x$ very large. As a consequence, we deduce that, up to subsequences,

$$\int_{-\infty}^{x} A_j(y)(-w_j'(y))\,dy \to \int_{-\infty}^{x} A(y)(-w'(y))\,dy,$$

and thus that w solves (22). By Theorem 2.3 (or a classical result if A is constant) there is a unique wave for such a problem which fulfills $w(0) = \lim_{j\to\infty} w_j(0)$. This shows that the whole sequence w_j converges uniformly to w. We know from Proposition 3.1 that w is decreasing.

The fact that the criticality condition (62) is preserved is obtained by estimating the integral at $+\infty$ using that $w_j(x) \le Ke^{-\gamma x}$. ■

4.2. The critical wave

We now turn to the proof of Theorem 4.1 (ii). It is divided into two main parts: we first build an approximated solution (z_n, w_n, c_n) (for a suitably truncated problem) through a fixed point argument, and secondly we pass to the limit on the mean-field game system in order to get a solution.

We assume here for the sake of simplicity that $\kappa = 1$; the general case is treated in the same way with the obvious modifications.

Part I. The approximated problem. For $n \in \mathbb{N}$, we consider the following approximated problem:

$$\begin{cases} w'' + cw' + w \displaystyle\int_{-\infty}^{x} A(y)(-w'(y))\,dy = 0, \quad x \in \mathbb{R}, \\ w > 0, \quad w(-\infty) = 1, \quad w(+\infty) = 0, \\ -z'' + (c-2)z' + (\rho - 1)z + Awz = 1 - [\sigma]_n, \quad x \in (-n, n), \\ z = 0 \text{ in } (-\infty, -n], \quad z = \dfrac{1}{\rho - 1} \text{ in } [n, +\infty), \\ [\sigma]_n(x) = \underset{s\in[0,1]}{\operatorname{argmax}} \Big\{ (1-s)e^x \\ \qquad\qquad + \alpha(s) \displaystyle\int_x^{+\infty} z(y)e^y w(y)\chi_n(y)\,dy \Big\}, \quad A := \alpha([\sigma]_n), \end{cases} \tag{63}$$

where

$$\chi_n(y) = \min\{1, e^{-2(y-n)}\}.$$

The role of χ_n is to prevent the case $A \equiv \alpha(1)$, which in the original problem (21) is excluded by the condition $ze^x w \in L^1(\mathbb{R})$; it actually guarantees that $[\sigma]_n(x), A(x) \to 0$ as $x \to +\infty$, uniformly with respect to c, w, z.

We will find a solution to (63) through a fixed point argument on the function $[\sigma]_n$. The latter is characterized by

$$\forall s \in (0,1), \;\; [\sigma]_n(x) = s \quad \Longleftrightarrow \quad \int_x^{+\infty} z(y)e^y w(y)\chi_n(y)\,dy = \frac{e^x}{\alpha'(s)}. \tag{64}$$

From this we immediately see that $[\sigma]_n$ is nonincreasing and satisfies $[\sigma]_n(x) = 1$ for $-x$ large and $[\sigma]_n(+\infty) = 0$. We therefore look for the fixed point in the set

$$X := \{\sigma \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) : \sigma \text{ is nonincreasing}, \ \sigma(x) = 1 \text{ for } -x \text{ large}, \ \sigma(+\infty) = 0\}.$$

At some point we will actually restrict to a closed subset of X with respect to the $L^\infty(\mathbb{R})$ topology.

We now define an operator T_n on X. Given $\sigma \in X$, we set $A := \alpha \circ \sigma$ and we call w_c the (unique) critical wave with speed $c \in (0, 2\sqrt{\alpha(1)})$ associated with A, provided by Theorem 2.3 ($\bar{A} = \alpha(1)$). Then we choose a speed c through a suitable normalization condition expressed in terms of the functions

$$\underline{w}_c(x) := \inf_{0<c'\leq c} w_{c'}(x).$$

Namely, we claim that there exists a unique $c \in (0, 2\sqrt{\alpha(1)})$ such that

$$\int_{-\infty}^{0} e^y \underline{w}_c(y)\, dy = \frac{1}{2}. \tag{65}$$

It is clear that the $\underline{w}_c$ are nonincreasing with respect to c. Moreover, thanks to Proposition 3.21, for any $\underline{x} \in \mathbb{R}$ where $A(\underline{x}) = \alpha(1)$ it holds that $w_c(\underline{x})$ is decreasing with respect to c. It follows that $\underline{w}_c(\underline{x}) = w_c(\underline{x})$ and that this value is strictly decreasing with respect to c. As a consequence, the mapping $G\colon (0, 2\sqrt{\alpha(1)}) \to \mathbb{R}$ defined by

$$G(c) := \int_{-\infty}^{0} e^y \underline{w}_c(y)\, dy$$

is decreasing. In addition, one checks that it is continuous using dominated convergence and the last part of Theorem 2.3. Furthermore, by the properties of X, there exists $x_0 \in \mathbb{R}$ such that $0 = A(+\infty) < A(x_0) < A(-\infty) = \alpha(1)$. Therefore, owing to the criticality condition

$$\mathcal{I}(w_c) := \int_{\mathbb{R}} A(y)(-w_c'(y))\, dy = \frac{c^2}{4},$$

the two inequalities in (34) (applied in x_0 by translation of the coordinate system, and with $\underline{A} = 0$, $\bar{A} = \alpha(1)$) imply that $w_c(x_0) \to 1$ as $c \searrow 0$ and $w_c(x_0) \to 0$ as $c \nearrow 2\sqrt{\alpha(1)}$. We then infer from Harnack's inequalities (24) that these convergences hold true locally uniformly in $\mathbb{R}$. The first one then implies that $G(c) \to 1$ as $c \searrow 0$, while the second one implies that

$$G(c) \leq \int_{-\infty}^{0} e^y w_c(y)\, dy \to 0 \quad \text{as } c \nearrow 2\sqrt{\alpha(1)}.$$

As a consequence, there exists a unique $c \in (0, 2\sqrt{\alpha(1)})$ such that $G(c) = 1/2$, i.e. (65) holds true. This normalization determines the choice of c employed to define T_n.

Now, given the above speed c and the associated critical wave w_c, we consider the solution z of the second equation in (63) with σ in place of $[\sigma]_n$ and the prescribed exterior conditions, which classically exists and is unique. The outcome $T_n(\sigma)$ is the function $[\sigma]_n$ generated by w_c and z as indicated in the last line of (63). Summing up, the operator T_n works as follows:

$$\sigma \in X \rightsquigarrow A := \alpha \circ \sigma$$
$$\rightsquigarrow (c, w_c) : \begin{cases} w'' + cw' + w \displaystyle\int_{-\infty}^{x} A(y)(-w'(y))\, dy = 0, \\ w(-\infty) = 1, \quad w(+\infty) = 0, \quad \displaystyle\int_{\mathbb{R}} A(-w') = \frac{c^2}{4}, \\ \displaystyle\int_{-\infty}^{0} e^y \underline{w}_c(y)\, dy = \frac{1}{2}, \end{cases}$$
$$\rightsquigarrow z : \begin{cases} -z'' + (c-2)z' + (\rho - 1)z + A w_c z = 1 - \sigma, \quad x \in (-n, n), \\ z = 0 \text{ in } (-\infty, -n], \quad z = \dfrac{1}{\rho - 1} \text{ in } [n, +\infty), \end{cases}$$
$$\rightsquigarrow T_n(\sigma) := [\sigma]_n(x) = \operatorname*{argmax}_{s \in [0,1]} \left\{ (1-s)e^x + \alpha(s) \int_x^{+\infty} z(y) e^y w_c(y) \chi_n(y)\, dy \right\}.$$

In the following lemma, we prove the existence of a fixed point for T_n.

