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New Insights on Portfolio Insurance Strategies for Financial and Longevity Risk Management

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Abstract

Portfolio insurance strategies, designed to safeguard a minimum level of wealth while participating in market gains, have become a linchpin of the asset management industry. The present research work focuses on various portfolio insurance strategies, such as *constant proportion portfolio insurance* (CPPI) strategy, *time–invariant portfolio protection* (TIPP) strategy, and *proportional portfolio insurance* (PPI) strategy, offering insights into their advantages and addressing their critical issues, including *cash–locked positions* and *gap risk*. In particular, the study explores modifications to the latter baseline strategies, focusing on improving market participation and capital protection, in a wide range of applications spanning from structured investment products with capital protection to pension funds. Notably, a new variant of the TIPP strategy, the *TIPP with guaranteed minimum equity exposure* (G–TIPP), is introduced, demonstrating its effectiveness in structured investment products with capital protection. The thesis also addresses the shortcomings of existing PI strategies in managing the accumulation phase of defined contribution (DC) pension funds, proposing an innovative approach that simultaneously protects the fund's members against market and *longevity risks*. Finally, the study investigates the determination of optimal exogenous parameters of a modified version of the PPI strategy in markets with frictions, such as jumps in risky asset dynamics. This new PPI strategy is able to immunize the investor from gap risk $\mathbb{P}$–a.s. over the entire investment horizon. The present dissertation comprises five chapters, each contributing to the understanding and advancements of PI strategies, offering valuable insights for academics and practitioners.

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Introduction and Overview

Portfolio insurance (PI) strategies are designed to guarantee a minimum level of wealth over a fixed time horizon while simultaneously participating in the potential gains of a reference portfolio. Despite their designation, PI strategies are not insurance contracts in which the investor pays a risk transfer premium to an insurer to limit losses from unfavourable market conditions. On the contrary, they are dynamic allocation strategies that can be pigeonholed into two main categories: *option-based portfolio insurance* (OBPI) and *constant proportion portfolio insurance* (CPPI). The OBPI strategy was pioneered after the seminal article by Black and Scholes [1973] while Leland and Rubinstein [1976] suggested using options for portfolio hedging. After a decade, Perold [1986] and Black and Jones [1988] developed the CPPI strategy. The latter is a simple dynamic allocation mechanism that splits the investor's wealth between two types of assets: a risky asset, such as a market index, and a reserve asset, such as a bond or a bank account, according to the following pre-established investment rule. First, the investor sets the *floor*, namely the present value of the guarantee, and represents the capital to be protected for each instant of the management period. Then, he/she invests an amount in the risky asset proportional to the *cushion*, defined as the difference between the current portfolio value and the floor, provided that this difference is positive. The proportionality factor is a positive constant, known as a *multiplier*, set at the beginning of the management period according to the investor's risk tolerance. Finally, since the CPPI strategy is self-financing, the remaining wealth is invested in the reserve asset.

During the years, the CPPI strategy has been the subject of several generalizations. The first is the so-called *time-invariant portfolio protection* conceived by Estep and Kritzman [1988]. The only difference between the CPPI and the TIPP strategy lies in the floor's rebalancing mechanism. The TIPP floor is defined as the maximum between the standard CPPI floor and the maximum of the historical TIPP portfolio values. Thanks to this particular floor's definition, the TIPP can lock in all past gains, and, unlike the CPPI, none of them are lost every time a market decline occurs. For this reason, the TIPP is perceived as an improved version of the standard CPPI strategy, providing better capital protection. The second generalization is the so-called *proportional portfolio insurance* (PPI) strategy in which the multiplier is no longer constant, but is set to be time-varying, to better adapt the strategy's exposure to market fluctuations, see, e.g. Lee et al. [2008], Hamidi et al. [2009], Ameur and Prigent [2014], Hamidi et al. [2014] and references therein.

Two crucial problems concerning PI strategies are the *cash-locked* positions and *gap risk*. Assuming that the dynamics of the underlying risky asset evolve according to a geometric Brownian motion and the rebalancing of the strategies occurs continuously over time, then the portfolio protection is efficient $\mathbb{P}$-a.s., that is, the strategies are always able to attain the guaranteed amount at maturity, see, e.g., Balder et al. [2009]. The only drawback within this framework is given by cash-locked positions, i.e. situations where the portfolio value hits the floor. According to the PI investment rule, from that moment on, the wealth is fully invested in the reserve asset until maturity, impeding obtaining equity market participation. As argued in Balder et al. [2009] and Cont and Tankov [2009], if we consider more realistic cases either allowing for discontinuities in the risky asset dynamics or trading restrictions, PI strategies are affected by gap risk. The latter can be considered as a particular case of the cash-locked position. A gap occurs when the portfolio manager cannot rebalance the strategy, due to a sudden drawdown in the market, causing the portfolio value to fall below the floor. The gap is the difference between the floor and the wealth. Thus, if a gap occurs, the CPPI portfolio cannot deliver the promised capital protection and equity market participation if the reference risky asset recovers. Part of the present thesis is focused on proposing modifications to the baseline versions of PI strategies that can eliminate or at least mitigate the effect of gap risk or cash-locked positions.

Despite such drawbacks, PI strategies gained momentum in the late eighties, after the collapse of the *Dow Jones Industrial Average* of the New York Stock Exchange and the *FT* 30 of the London Stock Exchange caused pension funds to withdraw. In particular, some authors, including Davis [2003], noted ex-post that the presence of PI strategies would convince investors not to leave the market, guaranteeing them the opportunity to benefit from the market upturn, an event that occurred just a couple of years later. More recently, after the financial crisis of 2008 and the crisis in 2020 due to COVID-19, we witnessed a substantial recovery of the equity market worldwide. However, these strong turbulences destroyed the trust of the market participants. Many investors in the US, Europe and Asia are increasingly interested in PI strategies due to their ability to provide capital protection and equity markets in case of favourable market conditions. Nowadays, such strategies represent a linchpin of the asset management industry, and their application area ranges from structured investment products with capital protection to pension fund management.

Structured investment products with capital protection guarantee a predetermined amount of initially invested capital plus participation in the stock market through options. More precisely, to guarantee such a payoff at maturity, the issuer has to invest a certain percentage of the initial wealth, the so-called *protection level*, in bonds and the remaining part, i.e. the *risk budget*, in plain vanilla European call options written on a market index. The number of call options the issuer can buy with the available risk budget is called the *participation rate*. Investors find these products attractive when they offer a high protection level and participation rate. However, the latter features are strongly correlated. Given an initial level of wealth, the only way to increase the protection level is to reduce the participation rate and vice versa. Such a situation worsens in a market characterized by low-interest rates and high volatility. The low interest rate level decreases the risk budget, and high market volatility increases the price of European call options, substantially reducing the participation rate. For this reason, some authors, including Albeverio et al. [2013, 2017], substituted plain vanilla European call options written on the market index with options written on PI strategies to keep a high participation rate regardless of the market conditions. However, as argued by Albeverio et al. [2013], the major challenge in embedding PI strategies in a structured product is cash-locked positions and gap risk. Indeed, after these events, the option written on the

PI strategy will end up out of the money $\mathbb{P}$-a.s., precluding the possibility of obtaining equity market participation. To address the issue, Di Persio et al. [2020] modify the CPPI allocation mechanism by including a further threshold, the guaranteed minimum equity exposure (GMEE). Such a new version of CPPI with GMEE is labeled G-CPPI. Specifically, the exposure of this new G-CPPI strategy is the maximum between the standard CPPI exposure and a predetermined percentage (the GMEE) of the portfolio value. Thanks to such a modification, such a predetermined percentage of the wealth remains invested in the risky asset even after a gap event or a cash-locked position, guaranteeing to benefit from market participation and thus making CPPI suitable for inclusion in a structured investment product with capital protection. The authors numerically show that options on G-CPPI are cheaper than options written on the stock, meaning that, *ceteris paribus*, the former increases the participation rate of structured investment products regardless of market conditions. However, what is missing in the current literature is a critical comparison between options written on G-CPPI and options written on other types of PI strategies that can be endowed with the GMEE, such as the TIPP, in order to check whether it is possible to improve further the attractiveness of structured investment products with capital protection.

A second application area of PI strategies is the accumulation phase of defined contribution (DC) pension funds. In DC pension funds, the contributions payable by the workers are a pre-specified percentage of their labour income. The amount of pension instalments after retirement depends on the size of the accumulated contributions and the fund's investment performance, meaning that the fund's member faces investment risk from market fluctuations. Despite such drawbacks, DC funds have become increasingly popular over the last decades due to the financial unsustainability of the defined benefits (DB) pension funds, especially in the presence of an ageing population. Therefore, Battocchio and Menoncin [2004] and Gao [2008] studied optimal portfolio selection problems, such as the DC fund delivering satisfactory benefits at the retirement date. However, such investment solutions cannot provide downside protections, exposing the fund members to the risk of receiving insufficient benefits after retirement. To overcome such drawbacks, Boulier et al. [2001], Deelstra et al. [2003], Guan and Liang [2014], Chen et al. [2017] and Tang et al. [2018] employed PI strategies in order to maximize the expected utility of the difference between the fund's final wealth and the price of a lifetime annuity at retirement. Such an annuity acts as a minimum guarantee. However, the value of the annuity price is contingent on members' survival probability, which, in the aforementioned works, is modelled through a deterministic force of mortality. Hence, such an approach ignores that the force of mortality can be stochastic, causing unexpected variations in the survival probability and, thus, in the value of the guarantee at retirement. Therefore, the resulting optimal PI strategy can only protect against investment risk and not against longevity risk. To the best of our knowledge, a PI strategy that can simultaneously immunize the fund's members from the market risk and unexpected fluctuations in the average life expectancy of the fund's participants has never been designed.

The third research topic addressed in the present thesis is the determination of the optimal multiplier in markets characterized by frictionalities, such as discontinuities into the dynamics of the underlying risky asset. Portfolio insurers can be modelled as expected utility maximizers with the additional constraint that the value of the PI strategy at maturity must be greater or equal to the guarantee.

As Kingston [1989] argued, CPPI and PPI strategies result from an expected utility maximization where the utility function is of the hyperbolic absolute risk aversion (HARA) type. As shown in Black and Perold [1992], if the risky asset follows a geometric diffusion process with constant drift and constant volatility, then the optimal PI strategy coincides with the CPPI one. In the latter case, the optimal constant multiplier is equal to the product of the instantaneous Sharpe ratio and the inverse of the investor's risk aversion. Moreover, when the dynamic of the underlying risky asset follows a geometric diffusion process with time-varying or stochastic volatility, under specific assumptions of the volatility risk premium, the optimal PI strategy is the PPI one. In such a framework, Zieling et al. [2014] found that the resulting time-varying optimal multiplier is the sum of two components. The first is the myopic demand, expressed in terms of Merton's solution, and the second is the intertemporal hedging demand caused by the correlation between the risky asset price and its volatility. However, the above considerations only apply in a frictionless market under the assumption that rebalancing of the strategies occurs continuously. The optimality of portfolio insurance strategies is hampered by the presence of jumps in the dynamics of the underlying risky asset, since they induce gap risk with a positive probability. When gap risk arises at some point in the investment horizon, the cushion becomes negative and, since the remaining wealth is fully invested in the reserve asset, continues to remain negative until maturity. As a result, the optimization problem above can no longer be solved, since the HARA utility function is no longer defined for a negative value of the cushion. In response to such a problem, some authors, including Ameur and Prigent [2014] and Hamidi et al. [2014], proposed alternative approaches to dynamically set the multiplier based on conditional expectations of future market drops. However, as discussed in Dichtl et al. [2017], such methods are entirely based on the forecasting ability of the PI insurers and may fail to deliver an effective gap risk control. Therefore, the optimal design of the PI strategy suitable to reach the guaranteed amount $\mathbb{P}$-a.s. at maturity and provide equity market participation when the market may experience downward jumps is still an open problem.
We aim to fill these gaps, introducing and developing new versions of PI strategies able to solve the aforementioned drawbacks and, at the same time, be suitable within a risk management context.
The dissertation is divided into five chapters, organized as follows. In Chapter 1, we will recall PI strategies' main properties and their analytical foundations, which will be subject to further elaboration in this thesis. Chapter 2 introduces a new exotic option written on a modified version of the TIPP strategy to be used within structured investment products with capital protection. Our approach suggests enriching the framework by including a threshold in the TIPP allocation mechanism to ensure a guaranteed minimum equity exposure at any point of the investment time horizon. We baptize such a new allocation mechanism G-TIPP. To test the goodness of our proposal, we provide an in-depth analysis of the prices of such new exotic options, assuming a Heston-Vasicek financial market model, and compare our results with other options used within structured products, such as options on G-CPPI introduced by Di Persio et al. [2020]. Our approach represents an interesting alternative for investors, as we numerically prove that options on G-TIPP have properties that significantly increase the participation rate of structured investment products with capital protection and, thus, their attractiveness on the market. The content of this chapter is based upon a joint work with Prof. I. Oliva, Prof. L. Di Persio, and Dr. K. Wallbaum, recently published in

Di Persio et al. [2023].
Chapter 3 concerns an optimal investment problem for a DC pension fund, which guarantees a minimum retirement saving in the form of a target annuity by assuming randomness in interest rates, labour income, and mortality. The fund manager aims to protect the retirement capital accumulation against investment and longevity risks by implementing an *ad hoc* PI strategy. More precisely, the manager considers the guaranteed lifetime annuity present value as the minimum wealth to hold at retirement, investing the residual wealth in a portfolio given by the combination of a rolling bond, a rolling longevity bond, and a stock. The goal is to determine the exogenous parameters of the PI strategy such that the expected utility of the difference between the final wealth and the target annuity value is maximized. We obtain a closed-form solution to the resulting stochastic optimal control problem and provide a numerical application to investigate how investment optimality is affected by longevity dynamics, highlighting the suitability of our proposal for defined contribution scheme management. The findings of this analysis are the results of collaborative work with Prof. I. Oliva, Prof. M. Di Giacinto, and Dr. M. Marino. The related paper is currently under review to an international journal.
In Chapter 4, we investigate an optimal investment problem associated with PPI strategies in the presence of jumps in the underlying dynamics. While PPI strategies are known to be free of downside risk in diffusion modelling framework with continuous trading, see, e.g., Cont and Tankov [2009], real market applications exhibit the non-negligible gap risk, which increases with the multiplier value. To address this issue, this paper analyzes an optimal investment problem for PPI strategies that maximize expected utility at maturity and, at the same time, minimize gap risk, even when the cushion turns negative, by continuing to maintain a positive exposure to the risky asset. In particular, two modifications to the standard literature have been introduced in formulating the problem: *(i)* we propose a generalization of PPI that admits the negative value of the multiplier (i.e. short-selling of the risky asset), and *(ii)* we consider a loss-averse investor whose preferences are described by an S-shaped utility function. We address the optimization problem by combining the worst-case martingale method with the concavification technique to simultaneously account for jumps in the underlying risky asset dynamics and the non-concavity of utility function. Our main results provide an algorithm for calculating the optimal multiplier in quasi-closed form, making our results easily applicable from a practical point of view. Interestingly, we have shown ex-post that the optimal multiplier is such that the gap risk never occurs $\mathbb{P}$-a.s. over the entire investment horizon. This last work results from a collaboration with Prof. I. Oliva and Prof. K. Colaneri and has been submitted to an international journal for publication. Finally, the present thesis considers a final discussion about the results achieved, emphasizing further suitable research projects in the domain of PI strategies.

Chapter 1

Some preliminaries on portfolio insurance strategies

Portfolio insurance (PI) strategies were devised by Leland and Rubinstein [1976] and Brennan and Schwartz [1976] after the stock market collapse in 1973-1974 that caused pension funds to withdraw. In particular, the authors observed that PI strategies would induce investors not to leave the market, allowing them to take advantage of the subsequent market upturn in 1975. Moreover, such PI strategies came back after the great financial crisis of 2008 and, nowadays, represent a linchpin of the asset management industry for institutional and retail investors. According to Grossman and Villa [1989] and Basak [2002], PI strategies aim to secure a predetermined minimum level of wealth G, the guaranteed amount, over a predetermined time horizon T and simultaneously guarantee equity market participation in a market upturn. These goals are achieved by combining a risky asset S, such as a market index, with a reserve asset P, such as a bond or a cash account. Depending on the underlying investment rule and their rebalancing disciplines, PI strategies can be classified into *buy-and-hold* (BH) strategy, *stop-loss* (SL) strategy, *synthetic put strategy*, and *constant proportion portfolio insurance* (CPPI) strategy. This thesis focuses on the latter approach, namely the CPPI strategy and its generalizations. We extensively review the main properties of the CPPI strategy and its fundamental drawback, the so-called *gap risk*. The remainder of this chapter is organized as follows. Section 1.1 provides a general overview of PI strategies. Section 1.2 delves into the modelling aspects of the CPPI strategy in different market environments, providing an in-depth analysis of gap risk and the metrics introduced in the literature for its quantification. Section 1.3 introduces the first extension of the CPPI strategy, namely the *time-invariant portfolio protection* (TIPP) strategy, and provides historical simulations on real data to highlight their main differences. Finally, Section 1.4 exploits the methodologies proposed in the literature to hedge against gap risk.

1.1 An overview of portfolio insurance strategies

We start by fixing a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and a finite time horizon T which coincides with the maturity of PI strategies. We also introduce a complete and right-continuous filtration $\mathbb{F} = \{\mathcal{F}(t),\ t \in [0, T]\}$, representing the global information flow, and assume that all processes defined below are adapted to $\mathbb{F}$. Each strategy mentioned above can be represented by a predictable process $\alpha^h = \left\{\alpha^h(t),\ t \in [0,T]\right\}$ for all $h \in \{BH,\ SL,\ SP,\ CPPI\}$. More specifically, α^h indicates the percentage of the portfolio at time t invested in the risky asset S. Furthermore, since these strategies are self-financing, meaning that money is neither injected nor withdrawn during the investment time horizon $[0,T]$, the percentage of wealth invested in the reserve asset is given by $1-\alpha^h$ for all $h \in \{BH,\ SL,\ SP,\ CPPI\}$. Hence, the portfolio value process dynamics satisfy

$$\begin{cases} \dfrac{\mathrm{d}W^h(t)}{W^h(t)} = \alpha^h(t)\dfrac{\mathrm{d}S(t)}{S(t)} + (1-\alpha(t))\,\dfrac{\mathrm{d}P(t)}{P(t)}, \quad t \in [0,T], \\ W^h(0) = w_0^h, \end{cases} \tag{1.1}$$

for all $h \in \{BH,\ SL,\ SP,\ CPPI\}$. For each strategy under consideration, we characterize the process α^h.

Buy-and-hold strategy. One of the simplest ways to protect investors' wealth from losses by guaranteeing at least G at maturity T is the buy-and-hold (BH) strategy. Denote by $G \cdot P(0)$ the present value of the guarantee at time $t = 0$, where, without loss of generality, $P(0)$ is the no-arbitrage price of a zero-coupon bond maturing at T. Then, the buy-and-hold strategy works as follows: at time $t = 0$, the present value of the guarantee is invested in the bond P, and the remaining part, that is, the surplus $W^{BH}(0) - G \cdot P(0)$ is invested in the risky asset S. Therefore, the percentage of wealth invested in the risky asset is given by

$$\alpha^{BH}(t) = \frac{W(0) - G \cdot P(0)}{W^{BH}(t)}\frac{S(t)}{S(0)}, \quad t \in [0,T]. \tag{1.2}$$

Although α^{BH} in equation (1.2) is stochastic, this strategy is static, meaning that there are no rebalancing decisions during the interval $(0,T]$. As argued in Balder et al. [2009], if one abstracts from the stochastic interest rate, this strategy reaches the guarantee G independently of the stochastic processes that generate asset prices.

Stop-loss strategy. Another prominent example to protect a portfolio from losses is the stop-loss (SL) strategy, see, e.g. Bird et al. [1988] for further details. This strategy works as follows: the entire value of portfolio $W^{SL} = \left\{W^{SL}(t),\ t \in [0,T]\right\}$ is invested in the risk asset. This position is held until the portfolio's market value exceeds the present value of the capital to be protected G. The present value of G is called the floor process $F = \{F(t),\ t \in [0,T]\}$ and is defined as

$$F(t) = G \cdot P(t), \quad t \in [0,T]. \tag{1.3}$$

Thus, the percentage of wealth invested in the risky asset $\alpha^{SL} = \left\{\alpha^{SL}(t),\ t \in [0,T]\right\}$ can be defined as

$$\alpha^{SL}(t) = \mathbb{1}_{W^{SL}(t) > F(t)}, \quad t \in [0,T]. \tag{1.4}$$

Hence, the entire wealth is invested in S until the difference between the current portfolio value W^{SL} and the floor F is exhausted. Indeed, if the value of the portfolio reaches or falls below the floor value, that is if $W^{SL}(t) < F(t)$, the portfolio is fully invested into the bond P until the end of the investment horizon T. Thus, as long as the portfolio does not fall below of the floor during the entire investment horizon $[0, T]$, the final value of the strategy will be above the guaranteed amount at maturity.

Synthetic put strategy. A third popular PI strategy is the synthetic put (SP) strategy devised by Rubinstein and Leland [1981]. This strategy uses the Black and Scholes [1973] option pricing formula to create a continuously adjusted synthetic put option on the risky asset S. Thus, for the description of the strategy, we put ourselves in the Black and Scholes [1973] framework, assuming that the dynamics of the risky asset S follow a geometric Brownian motion as in equation (1.12) and that the price of the bond P grows at a constant interest rate $r > 0$. The idea behind the synthetic put strategy is to combine the purchase of the risky asset S with the purchase of a European put option on S. This is equivalent to buy a continuously adjusted portfolio consisting of the risky asset S and the bond P. Hence, the percentage of wealth invested into the risky asset S for the synthetic put strategy is given by

$$\alpha^{SP}(t) = \frac{S(t) \cdot \mathcal{N}(d_1(t))}{S(t) \cdot \mathcal{N}(d_1) + K \cdot e^{-r(T-t)} \cdot \mathcal{N}(d_2)}, \quad t \in [0, T], \tag{1.5}$$

where K and T are the strike price of the synthetic European put option and the maturity of the synthetic put options (which coincides with the maturity of the strategy), respectively, and $\mathcal{N}(\cdot)$ is the standard cumulative normal distribution function where

$$d_1(t) = \frac{\ln(S(t)/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad t \in [0, T],$$

and

$$d_2(t) = d_1(t) - \sigma\sqrt{T-t}, \quad t \in [0, T],$$

with $\sigma > 0$ indicating the constant volatility of the underlying risk asset S. This strategy requires an increase (resp. a decrease) in the composition shares of the risky asset α^{SP} if the price of the risky asset increases (resp. decreases). In order to obtain the desired level of protection, the strike price of the European put option must be set such that

$$K = \frac{G}{W^{SP}(0)} \cdot (S(0) + Put(0)). \tag{1.6}$$

It is worth noting that the solution to equation (1.6) has to be determined through an iterative procedure since, it is necessary to know the strike price itself, to compute the no-arbitrage price of the European put option at $t = 0$. Given the assumption of the Black and Scholes [1973] framework, the portfolio must be readjusted continuously according to the rule depicted in equation (1.5) to maintain the desired level of protection over the entire time horizon $[0, T]$.

Constant proportion portfolio insurance strategy. In contrast to the synthetic put strategy, the constant proportion portfolio insurance (CPPI) strategy is a dynamic protection strategy not based on option pricing theory. The CPPI was proposed by Perold [1986], and later developed by Black and Jones [1987, 1988] for equity instruments and Perold and Sharpe [1988] for fixed-income instruments. For the CPPI strategy, the exposure to the risky asset S is a function of the cushion $C = \{C(t),\ t \in [0,T]\}$, defined as

$$C(t) := W^{CPPI} - F(t). \tag{1.7}$$

In particular, for each time $t \in [0,T]$, if $W^{CPPI}(t) > F(t)$, the exposure to the asset risky asset is given by

$$m \cdot C(t) = m \cdot \left(W^{CPPI}(t) - F(t)\right), \tag{1.8}$$

where $m > 0$ is the multiplier that is an exogenous parameter set at time $t = 0$ by the investor according to his/her risk aversion. If $W^{CPPI} \leq F(t)$, the remaining portfolio value is entirely invested in the bond P until the end of the investment horizon. Hence, the CPPI allocation rule prescribes that the percentage of wealth invested in the risky asset is given by

$$\alpha^{CPPI}(t) = m \cdot \frac{\left(W^{CPPI}(t) - F(t)\right)^+}{W^{CPPI}(t)}, \quad t \in [0,T]. \tag{1.9}$$

Remark 1.1.1. *Equation* (1.9) *implies that the CPPI strategy may lead to short positions in the reserve asset. Indeed, when the risky asset price S is high, or the price of the reserve asset P is low, the exposure may become larger than the portfolio's value. For this reason, in many commercial applications, the CPPI strategy is usually implemented so that the holding in the risky asset varies between 0% and 150% of the investment sum. In this case, equation* (1.9) *can be modified as*

$$\alpha^{CPPI}(t) = \min\left\{ m \cdot \frac{\left(W^{CPPI}(t) - F(t)\right)^+}{W^{CPPI}(t)},\ L_{max} \right\}, \quad t \in [0,T], \tag{1.10}$$

where $L_{max} \in [0, 1.5]$ is the so-called maximum leverage factor set by the investor at $t = 0$, according to his/her risk aversion.

Remark 1.1.2. *In order to rule out all arbitrage opportunities, we assume that the guarantee amount G cannot be reached by investing the initial wealth into the reserve asset P. Moreover, assuming a constant risk-free rate r, the no-arbitrage condition reads as $G \leq W^h(0)e^{rT}, \quad h \in \{BH,\ SL,\ SP,\ CPPI\}$. The guaranteed amount is chosen to be equal to a pre-specified percentage of the initial investments, that is $G = W^h(0) \cdot PL$, where $PL \in (0,1]$ is an exogenous parameter known as protection level.*

Given the strategic decision to implement a PI strategy, a strand of literature addresses the question of which particular strategy should be adopted. Several earlier studies evaluate and compare PI strategies in terms of their risk-return profile. The first one can be attributed to Garcia and Gould [1987], who argue that dynamic PI strategies cannot outperform static investment strategies such as buy-and-hold or constant mix. Then, using the Monte Carlo simulation approach, Benninga [1990] compares the stop-loss, synthetic put, and CPPI strategies. The authors report that the stop-loss strategy dominates the CPPI and synthetic put strategy in terms of both the expected terminal wealth and the Sharpe ratio. Bird et al. [1990] documents that portfolio insurance strategies are robust to various market conditions, including market crashes. Do [2002] compares the synthetic put strategy with the CPPI strategy using simulation analysis. Although the author claims that no strategy can be motivated from the point of view of loss minimization or gain participation, the CPPI strategy seems dominant in terms of floor protection and insurance costs. Subsequently, Cesari and Cremonini [2003] show that the relative performance of portfolio insurance strategies strongly depends on market conditions. In particular, using various performance measures, such as the downside deviation and the Sortino index, the authors show that the CPPI strategy dominates all other portfolio insurance strategies in bear and sideway markets.

Thereafter, using the stochastic dominance criteria, Annaert et al. [2009] compare portfolio insurance strategies with the buy-and-hold one. However, they cannot identify a stochastic dominance relationship between portfolio insurance strategies and the benchmark given by the buy-and-hold strategy. In particular, although all portfolio insurance strategies examined lead to lower returns than the buy-and-hold strategy, the authors conclude that their accompanying lower risk sufficiently compensates to make them attractive alternatives, at least for some investors.

Bertrand and Prigent [2011] argue that the Omega measure is the most adequate performance measure to compare portfolio insurance strategies as it captures the entire empirical distribution of their returns. Specifically, using a block-bootstrap simulation approach, the authors compare the CPPI strategy with the synthetic put strategy, reporting a dominance of the CPPI strategy in terms of the Omega ratio. Dichtl and Drobetz [2011] analyze portfolio insurance strategies within the framework of cumulative prospect theory. Their simulation results indicate that a loss aversion attitude makes most PI strategies preferred investment strategies, at least for a prospect theory investor.

Finally, Dichtl et al. [2017] compare different PI strategies using bootstrap simulations and relatively new risk-adjusted performance measures, such as the Farinelli-Tibiletti and Rachev ratios; see Farinelli et al. [2008] and Biglova et al. [2004] for further details. In particular, the first ratio considers the upper (resp. lower) partial moments of the portfolio insurance strategies' return distribution and measures positive (resp. negative) deviations from a predetermined target return. The second one is the Rachev ratio, defined as the ratio between the expected tail return above a given level and the expected tail loss below a given level. In particular, using these measures, the authors find that classical portfolio insurance strategies, such as the CPPI and synthetic put strategy, provide more downside protection than the simple stop-loss strategy and exhibit higher risk-adjusted performance in many instances.

Thus, there is strong agreement in the financial literature that classical PI strategies provide strong downside protection regardless of market conditions. However, it is unclear which strategy is best to adopt among the ones we have presented. Indeed, it strongly depends on several factors, such as the performance measures we decide to adopt to compare them, the market phase, and the type of investor we are considering.

1.2 Some modeling aspects of the CPPI strategy

1.2.1 Continuous-time CPPI strategy

To derive the basic properties of the continuous-time CPPI strategy, we consider a financial market model in which investment trading activity occurs continuously, and there are no transaction costs or taxes. Furthermore, we assume that in such a simplified market model, two assets are traded: the bond P and the risky asset S. More precisely, we assume that the bond price P grows at a constant risk-free rate $r > 0$, implying that its dynamics is given by

$$\begin{cases} \dfrac{\mathrm{d}P(t)}{P(t)} = r\mathrm{d}t, \quad t \in [0,T], \\ P(0) = e^{-rT}. \end{cases} \tag{1.11}$$

We assume that the evolution of the risky asset S, or the benchmark market index, is given by the following geometric Brownian motion

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t)} = \mu\mathrm{d}t + \sigma\mathrm{d}Z_S(t), \quad t \in [0,T], \\ S(0) = s_0, \end{cases} \tag{1.12}$$

where $\mu > r$ and $\sigma > 0$ are constants and $Z_S = \{Z_S(t),\ t \in [0,T]\}$ is a standard Brownian motion w.r.t. $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. As explained in Section 1.1, the basic idea behind the CPPI strategy is to obtain at least the guaranteed amount at maturity G by investing for each time $t \in [0,T]$ a constant multiple $m > 0$ of the current value of the cushion C into the risky asset S, provided the cushion is positive, and the remaining wealth in the reserve asset P. In this case, since we have assumed that the reserve asset evolves according to equation (1.11), the floor process in equation (1.3) can be rewritten as

$$F(t) = G \cdot e^{-r(T-t)}, \quad t \in [0,T], \tag{1.13}$$

meaning that it evolves according to the following ODE

$$\begin{cases} \dfrac{\mathrm{d}F(t)}{F(t)} = r\mathrm{d}t, \quad t \in [0,T], \\ F(0) = G \cdot e^{-r \cdot T}. \end{cases} \tag{1.14}$$

Within this framework, it is possible to derive a closed-form expression for the cushion process C and the corresponding CPPI portfolio value process as in the following proposition.

Proposition 1.2.1. *If the bond price process evolves according to equation* (1.11)*, and the risky asset price dynamics S evolves according to equation* (1.12)*, then the cushion process C evolves according to*

$$\begin{cases} \dfrac{\mathrm{d}C(t)}{C(t)} = \left[r + m\left(\mu - r\right)\right]\mathrm{d}t + m\sigma\mathrm{d}Z_S(t), \quad t \in [0,T], \\ C(0) = W^{CPPI}(0) - F(0), \end{cases} \tag{1.15}$$

whose solution is

$$C(t) = C(0) \cdot \exp\left\{ \left[r + m\left(\mu - r\right) - \frac{m^2\sigma^2}{2} \right] t + m\sigma Z_S(t) \right\}, \quad t \in [0, T]. \tag{1.16}$$

Moreover, the corresponding CPPI portfolio value evolves according to the following SDE

$$\begin{cases} \mathrm{d}W^{CPPI}(t) = rF(t)\mathrm{d}t + \left(W^{CPPI} - F(t)\right)\left\{\left[r + m\left(\mu - r\right)\right]\mathrm{d}t + m\sigma \mathrm{d}Z_S(t)\right\}, \quad t \in [0, T], \\ W^{CPPI}(0) = w_0^{CPPI}, \end{cases} \tag{1.17}$$

where

$$W^{CPPI}(t) = G \cdot e^{-r(T-t)} + \left(W^{CPPI}(0) - G \cdot e^{-rT}\right) \exp\left\{ \left[r - m\left(r - \frac{\sigma^2}{2}\right) - \frac{m^2\sigma^2}{2} \right] t \right\} \left(\frac{S(t)}{S(0)}\right)^m. \tag{1.18}$$

Proof. See, e.g., Black and Perold [1992] and Bertrand and Prigent [2005]. □

Equation (1.18) illustrates the basic properties of the CPPI strategy, which, under the assumption of continuous time trading, is path-independent. In particular, for each time $t \in [0, T]$, the value of the strategy consists of the present value of the guarantee G, i.e. the floor at time t, and a non-negative part proportional to $\left(\frac{S(t)}{S(0)}\right)^m$. In other words, the continuous-time CPPI strategy is equivalent to take a long position in a zero-coupon bond with a nominal value equal to the guaranteed amount at maturity T and investing the remaining wealth in a fictitious asset that has m times the excess return and the volatility of S, and is perfectly correlated with S. Moreover, the CPPI strategy has an additional degree of freedom compared to other portfolio insurance strategies introduced by the multiplier m. In particular, it governs the strategy exposure to the risky asset over the entire investment horizon, determining the shape of the strategy's payoff at maturity. Indeed, as shown in Figure 1.1 the payoff is linear for $m = 1$, and it is convex for $m \geq 2$. In the latter case, the CPPI strategy can be interpreted as a power claim and, most importantly, can obtain equity market participation in case of favourable market conditions.

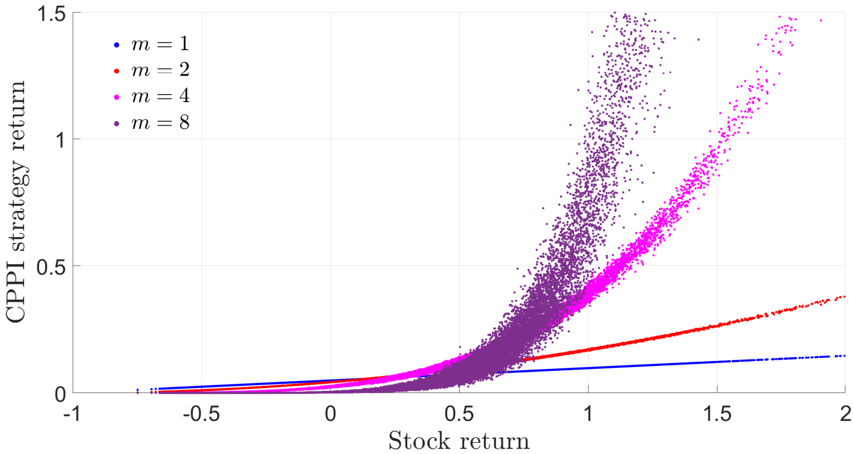


Figure 1.1 Payoff function of the CPPI strategy for different levels of multipliers.

Regardless of the multiplier's value, the portfolio protection given by the continuous-time CPPI strategy is efficient with probability one. Hence, the CPPI portfolio's terminal value will be at least equal to the value of the guarantee at maturity $\mathbb{P}$-a.s. One main issue in the continuous-time CPPI strategy is the *cash-lock* event. As shown in Figure 1.2, this is happen when the portfolio wealth hits the floor. Thus, according to the CPPI investment rule, from that moment until maturity, the portfolio remains fully invested in the reserve asset with no possibility of recovery. In this case, even if the guarantee G at maturity T is honoured, a cash-locked position prevents equity participation in the subsequent market upturn and is therefore considered a critical risk.

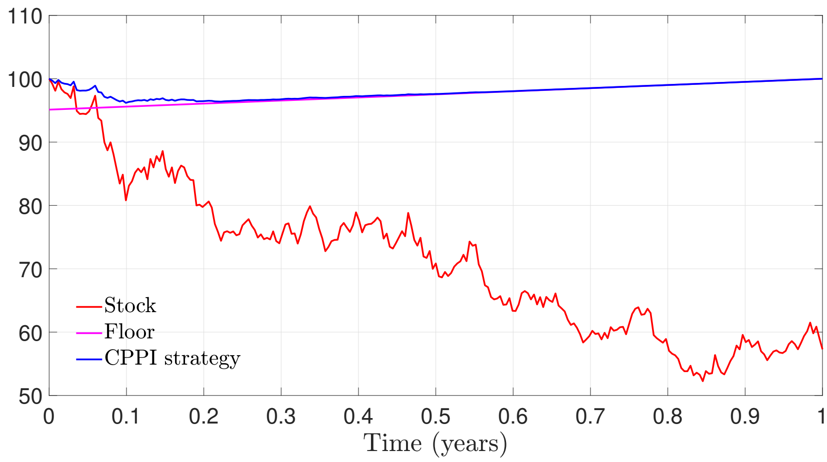


Figure 1.2 A simulated trajectory of the CPPI strategy exhibiting a cash-locked position for the following market configuration and strategy's parameters: $\mu = 0.085$, $r = 0.05$, $\sigma = 0.30$, $PL = 1$, $W^{CPPI}(0) = 100$, and $T = 1$ year.

Moreover, CPPI managers widely recognize that the portfolio value may break the floor. The risk of breaching the floor is referred to as gap risk. Indeed, the above considerations apply only in an idealized setting under the continuity assumption imposed on the CPPI strategy's rebalancing frequency and the dynamics of the underlying risky asset S. In practice, both assumptions are violated mainly for two reasons. The CPPI strategy is often written on an underlying risky asset characterized by low liquidity, meaning that its rebalancing frequency is typically weekly or even monthly; see Hirsa [2010] for more details. Hence, such trading restrictions make the risk of a floor violation non-negligible. It is widely documented that price trajectories of risky assets involve jumps; see, e.g., Andersen et al. [2002], Barndorff-Nielsen and Shephard [2006], Aït-Sahalia and Jacod [2009]. Such discontinuities in the dynamics introduce a risk of breaching the floor even under the assumption of continuous time trading.

