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The periodic *N* breather anomalous wave solution of the Davey–Stewartson equations; first appearance, recurrence, and blow up properties

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Abstract

The integrable focusing Davey–Stewartson (DS) equations, multidimensional generalizations of the focusing cubic nonlinear Schrödinger equation, provide ideal mathematical models for describing analytically the dynamics of $2+1$ dimensional anomalous (rogue) waves (AWs). In this paper (i) we construct the N-breather AW solution of Akhmediev type of the DS1 and DS2 equations, describing the nonlinear interaction of N unstable modes over the constant background solution. (ii) For the simplest multidimensional solution of DS2 we construct its limiting subcases, and we identify the constraint on its arbitrary parameters giving rise to blow up at finite time. (iii) We use matched asymptotic expansions to describe the relevance of the constructed AW solutions in the spatially doubly periodic Cauchy problem of DS2 for small initial perturbations of the background, in the case of one and two unstable modes. We also show, in the case of two unstable modes, that (i) no blow up takes place generically, although the AW amplitude can be arbitrarily large; (ii) the excellent

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agreement of our formulas, expressed in terms of elementary functions of the initial data, with numerical experiments.

Keywords: breathers, Davey–Stewartson, periodic Cauchy problem, exact solution, modulation instability, blow-up

1. Introduction

Davey–Stewartson (DS) type equations [21] describe the amplitude modulation of weakly nonlinear quasi monochromatic waves in $2+1$ dimensions, and are relevant in water waves, nonlinear optics, plasma physics and Bose condensates [1, 9, 21, 37, 55]. Only a sub class of these equations are integrable, for special choices of their constant parameters, and can be written in the form:

$$\begin{aligned} & i u_t + u_{xx} - \nu u_{yy} + 2\eta q u = 0, \ \ \eta = \pm 1, \ \ \nu = \pm 1, \\ & q_{xx} + \nu q_{yy} = (|u|^2)_{xx} - \nu (|u|^2)_{yy}, \\ & x, y, t \in \mathbb{R}^3, \ u = u(x,y,t) \in \mathbb{C}, \ q = q(x,y,t) \in \mathbb{R}, \end{aligned} \tag{1}$$

where u is the complex amplitude of the monochromatic wave, and the real field $q(x,y,t)$ is related to the mean flow. If $\nu = -1$ we have the DS1 equation (surface tension prevails on gravity in the water wave derivation); in this case the sign of η is irrelevant, since one can go from the equation with $\eta = -1$ to the equation with $\eta = 1$ via the changes $q \to -q$ and $x \leftrightarrow y$; therefore there exists only one DS1 equation [33]. If $\nu = 1$, gravity prevails on surface tension and we have the DS2 equations; in this case the sign of η cannot be rescaled away and we distinguish between focusing and defocusing DS2 equations for respectively $\eta = 1$ and $\eta = -1$. It turns out that the shallow water limit of the Benney–Roskes equations [9] leads to the DS1 and to the defocusing DS2 equations [4], and DS1 plays a relevant role in the description of the initial-boundary value problem for dromions [25, 26], exponentially localized solutions first discovered via Bäcklund transformations [14]. We also remark that DS2 plays a relevant role in the theory of immersion of surfaces in $\mathbb{R}^4$ [46, 47, 61, 64–66]. The DS equations (1) are integrable $2+1$ dimensional generalizations of the celebrated nonlinear Schrödinger (NLS) equations

$$i v_t + v_{xx} + 2\eta |v|^2 v = 0, \ \ \eta = \pm 1, \ \ x, t \in \mathbb{R}, \ v = v(x,t) \in \mathbb{C}, \tag{2}$$

reducing to them when there is no y dependence. The focusing ($\eta = 1$) NLS equation is the simplest nonlinear integrable model describing modulation instability (MI), and MI is considered the main physical mechanism for the creation of anomalous waves (AWs) in nature [24, 34, 42, 43, 56, 57].

Concerning the NLS Cauchy problem for initial perturbations of the unstable background, what we call the Cauchy problem for AWs, if such a perturbation is localized, then slowly modulated periodic oscillations described by the elliptic solution of (2) play a relevant role in the longtime regime [12, 13]. Using the finite-gap method, the NLS periodic Cauchy problem of AWs was recently solved to leading order [27, 31] in the case of a finite number of unstable modes, leading to a quantitative description of the recurrence properties of the dynamics in terms of the multi-breather generalization [39] of the Akhmediev breather (AB) solution [5]

$$Akh(x,t,\phi) := \mathrm{e}^{2it}\mathcal{A}(x,t,\phi),$$
$$\mathcal{A}(x,t,\phi) := \frac{\cosh[2\sin(2\phi)t + 2i\phi] - \sin(\phi)\cos[2\cos(\phi)x]}{\cosh[2\sin(2\phi)t] + \sin(\phi)\cos[2\cos(\phi)x]}, \tag{3}$$

where ϕ is an arbitrary real parameter. In the simplest case of one unstable mode only, this theory describes quantitatively a Fermi–Pasta–Ulam–Tsingou (FPUT) recurrence of AWs described by the AB (3) [27, 28]. In addition, a finite-gap perturbation theory for $1+1$ dimensional AWs has been also developed [15], see also [16, 17], to describe analytically the order one effects of physical perturbations of the NLS model on the AW dynamics. See also [28] for an alternative approach to the study of the AW recurrence, based on matched asymptotic expansions. See [32] for the study of the instability properties of the AB solution within the NLS dynamics, and [30] for a finite-gap model describing the numerical instabilities of the AB. See [29] for the analytic study of the phase resonances in the AW recurrence. See [18–20, 63] for the analytic study of the AW recurrence in other NLS type models: respectively the nonlocal PT-symmetric NLS equation [3], the discrete Ablowitz–Ladik model [2], and the relativistic massive Thirring model [72], showing analytically the universal features of the AW recurrence in the periodic setting. MI and AWs of integrable multicomponent NLS equations have also been investigated [6, 7, 22, 23]. We also remark that the NLS recurrence of AWs in the periodic setting has been investigated in several numerical and real experiments, see, f.i. [41, 54, 62, 74, 75], and qualitatively studied via a three-wave approximation of NLS [38, 73].

As it was discussed in [33], the integrable focusing DS2 equation (1) ($\nu = \eta = 1$) is the best mathematical model on which to construct an analytic theory of $2+1$ dimensional AWs, and a finite gap formalism allowing one to solve in principle, to leading order, the spatially doubly periodic Cauchy problem for AWs of the focusing DS2 equation has been recently constructed [33]. It was shown in particular that the associated Riemann surfaces are more general than the hyperelliptic curves of NLS. But they have in common with NLS the important property that the $O(\epsilon)$ initial perturbations cause a splitting of the resonant points associated with the background, generating $O(\epsilon)$ handles. Therefore the solutions constructed in our paper in terms of elementary functions correspond to closing completely such handles (the case of genus 0), and describe the leading order term of the ε-expansion of the solution of the Cauchy problem. The fact that degenerate Riemann surfaces may generate spatially doubly periodic and regular solutions of some particular multidimensional soliton equations like DS was discussed in the literature [67].

Although the physical relevance of DS2 is not clear at the moment, a $2+1$ dimensional generalization of the AW perturbation theory developed in [15–17] could be used in principle to treat non integrable physically relevant multidimensional NLS models with mean flow as perturbations of the integrable DS equations. This will be the subject of future investigation.

We remark that the homogeneous background solution $u_0 = a\mathrm{e}^{2\mathrm{i}\eta|a|^2t}$, $\quad q_0 = |a|^2$ of equation (1), where a is an arbitrary complex parameter, can be simplified to

$$u_0(x,y,t) = 1, \;\; q_0(x,y,t) = 0, \tag{4}$$

using the scaling symmetry and the gauge symmetry

$$u(x,y,t) \to u(x,y,t)\exp\left(-i\frac{\eta}{2}\int^t f(\tau)\mathrm{d}\tau\right), \;\; q(x,y,t) \to q(x,y,t) + f(t) \tag{5}$$

of (1) [33], where $f(t) \in \mathbb{R}$ is an arbitrary function of time, and in the rest of the paper we use such a background.

Some exact AW solutions of the DS equations are already known in the literature (see for example [10, 11, 50, 51, 58, 59, 71]) and, in contrast with the focusing NLS equation, DS2 solutions corresponding to smooth Cauchy data may blow up at finite time [44, 45, 59, 60, 68–70].

The paper is organized as follows. Section 2 is devoted to the construction and study of AW solutions of the DS equations. After investigating in section 2.1 the instability properties of the constant background solution under monochromatic wave perturbations, in section 2.2 we construct the N-breather solution of Akhmediev type using the Hirota method; this solution describes the nonlinear stage of such instability and the nonlinear interaction of N unstable modes over the background in terms of elementary functions. This solution is generically quasi-periodic, and the sub-class of solutions describing spatially doubly periodic AWs is also constructed. In section 2.3 we study the simplest solutions of DS2 for $N = 1, 2$. In particular, in the case of the simplest multidimensional AW solution, we identify the constraint on its arbitrary parameters giving rise to blow up at finite time. Section 2.4 is devoted to the study of some interesting limiting cases, including solutions rational in one space variable and circular in the other. In section 3, using matched asymptotic expansion techniques, we establish the basic role played by the solutions of section 2 in the spatially doubly periodic Cauchy problem of DS2 for initial perturbations of the background, in the case of one and two unstable modes. In particular we show how the first appearance and recurrence of AWs are described by elementary functions of the initial data, and we also show that blow up is not generic. In the last section 4 we list the research directions, opened by the above results, we plan to investigate in the near future.

2. MI and the *N*-breather solution of Akhmediev type

Here we study the MI properties of the DS equation (1) and we construct and study the exact solutions describing the nonlinear stage of such an MI.

2.1. MI

To study the linear instability properties of the background solution (4) of the DS equations, we slightly perturb it as follows

$$u = 1 + \epsilon(f + ig), \quad q = \epsilon w, \quad \epsilon \ll 1, f, g, w \in \mathbb{R}. \tag{6}$$

Then f, g, w satisfy the linear partial differential equations

$$f_t + g_{xx} - \nu g_{yy} = 0, \quad g_t - f_{xx} + \nu f_{yy} - 2\eta w = 0, w_{xx} + \nu w_{yy} = 2(f_{xx} - \nu f_{yy}). \tag{7}$$

Looking for a monochromatic perturbation

$$\begin{aligned} &f = U\mathrm{e}^{\mathrm{i}(kx+ly)+\sigma t} + cc, \; g = V\mathrm{e}^{\mathrm{i}(kx+ly)+\sigma t} + cc, \\ &w = W\mathrm{e}^{\mathrm{i}(kx+ly)+\sigma t} + cc, \quad k, l \in \mathbb{R}, \end{aligned} \tag{8}$$

one obtains the following system of homogeneous equations

$$\begin{pmatrix} \sigma & -(k^2 - \nu l^2) & 0 \\ k^2 - \nu l^2 & \sigma & -2\eta \\ -2(k^2 - \nu l^2) & 0 & k^2 + \nu l^2 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} = 0 \tag{9}$$

and the condition for the existence of nontrivial solutions gives

$$\sigma^2(k,l) = \frac{(k^2-\nu l^2)^2[4\eta-(k^2+\nu l^2)]}{k^2+\nu l^2}. \tag{10}$$

Therefore we have the following stability properties of the background (4).

