Wave propagation phenomena in nonlinear elastic metamaterials

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Abstract. The present paper provides a general method to deal with nonlinear integro-differential equations, based on statistical linearization and Fredholm's approach. In this context, the elastic metamaterial is characterised by long-range non-local interactions besides a nonlinear short-range constitutive relationship. Results are analytically obtained and unveil the birth of unconventional propagation.

Keywords: Elastic metamaterials, Nonlinearity, Long-range interaction, Wave propagation, Nonlocality.

1 Introduction

This work is aimed at presenting a general approach to nonlinear and long-range constitutive elastic relationships, based on statistical linearization and Fredholm's equation that enable the investigation of the dispersion relationship. Eventually, a simple example is introduced to show a first digression about the potential application of the method.

Metamaterials enjoy widespread attention due to the unexpected results they produce in many applications. In electromagnetics, metamaterials are frequently related to anomalous dissipation and diffraction properties of electromagnetic media that lead to negative group velocity, light stopping and fast light [1]. Superluminal propagation has been observed through an acoustic experimental setup [2-3]. In mechanics, metamaterials change the connectivity scheme of a structure and induce micropolar, higher-gradient and nonlocal elasticity. In this context, metamaterials are thought as conventional elastic materials equipped with long-range interactions, the source of integral contributions and nonlinear constitutive relationships. The effect of these long-range interactions has been investigated in [4-7], but never tackling the nonlinearity. Long-range interactions change the topology of the connections and represent a breakthrough in the conventional concept of particle-particle interaction between closest neighbours, which leads to classical wave propagation. When the connection is extended either to one-to-all particles or to all-to-all particles, the introduced modification becomes source of a more sophisticated propagation behaviour. Similar effects have been noticed only in quantum physics [8-9], in Vlasov's theory [10]. Another remarkable example is in acoustics, where long-range electrical interactions can control the acoustic fields and well-known examples are plasmas and charged gases.

In any case, the relation between nonlocality and wave propagation has not been intensively developed, and unconventional propagating phenomena are not as common as those appearing in electromagnetic applications.
Statistical linearization is a known procedure [11-13], often applied to random and complex systems [14]. An example is the analysis of a catenary anchor leg mooring (CALM) systems [15]. The chance to obtain analytical solutions is achieved by the definition of a specific kernel for the long-range interactions: indeed, they are modelled as elastic connections between far particles; however, the range of interaction is limited by a rectangular window, as in [7], but here analysed for its one-dimensional counterpart. Eventually, the obtained parametric problem is treated as a Fredholm's equation [16] and the dispersion relationship is calculated accordingly. Remarkable insights can be derived: i) the applied random force affects the propagation characteristics, namely the background noise modifies the phase and group velocity of the waveguide, ii) after stochastic linearization, the system shows periodic variation of the elastic parameters along the waveguide axis, iii) the nonlinear properties generates the wavenumbers coupling.

2 Prototype equation and statistical linearization

A mathematical model, based on the nonlocal elasticity theory of Eringen, is considered. For a three-dimensional, continuous, unbounded medium, equipped with long-range interactions, the Navier-Cauchy equation of motion becomes:

$$\rho \boldsymbol{u}_{tt}(\boldsymbol{x},t) + \frac{E}{2(1+\nu)}\left[\nabla^2 \boldsymbol{u}(\boldsymbol{x},t) - \frac{1}{1-2\nu}\nabla\big(\nabla \cdot \boldsymbol{u}(\boldsymbol{x},t)\big)\right] + \boldsymbol{g}(\boldsymbol{u}) + \int_{\boldsymbol{\xi}\in \boldsymbol{R}^3} \boldsymbol{F}(\boldsymbol{x},\boldsymbol{\xi})\, d\boldsymbol{V} = \boldsymbol{f}(\boldsymbol{x},t) \quad (1)$$

