Optimal Control Applications and Methods

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Optimal Control Allocation for 2D Reaction-Diffusion Equations With Multiple Locally Distributed Inputs

Andrea Cristofaro

Department of Computer, Control and Management Engineering, Sapienza University of Rome, Italy

Correspondence: Andrea Cristofaro (andrea.cristofaro@uniroma1.it)

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ABSTRACT

In this article, the problem of stabilization of a 2D unstable parabolic equation with multiple distributed inputs is addressed using a spectral decomposition approach. Furthermore the underlying redundancy of the actuation arrangement is exploited and actively used by introducing a suitable control allocation architecture. In particular, two optimal allocation policies have been considered: gradient descent and linear quadratic allocation. A simulation study supports and illustrates the theoretical findings.

1 | Introduction

In recent years, the control community has shown significant interest in partial differential equations (PDEs) due to their prevalence in models of infinite-dimensional systems, particularly in areas such as robotics (e.g., haptic controllers, flexible manipulators), industrial processes (e.g., manufacturing, reactors, heat transfer plants), and biomedical engineering (e.g., tissue engineering, hyperthermia). Reaction-diffusion equations, which are a special class of second-order parabolic partial differential equations, offer a suitable framework for describing phenomena such as chemical reactions, mechanical deformations, and population dynamics [1, 2].

In the context of reaction-diffusion equations, research has explored both boundary control (see for instance [3–7] and the references therein) and distributed control (see for instance [8–11] and the reference therein). In particular, several aspects and techniques have been investigated, including for example input-to-state stability, small gain theorems, backstepping, sliding-mode control or spectral decomposition. In this article, we will indeed reveal how the latter approach is well suited to be used in combination with the control allocation framework [12, 13]. Control allocation is a general setup that allows to handle input redundancy in a systematic way, typically defining secondary control objectives based on optimization criteria [14–16]. We consider here a 2D unstable reaction-diffusion equation, with a distributed actuation arrangement. Similarly to what has been done in [10] for 1D reaction-diffusion equations, by identifying a basis of eigenfunctions, the equation can be rewritten equivalently as an infinite collection of ODEs. Furthermore, only a finite number of those ordinary equations entails an unstable dynamics and this property enables for the design of finite-dimensional stabilizing controllers. We will consider here the case of actuation arrangements that are characterized by a redundancy with respect to the number of unstable modes in the system: namely, the number of inputs is assumed to be larger than the dimension of the unstable subsystem. This property allows in turn to introduce a suitable allocation scheme that aims at exploiting the *best control configuration*, according to prescribed optimal criteria, among all the configurations that provide the same response in the unstable subsystem. In particular, two allocation strategies will be described: gradient descent allocation and linear quadratic allocation.

The article is structured as follows. The basic setup is given in Section 2, addressing also the assumptions needed and providing the description of the problem to be investigated. The main results are given in Section 3, which covers the tasks of stabilization and input allocation. A simulation study in Section 4 supports and corroborates the theoretical results. Finally, conclusions are drawn in Section 5.

2 | Problem Setup

Consider the open domain $Q = (a, b) \times (c, d)$ and the interval $\mathcal{I} = (0, +\infty)$. In the parabolic cylinder $\Omega = \overline{Q} \times \overline{\mathcal{I}}$, we are interested in the Cauchy–Dirichlet problem

$$\begin{aligned} w_t &= \mu \Delta w + \kappa w + \sum_{j=1}^{m} \chi_j(x, y) u_j(t) \\ w(t, \cdot, \cdot) &= 0 \text{ on } \partial Q \\ w(0, x, y) &= w_0(x, y) \end{aligned} \tag{1}$$

where Δ denotes the 2D Laplacian operator $\Delta := \partial_{xx} + \partial_{yy}$, $\mu, \kappa > 0$ are positive scalars, and $\chi_j(x, y)$ are the indicator functions of a given family of disjoint closed sets $\Omega_j \subset Q$ of positive measure with, that is,

$$|\Omega_j| > 0, \quad \chi_j(x, y) = \begin{cases} 1 & (x, y) \in \Omega_j \\ 0 & (x, y) \notin \Omega_j \end{cases}$$

The functions of time $u_j(t)$, $j = 1, \dots, m$, are the available control inputs. It is worth stressing that the actuation arrangement can be very diversified and heterogeneous, as the one sketched in Figure 1.