For DS1 ($\nu=-1,\eta=1$): $\sigma^2(k,l)=\frac{(k^2+l^2)^2[4-(k^2-l^2)]}{k^2-l^2}$. If $k^2-l^2>4$, then $\sigma^2<0$, $\sigma\in i\mathbb{R}$, and the background is neutrally stable. If $0<k^2-l^2<4$, then $\sigma^2>0$ and the background is unstable with growth rate

$$\sigma(k,l) = \frac{(k^2+l^2)\sqrt{4-(k^2-l^2)}}{\sqrt{k^2-l^2}}. \tag{11}$$

For DS2 ($\nu=1$): $\sigma^2(k,l)=\frac{(k^2-l^2)^2[4\eta-(k^2+l^2)]}{k^2+l^2}$. If $\eta=-1$, then $\sigma^2<0$, $\sigma\in i\mathbb{R}$ and the background is neutrally stable. If $\eta=1$ we have two cases. If $k^2+l^2>4$, then $\sigma^2<0$ and the bachground is stable. If

$$|\vec{k}|^2 = k^2+l^2<4, \text{ and } k^2\neq l^2, \ \vec{k}=(k,l), \tag{12}$$

then $\sigma^2>0$ and the background is unstable with exponential growth rate

$$\sigma(k,l) = |\Omega(k,l)|, \ \ \Omega(k,l) = \frac{(k^2-l^2)\sqrt{4-(k^2+l^2)}}{\sqrt{k^2+l^2}}. \tag{13}$$

Therefore no AWs are associated with the defocusing DS2, while AWs are present in the focusing DS2 equation for sufficiently small wave vectors $\vec{k}$, in perfect analogy with the NLS case (see figure 1).

We observe that a convenient parametrization of the unstable modes of DS1 and DS2 reads as follows

$$\begin{aligned} &\text{DS1}: \ k=2\cos(\phi)\cosh(\theta), \ \ l=2\cos(\phi)\sinh(\theta), \ \Rightarrow \ \sigma=2\sin(2\phi)\cosh(2\theta),\\ &\text{DS2}: \ k=2\cos(\phi)\cos(\theta), \ \ l=2\cos(\phi)\sin(\theta), \ \Rightarrow \ \Omega=2\sin(2\phi)\cos(2\theta) \end{aligned} \tag{14}$$

with

$$\begin{aligned} &\text{DS1}: \ \phi=\arccos\left(\frac{\sqrt{k^2-l^2}}{2}\right), \quad \theta=\tanh^{-1}\left(\frac{l}{k}\right),\\ &\text{DS2}: \ \phi=\arccos\left(\frac{\sqrt{k^2+l^2}}{2}\right), \quad \theta=\arctan\left(\frac{l}{k}\right). \end{aligned} \tag{15}$$

2.2. *The N-breather solution of Akhmediev type*

After studying the linear instability properties of the background in section 2.1, now we construct the N-breather solution of Akhmediev type of the DS equation (1), describing in terms of elementary functions the nonlinear stage of MI. This solution, the $2+1$ dimensional generalizations of the N-breather solution of Akhmediev type of the focusing NLS equation [39],

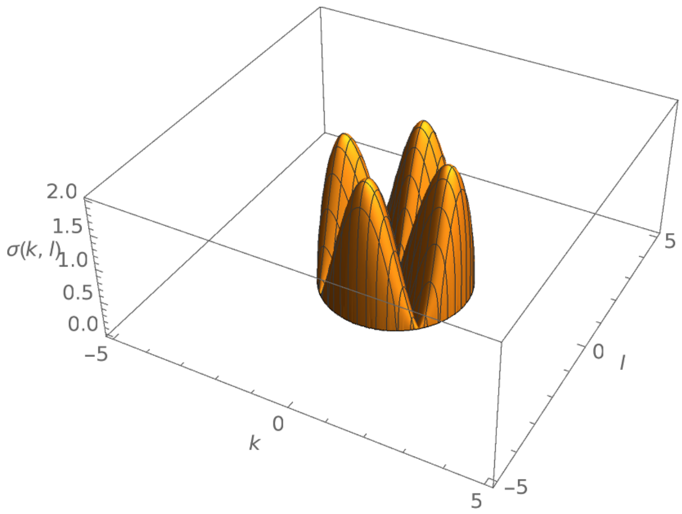

Figure 1. For the focusing DS2 equation, the growth rate $\sigma(k,l)$ in the instability region $k^2+l^2<4$.

oscillates in the space variables and decays exponentially over the unstable background in the remote past and in the far future.

Exact explicit solutions of integrable soliton equations can be constructed through different methods: a suitable genus zero degeneration of the finite gap method [8, 39, 48, 49], the Darboux transformations [53], dressing techniques [76–78], and the Hirota method [35, 36]. Here we use the Hirota method.

The Hirota bilinear form of the DS equation (1), corresponding to solutions over the background (4), reads [71, 79]:

$$\begin{aligned}&\left(iD_t+D_x^2-\nu D_y^2\right)G\cdot F=0,\\&\left(D_x^2+\nu D_y^2\right)F\cdot F=2\eta\left(|G|^2-F^2\right),\end{aligned}\tag{16}$$

where D_x is the Hirota derivative with respect to the generic independent variable x [36], so that

$$\begin{aligned}&D_xG\cdot F=G_x(x)F(x)-G(x)F_x(x),\\&D_x^2G\cdot F=G_{xx}(x)F(x)-2G_x(x)F_x(x)+G(x)F_{xx}(x),\end{aligned}\tag{17}$$

and functions G and F are related to the solution of equation (1) as follows:

$$\begin{aligned}&u(x,y,t)=\frac{G(x,y,t)}{F(x,y,t)},\quad F(x,y,t)\in\mathbb{R},\quad G(x,y,t)\in\mathbb{C},\\&q(x,y,t)=(\partial_x^2-\nu\partial_y^2)\log(F(x,y,t));\end{aligned}\tag{18}$$

in addition:

$$|u(x,y,t)|^2=1+(\partial_x^2+\nu\partial_y^2)\log(F(x,y,t)).\tag{19}$$

Then the N-breather AW solutions of the DS1 and DS2 equations are described by the following formulas in terms of elementary functions:

$$F(x,y,t) = \sum_{\substack{n_j=0,1\\ 1\leqslant j\leqslant 2N}} \exp\left(\sum_{j=1}^{2N} n_j\zeta_j(x,y,t) + \sum_{1\leqslant j<k\leqslant 2N} b_{jk}n_jn_k\right),$$

$$G(x,y,t) = \sum_{\substack{n_j=0,1\\ 1\leqslant j\leqslant 2N}} (-1)^{\sum_{j=1}^{2N} n_j} \exp\left(\sum_{j=1}^{2N} n_j(\zeta_j(x,y,t)+2i\hat{\phi}_j) + \sum_{1\leqslant j<k\leqslant 2N} b_{jk}n_jn_k\right). \tag{20}$$

For DS1:

$$\zeta_j(x,y,t) = \begin{cases} i[k_jx+l_jy+\zeta_j]+\Omega_j(t-\tau_j), & 1\leqslant j\leqslant N,\\ -i[k_{j-N}x+l_{j-N}y+\zeta_{j-N}]+\Omega_{j-N}(t-\tau_{j-N}), & N+1\leqslant j\leqslant 2N, \end{cases}.$$

$$k_j = 2\cos(\phi_j)\cosh(\theta_j),$$
$$l_j = 2\cos(\phi_j)\sinh(\theta_j),$$
$$\Omega_j = \Omega(k_j,l_j) = \frac{k_j^2+l_j^2}{\sqrt{k_j^2-l_j^2}}\sqrt{4-\left(k_j^2-l_j^2\right)} = 2\sin(2\phi_j)\cosh(2\theta_j), \tag{21}$$

$$b_{jk} = \begin{cases} \log\left(\frac{\cosh(\hat{\theta}_j-\hat{\theta}_k)-\cos(\hat{\phi}_j-\hat{\phi}_k)}{\cosh(\hat{\theta}_j-\hat{\theta}_k)+\cos(\hat{\phi}_j+\hat{\phi}_k)}\right), & 1\leqslant j<k\leqslant N,\ N+1\leqslant j<k\leqslant 2N,\\ \log\left(\frac{\cosh(\hat{\theta}_j-\hat{\theta}_k)+\cos(\hat{\phi}_j-\hat{\phi}_k)}{\cosh(\hat{\theta}_j-\hat{\theta}_k)-\cos(\hat{\phi}_j+\hat{\phi}_k)}\right), & 1\leqslant j\leqslant N \text{ and } N+1\leqslant k\leqslant 2N, \end{cases}.$$

$$\hat{\phi}_j = \begin{cases} \phi_j, & 1\leqslant j\leqslant N,\\ \phi_{j-N}, & N+1\leqslant j\leqslant 2N. \end{cases}. \tag{22}$$

For DS2:

$$\zeta_j(x,y,t) = \begin{cases} i[k_jx+l_jy+\zeta_j]+\Omega_j(t-\tau_j), & 1\leqslant j\leqslant N,\\ -i[k_{j-N}x+l_{j-N}y+\zeta_{j-N}]+\Omega_{j-N}(t-\tau_{j-N}), & N+1\leqslant j\leqslant 2N, \end{cases}.$$

$$k_j = 2\cos(\phi_j)\cos(\theta_j),$$
$$l_j = 2\cos(\phi_j)\sin(\theta_j),$$
$$\Omega_j = \Omega(k_j,l_j) = \frac{k_j^2-l_j^2}{\sqrt{k_j^2+l_j^2}}\sqrt{4-(k_j^2+l_j^2)} = 2\sin(2\phi_j)\cos(2\theta_j),\ \ \sigma_j = |\Omega_j|, \tag{23}$$

$$b_{jk} = \begin{cases} \log\left(\frac{\cos(\hat{\theta}_j-\hat{\theta}_k)-\cos(\hat{\phi}_j-\hat{\phi}_k)}{\cos(\hat{\theta}_j-\hat{\theta}_k)+\cos(\hat{\phi}_j+\hat{\phi}_k)}\right), & 1\leqslant j<k\leqslant N,\ N+1\leqslant j<k\leqslant 2N,\\ \log\left(\frac{\cos(\hat{\theta}_j-\hat{\theta}_k)+\cos(\hat{\phi}_j-\hat{\phi}_k)}{\cos(\hat{\theta}_j-\hat{\theta}_k)-\cos(\hat{\phi}_j+\hat{\phi}_k)}\right), & 1\leqslant j\leqslant N \text{ and } N+1\leqslant k\leqslant 2N, \end{cases}.$$

$$\hat{\phi}_j = \begin{cases} \phi_j, & 1\leqslant j\leqslant N,\\ \phi_{j-N}, & N+1\leqslant j\leqslant 2N, \end{cases}.\ \hat{\theta}_j = \begin{cases} \theta_j, & 1\leqslant j\leqslant N,\\ \theta_{j-N}, & N+1\leqslant j\leqslant 2N. \end{cases}. \tag{24}$$

(k_j,l_j), $j=1,\ldots,N$ are arbitrary real wave vectors inside the instability regions of section 2.1, and ζ_j,τ_j, $j=1,\ldots,N$ are arbitrary real parameters.

The proof of this result is by induction, and since it is long and tedious, but completely standard in the Hirota method philosophy, we omit it.

To the best of our knowledge, the solutions (18)–(24) is new for $N \geqslant 2$. If $N = 1$ it was constructed in [71, 79] and describes a straight line (one dimensional) AW that can be constructed from the AB solution (3) of NLS using elementary symmetry considerations. Other solutions obtained using the Hirota method describe the interaction of an AW with a dark-bright soliton, together with their rational limits [50, 51, 71]. Also rational AWs were constructed [58, 59].

The solutions (18)–(24) are quasi-periodic in space for generic values of the wave vectors (k_j, l_j), $j = 1, \ldots, N$ inside the instability regions, since the x wave numbers $\{k_j\}_{j=1}^N$ are generically incommensurable, as well as the y wave numbers $\{l_j\}_{j=1}^N$.

In view of the study, in section 3, of the relevance of these solutions in mathematically and physically sound Cauchy problems, we remark that the Cauchy problem for spatially quasi-periodic initial data presents several difficulties (in particular, a finite gap theory for it is not available). In addition, the linear operator expressing q in terms of $|u|^2$ in the second of equations (1) is unbounded, in the periodic setting, for the DS1 equation, implying very non-trivial analytic effects, and it is bounded for the focusing DS2 equation [33]. Therefore in the rest of the paper we shall limit our considerations to spatially doubly periodic exact solutions, and, in section 3, to the well-posed doubly periodic Cauchy problem for AWs of the DS2 equation.