Lemma 4.5. *The operator T_n has a fixed point in a closed subset $\widetilde{X}$ of X.*

As a consequence, problem (63) admits a solution $(c, w, z) = (c_n, w_n, z_n)$ with $c_n \in (0, 2\sqrt{\alpha(1)})$, z_n increasing in $(-n, n)$, w_n decreasing and satisfying in addition the following properties:

(i) *w_n is a critical wave corresponding to c_n, i.e.*

$$\int_{\mathbb{R}} A(-w_n')\, dy = c_n^2/4; \tag{66}$$

(ii) *there exists a constant $\underline{\vartheta} > 0$, only depending on $\alpha(1)$, such that*

$$w_n(0) \geq \underline{\vartheta} > 0 \quad \forall n \in \mathbb{N}; \tag{67}$$

(iii) *the following normalization condition holds true:*

$$\int_{-\infty}^{0} e^y \underline{w}_{c_n}(y)\, dy = \frac{1}{2}, \quad \textit{where } \underline{w}_{c_n}(x) := \inf_{0 < c' \leq c_n} w_{c'}(x).$$

Proof. The fixed point will be obtained as a consequence of Schauder's theorem. Some preliminary observations are in order, concerning the functions w_c, z associated with the definition of $T_n(\sigma)$ (cf. the previous scheme). We start with the monotonicities. We know from Proposition 3.1 that w_c, as well as any other wave for the KPP equation, is decreasing. On the other hand, the constant functions 0 and $\frac{1}{\rho-1}$ being respectively a sub- and a supersolution of the equation for z, the maximum principle yields $0 \le z \le \frac{1}{\rho-1}$ in $(-n, n)$; then $z'(\pm n) \ge 0$ and thus, applying the strong maximum principle to z', which satisfies (60), we infer that $z' > 0$ in $(-n, n)$.

Next we point out a lower bound for w_c. This is a crucial consequence of (65), which implies that

$$\frac{1}{2} \le \frac{1}{4} + \int_{-\ln 4}^{0} \underline{w}_c(y)\,dy \le \frac{1}{4} + (\ln 4)\underline{w}_c(-\ln 4) \le \frac{1}{4} + (\ln 4)w_c(-\ln 4).$$

Hence, by Harnack's inequality (24), for any $R > 0$ there exists a positive constant C_R, only depending on R and $\alpha(1)$ (recall that $c \in (0, 2\sqrt{\alpha(1)})$), such that

$$w_c(x) \ge C_R > 0 \quad \forall x \in [-R, R]. \tag{68}$$

In particular, (67) holds.

We now identify the compact set $\widetilde{X} \subset X$ where we apply Schauder's theorem and we separately check its hypotheses. For simplicity, hereafter we drop the index of T_n.

The invariant convex, compact set $\widetilde{X} \subset X$. Our goal is to show that $T(X)$ is contained in a compact subset of X with respect to the $L^\infty(\mathbb{R})$ norm; this will be our $\widetilde{X}$. Consider as before the functions w_c, z associated with $T(\sigma)$. We preliminarily observe that elliptic boundary estimates imply that the C^1 norm of z is controlled in terms of $\alpha(1)$ and ρ, and thus there exists a constant $\delta \in (0, 1)$, depending on $\alpha(1)$, ρ, such that $z \ge \frac{1}{2(\rho-1)}$ in $[n-\delta, n]$. Because of this and the lower bound (68) for w_c, we find that, for $x < n - \delta$,

$$e^{-x}\int_x^{+\infty} z(y)e^y w_c(y)\chi_n(y)\,dy \ge e^{-x}\int_{n-\delta}^{n} z(y)e^y w_c(y)\chi_n(y)\,dy \ge e^{-x}\frac{C_n\delta}{2(\rho-1)},$$

which is larger than $1/\alpha'(1)$ for x smaller than some x_n (possibly smaller than $-n$) only depending on n, $\alpha(1)$, ρ. Owing to the characterization (64), we derive

$$T(\sigma) = 1 \quad \text{on } (-\infty, x_n].$$

We are left to show the regularity of $[\sigma]_n = T(\sigma)$ and the uniform estimate as $x \to +\infty$. For these, we rewrite (64) as

$$[\sigma]_n(x) = (\alpha')^{-1}\left(\frac{e^x}{\int_x^{+\infty} z e^y w_c \chi_n\,dy}\right) \quad \forall x : [\sigma]_n(x) < 1. \tag{69}$$

Since $(\alpha')^{-1}$ is decreasing, this shows on one hand that $[\sigma]_n$ is strictly positive and non-increasing, and on the other, using $zw_c \le \frac{1}{\rho-1}$, that

$$[\sigma]_n(x) \le \omega_n(x) := (\alpha')^{-1}\left(\frac{e^x(\rho-1)}{\int_x^{+\infty} e^y \chi_n(y)\,dy}\right) \quad \forall x > y_n,$$

where y_n is the unique point where the right-hand side is equal to 1. Observe that $\omega_n(x) \to 0$ as $x \to +\infty$ because $\alpha'(0) = +\infty$.

As for the regularity, differentiating (69) we obtain (exactly as in (58))

$$[\sigma]_n'(x) = \frac{\alpha'([\sigma]_n(x))}{\alpha''([\sigma]_n(x))}\big(1 + \chi_n(x)z(x)w_c(x)\alpha'([\sigma]_n(x))\big) \quad \forall x : [\sigma]_n(x) < 1. \tag{70}$$

Then the strict concavity of α implies that $[\sigma]_n$ is a locally Lipschitz continuous function, whose $W^{1,\infty}$ norm remains bounded as long as $[\sigma]_n$ stays bounded away from 0. The lower bound follows from (69), namely, for any $x \geq n$ where $[\sigma]_n(x) < 1$,

$$[\sigma]_n(x) \geq (\alpha')^{-1}\left(\frac{e^x}{\int_x^{x+1} ze^y w_c \chi_n \, dy}\right) \geq (\alpha')^{-1}\left(\frac{(\rho - 1)e^{2(x+1-n)}}{w_c(x+1)}\right),$$

hence, by (68), $[\sigma]_n(x) \geq C(x)$, where $C(x)$ is a positive decreasing function depending on n, α, ρ, $\alpha(1)$. We deduce the existence of another positive decreasing function $\widetilde{C}(x)$, depending on the same terms, such that $\|[\sigma]_n\|_{W^{1,\infty}(-\infty,x)} \leq \widetilde{C}(x)$. Summing up, we have seen that

$$\begin{aligned} T(X) \subset \widetilde{X} := \big\{\sigma \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}) : \sigma \text{ is nonincreasing, } \sigma(x) = 1 \text{ for } x \leq x_n, \\ 0 \leq \sigma(x) \leq \omega_n(x) \text{ for } x \geq y_n, \\ \|\sigma\|_{W^{1,\infty}(-\infty,x)} \leq \widetilde{C}(x)\big\}, \end{aligned}$$

hence $\widetilde{X}$ is invariant under T. The set $\widetilde{X}$ is convex and one readily checks that it is compact in $L^\infty(\mathbb{R})$ using the Arzelà–Ascoli theorem and the conditions at $\pm\infty$.