2.1 | Background Results

Consider the operator $\mathcal{A} := \mu\Delta + \kappa \mathrm{Id}$, defined over $L^2(Q)$ with domain $\mathrm{dom}(\mathcal{A}) := \{v \in H_0^1(Q)\}$. Such operator is self-adjoint and has a compact resolvent, this implying that its spectrum is countable, with real eigenvalues having finite multiplicity [17]. Accordingly, there exists a Hilbert basis $\{e_j\}_{j=1}^{+\infty} \subset L^2(Q)$ composed by eigenfuctions of $\mathcal{A}$. Under the assumption $\mu, \kappa > 0$, the spectrum of the operator $\mathcal{A}$ is guaranteed to contain a finite number $N \in \mathbb{N} \cup \{0\}$ of non-negative eigenvalues, counted with multiplicity, and an infinite number of negative eigenvalues. In particular, arranging the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ in decreasing order, one has

$$\underbrace{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N}_{\in[0,+\infty)} > \underbrace{\lambda_{N+1} \geq \lambda_{N+2} \geq \ldots}_{\in(-\infty,0)}.$$

Accordingly, the parabolic equation

$$w_t = \mathcal{A}w$$

is characterized by a finite-dimensional unstable part and an infinite-dimensional stable part. The finite-dimensional subspace spanned by the first N eigenfunctions corresponds to the unstable part of the system, whereas the eigenfunctions e_j with $j \geq N + 1$ define the stable part of the system.

2.2 | Technical Assumptions

Let us now give a few assumptions to be used next.

Assumption 1. (boundedness and regularity). The initial condition is sufficiently regular, with $w_0(x, y) \in H_0^1(Q)$. The inputs $u_j(t)$ are assumed to be belong to the space of locally bounded and continuous functions, denoted by $\mathcal{U}(0, +\infty)$. The actuation sets Ω_j are assumed to have a Lipschitz boundary.

Assumption 2. (overactuation). We assume that the number of unstable modes N is less than the number of actuation regions m, that is,

$$N < m$$

Assumption 1 guarantees that, for any choice of the inputs within the admissible set $\mathcal{U}(0, +\infty)$, system (1) admits a unique mild solution [18]. In particular, for functions $w(t, x, y)$ satisfying (1) in the mild sense, the operation of differentiation under the integral sign is allowed. The condition in Assumption 2 entails instead, restricting the focus on the unstable states only, a redundancy in the system controllability, which in turn allows to allocate the actual control inputs according to some optimal criteria.

2.3 | Problem Statement

The problem studied and solved in this letter is twofold:

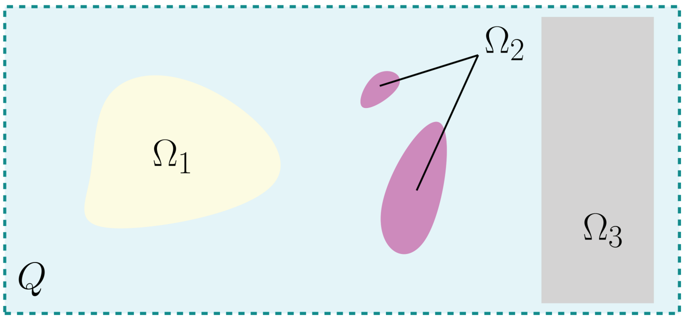

FIGURE 1 | Example of actuation arrangement with $m = 3$ sets.

- Globally stabilize the equilibrium $w(t,x,y)=0$ of Equation (1) with respect to the L^2-norm.
- Allocate the redundant control effort, hinging on Assumption 2, according to a given optimization policy.

3 | Control Design

3.1 | Spectral Decomposition

Let us now define explicitly the Hilbert basis of eigenfunctions of $\mathcal{A}$, providing an equivalent representation of (1) as the collection of infinitely many ODEs with decoupled drifts. An eigenpair $(\lambda, \varphi(x,y)) \in \mathbb{R} \times L^2(Q)$ must satisfy

$$\mathcal{A}\varphi = \mu\Delta\varphi + \kappa\varphi = \lambda\varphi$$

along with the homogeneous Dirichlet conditions on ∂Q. Observing that the function $\psi(z) = \sin(\alpha z - \beta)$ satisfies $\psi''(z) = -\alpha^2 \sin(\alpha z - \beta)$, we can easily see that $\varphi(x,y) := \gamma \sin(\alpha_1 x - \beta_1)\sin(\alpha_2 y - \beta_2)$ satisfies $\Delta\varphi = -(\alpha_1^2 + \alpha_2^2)\varphi$ for any $\gamma \neq 0$. Now, imposing the boundary conditions, we get the following expressions for the coefficients α_i, β_i :

$$\begin{aligned}
\alpha_1 = \alpha_1(r) &:= \frac{r\pi}{b-a}, \quad \beta_1 = \beta_1(r) := \frac{r\pi a}{b-a} \quad r \in \mathbb{N}\setminus\{0\} \\
\alpha_2 = \alpha_2(s) &:= \frac{s\pi}{d-c}, \quad \beta_2 = \beta_2(s) := \frac{s\pi c}{d-c} \quad s \in \mathbb{N}\setminus\{0\}
\end{aligned}$$