As it was shown in [33], in the well-posed periodic Cauchy problem for AWs of the DS2 equation, with periods L_x and L_y respectively in the x and y directions, the wave vectors of the above N-breather solution are quantized as follows

$$\vec{k}_{m,n} = (k_m, l_n), \;\; k_m = \tfrac{2\pi}{L_x} m, \; l_n = \tfrac{2\pi}{L_y} n, \;\; m, n \in \mathbb{Z}, \tag{25}$$

and lie on the rectangular lattice of figure 2, constrained by the instability condition

$$k_m^2 + l_n^2 < 4 \quad \Leftrightarrow \quad \left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2 < \frac{1}{\pi^2}, \;\; L_x \neq L_y. \tag{26}$$

The simplest possible instability configurations are, in order of complication, the following.
(1) The case in which there is only one unstable mode, the mode $\pm\vec{k}_{1,0} = \pm(k_1, 0)$ on the k axis, with:

$$1 < k_1 < 2, \;\; l_1 > 2 \quad \Leftrightarrow \quad \pi < L_x < 2\pi, \;\; L_y < \pi, \tag{27}$$

or the mode $\pm\vec{k}_{0,1} = \pm(0, l_1)$ on the l axis, with:

$$1 < l_1 < 2, \;\; k_1 > 2 \quad \Leftrightarrow \quad \pi < L_y < 2\pi, \;\; L_x < \pi; \tag{28}$$

see respectively the top left and top right pictures of figure 2.
(2) The case in which there are only the two unstable modes $\pm\vec{k}_{1,0}, \pm\vec{k}_{0,1}$, with

$$1 < k_1, l_1 < 2, \;\; k_1^2 + l_1^2 > 4 \quad \Leftrightarrow \quad \pi < L_x, L_y < 2\pi, \;\; \frac{1}{L_x^2} + \frac{1}{L_y^2} > \frac{1}{\pi^2}; \tag{29}$$

see the bottom left picture of figure 2.
(3) The case in which there are only the four unstable modes $\pm\vec{k}_{1,0}, \pm\vec{k}_{0,1}, \pm\vec{k}_{1,1}, \pm\vec{k}_{1,-1}$, with

$$1 < k_1, l_1 < 2, \;\; k_1^2 + l_1^2 < 4 \quad \Leftrightarrow \quad \pi < L_x, L_y < 2\pi, \;\; \frac{1}{L_x^2} + \frac{1}{L_y^2} < \frac{1}{\pi^2}; \tag{30}$$

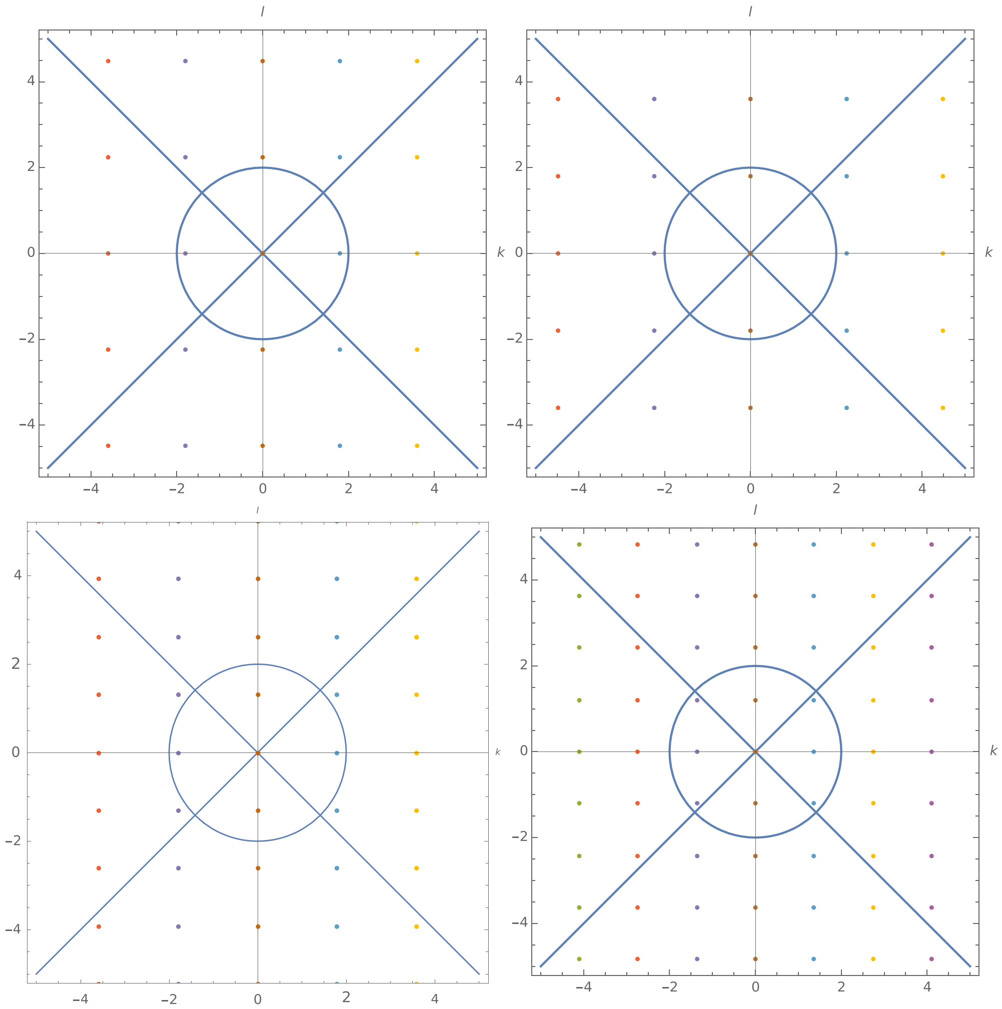

Figure 2. For the DS2 equation, the instability region in the $\vec{k} = (k, l)$ plane is the disk $k^2 + l^2 < 4$, with $k^2 - l^2 \neq 0$. The Fourier modes of the linearized theory are $\vec{k}_{m,n} = 2\pi(\frac{m}{L_x}, \frac{n}{L_y})$, where $m, n \in \mathbb{Z}$, and L_x and L_y are respectively the periods in the x and y directions. In the top left picture $L_x = 3.5,\ L_y = 2.8$, and there is only the unstable mode $\pm\vec{k}_{1,0}$. In the top right picture $L_x = 2.8,\ L_y = 3.5$, and there is only the unstable mode $\pm\vec{k}_{0,1}$. In the bottom left picture $L_x = 3.5,\ L_y = 4.8$, and there are only the two unstable modes $\pm\vec{k}_{1,0}, \pm\vec{k}_{0,1}$. In the bottom right picture $L_x = 4.6,\ L_y = 5.2$, and there are only the four unstable modes $\pm\vec{k}_{1,0}, \pm\vec{k}_{0,1}, \pm\vec{k}_{1,1}, \pm\vec{k}_{1,-1}$.

see the bottom right picture of figure 2. Increasing the periods L_x and L_y, higher order modes enter the instability region and the picture becomes more and more complicated. In this paper we limit our considerations to the first two cases (1) and (2), postponing to a subsequent paper the study of a higher number of unstable modes.

2.3. *The simplest cases*

2.3.1. $N = 1$. The solutions (18)–(22) of the DS2 equation reads, after some manipulation

$$u_1(x,y,t) = \frac{\cosh[\sigma_1(t-t_1)+2i\phi_1] - \sin(\phi_1)\cos[k_1x+l_1y+\zeta_1]}{\cosh[\sigma(t-t_1)] + \sin(\phi_1)\cos[k_1x+l_1y+\zeta_1]}, \tag{31}$$

$$q_1(x,y,t) = \cos(2\theta_1)(|u_1(x,y,t)|^2 - 1), \tag{32}$$

where

$$\sigma_1 = 2\sin(2\phi_1)\cos(2\theta_1), \;\; k_1 = 2\cos\phi_1\cos\theta_1, \;\; l_1 = 2\cos\phi_1\sin\theta_1, \tag{33}$$

and t_1, suitably related to τ_1 , is the arbitrary real parameter corresponding to the time translation symmetry of the DS equations.

This solution describes a straight line breather parallel to the line $x\cos(\theta_1)+y\sin(\theta_1)=0$ of arbitrary slope $\cot(\theta_1)$ (due to the arbitrariness of the parameter θ_1), repeated periodically in the x and y directions with periods $L_x = 2\pi/k_1$ and $L_y = 2\pi/l_1$, and decaying to the background (4) at $t \to \pm\infty$.

Since its appearance changes the phase of the background by the factor $4\phi_1$, it can be viewed as a quasi homoclinic solution as the AB solution (3) of NLS, and actually can be written in terms of the AB itself as follows:

$$u_1(x,y,t) = \mathcal{A}(\cos(\theta_1)x+\sin(\theta_1)y-x_1, \cos(2\theta_1)(t-t_1), \phi_1), \tag{34}$$

where $x_1 = -\zeta_1/(2\cos\phi_1)$.

In the doubly periodic Cauchy problem of the AWs, only its two limiting cases in which the solution depends on just one space variable play a role (see figure 2). The y dependence disappears if $l_1 = 0$ ($\theta_1 = 0$), and the DS2 solutions (31) and (32) reads:

$$u_{1,0}(x,t) = \mathcal{A}(x-x_1, t-t_1, \phi_1), \;\; q_{1,0}(x,t) = |u_{1,0}(x,t)|^2 - 1, \tag{35}$$

describing a straight line breather parallel to the y axis (top left picture in figure 2). The x dependence disappears if $k_1 = 0$ ($\theta_1 = \pi/2$), and the solutions (31) and (32) of DS2 reduces to

$$u_{0,1}(y,t) = \mathrm{e}^{2it}\mathcal{A}(y-x_1, -(t-t_1), \phi_1), \;\; q_{0,1}(y,t) = 1 - |u_{0,1}(y,t)|^2 \tag{36}$$

describing a straight line breather parallel to the x axis (top right picture in figure 2).

As in the NLS case, it is always possible to construct the rational limit of (31) and (32) when k_1, l_1 tend to zero (for $\phi_1 \to \pm\pi/2$). If $\phi_1 \to \pi/2$, then $u_1 \to -\exp 2it$. If $\phi_1 \to \pi/2$, up to an irrelevant minus sign, the breather tends to the following generalization of the Peregrine solution

$$
\begin{aligned}
u_{1P}(x,y,t) &= 1 - \frac{4+16\cos(2\theta_1)(t-t_1)}{1+16(t-t_1)^2\cos^2(2\theta_1)+4(\cos(\theta_1)x+\sin(\theta_1)y-x_1)^2}, \\
q_{1P}(x,y,t) &= 1 + \frac{8\cos(2\theta_1)[1+16\cos^2(2\theta_1)(t-t_1)^2-4(\cos(\theta_1)(x-x_1)+\sin(\theta_1)(y-y_1))^2]}{[1+16(t-t_1)^2\cos^2(2\theta_1)+4(\cos(\theta_1)x+\sin(\theta_1)y-x_1)^2]^2},
\end{aligned}
\tag{37}
$$

constant on the line $x\cos\theta_1 + y\sin\theta_1 = 0$ with arbitrary slope, and rationally localized over the background in any other direction.