The first point is discussed by Balder et al. [2009], who study the viability of the CPPI strategy under the assumption of discrete-time rebalancing. As the next Section outlines, the authors analyze a discretely rebalanced CPPI, maintaining the assumptions that the risky asset S evolves according to a geometric Brownian motion and that the reserve asset P evolves at a constant risk-free rate r. The second point is extensively addressed in Cont and Tankov [2009], which investigate the risks of CPPI by exclusively relaxing the continuity assumption of price processes. We analyze such last point more in-depth in Section 1.2.3.

1.2.2 Discrete-time CPPI strategy

Defining the discrete version of the CPPI strategy requires advancing some preliminary assumptions. First, as in Balder et al. [2009], we assume that the dynamics of the bond P and the risky asset S are given by equations (1.11) and (1.12), respectively. Moreover, we assume the following restriction on trading. Let $\pi^{\bar{n}}$ denote an equidistant refinement of the interval $[0,T]$, that is $\pi^{\bar{n}} = \{t_0^{\bar{n}} = 0 < t_1^{\bar{n}} \cdots < t_{\bar{n}-1}^{\bar{n}} < t_{\bar{n}}^{\bar{n}} = T\}$, where $t_{k+1}^{\bar{n}} - t_k^{\bar{n}} = \frac{T}{\bar{n}}$, for all $k = 1, \ldots, \bar{n}-1$. To simplify the notation, we omit the superscript $\bar{n}$ and denote the set of trading dates π instead of $\pi^{\bar{n}}$. We assume that trading is exclusively possible after $t_k \in \pi$, implying that the number of shares held in the risky asset S and in the bond P remains constant on the interval $(t_k, t_{k+1}]$ for all $k = 0, \ldots, \bar{n}-1$. However, the value of investments in the two assets changes as asset prices fluctuate.

Definition 1.2.2. *An investment strategy (ϕ_S, ϕ_P) is called discrete-time CPPI if*

$$\phi_S(t) = \frac{\alpha^{CPPI}(t_k) W^{CPPI}(t_k)}{S(t_k)} = \max\left\{\frac{mC(t_k)}{S(t_k)}, 0\right\}, \tag{1.19}$$

$$\phi_P(t) = \frac{W^{CPPI}(t_k) - S(t_k)\phi^S(t)}{P(t_k)}, \tag{1.20}$$

for all $t \in (t_k, t_{k+1}]$ and $k = 0, \ldots, \bar{n}-1$, and the self-financing condition holds, that is

$$\phi_S(t_k)S(t_{k+1}) + \phi_P(t_k)P(t_{k+1}) = \phi_S(t_{k+1})S(t_{k+1}) + \phi_P(t_{k+1})P(t_{k+1}), \tag{1.21}$$

for all $k = 0, \ldots, \bar{n}-1$.

Also in this case, it is possible to derive an explicit expression for the cushion process C as follows.

Proposition 1.2.3. *Let*

$$t_s := \min\left\{t_k \in \pi | W^{CPPI}(t_k) - F(t_k) < 0\right\}, \tag{1.22}$$

where $t_s = \infty$ if the minimum is not attained. Then, the following holds

$$C(t_{k+1}) = e^{r(t_{k+1} - \min\{t_s, t_{k+1}\})} \left(W^{CPPI}(t_0) - F(t_0)\right) \prod_{i=1}^{\min\{s,k+1\}} \left(m\frac{S(t_i)}{S(t_{i-1})} - (m-1)\, e^{r\frac{T}{\bar{n}}}\right), \tag{1.23}$$

for all $k = 0, \ldots, \bar{n}-1$.

Proof. See Balder et al. [2009]. □

Proposition 1.2.3 implies that, if $t_s < t_{\bar{n}} = T$, that is the CPPI portfolio falls below the floor in t_s, the corresponding price process $W^{CPPI}(t_s)$ switches from

$$W^{CPPI}(t_s) = \left(W^{CPPI}(0) - F(0)\right) \prod_{i=1}^{s} \left(m\frac{S(t_i)}{S(t_{i-1})} - (m-1)e^{r\frac{T}{\bar{n}}}\right) + F(t_s), \quad \forall s \leq k \tag{1.24}$$

to

$$W^{CPPI}(t_s) = W^{CPPI}(t_k)e^{r(t_s - t_k)}, \quad \forall s > k, \tag{1.25}$$

with $k = 1, \ldots, \bar{n}-1$. As shown in (1.25), when the gap risk occurs, the remaining value of the CPPI portfolio is invested in the reserve asset, meaning that the strategy cannot honour the guarantee G by realizing a loss equal to $\left(W^{CPPI}(t_s) - F(t_s)\right)$. Moreover, as the portfolio is fully monetized, the strategy can no longer capture equity market participation in the event of subsequent market rises. Balder et al. [2009] analyze the gap risk by computing the shortfall probability and the expected shortfall given default. Such risk measures, which determine the effectiveness of the discrete-time CPPI strategy, can be defined and computed as follows.

Definition 1.2.4. *The shortfall probability P^{SF} denotes the probability that the CPPI portfolio value is less than the guaranteed amount G at maturity, that is*

$$P^{SF} := \mathbb{P}\left(W^{CPPI}(T) < G\right) = \mathbb{P}\left(C(T) < 0\right). \tag{1.26}$$

The expected shortfall given default ESF is the expected amount of wealth lost when a shortfall occurs, that is

$$ESF := \mathbb{E}^{\mathbb{P}}\left[G - W^{CPPI}(T) | W^{CPPI}(T) < G\right] = \mathbb{E}^{\mathbb{P}}\left[-C(T)|C(T) < 0\right]. \tag{1.27}$$

Proposition 1.2.5. *Define E_1 and E_2 as*

$$\begin{aligned} E_1 &:= m e^{\mu \frac{T}{\bar{n}}} \mathcal{N}\left(d_3\right) - e^{r \frac{T}{\bar{n}}}\left(m-1\right)\mathcal{N}(d_4), \\ E_2 &:= e^{r\frac{T}{\bar{n}}}\left[1 + m\left(e^{(\mu - r)\frac{T}{\bar{n}}} - 1\right)\right] - E_1, \end{aligned}$$

with

$$d_3 := d_4 + \sigma\sqrt{\frac{T}{\bar{n}}}, \quad d_4 := \frac{\ln\left(\frac{m}{m-1}\right) + (\mu - r)\frac{T}{\bar{n}} - \frac{\sigma^2}{2}\frac{T}{\bar{n}}}{\sigma\sqrt{\frac{T}{\bar{n}}}}. \tag{1.28}$$

The shortfall probability and the expected shortfall given default are given by

$$P^{SF} = 1 - \left(1 - \mathcal{N}\left(-d_4\right)\right)^{\bar{n}}, \tag{1.29}$$

$$ESF = \frac{-\left(W^{CPPI}(0) - F(0)\right)e^{-r\frac{T}{\bar{n}}}E_2 \dfrac{e^{rT} - E_1^{\bar{n}}}{1 - E_1 e^{-r\frac{T}{\bar{n}}}}}{P^{SF}}, \tag{1.30}$$

respectively.

Proof. See Balder et al. [2009]. □

We conclude this Section with a sensitivity analysis of the risk measures. In particular, Figure 1.3 displays the shortfall probability and the expected shortfall given default as a function of the exogenous parameters of the strategy, namely, the multiplier m, and the number of rebalances $\bar{n}$ for a given market configuration.

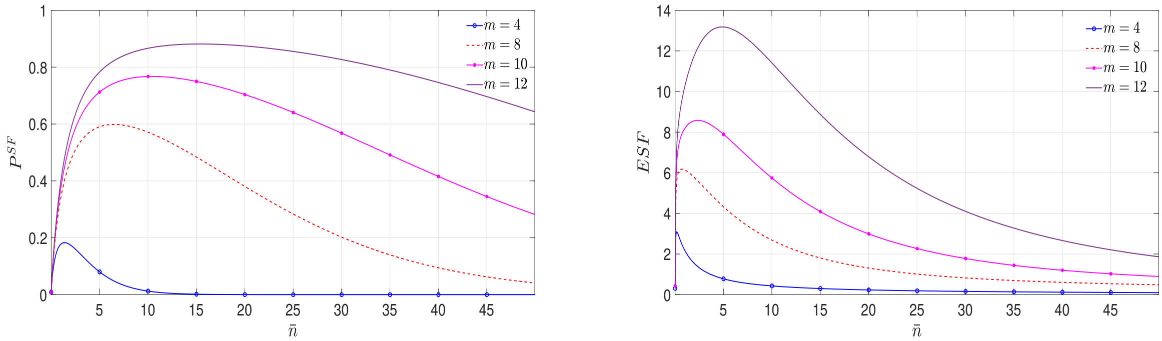


Figure 1.3 Shortfall probability (left chart) and expected shortfall given default (right chart) for the following market configuration and strategy's parameters: $\mu = 0.085$, $r = 0.05$, $\sigma = 0.30$, $PL = 1$, $W^{CPPI}(0) = 100$, and $T = 1$ year.

As expected, the probability of shortfall and the corresponding expected shortfall given default increase as the multiplier increases, regardless of the number of rebalances in the interval $[0, T]$. The most interesting result is the behaviour of these two risk metrics as a function of $\bar{n}$ for a given multiplier level. At first glance, one might be tempted to think that the shortfall probability is monotonically decreasing as $\bar{n}$ increases. However, as Figure 1.3 shows, this is only true once a sufficiently large number of rebalances has been reached. Let $\bar{n}^*$ be the number of rebalancings such that the shortfall probability increases as $\bar{n}$ increases for $\bar{n} < \bar{n}^*$ and decreases as $\bar{n}$ increases for $\bar{n} > \bar{n}^*$. Balder et al. [2009] argue that such a critical level $\bar{n}^*$ furnishes fundamental information: it can be interpreted as the minimum number of rebalancings required for the CPPI method to be effective for $m \geq 2$ in discrete time.

1.2.3 Continuous-time CPPI strategy in jump-diffusion models

In this Section, we discuss the effectiveness of CPPI strategy in the presence of jumps into the dynamics of the risky asset S maintaining the assumption of continuous-time rebalancing. In particular, we assume that the dynamics of S are driven by the following Lévy process

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t-)} = \mu \mathrm{d}t + \sigma \mathrm{d}Z_S(t) + \mathrm{d}J_S(t), \quad t \in [0, T], \\ S(0) = s_0, \end{cases} \tag{1.31}$$

where $J_S(t) = \sum_{j=1}^{N(t)} \left(e^{Y_j^S} - 1 \right)$ with $N = \{N(t),\ t \in [0, T]\}$ being an homogeneous Poisson process with intensity $\tilde{\lambda} > 0$. The jump sizes of log-returns $\left(Y_j^S \right)_{j \in \mathbb{N}}$ is a sequence of i.i.d. random variables with density function f, which is independent of Z_S and N. Moreover, as in Section 1.2.1, we assume that the reserve asset P evolves according to the constant risk-free rate $r > 0$, as in equation (1.11). Let $\tau := \inf \left\{ t \in [0, T] :\ W^{CPPI}(t) < F(t) \right\}$. Therefore, for any time $t \leq \tau$, the cushion dynamics depicted in equation (1.15) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}C(t)}{C(t-)} = \left[r + m\left(\mu - r \right) \right] \mathrm{d}t + m\sigma \mathrm{d}Z_S(t) + m\mathrm{d}J_S(t), \quad t \leq \tau, \\ C(0) = c_0. \end{cases} \tag{1.32}$$

Denoting by $\bar{C} = \left\{\bar{C}(t),\ t \in [0,T]\right\}$ the forward value of the cushion, that is $\bar{C}(t) = \dfrac{C(t)}{P(t)}$, and applying general Itô's formula for semi-martingales (cf. Øksendal and Sulem [2019]) we obtain

$$\begin{cases} \dfrac{\mathrm{d}\bar{C}(t)}{\bar{C}(t-)} = m\left(\mu - r\right)\mathrm{d}t + m\sigma \mathrm{d}Z_S(t) + m\mathrm{d}J_S(t), \quad t \leq \tau, \\ \bar{C}(0) = \bar{c}_0. \end{cases} \tag{1.33}$$

Defining $\tilde{L}(t) := \left(\mu - r\right)t + \sigma Z(t) + \sum_{j=1}^{N(t)}\left(e^{Y_j^S} - 1\right)$, equation (1.33) can be rewritten as

$$\frac{\mathrm{d}\bar{C}(t)}{\bar{C}(t-)} = m\tilde{L}(t)\mathrm{d}t, \quad t \leq \tau, \tag{1.34}$$

whose solution is given by

$$\bar{C}(t) = \bar{c}_0 \cdot \mathcal{E}\left(m\tilde{L}\right)(t), \quad t \leq \tau, \tag{1.35}$$

where $\mathcal{E}$ denotes the Doléans-Dade stochastic exponential. After τ, according to the definition of CPPI strategy, the remaining portfolio value is entirely invested in bond P in order to prevent further losses, that is

$$W^{CPPI}(t) = W^{CPPI}(\tau) \cdot \exp\left\{r\left(t - \tau\right)\right\}, \quad t \in (\tau, T]. \tag{1.36}$$

From equation (1.36), the cushion process after gap-risk reads as follows

$$C(t) = W^{CPPI}(\tau) \cdot \exp\left\{-r\left(t - \tau\right)\right\} - F(t), \quad t \in (\tau, T], \tag{1.37}$$

or

$$\bar{C}(t) = \frac{W^{CPPI}(\tau)}{G}\exp\left\{r\left(T - \tau\right)\right\} - 1, \quad t \in (\tau, T]. \tag{1.38}$$

Hence, we can introduce a new process $\bar{C}^* = \{C^*(t),\ t \in [0,T]\}$ defined as the stopped process of $\tilde{C}$ by the stopping time τ, that is

$$\bar{C}^*(t) = \bar{c}_0 \cdot \mathcal{E}(m\tilde{L})\left(t \wedge \tau\right), \quad t \in [0,T], \tag{1.39}$$

where $t \wedge \tau = \min\left\{t, \tau\right\}$. Within this framework, it is possible to find closed-form risk measures for evaluating gap-risk. In particular, as in Section 1.2.2, we can define and compute the shortfall probability and the expected shortfall given default as in the following Propositions.

Proposition 1.2.6. *The probability of breaching the floor is given by*

$$P^{SF} = \mathbb{P}\left(\exists t \in [0,T]:\ W^{CPPI}(t) < F(t)\right) = 1 - \exp\left\{-T\int_{-\infty}^{\ln\left(1-\frac{1}{m}\right)} \nu(x)\mathrm{d}x\right\}, \tag{1.40}$$

where ν is the Lévy measure of the process $\tilde{L}$, that is $\nu(x) = \tilde{\lambda}f(x)$ for all $x \in \mathbb{R} \setminus \{0\}$.

Proof. See Cont and Tankov [2009]. □

The process L can be decomposed as follows

$$\tilde{L}(t) = \tilde{L}_1(t) + \tilde{L}_2(t), \quad t \in [0,T], \tag{1.41}$$

where $\tilde{L}_1 = \left\{\tilde{L}_1(t),\ t \in [0,T]\right\}$ is a jump-diffusion process with Lévy measure $\nu(\mathrm{d}x)\mathbb{1}_{x>-\frac{1}{m}}$, and $\tilde{L}_2 = \left\{\tilde{L}_2(t),\ t \in [0,T]\right\}$ is a piecewise constant trajectories whose jumps satisfy $\nu(\mathrm{d}x)\mathbb{1}_{x\leq-\frac{1}{m}}$. Denote by $\tilde{\lambda}^\star := \tilde{\lambda}\int_{-\infty}^{-\frac{1}{m}} f(x)\mathrm{d}x$ the jump intensity of $\tilde{L}_2$, by τ the time of the first jump of $\tilde{L}_2$, and by $\tilde{L}_2 = \Delta\tilde{L}_2(\tau)$ the size of the first jump of $\tilde{L}_2$. Moreover, let $\tilde{\phi}(t)$ be the characteristic function of the Lévy process $\log\mathcal{E}(m\tilde{L}_1)(t)$ and $\tilde{\psi}(u) = \frac{1}{t}\ln\tilde{\phi}_t(u)$. The following result holds true.

Proposition 1.2.7. *Assuming that*

$$\int_1^\infty x\nu(\mathrm{d}x) < \infty,$$

then the expected shortfall given default is given by

$$ESF = \mathbb{E}\left[C^*(T)|\tau \leq T\right] = \frac{\tilde{\lambda}^* + m\int_{-1}^{-\frac{1}{m}} x\nu(x)\mathrm{d}x}{\tilde{\psi}(-i) - \tilde{\lambda}^*}\left(e^{-\tilde{\lambda}^* T}\tilde{\phi}_T(-i) - 1\right). \tag{1.42}$$

Proof. See Cont and Tankov [2009]. □

It is worth noting that equations (1.40) and (1.42) link the probability of the CPPI strategy falling below the floor and the expected loss given default to both market parameters and exogenous parameters that characterize the strategy, such as the multiplier m. Such a property makes P^{SF} and ESF fundamental risk management tools, providing a criterion for adjusting the multiplier according to the investor's risk aversion. Given the importance of gap risk measurement, the study of Cont and Tankov [2009] has been subject to several generalizations. In particular, in order to relax the assumption of time homogeneity, Weng [2013] generalizes the framework described by Cont and Tankov [2009] by including regime-switching in the Lévy exponential process that drives the dynamics of the underlying risky asset S, maintaining the assumption of trading in continuous time. More recently, Buccioli and Kokholm [2018] measure the gap risk by considering the self-contagious nature of asset prices. More precisely, to account for contagion while preserving the analytical traceability of risk measures, the authors introduce self-exciting jumps in the dynamics of the risky asset S employing the Hawkes [1971] process.

1.3 Time-invariant portfolio protection strategy

The standard results of CPPI strategies are based on the assumption that the floor F evolves like the reserve asset P, see equation (1.14). However, this assumption is quite stringent, as it makes the performance of the CPPI strategy highly price-dependent: any gain at any given time can be lost if the underlying asset price falls. In order to solve this problem, Estep and Kritzman [1988] proposed the time-invariant portfolio protection (TIPP) strategy, which is a modified version of the CPPI equipped with a ratchet mechanism, able to lock in a certain proportion of the upward performance. The TIPP is a self-financing strategy that dynamically allocates wealth between a reserve asset P

and a risky asset S, similar to the CPPI. In particular, the TIPP allocation rule prescribes that the percentage of wealth $\alpha^{TIPP} = \left\{\alpha^{TIPP}(t),\ t \in [0,T]\right\}$ invested in the risky asset is

$$\alpha^{TIPP}(t) = m \cdot \frac{\left(W^{TIPP}(t) - F^{TIPP}(t)\right)^+}{W^{TIPP}(t)}, \quad t \in [0,T], \tag{1.43}$$

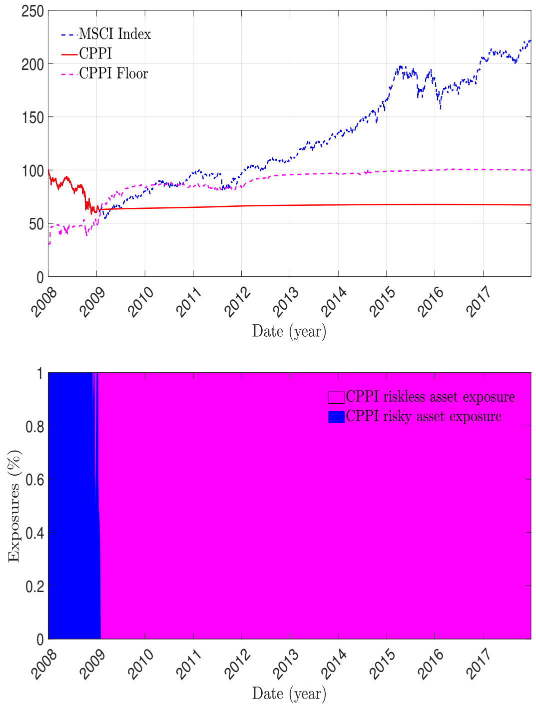
where $W^{TIPP} = \left\{W^{TIPP}(t),\ t \in [0,T]\right\}$ is the TIPP wealth process, and $F^{TIPP} = \{F^{TIPP}(t),\ t \in [0,T]\}$ is the TIPP floor. The main difference between the CPPI strategy and the TIPP strategy lies in the definition of the floor. Specifically, the TIPP floor is defined as the maximum between the usual CPPI floor F and a predetermined percentage of the maximum historical value of the TIPP portfolio W^{TIPP}, that is

$$F^{TIPP}(t) = \max\left\{F(t),\ PL \cdot \sup_{s \leq t} W^{TIPP}(s)\right\}, \quad t \in [0,T]. \tag{1.44}$$

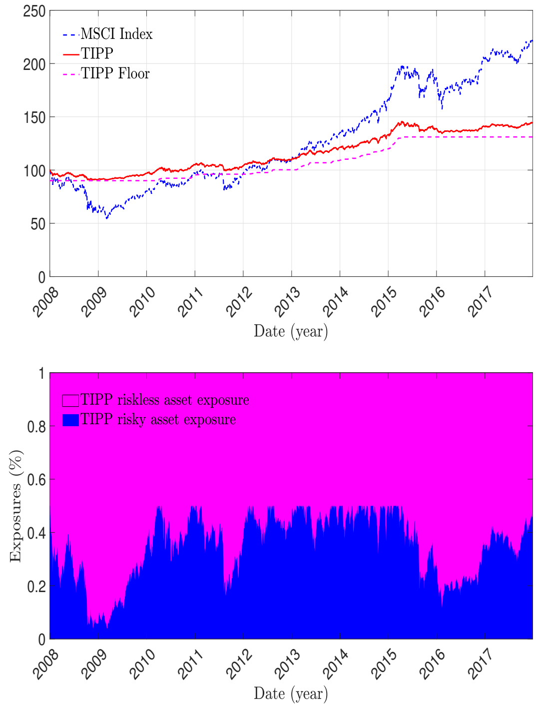
Hence, thanks to equation (1.44), the floor jumps up with the portfolio value to reduce exposure to the risky asset when the market peaks. Indeed, this new floor adjustment has some consequences on the allocation of the risky asset over time, especially in market scenarios where the risk-free interest rates are very low and sudden rises and falls of the risky asset might occur. Such a background sheds light on one of the main issues related to the use of CPPI: when the value of the risky asset increases, the CPPI portfolio value increases accordingly. If the risk-free interest rate level is low, then the growth rate of the CPPI portfolio will be higher than the corresponding growth rate of the floor. This implies that, after a very short period, there is no longer significant portfolio protection. Thanks to its particular floor definition, the TIPP allocation strategy overcomes such a drawback. Indeed, the growth rate of the TIPP floor can be considered comparable to the portfolio in each instant of time, even if the growth of the market index is sustained. Consequently, the TIPP exposure to the risky asset will be generally lower than the CPPI ones, and it will change more smoothly over time, furnishing better downside protection to the investor. However, in case of favourable market conditions, the overall return of the TIPP strategy will be generally lower w.r.t. the standard CPPI one. To better understand how the TIPP works and its advantages and shortcomings compared to the use of the CPPI, we perform a simulation study comparing the aforementioned strategies when the underlying risky asset is the MSCI World Net TR Eur index. Since portfolio insurance strategies' risk-returns are path-dependent, being also strongly influenced by market conditions, we perform historical simulation studies over two different time windows. The first one ranges from December 31, 2007, to December 28, 2017, while the second one covers the period from June 29, 2012, to June 29, 2022. We select such time windows since a market crash occurs either at the beginning (time window from December 31, 2007 to December 28, 2017) or at the end (time window from June 29, 2012, to June 29, 2022) of the period under investigation. As for the parameters' strategies, we set a multiplier of $m = 5$, a protection level at 100% (resp., 90%) of the capital initially invested for CPPI (resp., for TIPP), and a maximum leverage factor, L_{max}, equal to 100%. We also consider both daily and weekly rebalancing frequencies.

1.3.1 Historical simulations in a early-falling-and-rising trend market

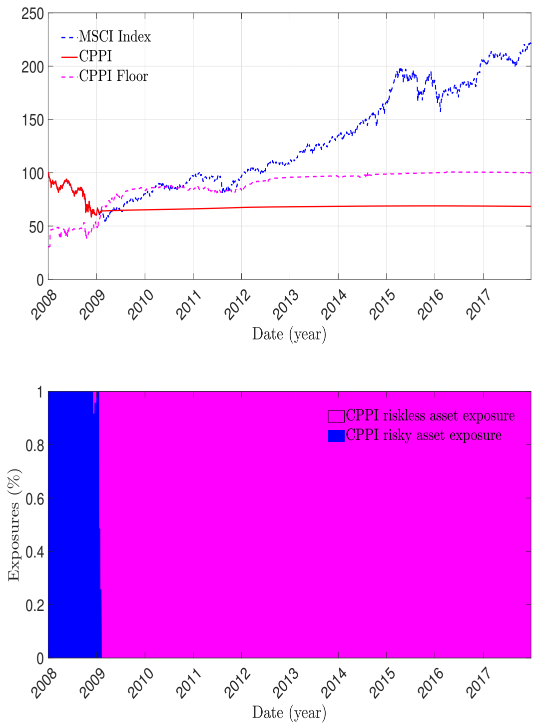
In this Section, we compare CPPI and TIPP strategies when the market shows an early falling and rising trend. Results concerning portfolio evolution and corresponding risky asset exposures for each investment strategy are displayed in Figure 1.4, w.r.t. different rebalancing disciplines. It is worth noticing that the beginning of the period under investigation is characterized by an initial turbulent phase due to the subprime crisis: at the beginning of 2009, the risky asset losses were approximately equal to 50% of its initial value. Subsequently, we can see considerable growth for the risky asset. At the end of 2014, the index gained more than 70% of its initial value, also showing a positive upward trend that was confirmed until the end of the time window. Moreover, such a trend generates a positive performance measured through an annualized total return and volatility equal to 8.24% and 17.67%, respectively, over the entire 10-year time horizon and a 1-day maximum loss of −8.56%. Hence, from the perspective of an investor whose goal is to minimize the risk of suffering losses, this type of equity investment might be too volatile. For this reason, we focus on the CPPI allocation strategy and its modified version, the TIPP. We start our analysis by considering the CPPI strategy with a daily rebalancing frequency. From Fig. 1.4a, we notice that, at the very beginning, the CPPI strategy exactly follows the trend of risky security: it exhibits a 100%-exposure to the risky asset. However, when the market deteriorates (due to the US subprime crisis), the CPPI portfolio flattens out on the floor. Such a circumstance sheds light on two main issues related to the CPPI mechanism. The first one is related to the initial level of the floor, which depends on the investment time horizon and the risk-free interest rate levels. Since we consider a 10-year frame with near-zero interest rate levels, we observe that the fluctuation of the floor at the beginning of the reference period will be strongly affected by this long time-to-maturity, whereby the floor will be significantly lower than the initial investment value. This low floor level determines that the entire portfolio will be invested in the risky asset, failing to guarantee protection, even in case the price of the risky asset decreases. The second issue is that the CPPI floor is not indexed to the evolution of the risky asset but depends exclusively on the risk-free rate level. As shown in Figure 1.4a, the risk-free interest rate plummeted at the onset of the subprime crisis, causing the floor to rise rapidly. The sharp rise in the floor, combined with the simultaneous fall in the market, causes the gap-risk event to occur, indicating an irreversible situation: the exposure to the risky security falls to 0%, so the initial investment will not be guaranteed at maturity. As a consequence of the drawbacks mentioned above, the CPPI strategy reaches an annualized return equal to −3.89%. The situation remains completely unchanged even if the frequency of rebalancing is reduced (weekly instead of daily observations), as shown in Figure 1.4c. A first attempt to overcome such drawbacks is considering an alternative floor definition. In other words, we move to the TIPP strategy. Looking at Figure 1.4b, we note that the initial TIPP exposure is 50%. The TIPP exposure is far lower than the CPPI one. However, after the market collapse, mainly because of the financial crisis, the exposure stands at 3% instead of cancelling. Thus, when the market gets back on track, i.e., between early 2009 and the end of 2018, the risky exposure gradually widens without exceeding 50% of the value of the entire TIPP portfolio. This behaviour of the TIPP mechanism, which is different from that of the CPPI strategy, is because the TIPP floor does not depend on the investment's time horizon, nor is it sensitive to the level of interest rates. More precisely, the TIPP floor depends on the level of protection,



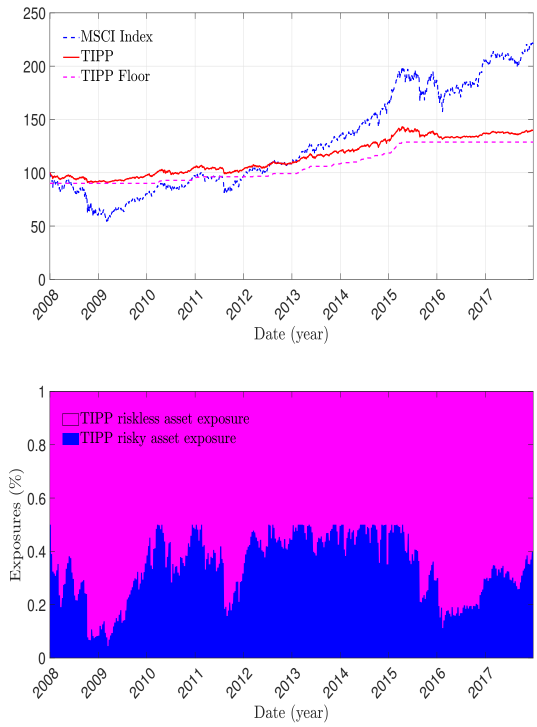
(a) CPPI Daily with 1-daily rebalancing frequency



(b) TIPP Daily with 1-daily rebalancing frequency



(c) CPPI 1-week rebalancing frequency



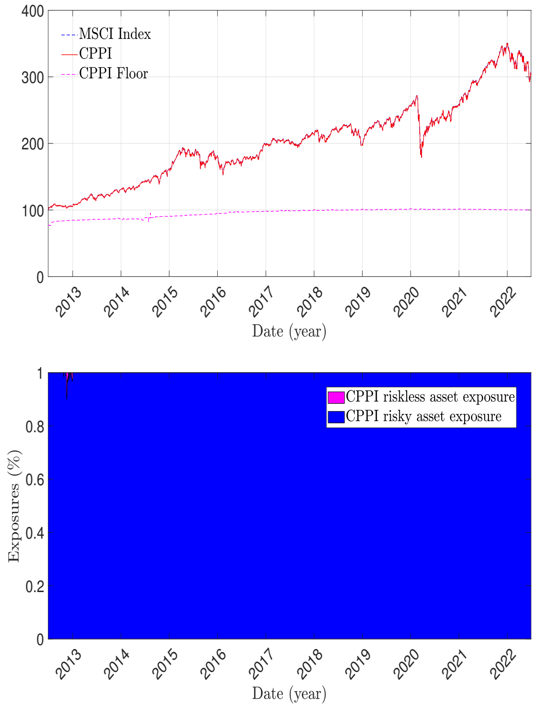
(d) TIPP 1-week rebalancing frequency

Figure 1.4 Historical simulation and investment exposures for the CPPI strategy (left charts) and the TIPP strategy (right charts) linked to the MSCI equity index. The time window ranges between December 31, 2007, and December 28, 2017. At the top of each sub-figure, we show the investment strategies' trend, while at the bottom we depict the assets' exposures.

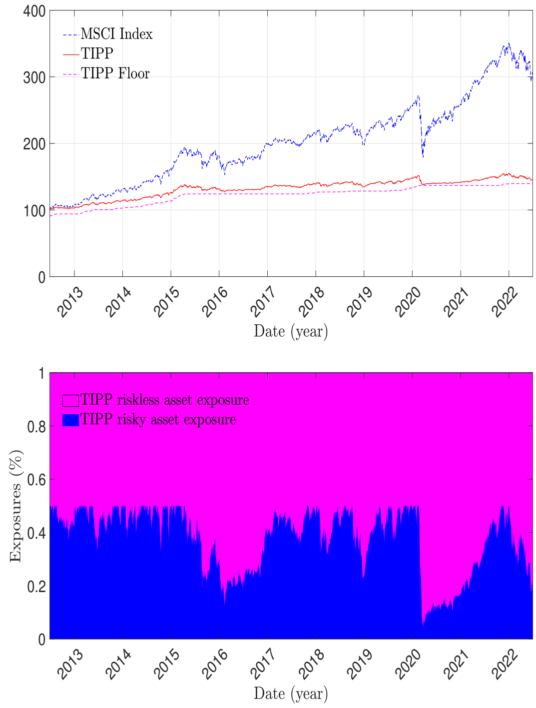
exogenously set to 90% of the initial investment, and on the risk index performance. Thus, at the beginning of the investment horizon, when the risky asset loses value under the blows of the financial crisis, the TIPP mechanism remains constant and never falls below 90%. As a result, the initial exposure is much lower than the levels obtained with the CPPI strategy. Therefore, even if a further sudden market crash occurs, the TIP strategy will be less exposed than the CPPI one. Moreover, in the recovery phase, the TIPP floor gradually increases at a rate comparable to the market, ensuring a slight but steady growth in exposure and, thus, in equity market participation. Again, the behaviour described above remains unchanged even if we vary the rebalancing frequency, as shown in Figure 1.4d.

1.3.2 Historical simulations in a late-falling-and-rising trend market

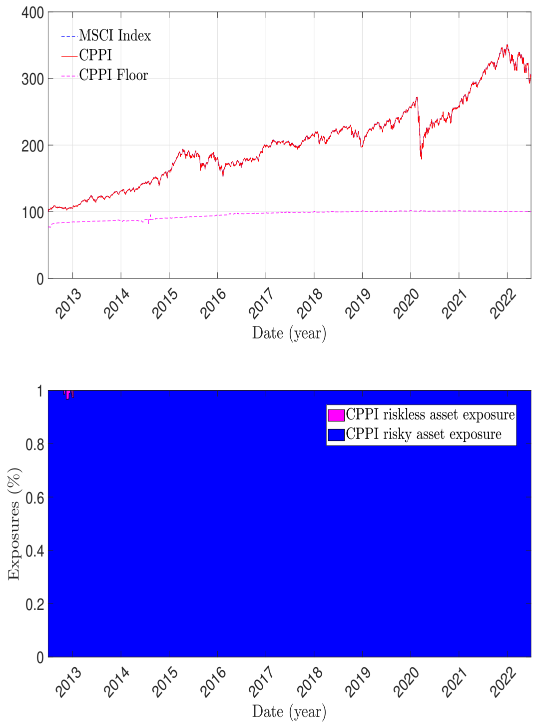
The analysis carried out in Section 1.3.1 suggests that introducing the TIPP strategy prevents gap risk. Unfortunately, this is not always true: as an example, we proceed by running a study on a different time horizon. More precisely, in this Section, we compare CPPI and TIPP strategies when the market shows a falling and rising trend at the end of the period under investigation. The results are depicted in Figure 1.5, where both the evolution of the portfolios and the corresponding exposure to the risky asset for each investment strategy are shown based on different rebalancing disciplines. This time horizon features two main characteristics: it depicts a long upward trend (from 2012 to early 2020), ending in a turbulent phase due to the COVID-19 outbreak, while, on the other hand, this decade is typified by constant risk-free interest rates close to zero. Because of these issues, the CPPI strategy behaves in a very particular way. Since the floor remains almost constant throughout the investment horizon and at low levels, the exposure to the risky asset is kept constant and equal to 100%, perfectly matching the pure risky asset investment. Moreover, due to the low floor level, the negative shock due to the pandemic outbreak does not affect the CPPI portfolio, guaranteeing a total investment in the risky security (and a consequent maximum exposure), as shown in Figure 1.5a. Thus, although the performance of the CPPI strategy is very high within such a market scenario, the level of protection the strategy should inherently provide is zero. Consequently, the CPPI strategy fails to meet the demands of potential investors seeking hedging against market risk. The same occurs with weekly rebalancing frequencies, as shown in Figure 1.5c. This issue is another reason for replacing the CPPI strategy with the TIPP one. Indeed, inspecting Figure 1.5b, we note that the TIPP portfolio no longer overlaps the risky security as with the CPPI portfolio. On the contrary, it always stays close to the floor. Even though the floor was reached during the peak of the crisis, the TIPP mechanism allows the portfolio to bounce off the floor (thanks also to a high rebalancing frequency). This behaviour allows the investor to breathe a sigh of relief: as proof of this, we note that the risky exposure is between 11% and 50% over the entire time horizon under consideration, with one exception: a 4%-downward spike coinciding with the lockdown in most European countries. After that, it nicely recovers to pre-Covid levels. Comparing the returns of the two proposed strategies, we also notice that the annualized return of TIPP is equal to 3.12%, i.e. slightly more than a quarter of the annualized return of CPPI (equal to 11.702%). In this case, the overall return is less than the CPPI one, but it provides consistent protection. For this reason, even when the interest rate level is zero, TIPP meets the needs of an investor whose only goal is to be



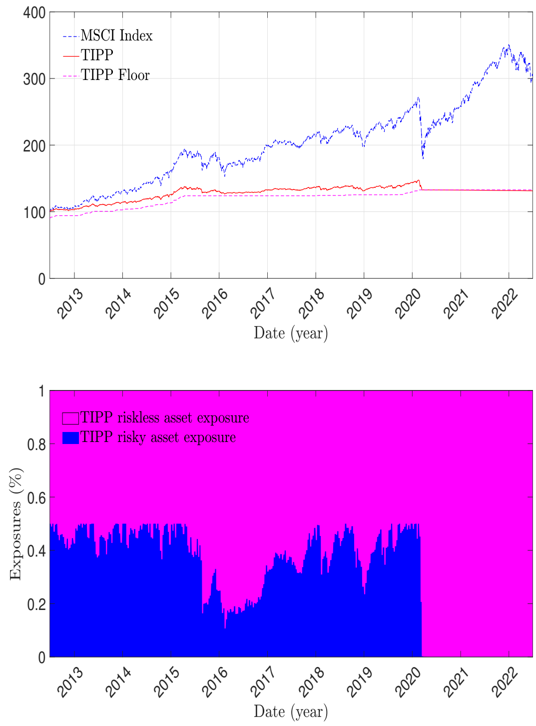
(a) CPPI 1-day rebalancing frequency



(b) TIPP 1-day rebalancing frequency



(c) CPPI 1-week rebalancing frequency



(d) TIPP 1-week rebalancing frequency

Figure 1.5 Historical simulation and investment exposures for the CPPI strategy (left charts) and the TIPP strategy (right charts) linked to the MSCI equity index. The time window ranges between June 29, 2012, and June 29, 2022. At the top of each sub-figure, we show the investment strategies' trend, while at the bottom we depict the assets' exposures.

always hedged from adverse market trends. On the other hand, when we decide to rebalance the portfolio once a week, we observe an even more extreme behaviour: the pandemic outbreak is once again and even more effectively a turning point in TIPP investments, the latter being subject to the gap risk. As shown in Figure 1.5d, the TIPP portfolio breaches through the floor. Consequently, the strategy can no longer guarantee the initial investment. Moreover, the remaining portfolio shifts entirely to the reserve asset, eliminating any equity market participation in the event of favourable market conditions, such as the market rise in early 2022.