Continuity of T. Let $(\sigma_j)_{j\in\mathbb{N}}$ in $\widetilde{X}$ converge uniformly to some σ. Then we have that $A_j := \alpha \circ \sigma_j$ uniformly converges to $A := \alpha \circ \sigma$; notice that the conditions in $\widetilde{X}$ imply that $A(x) = \alpha(1)$ for $x \leq x_n$ and that $0 \leq A(x) \leq \alpha(\omega_n(x)) \to 0$ as $x \to +\infty$, in particular the A_j do not trivialize in the limit. Let (c_j, w_j, z_j) be the triplet provided by the construction of $T(\sigma_j)$. Since the $(c_j)_{j\in\mathbb{N}}$ are in $(0, 2\sqrt{\alpha(1)})$, they converge, up to extraction of a subsequence, to some $c \in [0, 2\sqrt{\alpha(1)}]$. On one hand, the w_j are locally uniformly equibounded from below away from zero due to (68). On the other hand, the values $w_j(x_n)$ are bounded from above away from 1, because otherwise Proposition 3.21 and the second Harnack inequality in (24) would yield a contradiction with the normalization (65). We deduce that $(w_j(x_n))_{j\in\mathbb{N}}$ is contained in a compact subset of $(0, 1)$. Then, since by the criticality condition,

$$c_j^2/4 = \int_{\mathbb{R}} A_j(-w_j')\, dy \geq \int_0^{x_n} A_j(-w_j')\, dy = \alpha(1)(1 - w_j(x_n)),$$

we find that $c = \lim_{j\to+\infty} c_j > 0$. We can therefore apply Lemma 4.4 and infer that $(w_j)_{j\in\mathbb{N}}$ converges uniformly, up to subsequences, towards a decreasing solution w to (22), which in addition is critical, i.e. fulfills (62). We find as a by-product that $c < 2\sqrt{\alpha(1)}$. Indeed, by uniqueness of the critical wave, cf. Theorem 2.4, we have that w_j converges to w along the whole subsequence on which $c_j \to c$.

It remains to check that the normalization condition (65) is preserved up to subsequences. By dominated convergence, it is sufficient to show that the functions

$$\underline{w}_{j,c_j}(x) := \inf_{0<c'\le c_j} w_{j,c'}(x)$$

converge pointwise to

$$\underline{w}_c(x) := \inf_{0<c'\le c} w_{c'}(x),$$

where $w_{j,c'}$ and $w_{c'}$ are the critical waves with speed c' corresponding to the nonlinearity A_j and A respectively.

For this purpose, we observe that, for fixed $c' \in (0, 2\sqrt{\alpha(1)})$, the $w_{j,c'}$ converge uniformly, up to subsequences, to $w_{c'}$, thanks to Lemma 4.4 and the bounds (34) applied in x_n. This convergence holds true for the whole sequence $w_{j,c'}$ because of the uniqueness of the critical wave for fixed c'. Since $\underline{w}_{j,c_j}(x) \le w_{j,c'}(x)$ for any $c' < c_j$, and $c_j \to c$, we deduce that

$$\limsup_{j\to\infty} \underline{w}_{j,c_j}(x) \le w_{c'}(x) \quad \forall c' < c.$$

Recalling that the values $w_{c'}(x)$ are continuous with respect to c' owing to the last part of Theorem 2.3, this yields

$$\limsup_{j\to\infty} \underline{w}_{j,c_j}(x) \le \inf_{0<c'\le c} w_{c'}(x) = \underline{w}_c(x). \tag{71}$$

Conversely, we fix $x \in \mathbb{R}$ and, for any $\varepsilon > 0$ and $j \in \mathbb{N}$, we find $c_j^\varepsilon \in (0, c_j)$ such that

$$\underline{w}_{j,c_j}(x) \ge w_{j,c_j^\varepsilon}(x) - \varepsilon.$$

Without loss of generality, we can suppose that $c_j^\varepsilon \to \tilde{c}$ as $j \to \infty$, for some $\tilde{c} \in [0, c]$. If $\tilde{c} > 0$, Lemma 4.4 and (34) entail that $w_{j,c_j^\varepsilon}(x) \to w_{\tilde{c}}(x)$; while if $\tilde{c} = 0$, (34) yields $w_{j_\varepsilon,c_{j_\varepsilon}}(x) \to 1$. Hence, in any case,

$$\liminf_{j\to\infty} \underline{w}_{j,c_j}(x) \ge w_{\tilde{c}}(x) - \varepsilon \ge \underline{w}_c(x) - \varepsilon.$$

Since ε is arbitrary, the previous inequality together with (71) implies that $\underline{w}_{j,c_j}(x)$ converges to $\underline{w}_c(x)$, for every $x \in \mathbb{R}$. As we said above, this yields

$$\int_{-\infty}^0 e^y \underline{w}_c(y)\,dy = \lim_{j\to\infty} \int_{-\infty}^0 e^y \underline{w}_{j,c_j}(y)\,dy = \frac{1}{2},$$

i.e. (65) holds. As we have shown before, this condition uniquely characterizes c. We deduce that the whole sequence c_j converges to c (and consequently, the whole sequence $w_{j,c_j} \to w_c$).

The convergences of σ_j, c_j and w_j now imply, by standard stability in the second equation, that z_j converges uniformly to the unique z which solves $-z'' + (c-2)z' + (\rho-1)z + Aw_c z = 1 - \sigma$ in $(-n, n)$ with the given boundary conditions. Finally, we have proved that $(c_j, w_j, z_j) \to (c, w, z)$ uniformly, and the latter is the unique triple associated with $T(\sigma)$. We conclude using the characterization (64) that $T(\sigma_j) \to T(\sigma)$ in $L^\infty(\mathbb{R})$.

We can now invoke Schauder's theorem which provides us with a fixed point $\sigma_n \in \widetilde{X}$ such that $T_n(\sigma_n) = \sigma_n$. Associated with this function, we have $A_n := \alpha \circ \sigma_n$ and a unique triple (c_n, w_n, z_n) which therefore solves system (63). By construction, we have that w_n satisfies the conditions (i)–(iii). ∎

Part II. Passing to the limit in the approximation. Now we study the limit of the sequence $(c_n, w_n, z_n)_{n\in\mathbb{N}}$ of solutions to (63) provided by Lemma 4.5. We call σ_n the associated optimal functions $[\sigma]_n$, and $A_n := \alpha \circ \sigma_n$. We recall that w_n is decreasing and that z_n is increasing.