2.3.2. N = 2. The straight line breather solutions (31) and (32) and its rational limit (37) could have been constructed from the AB plus symmetry considerations. The simplest truly two dimensional AW describes the interaction of the horizontal $\pm\vec{k}_{1,0} = (k_1, 0)$ and vertical $\pm\vec{k}_{0,1} = (0, l_1)$ unstable modes (see the bottom left picture of figure 2), where

$$
k_1 = \frac{2\pi}{L_x} = 2\cos\phi_{1,0}, \;\; l_1 = \frac{2\pi}{L_y} = 2\cos\phi_{0,1}, \;\; \theta_{1,0} = 0, \; \theta_{0,1} = \pi/2, \tag{38}
$$

corresponding to the conditions

$$
\pi < L_x, L_y < 2\pi \;\Leftrightarrow\; 1 < k_1, l_1 < 2 \;\Leftrightarrow\; 0 < \phi_{1,0}, \phi_{0,1} < \pi/3. \tag{39}
$$

Then the solutions (18)–(22) reads, after some manipulation:

$$
u_2(x,y,t;\phi_{1,0},\phi_{0,1},x_0,y_0,t_{1,0},t_{0,1},\rho) = \frac{N(x,y,t)}{D(x,y,t)} e^{i\rho}, \tag{40}
$$

$$
\begin{aligned}
N(x,y,t) = {} & \cosh[\sigma_{1,0}(t-t_{1,0}) + \sigma_{0,1}(t-t_{0,1}) + 2i(\phi_{1,0}-\phi_{0,1})] \\
& + b_{12}^2\cosh[\sigma_{1,0}(t-t_{1,0}) - \sigma_{0,1}(t-t_{0,1}) + 2i(\phi_{1,0}+\phi_{0,1})] \\
& - 2b_{12}\Big(\sin\phi_{1,0}\cos(X_{1,0})\cosh[\sigma_{0,1}(t-t_{0,1}) - 2i\phi_{0,1}] \\
& + \sin\phi_{0,1}\cos(Y_{0,1})\cosh[\sigma_{1,0}(t-t_{1,0}) + 2i\phi_{1,0}] \\
& + \sin\phi_{1,0}\sin\phi_{0,1}\cos(X_{1,0})\cos(Y_{0,1})\Big),
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
D(x,y,t) = {} & \cosh[\sigma_{1,0}(t-t_{1,0}) + \sigma_{0,1}(t-t_{0,1})] + b_{12}^2\cosh[\sigma_{1,0}(t-t_{1,0}) - \sigma_{0,1}(t-t_{0,1}))] \\
& + 2b_{12}\Big(\sin\phi_{1,0}\cos(X_{1,0})\cosh[\sigma_{0,1}(t-t_{0,1})] + \sin\phi_{0,1}\cos(Y_{0,1})\cosh[\sigma_{1,0}(t-t_{1,0})] \\
& - \sin\phi_{1,0}\sin\phi_{0,1}\cos(X_{1,0})\cos(Y_{0,1})\Big),
\end{aligned}
\tag{42}
$$

where

$$X_{1,0} = k_1(x-x_0) = 2\cos(\phi_{1,0})(x-x_0), \quad Y_{0,1} = l_1(y-y_0) = 2\cos(\phi_{0,1})(y-y_0),$$
$$\sigma_{1,0} = k_1\sqrt{4-k_1^2} = 2\sin(2\phi_{1,0}), \quad \sigma_{0,1} = l_1\sqrt{4-l_1^2} = 2\sin(2\phi_{0,1}),$$
$$b_{12} = \frac{\cos(\phi_{1,0}-\phi_{0,1})}{\cos(\phi_{1,0}+\phi_{0,1})}, \tag{43}$$

ρ is the arbitrary real parameter connected to the elementary gauge symmetry, $t_{1,0}, t_{0,1}$ are arbitrary real parameters suitably connected to τ_1, τ_2, and $x_0 = -\zeta_1/k_1, y_0 = -\zeta_2/l_1$.

If, in addition,

$$\frac{1}{L_x^2} + \frac{1}{L_y^2} > \frac{1}{\pi^2} \quad \Leftrightarrow \quad k_1^2 + l_1^2 > 4 \quad \Leftrightarrow \quad \cos^2\phi_{1,0} + \cos^2\phi_{0,1} > 1, \tag{44}$$

then the modes $\pm\vec{k}_{1,1} = (k_1, l_1)$ and $\pm\vec{k}_{1,-1} = (k_1, -l_1)$ are stable and $b_{12} > 0$ (see the bottom left plot of figure 2); instead, if

$$\frac{1}{L_x^2} + \frac{1}{L_y^2} < \frac{1}{\pi^2} \quad \Leftrightarrow \quad k_1^2 + l_1^2 < 4 \quad \Leftrightarrow \quad \cos^2\phi_{1,0} + \cos^2\phi_{0,1} < 1, \tag{45}$$

then also the modes $\pm\vec{k}_{1,1} = (k_1, l_1)$ and $\pm\vec{k}_{1,-1} = (k_1, -l_1)$ are unstable and $b_{12} < 0$ (see the bottom right plot of figure 2). It is quite clear that the solutions (40)–(43) will be relevant in the periodic Cauchy problem for AWs only under the constraint (44).

While the parameters ρ, x_0, y_0 and one of the parameters $t_{1,0}, t_{0,1}$ (say, $t_{0,1}$) are associated with space-time translation and elementary gauge symmetries of DS2, the additional parameter $t_{1,0}$ is associated with the integrability of the model.

For generic parameters, the solutions (40)–(43) decays to the backgrounds $\exp[i(\rho \pm (\phi_{1,0} - \phi_{0,1}))]$ as $t \to \pm\infty$, and describes the nonlinear interaction between the horizontal and vertical unstable modes. Since the associated growth rates $\sigma_{1,0}, \sigma_{0,1}$ are generically different, it describes two consecutive appearances in time of $2+1$ dimensional doubly-periodic smooth bumps, both located at $(x_0 + L_x/2, y_0 + L_y/2)$ (see figure 3).

There are however two non generic choices of the parameter $t_{10} - t_{01}$:

$$t_{10} - t_{01} = \pm\Delta_c,$$
$$\Delta_c := \frac{1}{\sigma_{0,1}}\log(|B| + \sqrt{B^2-1}) - \frac{1}{\sigma_{1,0}}\log(|A| + \sqrt{A^2-1}), \tag{46}$$

where

$$A = \cos\phi_{1,0}\tan(\phi_{1,0}+\phi_{0,1}) - \frac{\sin\phi_{0,1}}{\cos(\phi_{1,0}-\phi_{0,1})},$$
$$B = \cos\phi_{0,1}\tan(\phi_{1,0}+\phi_{0,1}) - \frac{\sin\phi_{1,0}}{\cos(\phi_{1,0}-\phi_{0,1})}, \tag{47}$$

for which the solution (40) blows up at the critical times

$$t_c^{\pm} = t_{01} \pm \frac{1}{\sigma_{0,1}}\log\left(|B| + \sqrt{B^2-1}\right). \tag{48}$$

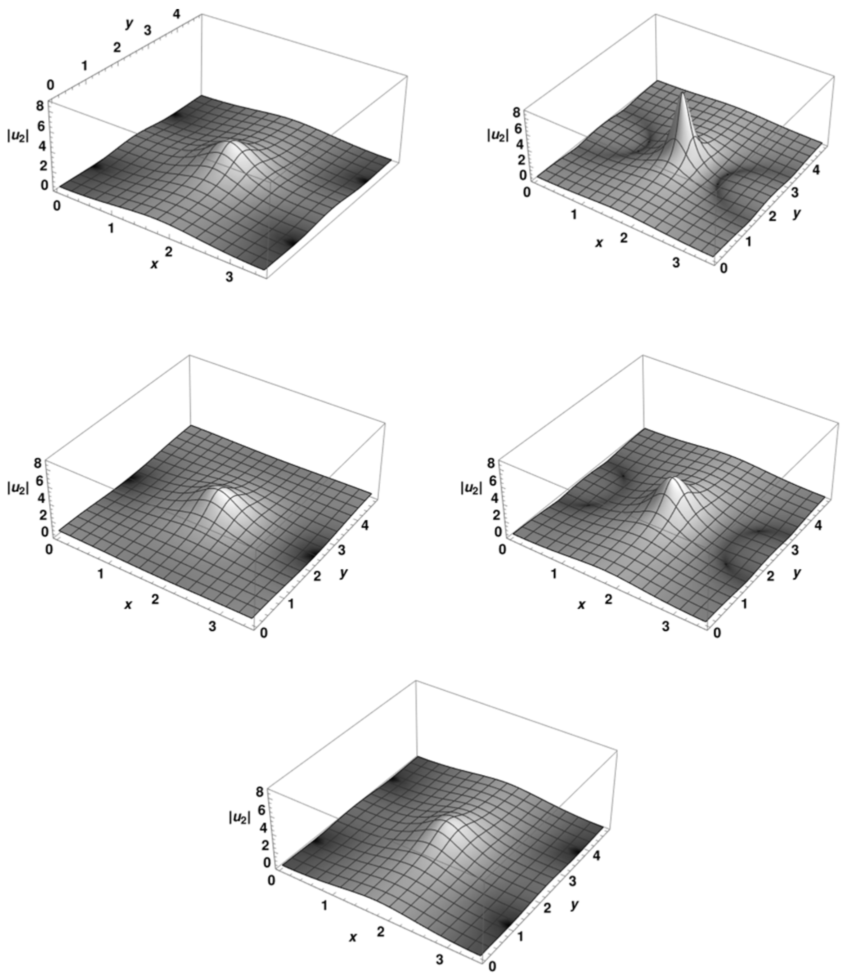

Figure 3. Five snapshots of the evolution of the absolute value of the two-breather AW solution (40) in a basic period ($L_x = 3.5$, $L_y = 4.8$), describing the nonlinear interaction of two unstable modes, one parallel to the x axis and the other parallel to the y axis with parameters $t_{1,0} = 0$ and $t_{0,1} = -0.1$. Top left: the growth of the AW from the background ($t = -1.2$); top right: the first emergence ($t = -0.83$); medium left: between two emergences ($t = -0.17$); medium right: second emergence ($t = 0.78$); bottom: the disappearance of the AW into the background ($t = 1.1$). For generic parameters, the solution is smooth and the two emergences, occurring both at $(x, y) = (x_0 + L_x/2, y_0 + L_y/2)$, are different. In this case $x_0 = y_0 = 0$.

More precisely, if $t_{10} - t_{01} = -\Delta_c$, the solution blows up at its first appearance in $(x_0, y_0, t = t_c^-)$; if $t_{10} - t_{01} = \Delta_c$, the solution blows up at its second appearance in $(x_0, y_0, t = t_c^+)$. It turns out that, at these two points in space time, not only the denominator is zero, but also the partial derivatives of the denominator with respect to the variables $\xi_{1,0} = \sigma_{1,0}(t - t_{1,0})$ and $\xi_{0,1} = \sigma_{0,1}(t - t_{0,1})$ are zero.

We observe that the second of equation (46) implies that, if $A^2 < 1$ and/or $B^2 < 1$, then Δ_c is not real, the two real parameters t_{10} and t_{01} cannot satisfy the first of equation (46), and blow up does not occur. The condition on the angles $\phi_{1,0}, \phi_{0,1}$ such that $A^2, B^2 > 1$ is

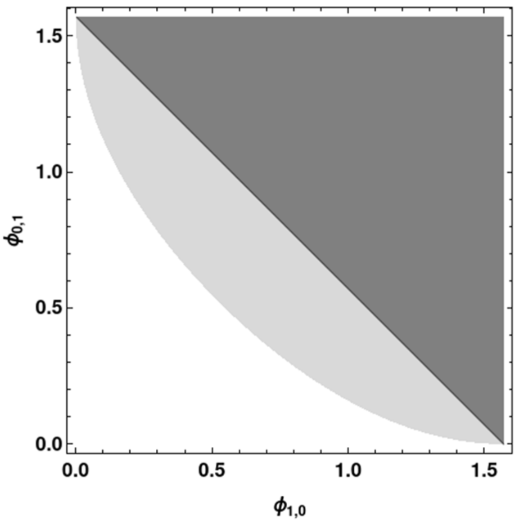

Figure 4. The regions of the plane ($\phi_{1,0}$,$\phi_{0,1}$) in which blow-up can occur. In the white region blow-up can never occur because (49) is not fulfilled. In the light and dark gray regions, instead, (49) is satisfied, $\Delta_c \in \mathbb{R}$, and blow-up can occur if $t_{1,0} - t_{0,1}$ is chosen to satisfy the first of equation (46). Light and dark gray regions, separated by the curve $\phi_{1,0} + \phi_{0,1} = \frac{\pi}{2}$, are respectively the regions where only two and more than two modes are unstable.

$$\sin\phi_{1,0} + \sin\phi_{0,1} > 1, \tag{49}$$

and corresponds to the gray regions of figure 4. Once (49) is satisfied, blow up takes place if the free parameters t_{10}, t_{01} satisfy the first of equation (46).