The long-range interaction appears as an integral term. It represents the summation of the long-range interaction forces, exerted on the particle originally at $\boldsymbol{x}$, due to all the particles at $\boldsymbol{\xi}$. $\boldsymbol{g}(x)$ resembles the possible sources of local nonlinearities and has differential nature.
Nonlinear equations can be attacked by the statistical linearization-SL approach [11,12]. Equation (1) can be written as:

$$\boldsymbol{L}(\boldsymbol{u}) + \boldsymbol{g}(\boldsymbol{u}) = \boldsymbol{f}(t) \quad (2)$$

where $\boldsymbol{L}$ and $\boldsymbol{g}$ are a linear integro-differential and a nonlinear differential operator, respectively, $\boldsymbol{u}(\boldsymbol{x},t)$ is the displacement of the elastic system, $\boldsymbol{f}(t)$ a random external force. The SL approach replaces the nonlinear term $\boldsymbol{g}(\boldsymbol{u})$ term with an equivalent linear operator $\boldsymbol{L_{eq}}$, leading to:

$$\boldsymbol{L}(\boldsymbol{u}) + \boldsymbol{L_{eq}}(\boldsymbol{u},\boldsymbol{p}) = \boldsymbol{f}(t). \quad (3)$$

The vector $\boldsymbol{p}$ is a set of parameters that can be suitably selected to make equation (3) as close as possible to equation (2). A direct comparison between (2) and (3) produces the *error equation*:

$$\boldsymbol{e}(\boldsymbol{u},\boldsymbol{p}) = \boldsymbol{L_{eq}}(\boldsymbol{u},\boldsymbol{p}) - \boldsymbol{g}(\boldsymbol{u}) \quad (4)$$

The SL requires the minimization of the mean-square $E\{\cdot\}$ of the *error equation* in terms of the parameters $\boldsymbol{p}$:

$$\frac{\partial E\{\boldsymbol{e}^T(\boldsymbol{p})\boldsymbol{e}(\boldsymbol{p})\}}{\partial \boldsymbol{p}} = 0. \tag{5}$$

The solution of this last equation determines the optimal vector $\boldsymbol{p}^*$and one can analyse the equivalent linear integro-differential equation $\boldsymbol{L}(\boldsymbol{u}) + \boldsymbol{L_{eq}}(\boldsymbol{u}, \boldsymbol{p}^*) = \boldsymbol{f}(t)$, instead of the nonlinear integro-differential equation (2).

3 Statistical linearization of the Navier-Cauchy equation with long-range forces

The analysis in the previous section leads to equation (5) the form of which is specifically investigated here under some simplified assumptions.
The equation of motion, for a one-dimensional system, with kernel $H(x-\xi)$ satisfying the action-reaction principle, i.e. $H(x-\xi) = -H(\xi - x)$, and decaying with the distance, namely $\lim_{|x-\xi|\to+\infty} H(x-\xi) = 0$, is of the form:

$$\rho u_{tt}(x,t) + E_0 u_{xx}(x,t) - 3E_1 u_x^2(x,t) u_{xx}(x,t) - \int \tilde{k}[u(x,t) - u(\xi,t)]H(x-\xi)d\xi = f(t)\delta(x) \tag{6}$$

meaning:

$$\boldsymbol{L}(\boldsymbol{u}) \equiv \rho u_{tt}(x,t) + E_0 u_{xx}(x,t) - \int \tilde{k}[u(x,t) - u(\xi,t)]H(x-\xi)d\xi \tag{7}$$

$$\boldsymbol{g}(\boldsymbol{u}) \equiv -3E_1 u_x^2(x,t) u_{xx}(x,t) \tag{8}$$

where $*$ indicates the convolution operation, and we assume:

$$\boldsymbol{L_{eq}}(\boldsymbol{p}) \equiv p u_{xx}(x,t) \tag{9}$$

$\tilde{k}$ resembles the stiffness modulation of the elastic connections. About the form of the kernel $H(x)$, it is not necessary at the moment to introduce any further specification.
The nonlinear term $3E_1 u_x^2(x,t) u_{xx}(x,t)$ is derived considering a nonlinear stress-strain relationship:

$$\sigma(\varepsilon) = E_0\varepsilon + E_1\varepsilon^3 \tag{10}$$

It is apparent then that the error equation is:

$$\boldsymbol{e}(\boldsymbol{p}) \equiv p u_{xx}(x,t) - 3E_1 u_x^2(x,t) u_{xx}(x,t) \tag{11}$$

and it is immediate to evaluate the parameter p through equation (5), which leads to:

$$p = 3E_1 \frac{E\{u_x^2 u_{xx}^2\}}{E\{u_{xx}^2\}} \tag{12}$$

The complete characterization of the parameter p requires the definition of the coefficients $E\{u_x^2 u_{xx}^2\}$ and $E\{u_{xx}^2\}$, which becomes simpler when it is reasonable to assume those variables equipped with a Gauss-like type of probability density function. For this reason, we limit our attention to the case f is a random force with flat power spectral density S_F and a Gauss-like distribution. Additionally, it is assumed to be a point force applied at the origin of the x axis.
A modal expansion of the solution helps in finding simpler results. Without loss of generality we assume $u(x,t) = \phi(x)q(t)$, including only one single mode. Under these conditions, equation (12) provides $p = 3E_1 {\phi'}^2(x)\sigma_q^2$.
Therefore, equation (3) assumes the form:

$$\rho\phi(x)\ddot{q}(t) - \left[E_0 + 3E_1\phi'^2(x)\sigma_q^2\right]\phi''(x)q(t) - q(t)\int \tilde{k}[\phi(x) - \phi(\xi)]H(x-\xi)d\xi = f(t)\delta(x). \quad (13)$$

To evaluate the coefficient σ_q^2, the following expression can be considered [17]:

$$\sigma_u^2(x) = \phi^2(x)\sigma_q^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_u\left(x,\omega,\sigma_q^2\right)d\omega = \frac{S_F}{2\pi}\int_{-\infty}^{+\infty}\left|FRF\left(x,\omega,\sigma_q^2\right)\right|^2 d\omega \quad (14)$$

with σ_q^2 the variance of $q(t)$, S_u the power spectral density of u, $FRF(x,\omega)$ the complex frequency response of the system at x when the input force is at $x = 0$. The frequency response can be easily obtained by equation (13). Eventually, equation (14) appears to be an equation in terms of the variance σ_q^2.

4 Parametric problem and Fredholm's equation

Once equation (14) is solved, the value σ_q^2 can be substituted into equation (13). Replacing in it $u(x,t) = \phi(x)q(t)$, and for a rectangular window on length a, around the origin, we obtain:

$$\rho\ddot{u}(x,t) - \left[E_0 + 3E_1\phi'^2(x)\sigma_q^2\right]u''(x,t) - 2\tilde{k}au(x,t) + H(x) * u(x,t) = f(t)\delta(x) \quad (15)$$

This is a linear integro-differential equation. However, the term $3E_1\phi'^2(x)\sigma_q^2 u''(x,t)$ exhibits a parametric dependence on x. This can be regarded as an extra-stiffness $E_p = 3E_1\phi'^2(x)\sigma_q^2$ that superimposes to E_0. For example, if $\phi(x)$ is the first mode, then $\phi'^2(x)$ is a square of a sinusoid, and it perturbates the constant stiffness E_0, as in Fig. 1.