To start with, we show that σ_n stays bounded away from 0. Indeed, by Lemma 4.3, for any $x \in \mathbb{R}$ we have $z_n(x+1) \geq C(1-\sigma_n(x))$, for some positive constant C depending on κ, ρ, $\alpha(1)$. Then for any given $x \in \mathbb{R}$, we find for $n > x+2$ that either $\sigma_n(x) = 1$, or by (64),

$$\begin{aligned}
\frac{1}{\alpha'(\sigma_n(x))} &= e^{-x}\int_x^{+\infty} z_n(y)e^y w_n(y)\chi_n(y)\,dy \\
&\geq e^{-x}\int_{x+1}^{x+2} z_n(y)e^y w_n(y)\,dy \\
&\geq C(1-\sigma_n(x))w_n(x+2)(e^2-e).
\end{aligned}$$

Owing to (68), this provides a positive lower bound for $\sigma_n(x)$ independent of n. We have thereby shown that

$$\liminf_{n\to\infty} \sigma_n(x) > 0 \quad \forall x \in \mathbb{R}. \tag{72}$$

Next we derive an upper bound for w_n. Namely, we claim that up to extraction of a subsequence, it holds that

$$w_n(0) \leq \overline{\vartheta} < 1 \quad \forall n \in \mathbb{N}. \tag{73}$$

To show this, we consider two (mutually excluding) possibilities. Either there exists some $\beta < 1$ such that $\sigma_n(0) \leq \beta < 1$ for all $n \in \mathbb{N}$; in this case the same computation as before yields, for any $\xi > 1$ and $n > \xi$,

$$\begin{aligned}
\frac{1}{\alpha'(\beta)} \geq \frac{1}{\alpha'(\sigma_n(0))} &\geq \int_1^{\xi} z_n(y)e^y w_n(y)\,dy \\
&\geq C(1-\sigma_n(0))w_n(\xi)(e^\xi - e) \\
&\geq C(1-\beta)w_n(\xi)(e^\xi - e),
\end{aligned}$$

which implies

$$w_n(\xi) \leq \frac{1}{e^\xi - e}\Big(\frac{1}{C(1-\beta)\alpha'(\beta)}\Big).$$

The right-hand side is smaller than 1 for ξ sufficiently large, and thus (73) follows from Harnack's inequality (24). Alternatively, there exists a subsequence (not relabeled) such that $\sigma_n(0) \to 1$ as $n \to \infty$. If this is the case, using the characterization (64) of σ_n we derive

$$\liminf_{n\to\infty} \int_{\ln\frac{2}{3}}^{+\infty} z_n(y)e^y w_n(y)\chi_n(y)\,dy \geq \liminf_{n\to\infty} \int_0^{+\infty} z_n(y)e^y w_n(y)\chi_n(y)\,dy \geq \frac{1}{\alpha'(1)},$$

which, owing to the same characterization, shows that $A_n(\ln\frac{2}{3}) = \alpha(1)$ for n sufficiently large. Hence, for such values of n, Proposition 3.21 yields $w_n = \underline{w}_{c_n}$ in $(-\infty, \ln\frac{2}{3}]$ and therefore, by (65),

$$\frac{1}{2} \geq \int_{-\infty}^{\ln\frac{2}{3}} e^y w_n(y)\,dy \geq \frac{2}{3} w_n(\ln\tfrac{2}{3}).$$

Namely, $w_n(\ln\frac{2}{3}) \leq \frac{3}{4}$, whence we deduce (73) because w_n is decreasing.

Henceforth, we reason up to subsequences and we suppose that c_n converges to some $c \in [0, 2\sqrt{\alpha(1)}]$ and that the functions w_n, z_n, σ_n, A_n converge, respectively, towards some w, z, σ, $A := \alpha(\sigma)$ locally uniformly in $\mathbb{R}$ (observe that the σ_n are equicontinuous on compact sets due to (70) and (72)).

We claim that these functions solve (21). To prove this we need to check that the various terms do not trivialize. This is done in the following items.

(a) $0 < w < 1$. This follows from the bounds (67) and (73) and Harnack's inequalities (24).

(b) $c > 0$. From the criticality condition (66) we obtain

$$c_n^2/4 = \int_{\mathbb{R}} A_n(-w_n')\,dy \geq \int_{-\infty}^{x} A_n(-w_n')\,dy \geq A_n(x)(1 - w_n(x)), \tag{74}$$

which implies $c > 0$ due to (72) and $w < 1$.

(c) $A > 0$, $A \not\equiv \alpha(1)$. We already know from (72) that $A > 0$. Then the lower bounds on c_n and $1 - w_n$ imply that w_n satisfies estimate (48) with a constant K independent of n, namely

$$w_n(x) \leq Kxe^{-\frac{c_n}{2}x} \quad \forall x \geq 1, \quad \forall n \in \mathbb{N}. \tag{75}$$

In particular, we see that $w(x) \to 0$ as $x \to \infty$. Now, suppose by contradiction that $A(x) \equiv \alpha(1)$. Applying (74) with x_0 arbitrarily large would show that $c = \lim_{n\to\infty} c_n = 2\sqrt{\alpha(1)}$. Since $\alpha(1) > 1$, together with (75) this would imply that $w_n(x)e^x$ are equi-integrable at $+\infty$ for n large enough. But then, from the characterization of σ_n in (64), we would have $A(x) < \alpha(1)$ for large x, which gives a contradiction.

(d) $0 < z < \frac{1}{\rho-1}$. As σ_n, A_n, w_n converge locally uniformly to σ, A, w respectively, we have by standard stability that the function z satisfies the equation in (21), i.e. (61). Moreover, z is nondecreasing and satisfies $0 \leq z \leq \frac{1}{\rho-1}$. Indeed, since the right-hand side in the equation is nonnegative, the strong maximum principle yields $z \equiv 0$ as soon as z vanishes somewhere. But this is impossible because

$z \equiv 0$ entails $\sigma \equiv 1$, i.e. $A \equiv \alpha(1)$, which was already excluded. This means in particular that $z > 0$.
A similar argument applies from above. The constant function $\frac{1}{\rho-1}$ is a supersolution of (61). Hence if z, which is less than or equal to $\frac{1}{\rho-1}$, attains the value $\frac{1}{\rho-1}$ somewhere, the strong maximum principle yields $z \equiv \frac{1}{\rho-1}$. Coming back to equation (61), we see that this is only possible if $\sigma \equiv 0$, i.e. $A \equiv 0$. But this has already been ruled out.