The AW solution (40) behaves as follows near the blow up:

$$\begin{aligned} &u_2(x,y,t;\phi_{1,0},\phi_{0,1},x_0,y_0,t_{0,1}\pm\Delta_c,t_{0,1},\rho) \\ &\quad = \frac{N(x_0+L_x/2,y_0+L_y/2,t_c^{\pm})e^{i\rho}}{a(t-t_c^{\pm})^2+b(x-x_0-L_x/2)^2+c(y-y_0-L_y/2)^2}(1+O(t-t_c^{\pm})), \\ &|x-x_0-L_x/2|,|y-y_0-L_y/2| = O(t-t_c^{\pm}),\ \ |t-t_c^{\pm}| \ll 1, \end{aligned} \tag{50}$$

where

$$\begin{aligned} a &= b_{12}\left[\sin\phi_{1,0}(|B|+\sin\phi_{0,1})\sigma_{1,0}^2 + \sin\phi_{0,1}\left(|A|+\sin\phi_{1,0}\right)\sigma_{0,1}^2\right] \\ &\quad + \left[|AB| + \sqrt{(A^2-1)(B^2-1)} + b_{12}^2\left(AB - \sqrt{(A^2-1)(B^2-1)}\right)\right]\sigma_{1,0}\sigma_{0,1}, \\ b &= b_{12}\sin\phi_{1,0}(|B|+\sin\phi_{0,1})k_1^2,\ \ c = b_{12}\sin\phi_{0,1}\left(|A|+\sin\phi_{1,0}\right)l_1^2. \end{aligned} \tag{51}$$

From these considerations one infers that, unlike the NLS case for which the amplitude of the N breather solution has a fixed maximum achieved when the interference among its N unstable modes is fully constructive, the amplitude of the AW u_2 can be arbitrarily large if the free parameters $t_{1,0}, t_{0,1}$ are such that $t_{1,0} - t_{0,1}$ is sufficiently close to $\pm\Delta_c$. We expect similar and even richer features for the exact solutions (18)–(24) coming from the interaction of more than two unstable modes, and a subsequent paper will be devoted to a systematic study of these behaviors.

2.4. Interesting limiting cases of the solutions (40)–(43)

2.4.1. The modes $\pm\vec{k}_{1,1}, \pm\vec{k}_{1,-1}$ tend to the instability circle. If $\pm\vec{k}_{1,1}, \pm\vec{k}_{1,-1}$ tend to the instability circle: $k_1^2 + l_1^2 \to 4$, then $\phi_{1,0} + \phi_{0,1} \to \frac{\pi}{2}$ and $b_{12} \to \infty$. In addition the two growth rates become the same:

$$\sigma_{1,0} = \sigma_{0,1} = 2\sin(2\phi) =: \sigma, \quad \phi := \phi_{1,0}. \tag{52}$$

To perform this limit we choose $e^{\sigma_{1,0}t_{1,0}}, e^{\sigma_{0,1}t_{0,1}} = O(b_{12})$, and we define the convenient $O(1)$ parameters

$$t_0 = \frac{1}{\sigma}\log\left(\frac{e^{\sigma_{1,0}t_{1,0}} + e^{\sigma_{0,1}t_{0,1}}}{\sqrt{e^{2\sigma_{1,0}t_{1,0}} + e^{2\sigma_{0,1}t_{0,1}}}}\right),\ \xi = \arccos\left(\frac{e^{\sigma_{0,1}t_{0,1}}}{\sqrt{e^{2\sigma_{1,0}t_{1,0}} + e^{2\sigma_{0,1}t_{0,1}}}}\right), \tag{53}$$

obtaining the following new solution in the limit

$$u_b(x,y,t) = \frac{N_b(x,y,t)}{D_b(x,y,t)} e^{i\rho}, \tag{54}$$

where:

$$\begin{aligned} N_b(x,y,t) = {} & \cosh(\sigma(t-t_0) + 2i\phi) - \sin\phi\sin\xi\cos(2\cos\phi(x-x_0)) \\ & + \cos\phi\cos\xi\cos(2\sin\phi(y-y_0)) \end{aligned} \tag{55}$$

and

$$\begin{aligned} D_b(x,y,t) = {} & \cosh(\sigma(t-t_0)) + \sin\phi\sin\xi\cos(2\cos\phi(x-x_0)) \\ & + \cos\phi\cos\xi\cos(2\sin\phi(y-y_0)). \end{aligned} \tag{56}$$

Since the two growth rates coincide in this limit, and we are left with only one time parameter t_0, the AW appearance consists of only one emergence. The blow up condition is achieved only if $\xi = \phi$.

2.4.2. Rational limit. As in the NLS case, the rational limit to Peregrine like solutions is achieved taking the long wave limit in both x and y directions. Then we introduce the following notation

$$k_1 = \epsilon\delta_x, \qquad l_1 = \epsilon\delta_y, \qquad \epsilon \ll 1, \tag{57}$$

and, correspondingly, we choose the angles in the first quadrant

$$\phi_{1,0} = \frac{\pi}{2} - \epsilon\frac{\delta_x}{2} + O(\epsilon^3), \quad \phi_{0,1} = \frac{\pi}{2} - \epsilon\frac{\delta_y}{2} + O(\epsilon^3), \tag{58}$$

to get the nontrivial rational limit

$$u_{2P}(x,y,t) = \frac{N_P(x,y,t)}{D_P(x,y,t)} e^{i\rho}, \tag{59}$$

where

$$N_{\mathrm{P}}(x,y,t) = \left(-3+4(x-x_0)^2+16(t-t_{1,0})(t-t_{1,0}-i)\right)\left(-3+4(y-y_0)^2 +16(t-t_{0,1})(t-t_{0,1}+i)\right)+64i\,(t_{1,0}-t_{0,1})-128(t-t_{1,0})(t-t_{0,1})-24 \tag{60}$$

and

$$D_{\mathrm{P}}(x,y,t) = (-3+4(x-x_0)^2+16(t-t_{1,0})^2)(-3+4(y-y_0)^2+16(t-t_{0,1})^2) +64(t_{1,0}-t_{0,1})^2+16((x-x_0)^2+(y-y_0)^2). \tag{61}$$

This $2+1$ dimensional generalization of the NLS Peregrine solution does not depend on the parameters δ_x and δ_y. It describes the nonlinear interaction of two rational Peregrine walls of the type (37), parallel to the x and y axes. Like the Peregrine solution, it decreases rationally to the background (4) as $t\to\pm\infty$; it decreases to the orthogonal walls for $x^2+y^2\gg 1$, unlike the Peregrine solution that decreases to the background in this limit. As the solutions (40)–(43), it appears twice, but now the two appearances are described by the same function. This rational solution, blowing up twice if $t_{1,0}=t_{0,1}$, in the space-time points $(x_0,y_0,t_{1,0}\pm\frac{\sqrt{3}}{4})$, should be a particular example of the general class of rational AWs solutions presented in [59].

2.4.3. Periodic-rational limit. If one performs the above long wave limit only, say, in the y direction, the solutions (40)–(43) leads to the following solution rational in y, periodic in x, and rational/hyperbolic in t:

$$u_{\mathrm{PR}}(x,y,t) = \frac{N_{\mathrm{PR}}(x,y,t)}{D_{\mathrm{PR}}(x,y,t)}\mathrm{e}^{\mathrm{i}\rho}, \tag{62}$$

where

$$N_{\mathrm{PR}}(x,y,t) = \cosh(\sigma_{1,0}(t-t_{1,0})+2i\phi_{1,0})(-7+16(i+t-t_{0,1})(t-t_{0,1})+4(y-y_1)^2 +4\csc^2\phi_{1,0})-8\cot\phi_{1,0}\sinh(\sigma_{1,0}(t-t_{1,0})+2i\phi_{1,0})(i+2(t-t_{0,1})) +\sin\phi_{1,0}\cos(k_1(x-x_1))(-3+16(i+t-t_{0,1})(t-t_{0,1})+4(y-y_1)^2) \tag{63}$$

and

$$D_{\mathrm{PR}}(x,y,t) = \cosh(\sigma_{1,0}(t-t_{1,0}))(1+16(t-t_{0,1})^2+4(y-y_1)^2+4\cot^2\phi_{1,0}) -16\cot\phi_{1,0}\sinh(\sigma_{1,0}(t-t_{1,0}))(t-t_{0,1})+ -\sin\phi_{1,0}\cos(k_1(x-x_1))(1+16(t-t_{0,1})^2+4(y-y_1)^2). \tag{64}$$

To the best of our knowledge also this solution is new.

3. MI and AW recurrence

In this section we use the matched asymptotic expansions technique introduced in [28] to describe the relevance of the above exact AW solutions in the DS2 doubly periodic Cauchy problem for AWs, in the case of one and two unstable modes.

The doubly periodic Cauchy problem for AWs of the focusing DS2 equation (1), $\eta = \nu = 1$, reads

$$\begin{aligned} &u(x+L_x,y,t) = u(x,y+L_y,t) = u(x,y,t),\\ &q(x+L_x,y,t) = q(x,y+L_y,t) = q(x,y,t),\\ &\quad u(x,y,0) = 1+\epsilon\, v(x,y),\;\; q(x,y,0) = \epsilon\, w(x,y),\;\; 0<\epsilon\ll 1, \end{aligned} \tag{65}$$

where the initial perturbations can be expanded in Fourier modes as follows:

$$v(x,y) = \sum_{\mu,\nu\in\mathbb{Z}} c_{\mu,\nu} \mathrm{e}^{\mathrm{i}(k_\mu x + l_\nu y)}. \tag{66}$$

For $|t| = O(1)$, the evolution is ruled by the linearized equation (7), and the solution is described by the following formulas

$$\begin{aligned} u(x,y,t) = 1+\epsilon \sum_{m,n\in\mathcal{D}} \Big(&\frac{|\alpha_{m,n}|}{\sin(2\phi_{m,n})}\cos(k_m x + l_n y - \arg(\alpha_{m,n}) - \pi/2)\mathrm{e}^{\Omega_{m,n}t + i\phi_{m,n}}\\ &+\frac{|\beta_{m,n}|}{\sin(2\phi_{m,n})}\cos(k_m x + l_n y + \arg(\beta_{m,n}) - \pi/2)\mathrm{e}^{-\Omega_{m,n}t - i\phi_{m,n}}\Big) + O(\epsilon)\text{-oscillations}, \end{aligned} \tag{67}$$

$$\begin{aligned} q(x,y,t) = \epsilon \sum_{m,n\in\mathcal{D}} \frac{\cos(2\theta_{m,n})}{\sin(\phi_{m,n})}\Big[&|\alpha_{m,n}|\cos(k_m x + l_n y - \arg(\alpha_{m,n}) - \pi/2)\mathrm{e}^{\Omega_{m,n}t}\\ &+|\beta_{m,n}|\cos(k_m x + l_n y + \arg(\beta_{m,n}) - \pi/2)\mathrm{e}^{-\Omega_{m,n}t}\Big] + O(\epsilon)\text{-oscillations}, \end{aligned} \tag{68}$$

where

$$\begin{aligned} &k_m = 2\cos\phi_{m,n}\cos\theta_{m,n},\;\; l_n = 2\cos\phi_{m,n}\sin\theta_{m,n},\\ \Rightarrow\;\; &\phi_{m,n} = \arccos\left(\frac{\sqrt{k_m^2 + l_n^2}}{2}\right), \theta_{m,n} = \arctan\left(\frac{l_n}{k_m}\right),\\ &\alpha_{m,n} = \mathrm{e}^{-\mathrm{i}\phi_{m,n}}\bar{c}_{m,n} - \mathrm{e}^{\mathrm{i}\phi_{m,n}} c_{-m,-n}, \beta_{m,n} = \mathrm{e}^{\mathrm{i}\phi_{m,n}}\bar{c}_{-m,-n} - \mathrm{e}^{-\mathrm{i}\phi_{m,n}} c_{m,n}, \end{aligned} \tag{69}$$

and

$$\mathcal{D} = \left\{ m\geqslant 1,\; n\in\mathbb{Z}, \left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2 < \frac{1}{\pi^2}\right\} \cup \left\{ m=0,\; n\geqslant 1,\; \left(\frac{n}{L_y}\right)^2 < \frac{1}{\pi^2}\right\}. \tag{70}$$

In (67) we do not describe explicitly the $O(\epsilon)$ oscillations because they are associated with the stable modes and remain $O(\epsilon)$ at later times; corrections to equation (67) are at $O(\epsilon^2)$.

