Proofs without words: focus on argumentation as a tool to investigate the link between visualization and generalization processes enacted by students

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The study presented in this paper is based on data collected within a teaching experiment aimed at the implementation of a didactical path focused on proofs without words. Our research aim is to investigate the link between visualization and generalization processes enacted by students during their work on the tasks that constitute the didactical path. To pursue our aim, we analyzed students' argumentative processes, documented within their written protocols and within excerpts from the video-recordings of small group activities and classroom discussions. Through this analysis, developed by means of Radford's theory of objectification and Duval's levels of apprehension, we highlight the role played by visualization in supporting generalization.

Keywords: Argumentation, explanation, generalization, proofs without words, visualization.

Background

A proof without words is a proof that makes use of a graphical artifact, a picture or other visual mean, to reconstruct a deductive process that justifies a given statement or an equation (Nelsen, 1993). There is debate around whether a proof without words really qualify as a proof (Gierdien, 2007, p.1); anyway research in Mathematics Education highlighted the powerful role of proofs without words in learning and teaching mathematics. Hanna (1989) showed that involving students in interpretation and analysis of activities on proofs without words could shift the focus of teaching from proofs that prove to proofs that explain.

In 2019 we designed and implemented a teaching experiment focused on proofs without words. It was a pilot study that involved three 10th grade classes of a scientific high school near Rome, with a total of 66 students. In this paper we will focus on the second stage of this experiment trying to highlight the link between the visualization process and the generalization one through the tool of argumentation. By argumentation we mean, according to Stylianides et al. (2016), the discourse or rhetorical means used by an individual or a group to convince others that a statement is true or false. It may involve exploration of examples, generation or refinement of conjectures, and production of arguments for these conjectures.

Theoretical framework

The analytical framework we refer to is made up of two main components, the apprehension levels described by Duval and the theory of objectification introduced by Radford.

Following Duval (2006, p.107) "Mathematical objects, in contrast to phenomena of astronomy, physics, chemistry, biology, etc., are never accessible by perception or by instruments". He goes on to state that "the only way to have access to them [mathematical objects] and deal with them is using signs and semiotic representations". This is the reason why "representation and visualization are at the core of understanding in mathematics" (Duval, 1999, p.1). Furthermore visualization, in the Duval framework, is a process that goes over the vision. "Unlike vision, which provides a direct access to the object, visualization is based on the production of a semiotic representation" (Duval, 1999, p.7). A semiotic representation does not show things as they are, it "shows relations or, better, organization of relations between representational units" (Duval, 1999, p.7). In our work students are asked to explain the relations they see between images of proofs. The *perceptual apprehension* refers to the recognition of figures and subfigures; *sequential apprehension*, is used when describing a construction of a figure based on mathematical properties; the ability to recognize such mathematical properties and explain them refers to the *discursive apprehension* while the *operative apprehension* is a process by which students, operating on the figure, identify the solution of a mathematical problem.

Following Radford (2001, p.81), generalization must be considered as "one of the more natural human semiotic process"; looking into the way students "deploy and mobilize signs to accomplish mathematical generalization", Radford points out three levels of generalization. *Factual generalization*, that is, "a generalization of numerical actions in the form of an operational scheme that remains bound to the numerical level" (p.82), at this level the semiotic tools for the objectification are the perceptual semiosis, the generative functions of language and the operational schemes. The factual generalization is achieved by students by means of rhythm in the utterances, ostensive gestures, adverbs like always, every, etc. and making reference to space and time (i.e. *the next figure*…). The second step in generalization is *contextual generalization*, at this level students are able to write explanations regarding their conjectures and mathematical operations, in these explanations they perform actions on abstract objects (*you take that square…*) but still have a *perspectival view* of mathematical objects. The last level is *symbolic generalization*, it occurs when students talk about mathematical objects in an impersonal way (*the sum of the first n squares…*, *the n-th figure…*) and using a more specific terminology, these aspects show that the symbolic generalization is reached when students succeed in a de-subjectification process.