1.4 Hedging against gap risk

As discussed in Section 1.3, the TIPP strategy is more conservative than the CPPI strategy, making it preferable for specific categories of investors. However, as shown in 1.5d, the TIPP is also subject to gap risk when sudden market crashes occur. Taking the latter risk into account is essential for at least two reasons. First, financial institutions issue this type of strategy by combining it with a guarantee for investors. Therefore, even if the floor is breached, the issuer of the PI strategy must pay the guaranteed amount at maturity to the investor who has suffered a loss. Secondly, pension fund managers often use PI strategies to protect the contributions paid by the workers during the accumulation phase of defined contribution pension funds. In the latter case, the occurrence of gap risk could cause the wealth of pension funds to fall below the desired level, i.e. the present value of pension obligations, making them underfunded; for more details, see Temocin et al. [2018]. For these reasons, several gap risk hedging methodologies and several extensions of the CPPI strategy have been proposed in the literature. In the present Section, we review some of them.

1.4.1 Option-based strategies for hedging gap risk

The issuer of portfolio insurance strategies usually incorporates a guarantee for the investor within this type of product. Consider a CPPI portfolio invested over a finite time investment horizon $[0, T]$ with $T > 0$, where the investors pay the initial value $W^{CPPI}(0)$ and are guaranteed to receive at least the value G at time T. If the portfolio value $W^{CPPI}(T)$ at time T is less than G, the issuer pays to the investor the shortfall amount $G - W^{CPPI}(T)$. Hence, the amount $\max\left\{W^{CPPI}(T), G\right\}$ such that

$$\max\left\{W^{CPPI}(T), G\right\} = G + \max\left\{C(T), 0\right\} = G + C(T)\mathbb{1}_{\{C(T)>0\}} \tag{1.45}$$

gives the final payoff for the investor, and the CPPI guarantor will be subject to a gap risk of $C(T)\mathbb{1}_{\{C(T)\leq 0\}}$. In practice, the issuer usually provides this guarantee against fee payment at $t = 0$. Such an initial fee is given by the present value of the CPPI loss, which, assuming a constant risk-free rate, equals

$$P^{Fee} = \mathbb{E}^{\mathbb{Q}}\left[\min\left\{C(T), 0\right\} e^{-rT}\right], \tag{1.46}$$

where $\mathbb{Q}$ is a risk-neutral probability measure. Given such a fee, the issuer constructs a hedging strategy to protect against gap risk events using options written on the same risky asset of the CPPI strategy. A first possible hedging strategy for CPPI can be constructed using short-maturity put options. This hedging strategy is based on the consideration that breaching the floor is equivalent to the cushion becoming negative. Assuming that this event never occurred until time t_k, using equation (1.23), we have that

$$C(t_{k+1}) < 0 \iff m\frac{S(t_{k+1})}{S(t_k)} - (m-1)\, e^{r\frac{T}{\bar{n}}} < 0, \qquad \text{for all } k = 0, \dots, \bar{n}-1. \tag{1.47}$$

Hence, hedging gap risk is equivalent to forcing the quantity in equation (1.47) to be strictly greater than 0. This can be done by buying at each rebalancing instant t_k a put option written on the same underlying of CPPI strategy with a strike price equal to $\left(1 - \frac{1}{m}\right) e^{r\frac{T}{\bar{n}}} S(t_k)$ and maturity given by the rebalancing frequency, that is $\frac{T}{\bar{n}}$. To hedge the entire CPPI portfolio, the issuer needs a number of put options equal to the strategy's total exposure to the risky asset, that is $m\frac{C(t_k)}{S(t_k)}$, for all $k = 0, \dots, \bar{n}-1$. Thus, the discounted payoff of the puts at each rebalancing instant is given by

$$e^{-r\frac{T}{\bar{n}}} C(t_k) \left((m-1)e^{r\frac{T}{\bar{n}}} - m\frac{S(t_{k+1})}{S_{t_k}} \right)^+, \tag{1.48}$$

and the corresponding hedging cost reads as

$$\text{Cost}(t_k) = m\frac{C(t_k)}{S(t_k)} \mathbb{E}^{\mathbb{Q}} \left[\left(\left(1 - \frac{1}{m}\right) e^{r\frac{T}{\bar{n}}} S(t_k) - S(t_{k+1}) \right)^+ \middle| \mathcal{F}(t_k) \right], \tag{1.49}$$

for all $k = 0, \dots, \bar{n}-1$. An alternative action for gap risk mitigation lies in using the *Gap Option*, defined as follows.

Definition 1.4.1. *Denote by*

$$r^S(t_k) = \frac{S(t_k)}{S(t_{k-1})} - 1$$

the standardize simple return of the risky asset S of the k-th period of the time interval $[0, T]$, $\tilde{\alpha}$ the return level which triggers the gap event, and $\tau^{\tilde{\alpha}}$ the time of the first gap event that is

$$\tau^{\tilde{\alpha}} := \inf \left\{ t_k \in [0, T] : 1 + r^S(t_k) \le \tilde{\alpha} \right\},$$

for all $k = 1, \dots, n$. The Gap Option pays to its holder an amount $f(1 + r^S(\tau))$ at time $\tau^{\tilde{\alpha}}$ if $\tau^{\tilde{\alpha}} \le T$ and 0 otherwise.

This kind of option, able to protect the investor against sudden and persistent market downside moves, have been extensively studied within the CPPI strategy by Tankov [2010] and Jessen [2014]. More precisely, the authors argue that the issuer of the CPPI strategy has to purchase a Gap Option whose threshold $\tilde{\alpha} = \frac{(m-1)e^{r\frac{T}{\bar{n}}}}{m}$ and whose payoff at $\tau^{\tilde{\alpha}} < T$ is given by

$$f^{\tilde{\alpha}}(1 + r^S(\tau)) = \tilde{\alpha} - \left(1 + r^S(\tau)\right),$$

in order to immunize himself from gap risk.

1.4.2 Some generalizations of the CPPI strategy

Another strand of literature introduces several extensions of the CPPI strategy to eliminate or at least mitigate the effects of gap risk. We briefly describe the most popular ones.

Proportional portfolio insurance strategies. The first extension is to allow the multiplier to vary over time. This modification leads to the so-called *proportional portfolio insurance* (PPI) strategies. As for CPPI, the PPI strategies are self-financing and dynamically allocate wealth $W^{PPI} = \left\{W^{PPI}(t),\ t \in [0,T]\right\}$ between the risky asset S and the reserve asset P according to the following allocation rule

$$\alpha^{PPI}(t) = m(t) \cdot \frac{\left(W^{PPI}(t) - F(t)\right)^+}{W^{PPI}(t)}, \quad t \in [0,T]. \tag{1.50}$$

Differently from equation (1.9), the multiplier $m = \{m(t),\ t \in [0,T]\}$ is no longer constant and fixed at time $t = 0$ based on the investor's risk aversion but varies over time in order to better adapt the strategy's exposure to market fluctuations. Depending on how the portfolio manager sets the multiplier, we can distinguish among several PPI strategies.
The first one is the *exponential proportion portfolio insurance* (EPPI) strategy, proposed by Lee et al. [2008] that allows the multiplier to vary over time according to a pre-specified rule which reads as

$$m(t) = \bar{\eta} + \exp\left\{\hat{a} \cdot \ln\left(\frac{S(t)}{\tilde{S}}\right)\right\}, \quad t \in [0,T], \tag{1.51}$$

where $\bar{\eta} > 1$ is an arbitrary constant, and $\exp\left\{\hat{a} \cdot \ln\left(\frac{S(t)}{\tilde{S}}\right)\right\}$ is the dynamic multiplier adjustment factor (DMAF). More precisely, $\hat{a}$ is a constant parameter strictly greater than one that accounts for the enlargement (resp. shrinkage) effect of DMAF in the case of bullish (resp. bearish) markets, and $\tilde{S}$ is a reference level for the risky asset set at $t = 0$ by the fund manager. The authors argue that a dynamic multiplier can enhance the convex nature of the CPPI strategy. Indeed, when the stock price increases, becoming greater than the reference level $\tilde{S}$, the multiplier increases as well leading to an increase in exposure and improving the upside capture of the strategy. By contrast, when the stock price decreases relative to the reference level, the same happens for the multiplier, reducing exposure and inducing a higher downside protection. Hence, thanks to this mechanism, the EPPI strategy should be able to mitigate the probability of gap risk occurrence and simultaneously provide equity market participation in the event of market upside.
However, as shown in Mancinelli and Oliva [2023] using a non-parametric simulation analysis, the EPPI strategy can only provide better equity market participation in bull markets than the CPPI and TIPP strategy and not better downside protection. Such a drawback is because such a time-varying multiplier is set based on a pre-specified rule, and it is independent of expectations of future market fluctuations. To accommodate the latter feature into the dynamic multiplier, several authors, including Jiang et al. [2009], Hamidi et al. [2009], Ameur and Prigent [2014], Hamidi et al. [2014], introduce the concept of *conditional multiplier* based on the following local quantile

guarantee condition

$$\mathbb{P}\left(C(t_{k+1}) > 0 | C(t_k) > 0\right) = \mathbb{P}\left(m(t_k)\frac{S(t_{k+1})}{S(t_k)} - (m(t_k) - 1) > 0\right) \geq 1 - \varepsilon, \tag{1.52}$$

for all $k = 0, \ldots, \bar{n} - 1$, where $\varepsilon \in [0, 1]$ is an upper bound on the shortfall probability exogenously set by the fund manager at $t = 0$. As shown in Hamidi et al. [2014], equation (1.52) implies an upper bound $\bar{m}(t_k)$ on the multiplier which is given by

$$\bar{m}(t_k) = \left| \mathbb{E}^{\mathbb{P}}\left[\frac{S(t_{k+1})}{S(t_k)}\middle|\mathcal{F}(t_k)\right] - 1 + \sqrt{\operatorname{Var}\left[\frac{S(t_{k+1})}{S(t_k)}\middle|\mathcal{F}(t_k)\right]}\tilde{F}_{t_k}^{-1}(\varepsilon)\right|^{-1}, \tag{1.53}$$

where $\tilde{F}_{t_k}(\cdot)$ is the cumulative distribution function of the standardized simple return $r^S(t_{k+1})$, for all $k = 0, \ldots, \bar{n} - 1$. Then the fund manager sets the multiplier for each time instant t_k in order to not exceed the time-varying upper bound $\bar{m}(t_k)$, which in this setup, affords the role of gap risk control. However, as argued by Dichtl et al. [2017], the superiority of these strategies in hedging against gap risk depends solely on the forecasting skills of the fund manager or, in other words, on the methodologies used to estimate the upper bound $\bar{m}(t_k)$ in equation (1.53).

CPPI with conditional floor. To hedge against gap risk or to avoid cash lock positions, Ameur and Prigent [2011] and Ameur and Prigent [2018] introduce some modifications of the CPPI strategy with dynamic floor evolving according to market conditions. In particular, the main idea behind such a new strategy is to introduce a *conditional floor* that is always higher than the target floor F, which characterizes the guarantee constraint G at maturity. Unlike the target floor, the conditional floor does not evolve according to the risk-free rate r. In contrast, it is set according to the fund manager expectations of future market developments. In this framework, the authors distinguish between two main operational methods. The first one, the *margin-based* CPPI strategy, prevents cash-locked positions. If the market is bearish, the conditional floor is reduced, so the strategies' exposure remains significantly positive. This reduction allows one to take advantage of potential future market rises, whereas, for the standard CPPI method, the final portfolio value can be equal only to the guaranteed amount G, leading to a return less than the riskless one. The second one is the CPPI with a *ratchet effect* that allows to lock in part of the past gains whatever the future significant market drawdowns. The working mechanism of the latter strategy is very similar to that of TIPP. However, unlike TIPP, Floor reappraisal is no longer based on a pre-specified rule (see equation (1.44)) but according to market fluctuations. More precisely, the floor is based on the quantile condition depicted in equation (1.52), which Ameur and Prigent [2018] specifically adapt to these two kinds of PI strategies.

Chapter 2

Time invariant PI strategies in structured products with GMEE

From the 2008 worldwide economic crisis, financial arenas have started to see an increase in the equity market along with a related decrease in interest rate levels until a generalized drop happened in 2018, then accompanied by abrupt volatility changes caused by the *Covid Crisis* in 2020. Consequently, many investors are increasingly interested in investment products providing strong capital protection at maturity. In particular, they have commenced looking at investment products to protect their portfolios from significant losses and, at the same time, to offer participation in equity (or equity indices) with an attractive participation rate. Structured investment products have been offered in the market to address these needs. These products provide capital protection and the opportunity to participate in equity markets through embedded financial derivatives. However, designing a structured investment product combining high capital protection with high participation is challenging because such features are highly correlated. Indeed, high capital protection negatively affects the amount of the available risk budget and, consequently, the amount that can be invested in financial derivatives. Moreover, the capital protection level is affected by market interest rates, and the participation rate is affected by both interest rate levels and market volatility. However, as argued by Allen and Gale [1994], Mouna and Anis [2016], and Albeverio et al. [2017], a combination of a low interest rate and high market volatility leads to a meagre participation rate, making the guarantee structure unattractive for investors.

For this reason, a provider of a structured investment product with capital protection is interested in increasing the attractiveness of his/her product to an investor by simultaneously providing a high protection level and market participation rate. Assuming that the protection level of a structured investment product is fixed, the main factor affecting the participation rate is the price of the embedded option. In order to obtain the above desirable features from the standpoint of a product provider, the price of the embedded option should not depend significantly on the market volatility, and it should decrease with decreasing market interest rate levels.

In the past, to obtain the above features, investment product providers replaced plain vanilla European options with more complex exotic derivatives written on the same underlying risky asset, such as barrier options or Asian options. More recently, a new family of structured investment products with capital protection emerged on the market. Their distinguishing characteristic is not an embedded option with an exotic payoff but a plain vanilla option linked to a modified underlying asset. In particular, traditional risky assets such as stock indices are replaced with volatility target (VolTarget) strategies. Investment mechanisms of this nature have been extensively utilized in portfolio management, as shown by Doan et al. [2018], and pension funds or annuity provisions, as evidenced by Olivieri et al. [2022]. In the context of financial applications, Albeverio et al. [2013] offer the first comprehensive analysis of the VolTarget strategies, characterizing them as the underlying asset in an option embedded in a structured investment product.

In the present chapter, we continue the aforementioned line of research by considering options written on particular dynamic asset allocation algorithms: the PI strategies. More precisely, we focus on the TIPP strategy, described in Section 1.3. However, although the TIPP strategy can be considered an improved version of the CPPI strategy, as argued in Dichtl et al. [2017] and shown in Figure 1.5d, it is characterized by path dependency and subject to gap risk. Indeed, in extreme market scenarios, when the value of a risky asset decreases significantly, the whole portfolio needs to be invested into a riskless asset. As a result, the investor loses the possibility to participate in potential upward movements of the risky asset. For this reason, the TIPP strategy cannot be viewed as the underlying of a European option embedded in a structured investment product with capital protection. Consequently, we propose modifying the standard TIPP strategy by adding an investment threshold in the risky asset allocation, the so-called *guaranteed minimum equity exposure* (GMEE). The GMEE has been introduced in Di Persio et al. [2020] within the CPPI framework to buffer the problem of equity market participation when a cash-lock event or gap risk occurs. Also in this case, we propose adding a guaranteed minimum equity exposure to the TIPP allocation mechanisms to overcome the risk of low market participation in a V-shaped market environment.

Hence, for our approach, we introduce and evaluate suitable equity-linked instruments, namely European-type OTC call options whose underlying is the TIPP strategy endowed with a GMEE mechanism. We evaluate these new types of exotic options and compare them with plain vanilla European call options and options written on G-CPPI. From a practical perspective, such a comparison makes a roundup of the possible proposals that an advisor may present to his clients according to the views on future market developments.

We present an accurate and comprehensive numerical analysis for option prices in a Vasicek-Heston framework, involving different time horizons for option maturities and various rebalancing frequencies without considering transaction costs. Our numerical findings, corroborated by an in-depth sensitivity analysis, reveal that the TIPP strategy endowed with GMEE is the cheapest procedure among those examined for short and medium time horizons, further confirming both the effectiveness and the usefulness of our new approach.

The rest of the chapter is organized as follows. Section 2.1 introduces the new TIPP strategy with guaranteed minimum equity exposure. In order to better clarify the advantages and shortcomings of this new PI strategy and emphasize the role of GMEE, the main basic definitions and formulae are supported by historical simulations. More general results are presented in Section 2.2, where the theoretical framework for pricing plain vanilla options on the stock, the CPPI, the TIPP and the new G-TIPP is described. Finally, Section 2.3 performs an in-depth numerical analysis using Monte

Carlo simulations to highlight the differences between the aforementioned options and proposes a detailed sensitivity analysis of our results concerning both market parameters and exogenous PI strategies' parameters.

2.1 TIPP with guaranteed minimum equity exposure

In the present Section, we propose a modified version of the TIPP strategy. In particular, as discussed in Section 1.3, the floor that characterizes the TIPP strategy partially solves the problem related to the cash locked position. Indeed, by exploiting the TIPP strategy, we can maintain a pre-specified percentage of past gains. Therefore, even if the cash lock event occurs at time $\tau^* < T$, the TIPP strategy can preserve the equity market participation obtained during the time interval $[0, \tau^*)$. However, the problem of obtaining equity market participation persists for each $t \in [\tau^*, T]$. Moreover, introducing the TIPP strategy cannot solve the gap risk. Indeed, as shown in Figure 1.5d, in the face of sudden collapses in the price of risky assets, the value of the TIPP portfolio may break the floor. In this case, even the TIPP may not return the guaranteed capital G at maturity and retain all equity market participation accumulated until the gap risk occurs.
We solve the problem of obtaining equity market participation by adding a threshold into the TIPP allocation mechanism called guaranteed minimum equity exposure (GMEE). We baptize the resulting strategy *time-invariant portfolio protection with guaranteed minimum equity exposure* (G-TIPP). As for the TIPP strategy, the G-TIPP is obtained by self-financing the investment the reserve asset P and in the risky asset S. Therefore, the dynamics of the G-TIPP portfolio $W^{G-TIPP} = \left\{W^{G-TIPP}(t),\ t \in [0,T]\right\}$ is given by

$$\begin{cases} \mathrm{d}W^{G-TIPP}(t) = W^{G-TIPP}(t) \left[\alpha^{G-TIPP}(t) \dfrac{\mathrm{d}S(t)}{S(t)} + \left(1 - \alpha^{G-TIPP}(t)\right) \dfrac{\mathrm{d}P(t)}{P(t)}\right], \quad t \in [0,T], \\ W^{G-TIPP}(0) = w_0^{G-TIPP}, \end{cases} \tag{2.1}$$

where

$$\alpha^{G-TIPP}(t) = \max\left\{\min\left\{L_{max},\ m \cdot \frac{\left(W^{G-TIPP}(t) - F^{TIPP}(t)\right)^+}{W^{G-TIPP}(t)}\right\},\ \alpha_{min}\right\}, \quad t \in [0,T], \tag{2.2}$$

with $\alpha_{min} \in [0,1]$ being the guaranteed minimum equity exposure. Thanks to this new allocation mechanism, we are sure that at least an α_{min} per cent of the G-TIPP remains invested into the risky asset S even though gap risk or cash-locked position occurs. Therefore, our new strategy can capture all potential market upturns by ensuring equity market participation.

Remark 2.1.1. *The G-TIPP strategy can be seen as a generalization of the standard TIPP strategy. Indeed, in the case of $\alpha_{min} = 0$, the G-TIPP strategy collapses into the standard TIPP strategy.*

Remark 2.1.2. *The introduction of the guaranteed minimum equity exposure (GMEE) is not exclusive to time-invariant strategies, as we have shown above. The construction of such an allocation mechanism has recently been introduced in the literature in the case of CPPI strategies,* *see, e.g., Di Persio et al. [2020]. Here we do not provide the details of the CPPI strategy with GMEE (which will be denoted as G-CPPI for the sake of notation uniformity). However, we deem it appropriate to consider it, as it is used as an additional benchmark for derivative pricing, as we will see in detail in Section 2.3.*

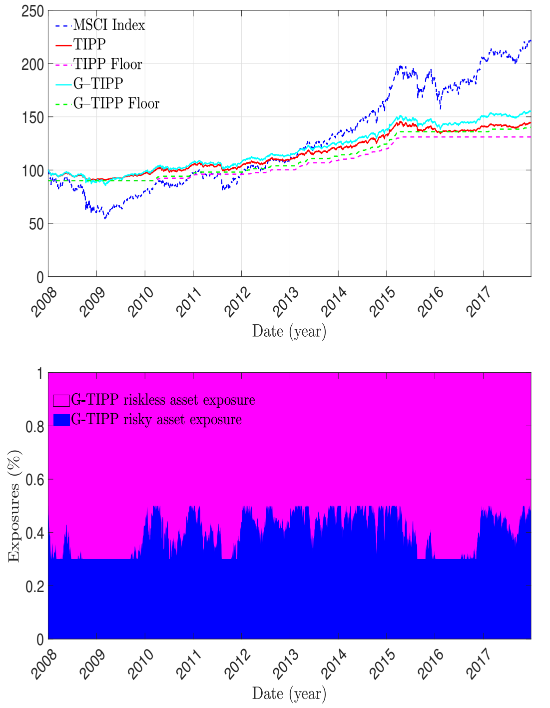
In order to better understand how the G-TIPP strategy works and its advantages and shortcomings, we perform a simulation study comparing such a new strategy with its standard version. Specifically, we perform the simulation of the G-TIPP strategy on the same data used in Section 1.3 for the historical simulation of the TIPP strategy. Moreover, to make this comparison meaningful, we set the same exogenous parameters used for the TIPP simulation in Section 1.3.

2.1.1 Historical simulations of G-TIPP

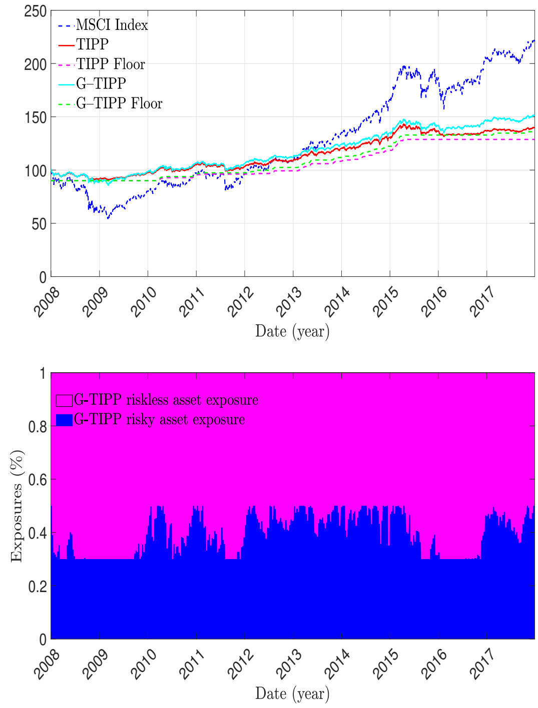
As discussed in Sections 1.3.1 and 1.3.2, the TIPP strategy's exposure to risky assets is generally lower than that of the CPPI, varying more regularly over time and leading to a higher level of protection. Furthermore, the historical simulation depicted in Figure 2.1d revealed that introducing the TIPP algorithm instead of the CPPI one is insufficient to avoid gap risk. Such a drawback implies the need to introduce the GMEE to make the TIPP suitable for inclusion in structured investment products with capital protection. We simulate the TIPP with GMEE over the same historical data used in Section 1.3 to highlight the effect of introducing such a new threshold. We choose a value for the guaranteed minimum equity exposure equal to $\alpha_{min} = 30\%$. The results are shown in Figure 2.1. We start by studying the 2007-2017 decade, assuming a daily portfolio rebalancing, see Figure 2.1a. The guaranteed minimum equity exposure causes the exposure to the risky asset of the G-TIPP to be higher than that of the standard TIPP. Such a more risky profile, coupled with the market crash during the 2008 financial crisis, led to the G-TIPP strategy breaking the floor. However, as shown in equation (2.2), the presence of the GMEE induces only transitory gap events. Indeed, such an intrinsic mechanism of the TIPP logic with the GMEE ensures that a pre-specified percentage of the portfolio always remains invested in the risky asset. Hence, when the market recovers, the G-TIPP can capture equity market participation better than the TIPP, resulting in a higher total return, see Table 2.1. The same pattern also occurs for the G-TIPP with weekly rebalancing, as shown in Figure 2.1b. The role played by GMEE in defending investment from preventing equity market participation is even more highlighted during the 2012-2022 decade.

Investment strategy	MSCI World	TIPP (d)	G-TIPP (d)	TIPP (w)	G-TIPP (w)
Annualized return (%)	5.62	2.55	3.07	2.33	2.88
Max loss 1-day (%)	−8.56	−1.54	−2.55	−1.89	−2.55
Annualized volatility (%)	17.67	4.79	6.10	4.91	6.15
Average Exposure (%)	–	35.03	39.70	34.26	39.49
Maximum average exposure (%)	–	50.00	50.00	50.00	50.00
Minimum average exposure (%)	–	3.90	30.00	4.46	30.00

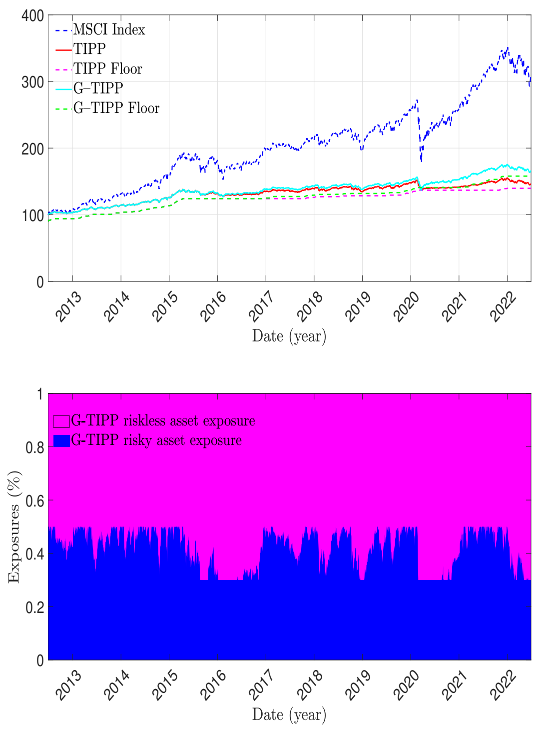
Table 2.1 Risk-return performances (in %) of MSCI World Net (Eur), TIPP strategy, G-TIPP strategy with daily (d) and weekly (w) rebalance frequencies. The time window ranges from December 31, 2007, to December 28, 2017.



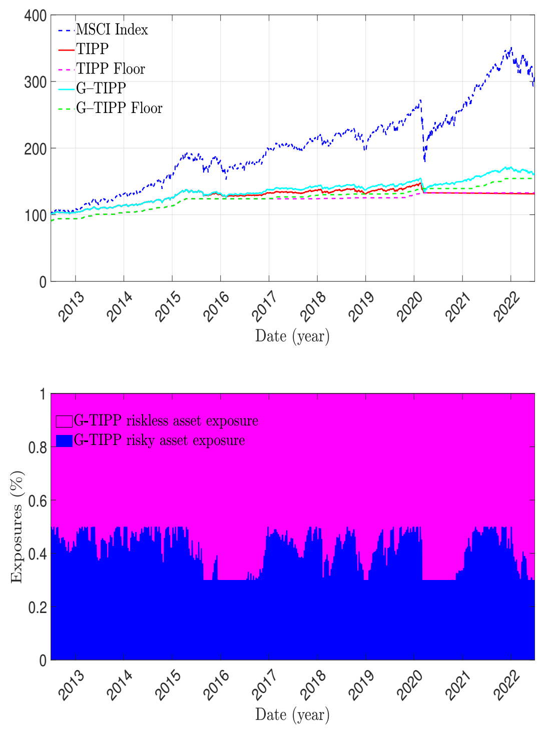
(a) G-TIPP with daily rebalancing frequency



(b) G-TIPP with weekly rebalancing frequency



(c) G-TIPP with daily rebalancing frequency



(d) G-TIPP with weekly rebalancing frequency

Figure 2.1 Historical simulation and risky exposures for TIPP with GMEE, linked to MSCI equity index. The time windows range between December 31, 2007, and December 28, 2017 (top charts), June 29, 2012, and June 29, 2022 (bottom charts).

As shown in Figure 2.1d, the standard TIPP with weekly rebalance is subject to gap risk due to market turbulence at the start of the COVID-19 pandemic. Hence, from that moment on, the exposure drops to zero, so the TIPP cannot capture the subsequent sharp rise in the market. This is not the case for the G-TIPP because when the floor is broken, the GMEE is triggered, causing 30% of the portfolio to remain invested in the index. As a result, the G-TIPP portfolio turned above the floor by capturing the market's strong bullishness and ending the investment with a strongly positive return. Therefore, the GMEE acts as a buffer and guarantees participation in bull markets. For ease of reading, we summarise the strategies' performances. More precisely, we report the 2007-2017 analysis in Table 2.1 and the 2012-2022 analysis in Table 2.2.

Investment strategy	MSCI World	TIPP (d)	G-TIPP (d)	TIPP (w)	G-TIPP (w)
Annualized return (%)	7.95	2.63	3.52	1.90	3.37
Max loss 1-day (%)	−8.09	−1.57	−2.43	−2.19	−2.43
Annualized volatility (%)	14.74	4.68	5.46	4.36	5.56
Average Exposure (%)	–	37.64	41.90	30.30	41.66
Maximum average exposure (%)	–	50.00	50.00	50.00	50.00
Minimum average exposure (%)	–	4.34	30.00	0	30

Table 2.2 Risk-return parameters (in %) of MSCI World Net (Eur), TIPP strategy and G-TIPP strategy with daily (d) and weekly (w) rebalance frequencies. The time window ranges from June 29, 2012, to June 29, 2022.

2.2 Option pricing with portfolio insurance strategies

The introduction of guaranteed minimum equity exposure within PI strategies solves the problem of equity market participation when the market recovers after the occurrence of gap risk or a cash-locked position. Indeed, as shown in Figures 2.1a and 2.1d, the G-TIPP can always go back above the floor in a bull market. However, if the market continues to fall after the event that generates the gap risk, the G-TIPP would realise a more significant loss than its standard version. For this reason, to guarantee both equity market participation and strong capital protection at maturity, we propose inserting the G-TIPP into a capital-guaranteed structured investment product. The risky part of such an investment product will be a European call option written on the G-TIPP. In the present Section, we describe structured investment products with guaranteed capital and introduce a pricing model for options written on the G-TIPP.

2.2.1 Investments with capital protection and market participation

We start by fixing a finite investment time horizon T which coincides with the maturity of the structured investment product with capital protection, and a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\mathbb{Q}$ represents a risk-neutral probability measure. We also introduce a complete and right-continuous filtration $\mathbb{F} = \{\mathcal{F}(t),\ t \in [0,T]\}$, representing the global information flow. In particular, we assume that all processes are adapted to $\mathbb{F}$.

A structured investment product with capital protection, say $A = \{A(t),\ t \in [0,T]\}$, is designed so that the initial investment $V(0)$ is split into two parts: the first one is invested in ZBCs to ensure a pre-specified capital protection level (PL), while what remains is invested in financial instruments allowing direct participation in a risky asset $S = \{S(t),\, t \in [0,T]\}$, like e.g. an equity index. The payoff at maturity T of such a structured investment product with capital protection is given by

$$A(T) = \max\left\{V(0)\cdot PL,\, V(0)\cdot PL\cdot\left(1+\xi\cdot\frac{S(T)-S(0)}{S(0)}\right)\right\}, \tag{2.3}$$

where $\xi > 0$ is the so-called *participation rate*; see, e.g., Albeverio et al. [2013] for further details. The payoff in equation (2.3) shows that at maturity T, the investor will receive at least a pre-determined percentage (PL) of the capital initially invested, or he/she will receive $PL \cdot V(0)$ plus a relative return on the principal investment $PL \cdot V(0)$ in the risky asset S over the investment horizon $[0,T]$, multiplied by the participation rate ξ, provided that the return on S is positive.

From (2.3), we find out that the higher the protection level, the higher the protection to the capital initially invested $V(0)$ in case of negative returns from the investment in an equity index. On the other hand, a high participation rate ξ increases the overall payoff of the structured investment product in case of positive returns from the risky asset investment. Thus, the issuer of a structured investment product with capital protection considers his/her product to be attractive for the investors if it either admits *(i)* a high capital protection level PL or *(ii)* a high participation rate ξ. However, the features mentioned above are strongly interrelated: a high capital protection level negatively affects the amount of the available risk budget, leading to a reduction in the participation rate ξ, especially within low interest rate levels. Therefore, it is crucial to derive an explicit expression for the participation rate. Using simple algebraic transformations, we can equivalently recast equation (2.3) as

$$A(T) = V(0)\cdot PL + V(0)\cdot PL\cdot\xi\cdot\max\left\{\frac{S(T)}{S(0)}-1,\, 0\right\} \tag{2.4}$$

The first term in equation (2.4), namely $PL \cdot V(0)$, is the zero-coupon bond value at maturity T, while the second term is the payoff of a European call option linked to the underlying S, with strike price $V(0)$. Thus, to guarantee the payoff (2.4) at maturity T, the issuer of such a structured investment product must invest at time $t = 0$ the amount $PL \cdot V(0) \cdot P(0,T)$ into the zero-coupon bond. The remaining part of the initial investment $V(0)$ represents the so-called risk budget (RB), namely

$$RB = V(0)\cdot(1 - PL\cdot P(0,T))\,. \tag{2.5}$$

The risk budget is invested at $t = 0$ in a European call option with maturity T, underlying risky asset S, and payoff that read as follows

$$g_{dp}(x) = \max\left\{\frac{V(0)}{S(0)}x - V(0),\, 0\right\} = V(0)\cdot f_{dp}(x), \quad \forall x > 0, \tag{2.6}$$

where

$$f_{dp}(x) = \max\left\{\frac{x}{S(0)} - 1,\, 0\right\}. \tag{2.7}$$

In particular, assuming $V(0) = S(0)$, the payoff function shown in equation (2.7) is that of a European call option with underlying risky asset S, maturity T, which is at-the-money at $t = 0$. We denote by $\mathcal{O}^S_{g_{dp}} = \left\{\mathcal{O}^S_{g_{dp}},\ t \in [0,T]\right\}$ (resp., $\mathcal{O}^S_{f_{dp}} = \left\{\mathcal{O}^S_{f_{dp}},\ t \in [0,T]\right\}$) the no-arbitrage price of the European call option with the payoff function g_{dp} (resp., f_{dp}). From eq. (2.6), it follows that

$$\mathcal{O}^S_{g_{dp}}(t) = V(0) \cdot \mathcal{O}^S_{f_{dp}}(t), \quad t \in [0,T]. \tag{2.8}$$

Definition 2.2.1. *The participation rate ξ is the number of units of the option with payoff g_{dp} that can be purchased, with the available risk budget, at time $t = 0$, given by*

$$\xi = \frac{RB}{\mathcal{O}^S_{g_{dp}}(0)}. \tag{2.9}$$

In particular, by substituting (2.5) *and* (2.8) *into* (2.9), *we have that*

$$\xi = \frac{1 - PL \cdot P(0,T)}{\mathcal{O}^S_{f_{dp}}(0)}. \tag{2.10}$$

From (2.10), we find that the participation rate can be increased by either increasing the risk budget or decreasing the price of the embedded ATM-European call option. More precisely, we observe that the former is mainly affected by the price of the ZCB maturing at T and the protection level (PL), and the latter is primarily driven by stock volatility. Therefore, in a market scenario characterized by low level interest rate levels (which correspond to high prices for the ZCBs) and high volatility of the underlying risky asset, issuers can only maintain a considerable participation rate ξ reducing the protection level PL. Unfortunately, such a choice would make these kinds of structured products less attractive to the investor. To overcome the previous drawback, the issuer could invest in derivative products, such that:

($\boldsymbol{C1}$) the price of the embedded option should not significantly depend on the volatility of the underlying asset and

($\boldsymbol{C2}$) the price of the embedded option should decrease with a decreasing market interest rate, and it should increase with an increasing market interest rate.

To meet ($\boldsymbol{C1}$) and ($\boldsymbol{C2}$), providing capital protection and guaranteeing market participation, we introduce a new structured product in which the embedded option is written on the TIPP strategy endowed with the guaranteed minimum equity exposure.