As time increases, the perturbation in (67) grows exponentially and, at $t = O(\log(1/\epsilon))$, it becomes order one and the dynamics is described by the fully nonlinear theory. It is when the exact solutions we constructed play a relevant role.

3.1. One unstable mode

In the case of one unstable mode we have the two cases (27) and (28).
(a) If $\vec{k}_{1,0} = (k_1, 0)$ is the only unstable mode, i.e.:

$$\begin{aligned}&\pi < L_x < 2\pi,\ L_y < \pi \quad \Leftrightarrow \quad 1 < k_1 < 2,\ \ l_1 > 2 \quad \Leftrightarrow\\&\theta_{1,0} = 0,\ k_1 = 2\cos\phi_{1,0},\ 0 < \phi_{1,0} < \pi/3,\\&\Omega_{1,0} = k_1\sqrt{4-k_1^2} = 2\sin(2\phi_{1,0}) = \sigma_{1,0},\end{aligned} \tag{71}$$

then equation (67) reduces to

$$\begin{aligned}u(x,y,t) = 1 + \epsilon[&\frac{1}{\sin(2\phi_{1,0})}(|\alpha_{1,0}|\cos(2\cos\phi_{1,0}x - \arg(\alpha_{1,0}) - \pi/2)\mathrm{e}^{\mathrm{i}\phi_{1,0}+\sigma_{1,0}t}\\&+|\beta_{1,0}|\cos(2\cos\phi_{1,0}x + \arg(\beta_{1,0}) - \pi/2)\mathrm{e}^{-\mathrm{i}\phi_{1,0}-\sigma_{1,0}t})] + O(\epsilon)\text{-oscillations.}\end{aligned} \tag{72}$$

Since the exact solution $u_{1,0}(x,y,t)$ in (35) describes the nonlinear instability of the mode $\pm\vec{k}_{1,0}$, it is the natural candidate to describe the first AW appearance at $t = O(\log(1/\epsilon))$. Then one chooses its appearance time $t^{(1)}$ as $t^{(1)} \equiv \frac{1}{\sigma_{1,0}}\log\frac{\gamma}{\epsilon}$, $\gamma > 0$, with γ to be fixed. In the intermediate time interval $1 \ll t \ll O(\log(1/\epsilon))$, (72) and $u_{1,0}(x,y,t)$ become

$$\begin{aligned}u(x,y,t) &\sim 1 + \frac{\epsilon|\alpha_{1,0}|}{\sin(2\phi_{1,0})}\cos[k_1x - \arg(\alpha_{1,0}) - \pi/2]e^{i\phi_{1,0}+\sigma_{1,0}t},\\u_{1,0}(x,y,t) &\sim \mathrm{e}^{\mathrm{i}\left(\rho^{(1)}-2\phi_{1,0}\right)}\left(1 + \frac{2\epsilon}{\gamma}\sin(2\phi_{1,0})\cos\left(k_1(x-x^{(1)})\right)\mathrm{e}^{\sigma_{1,0}t+i\psi_{1,0}}\right).\end{aligned} \tag{73}$$

Comparing the leading order asymptotics (73) one fixes all the free parameters of $u_{1,0}$ as follows

$$\rho^{(1)} = 2\phi_{1,0},\ x^{(1)} = \frac{\arg(\alpha_{1,0}) + \pi/2}{k_1},\ t^{(1)} = \frac{1}{\sigma_{1,0}}\log\left(\frac{2\sin^2(2\phi_{1,0})}{\epsilon|\alpha_{1,0}|}\right), \tag{74}$$

showing that the first AW appearance is described, to leading order and at $|t - t^{(1)}| = O(1)$, by

$$u(x,y,t) = \mathrm{e}^{2\mathrm{i}\phi_{1,0}}\mathcal{A}\left(x - x^{(1)}, t - t^{(1)}, \phi_{1,0}\right) + O(\epsilon), \tag{75}$$

an elementary function of the initial data. We remark that, although the initial perturbation is an arbitrary doubly periodic function of (x,y), since the only unstable mode is the horizontal mode $\pm\vec{k}_{1,0}$, the AW is one-dimensional and the y dependence is confined at $O(\epsilon)$.

To describe analytically the AW recurrence, we also construct the first AW appearance at negative times, following the same strategy, obtaining

$$\begin{aligned}&u(x,y,t) = \mathrm{e}^{-2\mathrm{i}\phi_{1,0}}\mathcal{A}(x - x^{(0)}, t - t^{(0)}, \phi_{1,0}) + O(\epsilon),\quad |t - t^{(0)}| = O(1),\\&x^{(0)} = \frac{-\arg(\beta_{1,0}) + \pi/2}{k_1},\quad t^{(0)} = -\frac{1}{\sigma_{1,0}}\log\left(\frac{2\sin^2(2\phi_{1,0})}{\epsilon|\beta_{1,0}|}\right).\end{aligned} \tag{76}$$

Then we compare the two consecutive appearances (75) and (76), and using the time translation property of the model, we infer that the dynamics is described by an FPUT recurrence of AWs, and that the *j*th appearance is described by

$$u(x,y,t) = \mathrm{e}^{\mathrm{i}\rho^{(j)}} \mathcal{A}\left(x - x_{1,0}^{(j)}, t - t_{1,0}^{(j)}, \phi_{1,0}\right) + O(\epsilon), \quad |t - t_{1,0}^{(j)}| = O(1), \quad j \geqslant 1, \tag{77}$$

where

$$\begin{aligned}
&\rho_{1,0}^{(j)} = \rho_{1,0}^{(1)} + (j-1)4\phi_{1,0}, \quad x_{1,0}^{(j)} = x_{1,0}^{(1)} + (j-1)\frac{\arg(\alpha_{1,0}\beta_{1,0})}{k_1}, \\
&t_{1,0}^{(j)} = t_{1,0}^{(1)} + (j-1)\frac{2}{\sigma_{1,0}} \log\left(\frac{2\sin^2(2\phi_{1,0})}{\epsilon\sqrt{|\alpha_{1,0}\beta_{1,0}|}}\right).
\end{aligned} \tag{78}$$

(b) If $\vec{k}_{0,1} = (0, l_1)$ is the only unstable mode, i.e.:

$$\begin{aligned}
&\pi < L_y < 2\pi, \; L_x < \pi \quad \Leftrightarrow \quad 1 < l_1 < 2, \; k_1 > 2 \quad \Leftrightarrow \\
&\theta_{1,0} = \pi/2, \; \theta_{0,1} = 0, \; l_1 = 2\cos\phi_{0,1}, \; 0 < \phi_{0,1} < \pi/3, \\
&\sigma_{0,1} = l_1\sqrt{4 - l_1^2} = 2\sin(2\phi_{0,1}) = -\Omega_{0,1},
\end{aligned} \tag{79}$$

now the nonlinear stages of MI are described, to leading order, by the exact solution $u_{0,1}$ and, proceeding as before, one can show that the solution of the Cauchy problem (65) is described by an FPUT recurrence of AWs, and that the *j*th appearance is described by

$$u(x,y,t) = \mathrm{e}^{\mathrm{i}\rho_{0,1}^{(j)}} \mathcal{A}(y - y_{0,1}^{(j)}, t - t_{0,1}^{(j)}, \phi_{0,1}) + O(\epsilon), \quad |t - t_{0,1}^{(j)}| = O(1), \quad j \geqslant 1, \tag{80}$$

where

$$\begin{aligned}
&\rho_{0,1}^{(j)} = \rho_{0,1}^{(1)} - (j-1)4\phi_{0,1}, \quad y_{0,1}^{(j)} = y_{0,1}^{(1)} - (j-1)\frac{\arg(\alpha_{0,1}\beta_{0,1})}{l_1}, \\
&t_{0,1}^{(j)} = t_{0,1}^{(1)} + (j-1)\frac{2}{\sigma_{0,1}} \log\left(\frac{2\sin^2(2\phi_{0,1})}{\epsilon\sqrt{|\alpha_{0,1}\beta_{0,1}|}}\right)
\end{aligned} \tag{81}$$

and

$$\rho_{0,1}^{(1)} = -2\phi_{0,1}, \; y_{0,1}^{(1)} = \frac{-\arg(\beta_{0,1}) + \pi/2}{l_1}, \; t_{0,1}^{(1)} = \frac{1}{\sigma_{0,1}} \log\left(\frac{2\sin^2(2\phi_{0,1})}{\epsilon\sqrt{|\beta_{0,1}|}}\right). \tag{82}$$

We remark that, in both cases, since we have only one growing mode (horizontal or vertical) in the overlapping region, and since the Akhmediev type solution, describing the growth of this unstable mode, contains enough free parameters for a successful matching, the remaining mismatch cannot affect the leading order behavior at the appearance. Therefore this stability argument plus uniqueness of the DS2 evolution imply that the appearance of the AW is described by the one dimensional Akhmediev solution, and the dependence on both x and y variables is hidden at $O(\epsilon)$.

3.2. Two unstable modes

The simplest truly two dimensional AW dynamics takes place when there are only the two unstable modes $\pm\vec{k}_{1,0}$ and $\pm\vec{k}_{0,1}$ (see the bottom left picture of figure 2), corresponding to the constraint

$$\begin{aligned}&\pi < L_x, L_y < 2\pi, \ \ \frac{1}{L_x^2}+\frac{1}{L_y^2} > \frac{1}{\pi^2}, \Leftrightarrow\\ &1 < k_1, l_1 < 2, \quad k_1^2 + l_1^2 > 4, \Leftrightarrow\\ &0 < \phi_{1,0}, \phi_{0,1} < \pi/3, \ \ \cos^2\phi_{1,0} + \cos^2\phi_{0,1} > 1.\end{aligned} \tag{83}$$

Then the linear stage of MI (67), for $|t| \leqslant O(1)$, reduces to

$$\begin{aligned}u(x,y,t) = 1 + \epsilon\Big[&\frac{1}{\sin(2\phi_{1,0})}(|\alpha_{1,0}|\cos(2\cos\phi_{1,0}x - \arg(\alpha_{1,0}) - \pi/2)\mathrm{e}^{\mathrm{i}\phi_{1,0}+\sigma_{1,0}t}\\ &+ |\beta_{1,0}|\cos(2\cos\phi_{1,0}x + \arg(\beta_{1,0}) - \pi/2)\mathrm{e}^{-\mathrm{i}\phi_{1,0}-\sigma_{1,0}t})\\ &+ \frac{1}{\sin(2\phi_{0,1})}\Big[|\alpha_{0,1}|\cos(l_1 y - \arg(\alpha_{0,1}) - \pi/2)\mathrm{e}^{-\sigma_{0,1}t+\mathrm{i}\phi_{0,1}}\\ &+ |\beta_{0,1}|\cos(l_1 y + \arg(\beta_{1,0}) - \pi/2)\mathrm{e}^{\sigma_{0,1}t-\mathrm{i}\phi_{0,1}}\Big] + O(\epsilon)\text{-oscillations}.\end{aligned} \tag{84}$$

Reasoning as before, since the exact solution u_2 of DS2 in (40)–(43) describes the nonlinear interaction of the unstable modes $\pm\vec{k}_{1,0}$, $\pm\vec{k}_{0,1}$, it is the natural candidate to characterize this nonlinear stage, and following exactly the same strategy as before, we find that the first AW appearance is described to leading order by the solution (40)

$$u(x,y,t) = u_2\left(x,y,t;\phi_{1,0},\phi_{0,1},x^{(1)},y^{(1)},t_{1,0}^{(1)},t_{0,1}^{(1)},\rho^{(1)}\right) + O(\epsilon), \tag{85}$$

where the solution parameters are expressed in terms of the initial data as follows

$$\begin{aligned}&\rho^{(1)} = 2(\phi_{1,0}-\phi_{0,1}), \ x^{(1)} = \frac{\arg(\alpha_{1,0})+\pi/2}{k_1}, \ y^{(1)} = \frac{-\arg(\beta_{0,1})+\pi/2}{l_1},\\ &t_{1,0}^{(1)} = \frac{1}{\sigma_{1,0}}\log\left(\frac{2b_{12}\sin^2(2\phi_{1,0})}{\epsilon|\alpha_{1,0}|}\right), \ t_{0,1}^{(1)} = \frac{1}{\sigma_{0,1}}\log\left(\frac{2b_{12}\sin^2(2\phi_{0,1})}{\epsilon|\beta_{0,1}|}\right).\end{aligned} \tag{86}$$

Therefore the first appearance of the AW in the Cauchy problem consists of the two emergences described by the exact solutions (40)–(43) (see figure 3), whose parameters are expressed in terms of the initial data through elementary functions.