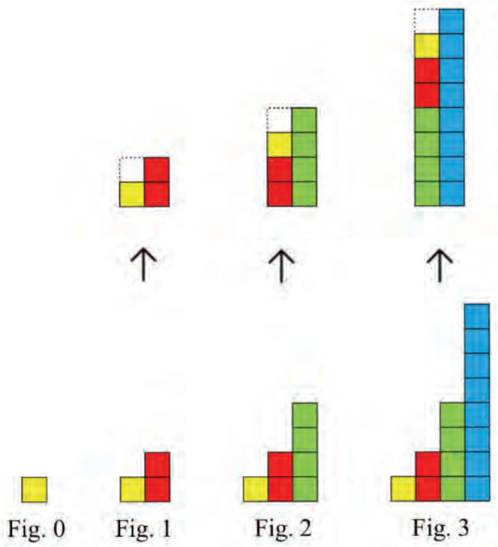
These two theorical frameworks start from different systems of principles since Duval takes constructivism as his point of departure, while Radford's theory stems from a sociocultural perspective on learning. Nevertheless, we decided to implement a *networking* (Prediger et al., 2008) of the two theories, integrating them locally in order to study the visualization process, by means of the apprehension levels, and the generalization process referring to Radford theory. This decision is motivated by the fact that, beyond the effectiveness of the level of apprehension in analysing the visualization processes, we noticed that students, explaining and justifying on the way they look at the figure, use the figure itself as an artifact of communication and signification. This is in line with the sociocultural perspective that grounded the theory of *semiotic mediation* in the sense of Vygotsky (Cole et al. 1978) and is one of the starting points for Radford's theory.

Context of the study

The study presented in this paper took place in Spring 2021 (totally 8 hours) and involved a 10th grade class of an Italian high school (26 students). Due to the pandemic period, the presence of students in the classroom was limited to 50%. Half of the students were present in the classroom, while the other half was at home connected using a video-communication service (Google Meet) and a Multimedia Interactive Whiteboard.

The students were divided into 6 non-homogeneous groups, in agreement with their teacher. Each meeting was articulated in an initial phase of small group activity, followed by a classroom discussion. The students were invited to discuss the questions as a group and to write down the answers that would be later collected by the teacher. After the small groups' work, the students were involved by the researchers (the authors of this paper) and by their teacher in a discussion regarding the strategies used by the different groups to deal with the tasks they had to face.

The didactical path was designed as a sequence of worksheets focused on questions based on examples of proofs without words. One of the worksheets is represented in Figure 1.



The number of squares in the previous figures can be represented as follows:

Figure	Number of squares
Fig. 0	$1 = 2^0$
Fig. 1	$1 + 2 = 2^0 + 2^1 = 3$
Fig. 2	$1 + 2 + 4 = 2^0 + 2^1 + 2^2 = 7$
Fig. 3	$1 + 2 + 4 + 8 = 2^0 + 2^1 + 2^2 + 2^3 = \ldots$

1. Imagine to continue the sequence of figures, what will be the number of squares (not considering the dashed one) in Figure 5? And in Figure 9?
2. Explain how to determine the total number of squares in any figure in the previous sequence.
3. Explain which features of the figures helped you in constructing answers to the previous questions.
4. These figures aim to justify a theorem on the sum of the powers of two. Try formulating this theorem.
5. Write a reasoning to justify the theorem you formulated in question 4.

Figure 1: Worksheet 3

All the worksheets included questions about a number sequence, as the one in Figure 1. The students were asked to find some elements of the number sequence under investigation (question 1, Figure 1) and to explain how to find out a generic element of the sequence (question 2, Figure 1). To make them reflect on the visualization processes they enacted, the students were asked to explain which features of the figures helped them in formulating their answers (question 3, Figure 1). Nevertheless we add a table with numbers (see figure 1) to let the students be free to choose the semiotic representation they felt more comfortable with: the picture that recalls a proof without words or the table that recalls a more traditional approach. From the analysis of data collected we could see that only one group preferred the table and did not use the picture in answering questions 1 and 2.

The last two questions aimed to prompt students identify the theorems behind the proofs without words proposed in the worksheets and reflect on the role played by figures in supporting the formulation of conjectures and in proving theorems (questions 4 and 5, Figure 1).

Research questions and methodology

Our research hypothesis is that argumentation could represent an effective tool through which the link between visualization and generalization could be explored. The analysis we developed was aimed at investigating this link. The research question that guided our analysis is: referring to the level of generalization and of apprehension reached by students, what characterizes the link between visualization and generalization when students face activities focused on proofs without words?

In order to answer this question, we collected the protocols written by students during the group work and the audio and video-recordings of all the activities (both face-to-face activities and activities carried out at distance), including group work and classroom discussions. We then analyzed the whole material, transcribing some excerpts we considered as interesting in order to establish a link between visualization and generalization.