Remark 2.2.2. *It is worth noting that the options on G-TIPP (as well as options on G-CPPI) are not traded on the market. However, the latter can be bought by an investment bank or a re-insurance partner, who guarantees the TIPP-GMEE strategy payoff. The issuer must achieve the latter by implementing a dynamic hedging strategy (according to either CPPI or TIPP mechanism), also considering the probability that the OTC option ends in the money or not. Hence, although the TIPP-GMEE is not traded on a public exchange, it is a full-fledged derivative product that different market participants and financial engineering groups can purchase .*

2.2.2 The model

We assume that the financial market consists of ZCBs with maturity T and a stock S. We model the instantaneous interest rate $r = \{r(t),\ t \in [0,T]\}$ according to the Vasicek model, whose dynamics under $\mathbb{Q}$ reads as

$$\begin{cases} \mathrm{d}r(t) = \alpha_r\left(\beta_r - r(t)\right)\mathrm{d}t + \sigma_r \mathrm{d}Z_r(t), & t \in [0,T], \\ r(0) = r_0. \end{cases} \tag{2.11}$$

where $\alpha_r > 0$ is the speed of reversion, $\beta_r \in \mathbb{R}$ is the long-run mean, and $\sigma_r > 0$ is the instantaneous volatility of the interest rate. Moreover, $Z_r = \{Z_r(t),\ t \in [0,T]\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$.

Proposition 2.2.3. *Let $r = \{r(t),\ t \in [0,T]\}$ be the instantaneous interest rate process with dynamics (2.11). Then, the arbitrage-free t-price of a ZCB paying one unit of currency at its expiration date T, denoted by $P(t,T)$, satisfies*

$$P(t,T) = \exp\left\{h_1(t,T) - h_2(t,T)r(t)\right\}, \quad t \in [0,T], \tag{2.12}$$

where

$$\begin{aligned} h_1(t,T) &:= \left(\beta_r - \frac{\sigma_r^2}{2\alpha_r^2}\right)\left[h_2(t,T) - (T-t)\right] - \frac{\sigma_r^2}{4\alpha_r} h_2^2(t,T), \\ h_2(t,T) &:= \frac{1 - e^{-\alpha_r(T-t)}}{\alpha_r}. \end{aligned}$$

Moreover, the ZCB price process satisfies the following backward stochastic differential equation (BSDE)

$$\begin{cases} \dfrac{\mathrm{d}P(t,T)}{P(t,T)} = r(t)\mathrm{d}t - \sigma_P(t,T)\mathrm{d}Z_r(t), & t \in [0,T], \\ P(T,T) = 1, \end{cases} \tag{2.14}$$

where $\sigma_P(t,T) := h_2(t,T)\sigma_r > 0$, $t \in [0,T]$, is the volatility of the ZCB price.

Proof. See Vasicek [1977]. □

The second asset in the financial market is a stock that evolves according to the Heston's stochastic volatility model, see Heston [1993] for further details. Its $\mathbb{Q}$-dynamics read as

$$\begin{cases} \mathrm{d}S(t) = S(t)r(t)\mathrm{d}t + S(t)\sqrt{v(t)}\mathrm{d}Z_S(t), & S(0) = s_0 \geq 0, \\ \mathrm{d}v(t) = k_v\left(\theta_v - v(t)\right)\mathrm{d}t + \sigma_v\sqrt{v(t)}\mathrm{d}Z_v(t), & v(0) = v_0 \geq 0, \end{cases} \tag{2.15}$$

for every $t \in [0,T]$. Moreover, $k_v > 0$ is the speed of reversion, $\theta_v > 0$ is the long-run mean, and $\sigma_v > 0$ is the standard deviation of the variance process $v = \{v(t),\ t \in [0,T]\}$. $Z_S = \{Z_S(t),\ t \in [0,T]\}$ and $Z_v = \{Z_v(t),\ t \in [0,T]\}$ are standard Brownian motions on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ such that

$$\langle \mathrm{d}Z_S(t), \mathrm{d}Z_v(t)\rangle = \rho\mathrm{d}t,$$

where $\rho \in [-1, 1]$ is the correlation coefficient. To ensure that v is strictly positive $\mathbb{Q}$-a.s., we assume that the Feller condition holds, that is $2k_v\theta_v \geq \sigma_v^2$, for all $t \in [0, T]$. Additionally, we assume that Z_r is independent of Z_S and Z_v.
We consider a European-type contingent claim with maturity T, underlying risky asset S, and payoff function f_{dp} given in (2.7). Since under the risk neutral measure $\mathbb{Q}$, the discounted dynamics of the risky asset $\tilde{S} = \left\{\tilde{S}(t),\ t \in [0, T]\right\}$, with

$$\tilde{S}(t) = \exp\left\{-\int_0^t r(s)\mathrm{d}s\right\} S(t), \quad t \in [0, T], \tag{2.16}$$

is a martingale, then the no-arbitrage price at $t = 0$ of such a contingent claim is given by

$$\mathcal{O}^S_{f_{dp}}(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)\mathrm{d}s} f_{dp}\left(S(T)\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[f_{dp}\left(\tilde{S}(T)\right)\right], \tag{2.17}$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation w.r.t. the risk-neutral measure $\mathbb{Q}$. For our purposes, in addition to the derivatives whose underlying is given by the stock S, we consider additional contingent claims with maturity T, whose underlying are the PI strategies introduced in Chapter 1 and Section 2.1, namely the CPPI, TIPP, G-CPPI, G-TIPP strategies. To make the notation as concise as possible, in the following we indicate with $W^h = \left\{W^h(t),\ t \in [0, T]\right\}$ the stochastic processes describing the evolution of such portfolios, with $h \in \{CPPI,\ G-CPPI,\ TIPP,\ G-TIPP\}$.
As discussed in Section 1.2 and formally proved in Schied [2014], to adequately account for gap risk, it is necessary, from a modelling perspective, to include discontinuities in the dynamics of stock S or to allow for trading restrictions in the interval $[0, T]$. In the present work, we consider the second approach. Therefore, as in Balder et al. [2009] and Hamidi et al. [2014], we assume that strategies can be rebalanced only at dates $0 = t_0 < t_1 < \cdots < t_{\bar{n}} = T$. As a consequence, the PI strategies are of buy-and-hold-type, for any $t \in [t_k, t_{k+1}),\ k = 0, \ldots, \bar{n}$. Hence, the portfolio process W^h can be written as

$$\begin{aligned} W^h(t) &= \frac{E^h(t_k)}{S(t_k)} S(t) + \frac{W^h(t_k) - E^h(t_k)}{P(t_k, T)} P(t, T) \\ &= \phi^h_S(t_k) S(t) + \phi^h_P(t_k) P(t, T), \quad t \in [t_k, t_{k+1}), \quad k = 0, \ldots, \bar{n}, \end{aligned}$$

where $E^h(t_k)$ represents the exposure to the risk asset S, and $\phi^h_S(t_k)$ (resp., $\phi^h_P(t_k)$) represents the shares of risky asset S (resp., the ZCBs P) held in the portfolios for all $t \in [t_k, t_{k+1})$, such that

$$\lim_{t \to t_k{}^-} W^h(t) = \lim_{t \to t_k{}^+} W^h(t), \quad k = 1, \ldots, \bar{n}, \tag{2.18}$$

for $h \in \{CPPI,\ G-CPPI,\ TIPP,\ G-TIPP\}$. Finally, we consider the discounted dynamics of the portfolio process $\tilde{W}^h = \left\{\tilde{W}^h(t),\ t \in [0, T]\right\}$, that is

$$\tilde{W}^h(t) = \exp\left\{-\int_0^t r(s)\mathrm{d}s\right\} W^h(t). \tag{2.19}$$

Due to the continuity property (2.18), it is straightforward to show that the discounted value of portfolio process $\tilde{W}^h$ is a martingale. Then, the no-arbitrage price of the contingent claim at time $t = 0$, with maturity T, underlying W^h, and payoff f_{dp} given in equation (2.7) reads as follows

$$\tilde{O}^h_{f_{dp}}(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)\mathrm{d}s} f_{dp}\left(W^h(T)\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[f_{dp}\left(\tilde{W}^h(T)\right)\right], \tag{2.20}$$

for all $h \in \{CPPI,\ G-CPPI,\ TIPP,\ G-TIPP\}$.

Remark 2.2.4. *Within this framework, the value of the G-TIPP portfolio can be explicitly written as*

$$\begin{aligned} W^{G-TIPP}(t) =& E^{G-TIPP}(t_k) \cdot \exp\left\{\int_{t_k}^t \left(r(s) - \frac{v(s)}{2}\right)\mathrm{d}s + \int_{t_k}^t \sqrt{v(s)}\mathrm{d}Z_S(s)\right\} \\ &+ \left(W^{G-TIPP}(t_k) - E^{G-TIPP}(t_k)\right) \cdot \frac{\exp\{h_1(t,T) - h_2(t,T)r(t)\}}{\exp\{h_1(t_k,T) - h_2(t_k,T)r(t_k)\}}, \end{aligned} \tag{2.21}$$

where

$$\begin{cases} E^{G-TIPP}(t_k) = \max\left\{\min\left\{L_{max} \cdot W^{G-TIPP}(t_k), m \cdot C^{G-TIPP}(t_k)\right\}, \alpha_{min} \cdot W^{G-TIPP}(t_k)\right\}, \\ C^{G-TIPP}(t_k) = \left(W^{G-TIPP}(t_k) - F^{G-TIPP}(t_k)\right)^+, \\ F^{G-TIPP}(t_k) = \max\left\{F^{G-CPPI}(t_k),\ PL \cdot \sup_{s \le t_k} W^{G-TIPP}(t_k)\right\}. \end{cases} \tag{2.22}$$

for any $t \in [t_k, t_{k+1})$, $k = 0, \ldots, \bar{n}$. *Analogously, the G-CPPI portfolio value reads as*

$$\begin{aligned} W^{G-CPPI}(t) =& E^{G-CPPI}(t_k) \exp\left\{\int_{t_k}^t \left(r(s) - \frac{v(s)}{2}\right)\mathrm{d}s + \int_{t_k}^t \sqrt{v(s)}\mathrm{d}Z_S(s)\right\} \\ &+ \left(W^{G-CPPI}(t_k) - E^{G-TIPP}(t_k)\right) \cdot \frac{\exp\{h_1(t,T) - h_2(t,T)r(t)\}}{\exp\{h_1(t_k,T) - h_2(t_k,T)r(t_k)\}}, \end{aligned} \tag{2.23}$$

where

$$\begin{cases} E^{G-CPPI}(t_k) = \max\left\{\min\left\{L_{max} \cdot W^{G-CPPI}(t_k), m \cdot C^{G-CPPI}(t_k)\right\}, \alpha_{min} \cdot W^{G-CPPI}(t_k)\right\}, \\ C^{G-CPPI}(t_k) = \left(W^{G-CPPI}(t_k) - F^{G-CPPI}(t_k)\right)^+, \\ F^{G-CPPI}(t_k) = PL \cdot W^{G-CPPI}(0) \cdot P(t_k, T). \end{cases} \tag{2.24}$$

for any $t \in [t_k, t_{k+1})$, $k = 0, \ldots, \bar{n}$. *Moreover, by setting* $\alpha_{min} = 0$ *in equations* (2.22) *and* (2.24), *we obtain, as special cases, the standard TIPP and CPPI portfolio value processes, respectively.*

2.3 Numerical results

In what follows, we rely on numerical methods to evaluate expressions (2.17) and (2.20), exploiting parameter values provided in Albeverio et al. [2017], see Table 2.3, in order to illustrate the benefits of using financial derivatives that are linked to portfolio insurance strategies. In particular, we compare the values of the ATM European call options on risky asset S with the ATM European call options when the underlying is either the CPPI strategy, the TIPP strategy, and their modified version, that is the G-CPPI and the G-TIPP strategies. More precisely, we consider Monte Carlo simulations, according to the following scheme:

(i) given a daily partition $\pi = \{t_0,\ t_1,\ \ldots,\ t_{\bar{n}}\}$ of $[0, T]$, with $\Delta t = \frac{T}{\bar{n}}$, we discretize the SDEs depicted in equations (2.11) and (2.15) according to the Euler scheme, to obtain $\bar{N}$ simulated paths for the short-term interest rate r, the ZCB P with maturity T, and the underlying risky asset S;

(ii) we simulate the trajectories of the PI strategies, according to the following discretization scheme

$$\begin{aligned} W_i^h(t_{k+1}) &= \frac{\bar{E}_i^h(t_k)}{S_i(t_k)} \cdot S_i(t_{k+1}) + \frac{W_i^h(t_k) - \bar{E}_i^h(t_k)}{P_i(t_k, T)} \cdot P_i(t_{k+1}, T), \\ C_i^h(t_{k+1}) &= \left(W_i^h(t_{k+1}) - F_i^h(t_{k+1})\right)^+, \\ E_i^h(t_{k+1}) &= \max\left\{\min\left\{L_{max} \cdot W_i^h(t_{k+1}),\ m \cdot C_i^h(t_{k+1})\right\},\ \alpha_{min} \cdot W_i^h(t_{k+1})\right\}, \\ \bar{E}_i^h(t_{k+1}) &= \begin{cases} E_i^h(t_{k+1}), & \text{if } k+1 \equiv 0 \mod \tilde{k}, \\ E_i^h(t_k), & \text{otherwise}, \end{cases} \end{aligned}$$

for all $k = 0, \ldots, \bar{n}-1$, $i = 1, \ldots, \bar{N}$, and $h \in \{CPPI,\ G-CPPI,\ TIPP,\ G-TIPP\}$. $\tilde{k}$ represents the pre-specified fixed rebalancing frequency;

(iii) we compute the price of the ATM European call option by using the trapezoidal rule to evaluate the stochastic discount factor (see Huynh et al. [2011]), that is

$$\mathcal{O}_{f_{dp}}^h(0) = \frac{1}{\bar{N}} \sum_{i=1}^{\bar{N}} \exp\left\{-\left(\frac{1}{2} r_i(t_0) + \sum_{k=1}^{\bar{n}-1} r_i(t_k) + \frac{1}{2} r_i(t_{\bar{n}})\right) \Delta t\right\} \left(W_i^h(t_{\bar{n}}) - W_i^h(t_0)\right)^+, \quad (2.25)$$

for all $k = 0, \ldots, \bar{n}-1$, $i = 1, \ldots, \bar{N}$, and $h \in \{CPPI,\ G-CPPI,\ TIPP,\ G-TIPP\}$.

Vasicek model		**Heston model**	
Parameter	Value	Parameter	Value
α_r	1.25	k_v	1.25
β_r	0.05	θ_v	0.04
σ_r	0.025	σ_v	0.2
-	-	ρ	-0.5

Table 2.3 Parameters used in the numerical experiments for the Vasicek's model and the Heston's model.

We compare the option prices by considering different levels of the guaranteed minimum equity exposure α_{min} and several maturities T. The results are displayed in Figure 2.2. Starting with an initial volatility level of 20% and an initial interest rate equal to 0%, we observe that the European call option prices written on stock are higher than the ones obtained by taking any portfolio insurance strategy as underlying, whatever the maturity T and the GMEE level α_{min}. Our results show that using European call options linked to portfolio insurance strategies within structured investment products with capital protection would lead to a significant increase in the participation rate, for a given protection level and a given risk budget. Furthermore, looking at the European call option prices written on portfolio insurance strategies, we notice that the TIPP options are always cheaper than the CPPI ones. Such a result hold for any maturity T. Therefore, it is more convenient to invest in a structured investment product with capital protection where the risky component is given by a European call option written on the TIPP strategy. Indeed, according to the reasoning described in Section 2.2.1, this improves the attractiveness for investors by increasing the related participation rate (see equation (2.10)). Such a result still holds even when we consider European call options associated with portfolio insurance strategies endowed with GMEE, i.e., the G-TIPP and the G-CPPI. The option on G-TIPP is cheaper than the one written on G-CPPI. We

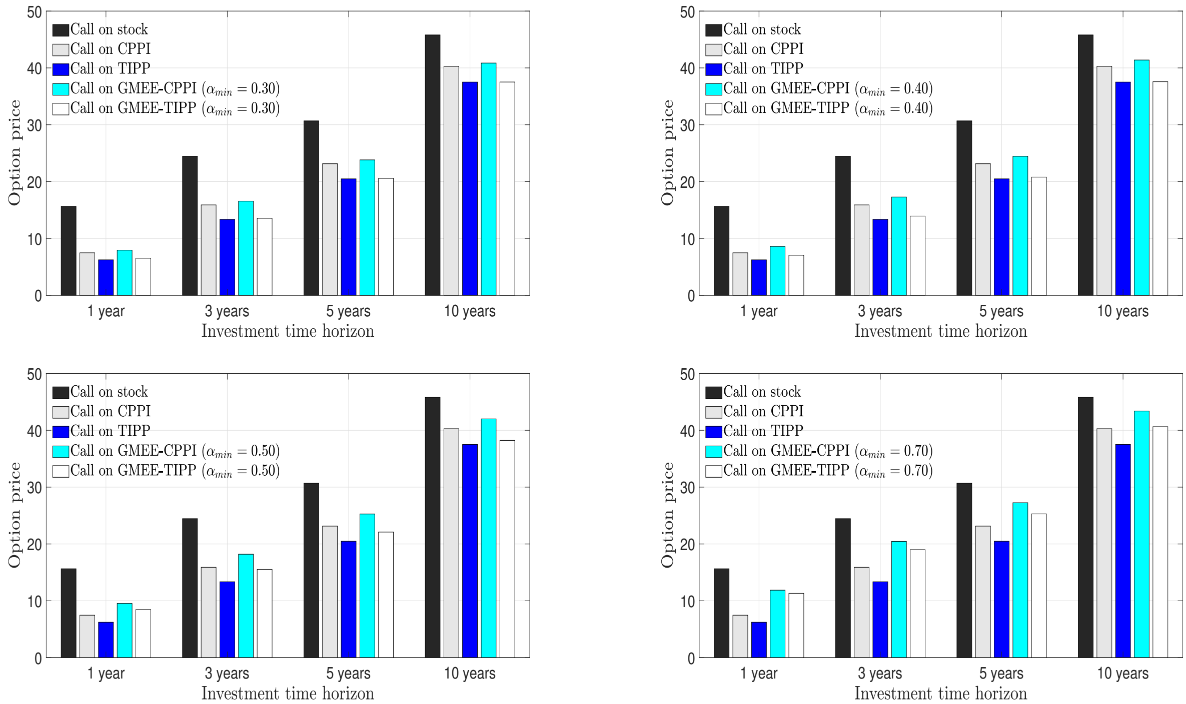


Figure 2.2 Comparison among ATM call option pricing under different underlyings, maturities, and level of guaranteed minimum equity exposure. The model parameters are: $v_0 = 20\%$, $k_v = 1.25$, $\theta_v = v_0^2$, $\sigma_v = 0.2$, $\rho = -0.5$ (Heston), $r_0 = 0.01$, $\alpha_r = 0.05$, $\beta_r = 0.02$, $\sigma_r = 0.025$ (Vasicek). The remaining parameters associated to the portfolio insurance strategies are: $L_{max} = 100\%$, $m = 4$, $PL = 90\%$, $\alpha_{min} = \{0.3, 0.4, 0.5, 0.7\}$. We performed 10^5 Monte Carlo simulations with $S_0 = 100$.

further observe that the G-TIPP (resp. G-CPPI) call option price is slightly higher than the option associated with the standard TIPP (resp. standard CPPI) logic. This is due to the introduction of the GMEE threshold, which triggers a larger number of paths ending above the strike price at maturity, leading to an increase in the corresponding option price at $t = 0$.

In conclusion, since the option written on G-TIPP solves the problem of obtaining an equity market participation and simultaneously leads to a smaller decrease in the participation rate (compared to G-CPPI) when it is used as underlying the European call option, it is more suitable to be included in a structured investment product with capital protection.

2.3.1 Sensitivity analysis w.r.t. the model parameters

Section 2.3 show that the highest participation rate is attained by considering structured products where the risky component given by European ATM call options linked to the G-TIPP. Furthermore, to confirm that this options is more suitable than those written on the risky asset S and G-CPPI to be included in structured investment product with capital protection, we must verify that properties **($\boldsymbol{C}$1)** and **($\boldsymbol{C}$2)** are fulfilled. The latter implies that the corresponding prices are less influenced by the variance of the risky asset and more influenced by the interest rate levels. Hence, we evaluate the prices of ATM call written on the risky asset S, the G-CPPI strategy, and the G-TIPP one for different combinations of initial interest rate and variance. Moreover, we quantify the dependence of

Panel A: Option on pure risky asset						
Initial interest rate (r_0)	**Initial annual volatility (v_0)**					**RPR^v**
	0.10	0.20	0.30	0.40	0.50	
0.01	10.31	14.98	18.73	22.32	25.13	1.44
0.03	11.19	15.69	19.85	22.71	25.95	1.32
0.05	12.10	16.93	20.37	23.44	26.54	1.19
0.07	12.99	17.33	20.97	24.42	27.46	1.11
0.10	14.37	18.76	22.30	25.52	28.46	0.98
RPR^r	0.39	0.25	0.19	0.14	0.13	
Panel B: Option on G-CPPI						
Initial interest rate (r_0)	**Initial annual volatility (v_0)**					**RPR^v**
	0.10	0.20	0.30	0.40	0.50	
0.01	4.76	6.83	8.48	10.01	10.93	1.29
0.03	6.05	8.05	10.02	11.09	12.45	1.06
0.05	7.42	9.75	11.19	12.40	13.67	0.84
0.07	8.83	10.82	12.42	14.03	15.21	0.72
0.10	10.91	12.92	14.52	15.99	17.05	0.56
RPR^r	1.29	0.89	0.71	0.60	0.56	
Panel C: Option on G-TIPP						
Initial interest rate (r_0)	**Initial annual volatility (v_0)**					**RPR^v**
	0.10	0.20	0.30	0.40	0.50	
0.01	4.05	5.63	6.84	7.98	8.91	1.20
0.03	5.15	6.66	8.01	8.94	9.97	0.94
0.05	6.36	7.94	9.03	10.01	11.04	0.73
0.07	7.68	8.98	10.13	11.18	12.19	0.59
0.10	9.88	11.00	12.01	12.99	13.89	0.41
RPR^r	1.44	0.96	0.76	0.63	0.56	

Table 2.4 ATM call option prices on risky asset, G-CPPI and G-TIPP, for different values of initial interest rate (r_0) and initial annual volatility (v_0). The model parameters are: $k_v = 1.25$, $\theta_v = v_0^2$, $\sigma_v = 0.2$, $\rho = -0.5$ (Heston), $\alpha_r = 1.25$, $\beta_r = r_0$, $\sigma_r = 0.025$ (Vasicek). The remaining parameters associated to the portfolio insurance strategies are: $L_{max} = 100\%$, $m = 4$, $\alpha_{min} = 30, \%$ $PL = 90\%$. We performed 10^5 MC simulations with $S_0 = 100$.

option prices on both the initial variance level of the underlying risky asset and the initial interest rate level. To do this, as in Albeverio et al. [2017], we resort to the so-called *relative range of option prices* (RPR). We first define the relative range of option prices for a fixed initial interest rate level as

$$RPR^v := \frac{\mathcal{O}^{\tilde{h}}_{f_{dp}}(0; v_0^{max}) - \mathcal{O}^{\tilde{h}}_{f_{dp}}(0; v_0^{min})}{\mathcal{O}^{\tilde{h}}_{f_{dp}}(0; v_0^{min})}, \tag{2.26}$$

for $\tilde{h} \in \{S,\ CPPI,\ G-CPPI,\ TIPP,\ G-TIPP\}$, where v_0^{max} (resp., v_0^{min}) is the highest (resp., the smallest) variance level we consider in the numerical analysis. The results are depicted in the last column of Table 2.4. We observe the following:

(i) the highest RPR^v value is obtained for call options linked to the pure risky asset for any initial interest rate. Therefore, we conclude that the options linked to portfolio insurance strategies are less affected by the initial variance of the underlying risky asset;

(ii) the RPR^v associated with G-TIPP options is always lower than the RPR^v for G-CPPI options. Such a result stems from the floor rebalancing mechanism within the G-TIPP strategy, which implies a smaller exposure to the risky asset. Therefore, we get an overall lower variance for the G-TIPP strategy than the G-CPPI one, making the G-TIPP options less affected by the level of the initial variance.

We now study the connection between the option prices and the interest rate initial level. Looking at Table 2.4, we note that option prices increase as the initial level of the interest rate rises, regardless of the type of underlying and the initial level of variance. To quantify the dependence of the option price on the interest rate levels, we compute the relative range of option prices for a fixed initial variance level as

$$RPR^r := \frac{\mathcal{O}^{\tilde{h}}_{f_{dp}}(0; r_0^{max}) - \mathcal{O}^{\tilde{h}}_{f_{dp}}(0; v_0^{min})}{\mathcal{O}^{\tilde{h}}_{f_{dp}}(0; r_0^{min})}, \tag{2.27}$$

for $\tilde{h} \in \{S,\ CPPI,\ G-CPPI,\ TIPP,\ G-TIPP\}$, where r_0^{max} (resp., r_0^{min}) is the highest (resp., the smallest) interest rate level we consider in the numerical analysis. The results are depicted in the last row of each panel in Table 2.4. In particular, for a fixed level of initial variance, the lowest RPR^r is attained for options on purely risky stocks. Therefore, the price of stock options is much less sensitive to interest rate increases. We also observe that the RPR^r associated with options written on the G-CPPI strategy is better than those written the stock and lower than those written on G-TIPP. Therefore, the option that best satisfies condition **(C1)** and **(C2)** and simultaneously provides the highest participation rate is the one written on the G-TIPP. Thus, introducing the G-TIPP in structured investment products with capital protection makes this product more attractive for investors.

2.3.2 Sensitivity analysis w.r.t. the portfolio insurance strategies parameters

The results in Section 2.3.1 show that the G-TIPP option is better suited than those written on stock and on G-CPPI to be included in structured investment product with capital protection. However, the price of this type of option is highly influenced by the exogenous parameters characterizing the strategy. To strengthen the results obtained in Section 2.3.1, we show that the lowest prices are achieved when we consider the G-TIPP strategy as underlying, regardless of the protection level, the multiplier, or the rebalancing frequency, further proving that the properties $(\boldsymbol{C1})$ and $(\boldsymbol{C2})$ are still preserved.

The role of the multiplier. We consider the European call option prices as a function of the multiplier, for different values of the initial volatility. The results are depicted in Table 2.5.

Panel A: Option on pure risky asset						
Multiplier (M)	**Initial annual volatility (v_0)**					RPR^v
	0.10	0.20	0.30	0.40	0.50	
-	10.31	14.98	18.73	22.32	25.13	1.44
Panel B: Option on G-CPPI						
Multiplier (M)	**Initial annual volatility (v_0)**					RPR^v
	0.10	0.20	0.30	0.40	0.50	
4	4.76	6.83	8.48	10.01	10.93	1.29
5	5.65	7.81	9.41	10.80	11.91	1.11
6	6.36	8.49	9.96	11.22	12.21	0.92
10	7.69	9.49	10.64	11.72	12.55	0.63
Panel C: Option on G-TIPP						
Multiplier (M)	**Initial annual volatility (v_0)**					RPR^v
	0.10	0.20	0.30	0.40	0.50	
4	4.05	5.63	6.84	7.98	8.91	1.20
5	4.76	6.48	7.74	8.92	9.86	1.07
6	5.43	7.24	8.49	9.68	10.20	0.88
10	7.54	9.27	10.20	11.01	11.42	0.51

Table 2.5 ATM call option prices on pure risky asset, on the G-CPPI and the G-TIPP strategies, for different values of multiplier (m) and initial annual volatility (v_0). The model parameters are: $k_v = 1.25$, $\theta_v = v_0^2$, $\sigma_v = 0.2$, $\rho = -0.5$ (Heston), $\alpha_r = 1.25$, $\beta_r = r_0$, $\sigma_r = 0.025$ (Vasicek). The remaining parameters associated to the portfolio insurance strategies are: $PL = 90\%$, $L_{max} = 100\%$, $\alpha_{min} = 30\%$. We performed 10^5 MC simulations with $S_0 = 100$.

As a preliminary comment, we observe that for any v_0 value, the greater the multiplier, the more expensive the European call option price, whatever the portfolio strategy involved. This is not surprising, since the multiplier m is an amplifying factor for the risk budget, and directly affects the risky exposure. However, we observe that G-TIPP options (Panel C) are always less expensive than the G-CPPI one (Panel B) for each multiplier level considered. Therefore, this reinforces the previous results: including the G-TIPP options within a structured investment product with capital protection can increase the corresponding participation rate, regardless of the multiplier set at $t = 0$ by the fund manager. Furthermore, by analyzing the RPR^v, we note that such an indicator is the smallest when considering the G-TIPP strategy as underlying for the ATM European call option. This implies that the option on G-TIPP is less affected by the variance levels of the underlying risky, regardless of the multiplier value.

The role of the protection level. We consider the European call option prices as a function of the initial volatility and the protection level of the strategies. The results are depicted in Table 2.6.

Again, the protection level impacts the risk budget. In particular, the lower the protection level,

Panel A: Option on pure risky asset						
Protection level (PL)	**Initial annual volatility (v_0)**					RPR^v
	0.10	0.20	0.30	0.40	0.50	
-	10.31	14.98	18.73	22.32	25.13	1.44
Panel B: Option on G-CPPI						
Protection level (PL)	**Initial annual volatility (v_0)**					RPR^v
	0.10	0.20	0.30	0.40	0.50	
0.70	10.01	13.88	16.59	18.95	20.59	1.06
0.75	9.38	12.75	15.10	17.15	18.52	0.98
0.80	8.16	11.07	13.13	14.91	16.25	0.99
0.85	6.52	8.99	10.82	12.43	13.44	1.06
0.90	4.76	6.83	8.48	10.01	10.93	1.29
Panel C: Option on G-TIPP						
Protection level (PL)	**Initial annual volatility (v_0)**					RPR^v
	0.10	0.20	0.30	0.40	0.50	
0.70	10.01	13.88	16.59	18.94	20.57	1.06
0.75	9.29	12.60	14.87	16.91	18.24	0.96
0.80	7.48	10.13	11.98	13.64	14.82	0.98
0.85	5.71	7.77	9.26	10.61	11.65	1.04
0.90	4.05	5.63	6.84	7.98	8.91	1.20

Table 2.6 ATM call option prices on pure risky asset for different values of initial annual volatility (v_0). ATM call option prices on G-CPPI and G-TIPP for different values of protection level (PL) and initial annual volatility (v_0). $k_v = 1.25$, $\theta_v = v_0^2$, $\sigma_v = 0.2$, $\rho = -0.5$ (Heston), $\alpha_r = 1.25$, $\beta_r = r_0$, $\sigma_r = 0.025$ (Vasicek). The remaining parameters associated to the portfolio insurance strategies are: $m = 4$, $L_{max} = 100\%$, $\alpha_{min} = 30\%$. We performed 10^5 MC simulations with $S_0 = 100$.

the greater the risk budget and the exposure to the risky asset. Therefore, the European call option prices increase as the protection level decreases, whatever the portfolio insurance strategy involved. Also in this case we observe that the European call options written on G-TIPP are *(i)* less affected by the initial variance of the underlying risky asset, and *(ii)* the cheapest for any combination of protection level and initial volatility. For this reason, also in this case they represent the best choice among the contingent claims we have analyzed to be included in a structured investment product with capital protection.

The role of the rebalancing frequency. We consider the European call option prices as a function of the initial volatility and the rebalancing frequencies. The results are depicted in Table 2.7. Up to now, we have only considered daily rebalancing for the portfolio strategies involved. However, these assumptions might be violated in practice: due to transaction costs and liquidity constraints, it is not always possible to re-balance the portfolio daily. For this reason, portfolio managers might also consider lower re-balancing frequencies. However, a lower rebalancing frequency accentuates the risk of a floor violation between two consecutive trading dates. Hence, we also consider different frequencies for portfolio rebalancing: weekly (w), biweekly (2w), monthly (m), quarterly (3m), and four-monthly (4m). We observe that the portfolio rebalancing frequency impacts the European

Panel A: Option on pure risky asset						
Rebalancing frequencies	**Initial annual volatility (v_0)**					**RPR**
	0.10	0.20	0.30	0.40	0.50	
-	10.31	14.98	18.73	22.32	25.13	1.44
Panel B: Option on G-CPPI						
Rebalancing frequencies	**Initial annual volatility (v_0)**					**RPR**
	0.10	0.20	0.30	0.40	0.50	
d	4.76	6.83	8.48	10.01	10.93	1.29
w	4.80	6.89	8.60	10.22	11.19	1.33
2w	4.83	6.98	8.75	10.40	11.46	1.37
m	4.86	7.07	8.86	10.56	11.64	1.40
3m	4.96	7.32	9.18	10.93	12.18	1.45
4m	4.97	7.34	9.21	11.58	12.25	1.46
Panel C: Option on G-TIPP						
Rebalancing frequencies	**Initial annual volatility (v_0)**					**RPR**
	0.10	0.20	0.30	0.40	0.50	
d	4.05	5.63	6.84	7.98	8.91	1.20
w	4.06	5.64	6.86	8.01	8.96	1.21
2w	4.07	5.67	6.90	8.04	9.00	1.21
m	4.08	5.69	6.93	8.09	9.05	1.22
3m	4.15	5.82	7.12	8.33	9.33	1.25
4m	4.18	5.88	7.20	8.44	9.44	1.26

Table 2.7 ATM call option prices on G-TIPP and G-CPPI for different rebalancing disciplines (f) and and initial annual volatility (v_0). The model parameters are: $v_0 = 20\%$, $k_v = 1.25$, $\theta_v = v_0^2$, $\sigma_v = 0.2$, $\rho = -0.5$ (Heston), $r_0 = 0.01$, $\alpha_r = 1.25$, $\beta_r = r_0$, $\sigma_r = 0.025$ (Vasicek). The remaining parameters associated to the portfolio insurance strategies are: $PL = 90\%$, $L_{max} = 100\%$, $m = 4$, $\alpha_{min} = 30\%$. The rebalancing frequency is daily (d), weekly (w), biweekly (2w), monthly (m). We performed 10^5 MC simulations with $S_0 = 100$.

option prices written on portfolio insurance strategies: options get cheaper as the rebalancing frequency intensifies, due to the smaller premium required by the issuer. We focus on comparing G-TIPP and G-CPPI option prices (Panel C and B, respectively). Our findings show that even using different rebalancing disciplines, the G-TIPP call option is always lower-priced than the G-CPPI one. Such a result might be clarified by considering the following reasoning: by construction, the European option prices on portfolio insurance strategies heavily depend on the strategy's risky exposure, which is seen as a metric to evaluate the probability of gap risk or cash-lock event during the investment time horizon. The latter increases as the rebalancing frequency decreases. As seen in Section 2.1, the average exposure to the risky asset for the G-TIPP strategy is lower than for the G-CPPI one. Hence, the gap risk, or a cash-lock event, is less likely to occur. This turns into a lower price for G-TIPP. Even in this case, it is more convenient to include the G-TIPP within a structured investment product capital as there will be an increase in the participation rate. This property is fulfilled for any combination of the rebalancing frequency and the initial level of variance. Furthermore, by looking at the corresponding RPR^v we observe that the European options linked to the G-TIPP strategy are less affected by the initial variance level of the underlying risky asset.

Chapter 3

Pension fund with longevity risk: an optimal portfolio insurance approach

Pension funds are financial intermediaries whose primary goal consists in guaranteeing an income to participants during their retirement life. Accordingly, pension funds are linchpins in supporting both retirees' consumption and savings, and more in general, the growth of national economics. In recent decades, economic cycles, financial market fluctuations, and the progressive ageing of the population have involved challenges in pension funds' management and sustainability.

The pension fund manager is responsible for handling financial resources over two different periods: the *accumulation* and the *decumulation* phases. During the accumulation phase, contributions are made, while the fund's manager invests them to built up funds that will be available at the retirement date. Once this date is reached, the decumulation period begins, meaning that the accumulated funds are naturally employed to purchase life annuities for each retiree, ensuring financial support throughout their remaining lifetime. It is worth noting that the accumulation phase is affected by both the randomness in members' labour income and the downside investment risk, whilst the decumulation phase is exposed to longevity risk. Depending on the type of pension scheme, the investment risk may be borne by either the fund or the member. Specifically, there are two main categories of pension funds, namely, defined benefit (DB) and defined contribution (DC) schemes. Nowadays, due to the financial unsustainability of DB systems, DC-based policies represent the reference pension scheme. For further details, refer to Cocco and Gomes [2012]. The present chapter focuses on the accumulation phase, leaving the interested reader to Blake [2006] for a comprehensive overview of the economics of pension funds.

In DC schemes, contributions are predetermined, and benefits depend on the accumulated capital until retirement. Consequently, the pension fund's members bear the investment risks associated with market fluctuations during the accumulation period, while the pension fund endures the longevity risk during the decumulation phase. To dominate the uncertainties that impact the growth of retirement capital, the fund's manager needs to implement a suitable investment strategy taking into account protection against adverse market conditions.

In the present chapter we are focused in deriving explicit optimal strategies for a defined contribution pension fund within a continuous-time framework, accounting for the presence of a minimum guarantee. Our work contributes to the existing literature in three directions. From an actuarial perspective, we incorporate the demographic risk factor during the accumulation phase in a DC pension plan. From a financial standpoint, we adopt non-standard investment strategies throughout the accumulation stage addressing specific investor purposes and providing a minimum guarantee. From a numerical point of view, we screen different mortality scenarios to survey the impact of a longer life expectancy on investment choices for DC pension funds. To the best of our knowledge, this work brings together all of the above features into a unified study for the first time in the literature.

A broad research arena on the optimal asset allocation problem with the presence of a minimum guarantees for pension funds has been developed. For instance, Boulier et al. [2001] face the optimal investment problem for DC pension plans considering both stochastic interest rates and the presence of a minimum benefit guarantee. Employing the dynamic programming approach, the resulting investment strategy maximizes the expected terminal utility of the surplus between the final wealth of the fund and the guaranteed amount over a finite horizon. Deelstra et al. [2003] also take into account a minimum guarantee for the terminal wealth of the fund in the context of asset-liability management for a DC plan. Their objective is to maximize the expected utility of terminal wealth over a finite horizon, when the contribution process is stochastic. Di Giacinto et al. [2011] take the perspective of a fund manager aiming to maximize expected utility of wealth over an infinite time horizon. In their analysis, they consider the constraint that the fund's wealth must remain above a "solvency level", and explore the fund's behavior when it reaches this minimum wealth threshold. Their model is naturally stated as an optimal stochastic control problem with state and control constraints, and analyzed using the dynamic programming approach. Han and Hung [2012] focus on the optimal asset allocation problem in a DC pension fund when randomness in both inflation and labour income is involved. The authors consider a minimum guarantee on purchasing an inflation-linked annuity at retirement. Furthermore, the literature provides research works wherein both a minimum guarantee and longevity dynamics influence optimal investment strategies. Among others, Guan and Liang [2014, 2015], Sun et al. [2016], and Wang et al. [2021] examine how the pension fund's wealth is optimally allocated during the accumulation phase, when a target annuity value serves as a minimum guarantee. The residual lifetime of the fund's members is shaped according to suitable deterministic mortality laws. Nevertheless, a stochastic mortality behaviour is desirable to properly account for the longevity risk.