As for the case of one unstable mode, we remark that, since we have only two growing modes in the overlapping time region, and since the exact solutions (40)–(43), describing the growth and the nonlinear interaction of these unstable modes, contains enough free parameters for a successful matching, the remaining mismatch cannot affect the leading order behavior. Therefore this stability argument plus uniqueness of the DS2 evolution imply that the first appearance of the AW is described by the solutions (85) and (86), an elementary function of the initial data.

To have an idea of how well the analytic solution u_2 in (85) and (86) describe the first appearance of the AW in the AW Cauchy problem, we evaluate the uniform distance between

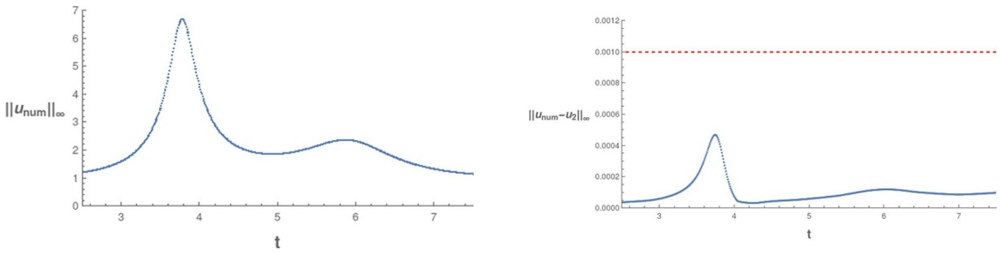

Figure 5. Here we study the two emergences of AWs in the time interval of the first appearance, for the initial data $\epsilon = 10^{-3}$, $c_{1,0} = 0.8 + i0.4$, $c_{-1,0} = 1.2 - i0.1$, $c_{0,1} = -0.64 - i0.3$, $c_{0,-1} = 0.5 + i0.2$. In the left picture we plot the max of the amplitude of the AW $\|u_{\mathrm{num}}\|_\infty(t)$ as function of time, where u_{num} is the numerical solution; the first emergence at $(x, y, t) = (3.24019, 1.227442, 3.780)$ with a peak of height 6.6786; the second emergence at $(x, y, t) = (3.24019, 1.227442, 5.868)$ with a peak of smaller height 2.3631. Right picture: the uniform distance $\|u_{\mathrm{num}} - u_2\|_\infty(t)$ between the analytic solution u_2 (85) and (86) and the numerical solution u_{num}; the two peaks of the distance correspond exactly to the two AW emergences of the left picture, and the distance remains always $\leqslant 5 \cdot 10^{-4}$, smaller than the estimated error from theoretical considerations $O(10^{-3})$, indicated by the horizontal dotted line.

u_2 and the numerical solution u_{num} (obtained using the 4th order split step Fourier method [40]):

$$\|u_{\mathrm{num}} - u_2\|_\infty(t) := \sup_{x\in[0,L_x], y\in[0,L_y]} |u_{\mathrm{num}}(x,y,t) - u_2(x,y,t)|, \tag{87}$$

in the time interval in which the AW first appears, see figure 5. The agreement is excellent, since the error is much smaller than expected from theoretical considerations.

We end this paper with some considerations on the possibility of blow up in the first appearance of the AW (85) and (86). If we compare the difference between the time parameters $t^{(1)}_{1,0} - t^{(1)}_{0,1}$ coming from the Cauchy problem

$$t^{(1)}_{1,0} - t^{(1)}_{0,1} = \frac{1}{\sigma_{1,0}} \log\left(\frac{2b_{12}\sin^2(2\phi_{1,0})}{\epsilon|\alpha_{1,0}|}\right) - \frac{1}{\sigma_{0,1}} \log\left(\frac{2b_{12}\sin^2(2\phi_{0,1})}{\epsilon|\beta_{0,1}|}\right) \tag{88}$$

with the critical difference (46) and (47) corresponding to the blow up of the solution (40), we infer that we have blow up if one of the following two equations are satisfied

$$\left(\frac{b_{12}\sigma^2_{1,0}(B+\sqrt{B^2-1})}{2\epsilon|\alpha_{1,0}|}\right)^{\sigma_{1,0}} = \left(\frac{b_{12}\sigma^2_{0,1}(A+\sqrt{A^2-1})}{2\epsilon|\beta_{0,1}|}\right)^{\sigma_{0,1}}, \tag{89}$$

$$\left(\frac{b_{12}\sigma^2_{1,0}}{2\epsilon|\alpha_{1,0}|(B+\sqrt{B^2-1})}\right)^{\sigma_{1,0}} = \left(\frac{b_{12}\sigma^2_{0,1}}{2\epsilon|\beta_{0,1}|(A+\sqrt{A^2-1})}\right)^{\sigma_{0,1}}. \tag{90}$$

If (89) holds, then blow up occurs at the first emergence; if (90) holds, then blow up occurs at the second emergence.

Equations (89) and (90) depend on the initial data parameters $\epsilon, c_{m,n}$, and on the unstable mode parameters $\phi_{1,0}, \phi_{0,1}$. If, for instance, we fix the initial condition parameters, then the

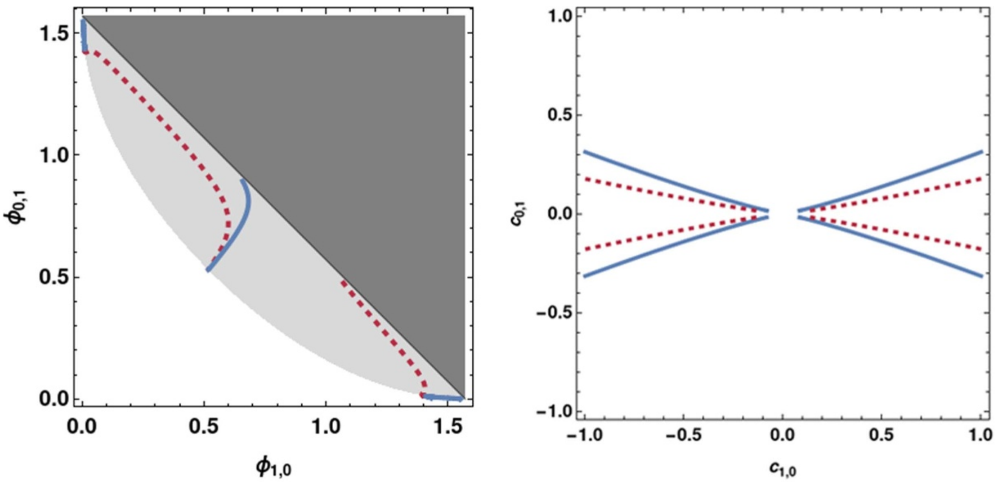

Figure 6. The left picture shows the solid and dashed curves in the $(\phi_{1,0}, \phi_{0,1})$ plane on which the blow-up conditions (89) and respectively (90) are satisfied, for the initial data $c_{1,0} = 0.8 + i0.4$, $c_{-1,0} = 1.2 - i0.1$, $c_{0,1} = -0.64 - i0.3$, $c_{0,-1} = 0.5 + i0.2$ and $\epsilon = 10^{-3}$. The light and dark gray regions (for which $\sin\phi_{1,0} + \sin\phi_{0,1} > 1$) are the regions where blow-up can occur; light and dark tones color indicate respectively the regions where only two modes and more than two modes are unstable. The right picture shows the solid and dashed curves in the $(c_{1,0}, c_{0,1})$ plane on which the blow-up conditions (89) and respectively (90) are satisfied, for $\phi_{1,0} = 0.7$, $\phi_{0,1} = 0.5$, and $\epsilon = 10^{-3}$.

blow up regions in the $(\phi_{1,0}, \phi_{0,1})$ plane are curves (see the left picture in figure 6). If we fix instead the unstable mode parameters $(\phi_{1,0}, \phi_{0,1})$, in the space of real initial data of the type

$$u(x,y,0) = 1 + 2\epsilon[c_{1,0}\cos(2\cos\phi_{1,0}x) + c_{0,1}\cos(2\cos\phi_{0,1}y)], \quad c_{1,0}, c_{0,1} \in \mathbb{R}, \tag{91}$$

the blow up regions in the $(c_{1,0}, c_{0,1})$ plane are again curves (see the right picture in figure 6).

Since curves have zero measure in the plane, we conclude that, generically, the first appearance of the AW does not give rise to blow up. But the amplitude of the AW can be arbitrarily large if the parameters are sufficiently close to the singular curves, and situations of this type are expected to take place also at later times, during the recurrence.

Similar considerations are expected to be valid for a generic Cauchy problem of AWs involving more than two unstable modes, and will be the subject of future investigation.

4. Future perspectives

The results of this paper and of paper [33] open several research directions we plan to follow in the near future. (1) The proper implementation of the finite gap formalism developed in [33] to solve, in terms of elementary functions of the initial data, the generic spatially doubly periodic Cauchy problem of AWs for the DS2 equation, in the case of a higher, but finite, number of unstable nonlinear modes, since matched asymptotic expansions are not adequate to study AW recurrence in this case [28]. (2) The use of the analytic solution of the AW Cauchy problem to study the probability of generating multidimensional AWs of amplitude greater than a certain critical value in a given time interval. (3) The generalization to the $2+1$ dimensional DS2 equation of the perturbation theory of AWs developed for $1+1$ dimensional NLS type equations in [15–17, 19], to describe analytically the order one effects of physical perturbations of the DS2 equation on the AW dynamics.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Ablowitz M J, Biondini G and Blair S 2001 Nonlinear Schrödinger equations with mean terms in nonresonant multidimensional quadratic materials *Phys. Rev.* E **63** 046605
[2] Ablowitz M J and Ladik J F 1975 Nonlinear differential-difference equations *J. Math. Phys.* **16** 598–603
[3] Ablowitz M J and Musslimani Z H 2013 Integrable nonlocal nonlinear Schrödinger equation *Phys. Rev. Lett.* **110** 064105
[4] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (SIAM)
[5] Akhmediev N N, Eleonskii V M and Kulagin N E 1985 Generation of periodic trains of picosecond pulses in an optical fiber: exact solutions *Sov. Phys - JETP* **62** 894–9
[6] Baronio F, Conforti M, Degasperis A, Lombardo S, Onorato M and Wabnitz S 2014 Vector rogue waves and baseband modulation instability in the defocusing regime *Phys. Rev. Lett.* **113** 034101
[7] Baronio F, Degasperis A, Conforti M and Wabnitz S 2012 Solutions of the vector nonlinear Schrödinger equations: evidence for deterministic rogue waves *Phys. Rev. Lett.* **109** 44102
[8] Belokolos E D, Bobenko A I, Enolski V Z, Its A R and Matveev V B 1994 *Algebro-Geometric Approach in the Theory of Integrable Equations* (*Springer Series in Nonlinear Dynamics*) (Springer)
[9] Benney D J and Roskes G J 1969 Wave instabilities *Stud. Appl. Math.* **48** 377–85
[10] Biondini G and Kireyev D 2022 On the periodic solutions of the Davey Stewartson system *East Asian J. Appl. Math.* **12** 1–21
[11] Biondini G, Kireyev D and Maruno K 2022 Soliton resonance and web structure in the Davey-Stewartson system *J. Phys. A: Math. Theor.* **55** 305701
[12] Biondini G and Kovacic G 2014 *J. Math. Phys.* **55** 031506
[13] Biondini G, Li S and Mantzavinos D 2016 *Phys. Rev.* E **94** 060201(R)
[14] Boiti M, Leon J, Martina L and Pempineili F 1988 Scattering of localized solitons in the plane *Phys. Lett.* A **132** 432–9
[15] Coppini F, Grinevich P G and Santini P M 2020 The effect of a small loss or gain in the periodic NLS anomalous wave dynamics I *Phys. Rev.* E **101** 032204
[16] Coppini F, Grinevich P G and Santini P M 2021 Periodic rogue waves and perturbation theory *Encyclopedia of Complexity and Systems Science* ed R A Meyers (Springer)