This shows that all eigenpairs of the operator $\mathcal{A}$ are given by $(\tilde{\lambda}_{r,s}, \varphi_{r,s}(x,y))$ with

$$\begin{aligned}
\tilde{\lambda}_{r,s} &= -\mu(\alpha_1^2(r) + \alpha_2^2(s)) + \kappa \\
\varphi_{r,s}(x,y) &= \gamma_{r,s}\sin(\alpha_1(r)x - \beta_1(r))\sin(\alpha_2(s)y - \beta_s(s))
\end{aligned}$$

for $(r,s) \in \mathbb{N}^2$ and where $\gamma_{r,s}$ are normalization constants. Observing that the coefficients $\alpha_1(r), \alpha_2(s)$ are strictly increasing with respect to r, s, for any given $\kappa > 0$ we have that only a finite number of choices $(r,s) \in \mathbb{N}$ may lead to a positive eigenvalue $\tilde{\lambda}_{r,s} > 0$, this being consistent with the property of the unstable part of the equation of being finite-dimensional. Sorting the eigenvalues in decreasing order, we can set

$$\begin{aligned}
\lambda_1 &:= \tilde{\lambda}_{1,1} \\
\lambda_2 &:= \tilde{\lambda}_{r_2,s_2} \text{ where } r_2 + s_2 = 3 \\
\lambda_3 &:= \tilde{\lambda}_{r_3,s_3} \text{ where } r_3 + s_3 = 3 \wedge 4 \\
\lambda_4 &:= \tilde{\lambda}_{r_4,s_4} \text{ where } r_4 + s_4 = 3 \wedge 4 \wedge 5 \\
&\vdots \\
\lambda_N &:= \tilde{\lambda}_{r_N,s_N} \text{ where } r_N + s_N = 3 \wedge \cdots \wedge N+1 \\
\lambda_{N+1} &:= \tilde{\lambda}_{r_{N+1},s_{N+1}} \text{ where } r_{N+1} + s_{N+1} = 3 \wedge \cdots \wedge N+2 \\
&\vdots \\
&\vdots
\end{aligned}$$

with N such that $\lambda_N \geq 0$ and $\lambda_j < 0$ for any $j \geq N+1$. Accordingly, the equation can be recast in the diagonal form

$$\Sigma_+ \begin{cases}
\dot{w}_1(t) = \lambda_1 w_1(t) + \sum_{j=1}^{m} b_{1,j}u_j(t) \\
\dot{w}_2(t) = \lambda_2 w_2(t) + \sum_{j=1}^{m} b_{2,j}u_j(t) \\
\vdots \\
\dot{w}_N(t) = \lambda_N w_N(t) + \sum_{j=1}^{m} b_{N,j}u_j(t)
\end{cases}$$

$$\Sigma_- \begin{cases}
\dot{w}_{N+1}(t) = \lambda_{N+1} w_{N+1}(t) + \sum_{j=1}^{m} b_{N+1,j}u_j(t) \\
\vdots \\
\vdots
\end{cases}$$

where, for $i = 1, 2, \ldots, N, N+1, \ldots$, one has

$$w_i(t) := \int_Q w(t,x,y)\varphi_{r_i,s_i}(x,y)dxdy \tag{2}$$

$$b_{i,j} := \int_Q \chi_j(x,y)\varphi_{r_i,s_i}(x,y)dxdy = \int_{\Omega_j} \varphi_{r_i,s_i}(x,y)dxdy \tag{3}$$

We can divide the ODEs in two groups, denoted by Σ_+ and Σ_- and indicating, respectively, the unstable and the stable dynamics. In particular, Σ_+ corresponds to the finite-dimensional system governing the dynamics of the first N states $\{w_1(t), \ldots, w_N(t)\}$.

3.2 | Stabilizing Control Law

The stabilization of the system can be carried out by focusing on the unstable subsystem Σ_+ only. Furthermore, a finite-dimensional controller can be used. To this end, let us rewrite in compact form the dynamics of Σ_+ as follows

$$\dot{w}_+ = \Lambda_+ w_+ + Bu \tag{4}$$

where we have set $w_+(t) := [w_1(t) \cdots w_N(t)]^\top$, $u(t) := [u_1(t) \cdots u_m(t)]^\top$ and

$$\Lambda_+ := \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} B := \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N,1} & b_{N,2} & \cdots & b_{N,m} \end{bmatrix}$$

Assumption 3. (stabilizability). The pair of matrices (Λ_+, B) is controllable.

It is easy to check that a sufficient condition for controllability is that the eigenvalues $\lambda_1, \ldots, \lambda_N$ are all distinct, and that for any $\ell = 1, \ldots, N$ there exists $i_0 = 1, \ldots, m$ such that $b_{\ell,i_0} \neq 0$. In the following counterexample we will show that the controllability of Σ_+ is not a trivial assumption, as there exist critical choices of system parameters and actuation sets that lead to uncontrollable unstable dynamics.