For our analysis, we referred to the main components of our theoretical framework: Duval's levels of apprehension and the generalization categorization proposed by Radford.

In this paper we will analyse the work of only two groups. The reason for this choice is that the students in these groups felt the need to explain in depth their written protocols during the class discussion even if they didn't grasp the whole solution of the task. This didn't happen in other groups, probably because students were not used to argumentation neither written nor oral. Sometimes (it is the case of one of the groups of our experiment) the solution of the task was easily grasped, and the visualization process remained implicit.

We follow the evolution of each group along three steps: (1) the discussion in the small group, (2) the students' written protocols, and (3) the classroom discussion with the teacher and the authors. We chose to focus our analysis on the excerpts regarding questions 2 and 3 of the worksheet in Figure 1, since they enable us to focus on the way students formulate their generalizations (question 2) and explain how they looked at the figures, that is the visualization processes that supported their generalizations (question 3).

Regarding step (2), we should note that the condition under which the experiment was conducted (remote meeting) strongly limited the semiotic potential of the interaction between students. In particular, it was not possible to collect any data about students' use of gestures or facial expressions during the small group discussions.

The final discussion in class which is reported here was led by one of the authors with the aim to foster students' explication of their visualization and generalization processes.

Data analysis

Analysis of data taken from group 1

The first excerpt is taken from the discussion of group 1. They are working together during a remote meeting. One of the participants share the screen with the figures of worksheet 3 (Figure 1). The group is trying to answer the number 2 question while they look at the figures on the screen.

20 Giada: …But…I think I understood that the next figure is always equal to twice the previous figure plus one.

21 Giorgia: Yes, it's true.

22 Lea: Giada, can you repeat what you said?

23 Giada: So… for instance figure 5 is equal to twice the figure 4 plus 1.

24 Lea: Ah, ok, I got it, so you suggest writing down this instead of the whole formula?

In line 20 Giada comes up with an idea, "the next figure is always equal to twice the previous figure plus one", and she shares the idea with the group. The group discusses and they write it in their protocol. From these data it emerges, according to Radford theoretical framework, a *contextual generalization*, it goes over the numerical level; they find out a property that holds for each figure. However, we argue that they still have a *perspectival view* of the mathematical objects, seen in the expression "the next figure …the previous figure", so the generalization process is not complete, they did not reach to the symbolic level.

In their written answer to question 2, the students wrote: "*The number of squares of the next figure, with respect to the figure we are considering, is doubled plus one*". And when they are asked to clarify the visualization process (question 3) they discuss and then write: "*Colours that helps to differentiate the different columns, the increasing shape, the figure number*".

During the discussion in class, we tried to push forward the investigation about the visualization, asking the students for explanations about the written answers. This can be seen in the following excerpt involving one of the authors and a student from group 1.

13 Giorgia: Given a figure of the sequence, to get the number of squares of the following figure, we doubled the number of squares of the given figure and add 1.

14 Author: Where did you see it in the figure?

15 Giorgia: … for instance [*pointing at figure 0 of the worksheet displayed on the whiteboard*] the figure 0… 1 times 2 is 2, plus 1 is 3, and so on…

16 Author: Did the figure help you in discovering that?

17 Giorgia: Yes.

18 Author: How?

19 Giorgia: It helped us because we saw that … this one [*pointing at the yellow square*]… in order to get to the red one [*pointing at the two red squares*] … we multiply times two and then we add the one that we had before.

20 Author: Which one?

21	Giorgia:	The yellow one.
22	Author:	And what about the next steps?
23	Giorgia:	… all this stuff [*pointing at figure 1, lower part*] times 2 is 6, the same as 1,2,3,4,5,6… the red and green squares [*pointing at the red and green squares of figure 2, lower part*] plus 1 that is, once again, the yellow one.

We think that, in the previous excerpt, Giorgia has carried out a very peculiar visualization process. Even if she didn't use the hint given by the upper part of the sequence of figures of the worksheet, she is able to perform an advanced process of visualization. She actually doesn't stop herself to the visual perception of the figures, she goes over, seeing the "relations", and the "organization of relations between representational units" (Duval, 1999). In this case the representational units are the squares. She highlighted the relation between the yellow square of figure 0 and the two red squares of figure 1 (the yellow square doubled and "became" the two red squares of figure 1) and did the same for each figure in relation with the next one. Moreover, in line 19, Giorgia describes the construction of any figure of the sequence starting from the previous one (*sequential apprehension*).