[17] Coppini F and Santini P M 2020 The Fermi-Pasta-Ulam-Tsingou recurrence of periodic anomalous waves in the complex Ginzburg-Landau and in the Lugiato-Lefever equations *Phys. Rev.* E **102** 062207
[18] Coppini F and Santini P M 2024 Modulation instability, periodic anomalous wave recurrence and blow up in the Ablowitz-Ladik lattices *J. Phys. A Math. Theor.* **57** 015202
[19] Coppini F and Santini P M 2023 The effect of loss/gain and Hamiltonian perturbations on the recurrence of periodic anomalous waves of the Ablowitz-Ladik lattice (arXiv: 2305.07339)
[20] Coppini F, Santini P M 2023 Anomalous waves in the massive thirring relativistic field theory (in preparation)
[21] Davey A and Stewartson K 1974 On three-dimensional packets of surface waves *Proc. R. Soc. London* A **338** 101–10
[22] Degasperis A, Lombardo S and Sommacal M 2017 Integrability and linear stability of nonlinear waves *J. Nonlinear Sci.* **28** 1251–1291
[23] Degasperis A, Lombardo S and Sommacal M 2019 Rogue wave type solutions and spectra of coupled nonlinear Schrödinger equations *Fluids* **4** 57
[24] Dysthe K B and Trulsen K 1999 Note on breather type solutions of the NLS as models for Freak-waves *Phys. Scr.* **T82** 48–52
[25] Fokas A S and Santini P M 1989 Coherent structures in multidimensions *Phys. Rev. Lett.* **63** 1329–33
[26] Fokas A S and Santini P M 1990 Dromions and a boundary-value problem for the Davey-Stewartson equation *Physica* D **44** 99–130
[27] Grinevich P G and Santini P M 2018 The finite gap method and the analytic description of the exact rogue wave recurrence in the periodic NLS Cauchy problem. 1 *Nonlinearity* **31** 5258–308
[28] Grinevich P G and Santini P M 2018 The exact rogue wave recurrence in the NLS periodic setting via matched asymptotic expansions, for 1 and 2 unstable modes *Phys. Lett.* A **382** 973–9
[29] Grinevich P G and Santini P M 2018 Phase resonances of the NLS rogue wave recurrence in the quasi-symmetric case *Theor. Math. Phys.* **196** 1294–306
[30] Grinevich P G and Santini P M *et al* 2019 Numerical instability of the Akhmediev breather and a finite gap model of it *Recent Developments in Integrable Systems and Related Topics of Mathematical Physics, Proms* ed V M Buchstaber (Springer) pp 3–23
[31] Grinevich P G and Santini P M 2019 The finite-gap method and the periodic NLS Cauchy problem of anomalous waves for a finite number of unstable modes *Russ. Math. Surv.* **74** 211–63
[32] Grinevich P G and Santini P M 2021 The linear and nonlinear instability of the Akhmediev breather *Nonlinearity* **34** 8331–58
[33] Grinevich P G and Santini P M 2022 The finite-gap method and the periodic Cauchy problem for (2+1)-dimensional anomalous waves for the focusing Davey-Stewartson 2 equation *Russ. Math. Surv.* **77** 102–1059
[34] Henderson K L, Peregrine D H and Dold J W 1999 Unsteady water wave modulations: fully nonlinear solutions and comparison with the nonlinear Schrödinger equtation *Wave Motion* **29** 341
[35] Hirota R 1971 Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons *Phys. Rev. Lett.* **27** 1192
[36] Hirota R 1976 *Direct Methods for Finding Exact Solutions of Nonlinear Evolution Equations* (*Lecture Notes in Mathematics* vol 515) (Springer)
[37] Huang G, Deng L and Hang C 2005 Davey-Stewartson description of two-dimensional nonlinear excitations in Bose-Einstein condensates *Phys. Rev.* E **72** 036621
[38] Infeld E 1981 Quantitive theory of the Fermi-Pasta-Ulam recurrence in the nonlinear Schrödinger equation *Phys. Rev. Lett.* **47** 717–8
[39] Its A R, Rybin A V and Sall M A 1988 Exact integration of nonlinear Schrödinger equation *Theor. Math. Phys.* **74** 20–32
[40] Javanainen J and Ruostekoski J 2006 Symbolic calculation in development of algorithms: split-step methods for the Gross-Pitaevskii equation *J. Phys. A: Math. Gen.* **39** L179–84 Javanainen J and Ruostekoski J 2004 Split-step Fourier methods for the Gross-Pitaevskii equation 3 (arXiv: cond-mat/0411154)
[41] Kimmoun O *et al* 2016 Modulation instability and phase-shifted Fermi-Pasta-Ulam recurrence *Sci. Rep.* **6** 28516
[42] Kharif C and Pelinovsky E 2004 Physical mechanisms of the rogue wave phenomenon *Eur. J. Mech. B/Fluids J. Mech.* **22** 603–34

[43] Kharif C and Pelinovsky E 2001 Focusing of nonlinear wave groups in deep water *JETP Lett.* **73** 170–5
[44] Klein C and Saut J-C 2014 A numerical approach to blow-up issues for Davey-Stewartson II systems *Commun. Pure Appl. Anal.* **14** 1443–1467
[45] Klein C and Saut J-C 2015 IST versus PDE: a comparative study *Hamiltonian Partial Differential Equations and Applications* (*Fields Institute Communications*) vol 75 (Fields Institute for Research in Mathematical Sciences) pp 383–449
[46] Konopelchenko B G 1996 Induced surfaces and their integrable dynamics *Stud. Appl. Math.* **96** 9–51
[47] Konopelchenko B G 2000 Weierstrass representations for surfaces in 4D spaces and their integrable deformations via DS hierarchy *Ann. Global Anal. Geom.* **18** 61–74
[48] Krichever I M 1989 Spectral theory of two-dimensional periodic operators and its applications *Russ. Math. Surv.* **44** 145–225
[49] Krichever I M 1992 Perturbation theory in periodic problems for two-dimensional integrable systems *Sov. Sci. Rev.* C **9** 1–103
[50] Liu C, Wang C, Dai Z and Liu J 2014 New rational homoclinic and rogue waves for Davey-Stewartson equation *Abstr. Appl. Anal.* **2014** 572863
[51] Liu Y, Qian C, Mihalache D and He J 2019 Rogue waves and hybrid solutions of the Davey-Stewartson I equation *Nonlinear Dyn.* **95** 839–57
[52] Matuev R M and Taimanov I A 2016 The Moutard transformation of two-dimensional Dirac operators and the conformal geometry of surfaces in four-dimensional space *Math. Notes* **100** 835–46
[53] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (*Springer Series in Nonlinear Dynamics*) (Springer)
[54] Mussot A, Naveau C, Conforti M, Kudlinski A, Szriftgiser P, Copie F and Trillo S 2018 Fibre multiwave mixing combs reveal the broken symmetry of Fermi-Pasta-Ulam recurrence *Nat. Photon.* **12** 303–8
[55] Nishinari K, Abe K and Satsuma J 1994 A new type of soliton behavior of the Davey-Stewartson equations in a plasma system *Theor. Math. Phys.* **99** 745–53
[56] Onorato M, Residori S, Bortolozzo U, Montina A and Arecchi F T 2013 Rogue waves and their generating mechanisms in different physical contexts *Phys. Rep.* **528** 47–89
[57] Osborne A, Onorato M and Serio M 2000 The nonlinear dynamics of rogue waves and holes in deep-water gravity wave trains *Phys. Lett.* A **275** 386
[58] Ohta Y and Yang J 2012 Rogue waves in the Davey-Stewartson equation *Phys. Rev.* E **86** 036604
[59] Ohta Y and Yang J 2013 Dynamics of rogue waves in the Davey-Stewartson II equation *J. Phys. A: Math. Theor.* **46** 105202
[60] Ozawa T 1992 Exact blow-up solutions to the Cauchy problem for the Davey-Stewartson systems *Proc. R. Soc. London* A **436** 345–9
[61] Pedit F and Pinkall U 1998 Quaternionic analysis on Riemann surfaces and differential geometry *Doc. Math., J. DMV Extra Vol. ICM* **II** 389–400
[62] Pierangeli D, Flammini M, Zhang L, Marcucci G, Agranat A J, Grinevich P G, Santini P M, Conti C and DelRe E 2018 Observation of exact Fermi-Pasta-Ulam-Tsingou recurrence and its exact dynamics *Phys. Rev.* X **8** 041017
[63] Santini P M 2018 *J. Phys. A: Math. Theor.* **51** 495207 (21)
[64] Taimanov I A 1997 Modified Novikov-Veselov equation and differential geometry of surfaces *Am. Math. Soc. Transl.* 2 **179** 133–51
[65] Taimanov I A 1997 The global Weierstrass representation and its spectrum *Russ. Math. Surv.* **52** 1330–2
[66] Taimanov I A 2006 Two-dimensional Dirac operator and the theory of surfaces *Russ. Math. Surv.* **61** 79–159
[67] Taimanov I A 2011 Singular spectral curves in finite-gap integration *Russ. Math. Surv.* **66** 107
[68] Taimanov I A 2015 Blowing up solutions of the modified Novikov-Veselov equation and minimal surfaces *Theor. Math. Phys.* **182** 173–81
[69] Taimanov I A 2021 The Moutard transformation for the Davey-Stewartson II equation and its geometrical meaning *Math. Notes* **110** 754–66
[70] Taimanov I A and Tsarëv S P 2008 Blowing up solutions of the Novikov-Veselov equation *Dokl. Math., Math. Phys.* **77** 467–8
[71] Tajiri M and Arai T 1999 Growing-and-decaying mode solution to the Davey-Stewartson equation *Phys. Rev.* E **60** 2297–305

[72] Thirring W E 1958 A soluble relativistic field theory *Ann. Phys., NY* **3** 91–112
[73] Trillo S and Wabnitz S 1991 Dynamics of the nonlinear modulational instability in optical fibers *Opt. Lett.* **16** 986–8
[74] Yuen H C and Ferguson W E 1978 Relationship between Benjamin-Feir instability and recurrence in the nonlinear Schrödinger equation *Phys. Fluids* **21** 1275–8
[75] Yuen H C and Lake B 1982 Nonlinear dynamics of deep-water gravity waves *Adv. Appl. Mech.* **22** 67–229
[76] Zakharov V E and Manakov S V 1985 Construction of higher-dimensional nonlinear integrable systems and of their solutions *Funct. Anal. Appl.* **19** 89–101
[77] Zakharov V E and Mikhailov A V 1978 Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method *Sov. Phys. - JETP* **47** 1017–27
[78] Zakharov V E and Shabat A B 1974 A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering transform I *Funct. Anal. Appl.* **8** 226–35
[79] Zhang J, Guo B and Shen S 2016 Homoclinic orbits of the doubly periodic Davey-Stewartson equation *Prog. Nat. Sci.* **14** 1031–2