Example 1. Let us consider the domain $Q = [0,1]^2$, and the reaction diffusion-equation with $\mu = 1$, $\kappa = 30$ and $m = 4$

actuation sets. The system admits three positive eigenvalues $\lambda_1 = -2\pi^2 + 30,\ \lambda_2 = \lambda_3 = -3\pi^2 + 30$, whereas all other eigenvalues, which are of the form $\lambda_j = -n_j\pi^2 + 20$ with $n_j \geq 4$, are negative. The eigenfunctions associated to $\lambda_i, i = 1, 2, 3$, are

$$\varphi_{1,1}(x,y) = 2\sin(\pi x)\sin(\pi y)$$
$$\varphi_{2,1}(x,y) = 2\sin(2\pi x)\sin(\pi y)$$
$$\varphi_{1,2}(x,y) = 2\sin(\pi x)\sin(2\pi y)$$

Then selecting the numbers $0 < x_1 < x_2 < z_2 < x_3 < z_3 < x_4 < z_4 < 1$ and $\epsilon \in (0, \frac{1}{2})$, it is straightforward to verify that the actuation sets

$$\Omega_1 := [-x_1, x_1] \times [\epsilon, 1-\epsilon]$$
$$\Omega_2 := \{[-x_2, -z_2] \cup [z_2, x_2]\} \times [\epsilon, 1-\epsilon]$$
$$\Omega_3 := \{[-x_3, -z_3] \cup [z_3, x_3]\} \times [\epsilon, 1-\epsilon]$$
$$\Omega_4 := \{[-x_4, -z_4] \cup [z_4, x_4]\} \times [\epsilon, 1-\epsilon]$$

satisfy $\int_{\Omega_j} \varphi_{2,1}(x,y)dxdy = 0$, thus yielding $b_{2,j} = 0$ for any $j = 1, \ldots, 4$ and preventing the system from being controllable.

The previous counterexample highlights the importance of a good shape and a good placement for the actuation sets Ω_j. Now, under Assumption 3, a stabilizing control law for the unstable subsystem Σ_+ can be obtained, for instance, with a traditional assignment of poles by selecting a gain matrix $K \in \mathbb{R}^{m\times N}$ such that

$$\sigma(\Lambda_+ - BK) \subset \mathbb{C}^- \tag{5}$$

By this choice, the stability of the whole system can be inferred by the cascade arguments exploited in the following fact (see also Figure 2). For the sake of simplicity, we introduce the compact notation $b_{i,*} := [b_{i,1} \ \cdots \ b_{i,m}]$, $i \geq N+1$, to indicate the input matrices appearing in the stable subsystem.

Fact 1. The adoption of the feedback control $u = -Kw_+$ entails an exponentially stable closed-loop dynamics for the finite-dimensional subsystem, denoted from now on by Σ'_+ to avoid confusion with the open-loop dynamics, with

$$\dot{w}_+ = (\Lambda_+ - BK)w_+ \tag{6}$$

The overall system dynamics can be then seen as a countable collection of decoupled cascade interconnections of (6) with one-dimensional exponentially stable equations (with $\lambda_i < 0$ and gains $b_{i,*}K$ for $i = N+1, N+2, \ldots$). By the stability of cascade interconnections of stable systems, each element of this collection is a stable system and therefore stability of the whole dynamics is guaranteed.

Now, based on Fact 1 and recalling the isometry relationship $\iota : L^2(Q) \to \ell_2$ defined by

$$\iota(w(x,y)) = \{w_i\}_{i\in\mathbb{N}},\ w_i = \int_Q w(x,y)\varphi_{r_i,s_i}(x,y)dxdy$$

and fulfilling

$$\|w\|^2_{L_2(Q)} = \sum_{i\in\mathbb{N}} w_i^2 \tag{7}$$

the exponential decay of $w_{r,s}(t)$ implies the asymptotic convergence to zero of the L^2-norm. Let us convey the above discussion in a formal statement.

Theorem 1. *Consider the reaction-diffusion system (1), under Assumptions 1, 2, and 3. Pick a feedback gain $K \in \mathbb{R}^{m\times N}$ such that the condition (5) is satisfied. Then the finite-dimensional inputs*

$$u_j(t) := -\sum_{i=1}^{N} K_{j,i} \int_Q w(t,x,y)\varphi_{r_i,s_i}(x,y)dxdy$$

provide global exponential stabilization of (1) in the sense of the L^2-norm, that is,

$$\lim_{t\to+\infty} \|w(t,\cdot,\cdot)\|_{L^2(Q)} = 0$$

Proof. Focusing on the finite-dimensional subsystem Σ_+ in (6), let us denote by

$$\bar{\varrho} := \min_{\varrho\in\sigma(\Lambda_+ - BK)} |\mathrm{Re}(\varrho)|$$