Analysis of data taken from group 2

We now move to the analysis of the material of group 2. We will notice that, even if the two groups obtained two different formulas and looked at the figures in two different ways, the characterization of the link between the visualization and the generalization process is similar.

7	Francesco:	…Anyway… these figures…are simply increasing…every next column is one plus the previous one and then…I didn't get why… they … put them on top of each other...
8	Federica:	I think they go like 2 x 2 is 4, next 4 x 2 is 8, the last column, so 8 x 2 I think … so figure 4 will be 16, the last column and so…
9	Francesco:	Are you talking about…the one that has been added?
10	Federica:	Yes, the blue one...
11	Francesco:	Ok, yes, it is multiplied by 2
…		
16	Francesco:	… anyway… the column that has been added is a number… and the other columns equal that number minus 1, so… for instance, in figure 5 they will be 32 plus 31…
17	Federica:	Yes, is it so.

Francesco, in line 16, makes an effort towards generalization. In this moment he is contributing to the work of the group with an idea that goes over the numerical level. After the discussion, the group wrote down their idea in the written answer to question 2 as follows:

Last column times 2, minus 1

We notice that this protocol is different from the formula given by group 1. Even the generalization process is slightly different because group 2 explained how to compute the number of squares of any figure without any reference to the previous figure, so they go over the *perspectival view*. Anyway, the generalization level of group 2 is still a *contextual generalization* because they did not proceed to the *des-embodiment* in their mathematical description; they actually need to know the number of squares of the *last column* in order to compute the number of squares of the whole figure. Since we want to investigate the link between this generalization process and the visualization, we analyze the written answer to question 3 given by group 2. It was:

It helped us the way the squares are arranged in the upper part of the figures.

Therefore, the way students in group 2 look at the figure is also different with respect to group 1; specifically, students of group 2 look at the upper part of the figures while students from group 1 look at the lower part. In the next excerpt, taken from the class discussion with one of the authors, the visualization process fielded by the students in answering question 2 emerges.

41	Francesco:	A way to find the number of squares is taking the last column, multiply such a column times 2 and subtract 1.
42	Author:	What tells you that?
43	Francesco:	We can argue that from the figure here above, i mean … you see that it always misses a square… is this one times 2 minus 1 [*pointing at figure 1*]
44	Author:	Ok
45	Francesco:	…Or this times 2 minus 1 [*pointing at figure 3*]…because…I mean…you can tell it by this figure [*pointing at the upper part of figure 3*]

In line 41 Francesco describes an action on abstract objects (*taking the last column*); this kind of operation, characteristic of the *contextual generalization*, is suggested by the figure, as Francesco explained in line 43. Furthermore, from line 45 we argue that the description of Francesco is indeed the description of the construction of the whole sequence of mathematical figures, as pointed out by Duval describing the *sequential apprehension*.

Conclusions

Generalization and visualization are two essential components of the mathematics learning and teaching process. Often the link between these two components is obscured by many factors. We refer to the attitude of students toward argumentation (as we already discussed) and to the peculiar conditions in which the experiment took place (remote meetings) which affect the social-communicative dimension of the interaction between students. In the small-group activities mediated by screens, students' semiotic assets are restricted. The fact that *implicit and mutual agreement of face-to-face interaction must be replaced by objective elements of social interaction*, according to Radford (2001), guides the activity toward the generalization. Thus, the balance between visualization and generalization it's unlikely to arise. However, the two examples discussed in this paper show that pushing forward the argumentation process is an effective way to clarify the connection between the visualization and the generalization processes fielded by students when they approach a mathematical problem. The link between visualization and generalization process was outlined in other studies (e.g., Barbosa, 2011). The contribution of this paper is to characterize, in terms of level of apprehension, the generalization achieved by 10th grade students who do not succeed in the whole task. In the analysis we outlined that, even if the two groups looked at the figure in different ways and produced different formulas, their contextual generalization was guided by a sequential apprehension. So, in these two cases, the link between the visualization and the generalization process is characterized by the match of sequential apprehension and contextual generalization. It seems to us to be worthy of further study to investigate the visualization process of groups of students that reach a symbolic generalization in solving similar tasks.

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