In a continuous-time framework, the stochastic force of mortality is typically modeled by affine stochastic processes, exploiting their befitting properties for mortality analysis and their suitability for the market-consistent evaluation of mortality/longevity-linked securities. See, e.g., Dahl [2004], Biffis [2005], Luciano and Vigna [2005], Schrager [2006], Russo et al. [2011], Zeddouk and Devolder [2020]. As the force of mortality is considered stochastic, and the minimum guarantee consists of a target annuity value, longevity risk becomes a significant factor affecting the DC fund's investments during the accumulation period. The annuity value as well as the contribution payments depend on the member's lifetime and the fund's manager has to acquire longevity-linked securities to hedge against longevity risk and diversify the portfolio. The first insight into longevity-linked assets was provided by Blake and Burrows [2001], afterwards deepened by Blake et al. [2006]. The authors introduced the so-called survivor or longevity bond, whose coupon amounts are defined according to the number of survivors in a reference population. Currently, the longevity-linked securities market is still evolving since the hedging offered by index-based products, such as longevity bonds, may be limited by basis risk, credit risk, and liquidity risk. For further details, see, e.g., Blake et al. [2019]. Most frequently, longevity-linked securities are traded over-the-counter (OTC), and they are customized to fulfil the needs of the annuity provider. Alongside these agreements, the capital market for such financial instruments is growing due to their usefulness in hedging longevity risk. See, e.g., Ngai and Sherris [2011], Tsai et al. [2011], Blake et al. [2014], De Rosa et al. [2017]. In addition, several research studies show the relevant role of longevity bonds in optimal investment problems within a stochastic mortality framework. See, e.g., Menoncin [2008], De Kort and Vellekoop [2017], Menoncin and Regis [2017], Agarwal et al. [2023].

Within the aforementioned setting, a crucial role is played by the investment strategy. A wide strand of literature investigates dynamic portfolio insurance (PI) strategies in the accumulation phase of DC pension schemes. The rationale behind such a choice is to protect the contributions paid by the pension fund's members to eliminate or reduce the impact of the risks above-mentioned, while still allowing for equity market participation in favourable market conditions. An explicit implementation of a PI strategy in the accumulation phase of a DC pension plan can be found in Temocin et al. [2018]. Specifically, the authors employ the CPPI strategy in a pension plans framework. By assuming a constant interest rate and a stock price dynamic driven by a geometric Brownian motion, Temocin et al. [2018] derive the optimal multiplier maximizing the CRRA utility of the terminal surplus, also known as the cushion, and intended as the difference between the portfolio value and the guaranteed amount.

In the present chapter, we intend to generalize the framework proposed by Temocin et al. [2018], deriving the optimal design of the PPI strategy for the accumulation phase of a DC pension scheme in the presence of market/investment risk, interest rate risk, and longevity risk. Due to the presence of these three sources of risk, the standard version of the PPI strategy cannot be applied, as it is unable to hedge against adverse movements in the stochastic interest rate and the force of mortality. To overcome this issue, we propose a modified version of the PPI strategy following the approach outlined in the recent work of Chen et al. [2022]. By adopting a target lifetime annuity as the guaranteed amount, the minimum portfolio value – the so-called floor – to be protected during the entire investment horizon becomes no longer deterministic. Accordingly, the present value of the guaranteed lifetime annuity is regarded as the investment strategy's floor, while the residual wealth is employed in a purpose-oriented diversified portfolio, namely, a combination of bonds, longevity bonds, and stocks. We refer to our approach as *purpose-oriented* proportional portfolio insurance (PO-PPI) strategy.

From the perspective of the fund's manager, the goal is to determine the exogenous parameter of the proposed PI strategy, known as the multiplier, along with the optimal composition of the purpose-oriented portfolio surplus between the fund's terminal wealth and the target annuity value at retirement date. To obtain explicit solutions, we posit a CRRA-type expected utility maximization problem over the terminal value of the cushion, that is, the surplus between the fund's terminal wealth and the target annuity value at retirement date. The associated wealth

process fails to be self-financing, and consequently, we reformulate the problem in terms of an equivalent single-investment process, in compliance with Boulier et al. [2001]. To solve the resulting stochastic optimization problem, we apply standard dynamic programming techniques, and we derive closed-form solutions both for the auxiliary and the original problems. Interestingly, the resulting optimal multiplier is genuinely dynamic, different than the standard version of the CPPI. Our theoretical proposal is supported by a comprehensive numerical application that focuses on homogeneous cohorts in terms of gender, age, and economic status. As a baseline study, we run a simulation analysis of the investment strategies as determined by setting the controls to the optimum, namely, the optimal proportions of the fund's wealth invested in each asset. Our findings indicate that for both genders, optimality requires a switch in investment aptitude, which is consistent with the financial life cycle theory.
The numerical experiments exhibit a two-mode behavior, depending on the time period under consideration. As expected, in the early accumulation phase, the manager executes an aggressive investment policy to boost capital accretion, investing primarily in bonds and longevity bonds. As retirement approaches, a shift towards more conservative investments becomes essential to preserve the accumulated capital. Remarkably, the baseline case highlights the role of the multiplier in explaining gender-based differences in optimal investments. As the females have a longer life expectancy than males, the value of their annuity becomes larger, increasing the floor value, and consequently, reducing the cushion available for investments. Thus, the fund manager needs to push the investment leverage via the multiplier to ensure that, at the retirement date, the accrued amount provides at least the target annuity. The comparison between optimal gender-based multipliers highlights a gap in quantifying the financial impact of the gender-based longevity gap on retirement capital accumulation. More generally, the longer the fund member's lifetime manifesting in the decumulation phase, the greater the multiplier required during the accumulation phase. We emphasize that our proposal allows us to establish a connection between the accumulation and decumulation phases, meaning how future longevity dynamics influence the optimal PI strategy through the multiplier. Such a result represents a novel contribution to the existing literature. To investigate variations in optimal asset allocation, we further carry out a sensitivity analysis based on changes in financial and mortality factors, namely, the risk aversion coefficient, the market price of the longevity risk, the time to maturity of the rolling longevity bond, and the wage replacement ratio. Finally, we examine the impact of the mortality phenomenon on optimal investment proportions by considering both higher mortality and higher longevity scenarios. Our findings are in line with the results stemming from comparable research works, such as Menoncin and Regis [2017] and Agarwal et al. [2023], confirming the crucial role of longevity bonds in building and protecting the retirement capital.
The remainder of this chapter is organized as follows. Section 3.1 builds the modelling setting of our proposal. The stochastic models describing mortality, financial, and labour income dynamics, and the mathematical representation of the pension scheme's management are presented. In Section 3.2, we introduce the purpose-oriented PPI for a DC pension fund, and formulate and solve the optimal asset allocation problem of the fund's manager. Section 3.3 shows a numerical application consisting of both simulation and sensitivity analysis about our theoretical framework.

3.1 The model

We look at the challenge faced by a pension plan manager who has to determine investment strategy decisions during the accumulation phase. The primary objective is to ensure that the pension wealth remains sufficient to provide lifetime annuities for the surviving pension plan members at retirement. The value of these annuities, which acts as a minimum guarantee, depends on the remaining life expectancy of the pension plan members. As a result, the pension plan's wealth becomes vulnerable to uncertainties arising from both financial market dynamics and the longevity risk of its members. Therefore, in line with previous research, such as Biffis [2005], Ceci et al. [2020], we construct a *combined financial-insurance* market model by applying the progressive enlargement of the filtration approach. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a complete and right-continuous filtration $\mathbb{F} = \{\mathcal{F}(t),\ t \in [0,T]\}$, where $0 < T < +\infty$, represents the retirement date. Within this filtered probability, we introduce three independent one-dimensional $(\mathbb{F}, \mathbb{P})$-Wiener processes $Z_i = \{Z_i(t),\ t \in [0,T]\}$, $i = \{r, \lambda, S\}$, representing the sources of uncertainty driving the financial market dynamics and the mortality intensity of the pension plan members. For further details refer to Section 3.1.1 and Section 3.1.2, respectively. Furthermore, let $\mathbb{F}^{Z_i} = \left\{\mathcal{F}^{Z_i}(t),\ t \in [0,T]\right\}$ be the canonical filtrations of Z_i, $i = \{r, \lambda, S\}$, with $\mathcal{F}^{Z_i}(t) := \sigma\left\{Z_i(z),\ 0 \le z \le t\right\}$, $i = \{r, \lambda, S\}$, for every $t \in [0,T]$, respectively. Additionally, we set $\tilde{\mathbb{F}} = \left\{\tilde{\mathcal{F}}(t),\ t \in [0,T]\right\}$, where

$$\tilde{\mathcal{F}}(t) := \mathcal{F}^{Z_r}(t) \vee \mathcal{F}^{Z_\lambda}(t) \vee \mathcal{F}^{Z_S}(t), \quad t \in [0,T].$$

We postulate that the reference filtration $\mathbb{F}$ completed by all $\mathbb{P}$-null sets, i.e., $\mathbb{F} = \tilde{\mathbb{F}}$, characterizes the information flow related to the financial market, without accounting for the survival time of pension plan members.

3.1.1 The mortality model

We posit that the pension plan members form a cohort of n identical workers who share the same age, gender, health, and social status. Let $\lambda := \{\lambda(t),\ t \in [0,T]\}$ be an $\mathbb{F}$-adapted process on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, representing the mortality intensity of pension plan members. It satisfies the condition $\mathbb{E}\left[\int_0^t \lambda(z)\mathrm{d}z\right] < +\infty$, for every $t \in [0,T]$. If we define $p(t)$ as the number of pension plan members who have survived from time 0 to t, we can express it as

$$p(t) = p_0 \cdot \exp\left\{-\int_0^t \lambda(z)\mathrm{d}z\right\}, \quad t \in [0,T],$$

where $p_0 =: p(0)$. Note that if we normalize p_0 to one, then $p(\cdot)$ corresponds to the survival probability of the pension plan members. We suppose that λ evolves according to a moving-target, mean-reverting, square-root process. Its $\mathbb{P}$-dynamics read as

$$\begin{cases} \mathrm{d}\lambda(t) = \rho_\lambda\left(\beta_\lambda(t) - \lambda(t)\right)\mathrm{d}t + \sigma_\lambda\sqrt{\lambda(t)}\mathrm{d}Z_\lambda(t), & t \in [0,T], \\ \lambda(0) = \lambda_0 > 0. \end{cases} \tag{3.1}$$

Here $\rho_\lambda > 0$ is the speed of mean reversion of mortality intensity, $\beta_\lambda(\cdot)$ is the long-run, time-dependent, positive mean value of the mortality intensity, and $\sigma_\lambda > 0$ is the volatility of mortality intensity. To ensure that λ is strictly positive $\mathbb{P}$-a.s., we hypothesize that the *extended* Feller condition holds, i.e., $2\rho_\lambda\beta_\lambda(t) \geq \sigma_\lambda^2$, for every $t \in [0, T]$. For further details refer to De Kort and Vellekoop [2017]. Furthermore, as in Menoncin [2009], Zeddouk and Devolder [2020], in order for λ to assume reasonable values, we require, for any $t \in [0, T]$, that its expected value equals the corresponding deterministic Gompertz-Makeham mortality intensity. To achieve this goal, we set

$$\beta_\lambda(t) = \phi_\lambda + \frac{1}{b_\lambda}\left(\frac{1}{b_\lambda}\frac{1}{\rho_\lambda} + 1\right) e^{\frac{\iota + t - l_\lambda}{b_\lambda}}, \quad t \in [0, T], \tag{3.2}$$

where ι is the age of the pension plan members at $t = 0$, $\phi_\lambda > 0$ quantifies the age-independent component of mortality intensity, and $l_\lambda, b_\lambda > 0$ represent the modal value and the dispersion parameter of the age-at-death distribution, respectively.

At this point, we construct a stochastic random time of death for the pension plan members. Mimicking the reduced-form credit risk models, we adopt the canonical construction of defining a random time using a specified hazard process. Given the $\mathbb{F}$-mortality intensity process λ, the corresponding $\mathbb{F}$-hazard process within the time interval $[0, T]$ is $\int_0^t \lambda(z)\mathrm{d}z$, for every $t \in [0, T]$. By introducing a random variable Θ_λ which follows a unit exponential distribution and is independent on $\mathcal{F}(T)$, we define the stochastic random time of death τ^λ as

$$\tau^\lambda := \inf\left\{t \geq 0 : \int_0^t \lambda(z)\mathrm{d}z \geq \Theta^\lambda\right\}.$$

By exploiting the $\mathbb{P}$-independence on Θ^λ and the $\mathcal{F}(t)$-measurability of the hazard process, for every $t \in [0, T]$, we have

$$\mathbb{P}(\tau^\lambda > t|\mathcal{F}(t)) = \mathbb{E}\left[p(t)|\mathcal{F}(t)\right] = \mathbb{P}\left(\int_0^t \lambda(z)\mathrm{d}z < \Theta^\lambda \middle| \mathcal{F}(t)\right) = e^{-\int_0^t \lambda(z)\mathrm{d}z}, \quad t \in [0, T].$$

It is worth pointing out that the stochastic random time of death τ^λ is not a stopping time w.r.t. the reference filtration $\mathbb{F}$. Therefore, we have to introduce an enlarged filtration to satisfy the above property. To this end, we introduce the death occurrence process $H^\lambda = \left\{H^\lambda(t),\ t \in [0, T]\right\}$ defined as

$$H^\lambda(t) := \mathbb{1}_{\left\{\tau^\lambda \leq t\right\}}, \quad t \in [0, T],$$

which records whether the pension plan member has died at every given time $t \in [0, T]$. Accordingly, we set $\mathcal{F}^{H^\lambda}(t) := \sigma\left\{H^\lambda(z) : 0 \leq z \leq t\right\}$, for every $t \in [0, T]$ express the enlarged filtration $\mathbb{G} = \{\mathcal{G}(t),\ t \in [0, T]\}$ as

$$\mathcal{G}(t) := \mathcal{F}(t) \vee \mathcal{F}^{H^\lambda}(t), \quad t \in [0, T].$$

Consequently, $\mathbb{G}$ is the smallest filtration containing $\mathbb{F}$ such that τ^λ is a $\mathbb{G}$-stopping time. For this reason, $\mathbb{G}$ incorporates information about both the financial market and the survival time of pension plan members. By the canonical construction of the stochastic random time of death τ^λ, we observe that: *(i)* every local $(\mathbb{F}, \mathbb{P})$-martingale is also a local $(\mathbb{G}, \mathbb{P})$-martingale (for further details, see Brémaud and Yor [1978]), and *(ii)* the process $\tilde{H}^\lambda = \left\{\tilde{H}^\lambda(t),\ t \in [0, T]\right\}$ defined as

$\tilde{H}^\lambda(t) := H^\lambda(t) - \int_0^{t\wedge\tau^\lambda} \lambda(z)\mathrm{d}z$, $t \in [0,T]$ is a $(\mathbb{G},\mathbb{P})$-martingale.

3.1.2 The combined financial-insurance model

In the present section, we will construct a combined financial-insurance market model defined on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ with $\mathcal{G} = \mathcal{G}(T)$. Our focus is on determining the price processes of the financial assets within this market, which requires the characterization of the risk-neutral probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$. To achieve this, we define the probability measure $\mathbb{Q}$ by setting

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}(T)} := \Pi(T), \quad \mathbb{P}\text{-a.s.}, \quad \text{such that} \quad \mathbb{E}_{\mathbb{P}}[\Pi(T)] = 1,$$

where the Radon-Nikodým density process $\pi = \{\pi(t),\ t \in [0,T]\}$ satisfies the following stochastic differential equation (SDE)

$$\begin{cases} \dfrac{\mathrm{d}\Pi(t)}{\Pi(t-)} = -\Xi_r(t)\mathrm{d}Z_r(t) - \Xi_\lambda(t)\mathrm{d}Z_\lambda(t) - \Xi_S(t)\mathrm{d}Z_S(t) + \Phi(t)\mathrm{d}\tilde{H}^\lambda(t), \quad t \in [0,T], \\ \pi(0) = 1, \end{cases} \tag{3.3}$$

for some $\mathbb{G}$-predictable processes $\Xi_i = \{\Xi_i(t),\ t \in [0,T]\}$, $i = \{r, \lambda, S\}$, and $\Phi = \{\Phi(t),\ t \in [0,T]\}$ such that $\Phi(t) > -1$, for every $t \in [0,T]$. Processes $\Xi_i(\cdot)$, $i = \{r, \lambda, S\}$, represent the market prices of interest rate risk, systematic longevity risk, and stock price risk, respectively, while $\Phi(\cdot)$ represents the market price of idiosyncratic longevity risk. A straightforward application of Itô's product rule shows that the unique solution to (3.3) is $\Pi(t) = \Pi_1(t)\Pi_2(t)$, where Π_1 and Π_2 are the unique solutions to

$$\begin{cases} \dfrac{\mathrm{d}\Pi_1(t)}{\Pi_1(t)} = -\Xi_r(t)\mathrm{d}Z_r(t) - \Xi_\lambda(t)\mathrm{d}Z_\lambda(t) - \Xi_S(t)\mathrm{d}Z_S(t), \quad t \in [0,T], \\ \Pi_1(0) = 1, \end{cases} \tag{3.4a}$$

and

$$\begin{cases} \mathrm{d}\Pi_2(t) = \Pi_2(t-)\Phi(t)\mathrm{d}\tilde{H}^\lambda(t), \quad t \in [0,T], \\ \Pi_2(0) = 1, \end{cases} \tag{3.4b}$$

given by

$$\begin{aligned} \Pi_1(t) = \exp\Bigg\{ & -\int_0^t \Xi_r(z)\mathrm{d}Z_r(z) - \int_0^t \Xi_\lambda(z)\mathrm{d}Z_\lambda(z) - \int_0^t \Xi_S(z)\mathrm{d}Z_S(z) \\ & -\frac{1}{2}\int_0^t \left(\Xi_r^2(z) + \Xi_\lambda^2(z) + \Xi_S^2(z)\right)\mathrm{d}z \Bigg\},\ t \in [0,T], \end{aligned}$$

and

$$\Pi_2(t) = 1 + \int_0^t \Phi(z)\Pi_2(z-)\mathrm{d}\tilde{H}^\lambda(z), \quad t \in [0,T],$$

respectively. A straightforward application of Girsanov's theorem implies that the processes

$$Z_i^{\mathbb{Q}}(t) = Z_i(t) + \int_0^t \Xi_i(z)\mathrm{d}z, \quad t \in [0,T], \quad i = \{r, \lambda, S\}, \tag{3.5}$$

are $(\mathbb{G}, \mathbb{Q})$-Brownian motions and the process

$$\tilde{H}^{\lambda,\mathbb{Q}}(t) = \tilde{H}^{\lambda}(t) - \int_0^{t\wedge\tau^\lambda} \Phi(z)\lambda(z)\mathrm{d}z = H^{\lambda}(t) - \int_0^{t\wedge\tau^\lambda} (1+\Phi(z))\,\lambda(z)\mathrm{d}z, \quad t \in [0,T],$$

is a $(\mathbb{G}, \mathbb{Q})$-martingale. Following Luciano et al. [2012], Agarwal et al. [2023], we assume that the idiosyncratic component of risk is diversifiable, implying a zero risk premium for individual death occurrences. Consequently, in our framework the risk-neutral probability measure $\mathbb{Q}$ is given by

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}(T)} = \Pi_1(T).$$

We suppose that the combined financial-insurance market consists of a cash account, a stock, a rolling zero-coupon bond (ZCB), and a rolling longevity ZCB. The price process of the cash account $S_0 = \{S_0(t),\ t \in [0,T]\}$ is described by the following ordinary differential equation (ODE)

$$\begin{cases} \dfrac{\mathrm{d}S_0(t)}{S_0(t)} = r(t)\mathrm{d}t, \quad t \in [0,T], \\ S_0(0) = 1, \end{cases} \tag{3.6}$$

where $r = \{r(t),\ t \in [0,T]\}$ is the instantaneous interest rate. Because of the investment horizon of a DC pension scheme spans several decades, we model r using a square-root mean-reverting process. In particular, its $\mathbb{P}$-dynamics are given as follows

$$\begin{cases} \mathrm{d}r(t) = \alpha_r\left(\beta_r - r(t)\right)\mathrm{d}t + \sigma_r\sqrt{r(t)}\mathrm{d}Z_r(t), \quad t \in [0,T], \\ r(0) = r_0 > 0, \end{cases} \tag{3.7}$$

where $\beta_r > 0$ is the long-run mean of the interest rate, $\alpha_r > 0$ is the speed of mean reversion, and $\sigma_r > 0$ is the interest rate volatility. Furthermore, we assume that the Feller condition $2\alpha_r\beta_r \geq \sigma_r^2$ holds such that $r(t) \geq 0$ $\mathbb{P}-$a.s., for every $t \in [0,T]$. As in Wang et al. [2021], we hypothesize that the dynamics of the stock price process $S = \{S(t),\ t \in [0,T]\}$ are

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t)} = r(t)\mathrm{d}t + \sigma_{S,r}\sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right) + \sigma_S\left(\mathrm{d}Z_S(t) + \xi_S\mathrm{d}t\right), \quad t \in [0,T], \\ S(0) = s > 0, \end{cases} \tag{3.8}$$

where $\sigma_{S,r} > 0$ measures the effect of interest rate volatility on the stock price and $\sigma_S > 0$ represents the stock price volatility depending on factors different from the interest rate. We further assume that: *(i)* the market price of interest rate risk $\Xi_r(t)$ is proportional to the square root of the interest rate, i.e., $\Xi_r(t) = \xi_r\sqrt{r(t)}$ with $\xi_r < 0$, for every $t \in [0,T]$, and *(ii)* the market price of stock price risk $\Xi_S(t)$ is constant, namely $\Xi_S(t) = \xi_S > 0$, for every $t \in [0,T]$. To manage the interest rate risk, we introduce ZCBs within the financial market and the following proposition provides the corresponding arbitrage-free price.

Proposition 3.1.1. *Let* $r = \{r(t), t \in [0,T]\}$ *be the instantaneous interest rate process with dynamics* (3.7). *Then, the arbitrage-free t-price of a ZCB paying one unit of currency at its expiration date* $s \geq t$*, denoted by* $P(t,s)$*, satisfies*

$$P(t,s) = \exp\left\{h_1(t,s) - h_2(t,s)r(t)\right\}, \quad 0 \leq t \leq s,$$

where

$$h_1(t,s) := \frac{2\alpha_r\beta_r}{\sigma_r^2}\ln\left(\frac{2k_r e^{\frac{(\alpha_r+\sigma_r\xi_r+k_r)(s-t)}{2}}}{(\alpha_r+\sigma_r\xi_r+k_r)\left(e^{k_r(s-t)}-1\right)+2k_r}\right), \tag{3.9a}$$

$$h_2(t,s) := \frac{2\left(e^{k_r(s-t)}-1\right)}{(\alpha_r+\sigma_r\xi_r+k_r)\left(e^{k_r(s-t)}-1\right)+2k_r}, \tag{3.9b}$$

and $k_r := \sqrt{(\alpha_r+\sigma_r\xi_r)^2+2\sigma_r^2}$. *Moreover, the* $\mathbb{P}$*-dynamics of the ZCB price process satisfy*

$$\begin{cases} \dfrac{\mathrm{d}P(t,s)}{P(t,s)} = r(t)\mathrm{d}t - \sigma_P(t,s)\sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right), & 0 \leq t \leq s, \\ P(s,s) = 1, \end{cases} \tag{3.10}$$

where $\sigma_P(t,s) := h_2(t,s)\sigma_r > 0$, $0 \leq t \leq s$, *is the volatility of the ZCB price.*

Proof. See [Cox et al., 1985, p. 393]. □

We notice that the time to maturity $s-t$, $t \in [0,s]$ of the ZCB varies over time. Besides, as argued in Boulier et al. [2001], the market is not able to provide a ZCB maturing at s for any given time $t \in [0,s]$. To overcome such a drawback, we introduce a rolling ZCB with constant time to maturity K, whose price process $P_K = \{P_K(t), t \in [0,T]\}$, according to (3.10), exhibits the following $\mathbb{P}$-dynamics

$$\begin{cases} \dfrac{\mathrm{d}P_K(t)}{P_K(t)} = r(t)\mathrm{d}t - \sigma_K\sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right), & t \in [0,T], \\ P_K(0) = p_K > 0, \end{cases} \tag{3.11}$$

where

$$\sigma_K = \frac{2\sigma_r\left(e^{k_rK}-1\right)}{(\alpha_r+\sigma_r\xi_r+k_r)\left(e^{k_rK}-1\right)+2k_r} > 0$$

is the volatility of the rolling ZCB price. It is well-known that in a complete market a ZCB issued at time t with a maturity of s can be replicated by investing in the rolling ZCB and the cash account as follows

$$\frac{\mathrm{d}P(t,s)}{P(t,s)} = \frac{\sigma_P(t,s)}{\sigma_K}\frac{\mathrm{d}P_K(t)}{P_K(t)} + \left(1 - \frac{\sigma_P(t,s)}{\sigma_K}\right)\frac{\mathrm{d}S_0(t)}{S_0(t)}, \; 0 \leq t \leq s.$$

Similar to the approach applied for the interest rate risk, to manage the longevity risk we introduce suitable longevity ZCBs within the financial market. Specifically, we adopt the definition provided by De Kort and Vellekoop [2017], which can be stated as follows.

Definition 3.1.2. *A longevity bond is a financial security that pays an amount*

$$p(s) = \exp\left\{-\int_0^s \lambda(u)\mathrm{d}u\right\} \tag{3.12}$$

at its expiration date $s > 0$, that is, the expected fraction of survivors at time s among individuals in a large reference population sharing a common mortality intensity.

If we assume that: (i) the mortality intensity of the pension plan members is given by (3.1), and (ii) the market price of longevity risk $\Xi_\lambda(t)$ is proportional to the square root of mortality intensity, that is, $\Xi_\lambda(t) = \xi_\lambda\sqrt{\lambda(t)}$, with $\xi_\lambda < 0$, for every $t \in [0, T]$, then the no-arbitrage price of the longevity ZCB can be explicitly expressed as follows.

Proposition 3.1.3. *Let $\lambda = \{\lambda(t),\, t \in [0, T]\}$ be the instantaneous force of mortality process with dynamics* (3.1)*. Then, the no-arbitrage t-price $L_B(t, s)$ of longevity ZCB issued at time $t \in [0, s]$, and expiring at $s \geq t$ is given by*

$$L_B(t,s) = \exp\left\{-\int_0^t \lambda(z)\mathrm{d}z + h_1(t,s) - h_2(t,s)r(t) + \eta_1(t,s) - \eta_2(t,s)\lambda(t)\right\}, \quad 0 \leq t \leq s, \tag{3.13}$$

where

$$\eta_1(t,s) := -\rho_\lambda \int_t^s \beta_\lambda(z)\eta_2(z,s)\mathrm{d}z, \tag{3.14a}$$

$$\eta_2(t,s) := \frac{2\left(e^{k_\lambda(s-t)} - 1\right)}{(\rho_\lambda + \sigma_\lambda\xi_\lambda + k_\lambda)\left(e^{k_\lambda(s-t)} - 1\right) + 2k_\lambda}, \tag{3.14b}$$

and $k_\lambda := \sqrt{(\rho_\lambda + \sigma_\lambda\xi_\lambda)^2 + 2\sigma_\lambda^2}$. Additionally, denoting by $\sigma_{L_B}(t,s) := \eta_2(t,s)\sigma_\lambda$, $0 \leq t \leq s$, the $\mathbb{P}$-dynamics of the longevity ZCB price process satisfy

$$\begin{cases} \dfrac{\mathrm{d}L_B(t,s)}{L_B(t,s)} = r(t)\mathrm{d}t - \sigma_P(t,s)\sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right) \\ \qquad\qquad - \sigma_{L_B}(t,s)\sqrt{\lambda(t)}\left(\mathrm{d}Z_\lambda(t) + \xi_\lambda\sqrt{\lambda(t)}\mathrm{d}t\right), \quad 0 \leq t \leq s, \\ L_B(s,s) = e^{-\int_0^s \lambda(z)\mathrm{d}z}. \end{cases} \tag{3.15}$$

Proof. For any $t \in [0, T]$, under the risk-neutral probability measure $\mathbb{Q}$, the dynamics of the mortality intensity process and the instantaneous interest rate process are, respectively, given by

$$\mathrm{d}\lambda(t) = \rho_\lambda^{\mathbb{Q}}\left(\beta_\lambda^{\mathbb{Q}}(t) - \lambda(t)\right)\mathrm{d}t + \sigma_\lambda\sqrt{\lambda(t)}\mathrm{d}Z_\lambda^{\mathbb{Q}}(t), \quad t \in [0,T],$$
$$\mathrm{d}r(t) = \alpha_r^{\mathbb{Q}}\left(\beta_r^{\mathbb{Q}} - r(t)\right)\mathrm{d}t + \sigma_r\sqrt{r(t)}\mathrm{d}Z_r^{\mathbb{Q}}(t), \quad t \in [0,T],$$

where

$$\begin{aligned}
\rho_\lambda^{\mathbb{Q}} &= \rho_\lambda + \sigma_\lambda \xi_\lambda, \\
\beta_\lambda^{\mathbb{Q}}(t) &= \rho_\lambda \frac{\phi_\lambda + \frac{1}{b_\lambda}\left(\frac{1}{\rho_\lambda}\frac{1}{b_\lambda} + 1\right) e^{\frac{\iota + t - l_\lambda}{b_\lambda}}}{\rho_\lambda + \sigma_\lambda \xi_\lambda}, \quad t \in [0, T], \\
\alpha_r^{\mathbb{Q}} &= \alpha_r + \sigma_r \xi_r, \\
\beta_r^{\mathbb{Q}} &= \frac{\alpha_r \beta_r}{\alpha_r + \sigma_r \xi_r}.
\end{aligned}$$

Now we consider a longevity ZCB as in Definition 3.1.2. Standard arguments ensure that its risk-neutral value is

$$\begin{aligned}
L_B(t,s) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{p(s)}{p(0)} e^{-\int_t^s r(z)\mathrm{d}z} \middle| \mathcal{G}(t)\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^s \lambda(z)\mathrm{d}z} e^{-\int_t^s r(z)\mathrm{d}z} \middle| \mathcal{G}(t)\right] \\
&= e^{-\int_0^t \lambda(z)\mathrm{d}z} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s (\lambda(z)+r(z))\mathrm{d}z} \middle| \mathcal{G}(t)\right]. \qquad (3.16)
\end{aligned}$$

We set $N^L(t,s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s (\lambda(z)+r(z))\mathrm{d}z} \middle| \mathcal{G}(t)\right]$. Hence, to determine $L_B(t,s)$ we need to compute $N^L(t,s)$. We notice that $\tilde{N}^L(t,s) = e^{-\int_0^t (r(u)+\lambda(u))\mathrm{d}u} N^L(t,s)$ is a martingale under $\mathbb{Q}$. From Itô's formula we get

$$\begin{aligned}
\mathrm{d}\tilde{N}^L(t,s) =& e^{-\int_0^t (r(z)+\lambda(z))\mathrm{d}z} \Bigg\{ \Bigg[-(\lambda(t)+r(t))\, N(t,s) + \frac{\partial}{\partial t} N^L(t,s)\mathrm{d}t + \frac{\partial}{\partial r} N^L(t,s) \alpha_r^{\mathbb{Q}} \left(\beta_r^{\mathbb{Q}} - r(t)\right) \\
&+ \frac{\sigma_r^2}{2} \frac{\partial^2}{\partial r^2} N^L(t,s) r(t) + \frac{\partial}{\partial \lambda} N^L(t,s) \rho_\lambda^{\mathbb{Q}} \left(\beta_\lambda^{\mathbb{Q}}(t) - \lambda(t)\right) + \frac{\sigma_\lambda^2}{2} \frac{\partial^2}{\partial \lambda^2} N^L(t,s) \lambda(t) \Bigg] \mathrm{d}t \\
&+ \left(\frac{\partial}{\partial r} N^L(t,s) \sigma_r \sqrt{r(t)} \mathrm{d}Z_r^{\mathbb{Q}}(t) + \frac{\partial}{\partial \lambda} N^L(t,s) \sigma_\lambda \sqrt{\lambda(t)} \mathrm{d}Z_\lambda^{\mathbb{Q}}(t) \right) \Bigg\}. \qquad (3.17)
\end{aligned}$$

Setting the dt term of (3.17) to zero leads to the following PDE

$$\begin{aligned}
&\frac{\partial}{\partial t} N^L(t,s)\mathrm{d}t + \frac{\partial}{\partial r} N^L(t,s) \alpha_r^{\mathbb{Q}} \left(\beta_r^{\mathbb{Q}} - r(t)\right) + \frac{\sigma_r^2}{2} \frac{\partial^2}{\partial r^2} N^L(t,s) r(t) \\
&\quad + \frac{\partial}{\partial \lambda} N^L(t,s) \rho_\lambda^{\mathbb{Q}} \left(\beta_\lambda^{\mathbb{Q}}(t) - \lambda(t)\right) + \frac{\sigma_\lambda^2}{2} \frac{\partial^2}{\partial \lambda^2} N^L(t,s) \lambda(t) = (r(t)+\lambda(t))\, N^L(t,s), \quad (3.18)
\end{aligned}$$

with terminal condition $N^L(s,s) = 1$. Since $r(\cdot)$ and $\lambda(\cdot)$ are driven by independent Brownian motions, and since we assume affine dynamics for both interest rate and force of mortality, we are able to obtain the solution to (3.18) as

$$N^L(t,s) = e^{h_1(t,s) - h_2(t,s) r(t)} e^{\eta_1(t,s) - \eta_2(t,s)\lambda(t)}, \qquad (3.19)$$

with terminal conditions $\eta_1(s,s) = \eta_2(s,s) = 0$. From (3.18) and (3.19), we derive

$$\left(\frac{\partial}{\partial t}h_1(t,s) - \frac{\partial}{\partial t}h_2(t,s)r(t) + \frac{\partial}{\partial t}\eta_1(t,s) - \lambda(t)\frac{\partial}{\partial t}\eta_2(t,s)\right) - h_2(t,s)\alpha_r^{\mathbb{Q}}\left(\beta_r^{\mathbb{Q}} - r(t)\right) + \frac{\sigma_r^2}{2}h_2^2(t,s)r(t)$$
$$- \eta_2(t,s)\rho_\lambda^{\mathbb{Q}}\left(\beta_\lambda^{\mathbb{Q}}(t) - \lambda(t)\right) + \frac{\sigma_\lambda^2}{2}\eta_2^2(t,s)\lambda(t) = r(t) + \lambda(t).$$

By collecting r and λ above, we get the following two systems of ODEs

$$\begin{cases} 0 = \frac{\partial}{\partial t}h_2(t,s) - \alpha_r^{\mathbb{Q}}h_2(t,s) - \frac{\sigma_r^2}{2}h_2^2(t,s) + 1, \\ 0 = \frac{\partial}{\partial t}h_1(t,s) - h_2(t,s)\alpha_r^{\mathbb{Q}}\beta_r^{\mathbb{Q}}, \end{cases} \tag{3.20a}$$

$$\begin{cases} 0 = \frac{\partial}{\partial t}\eta_1(t,s) - \rho_\lambda^{\mathbb{Q}}\beta_\lambda^{\mathbb{Q}}(t)\eta_2(t,s), \\ 0 = \frac{\partial}{\partial t}\eta_2(t,s) - \rho_\lambda^{\mathbb{Q}}\eta_2(t,s) - \frac{\sigma_\lambda^2}{2}\eta_2^2(t,s) + 1, \end{cases} \tag{3.20b}$$

whose solutions are given by (3.9) and, (3.14), respectively. Substituting (3.19) into (3.16), we obtain equation (3.13). Applying Itô's Lemma to (3.13), we derive the $\mathbb{Q}$-dynamics of the longevity ZCB price satisfying

$$\begin{aligned}\frac{\mathrm{d}L_B(t,s)}{L_B(t,s)} =& \Bigg[-\lambda(t) + \left(\frac{\partial}{\partial t}h_1(t,s) - h_2(t,s)\alpha_r^{\mathbb{Q}}\beta_r^{\mathbb{Q}}\right) - \left(\frac{\partial}{\partial t}h_2(t,s) - \alpha_r^{\mathbb{Q}}h_2(t,s) - \frac{\sigma_r^2}{2}h_2^2(t,s)\right)r(t) \\ &+ \left(\frac{\partial}{\partial t}\eta_1(t,s) - \eta_2(t,s)\rho_\lambda^{\mathbb{Q}}\beta_\lambda^{\mathbb{Q}}(t)\right) - \left(\frac{\partial}{\partial t}\eta_2(t,s) - \eta_2(t,s)\rho_\lambda^{\mathbb{Q}} - \frac{\sigma_\lambda^2}{2}\eta_2^2(t,s)\right)\lambda(t)\Bigg]\mathrm{d}t \\ &- \sigma_P(t,s)\sqrt{r(t)}\mathrm{d}Z_r^{\mathbb{Q}}(t) - \eta_2(t,s)\sigma_\lambda\sqrt{\lambda(t)}\mathrm{d}Z_\lambda^{\mathbb{Q}}(t),\end{aligned}$$