By standard arguments from linear systems, there exists one positive constant C_0 such that

$$|w_+(t)| \leq C_0 e^{-\bar{\varrho}t}|w_+(0)|$$

For each of the states $w_i(t)$ with $i \geq N+1$, we can instead compute the following bound[1]:

$$|w_i(t)| \leq C_j e^{-|\lambda_i|t}|w_i(0)| + |b_{i,*}||K|C_0(|\lambda_i| - \bar{\varrho})^{-1}(e^{-\bar{\varrho}t} - e^{-|\lambda_i|t})|w_+(0)|$$

Now, recalling that by construction $\sum_{i=N+1}^{+\infty}|b_{i,*}|^2 \leq \sum_{j=1}^{m}|\Omega_j|^2$, by simple algebraic manipulations we obtain that $\sum_{i=1}^{+\infty}|w_i(t)|^2$ vanishes exponentially as $t \to +\infty$. Finally, invoking the isometry (7), the conclusion follows. □

The previous result delivers the recipe for designing a closed-form stabilizing controller for (1), thus addressing item (i) of Section 2.3.

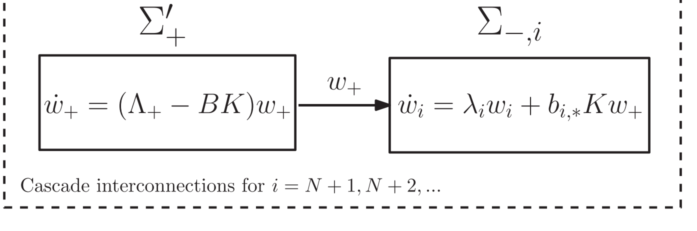

FIGURE 2 | Sketch of the collection of cascade interconnections.

3.3 | Optimal Input Allocation

Next step will be related to exploiting the overactuation of the system, hinging on Assumption 2 which results in the redundancy condition

$$p := \mathrm{rank}(B) \leq m < N \tag{8}$$

The previous rank condition implies that a full column rank matrix $B_\perp \in \mathbb{R}^{N\times q}$, with $q := N - p \geq 1$, exists such that

$$BB_\perp v = 0 \ \forall v \in \mathbb{R}^q \tag{9}$$

As a consequence, the control input

$$u = -Kw_+ + B_\perp v$$

will produce exactly the same response in the upper subsystem, no matter what the signal v actually is. On the other hand, the signal v may in principle affect the dynamics of the stable subsystems $\Sigma_{-,i}$ due to cascade effects. In fact, in general, one must expect $b_{i,\star}B_\perp v \neq 0$ for $i \geq N+1$. We will now discuss possible criteria to select this additional signal v in order to *optimize* the response of $\Sigma_{-,i}$, this being the goal described in item ii) of Section 2.3. In particular, we may design v either as a feedback (static or dynamic) from the states w_+ only, or as a function of some of the w_i too, with $i \geq N+1$. We will address two different optimization problems, namely *gradient based allocation* and *linear quadratic allocation*, covering both feedback structures.

3.3.1 | Gradient Based Allocation

Looking at the dynamics of the subsystems $\Sigma_{-,i}$, the input $u = -Kw_+ + B_\perp v$ acts as a perturbation and it would make sense to try to keep it as small as possible. Towards this goal, let us consider a cost function $\mathcal{S}(u)$, where $\mathcal{S}$ is convex, continuously differentiable and satisfies $\mathcal{S}(0) = 0$. Accordingly, by using the gradient descent principle, we can define the following dynamic allocation scheme

$$\begin{aligned} u &= -Kw_+ + B_\perp v \\ \dot{v} &= -\eta B_\perp^\top \nabla \mathcal{S}(-Kw_+ + B_\perp v) \end{aligned} \tag{10}$$

where $\eta > 0$ is a descent rate parameter. The aim of the allocation policy (10) is to make the auxiliary input v dynamically track the value that minimizes the overall control effort. In this regard let us stress that, thanks to the convexity of $\mathcal{S}(\cdot)$, the dynamics of v in (10) is stable and therefore the cascade argument used in Theorem 1 still holds for $u = -Kw_+ + B_\perp v$. It is worth noticing that, in the simple case of $\mathcal{S}(u) = |u|^2$, the allocation scheme reduces to the linear dynamics

$$\begin{aligned} u &= -Kw_+ + B_\perp v \\ \dot{v} &= 2\eta B_\perp^\top K w_+ - 2\eta B_\perp^\top B_\perp v \end{aligned} \tag{11}$$