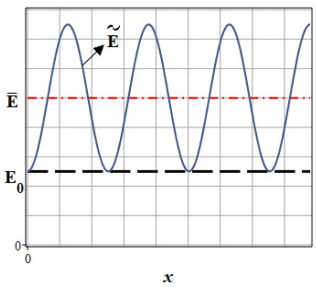

Fig. 1 – Effective stiffness $\tilde{E}$

We present here a method based on the Fredholm's integral equation to approach equation (15). Applying the Fourier transform both in space and time, we obtain:

$$-\rho\omega^2\hat{u} + k^2E_0\hat{u} + 3E_1\hat{\phi}'^2\sigma_q^2 * \left(k^2\hat{u}\right) - 2\tilde{k}a\hat{u} + \tilde{k}\hat{u}\hat{H} = \hat{F} \tag{16}$$

where $\hat{H} = 2\tilde{k}sin(ka)/k$ for the chosen window. For any $x \neq 0$ we have the homogeneous integral equation:

$$-\rho\omega^2\hat{u} + k^2E_0\hat{u} + 3E_1\hat{\phi}'^2\sigma_q^2 * \left(k^2\hat{u}\right) - 2\tilde{k}a\hat{u} + \tilde{k}\hat{u}\hat{H} = 0 \tag{17}$$

whose explicit form is:

$$3E_1\sigma_q^2\int_{-\infty}^{+\infty}\hat{\phi}'^2(k-k')k'^2\hat{u}(k')dk' = \left[\rho\omega^2 - k^2E_0 + 2\tilde{k}a - \tilde{k}\hat{H}\right]\hat{u}. \tag{18}$$

The discretization of the previous equation leads to:

$$\sum_j 3E_1\sigma_q^2\hat{\phi}'^2\left(k_i - k_j\right)k_j^2\hat{u}\left(k_j\right)\Delta k - \left[\rho\omega^2 - k_i^2E_0 + 2\tilde{k}a - \tilde{k}\hat{H}\right]\hat{u}(k_i) = 0 \tag{19}$$

namely,

$\boldsymbol{Au} = \boldsymbol{0}$, where $\boldsymbol{u} = \begin{Bmatrix} u(k_1) \\ u(k_2) \\ \vdots \\ u(k_n) \end{Bmatrix}$, with

$$[\boldsymbol{A}]_{ij} = 3E_1\sigma_q^2\hat{\phi}'^2\left(k_i - k_j\right)k_j^2\Delta k - \left(\rho\omega^2 - k_j^2E_0 + 2\tilde{k}a - \tilde{k}\hat{H}\right)\delta_{ij} \tag{20}$$

We are interested in non-trivial and not identically to zero solutions. This technically happens if the *Fredholm determinant* associated to $\boldsymbol{A}$ vanishes, from which we can derive the dispersion relationship.
The condition for the determinant to vanish does not lead to equations independent for each k_i and the wavenumbers mix up. Given a value for the frequency, the vanishing determinant condition does not imply each k_i is determined. The determinant expression is:

$$\boldsymbol{detA} = \boldsymbol{f}\left(k_1, k_2, .., k_n, \omega, \sigma_q^2, \hat{H}\right) = 0 \tag{21}$$

This means that, assigned a value for ω, σ_q^2 and $\widehat{H}$, the values for the wavenumbers are not uniquely determined. Rather, one can select arbitrarily $N-1$ wavenumber values and determine consequently the value for the N-th remaining wavenumber. Since, at any frequency, the values for the N-1 wavenumbers can be arbitrarily assigned, the N-th can range in a set, describing a segment at each frequency.
The dispersion relationship depends on: i) the presence of $\widehat{H}$ due to long-range interactions, ii) the presence of σ_q^2 as an effect of the background noise in the structure.
From equation (21) we can learn what follows: firstly it is not possible to excite individual and independent wavenumbers with a given frequency, and this is reminiscent of nonlinear systems (e.g super-harmonics and sub-harmonics); secondly, the level of the random excitation $\boldsymbol{f}$ affects the propagation characteristics since it alters the value of the coefficient by σ_q^2. Accordingly, Fig. 2 shows the effect of this coefficient on the parametric stiffness E_p.