Setting $\sigma_{L_B}(t,s) = \eta_2(t,s)\sigma_\lambda$ and thanks to (3.20), the above $\mathbb{Q}$-dynamics can be simplified as follows

$$\frac{\mathrm{d}L_B(t,s)}{L_B(t,s)} = r(t)\mathrm{d}t - \sigma_P(t,s)\sqrt{r(t)}\mathrm{d}Z_1^{\mathbb{Q}}(t) - \sigma_{L_B}(t,s)\sqrt{\lambda(t)}\mathrm{d}Z_2^{\mathbb{Q}}(t), \tag{3.21}$$

with final condition $L_B(s,s) = e^{-\int_0^s \lambda(z)\mathrm{d}z}$. Hence, its corresponding $\mathbb{P}$-dynamics are the ones depicted in equation (3.15). The proof is now completed. □

In accordance with the financial ZCB, we consider a rolling longevity ZCB $L_{K_1} = \{L_{K_1}(t),\, t \in [0,T]\}$ with constant time to maturity K_1, whose $\mathbb{P}$-dynamics satisfy

$$\begin{cases} \frac{\mathrm{d}L_{K_1}(t)}{L_{K_1}(t)} = r(t)\mathrm{d}t - \sigma_{K_1}\sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right) - \sigma_{L_{K_1}}\sqrt{\lambda(t)}\left(\mathrm{d}Z_\lambda(t) + \xi_\lambda\sqrt{\lambda(t)}\mathrm{d}t\right), \\ L_{K_1}(0) = l_{K_1} > 0, \end{cases} \tag{3.22}$$

with

$$\sigma_{K_1} := \frac{2\sigma_r\left(e^{k_r K_1} - 1\right)}{\left(\alpha_r + \sigma_r\xi_r + k_r\right)\left(e^{k_r K_1} - 1\right) + 2k_r} > 0,$$

$$\sigma_{L_{K_1}} := \frac{2\sigma_\lambda \left(e^{k_\lambda K_1} - 1\right)}{(\rho_\lambda + \sigma_\lambda \xi_\lambda + k_\lambda)\left(e^{k_\lambda K_1} - 1\right) + 2k_\lambda} > 0.$$

For further details, see, e.g., Menoncin [2008]. A longevity ZCB issued at time t and with maturity s can be replicated using rolling bonds, rolling longevity bonds, and cash as follows

$$\begin{aligned}\frac{\mathrm{d}L_B(t,s)}{L_B(t,s)} &= \left[1 - \frac{\sigma_P(t,s)}{\sigma_K} - \frac{\sigma_{L_B}(t,s)}{\sigma_{L_{K_1}}}\left(1 - \frac{\sigma_{K_1}}{\sigma_K}\right)\right]\frac{\mathrm{d}S_0(t)}{S_0(t)} \\ &+ \left(\frac{\sigma_P(t,s)}{\sigma_K} - \frac{\sigma_{K_1}}{\sigma_K}\frac{\sigma_{L_B}(t,s)}{\sigma_{L_{K_1}}}\right)\frac{\mathrm{d}P_{K_1}(t)}{P_{K_1}(t)} + \frac{\sigma_{L_B}(t,s)}{\sigma_{L_{K_1}}}\frac{\mathrm{d}L_{K_1}(t)}{L_{K_1}(t)}, \quad 0 \le t \le s \le T.\end{aligned}$$

3.1.3 Contributions and benefits

In a DC pension plan, each member pays a flow to the pension account. Along the lines of Deelstra et al. [2003, 2004], we assume that such flow consists of a lump sum at date $t = 0$, denoted by w_0, and a continuously paid premium, given by a constant percentage of their labour income. As in Guan and Liang [2014] and Wang et al. [2021], we assume that the labour income of the representative pension plan's member is a stochastic process $L = \{L(t),\, t \in [0,T]\}$ with the following $\mathbb{P}$-dynamics

$$\begin{cases}\dfrac{\mathrm{d}L(t)}{L(t)} = \zeta \mathrm{d}t + \sigma_{L,r}\sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right) + \sigma_L\left(\mathrm{d}Z_S(t) + \xi_S \mathrm{d}t\right), & t \in [0,T], \\ L(0) = l_0 > 0, \end{cases}$$

where $\zeta \in \mathbb{R}$ is the labour income appreciation rate, while $\sigma_{L,r} > 0$ and $\sigma_L > 0$ are two factors measuring the effects of the interest rate and the stock market risks on the labour income, respectively. The instantaneous flow of contribution $C = \{C(t),\, t \in [0,T]\}$ of the representative pension plan's member is $C(t) = \gamma L(t)$, for every $t \in [0,T]$. Here, $\gamma \in (0,1)$ is the constant percentage of the labour income allocated to the pension plan/deposited into the pension account. Evidently, the $\mathbb{P}$-dynamics of his/her flow of contribution C are

$$\begin{cases}\dfrac{\mathrm{d}C_L(t)}{C_L(t)} = \zeta \mathrm{d}t + \sigma_{L,r}\sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right) + \sigma_L\left(\mathrm{d}Z_S(t) + \xi_S \mathrm{d}t\right), & t \in [0,T], \\ C_L(0) = \gamma l_0 =: c_L. \end{cases} \tag{3.23}$$

In our framework, the aim of the pension plan manager is to determine the investment strategy under the constraint that the pension plan wealth at retirement T exceeds the value of $np(T)$ lifetime annuities, which play the role of a *minimum guarantee,* along the lines of Deelstra et al. [2003], Di Giacinto et al. [2011], Han and Hung [2012], Guan and Liang [2014], Zhang et al. [2020] and Wang et al. [2021], among others. For this purpose, we first set the flow of benefits level that the annuity must provide. As in Wang et al. [2021], we assume that the flow of benefits $b = \{b(t),\, t \in [T, \omega - \iota]\}$ is

$$b(t) = b_T \cdot \exp\left\{\tilde{j}(t - T)\right\},$$

where $b_T > 0$ is the benefit at retirement date, ω is the maximal survival age, ι is the age of the individual as in (3.2), and $\tilde{j} \ge 0$ is the inflation rate associated with the cost of living. In such a framework, we only have to determine the level of b_T. However, the level of benefit at retirement b_T and the contribution rate γ cannot be set separately: the pension plan members choose either the contribution rate γ or b_T, while the remaining parameter can be obtained via the so-called *feasible condition*. Along the lines of El Karoui and Jeanblanc-Picqué [1998] and Deelstra et al. [2003, 2004], the latter reads as

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^{\omega-\iota} n\left(C_L(z)\mathbb{1}_{z<T} - b(z)\mathbb{1}_{z\geq T}\right)e^{-\int_0^z r(q)\mathrm{d}q}p(z)\mathrm{d}z\right] = 0. \tag{3.24}$$

Condition (3.24) states that the expected present value of contribution streams, paid by the pension plan members during the accumulation phase, must be equal to the expected present value of the flow of benefits delivered by the annuity during the decumulation phase. With the aid of equation (3.24), we are able to characterize a feasible pair (γ, b_T) of contribution rate γ and flow of benefits b_T at retirement date T by the following relation

$$\frac{\gamma}{b_T} = \frac{\int_T^{\omega-\iota} e^{\tilde{j}(z-T)}L_B(0,z)\mathrm{d}z}{\int_0^T \mathbb{E}^{\mathbb{Q}}\left[L(z)e^{-\int_0^z r(q)\mathrm{d}q}p(z)\right]\mathrm{d}u}. \tag{3.25}$$

As in Agarwal et al. [2023], we assume that the annuity provider incorporates the mortality behaviour of the pension plan members through the mortality intensity λ given in (3.1). Hence, the fair price of the lifetime annuity at retirement date T is

$$B(T) = \mathbb{E}^{\mathbb{Q}}\left[\int_T^{\omega-\iota} b(z)e^{-\int_T^z r(q)\mathrm{d}q}\frac{p(z)}{p(T)}\mathrm{d}z\middle|\mathcal{G}(T)\right].$$

Remark 3.1.4. *It is well known that annuity providers have different risk-neutral probabilities regarding the mortality intensity process when compared to policyholders, see Biffis and Millossovich [2006] for further details. To keep the picture as simple as possible, in the present work, we assume that the annuity providers and the policyholders share the same risk-neutral probabilities, leaving the study of the more general issue to future research.*

Finally, we introduce the *minimum guarantee* used to purchase the lifetime annuities for surviving members at retirement time T. In particular, its value at retirement time is given by

$$G(T) = np(T)B(T) = n\mathbb{E}^{\mathbb{Q}}\left[\int_T^{\omega-\iota} b(z)e^{-\int_T^z r(q)\mathrm{d}q}p(z)\mathrm{d}z\middle|\mathcal{G}(T)\right]. \tag{3.26}$$

3.1.4 The PO-PPI strategy for the accumulation phase

We explore the investment strategy deployed by the pension fund manager during the accumulation phase. In particular, starting from time $t = 0$ and until the retirement date T the pension plan manager shall invest the total contributions' streams according to a pre-specified allocation mechanism. Since the aim of the pension plan manager is to obtain at least the guaranteed amount $G(T)$ at time T, such an allocation mechanism should both preserve the wealth against downward losses and offer equity market participation in case of favorable market conditions. To fulfil such properties, we generalize the framework of Temocin et al. [2018] employing in the management of

DC pension plan an allocation mechanism belonging to the PPI strategy. As shown in Section 1.4.2, the PPI strategy is a dynamic allocation mechanism built upon the floor, which is the minimum acceptable value of the total wealth during the accumulation phase. More precisely, it is the process $G = \{G(t), t \in [0,T]\}$ given by the present value of the minimum guarantee $G(T)$ at the retirement date T

$$G(t) = \mathbb{E}^{\mathbb{Q}}\left[G(T)e^{-\int_t^T r(z)\mathrm{d}z}|\mathcal{G}(t)\right] = n\int_T^{\omega-\iota} b(z)L_B(t,z)\mathrm{d}z, \quad t \in [0,T], \tag{3.27}$$

whose $\mathbb{P}$-dynamics read as

$$\begin{aligned}\frac{\mathrm{d}G(t)}{G(t)} =& r(t)\mathrm{d}t - \frac{n}{G(t)}\left(\int_T^{\omega-\iota} b(z)\sigma_P(t,z)L_B(t,z)\mathrm{d}z\right)\sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right)\\ &- \frac{n}{G(t)}\left(\int_T^{\omega-\iota} b(z)\sigma_{L_B}(t,z)L_B(t,z)\mathrm{d}z\right)\sqrt{\lambda(t)}\left(\mathrm{d}Z_\lambda(t) + \xi_\lambda\sqrt{\lambda(t)}\mathrm{d}t\right), \quad t \in [0,T],\end{aligned} \tag{3.28}$$

The exposure to the risky asset S is proportional to the cushion, that is, the difference between the total wealth process $W^{PPI} = \{W^{PPI}(t),\ t \in [0,T]\}$ and the floor process. The proportionality factor is the multiplier m which in the case of the PPI strategy is time-varying, i.e. $m = \{m(t),\ t \in [0,T]\}$. As the PPI strategy is self-financing, the remaining wealth is invested in the cash account S_0. For our purposes, we make the following assumptions.

Assumption 3.1.1. *If a pension plan member dies before the retirement date T, their contributions will no longer be paid out, and their heirs receive no benefits (see, [Agarwal et al., 2023, p. 13]); in turn, the resources already accumulated are redistributed among the surviving members of the pension plan. This can be seen as a form of cross-subsidization because the contributions of those who die earlier are used to bolster the financial security of the surviving members.*

Given Assumption 3.1.1 and, according to the PPI strategy, the wealth $\mathbb{P}$-dynamics are given by

$$\begin{cases}\mathrm{d}W^{PPI}(t) = m(t)\left(W^{PPI}(t) - G(t)\right)\dfrac{\mathrm{d}S(t)}{S(t)} + \left[W^{PPI}(t) - m(t)\left(W^{PPI}(t) - G(t)\right)\right]\dfrac{\mathrm{d}S_0(t)}{S_0(t)}\\ \qquad\qquad + np(t)C_L(t)\mathrm{d}t, \quad t \in [0,T],\\ W^{PPI}(0) = n \cdot w_0 > 0.\end{cases} \tag{3.29}$$

In our modeling framework, the pension plan wealth is exposed to both longevity and interest rate risks. Indeed, as shown in equations (3.26) and (3.27), the level of the minimum guarantee will be higher than expected if *(i)* the actual survival rate of the pension plan members exceeds its expectation or *(ii)* the actual level of interest rates is lower than expected. Therefore, given the need to hedge against the aforementioned risks, the standard version of the PPI strategy can no longer be used. To deal with the latter scenario, along the lines of Chen et al. [2022], we exploit a modified version of the PPI strategy, termed *purpose-oriented* PPI (PO-PPI) strategy. The main difference between PPI and PO-PPI is the underlying portfolio of risky assets to which the latter is exposed. As shown in equation (3.29), the exposure of the PPI strategy is invested in a stock market index. By contrast, the exposure of PO-PPI is invested in a purpose-oriented and diversified fund. More precisely, we consider a linear combination of assets related to the specific risks the pension plan manager is intended to hedge against. According to the financial-insurance market model proposed in Section 3.1, the diversified index is a linear combination of a rolling longevity bond (to safeguard against longevity risk), a rolling bond (to protect against interest rate risk), and a risky asset (to hedge against market/investment risk and provide an equity market participation in case of favorable market conditions). Denoting by $I = \{I(t),\ t \in [0,T]\}$ the price process of the purpose-oriented fund, the corresponding $\mathbb{P}$-dynamics are

$$\begin{cases} \dfrac{\mathrm{d}I(t)}{I(t)} = \alpha_{P_K}(t)\dfrac{\mathrm{d}P_K(t)}{P_K(t)} + \alpha_{L_{K_1}}(t)\dfrac{\mathrm{d}L_{K_1}(t)}{L_{K_1}(t)} + \alpha_S(t)\dfrac{\mathrm{d}S(t)}{S(t)}, \quad t \in [0,T], \\ I(0) = i_0, \end{cases}$$

where $\alpha_i \colon [0,T] \times \Omega \to \mathbb{R}$, $i = \{P_K,\ L_{K_1},\ S\}$, represents the share invested in the rolling ZCB, the rolling longevity ZCB, and the stock, respectively, such that

$$\alpha_{P_K}(t) + \alpha_{L_{K_1}}(t) + \alpha_S(t) = 1 \ \mathbb{P}\text{-a.s.}, \quad \forall t \in [0,T].$$

This means that given the investments in the rolling ZCB and rolling longevity ZCB, the allocation to the risky asset is automatically determined. Thus, the $\mathbb{P}$-dynamics of pension plan wealth $W^{PO-PPI} = \left\{W^{PO-PPI}(t),\ t \in [0,T]\right\}$, managed according to PO-PPI strategy, is given by

$$\begin{cases} \mathrm{d}W^{PO-PPI}(t) = m(t)\left(W^{PO-PPI}(t) - G(t)\right)\left[\alpha_{P_K}(t)\dfrac{\mathrm{d}P_K(t)}{P_K(t)} + \alpha_{L_{K_1}}(t)\dfrac{\mathrm{d}L_{K_1}(t)}{L_{K_1}(t)} + \alpha_S(t)\dfrac{\mathrm{d}S(t)}{S(t)}\right] \\ \qquad + \left[W^{PO-PPI}(t) - m(t)\left(W^{PO-PPI}(t) - G(t)\right)\right]\dfrac{\mathrm{d}S_0(t)}{S_0(t)} + np(t)C_L(t)\mathrm{d}t, \quad t \in [0,T], \\ W^{PO-PPI}(0) = n \cdot w_0 > 0. \end{cases} \tag{3.30}$$

Injecting equations (3.6), (3.8), (3.11), and (3.22) into equation (3.30), the wealth $\mathbb{P}$-dynamics can be rewritten as

$$\begin{cases} \mathrm{d}W^{PO-PPI}(t) = W^{PO-PPI}(t)r(t)\mathrm{d}t + np(t)C_L(t)\mathrm{d}t + m(t)\left(W^{PO-PPI}(t) - G(t)\right) \\ \qquad \Big\{ \left[\left(1 - \alpha_{P_K}(t) - \alpha_{L_{K_1}}(t)\right)\sigma_{S,r} - \alpha_{P_K}(t)\sigma_K - \alpha_{L_{K_1}}(t)\sigma_{K_1}\right] \\ \qquad \cdot \sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right) - \alpha_{L_{K_1}}(t)\sigma_{L_{K_1}}\sqrt{\lambda(t)}\left(\mathrm{d}Z_\lambda(t) + \xi_\lambda\sqrt{\lambda(t)}\mathrm{d}t\right) \Big\} \\ \qquad + \left(1 - \alpha_{P_K}(t) - \alpha_{L_{K_1}}(t)\right)\sigma_S\left(\mathrm{d}Z_S(t) + \xi_S\mathrm{d}t\right), \quad t \in [0,T], \\ W^{PO-PPI}(0) = n \cdot w_0 > 0 \end{cases} \tag{3.31}$$

Set $\boldsymbol{u}(\cdot) := \left(\alpha_{P_K}(\cdot), \alpha_{L_{K_1}}(\cdot), m(\cdot)\right)^\top$. For generic initial time $t \in [0,T]$, with $W^{PO-PPI}(t) = w$, $r(t) = r$, and $\lambda(t) = \lambda$, we define the set of admissible strategies $\mathcal{U}_{ad}(t,w,r,\lambda)$ depending on the initial data $(t, w, r, \lambda) \in [0, T] \times (0, +\infty)^3$ as

$$\mathcal{U}_{ad}(t, w, r, \lambda) := \Big\{ \boldsymbol{u} \colon [0, T] \times \Omega \to \mathbb{R}^3 \mid \boldsymbol{u}(\cdot) \text{ is progressively measurable w.r.t. } (\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P}) \mid \mathbb{E}\left[\int_0^T \Sigma(z) dz\right] < +\infty, \; W^{PO-PPI}(T) \geq G(T) \; \mathbb{P}\text{-a.s.} \Big\},$$

where

$$\Sigma(t) := m^2(t) \left(W^{PO-PPI}(t) - G(t)\right)^2 \Big\{ \Big[\left(1 - \alpha_{P_K}(t) - \alpha_{L_{K_1}}(t)\right) \sigma_{S,r} - \alpha_{P_K}(t) \sigma_K - \alpha_{L_{K_1}}(t) \sigma_{K_1} \Big]^2 \cdot r(t) + \left(1 - \alpha_{P_K}(t) - \alpha_{L_{K_1}}(t)\right) \sigma_S^2 + \alpha_{L_{K_1}}^2(t) \sigma_{L_{K_1}}^2 \lambda(t) \Big\},$$

for every $t \in [0, T]$.

3.2 The optimization problem

The aim of the pension plan manager is to maximize the expected utility of the terminal cushion, defined as the difference between the total wealth $W^{PO-PPI}(T)$ and the minimum guarantee $G(T)$ at the retirement date T. Additionally, we assume that the pension plan manager receives a predetermined fraction of the terminal surplus $W^{PO-PPI}(T) - G(T)$ as remuneration. We further assume that the fund manager's preferences align perfectly with those of the surviving members when maximizing the expected utility, and are described by a constant relative risk aversion (CRRA) utility function $U : (0, +\infty) \to \mathbb{R}$ defined as

$$U(x) := \frac{x^{1-\delta}}{1-\delta}, \tag{3.32}$$

where $\delta > 0$ and $\delta \neq 1$. The resulting stochastic optimal control problem can be naturally solved by the dynamic programming method and stated as follows.

Problem 3.2.1. *For generic initial data* $(t, w, r, \lambda) \in [0, T] \times (0, +\infty)^3$

$$\textit{maximize} \quad J(t, w, r, \lambda; \boldsymbol{u}(\cdot)) := \mathbb{E}\left[\frac{1}{1-\delta}\left(W^{PO-PPI}(T) - G(T)\right)^{1-\delta}\right] \quad \textit{over} \quad \boldsymbol{u}(\cdot) \in \mathcal{U}_{ad}(t, w, r, \lambda). \tag{3.33}$$

Here $\mathbb{E}[\cdot]$ denotes the conditional expectation w.r.t. $\mathcal{G}(t)$. Necessary and sufficient conditions for $W^{PO-PPI}(T) - G(T) \geq 0$, which ensure the well-posedness of optimization problem 3.2.1, will be outlined later.

3.2.1 Solution to the optimization problem

The optimization problem 3.2.1 is not a typical *single investment problem* due to the inclusion of both the contributions' stream paid by the pension plan members, which makes the dynamics of the wealth process not self-financing, and the terminal wealth constraint. To address this challenge, in the following we introduce an *auxiliary process* and recast an optimization problem equivalent to Problem 3.2.1. In line with the methodology employed by Han and Hung [2012], Guan and Liang [2014], we replicate the inflow $np(\cdot)C_L(\cdot)$ according to the following procedure. Under the assumption of the absence of arbitrage and market completeness, we may construct a fictitious financial instrument that provides payments proportional to the expected fraction of surviving pension plan members $p(u)$ at time $u \in [t,T]$. Of course, the proportionality factor is given by $nC_L(u)$. The value of this asset at time t, $t \le u \le T$, will be denoted by $D(t,u)$.

Proposition 3.2.2. *The arbitrage-free t-price of a financial security issued a time $t \in [0,s]$ and paying a face amount equal to $nC_L(s)p(s)$ at time $s \in [t,T]$ is given by*

$$D(t,s) = nC_L(t)\exp\left\{-\int_0^t \lambda(z)\mathrm{d}z + f_2(t,s) - f_1(t,s)r(t) + \eta_1(t,s) - \eta_2(t,s)\lambda(t)\right\}, \quad 0 \le t \le s \le T, \tag{3.34}$$

where

$$f_1(t,s) := \frac{2\left(e^{k_f(s-t)}-1\right)}{\left[\alpha_r + \sigma_r\left(\xi_r - \sigma_{L,r}\right) + k_f\right]\left(e^{k_f(s-t)}-1\right) + 2k_f}, \tag{3.35a}$$

$$f_2(t,s) := \zeta(s-t) + \frac{2\alpha_r\beta_r}{\sigma_r^2}\ln\left(\frac{2k_f e^{\frac{\left[\alpha_r+\sigma_r\left(\xi_r-\sigma_{L,r}\right)+k_f\right](s-t)}{2}}}{\left[\alpha_r + \sigma_r\left(\xi_r - \sigma_{L,r}\right) + k_f\right]\left(e^{k_f(s-t)}-1\right) + 2k_f}\right), \tag{3.35b}$$

$$\eta_1(t,s) := -\rho_\lambda \int_t^s \beta_\lambda(z)\eta_2(z,s)\mathrm{d}z \tag{3.35c}$$

$$\eta_2(t,s) := \frac{2\left(e^{k_\lambda(s-t)}-1\right)}{\left(\rho_\lambda + \sigma_\lambda\xi_\lambda + k_\lambda\right)\left(e^{k_\lambda(s-t)}-1\right) + 2k_\lambda}, \tag{3.35d}$$

with

$$k_f := \sqrt{\left[\alpha_r + \sigma_r\left(\xi_r - \sigma_{L,r}\right)\right]^2 + 2\sigma_r^2}, \qquad k_\lambda := \sqrt{\left(\rho_\lambda + \sigma_\lambda\xi_\lambda\right)^2 + 2\sigma_\lambda^2}.$$

Moreover, the $\mathbb{P}$-dynamics of $D(t,s)$ satisfy

$$\begin{cases} \dfrac{\mathrm{d}D(t,s)}{D(t,s)} = r(t)\mathrm{d}t + \left(\sigma_{L,r} - f_1(t,s)\sigma_r\right)\sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right) \\ \qquad - \sigma_{L_B}(t,s)\sqrt{\lambda(t)}\left(\mathrm{d}Z_\lambda(t) + \xi_\lambda\sqrt{\lambda(t)}\mathrm{d}t\right) + \sigma_L\left(\mathrm{d}Z_S(t) + \xi_S\mathrm{d}t\right), \quad 0 \le t \le s \le T, \\ D(s,s) = ne^{-\int_0^s \lambda(z)\mathrm{d}z}C_L(s). \end{cases} \tag{3.36}$$

Proof. Under the risk-neutral probability measure $\mathbb{Q}$, the dynamics of the instantaneous flow of contributions C_L is given by

$$\frac{\mathrm{d}C_L(t)}{C_L(t)} = \zeta\mathrm{d}t + \sigma_{L,r}\sqrt{r(t)}\mathrm{d}Z_r^{\mathbb{Q}}(t) + \sigma_L\mathrm{d}Z_S^{\mathbb{Q}}(t), \quad t \in [0,T].$$

The fundamental theorem of asset pricing allows to determine the risk-neutral value of the fictitious financial investment

$$D(t,s)=\mathbb{E}^{\mathbb{Q}}\left[nC_L(s)\frac{p(s)}{p(0)}e^{-\int_t^s r(z)\mathrm{d}z}\middle|\mathcal{G}(t)\right]=n\mathbb{E}^{\mathbb{Q}}\left[C_L(s)e^{-\int_0^s \lambda(z)\mathrm{d}z}e^{-\int_t^s r(z)\mathrm{d}z}\middle|\mathcal{G}(t)\right]$$

$$=ne^{-\int_0^t \lambda(z)\mathrm{d}z}\mathbb{E}^{\mathbb{Q}}\left[C_L(s)e^{-\int_t^s (\lambda(z)+r(z))\mathrm{d}z}\right]. \tag{3.37}$$

We set $M(t,s)=\mathbb{E}^{\mathbb{Q}}\left[C_L(s)e^{-\int_t^s (\lambda(z)+r(z))\mathrm{d}z}\right]$. Hence, to determine $D(t,s)$, we need to compute $M(t,s)$. We notice that $\tilde{M}(t,s)=e^{-\int_0^t (r(z)+\lambda(z))\mathrm{d}z}M(t,s)$ is a martingale under $\mathbb{Q}$. From Itô's formula, we have

$$\begin{aligned}
\mathrm{d}\tilde{M}(t,s)=&e^{-\int_0^t (r(z)+\lambda(z))\mathrm{d}z}\Bigg\{\Bigg[-\left(r(t)+\lambda(t)\right)M(t,s)+\frac{\partial}{\partial t}M(t,s)+\zeta C_L(t)\frac{\partial}{\partial C_L}M(t,s)\\
&+\frac{1}{2}\left(r(t)\sigma_{L,r}^2+\sigma_L^2\right)C_L^2(t)\frac{\partial^2}{\partial C_L^2}M(t,s)+\alpha_r^{\mathbb{Q}}\left(\beta_r^{\mathbb{Q}}-r(t)\right)\frac{\partial}{\partial r}M(t,s)+\frac{\sigma_r^2}{2}r(t)\frac{\partial^2}{\partial r^2}M(t,s)\\
&+\sigma_r\sigma_{L,r}r(t)C_L(t)\frac{\partial^2}{\partial r\partial C_L}M(t,s)+\rho_\lambda^{\mathbb{Q}}\left(\beta_\lambda^{\mathbb{Q}}(t)-\lambda(t)\right)\frac{\partial}{\partial\lambda}M(t,s)+\frac{\sigma_\lambda^2}{2}\lambda(t)\frac{\partial^2}{\partial\lambda^2}M(t,s)\Bigg]\mathrm{d}t\\
&+\Bigg[C_L(t)\frac{\partial}{\partial C_L}M(t,s)\left(\sqrt{r(t)}\sigma_{L,r}\mathrm{d}Z_r^{\mathbb{Q}}(t)+\sigma_L\mathrm{d}Z_S^{\mathbb{Q}}(t)\right)+\sigma_r\frac{\partial}{\partial r}M(t,s)\sqrt{r(t)}\mathrm{d}Z_r^{\mathbb{Q}}(t)\\
&+\sigma_\lambda\frac{\partial}{\partial\lambda}M(t,s)\sqrt{\lambda(t)}\mathrm{d}Z_\lambda^{\mathbb{Q}}(t)\Bigg]\Bigg\}
\end{aligned}$$

Setting the dt terms equal to zero, we obtain the following PDE

$$\begin{aligned}
&\frac{\partial}{\partial t}M(t,s)+\frac{1}{2}\left(r(t)\sigma_{L,r}^2+\sigma_L^2\right)C_L^2(t)\frac{\partial^2}{\partial C_L^2}M(t,s)+\frac{\sigma_r^2}{2}r(t)\frac{\partial^2}{\partial r^2}M(t,s)\\
&\quad+\sigma_r\sigma_{L,r}r(t)C_L(t)\frac{\partial^2}{\partial r\partial C_L}M(t,s)+\rho_\lambda^{\mathbb{Q}}\left(\beta_\lambda^{\mathbb{Q}}(t)-\lambda(t)\right)\frac{\partial}{\partial\lambda}M(t,s)+\zeta C_L(t)\frac{\partial}{\partial C_L}M(t,s)\\
&\quad+\alpha_r^{\mathbb{Q}}\left(\beta_r^{\mathbb{Q}}-r(t)\right)\frac{\partial}{\partial r}M(t,s)-\left(r(t)-\lambda(t)\right)M(t,s)=0,
\end{aligned}\tag{3.38}$$

with terminal condition $M(s,s)=C_L(s)$. Being $C_L(t)$, $r(t)$ and $\lambda(t)$ affine processes, and thanks to the independence of the Brownian motions driving the interest rate and the force of mortality processes, the solution to (3.38) is

$$M(t,s)=C_L(t)e^{f_2(t,s)-f_1(t,s)r(t)}e^{\eta_1(t,s)-\eta_2(t,s)\lambda(t)}, \tag{3.39}$$

with terminal conditions $f_1(s,s)=f_2(s,s)=\eta_1(s,s)=\eta_2(s,s)=0$. From (3.38) and (3.39), we derive

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}f_2(t,s)-\frac{\partial}{\partial t}f_1(t,s)r(t)+\frac{\partial}{\partial t}\eta_1(t,s)-\frac{\partial}{\partial t}\eta_2(t,s)\lambda(t)\right)+\zeta-\alpha_r^{\mathbb{Q}}\left(\beta_r^{\mathbb{Q}}-r(t)\right)f_1(t,s)\\
&+r(t)\frac{\sigma_r^2}{2}f_1^2(t,s)-r(t)\sigma_r\sigma_{L,r}f_1(t,s)-\rho_\lambda^{\mathbb{Q}}\left(\beta_\lambda^{\mathbb{Q}}(t)-\lambda(t)\right)\eta_2(t,s)+\lambda(t)\frac{\sigma_\lambda^2}{2}\eta_2^2(t,s)-r(t)-\lambda(t)=0,
\end{aligned}$$

By collecting r and λ terms, we derive the following two systems of ODEs

$$\begin{cases} 0 = \dfrac{\partial}{\partial t} f_1(t,s) - \left(\alpha_r^{\mathbb{Q}} - \sigma_r \sigma_{L,r}\right) f_1(t,s) - \dfrac{\sigma_r^2}{2} f_1^2(t,s) + 1, \\ 0 = \dfrac{\partial}{\partial t} f_2(t,s) - \alpha_r^{\mathbb{Q}} \beta_r^{\mathbb{Q}} f_1(t,s) + \zeta, \end{cases} \tag{3.40}$$

and

$$\begin{cases} 0 = \dfrac{\partial}{\partial t} \eta_1(t,s) - \rho_\lambda^{\mathbb{Q}} \beta_\lambda^{\mathbb{Q}}(t) \eta_2(t,s), \\ 0 = \dfrac{\partial}{\partial t} \eta_2(t,s) - \rho_\lambda^{\mathbb{Q}} \eta_2(t,s) - \dfrac{\sigma_\lambda^2}{2} \eta_2^2(t,s) + 1, \end{cases} \tag{3.41}$$

whose solutions are given in (3.35). Injecting (3.39) in (3.37), we obtain the result depicted in equation (3.34). Applying Itô's Lemma to (3.34), we derive the $\mathbb{Q}$-dynamics of $D(t,s)$ which satisfy

$$\begin{aligned} \frac{\mathrm{d}D(t,s)}{D(t,s)} = & -\lambda(t)\mathrm{d}t + \left\{ \frac{\partial}{\partial t}\eta_1(t,s) - \rho_\lambda^{\mathbb{Q}} \beta_\lambda^{\mathbb{Q}}(t) \eta_2(t,s) - \left(\frac{\partial}{\partial t}\eta_2(t,s) - \rho_\lambda^{\mathbb{Q}} \eta_2(t,s) - \frac{\sigma_\lambda^2}{2}\eta_2^2(t,s) \right) \lambda(t) \right. \\ & \left. + \frac{\partial}{\partial t} f_2(t,s) - \alpha_r^{\mathbb{Q}} \beta_r^{\mathbb{Q}} f_1(t,s) + \zeta - \left[\frac{\partial}{\partial t} f_1 - \left(\alpha_r^{\mathbb{Q}} - \sigma_r\sigma_{L,r}\right) f_1(t,s) - \frac{\sigma_r^2}{2} f_1^2(t,s) \right] r(t) \right\} \mathrm{d}t \\ & + \left(\sigma_{L,r} - f_1(t,s)\sigma_r\right) \sqrt{r(t)}\mathrm{d}Z_r^{\mathbb{Q}}(t) - \eta_2(t,s)\sigma_\lambda \sqrt{\lambda(t)}\mathrm{d}Z_\lambda^{\mathbb{Q}}(t) + \sigma_L \mathrm{d}Z_S^{\mathbb{Q}}(t). \end{aligned}$$

Setting $\sigma_{L_B}(t,s) = \eta_2(t,s)\sigma_\lambda$, and applying (3.41)-(3.40), the above $\mathbb{Q}$-dynamics simplify as

$$\frac{\mathrm{d}D(t,s)}{D(t,s)} = r(t)\mathrm{d}t + \left(\sigma_{L,r} - f_1(t,s)\sigma_r\right)\sqrt{r(t)}\mathrm{d}Z_r^{\mathbb{Q}}(t) - \sigma_{L_B}(t,s)\sqrt{\lambda(t)}\mathrm{d}Z_\lambda^{\mathbb{Q}}(t) + \sigma_L \mathrm{d}Z_S^{\mathbb{Q}}(t), \tag{3.42}$$

with final condition $D(s,s) = ne^{-\int_0^s \lambda(z)\mathrm{d}z} C_L(s)$. Hence, the corresponding $\mathbb{P}$-dynamics are the ones depicted in equation (3.36). This completes the proof. □

Now, we can define $K(t,T) := \int_t^T D(t,z)\mathrm{d}z$, namely, the t-price of a coupon bond expiring in T and paying an instantaneous coupon rate equal to $np(\cdot)C_L(\cdot)$ in the time interval $[t,T]$. By applying the Leibniz integral rule, we can further derive its $\mathbb{P}$-dynamics, which is given by

$$\begin{aligned} \mathrm{d}K(t,T) = & -np(t)C_L(t)\mathrm{d}t + r(t)K(t,T)\mathrm{d}t \\ & + \left(\int_t^T D(t,z)\left(\sigma_{L,r} - f_1(t,z)\sigma_r\right)\mathrm{d}z \right) \sqrt{r(t)} \left(\mathrm{d}Z_r(t) + \xi_r \sqrt{r(t)}\mathrm{d}t \right) \\ & - \left(\int_t^T D(t,z)\sigma_{L_B}(t,z)\mathrm{d}z \right) \sqrt{\lambda(t)} \left(\mathrm{d}Z_\lambda(t) + \xi_\lambda \sqrt{\lambda(t)}\mathrm{d}t \right) \\ & + K(t,T)\sigma_L \left(\mathrm{d}Z_S(t) + \xi_S \mathrm{d}t\right), \quad t \in [0,T], \end{aligned} \tag{3.43}$$

Although $K(\cdot,T)$ is not a financial security traded on the financial market, completeness of the market allows the replication of its corresponding return process by a combination of the traded assets.

Proposition 3.2.3. *The process $K(\cdot,T)$, can be replicated using a combination of cash, stock,* *rolling ZCB, and rolling longevity ZCB as follows*

$$\frac{\mathrm{d}K(t,T)+np(t)C_L(t)\mathrm{d}t}{K(t,T)} = \frac{\mathrm{d}S_0(t)}{S_0(t)}\nu^K_{S_0}(t)+\frac{\mathrm{d}P_K(t)}{P_K(t)}\nu^K_{P_K}(t)+\frac{\mathrm{d}L_{K_1}(t)}{L_{K_1}(t)}\nu^K_{L_{K_1}}(t)+\frac{\mathrm{d}S(t)}{S(t)}\nu^K_S(t), \quad t\in[0,T], \tag{3.44a}$$

where

$$\begin{pmatrix}\nu^K_{S_0}(t)\\ \nu^K_{P_K}(t)\\ \nu^K_{L_{K_1}}(t)\\ \nu^K_S(t)\end{pmatrix} = \begin{pmatrix} 1-\nu^K_{P_K}(t)-\nu^K_{L_{K_1}}(t)-\nu^K_S(t)\\ \frac{\sigma_L}{\sigma_S}\frac{\sigma_{S,r}}{\sigma_K}-\frac{\int_t^T D(t,z)\left(\sigma_{L,r}-f_1(t,z)\sigma_r\right)\mathrm{d}z}{\sigma_K K(t,T)}-\frac{\sigma_{K_1}\int_t^T D(t,z)\sigma_{L_B}(t,z)\mathrm{d}z}{\sigma_K\sigma_{L_{K_1}}K(t,T)}\\ \frac{\int_t^T D(t,z)\sigma_{L_B}(t,z)\mathrm{d}z}{\sigma_{L_{K_1}}K(t,T)}\\ \frac{\sigma_L}{\sigma_S}\end{pmatrix} \tag{3.44b}$$

are the shares invested in each asset.

Proof. By comparing (3.43) with (3.8), (3.11), and (3.22) we can straightforwardly obtain (3.44). □

At this stage, we are able to replicate the present value of the minimum guarantee $G(t)$ in (3.28).