(e) $2 < c < 2\sqrt{\alpha(1)}$. Recall that $0 < w < 1$, $c > 0$ and $A \not\equiv 0$. We can then apply Lemma 4.4 and infer that w solves (22) and fulfills the critical identity (62). In particular, since $A \not\equiv \alpha(1)$, as seen in (c), we deduce that $c < 2\sqrt{\alpha(1)}$.
Finally, we are left to show that $c > 2$. For this purpose, we observe that Proposition 3.12, together with (67), implies that

$$w_n(x) \geq \underline{\vartheta} e^{-\frac{c_n}{2}x} \quad \forall x > 0.$$

Assume by contradiction that $c \leq 2$. We then have

$$w(x) \geq \underline{\vartheta} e^{-x} \quad \forall x > 0.$$

For any arbitrary $x_0 < x_1$, we find that

$$\int_{x_0}^{x_1} z(y) e^y w(y)\, dy \geq z(x_0)\underline{\vartheta}(x_1 - x_0).$$

In particular, because $z(x_0) > 0$ by (b), there exists x_1 (depending on x_0) such that

$$\int_{x_0}^{x_1} z(y) e^y w(y)\, dy > \frac{e^{x_0}}{\alpha'(1)},$$

and this inequality holds true for z_n, w_n when n is sufficiently large. It follows from (64) that, for such values of n (that we can assume without loss of generality are larger than x_1), $\sigma_n(x_0) = 1$. This means that $A \equiv \alpha(1)$, but this case has been excluded in (c).

Summing up, we have shown that c, w, z solve the equations and constraints in (21) with $A := \alpha \circ \sigma$. It remains to prove that σ is indeed the optimal function associated with w, z. This follows from the fact that, as $n \to \infty$, the integral equivalence in (64) reduces by dominated convergence (recall that the decays (75) hold with $c_n \to c > 2$) to the characterization (56) of σ. This concludes the proof of Theorem 4.1 (ii).

4.3. Waves with supercritical speed

We now deal with Theorem 4.1 (iii), namely, we construct other traveling waves for system (21), with speeds which are faster than $2\kappa\sqrt{\alpha(1)}$. As in the previous subsection, we assume for simplicity that $\kappa = 1$. For each speed, we are able to attain any arbitrary normalization $\ell_0 \in (0, 1)$ for w at a given point $x_0 \in \mathbb{R}$.

We start with a lemma on the shooting method for the nonlocal problem, similar to Lemma 3.3.

Lemma 4.6. *Let $A \in W^{1,\infty}_{\rm loc}(\mathbb{R})$ be a bounded, nonnegative, nonincreasing function satisfying $\bar{A} := A(-\infty) > 0$. Consider the problem*

$$\begin{cases} w'' + cw' + w\left(A(a) - Aw + \int_a^x A'(y)w(y)\,dy\right) = 0, & x > a, \\ w(a) = \gamma, \\ w'(a) = 0, \end{cases} \tag{76}$$

with $c \geq 2\sqrt{\bar{A}}$, $a \in \mathbb{R}$ and $\gamma \in (0,1)$. Then we have

(i) *problem (76) admits a unique solution $w(\cdot;\gamma)$, which in addition is decreasing and positive in $(0,+\infty)$, with $w \to 0$ as $x \to +\infty$;*

(ii) *for any $x_0 > a$ and $\ell_0 \in (0,1)$, there exists a unique γ_0 such that $w(x_0;\gamma_0) = \ell_0$.*

Proof. For given γ, local existence and uniqueness of w is provided by Lemma 3.3, and w is nonincreasing from Lemma 3.4, because the constant γ is a subsolution. In fact, if we observe that $w''(x) < 0$ if ever $x \geq a$ and $w'(x) = 0$, we conclude that w is actually decreasing. Consider now the unique wave ψ_γ of the classical KPP equation

$$\begin{cases} \psi'' + c\psi' + \bar{A}\psi(1-\psi) = 0, & x \in \mathbb{R}, \\ \psi(-\infty) = 1, \quad \psi(+\infty) = 0, & \psi(a) = \gamma, \end{cases} \tag{77}$$

which exists because $c \geq 2\sqrt{\bar{A}}$. Since we have

$$\begin{aligned} A(a) - A\psi + \int_a^x A'(y)\psi(y)\,dy &= A(a)(1-\psi(a)) - \int_a^x A(y)\psi'(y)\,dy \\ &\leq A(a)(1-\psi) \leq \bar{A}(1-\psi), \end{aligned}$$

then ψ is a supersolution of problem (76), with $\psi'(a) < 0$. By the comparison principle of Lemma 3.4 we deduce that $w(\cdot;\gamma) \geq \psi_\gamma$. Hence w exists for all times and admits a limit as $x \to \infty$. We observe that the equation reads

$$(w'e^{cx})' = -we^{cx}\left(A(a)(1-\gamma) - \int_a^x A(y)w'(y)\,dy\right) = -we^{cx}g(x),$$

where $g(x)$ is an increasing function which admits a bounded limit as $x \to \infty$; then necessarily we deduce that $w' \to 0$ and $w \to 0$ as $x \to \infty$. Indeed, if $w(x)$ has a positive limit at infinity, then $(w'e^{cx})' \sim -\vartheta e^{cx}$ for some $\vartheta > 0$, in which case w' converges to a negative constant at infinity. But this is impossible, so $w(x) \to 0$ and in turn $w'(x) \to 0$ as well, for $x \to \infty$. This proves (i).

Now, for $x_0 > a$, we consider the map $\gamma \mapsto w(x_0;\gamma)$; this is continuous and nondecreasing due to Lemmas 3.3 and 3.4 respectively. But the monotonicity is actually strict; as we observed before, for $\gamma_1 < \gamma_2$, we have $w''_{\gamma_1}(a) < w''_{\gamma_2}(a)$, so $w'_{\gamma_1} < w'_{\gamma_2}$ for $x > a$ and we get $w(x_0;\gamma_1) < w(x_0,\gamma_2)$ by Lemma 3.4 again. Therefore, the range of $\gamma \mapsto w(x_0;\gamma)$ is an interval. Clearly we have $w(x_0;\ell_0) < \ell_0$ because w is decreasing. On the other hand, there exists a wave ψ_0 for the KPP equation (77) such that $\psi_0(x_0) = \ell_0$; if we take $\gamma = \psi_0(a)$, by comparison we know that $w(x_0; \gamma) > \psi_0(x_0) = \ell_0$. Therefore, we deduce the existence of a unique $\gamma_0 \in (\ell_0, \psi_0(a))$ such that $w(x_0; \gamma_0) = \ell_0$. ■

Similarly to the previous section, we use a fixed point argument to build an approximation of the traveling wave in the compact set $[-n, n]$. However, the approximated problem slightly differs from (63).