3.3.2 | Linear Quadratic Allocation

Consider now a different problem. Pick a number $M \geq 1$ and set $\tilde{w} := [w_{N+1} \ \cdots \ w_{N+M}]^\top$. The coupled (finite-dimensional) dynamics of $(w_+, \tilde{w})$, under the control $u = -Kw_+ + B_\perp v$, is

$$\begin{bmatrix} \dot{w}_+ \\ \dot{\tilde{w}} \end{bmatrix} = \underbrace{\begin{bmatrix} \Lambda_+ - BK & 0 \\ -B_M K & \tilde{\Lambda}_M \end{bmatrix}}_{=:A_\sharp} \begin{bmatrix} w_+ \\ \tilde{w} \end{bmatrix} + \underbrace{\begin{bmatrix} 0_{N\times q} \\ B_M B_\perp \end{bmatrix}}_{=:B_\sharp} v$$

where

$$\tilde{\Lambda}_M := \begin{bmatrix} \lambda_{N+1} & 0 & \cdots & 0 \\ 0 & \lambda_{N+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N+M} \end{bmatrix} B_M := \begin{bmatrix} b_{N+1,*} \\ b_{N+2,*} \\ \vdots \\ b_{N+M,*} \end{bmatrix}$$

Pick a positive semi-definite matrix $Y = Y^\top \succcurlyeq 0$ with $Y \in \mathbb{R}^{M\times M}$ and a positive definite matrix $R = R^\top \succ 0$ with $R \in \mathbb{R}^{q\times q}$, and define the quadratic cost function

$$\mathcal{J}(v) = \int_0^{+\infty} (\tilde{w}^\top Y \tilde{w} + v^\top R v) dt \tag{12}$$

In this way, the allocation problem can be recast as a LQR problem for $\tilde{w}$ coupled with w_+, which acts as the state of a stable exosystem. To this end, let $P = P^\top \succ 0$, with $P \in \mathbb{R}^{(N+M)\times(N+M)}$, be the solution to the algebraic Riccati equation

$$PB_\sharp R^{-1} B_\sharp^\top P - PA_\sharp - A_\sharp^\top P - \mathrm{blkdg}(0_{N\times N}, Y) = 0 \tag{13}$$

Then the following result holds true.

Proposition 1. *Consider the coupled subsystem for $(w_+, \tilde{w})$ given by the N-dimensional unstable subsystem and an arbitrary—but finite—choice of M components of the stable subsystem. Let K be defined as in Theorem 1, and $B_\perp$ defined as in (9). Consider the allocation policy*

$$u = -Kw_+ + B_\perp(G_+ w_+ + \tilde{G}\tilde{w}) \tag{14}$$

where $[G_+ \ \tilde{G}] := -R^{-1}B_\sharp^\top P$, with $P = P^\top \succ 0$ solution to the ARE (13). Then the closed-loop system is exponentially stable, and the cost function $\mathcal{J}(v)$ in (12) is minimized.

Proof. By construction, that is, due to $BB_\perp = 0$, the control law in (14) preserves the closed-loop dynamics (6) for w_+, which is exponentially stable thanks to the choice of the gain K. Moreover, it is well known that the positive-definite solution to the ARE is stabilizing for the ensuing control system [19, chapt. 8.4], and therefore the dynamics of the extended system $(w_+, \tilde{w})$ is still exponentially stable, with a feedback gain $[G_+ \ \tilde{G}]$ guaranteed to be optimal for the cost function $\mathcal{J}(v)$. To conclude, it is enough to invoke a cascade argument similar to the one used in the proof of Theorem 1, applied to the cascade interconnections of $(w_+, \tilde{w})$ with each of the subsystems $\Sigma_{-,i}$, for $i \geq N+M+1$. □

3.3.3 | Possible Extensions

In the previous subsections, we have illustrated two alternative allocation policies based on optimization criteria. However, it is worth stressing that other policies can be adopted as well, for example to deal with:

- Saturation constraints; the presence of saturation constraints [10], in the case of redundant actuation sets $m < N$, can also be dealt with via control allocation methods, for example using the redistributed pseudoinverse technique [13].

- Fault-tolerant control; the redundancy can be exploited as a valuable asset for resilience of the control action. For instance, the methods proposed in [20] can be readily implemented also in (4), with the aim of preventing possible actuation faults from hindering the stabilization of the unstable subsystem.
- Set-point regulation; the proposed machinery can be easily extended to the case where the nominal controller, instead of being a simple stabilizing feedback $u_0 = -Kw_+$, includes also a feedforward term depending on a given reference signal $\psi \in \mathbb{R}^{n_\psi}$, that is,

$$u_0 = -Kw_+ + K_\psi \psi$$

for a suitable gain $K_\psi \in \mathbb{R}^{m \times n_\psi}$.