Proposition 3.2.4. *The present value of the minimum guarantee G is correlated with both the interest rate process r and the mortality intensity process λ, and can be replicated using a combination of cash, rolling ZCB, and rolling longevity ZCB as follows*

$$\frac{\mathrm{d}G(t)}{G(t)} = \frac{\mathrm{d}S_0(t)}{S_0(t)}\nu^G_{S_0}(t)+\frac{\mathrm{d}P_K(t)}{P_K(t)}\nu^G_{P_K}(t)+\frac{\mathrm{d}L_{K_1}(t)}{L_{K_1}(t)}\nu^G_{L_{K_1}}(t), \quad t\in[0,T], \tag{3.45a}$$

where

$$\begin{pmatrix}\nu^G_{S_0}(t)\\ \nu^G_{P_K}(t)\\ \nu^G_{L_{K_1}}(t)\end{pmatrix} = \begin{pmatrix} 1-\nu^G_{P_K}(t)-\nu^G_{L_{K_1}}(t)\\ \frac{n\int_T^{\omega-\iota} b(z)\sigma_P(t,z)L_B(t,z)\mathrm{d}z}{\sigma_K G(t)}-\frac{n\sigma_{K_1}\int_T^{\omega-\iota} b(z)\sigma_{L_B}(t,z)L_B(t,z)\mathrm{d}z}{\sigma_K\sigma_{L_{K_1}}G(t)}\\ \frac{n\int_T^{\omega-\iota} b(z)\sigma_{L_B}(t,z)L_B(t,z)\mathrm{d}z}{\sigma_{L_{K_1}}G(t)}\end{pmatrix} \tag{3.45b}$$

are the shares allocated in each asset.

Proof. By juxtaposing (3.28) with both (3.11) and (3.22), we can easily derive (3.45). □

Next, with the help of $K(\cdot,T)$ and $G(\cdot)$, we can transform the non-self-financing constrained optimization problem stated in Problem 3.2.1 into a single investment one, by defining the *surplus* process $Y=\{Y(t),t\in[0,T]\}$ as

$$Y(t):=W^{PO-PPI}(t)+K(t,T)-G(t), \quad t\in[0,T],$$

with $\mathbb{P}$-dynamics

$$\mathrm{d}Y(t) = \mathrm{d}W^{PO-PPI}(t) + \mathrm{d}K(t,T) - \mathrm{d}G(t), \quad t \in [0,T]. \tag{3.46}$$

Injecting (3.28), (3.31), and (3.43) into (3.46), equation (3.46) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}Y(t)}{Y(t)} = r(t)\mathrm{d}t - \left(\sigma_K \alpha_{Y,P_K}(t) + \sigma_{K_1}\alpha_{Y,L_{K_1}}(t) - \sigma_{S,r}\alpha_{Y,S}(t)\right)\sqrt{r(t)}\left(\mathrm{d}Z_r(t) + \xi_r\sqrt{r(t)}\mathrm{d}t\right) \\ \qquad - \sigma_{L_{K_1}}\alpha_{Y,L_{K_1}}(t)\sqrt{\lambda(t)}\left(\mathrm{d}Z_\lambda(t) + \xi_\lambda\sqrt{\lambda(t)}\mathrm{d}t\right) + \sigma_S\alpha_{Y,S}(t)\left(\mathrm{d}Z_S(t) + \xi_S\mathrm{d}t\right), \ t \in [0,T], \\ Y(0) = W^{PO-PPI}(0) + K(0,T) - G(0), \end{cases} \tag{3.47a}$$

where

$$\begin{pmatrix} \alpha_{Y,S_0}(t) \\ \alpha_{Y,P_K}(t) \\ \alpha_{Y,L_{K_1}}(t) \\ \alpha_{Y,S}(t) \end{pmatrix} = \begin{pmatrix} 1 - \alpha_{Y,P_K}(t) - \alpha_{Y,L_{K_1}}(t) - \alpha_{Y,S}(t) \\ \dfrac{m(t)(Y(t) - K(t,T))\alpha_{P_K}(t) + K(t,T)\nu^K_{P_K}(t) - G(t)\nu^G_{P_K}(t)}{Y(t)} \\ \dfrac{m(t)(Y(t) - K(t,T))\alpha_{L_{K_1}}(t) + K(t,T)\nu^K_{L_{K_1}}(t) - G(t)\nu^G_{L_{K_1}}(t)}{Y(t)} \\ \dfrac{m(t)(Y(t) - K(t,T))\left(1 - \alpha_{P_K}(t) - \alpha_{L_{K_1}}(t)\right) + K(t,T)\nu^K_S(t)}{Y(t)} \end{pmatrix} \tag{3.47b}$$

are the shares allocated in cash, rolling ZCB, rolling longevity ZCB, and stock, respectively. By construction, $K(T,T) = 0$. This implies that the surplus at retirement date T fullfills $Y(T) = W^{PO-PPI}(T) - G(T)$, and thus the original constraint $W^{PO-PPI}(T) \geq G(T)$ can be recast into $Y(T) \geq 0$. In such a framework, the feasible pair (γ, b_T) fullfills by construction equation (3.25). This is equivalent to say that

$$\underbrace{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T nC_L(z)e^{-\int_0^z r(q)\mathrm{d}q}\frac{p(z)}{p(0)}\mathrm{d}z\right]}_{:=K(0,T)} - \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\int_T^{\omega-\iota} nb(z)e^{-\int_0^z r(q)\mathrm{d}q}\frac{p(z)}{p(0)}\mathrm{d}z\right]}_{:=G(0)} = 0, \tag{3.48}$$

is satisfied, or equivalently $Y(0) = W^{PO-PPI}(0) > 0$. As argued in Guan and Liang [2014] and Agarwal et al. [2023], since the auxiliary surplus process (3.47a) is self-financing, then $Y(0) > 0$ is a necessary and sufficient condition to ensure the existence of investment strategies such that $Y(T) \geq 0$. It is worth noting that the condition stated in equation (3.48) ensures that the set $\mathcal{U}_{\mathrm{ad}}$ of admissible strategies is non-empty. Finally, let

$$\begin{aligned} \boldsymbol{\ell}: \quad \mathbb{R}^3 \quad &\longrightarrow \quad \mathbb{R}^3, \\ \begin{pmatrix} \alpha_{P_K} \\ \alpha_{L_{K_1}} \\ m \end{pmatrix} &\longmapsto \frac{1}{Y}\begin{pmatrix} m(W^{PO-PPI} - G)\alpha_{P_K} + K\nu^K_{P_K} - G\nu^G_{P_K} \\ m(W^{PO-PPI} - G)\alpha_{L_{K_1}} + K\nu^K_{L_{K_1}} - G\nu^G_{L_{K_1}} \\ m(W^{PO-PPI} - G)(1 - \alpha_{P_K} - \alpha_{L_{K_1}}) + K\nu^K_S \end{pmatrix}, \end{aligned}$$

and set $\boldsymbol{u}_Y(\cdot) := \left(\alpha_{Y,P_K}(\cdot), \alpha_{Y,L_{K_1}}(\cdot), \alpha_{Y,S}(\cdot)\right)^\top$. For generic initial time $t \in [0,T]$, with $Y(t) = y$, $r(t) = r$, and $\lambda(t) = \lambda$, define the set of admissible strategies $\widetilde{\mathcal{U}}_{\mathrm{ad}}(t,y,r,\lambda)$ depending on the initial data $(t, y, r, \lambda) \in [0, T] \times (0, +\infty)^3$ as

$$\widetilde{\mathcal{U}}_{\text{ad}}(t, y, r, \lambda) := \left\{ \boldsymbol{u}_Y : [0, T] \times \Omega \to \mathbb{R}^3 \mid \boldsymbol{u}_Y(\cdot) = \boldsymbol{\ell}(\boldsymbol{u}(\cdot)),\ \forall \boldsymbol{u} \in \mathcal{U}_{\text{ad}}(t, w, r, \lambda) \right\}.$$

Then, we can transform the original Problem 3.2.1 into the following equivalent unconstrained optimization problem.

Problem 3.2.5. *For generic initial data* $(t, y, r, \lambda) \in [0, T] \times (0, +\infty)^3$

$$\textit{maximize} \quad \tilde{J}(t, y, r, \lambda; \boldsymbol{u}_Y(\cdot)) = \mathbb{E}\left[\frac{(Y(T))^{1-\delta}}{1-\delta} \right] \quad \textit{over } \boldsymbol{u}_Y(\cdot) \in \widetilde{\mathcal{U}}_{ad}(t, y, r, \lambda) \tag{3.49}$$

In this context, $\mathbb{E}[\cdot]$ represents the conditional expectation w.r.t. $\mathcal{G}(t)$. To solve Problem 3.2.5, we apply the dynamic programming method and define the value function as follows

$$\begin{cases} V(t, y, r, \lambda) := \sup\limits_{\boldsymbol{u}_Y(\cdot) \in \widetilde{\mathcal{U}}_{\text{ad}}(t, y, r, \lambda)} \tilde{J}(t, y, r, \lambda; \boldsymbol{u}_Y(\cdot)), \\ V(T, Y(T), r(T), \lambda(T)) := \dfrac{(Y(T))^{1-\delta}}{1-\delta}. \end{cases}$$

For every initial data (t, y, r, λ), let $\mathcal{L}^{\boldsymbol{u}_Y}$ be the infinitesimal generator of the diffusion processes (3.47), (3.7), and (3.1), that is,

$$\begin{aligned} \mathcal{L}^{\boldsymbol{u}_Y} v(t, y, r, \lambda) = {} & \frac{1}{2} y^2 \left[\left(\sigma_K \alpha_{Y,P_K} + \sigma_{K_1} \alpha_{Y,L_{K_1}} - \sigma_{S,r} \alpha_{Y,S} \right)^2 r + \sigma_{L_{K_1}}^2 \lambda \alpha_{Y,2}^2 + \sigma_S^2 \alpha_{Y,S}^2 \right] \frac{\partial^2}{\partial y^2} v(t, y, r, \lambda) \\ & + \frac{1}{2} \sigma_r^2 r \frac{\partial^2}{\partial r^2} v(t, y, r, \lambda) + \frac{1}{2} \sigma_\lambda^2 \lambda \frac{\partial^2}{\partial \lambda^2} v(t, y, r, \lambda) \\ & - y r \sigma_r \left(\sigma_K \alpha_{Y,P_K} + \sigma_{K_1} \alpha_{Y,L_{K_1}} - \sigma_{S,r} \alpha_{Y,S} \right) \frac{\partial^2}{\partial y \partial r} v(t, y, r, \lambda) \\ & - y \sigma_\lambda \lambda \sigma_{L_{K_1}} \alpha_{Y,L_{K_1}} \frac{\partial^2}{\partial y \partial \lambda} v(t, y, r, \lambda) \\ & + y \left[r - \left(\sigma_K \alpha_{Y,P_K} + \sigma_{K_1} \alpha_{Y,L_{K_1}} - \sigma_{S,r} \alpha_{Y,S} \right) \xi_r r - \sigma_{L_{K_1}} \lambda \xi_\lambda \alpha_{Y,L_{K_1}} + \sigma_S \xi_S \alpha_{Y,S} \right] \\ & \cdot \frac{\partial}{\partial y} v(t, y, r, \lambda) + \alpha_r (\beta_r - r) \frac{\partial}{\partial r} v(t, y, r, \lambda) + \rho_\lambda (\beta_\lambda(t) - \lambda) \frac{\partial}{\partial \lambda} v(t, y, r, \lambda), \end{aligned} \tag{3.50}$$

where v is a function in $C^{1,2}([0, T) \times (0, +\infty)^3; \mathbb{R}) \cap C^0([0, T] \times (0, +\infty)^3; \mathbb{R})$. By standard arguments of stochastic control (see, e.g., [Yong and Zhou, 1999, Chapter 4]), the Hamilton-Jacobi-Bellman equation (HJB) associated to the auxiliary Problem 3.2.5 is

$$-\frac{\partial}{\partial t} v(t, y, r, \lambda) - \sup_{\boldsymbol{u}_Y \in \mathbb{R}^3} \mathcal{L}^{\boldsymbol{u}_Y} v(t, y, r, \lambda) = 0, \quad (t, y, r, \lambda) \in [0, T) \times (0, +\infty), \tag{3.51a}$$

subject to the terminal condition

$$v(T, y, r, \lambda) = \frac{y^{1-\delta}}{1-\delta}, \quad y \in (0, +\infty), \quad \delta > 0 \wedge \delta \neq 1. \tag{3.51b}$$

In the following, our aim is to characterize the value function as a solution to HJB equation (3.51) and identify the optimal control strategy. We begin by stating and proving the following lemma.

Lemma 3.2.6. *Let*

$$\delta > \max\left\{\frac{\sigma_r^2\xi_r^2 + 2\sigma_r\xi_r\alpha_r + 2\sigma_r^2}{\sigma_r^2\xi_r^2 + \alpha_r + 2\sigma_r\xi_r\alpha_r + 2\sigma_r^2}, \frac{\sigma_\lambda\xi_\lambda\left(\sigma_\lambda\xi_\lambda + 2\rho_\lambda\right)}{\left(\sigma_\lambda\xi_\lambda + \rho_\lambda\right)^2}, 0\right\}, \tag{3.52}$$

and $a, d, g \in C^1([0,T];\mathbb{R})$ *such that*

$$\begin{cases} a(t) = \dfrac{1}{2\delta}(1-\delta)\xi_S^2(T-t) + \alpha_r\beta_r\displaystyle\int_t^T d(z)\mathrm{d}z + \rho_\lambda\int_t^T \beta_\lambda(z)g(z)\mathrm{d}z, & t\in[0,T], \\ d(t) = \dfrac{2\Theta_2\left(e^{k_d(T-t)}-1\right)}{(-\Theta_1+k_d)\left(e^{k_d(T-t)}-1\right)+2k_d}, & t\in[0,T], \\ g(t) = \dfrac{2\Theta_4\left(e^{k_g(T-t)}-1\right)}{(-\Theta_3+k_g)\left(e^{k_g(T-t)}-1\right)+2k_g}, & t\in[0,T], \end{cases} \tag{3.53}$$

where

$$k_d := \sqrt{\Theta_1^2 - \frac{2\sigma_r^2\Theta_2}{\delta}}, \qquad k_g := \sqrt{\Theta_3^2 - \frac{2\sigma_\lambda^2\Theta_4}{\delta}},$$

$$\Theta_1 := \left(\frac{1-\delta}{\delta}\xi_r\sigma_r - \alpha_r\right), \quad \Theta_2 := \left(\frac{1-\delta}{2\delta}\xi_r^2 + 1 - \delta\right), \quad \Theta_3 := \left(\frac{1-\delta}{\delta}\sigma_\lambda\xi_\lambda - \rho_\lambda\right), \quad \Theta_4 := \frac{1-\delta}{2\delta}\xi_\lambda^2.$$

Then, the following function

$$v(t,y,r,\lambda) = \frac{y^{1-\delta}}{1-\delta}\psi(t,r,\lambda), \quad (t,y,r,\lambda)\in[0,T]\times(0,+\infty)^3, \tag{3.54a}$$

with

$$\psi(t,r,\lambda) := \exp\left\{a(t) + d(t)r + g(t)\lambda\right\}, \quad (t,r,\lambda)\in[0,T]\times(0,+\infty)^2, \tag{3.54b}$$

is a classical solution to HJB equation (3.51).

Proof. It turns out that for the power utility functions of CRRA-type, as considered originally in Merton [1969], we can find a smooth solution to HJB equation (3.51). We look for a candidate solution of the form

$$v(t,y,r,\lambda) = \frac{y^{1-\delta}}{1-\delta}\psi(t,r,\lambda), \quad (t,y,r,\lambda)\in[0,T]\times(0,+\infty)^3,$$

and for some positive function ψ such that $\psi(T,r,\lambda) = 1$. Injecting this candidate into (3.50) and taking the first order conditions, the following linear system results

$$\begin{pmatrix} \sigma_K r & \sigma_{K_1} r & -\sigma_{S,r} r \\ \sigma_K\sigma_{K_1} r & \sigma_{K_1}^2 r + \sigma_{L_{K_1}}^2\lambda & -\sigma_{S,r}\sigma_{K_1} r \\ -\sigma_{S,r}\sigma_K r & -\sigma_{S,r}\sigma_{K_1} r & \sigma_S^2 + \sigma_{S,r}^2 r \end{pmatrix}\begin{pmatrix} \alpha_{Y,P_K} \\ \alpha_{Y,L_{K_1}} \\ \alpha_{Y,S} \end{pmatrix}$$

$$= \frac{1}{y\frac{\partial^2}{\partial y^2}v(t,y,r,\lambda)}$$

$$\cdot \begin{pmatrix} \sigma_r r \frac{\partial^2}{\partial y \partial r} v(t,y,r,\lambda) + \xi_r r \frac{\partial}{\partial y} v(t,y,r,\lambda) \\ \sigma_\lambda \sigma_{L_{K_1}} \lambda \frac{\partial^2}{\partial y \partial \lambda} v(t,y,r,\lambda) + \sigma_r \sigma_{K_1} r \frac{\partial^2}{\partial y \partial r} v(t,y,r,\lambda) + (\sigma_{K_1}\xi_r r + \sigma_{L_{K_1}}\xi_\lambda \lambda)\frac{\partial}{\partial y} v(t,y,r,\lambda) \\ -\sigma_r \sigma_{S,r} r \frac{\partial^2}{\partial y \partial r} v(t,y,r,\lambda) - (\sigma_S \xi_S + \sigma_{S,r}\xi_r r)\frac{\partial}{\partial y} v(t,y,r,\lambda) \end{pmatrix}.$$

Thus, the unique maximum point is attained at

$$\begin{aligned} \bar{\boldsymbol{u}}_Y(t,r,\lambda) = \begin{pmatrix} \bar{\alpha}_{Y,P_K}(t,r,\lambda) \\ \bar{\alpha}_{Y,L_{K_1}}(t,r,\lambda) \\ \bar{\alpha}_{Y,S}(t,r,\lambda) \end{pmatrix} &:= \arg\max_{\boldsymbol{u}_Y \in \mathbb{R}^3} \mathcal{L}^{\boldsymbol{u}_Y} v(t,y,r,\lambda) \\ &= \begin{pmatrix} -\frac{\sigma_r}{\delta\sigma_K}\frac{\frac{\partial}{\partial r}\psi(t,r,\lambda)}{\psi(t,r,\lambda)} + \frac{\sigma_\lambda\sigma_{K_1}}{\delta\sigma_K\sigma_{L_{K_1}}}\frac{\frac{\partial}{\partial\lambda}\psi(t,r,\lambda)}{\psi(t,r,\lambda)} + \frac{\sigma_{K_1}\xi_\lambda}{\delta\sigma_K\sigma_{L_{K_1}}} + \frac{\sigma_{S,r}\xi_S}{\delta\sigma_K\sigma_S} - \frac{\xi_r}{\delta\sigma_K} \\ -\frac{\sigma_\lambda}{\delta\sigma_{L_{K_1}}}\frac{\frac{\partial}{\partial\lambda}\psi(t,r,\lambda)}{\psi(t,r,\lambda)} - \frac{\xi_\lambda}{\delta\sigma_{L_{K_1}}} \\ \frac{\xi_S}{\delta\sigma_S} \end{pmatrix}. \end{aligned} \tag{3.55}$$

Plugging (3.55) into the HJB equation (3.51a), we then obtain

$$\begin{aligned} 0 =& \frac{\partial}{\partial t}\psi(t,r,\lambda) + (1-\delta)r\psi(t,r,\lambda) + \frac{1}{2\delta}(1-\delta)r\xi_r^2\psi(t,r,\lambda) + \frac{1}{\delta}(1-\delta)r\xi_r\sigma_r\frac{\partial}{\partial r}\psi(t,r,\lambda) \\ &+ \frac{1}{2\delta}(1-\delta)\xi_S^2\psi(t,r,\lambda) + \frac{1}{\delta}(1-\delta)\lambda\sigma_\lambda\xi_\lambda\frac{\partial}{\partial\lambda}\psi(t,r,\lambda) + \frac{1}{2\delta}(1-\delta)\lambda\xi_\lambda^2\psi(t,r,\lambda) \\ &+ \frac{1}{2\delta}(1-\delta)r\sigma_r^2\frac{\frac{\partial}{\partial r}\psi^2(t,r,\lambda)}{\psi(t,r,\lambda)} + \frac{1}{2\delta}(1-\delta)\lambda\sigma_\lambda^2\frac{\frac{\partial}{\partial\lambda}\psi^2(t,r,\lambda)}{\psi(t,r,\lambda)} + \alpha_r(\beta_r - r)\frac{\partial}{\partial r}\psi(t,r,\lambda) \\ &+ \frac{\sigma_r^2}{2}r\frac{\partial^2}{\partial r^2}\psi(t,r,\lambda) + \rho_\lambda(\beta_\lambda(t)-\lambda)\frac{\partial}{\partial\lambda}\psi(t,r,\lambda) + \frac{1}{2}\lambda\sigma_\lambda^2\frac{\partial^2}{\partial\lambda^2}\psi(t,r,\lambda). \end{aligned} \tag{3.56}$$

We conjecture that

$$\psi(t,r,\lambda) = \exp\{a(t) + d(t)r + g(t)\lambda\}, \quad (t,r,\lambda) \in [0,T]\times(0,+\infty)^2.$$

Substituting this into (3.56) and considering terminal condition (3.51b) results in a linear equation in r and λ. Setting the coefficients of the terms r, λ and the independent term to zero yields that the triplet of functions $a, d, g \in C^1([0,T];\mathbb{R})$ must satisfy the following system of ODEs

$$\begin{cases} \dot{a}(t) + \frac{1}{2\delta}(1-\delta)\xi_S^2 + \alpha_r\beta_r d(t) + \rho_\lambda\beta_\lambda(t)g(t) = 0, \quad t\in[0,T], \\ \dot{d}(t) + \left[\frac{1}{\delta}(1-\delta)\xi_r\sigma_r - \alpha_r\right]d(t) + \frac{\sigma_r^2}{2\delta}d^2(t) + 1 - \delta + \frac{1}{2\delta}(1-\delta)\xi_r^2 = 0, \quad t\in[0,T], \\ \dot{g}(t) + \left(\frac{1}{\delta}(1-\delta)\sigma_\lambda\xi_\lambda - \rho_\lambda\right)g(t) + \frac{1}{2\delta}\sigma_\lambda^2 g^2(t) + \frac{1}{2\delta}(1-\delta)\xi_\lambda^2 = 0, \quad t\in[0,T], \\ a(T) = 0 \quad d(T) = 0 \quad g(T) = 0. \end{cases}$$

Under the hypothesis that (3.52) holds, that is,

$$\delta > \max\left\{ \frac{\sigma_r^2\xi_r^2 + 2\sigma_r\xi_r\alpha_r + 2\sigma_r^2}{\sigma_r^2\xi_r^2 + \alpha_r + 2\sigma_r\xi_r\alpha_r + 2\sigma_r^2},\ \frac{\sigma_\lambda\xi_\lambda\left(\sigma_\lambda\xi_\lambda + 2\rho_\lambda\right)}{\left(\sigma_\lambda\xi_\lambda + \rho_\lambda\right)^2},\ 0 \right\},$$

the unique solution to the above system is provided by equation (3.53). This completes the proof. □

Now, we are ready to deliver a verification result and an optimal Markovian control strategy as a byproduct.

Theorem 3.2.7. *Consider a pension fund manager endowed with a CRRA utility function such that the risk-aversion parameter is given by* (3.52)*, i.e.,*

$$\delta > \max\left\{ \frac{\sigma_r^2\xi_r^2 + 2\sigma_r\xi_r\alpha_r + 2\sigma_r^2}{\sigma_r^2\xi_r^2 + \alpha_r + 2\sigma_r\xi_r\alpha_r + 2\sigma_r^2}, \frac{\sigma_\lambda\xi_\lambda\left(\sigma_\lambda\xi_\lambda + 2\rho_\lambda\right)}{\left(\sigma_\lambda\xi_\lambda + \rho_\lambda\right)^2}, 0 \right\}.$$

Then, the function v defined in (3.54) *with a, d, and g given by* (3.53) *is the value function V associated to Problem 3.2.5, namely,*

$$V(t,y,r,\lambda) = v(t,y,r,\lambda) = \frac{y^{1-\delta}}{1-\delta}\exp\left\{a(t) + d(t)r + g(t)\lambda\right\}.$$

Furthermore, for the initial data $(t,y,r,\lambda) \in [0,T]\times(0,+\infty)^3$, the control strategy given by

$$\boldsymbol{u}_Y^\star(s) = \begin{pmatrix} \alpha_{P_K}^\star(s) \\ \alpha_{L_{K_1}}^\star(s) \\ \alpha_S^\star(s) \end{pmatrix} = \begin{pmatrix} -\frac{\sigma_r}{\delta\sigma_K}d(s) + \frac{\sigma_{K_1}\sigma_\lambda}{\delta\sigma_K\sigma_{L_{K_1}}}g(s) + \frac{\sigma_{K_1}\xi_\lambda}{\delta\sigma_K\sigma_{L_{K_1}}} + \frac{\sigma_{S,r}\xi_S}{\delta\sigma_K\sigma_S} - \frac{\xi_r}{\delta\sigma_K} \\ -\frac{\sigma_\lambda}{\delta\sigma_{L_{K_1}}}g(s) - \frac{\xi_\lambda}{\delta\sigma_{L_{K_1}}} \\ \frac{\xi_S}{\delta\sigma_S} \end{pmatrix}, \quad s\in[t,T], \tag{3.57}$$

lies in $\widetilde{\mathcal{U}}_{ad}(t,y,r,\lambda)$ and is unique. Accordingly, the surplus process $Y^\star(\cdot;t,y,r,\lambda,\boldsymbol{u}_Y^\star(\cdot))$ starting from (y,r,λ) at t and controlled by $\boldsymbol{u}_Y^\star(\cdot)$ is the optimal surplus trajectory and the unique solution to the associated closed-loop equation.

Proof. For any initial data $(t,y,r,\lambda)\in[0,T]\times(0,+\infty)$ and $\boldsymbol{u}_Y(\cdot)\in\widetilde{\mathcal{U}}_{\mathrm{ad}}(t,y,r,\lambda)$, state equation (3.47) admits a unique (strong) solution on $(\Omega,\mathcal{G},\mathbb{G},\mathbb{P})$ that we denote by $Y(\cdot;t,y,r,\lambda,\boldsymbol{u}_Y(\cdot))$, that is, the surplus process starting from (y,r,λ) at t and controlled by $\boldsymbol{u}_Y(\cdot)$. Let $Y(\cdot) := Y(\cdot;t,y,r,\lambda,\boldsymbol{u}_Y(\cdot))$, $r(\cdot)$, and $\lambda(\cdot)$ be the unique solutions to the diffusion processes (3.1), (3.7), and (3.47), respectively, and v the solution given in (3.54) to HJB equation (3.51). By applying Dynkin's formula to the function $(t,y,r,\lambda)\mapsto v(t,Y(t),r(t),\lambda(t))$ with processes $Y(\cdot)$, $r(\cdot)$, and $\lambda(\cdot)$ defined in (3.47), (3.7) and (3.1), respectively, we obtain

$$\begin{aligned}&\mathbb{E}\left[v(T,Y(T),r(T),\lambda(T))\right]\\ &\qquad = v(t,y,r,\lambda) + \mathbb{E}\left[\int_t^T\left(\frac{\partial}{\partial t}v(t,Y(t),r(t),\lambda(t)) + \mathcal{L}^{\boldsymbol{u}_Y}v(z,Y(z),r(z),\lambda(z))\right)\mathrm{d}z\right],\end{aligned}$$

that is,

$$\mathbb{E}\left[\frac{1}{1-\delta}\left(Y(T)\right)^{1-\delta}\right]$$
$$= v(t,y,r,\lambda) + \mathbb{E}\left[\int_t^T \left(-\sup_{\boldsymbol{u}_Y\in\mathbb{R}^3} \mathcal{L}^{\boldsymbol{u}_Y} v(z,Y(z),r(z),\lambda(z)) + \mathcal{L}^{\boldsymbol{u}_Y} v(z,Y(z),r(z),\lambda(z))\right) \mathrm{d}z\right].$$

Rearranging the terms we then have

$$v(t,y,r,\lambda) = \tilde{J}(t,w,r,\lambda;\boldsymbol{u}_Y(\cdot)) + \mathbb{E}\left[\int_t^T \left(\sup_{\boldsymbol{u}_Y\in\mathbb{R}^3} \mathcal{L}^{\boldsymbol{u}_Y} v(z,Y(z),r(z),\lambda(z)) - \mathcal{L}^{\boldsymbol{u}_Y} v(z,Y(z),r(z),\lambda(z))\right) \mathrm{d}z\right]. \quad (3.58)$$

As equation (3.58) holds for every $\boldsymbol{u}_Y(\cdot) \in \widetilde{\mathcal{U}}_{\mathrm{ad}}(t,y,r,\lambda)$, and since $\sup_{\boldsymbol{u}_Y\in\mathbb{R}^3} \mathcal{L}^{\boldsymbol{u}_Y} v \geq \mathcal{L}^{\boldsymbol{u}_Y} v$ for every $\boldsymbol{u}_Y \in \mathbb{R}^3$, it follows that

$$v(t,x,r,\lambda) \geq V(t,x,r,\lambda).$$

Consider now the feedback map corresponding to the maximization of the operator $\mathcal{L}^{\boldsymbol{u}_Y} v$, with v defined in (3.54). It is provided in (3.55) and results in the following related closed-loop equation

$$\begin{cases} \dfrac{\mathrm{d}Y(s)}{Y(s)} = r(s)\mathrm{d}s + \Big(\bar{\alpha}_{Y,S}(s,r(s),\lambda(s))\sigma_{S,r} - \bar{\alpha}_{Y,P_K}(s,r(s),\lambda(s))\sigma_K \\ \qquad - \bar{\alpha}_{Y,L_{K_1}}(s,r(s),\lambda(s))\sigma_{K_1}\Big)\sqrt{r(s)}\left(\mathrm{d}Z_r(s) + \xi_r\sqrt{r(s)}\mathrm{d}s\right) \\ \qquad - \bar{\alpha}_{Y,L_{K_1}}(s,r(s),\lambda(s))\sigma_{L_{K_1}}\sqrt{\lambda(s)}\left(\mathrm{d}Z_\lambda(s) + \xi_\lambda\sqrt{\lambda(s)}\mathrm{d}s\right) \\ \qquad + \bar{\alpha}_{Y,S}(s,r(s),\lambda(s))\sigma_S\Big(\mathrm{d}Z_S(s) + \xi_S\mathrm{d}s\Big), \quad s\in[t,T], \\ Y(t) = W(t) + K(t,T) - G(t) =: y, \quad y\in(0,+\infty), \end{cases} \quad (3.59)$$

which is linear and admits a unique and strictly positive solution $\bar{Y}(\cdot;t,y,r,\lambda)$. Accordingly, the feedback strategy $\boldsymbol{u}_Y^\star(\cdot) := (\alpha_{Y,1}^\star(\cdot),\alpha_{Y,2}^\star(\cdot),\alpha_{Y,3}^\star(\cdot))^\top$ associated to the feedback map (3.55), i.e.,

$$\boldsymbol{u}_Y^\star(s) = \begin{pmatrix} \alpha_{Y,P_K}^\star(s) \\ \alpha_{Y,L_{K_1}}^\star(s) \\ \alpha_{Y,S}^\star(s) \end{pmatrix} := \begin{pmatrix} \bar{\alpha}_{Y,P_K}(s,r(s),\lambda(s)) \\ \bar{\alpha}_{Y,L_{K_1}}(s,r(s),\lambda(s)) \\ \bar{\alpha}_{Y,S}(s,r(s),\lambda(s)) \end{pmatrix}$$
$$= \begin{pmatrix} -\dfrac{\sigma_r}{\delta\sigma_K}d(s) + \dfrac{\sigma_{K_1}\sigma_\lambda}{\delta\sigma_K\sigma_{L_{K_1}}}g(s) + \dfrac{\sigma_{K_1}\xi_\lambda}{\delta\sigma_K\sigma_{L_{K_1}}} + \dfrac{\sigma_{S,r}\xi_S}{\delta\sigma_K\sigma_S} - \dfrac{\xi_r}{\delta\sigma_K} \\ -\dfrac{\sigma_\lambda}{\delta\sigma_{L_{K_1}}}g(s) - \dfrac{\xi_\lambda}{\delta\sigma_{L_{K_1}}} \\ \dfrac{\xi_S}{\delta\sigma_S} \end{pmatrix}, \quad s\in[t,T]. \quad (3.60)$$

is admissible, namely, $\boldsymbol{u}_Y^\star(s) \in \widetilde{\mathcal{U}}_{\mathrm{ad}}(t,y,r,\lambda)$. Furthermore, by denoting $Y^\star(\cdot) := Y^\star(\cdot;t,y,r,\lambda,\boldsymbol{u}_Y^\star(\cdot))$ the surplus process starting from (y,r,λ) at t and controlled by $\boldsymbol{u}_Y^\star(\cdot)$, the uniqueness of the solution to the closed-loop equation implies that $Y^\star(\cdot) = \bar{Y}(\cdot; t, y, r, \lambda)$. At this point, we observe that the feedback control $\boldsymbol{u}_Y^\star(\cdot)$ and surplus process $Y^\star(\cdot)$ applied to (3.58) implies $v(t, y, r, \lambda) = \tilde{J}(t, w, r, \lambda; \boldsymbol{u}_Y^\star(\cdot))$ and therefore

$$V(t, y, r, \lambda) \geq \tilde{J}(t, y, r, \lambda; \boldsymbol{u}_Y^\star(\cdot)) = v(t, y, r, \lambda) \geq V(t, y, r, \lambda).$$

This shows that $v(t, y, r, \lambda) = V(t, y, r, \lambda)$, and as a byproduct, that the couple $(\boldsymbol{u}_Y^\star(\cdot), Y^\star(\cdot))$ constitutes the optimal strategy and the optimal surplus trajectory starting at (t, y, r, λ), respectively. Finally, the uniqueness of the optimal strategy and the optimal surplus trajectory is a consequence of both the characterization (3.60) and the uniqueness of the solution to the closed-loop equation (3.59). This concludes the proof. □

Finally, we are able to provide the optimal strategy $\boldsymbol{u}^\star(\cdot) = \left(\alpha^\star_{P_K}(\cdot), \alpha^\star_{L_{K_1}}(\cdot), m^\star(\cdot)\right)^\top$, for the initial Problem 3.2.1.

Corollary 3.2.8. *The optimal multiplier of the PO-PPI strategy and the optimal composition shares of the purpose-oriented fund I are given by*

$$\begin{pmatrix} \alpha^\star_{P_K}(s) \\ \alpha^\star_{L_{K_1}}(s) \\ \alpha^\star_S(s) \\ m^\star(s) \end{pmatrix} = \begin{pmatrix} \dfrac{Y^\star(s)\alpha^\star_{Y,P_K}(s) - K(s,T)\nu^K_{P_K}(s) + G(s)\nu^G_{P_K}(s)}{(1-\alpha^\star_{Y,S_0}(s))Y^\star(s) - \left(1-\nu^K_{S_0}(s)\right)K(s,T) + \left(1-\nu^G_{S_0}(s)\right)G(s)} \\ \dfrac{Y^\star(s)\alpha^\star_{Y,L_{K_1}}(s) - K(s,T)\nu^K_{L_{K_1}}(s) + G(s)\nu^G_{L_{K_1}}(s)}{(1-\alpha^\star_{Y,S_0}(s))Y^\star(s) - \left(1-\nu^K_{S_0}(s)\right)K(s,T) + (1-\nu^G_{S_0}(s))G(s)} \\ \dfrac{Y^\star(t)\alpha^\star_S(s) - K(s,T)\nu^K_S(s)}{(1-\alpha^\star_{Y,S_0}(s))Y^\star(s) - \left(1-\nu^K_{S_0}(s)\right)K(s,T) + \left(1-\nu^G_{S_0}(s)\right)G(s)} \\ \dfrac{(1-\alpha^\star_{Y,S_0})Y^\star(s) - \left(1-\nu^K_{S_0}(s)\right)K(s,T) + (1-\nu^G_{S_0}(s))G(s)}{Y^\star(s) - K(s,T)} \end{pmatrix}, \quad s \in [t,T]. \tag{3.61}$$

The optimal proportions of the wealth invested into the cash account, the rolling ZCB, the rolling longevity ZCB, and the stock, respectively, are given by

$$\begin{pmatrix} \bar{\alpha}^\star_{S_0}(s) \\ \bar{\alpha}^\star_{P_K}(s) \\ \bar{\alpha}^\star_{L_{K_1}}(s) \\ \bar{\alpha}^\star_S(s) \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha^\star_{Y,S_0}(s)Y^\star(s) - \nu^K_{S_0}(s)K(s,T) + \nu^G_{S_0}(s)G(s)}{Y^\star(s) - K(s,T) + G(s)} \\ \dfrac{\alpha^\star_{Y,P_K}(s)Y^\star(s) - \nu^K_{P_K}(s)K(s,T) + \nu^G_{P_K}(s)G(s)}{Y^\star(s) - K(s,T) + G(s)} \\ \dfrac{\alpha^\star_{Y,L_{K_1}}(s)Y^\star(s) - \nu^K_{L_{K_1}}(s)K(s,T) + \nu^G_{L_{K_1}}(s)G(s)}{Y^\star(s) - K(s,T) + G(s)} \\ \dfrac{\alpha^\star_{Y,S}(s)Y^\star(s) - \nu^K_S(s)K(s,T) + \alpha^G_{Y,S}(s)G(s)}{Y^\star(s) - K(s,T) + G(s)} \end{pmatrix}, \quad s \in [t,T], \tag{3.62}$$

where $Y^\star(\cdot) := Y^\star(\cdot; t, y, r, \lambda, \boldsymbol{u}_Y^\star(\cdot)) = \bar{Y}(\cdot; t, y, r, \lambda)$, is the unique solution to the closed loop equation (3.59), and $Y^\star(\cdot; t, y, r, \lambda, \boldsymbol{u}_Y^\star(\cdot))$ denotes the optimal surplus process starting from (y, r, λ) at t and controlled by the optimal strategy $\boldsymbol{u}_Y^\star(\cdot)$.

Proof. It results from a trivial computation taking into consideration (3.47b) and (3.57). □

3.3 Numerical application

In the following, we delve into the proposed theoretical framework through a numerical application to scrutinize how financial and mortality factors affect the optimal investment strategy during retirement capital accumulation. As a baseline study, we provide a simulation analysis concerning the optimal strategy (3.61) and the optimal proportion (3.62). Normalizing the fund size to $n = 1$, the fund member is a $\iota = 40$ years old person adhering to the pension plan at the time $t = 0$, and he will retire at $T = 25$. The analysis is performed separately for male and female members, highlighting gender-based differences in optimal investments induced by disparity in human lifespans.