Lemma 4.7. *Assume that hypotheses (8)–(11) hold true and let $c \geq 2\sqrt{\alpha(1)}$. For $n \in \mathbb{N}$ larger than $\alpha'(1)$, $|x_0| < n$ and $\ell_0 \in (0, 1)$, there exists a solution (w_n, z_n) of the problem*

$$\begin{cases} w'' + cw' + w\left(A(-n) - Aw + \displaystyle\int_{-n}^{x} A'(y)w(y)\,dy\right) = 0, \quad x \in [-n, n], \\ 0 < w < 1, \quad w(x_0) = \ell_0, \quad w' < 0, \\ -z'' + (c-2)z' + (\rho - 1)z + Awz = 1 - [\sigma]_n, \quad x \in [-n, n], \\ z(-n) = 0, \quad z(n) = \dfrac{1}{\rho - 1}, \quad z' > 0, \\ [\sigma]_n(x) = \underset{s \in [0,1]}{\operatorname{argmax}} \Big\{ (1-s)e^x \\ \qquad\qquad + \alpha(s)\left(\displaystyle\int_x^n z(y)e^y w(y)\,dy + \frac{1}{n}\right)\Big\}, \quad A := \alpha \circ [\sigma]_n. \end{cases} \tag{78}$$

Proof. We consider the following subset of $C^0([-n, n])$:

$$X := \left\{\sigma \in W^{1,\infty}([-n, n]) : \sigma \text{ is nonincreasing}, \ (\alpha')^{-1}(ne^n) \leq \sigma \leq 1\right\}$$

(observe that $(\alpha')^{-1}(ne^n) \leq (\alpha')^{-1}(n) < 1$ by hypothesis). We define a map T on X in the following way. Take $\sigma \in X$ and call $A := \alpha \circ \sigma$. First, we let w be the unique solution of the problem

$$\begin{cases} w'' + cw' + w\left(A(-n) - Aw + \displaystyle\int_a^x A'(y)w(y)\,dy\right) = 0, \quad x \in [-n, n], \\ w(x_0) = \ell_0, \\ w'(-n) = 0. \end{cases}$$

Existence and uniqueness of w are given by Lemma 4.6, which additionally ensures that w is decreasing and satisfies $0 < w < 1$. Next, given σ, A and w, we consider the unique solution of the linear elliptic problem in (78), with $[\sigma]_n$ replaced by σ. We have seen in the proof of Lemma 4.5 that z is increasing. We finally define $T(\sigma) := [\sigma]_n$ from the last line of (78). This function is nonincreasing and fulfills an analogous characterization to the one derived in the previous section:

$$[\sigma]_n(x) = (\alpha')^{-1}\left(\frac{e^x}{\int_x^n ze^y w\,dy + \frac{1}{n}}\right) \quad \forall x \in [-n, n] : [\sigma]_n(x) < 1. \tag{79}$$

This yields $[\sigma]_n(x) \geq (\alpha')^{-1}(ne^n)$ and moreover, by an analogous computation to (58),

$$[\sigma]_n'(x) = \frac{\alpha'([\sigma]_n(x))}{\alpha''([\sigma]_n(x))}\big(1 + z(x)w(x)\alpha'([\sigma]_n(x))\big) \quad \forall x : [\sigma]_n(x) < 1. \tag{80}$$

By the boundedness of w, z and the regularity of α, as well as the positive lower bound for $[\sigma]_n$, we eventually deduce $|[\sigma]_n'| \leq C$ for some positive constant C only depending on c, ρ, α, n. We have thereby shown that $T(X) \subset X$.

Actually, we have shown that $T(X) \subset \widetilde{X}$, with

$$\widetilde{X} := \big\{\sigma \in W^{1,\infty}([-n,n]) : \sigma \text{ is nonincreasing}, \ (\alpha')^{-1}(ne^n) \leq \sigma \leq 1, \ |\sigma'| \leq C\big\}.$$

This is a compact, convex subset of $C^0([-n,n])$.

We now prove that T admits a fixed point in $\widetilde{X}$. Let us check the continuity of T. Consider a sequence $(\sigma_j)_{j\in\mathbb{N}}$ in $\widetilde{X}$ converging uniformly to some σ. Call $A_j := \alpha \circ \sigma_j$ and w_j, z_j the associated functions used in the definition of T. Integrating the term in the equation for w_j by parts (in order to get rid of the term A_j'), using elliptic estimates, and then integrating back, we find a subsequence of $(w_j)_{j\in\mathbb{N}}$ converging uniformly to a solution w of the same equation, with $A := \alpha \circ \sigma$, which satisfies in addition $0 \leq w \leq 1$ and $w'(-n) = 0$, $w(x_0) = \ell$. By Lemma 4.6 there is a unique such solution, hence the whole sequence $(w_j)_{j\in\mathbb{N}}$ converges towards it. Likewise, $(z_j)_{j\in\mathbb{N}}$ converges uniformly to the unique solution of the corresponding equation with A and σ. Then, using the characterization (79), we deduce that $(T(\sigma_j))_{j\in\mathbb{N}}$ converges uniformly to the function $[\sigma]_n$ defined as in (78). This is precisely $T(\sigma)$.

We can therefore invoke Schauder's theorem and conclude that the map T has a fixed point in $\widetilde{X}$, which is by construction a solution of (78). ■

We finally analyze the limit as $n \to \infty$, in order to get the wave for the system on the whole line. It is here that we face the question whether we^x is integrable at $+\infty$.

Proof of Theorem 4.1 (iii). Fix $c \in [2\sqrt{\alpha(1)}, \alpha(1) + 1)$. Let w_n, z_n be a solution of system (78), provided by Lemma 4.7, and let $[\sigma]_n$ and $A_n := \alpha \circ [\sigma]_n$ be the associated function from the last line of (78). First of all, by elliptic estimates, both w_n and z_n are locally bounded in C^2 norm, hence they converge (up to subsequences) in $C^1_{\text{loc}}(\mathbb{R})$ to some functions w, z. We claim that

$$\sup_{n\in\mathbb{N}} \int_{-n}^{n} z_n w_n e^y \, dy < +\infty. \tag{81}$$

To show this, assume first by contradiction that there exists $x \in \mathbb{R}$ such that (up to subsequences)

$$p_n(x) := \int_x^n z_n w_n e^y \, dy \to +\infty.$$

Then, since $p_n'(x)$ is locally uniformly bounded, this must be true for all $x \in \mathbb{R}$. Recalling the characterization (79) for $[\sigma]_n$, we deduce that $[\sigma]_n(x) = 1$ for n sufficiently large, depending on x, hence $A_n(x) = \alpha(1)$ for large n, and this actually holds uniformly in $(-\infty, x]$, by monotonicity. It follows that the limit w of the w_n is a solution of

$$w'' + cw' + \alpha(1)w(1-w) = 0,$$

such that $w(x_0) = \ell_0 \in (0,1)$, i.e. w is a wave for the standard KPP equation. We deduce that w tends to 0 as $x \to +\infty$. In particular, for $\varepsilon \in (0,1)$ to be chosen later, we can find $x_1 > 0$ such that $w(x_1) < \varepsilon/2$. Now we come back to w_n; by pointwise convergence, we take n large enough so that $w_n(x_1) < \varepsilon$. For $x > x_1$, we estimate

$$\begin{aligned} 0 &= w_n'' + cw_n' + w_n(x)\left(A_n(-n) - A_n w + \int_{-n}^{x} A_n'(y)w(y)\,dy\right) \\ &\geq w_n'' + cw_n' + w_n(x)A_n(x)(1 - w_n(x)) \\ &\geq w_n'' + cw_n' + A_n(x)w_n(x)(1-\varepsilon). \end{aligned}$$