4 | Simulations

Let us consider again the system described in Example 1, with coefficients $\mu = 1$, $\kappa = 30$ and $m = 4$ actuation sets. Let these

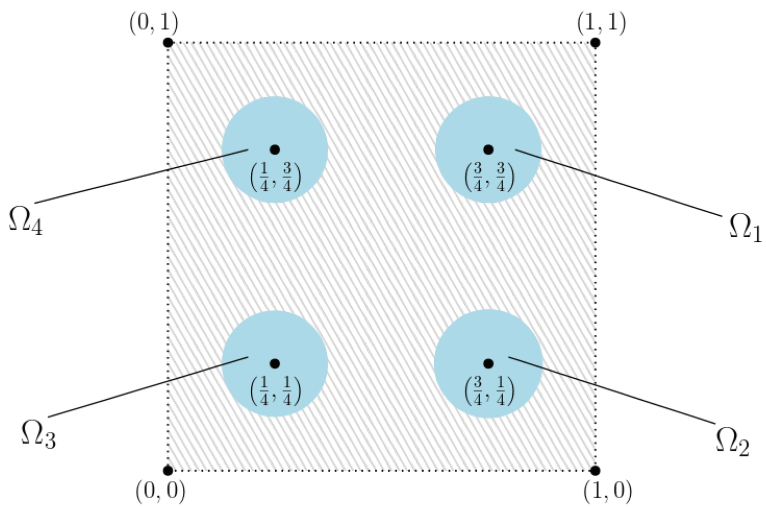

FIGURE 3 | Actuation arrangement for the considered example: the dashed region is unactuated, while the cyan disks indicate the four actuation regions Ω_j, $j = 1, \dots, 4$.

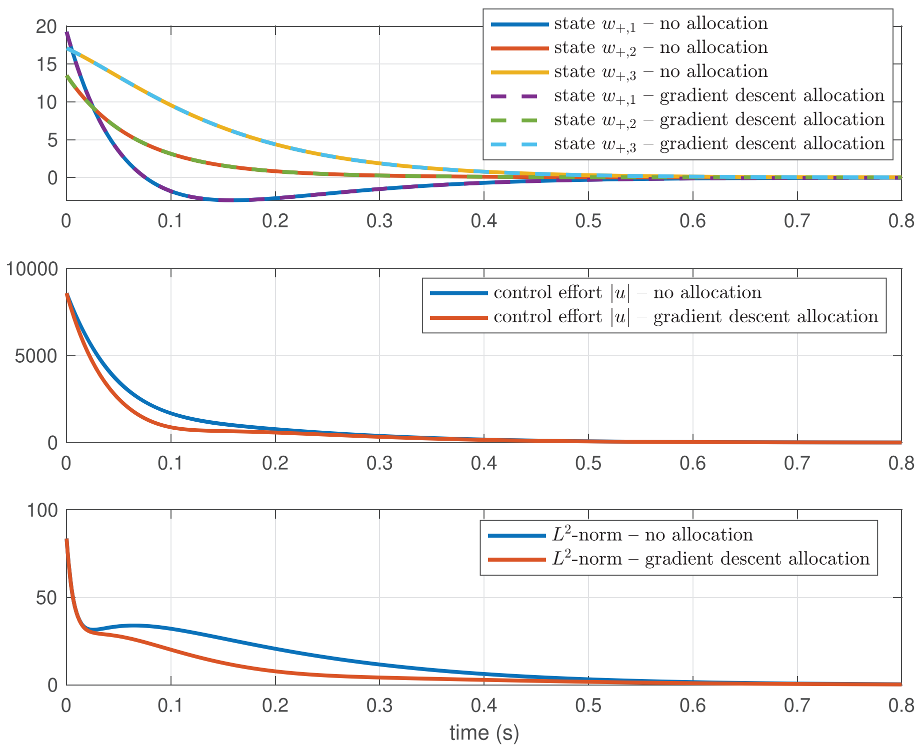

FIGURE 4 | Allocation using gradient descent policy.