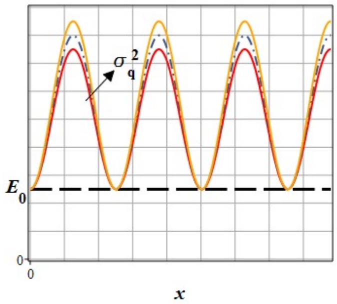

Fig. 2 – Effect of σ_q^2on $\tilde{E}$

A last consideration can be extrapolated by both Fig. 1 and Fig. 2. The effective stiffness $\tilde{E}$, taking into account both the standard elastic modulus E_0 and its modulation E_p due to the nonlinearities, is a periodic function. This suggests a localization phenomenon, namely the Anderson localization, related to stop band, low pass and high pass behaviours.

5 Homogenization of the stiffness and parametric dispersion relationship

To complete the investigation, we report here the analysis of the dispersion relationship associated to equation (15). The aim is to highlight the effect of the random external force on the propagation characteristics. It is then reasonable to consider the homogenised value of the effective stiffness $\tilde{E}$, i.e. its average value $\bar{E}$, deduced by Fig. 1. Using the same type of long-range interactions, the non-dimensional dispersion relationship is:

$$\Omega^2 = K^2 - 2\chi(1 - sincK) \qquad (22)$$

where $\Omega = \omega a\sqrt{\rho/\bar{E}}$ is the non-dimensional frequency, $K = ka$ the non-dimensional wavenumber and $\chi = \tilde{k}a^3/\bar{E}$ is the non-dimensional parameter comparing the effect of the long-range interactions and of the nonlinearities. The advantage of such type of formulation stands in the chance to discuss the dynamic behaviour in terms of the parameter χ only. Fig. 3 shows the trend of the dispersion relationship varying with χ, compared with the standard one. All curves start with a very high slope to converge, for high values of the non-dimensional wavenumber, to the conventional D'Alembert propagation. Since the derivative of the dispersion relationship is the group velocity, this implies that the steeper the slope, the higher the group velocity, disclosing super-fast propagation of the envelope, till the limit of superluminal propagation when the curve has a vertical tangent, i.e. the case of σ_{q4}^2. Moreover, since $\bar{E}$ is a function of the random force, through the coefficient σ_q^2, it can be discussed also in terms of σ_q^2: the higher the value of σ_q^2, the higher the homogenised stiffness $\bar{E}$ and the higher the value of σ_q^2, the closer is the behaviour to the standard propagation. Eventually, the smaller is σ_q^2, the higher is the group velocity.

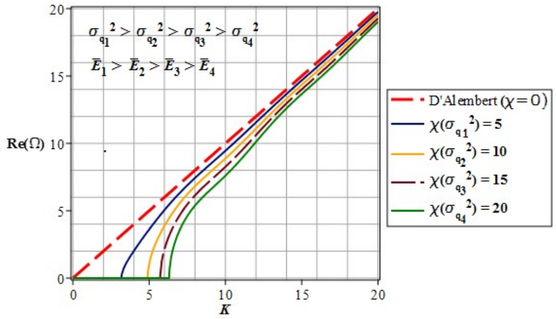

Fig. 3 – Non-dimensional dispersion relationship

6 Conclusions

In this paper, the case of a one-dimensional waveguide, in which the presence of long-range interactions together with the nonlinear nature of the constitutive law, is investigated. The long-range interactions are modelled as elastic connections occurring between distant particles but only within a delimited region of length a. The nonlinear constitutive law accounts not only for the first order of the strain but it includes also the third. The resulting equation of motion, approached with the statistical linearization process, has space-dependent coefficients. To analyse the obtained parametric problem, the Fredholm's approach is applied and by solving the associated determinant the dispersion relationship is found. It depends on the nature of the long-range interaction, through the term $\hat{H}$, and of the nonlinearity, related to the coefficient σ_q^2. This brings to light important insights: i) the level of the random excitation directly affects the propagation characteristics, ii) as recurring of nonlinear systems, it is not possible with a single frequency to excite independent and individual wavenumbers. Moreover, the shape of the equivalent stiffness unveils Anderson localization and thus, possible wave-stopping phenomena.

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