Since we are assuming $A_n(x) \to \alpha(1)$, this implies, for n sufficiently large,

$$w_n'' + cw_n' + (\alpha(1) - \varepsilon)(1-\varepsilon)w_n \leq 0.$$

Then the function $q := -w_n'/w_n$ satisfies

$$q' \geq q^2 - cq + (\alpha(1) - \varepsilon)(1-\varepsilon) = (q - \lambda_\varepsilon^-)(q - \lambda_\varepsilon^+),$$

where

$$\lambda_\varepsilon^\pm := \frac{c}{2} \pm \sqrt{\frac{c^2}{4} - (\alpha(1) - \varepsilon)(1-\varepsilon)}.$$

We infer that

$$\liminf_{x \to +\infty} q(x) \geq \lambda_\varepsilon^-.$$

But the condition $2 < c < \alpha(1) + 1$ implies $\lim_{\varepsilon \to 0^+} \lambda_\varepsilon^- > 1$, hence we can choose ε small enough so that $\lambda_\varepsilon^- > 1$. Reverting to the function w_n, we derive

$$w_n(x) \leq Ce^{-\lambda x} \quad \forall x > x_1,$$

for some $C > 0$ and $\lambda > 1$. This estimate and the bound on z imply

$$p_n(x_1) = \int_{x_1}^{+\infty} z_n w_n e^y\,dy \leq \frac{C}{\rho - 1} \int_{x_1}^{+\infty} e^{(1-\lambda)y}\,dy,$$

so $p_n(x_1)$ cannot blow up. This contradicts the fact that $p_n \to +\infty$ pointwise. We have thus shown that $p_n(x)$ remains bounded at any given x. We improve this to the bound (81) by noticing that

$$\int_{-n}^{n} z_n w_n e^y\,dy = \int_{-n}^{0} z_n w_n e^y\,dy + p_n(0) \leq \frac{1}{\rho - 1} + p_n(0).$$

We now refine the above argument to show a uniform decay for w_n at infinity. Observe that (81), together with (79), implies that the function $[\sigma]_n$ in (78) associated with (z_n, w_n)

does not tend to 1 as $n \to \infty$ at a sufficiently large point x. Thus, by Lemma 4.3, the limit z of (a subsequence of) z_n is positive for x sufficiently large. We infer that $zwe^y \in L^1(\mathbb{R})$ as a consequence of (81) and Fatou's lemma. Now consider the functions $q(x) := -\frac{w'(x)}{w(x)}$ and $q_n(x) := -\frac{w_n'}{w_n}$. Since $we^y \in L^1$ there must be a point $\bar{x}$ where $q(\bar{x}) > 1$ (otherwise, if $-\frac{w'}{w} \le 1$ for every x, then we^y is not integrable at infinity). By pointwise convergence, we can assume that $q_n(\bar{x}) \ge (1+\varepsilon)$ for some $\varepsilon > 0$, and for all $n \in \mathbb{N}$. But since

$$q_n' = q_n^2 - cq_n + \left(A_n(-n) - A_n w_n + \int_{-n}^{x} A_n'(y) w_n(y)\, dy\right),$$

the same argument used to prove Proposition 3.12 shows that q_n is increasing in $(0, n)$; hence we deduce that $q_n(x) \ge (1+\varepsilon)$ for every $x \in (\bar{x}, n)$. Recalling that $q_n = -\frac{w_n'}{w_n}$, integrating we get

$$w_n(x) \le w_n(\bar{x}) e^{-(1+\varepsilon)(x-\bar{x})} \le C e^{-(1+\varepsilon)x}, \quad x \in (\bar{x}, n).$$

Thanks to this estimate, we can use the dominated convergence theorem and we conclude that

$$\int_x^n z_n w_n e^y\, dy \xrightarrow{n\to\infty} \int_x^{+\infty} zwe^y\, dy.$$

This implies that $[\sigma]_n$ characterized by (79) pointwise converges to σ characterized by (56), i.e.

$$\sigma(x) := \operatorname*{argmax}_{s\in[0,1]} \left\{(1-s)e^x + \alpha(s) \int_x^{+\infty} z(y) e^y w(y)\, dy\right\}.$$

Call $A := \alpha \circ \sigma$. We immediately deduce that z solves the equation in (21). Next, the uniform positive lower bound on $z_n(x)$ for x large, which is also true for $w_n(x)$ due to $w_n(x_0) = \ell_0$ and the Harnack inequality, implies that $A_n = A = \alpha(1)$ on some half-line $(-\infty, \bar{x}]$. For $-n < \bar{x}$ we then find that

$$\begin{aligned}
-w_n'' - cw_n' &= \alpha(1) - A_n w_n + \int_{\bar{x}}^{x} A_n'(y) w_n(y)\, dy \\
&= \alpha(1)(1 - w_n(\bar{x})) - \int_{\bar{x}}^{x} A_n(y) w_n'(y)\, dy \\
&\xrightarrow{n\to\infty} \alpha(1)(1 - w(\bar{x})) - \int_{\bar{x}}^{x} A(y) w'(y)\, dy \\
&= \alpha(1)(1 - w(-\infty)) - \int_{-\infty}^{x} A(y) w'(y)\, dy.
\end{aligned}$$

Hence w satisfies the differential inequality (28) used in the proof of Lemma 3.2 to derive $w(-\infty) = 1$ and $w(+\infty) = 0$. This shows that w solves (22). In the end, (z, w) are proved to be solutions of problem (21). ∎

This concludes the proof of Theorem 4.1.

4.4. From traveling waves to BGP solutions

We now deduce Theorem 2.2 from the results obtained in Theorem 4.1 on the traveling waves. We only need to establish first a rigorous connection between solutions of (21) and BGP solutions of the mean-field game system (2), which are defined as in Definition 2.1.

Proposition 4.8. *Assume conditions* (8)*,* (9)*,* (11)*,* (12) *hold. If* (f, v, s^*) *is a BGP with growth* c*, then* $w(x) := \int_x^{+\infty} \varphi(y)\,dy$ *and* $z(x) := \nu'(x)e^{-x}$ *are solutions of the traveling wave system* (21)*.*

Conversely, let $c < \rho$ *and* (w, z) *be solutions of* (21)*. Then the triple given by*

$$f = -w'(x - ct), \quad v = e^{ct}\bigg(\int_{-\infty}^{x-ct} z(y)e^y\,dy + K\bigg), \quad s^* = \sigma(x - ct) \tag{82}$$

is a BGP solution of system (2)*, with* $K = \frac{\alpha(1)}{\rho - c}\int_{\mathbb{R}} e^y z(y)w(y)\,dy$*.*

Finally, there are no possible BGPs with growth $c \geq \rho$*.*

Proof. Suppose we are given a BGP solution (f, v, s^*) of (2), hence $f(t,x) = \varphi(x - ct)$, $v = e^{ct}\nu(x - ct)$, $s^* = \sigma(x - ct)$ for some $c > 0$. Thus, as shown in Section 2.1, the functions $w(x) := \int_x^{+\infty}\varphi(y)\,dy$, $z(x) := \nu'(x)e^{-x}$ are solutions to the traveling wave system (21). Finally, according to Definition 2.1, we also have $we^x \in L^1(\mathbb{R})$, while z is nonnegative and bounded. Hence $zwe^x \in L^1(\mathbb{R})$ and therefore all conditions in (21) are fulfilled.