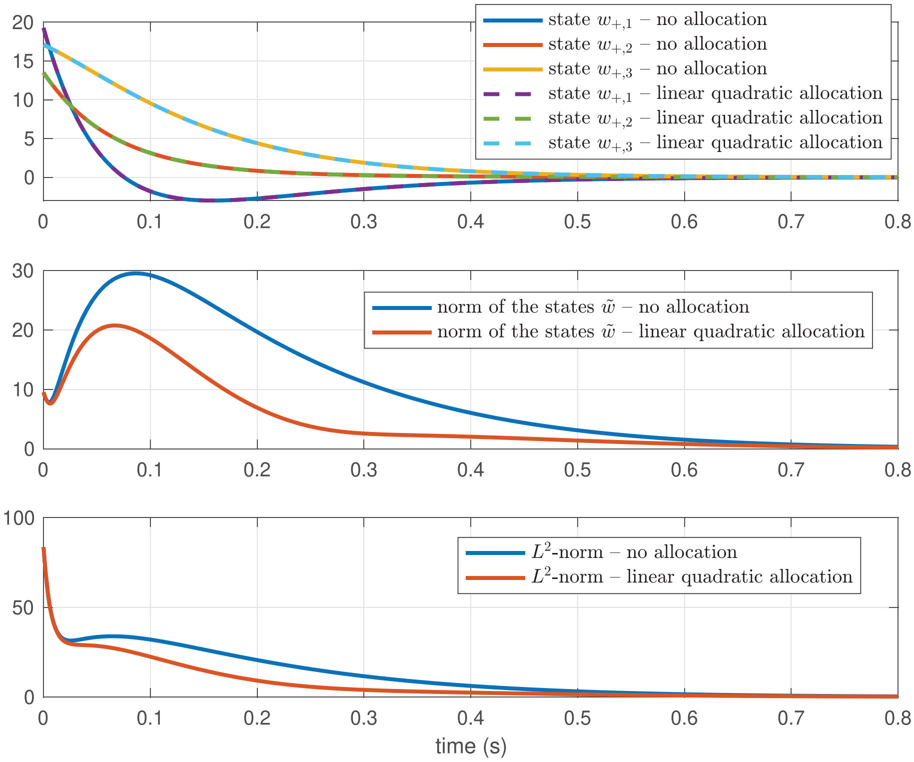

FIGURE 5 | Allocation using linear quadratic optimization policy.

sets be defined by four disks located at different sectors in $Q = [0, 1]^2$:

$$\Omega_1 := \{(x, y) : \sqrt{(x - 3/4)^2 + (y - 3/4)^2} \leq 1/8\}$$
$$\Omega_2 := \{(x, y) : \sqrt{(x - 3/4)^2 + (y - 1/4)^2} \leq 1/8\}$$
$$\Omega_3 := \{(x, y) : \sqrt{(x - 1/4)^2 + (y - 1/4)^2} \leq 1/8\}$$
$$\Omega_4 := \{(x, y) : \sqrt{(x - 1/4)^2 + (y - 3/4)^2} \leq 1/8\}$$

The actuation arrangement is also illustrated in Figure 3.

Accordingly, the matrix B satisfies $\text{rank}(B) = 3$ and reads as

$$B = \begin{bmatrix} 0.0472 & 0.0472 & 0.0472 & 0.0472 \\ -0.0629 & -0.0629 & 0.0629 & 0.0629 \\ -0.0629 & 0.0629 & 0.0629 & -0.0629 \end{bmatrix}$$

so that the stabilizability Assumption 3 is fulfilled. A quite aggressive stabilizing controller $u = -Kw_+$ has been selected, with K such that

$$\sigma(\Lambda_+ - BK) = \{-15.4735, -9.2134, -8.3912\}$$

thus resulting in a fast decay of the states w_+. Both optimal control allocation schemes have been considered. In particular, Figure 4 shows the results obtained with the gradient based allocation policy, using the cost function $S(u) = |u|^2$ and the rate coefficient $\eta = 30$, which lead to a dynamic allocation in the form (11). The top plot in Figure 4 confirms that the allocation procedure delivers signals that are completely invisible for the dynamics of w_+. The middle and bottom plots highlight instead the advantages of the allocation policy (11) in lowering the control effort and, as a byproduct, also in reducing the L^2-norm of the full system. Conversely, the results obtained with the linear quadratic allocation policy are collected in Figure 5, with $M = 4$ additional states $\tilde{w}$ and weighting matrices selected as

$$Y = 10^4 I_{4\times 4}, \quad R = 1$$

This choice of weights reflects a priority in the minimization of the states $\tilde{w}$ over the minimization of the control effort. Accordingly, the control input is allocated by (14), with P being the stabilizing solution of the ARE (13). The top plot in Figure 5 shows once again the invisibility of the allocation scheme with respect to the dynamics of w_+. The middle plot illustrate the efficacy of the allocation policy in shrinking the size of the states $\tilde{w}$, consistently with the minimization of the cost index $\mathcal{J}(u)$ defined as in (12). Finally, the bottom plot reports the evolution of the L^2-norm of the complete system, which is characterized by a moderate reduction when the allocation scheme is implemented.

5 | Conclusions

The stabilization of a 2D unstable parabolic equation with redundant distributed inputs has been considered and tackled based on spectral decomposition using a suitable Hilbert basis made by eigenfunctions. Borrowing the tools of the well-established control allocation framework, the benefits of the underlying redundancy of the actuation arrangement have been exploited, following two possible optimization pathways. The first policy, *gradient descent allocation* aims at lowering the overall control effort, whereas the second policy, *linear quadratic allocation*, guarantees the minimization of a quadratic cost function involving both the control input and the state of a finite-dimensional subsystem. The efficacy of the two considered allocation policies has been showcased though a numerical simulation study.

The proposed architecture could be further generalized to 3D parabolic equations, including also additional allocation criteria such as those reported at the end of Section 3.3. An interesting future extension could also be devoted to merging the proposed allocation framework with more general synthesis algorithms for finite-dimensional controllers like, for example, the one recently proposed in [21].

Author Contributions

Andrea Cristofaro: idea, methodology, simulation, writing.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Endnotes

[1] It is assumed here $|\lambda_j| \neq \varrho$. In the special case of equality among the two values, a slightly different bound can be inferred.

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