Conversely, assume that (w, z, σ) is a solution of (21) and define (f, v, s^*) from (82). Set $\varphi(r) := -w'(r)$ and $\nu(r) := \int_{-\infty}^r z(y)e^y\,dy + K$, where

$$K := \frac{\alpha(1)}{\rho - c}\int_{\mathbb{R}} e^y z(y)w(y)\,dy. \tag{83}$$

It follows that $f = \varphi(x - ct)$, $v = e^{ct}\nu(x - ct)$ with $\varphi, \nu \in C^2(\mathbb{R})$, as required in Definition 2.1. We further know that $\sigma \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ by Proposition 4.2. Still from Proposition 4.2, we know that $z > 0$ and it has a bounded positive limit as $r \to +\infty$; this implies that ν is increasing, nonnegative and $\nu'e^{-x} \in L^\infty(\mathbb{R})$. Therefore, the condition $zwe^y \in L^1(\mathbb{R})$ implies that $e^r w(r) = e^r\int_r^{+\infty}\varphi(y)\,dy \in L^1(\mathbb{R})$. By Proposition 3.12, we also know that $\frac{-w'(r)}{w}(r) \to \lambda > 0$ as $r \to +\infty$; hence we deduce that $w'e^y \in L^1(\mathbb{R})$ as well.

So far, we have checked that the first three conditions in Definition 2.1 hold. We are left to show that (f, v) solve system (2). We preliminarily observe that the definition of σ in (21) yields through the computations (20) and (17), that $s^* = \sigma(x - ct)$ satisfies

$$\begin{aligned} s^* &= \operatorname*{argmax}_{s\in[0,1]}\bigg\{(1 - s)e^x + \alpha(s)\int_x^{+\infty} v_x(t,y)\int_y^{+\infty} f(t,r)\,dr\,dy\bigg\} \\ &= \operatorname*{argmax}_{s\in[0,1]}\bigg\{(1 - s)e^x + \alpha(s)\int_x^{+\infty}[v(t,y) - v(t,x)]f(t,y)\,dy\bigg\}, \end{aligned}$$

i.e. s^* is given by the formula in (2). As far as v is concerned, taking the derivative in the equation for w, and using that $A(x-ct) = \alpha(s^*(t,x))$, it readily follows that $f = \varphi(x-ct)$ is a traveling wave solution of the Fokker–Planck equation. Finally, in order to derive the equation for v, we consider the Hamiltonian

$$\begin{aligned} H(t,x;v) &:= \max_{s\in[0,1]}\left[(1-s)e^x + \alpha(s)\int_x^{+\infty}[v(t,y)-v(t,x)]f(t,y)\,dy\right] \\ &= \max_{s\in[0,1]}\left[(1-s)e^x + \alpha(s)\int_x^{+\infty} v_x(t,y)\int_y^{+\infty} f(t,r)\,dr\,dy\right], \end{aligned}$$

where the equality follows from (17). We know that the above maxima are attained at $s = s^*(t,x)$, and actually this is the unique maximizer because the expressions are concave in s, because $v_x = e^x z(x-ct) > 0$. Applying the envelope theorem as in Section 2.1, we notice that H is differentiable in x with $\partial_x H = (1-s^*)e^x - \alpha(s^*)v_x \int_x^{+\infty} f(t,y)\,dy$. This means that

$$(1-\sigma(x-ct)) - \alpha(\sigma(x-ct))w(x-ct)z(x-ct) = e^{-x}\partial_x H(t,x;v).$$

Inserting this information into the equation for z in (21), we conclude that $v_x = e^x z(x-ct)$ satisfies the differential equation

$$-\partial_t v_x - \kappa^2 \partial_{xx} v_x + \rho v_x = \partial_x H(t,x;v).$$

Now we integrate this equation in the interval $(-\infty, x)$. We observe that

$$\begin{aligned} H(t,-\infty;v) &= \alpha(1)\int_{\mathbb{R}} v_x(t,y)\int_y^{+\infty} f(t,r)\,dr\,dy \\ &= e^{ct}\alpha(1)\int_{\mathbb{R}} e^y z(y)w(y)\,dy = K(\rho-c)e^{ct}, \end{aligned}$$

with K defined by (83), while, by definition of v, we have

$$v(t,-\infty) = e^{ct}K, \quad \partial_t v(t,-\infty) = ce^{ct}K.$$

Finally, using that $v_{xx} = e^x(z(x-ct) + z'(x-ct)) \to 0$ as $x \to -\infty$, we conclude that

$$\begin{aligned} 0 &= \int_{-\infty}^x \{-\partial_t v_x - \kappa^2\partial_{xx} v_x + \rho v_x - \partial_x H(t,x;v)\} \\ &= -\partial_t v - \kappa^2 \partial_{xx} v + \rho v - H(t,x;v) + \partial_t v(t,-\infty) - \rho v(t,-\infty) + H(t,-\infty;v) \\ &= -\partial_t v - \kappa^2\partial_{xx} v + \rho v - H(t,x;v), \end{aligned}$$

which means that v is a solution of the Bellman equation. Therefore, we have proved that (f, v, s^*) is a BGP solution of system (2).

We conclude by observing that $c < \rho$ is necessary for a BGP to exist. Indeed, if a BGP exists, we have established so far that it is of the form (82) for some (z, w, σ) solution of

(21) and for some constant $K \in \mathbb{R}$. By properties of z and σ, given in Proposition 4.2, we deduce that $v_{xx} \to 0$ as $x \to -\infty$, owing to elliptic estimates, while $s^* = \sigma(x - ct) \to 1$ as $x \to -\infty$. Then, writing the equation for z in (21) in terms of $v_x = e^x z(x - ct)$, integrating it on $(-\infty, x)$ and using that v is a solution of (2), with the same computation as before we get

$$(\rho - c)K = \alpha(1) \int_{\mathbb{R}} e^y z(y) w(y)\, dy.$$

Since the right-hand side is positive, and $K \geq 0$ because otherwise v would be negative for $-x$ large, we deduce that $\rho > c$. ■

We can finally conclude with the proof of our main result.

Proof of Theorem 2.2. (i) If there exists a BGP solution, then we have the necessary condition $c < \rho$ from Proposition 4.8. In addition, a BGP solution yields a traveling wave (z, w) solution of (21). Hence c must satisfy the conditions in Theorem 4.1 (i).

(ii) By Theorem 4.1 (ii), there exists a traveling wave with speed $c \in (2\kappa^2, 2\kappa\sqrt{\alpha(1)})$ which is also critical. If $\rho \geq 2\kappa\sqrt{\alpha(1)}$, then $c < \rho$ so by Proposition 4.8 this yields a BGP solution of (2), which is critical as well, i.e. fulfills (4).

(iii) Putting together Theorem 4.1 (iii) and Proposition 4.8, for every $c \in [2\kappa\sqrt{\alpha(1)}, \alpha(1) + \kappa^2)$ such that $c < \rho$, we have a BGP solution of (2) with growth c, and with arbitrary normalization at any given point. ■

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