Parameter	Female	Male
ρ_λ	0.9058	0.7460
ϕ_λ	0.00398	0.00138
b_λ	10.97	13.59
l_λ	85.18	76.36
σ_λ	0.0230911	0.0194941

Table 3.1 Parameters estimate for the instantaneous force of mortality.

The parameters characterizing the instantaneous force of mortality in (3.1) are obtained through the estimation procedure proposed in Menoncin [2021], using the Human Mortality Database's cohort life tables of US males and females born in 1950. From the latter, we observe an extremal age $\omega = 114$. The resulting estimates of mortality intensity parameters are reported in Table 3.1. We notice that females tend to live longer than males, also showing a higher speed of reversion to the Gompertz-Makeham law-based mean. To some extent, we expect that the optimal investment strategy for females is marked by higher proportions of wealth invested in each asset class, or larger leverage effects, to support a greater retirement capital w.r.t. the male case.

Interest rate		**Stock**		**Contributions**	
Parameter	Value	Parameter	Value	Parameter	Value
β_r	0.0621328	$\sigma_{S,r}$	0.0046306	ζ	0.04
α_r	0.0904668	σ_S	0.14926	$\sigma_{L,r}$	0.05978
σ_r	0.0543625	ξ_S	0.1108301	σ_L	0.18
ξ_r	-0.5590635	-	-	-	-

Table 3.2 Parameters estimate for interest rate, stock and contribution processes.

According to our proposal, we assume the existence of a continuously-rolled ZCB with maturity $K = 10$ years and a rolling longevity ZCB with maturity $K_1 = 10$ years whose underlying is the mortality intensity for the considered US male (or female) population. Parameters featuring financial dynamics in (3.7) and (3.8) are taken from Menoncin and Regis [2017], while parameters shaping the contribution process in (3.23) stem from Han and Hung [2012], and are displayed in Table 3.2. From the latter, we stress that a negative market price of the interest rate risk stems from $\xi_r = -0.5590635$. Indeed, as argued in Menoncin [2021], the market price of interest rate risk must be lower than 0: the semi-elasticity of any bond w.r.t. the interest rate is negative. Moreover, the market price of longevity risk is defined according to $\xi_\lambda = -0.1$. Then, as in Menoncin [2021] we assume that the rolling longevity bond has the same behaviour as the rolling bond, that is it negatively responds to increases in $\lambda(t)$, for every $t \in [0, 25]$. The opposite situation occurs for the stock since $\xi_S = 0.118301$. As a risk aversion parameter, we adopt $\delta = 2.5$, and a constant inflation rate $\tilde{j} = 0,1\%$ is assumed. In addition, we pose $b_T = 750$ so that, by the feasible condition (3.25), we obtain $\gamma = 0.0442$ for males and $\gamma = 0.0584$ for females. Figure 3.1 depicts simulated paths for the optimal controls over the accumulation period, distinguished by gender. As a general statement, we observe that the behavior of optimal exposures appears to be gender-invariant. In particular, looking at the multiplier, the optimal investment strategy appears aggressive in the early periods of the accumulation phase, accelerating capital accumulation and contrasting the interest rate risk.

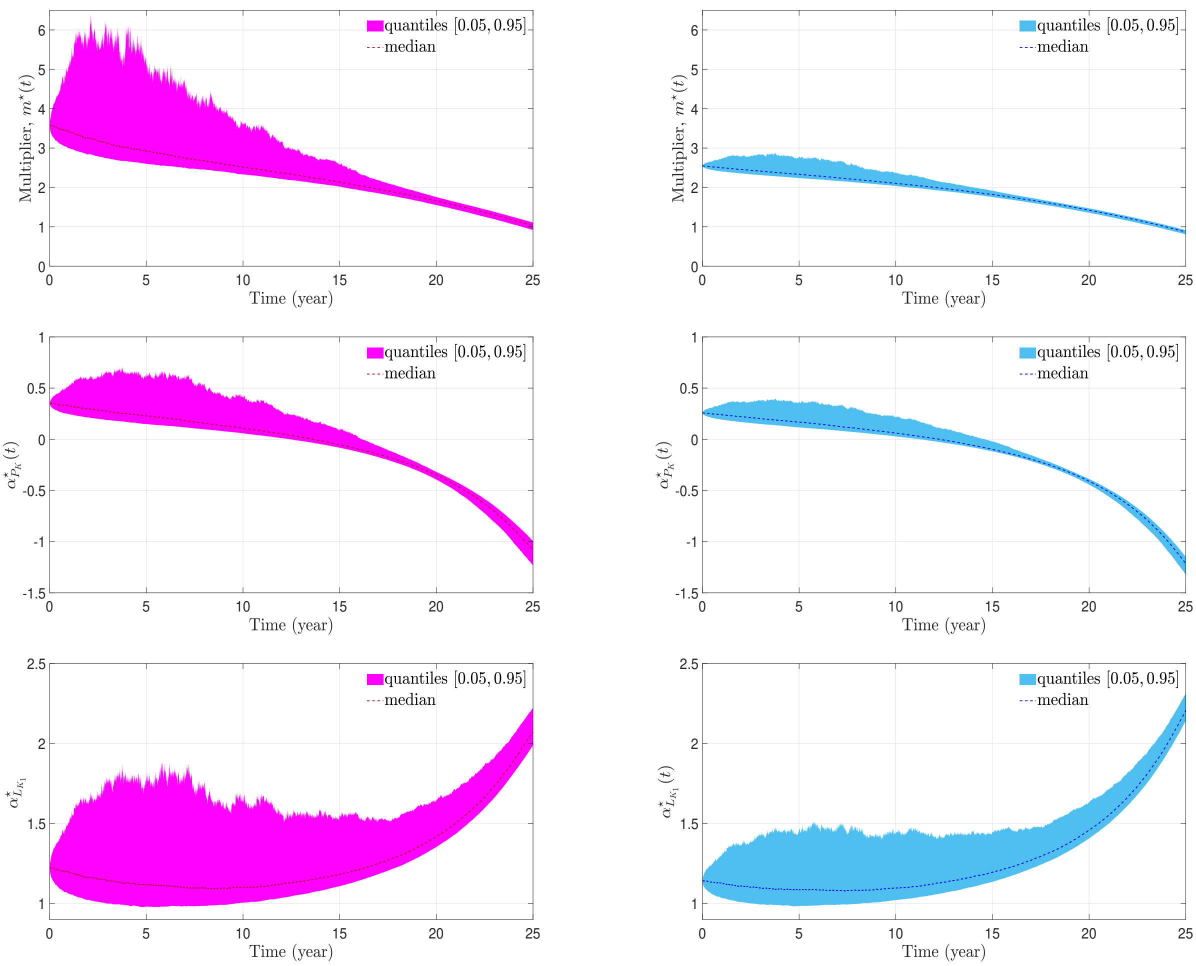


Figure 3.1 Simulated optimal controls over the accumulation accumulation phase for US females (lhs) and males (rhs). Sample paths are simulated according to the parameters gathered in Table 3.1 and Table 3.2. In addition, we recall that $\xi_r = -0.5590635$, $\xi_\lambda = -0.1$, $\xi_S = 0.118301$, $K = K_1 = 10$, $\delta = 2.5$ and $i = 0.1\%$. The Figure depicts paths ranging between the 0.5-quantile and the 0.95-quantile.

Afterwards, the multiplier decreases, implying that the investment strategy becomes more conservative compared to the early periods of the accumulation phase. In addition, when retirement is approaching, the synthetic index's exposure to the rolling longevity bond increases, implying that the fund manager orients investments towards longevity risk coverage. Therefore, the optimal investment strategy implies time-decreasing exposures to rolling bond and stock, as well as time-increasing exposures to rolling longevity bond, and complementarily, to the cash account.

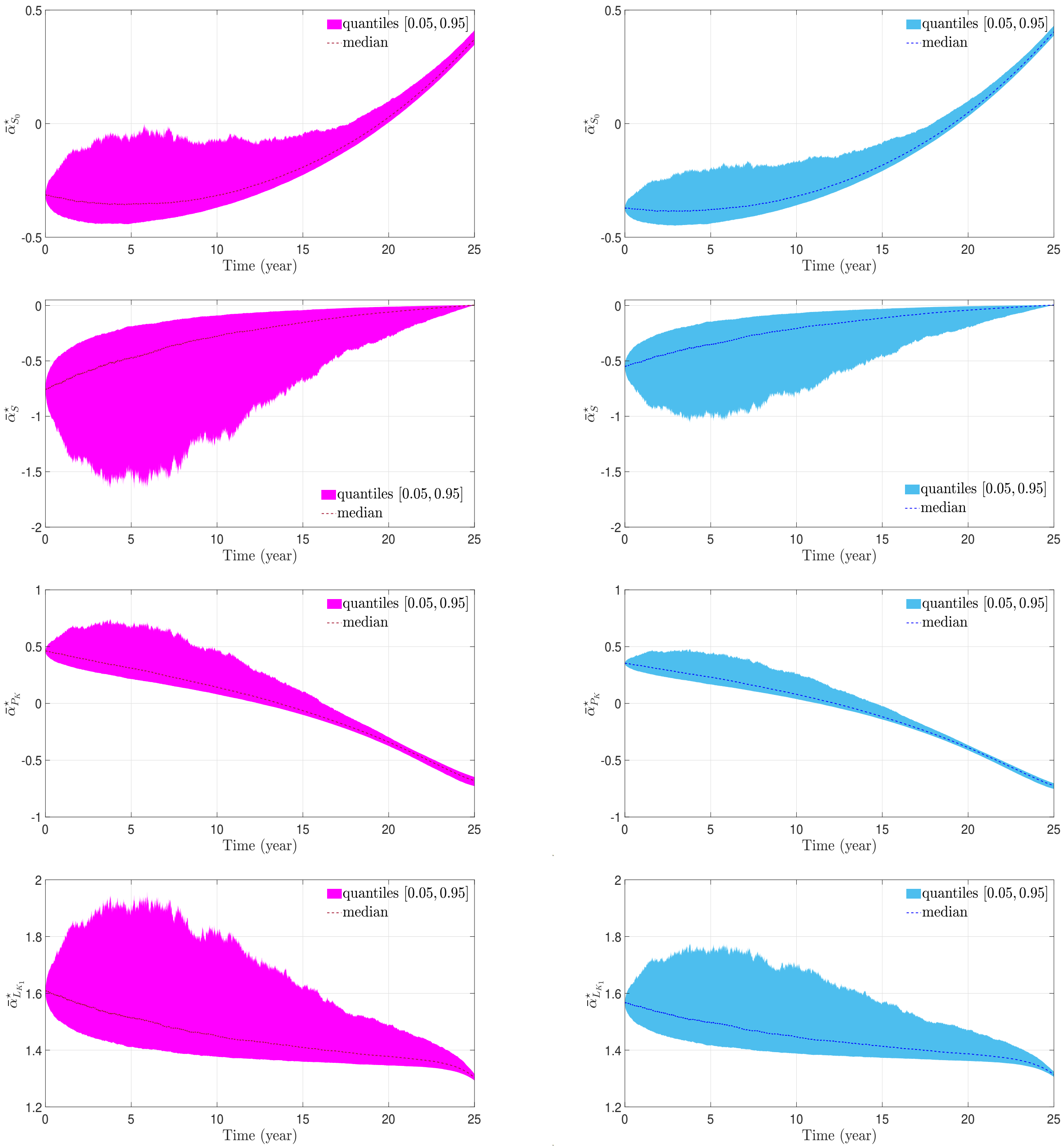


Figure 3.2 Optimal proportions invested in risky assets and the risk-free asset for females (lhs) and males (rhs). Adopting a top-down view, on the y-axis are considered the optimal share for the cash account S_0, the stock S, the rolling bond P_K, and the rolling longevity bond P_{K_1}.

The above considerations apply to both males and females. Going deeper into the comparison between genders, we note that optimal controls' paths for females are greater than optimal exposures for males and more volatile, due to the higher value of the diffusion parameter σ_λ (see Table 3.1). The former evidence is particularly relevant for the optimal multiplier $m^\star(\cdot)$. As the females live longer, the value of their annuity becomes larger than for males members. In terms of portfolio insurance strategy, longevity increases the floor value, and consequently, reduces the cushion available for investments. Then, the fund manager needs to boost the investment leverage, i.e. the multiplier, to accrue at least the annuity value at the retirement date. The comparison between optimal multipliers highlights a *gender-based multiplier gap*, measuring the financial impact of the gender-based longevity gap on retirement capital accumulation. More generally, the longer the fund member's lifetime, manifesting in the decumulation phase, the greater the financial leverage needed during the accumulation phase, and vice versa. The portfolio insurance strategy can capture such a link between accumulation and decumulation phases through the multiplier, which financially reflects the longer (or shorter) member's lifespan.
Figure 3.2 concerns the simulated paths of the optimal investment proportions into the cash account, the stock, the rolling bond, and the rolling longevity bond. It is interesting to note that our optimal investment strategy requires a change in the fund manager's aptitude in allocating wealth among assets. In particular, in the early accumulation stage, the optimal asset allocation reflects the aggressive investment aptitude of the fund manager. The optimal proportions employed in the cash account are negative and time-increasing, as well as time-decreasing short positions (in absolute value) on the stock are taken. In particular, the fund manager initially borrows money and uses the stock to finance investment in both rolling bond and rolling longevity bond. As the retirement date is approached, the optimal asset allocation becomes conservative, since the fund manager aims to preserve the accumulated capital through investments in safer assets. Therefore, the last accumulation years are characterized by short positions in the rolling bond to support allocations in both cash account and rolling longevity bond. We observe that only the optimal investment proportion in longevity bond are positive over the entire accumulation period. Such a piece of evidence highlights, on one side, the need to contrast longevity impact on capital retirement accretion and, on the other, the suitability of longevity bonds in building (and hedging) pension savings. Comparing optimal investment proportions by gender, Figure 4.2 reflects the meaning of the aforementioned gender-based multiplier gap. Since females live longer than males, their optimal capital accumulation requires heavier short positions in cash account and stock in the early stage, funding both rolling bond and rolling longevity bond long positions. Afterwards, when the retirement is closer, a faster time increase in cash account occurs, and in contrast both rolling bond and rolling longevity bond proportions decrease. Then, a greater optimal multiplier for females acts mainly at the beginning of the accumulation stage, endorsing, together with short positions in cash account and stock, the purchase of both rolling bond and rolling longevity bonds.

3.3.1 Sensitivity analysis

Along the lines of Menoncin and Regis [2017] and Agarwal et al. [2023], we perform a sensitivity analysis examining how the relevant financial and mortality parameters affect the optimal proportion of wealth invested in both risky and risk-free assets. To this end, we consider as the base scenario for optimal shares the median trajectory stemming from the previous simulation study (see Figure 3.2). Consequently, we conduct a numerical study to examine how the median-based optimality varies according to changes in financial and mortality parameters.

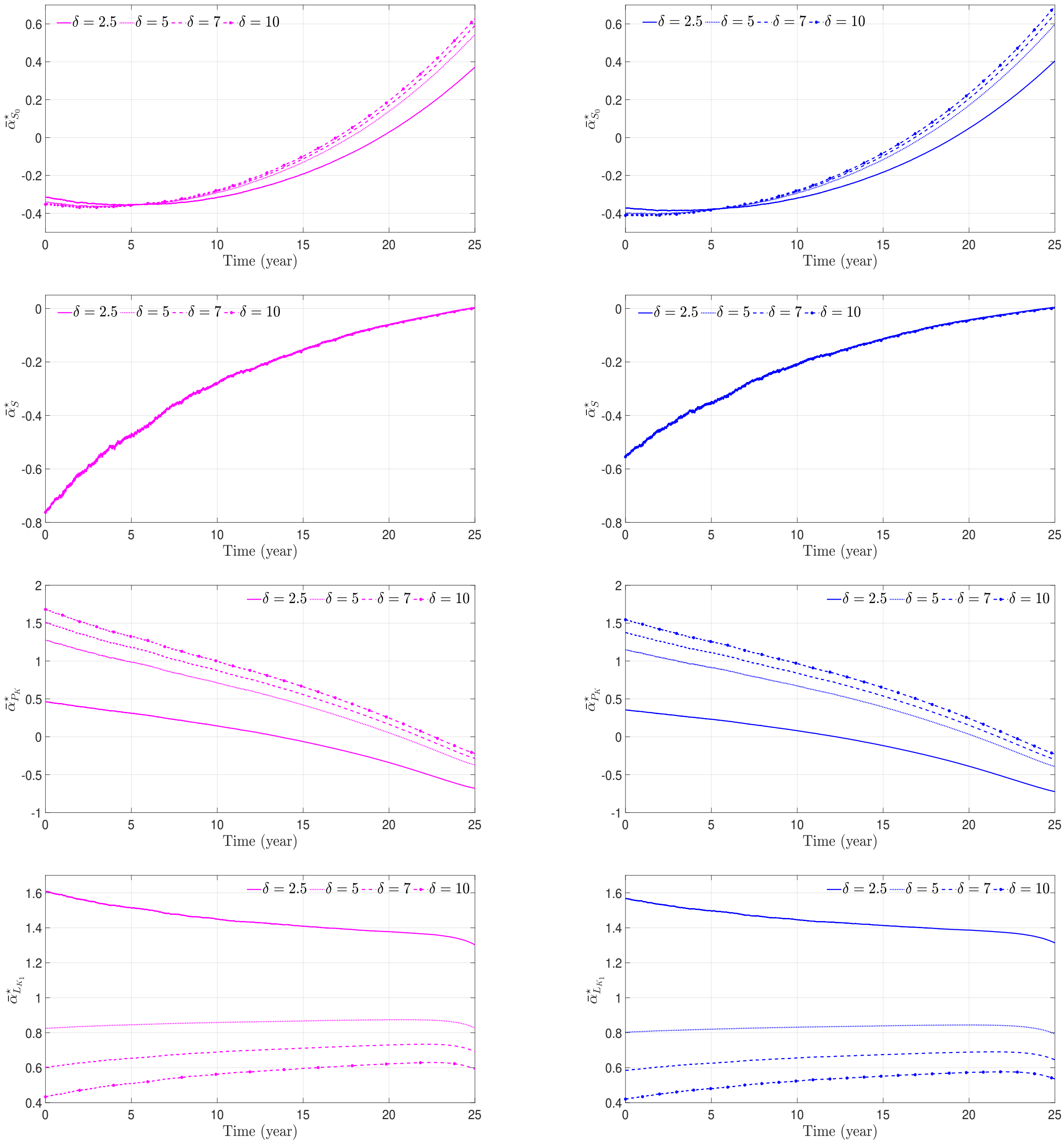


Figure 3.3 Sensitivity of optimal proportions for females (lhs) and males (rhs) with respect to the risk aversion parameter δ. The base scenario is characterized by $\delta = 2.5$.

Figure 3.3 shows the behaviour of the optimal investment proportions when the risk aversion parameter increases, i.e. when $\delta = 2.5, 5, 7, 10$. Using the CRRA utility function, we stress that the δ parameter measures the fund's manager risk aversion: a more risk-averse manager tends to allocate fund's wealth into assets presenting lower risks and (possibly) higher guaranteed returns. As a result, when δ grows, the fund manager is encouraged to augment both cash account and rolling bond proportions, dropping rolling longevity bond shares. Such evidence holds for both genders and involves the entire accumulation phase, although optimal shares in rolling longevity bonds remain positive and approximately time increasing. Reflecting the conservative approach during the latest accumulation times, also a higher risk-averse fund manager maintains short positions on the rolling bond to support long positions on the rolling longevity bond and cash account.

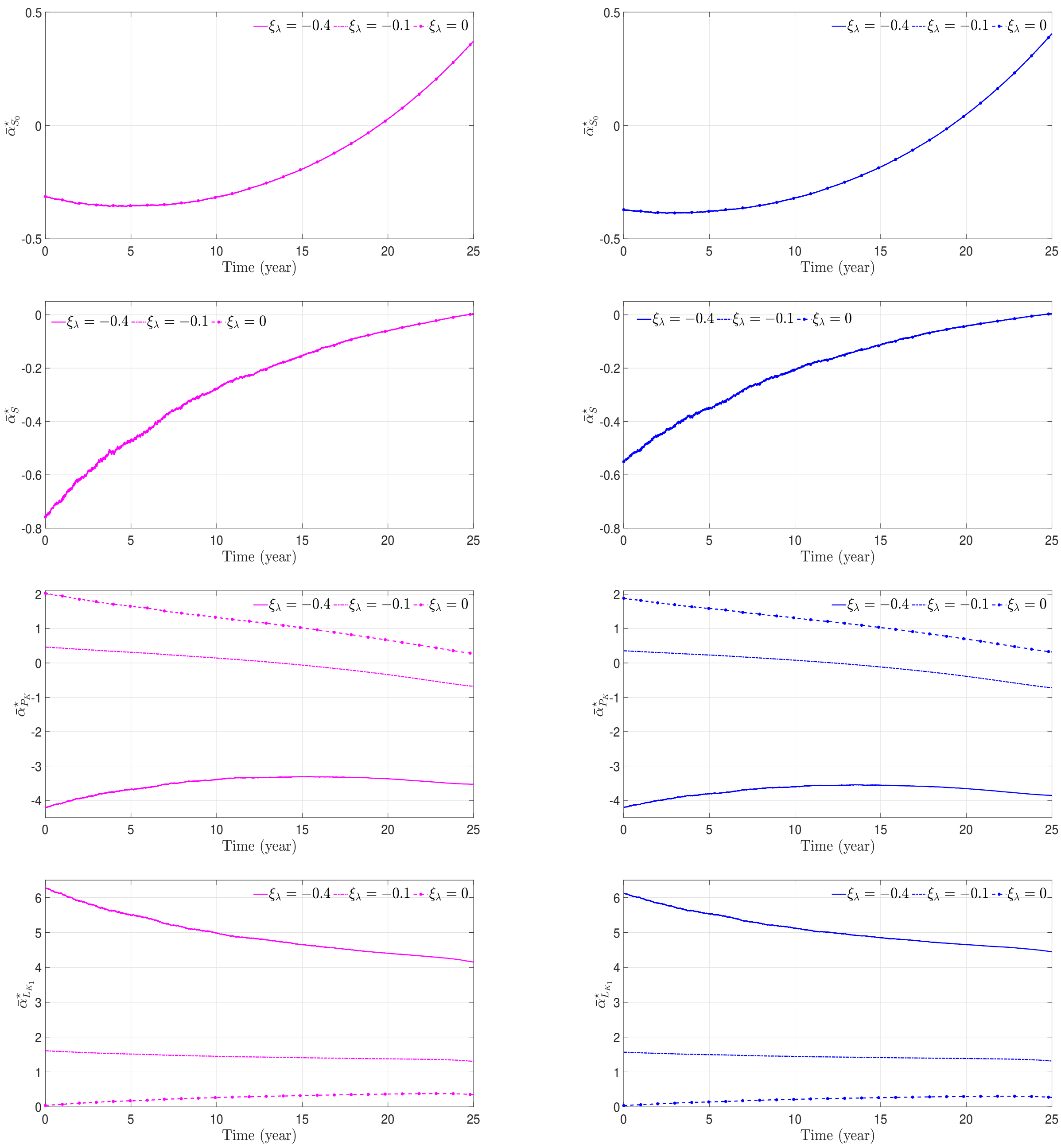


Figure 3.4 Sensitivity of optimal proportions for females (lhs) and males (rhs) with respect to the market price of longevity risk. The base scenario is characterized by $\xi_\lambda = 0.1$.

An additional insight from Figure 3.3 concerns the optimal proportions invested in stock, which appears uncaring to changes in δ. Although this outcome could be surprising, it reveals, at the same time, the importance of short positions in stock during the aggressive cycle of the optimal investment strategy. To some extent, the fund's manager is constrained in using the stock as a leverage tool in financing bond allocations, independently of the risk aversion level.

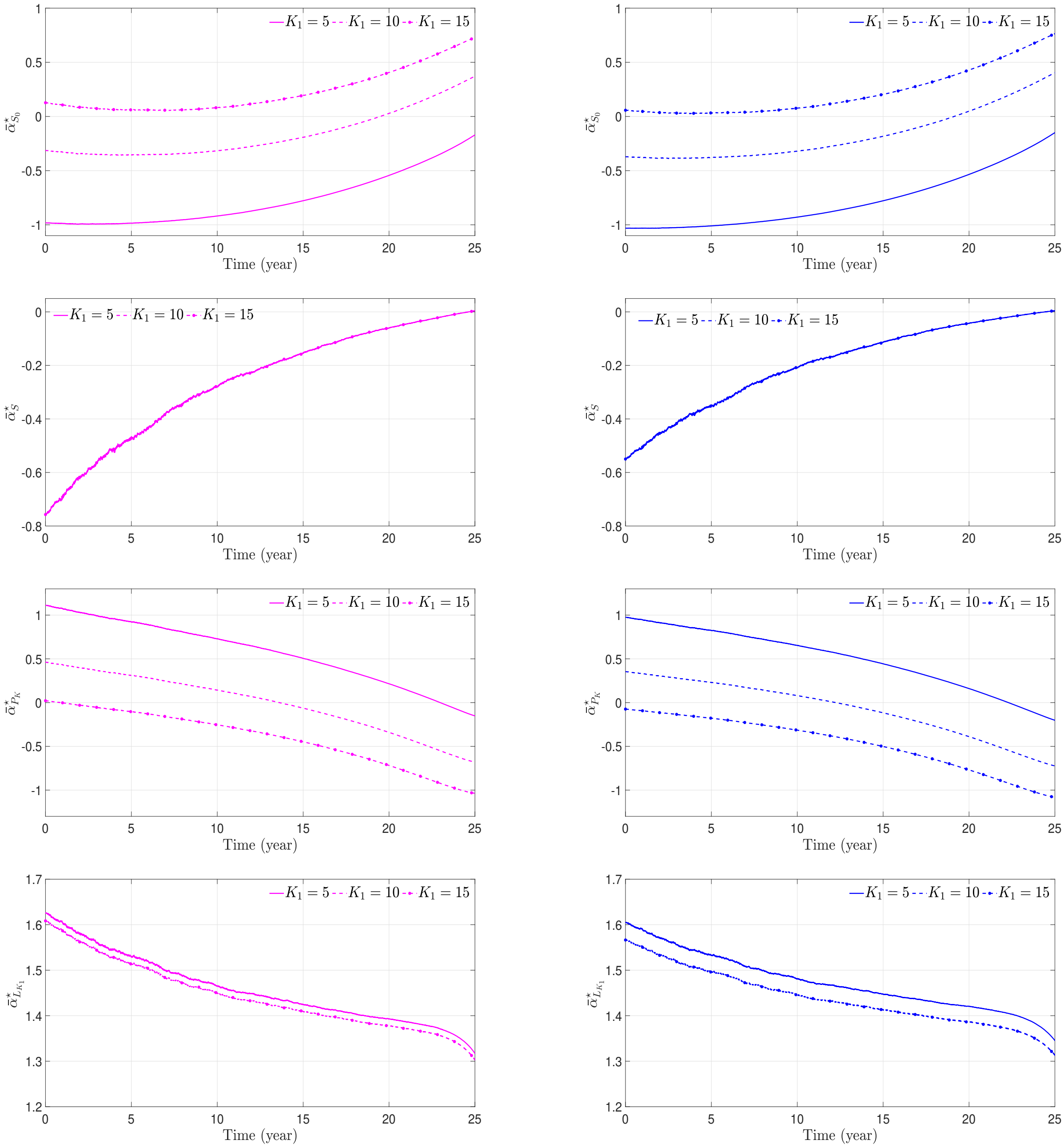


Figure 3.5 Sensitivity of optimal proportions for females (lhs) and males (rhs) with respect to the rolling longevity bond maturity. The base scenario is characterized by $K_1 = 10$.

Figure 3.4 represents the sensitivity of optimal proportions w.r.t. variations in the market price of longevity risk, which is defined according to $\xi_\lambda = -0.1$ in the base scenario. Then, we consider two alternative values, namely, $\xi_\lambda = -0.4$ and $\xi_\lambda = 0$. It is straightforward to observe that both cash account and stock shares are insensitive to changes in ξ_λ. Instead, the optimal weights for the rolling longevity bonds grow with $-\xi_\lambda$, and the opposite situation occurs for the optimal shares concerning the rolling bond. In particular, for $\xi_\lambda = -0.4$ the rolling longevity bond attractiveness dramatically increases, meaning that such an asset is helpful to accommodate both the aggressive and the conservative investment aptitude. As a result, the longevity bond is the asset dominating the fund portfolio over the entire accumulation phase.

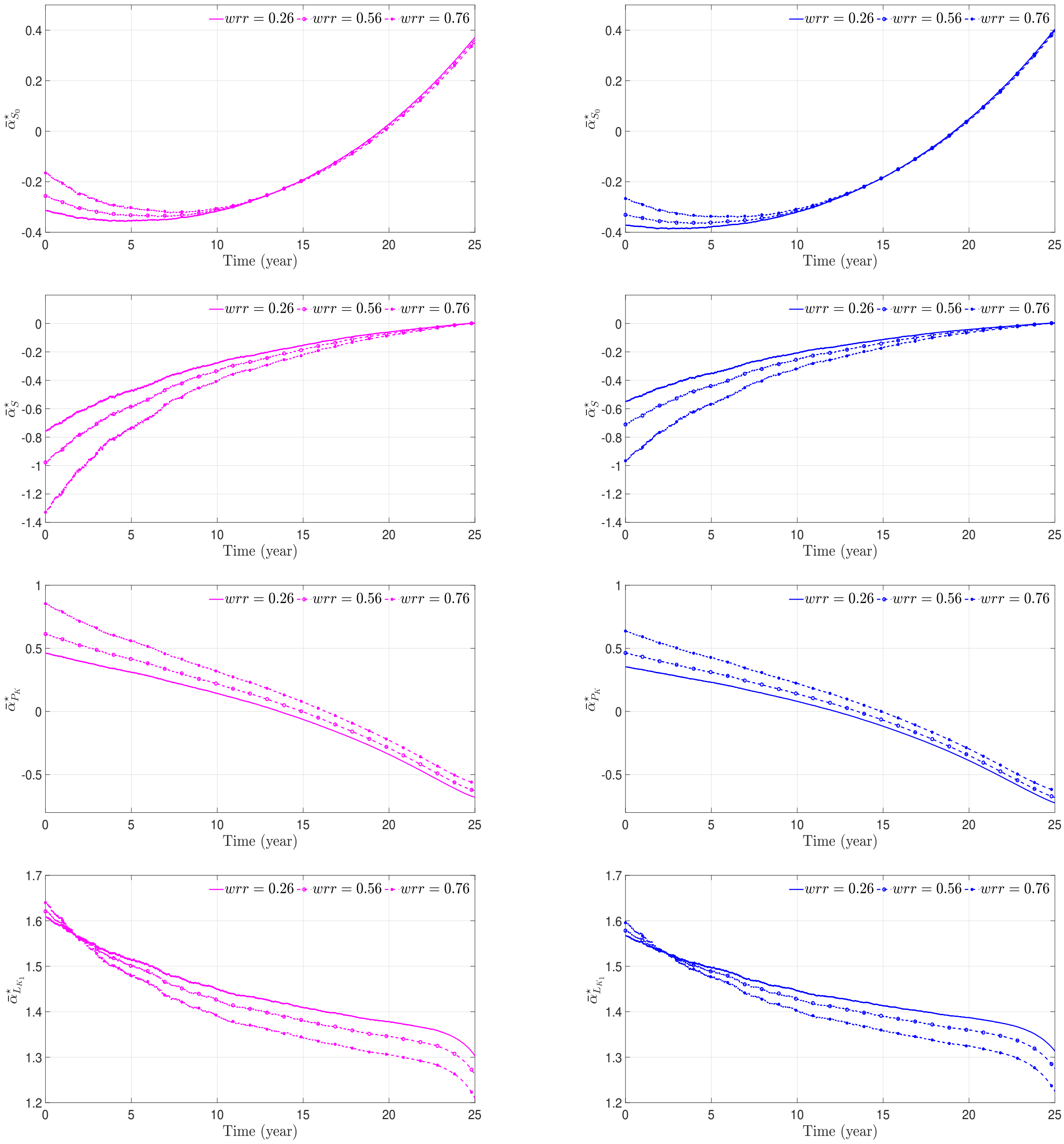


Figure 3.6 Sensitivity of optimal proportions for females (lhs) and males (rhs) with respect to the wage replacement ratio. The base scenario is characterized by $wrr = 0.26$.

Obviously, the rolling bond shares exhibit opposing dynamics, making them a suitable asset for financing long positions in longevity bonds. This trend also holds for the cash account as the retirement date approaches. When $\xi_\lambda = 0$, the accumulation of retirement capital is governed by larger allocations in the rolling bond, which gradually decrease over time to favor allocations in the cash account as the fund approaches the retirement date. Despite the null risk premia, slight and time-increasing longevity bond allocations are present for both males and females, highlighting the relevance of longevity risk perceived by the fund manager.

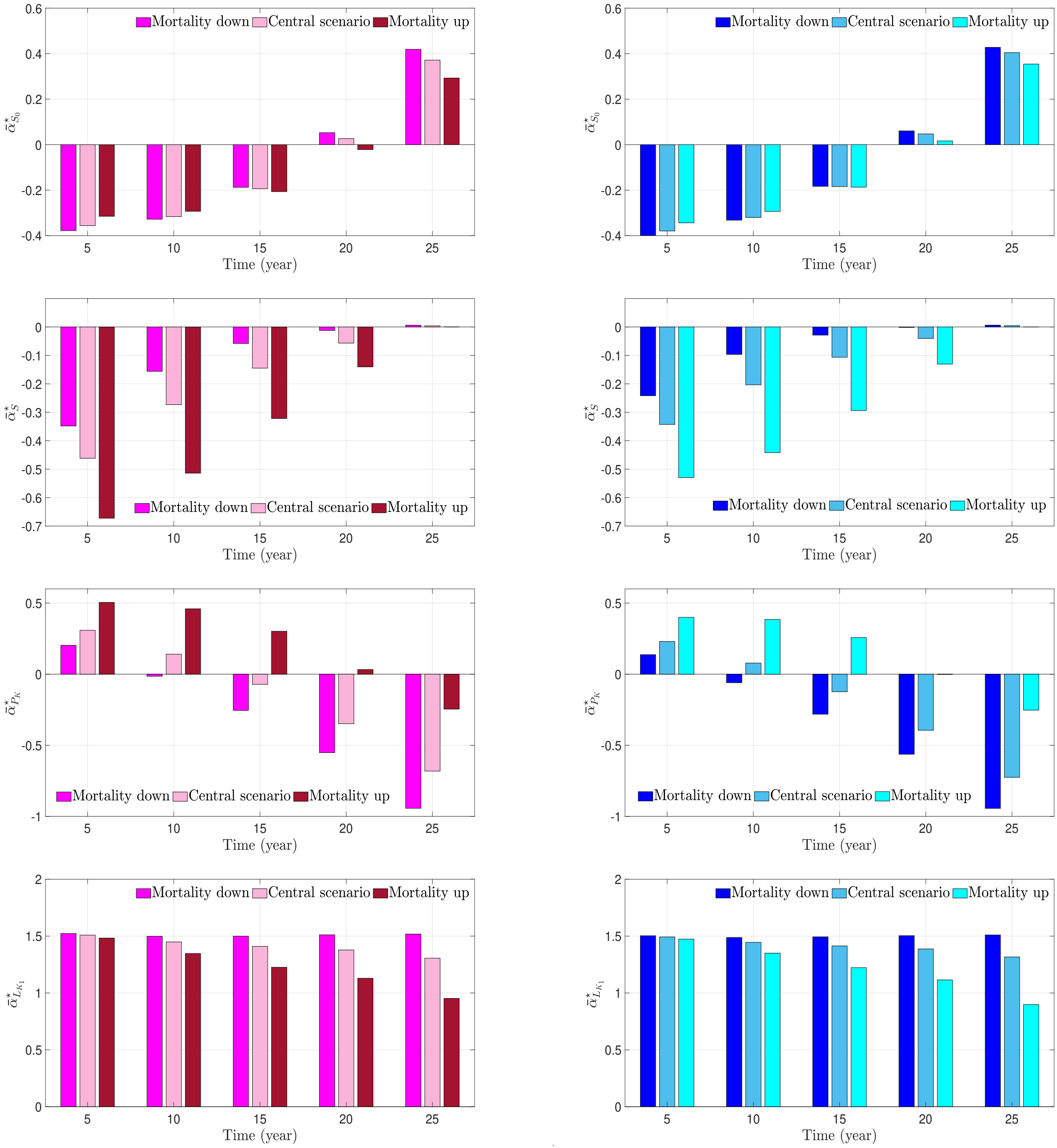


Figure 3.7 Sensitivity of optimal proportions for females (lhs) and males (rhs) with respect to the "mortality down" scenario and the "mortality up" scenario.

In Figure 3.5 changes in optimal proportions to variations in the maturity of the rolling longevity bond are considered. In this case, we aim to investigate how optimal shares vary according to $K_1 = 5$ or $K_1 = 15$, where $K_1 = 10$ is assumed for the base scenario. While for stock we do not observe modifications, the remaining assets manifest optimal shares significantly altered. For the longer longevity bond maturity, we record a substantially parallel shift for both cash account and rolling bond proportions (ZCB and longevity ZCB): in the former case the shift is increasing, while in the latter is decreasing. Then, with $K_1 = 15$ the fund manager sets short positions in the rolling bond to offset long positions in both the cash account and rolling longevity bond throughout the entire accumulation phase. However, optimal proportions in the rolling longevity bonds are the same for $K_1 = 10$ and $K_1 = 15$. As described in Agarwal et al. [2023], longer maturities result in more uncertainty in the rolling longevity bonds, and the risk-averse manager prefers to avoid exposing the capital accumulation to bonds with longer maturities. On the other hand, shortening the time to maturity, the capital accumulation is shaped according to higher allocation in both rolling bonds and rolling longevity bonds, requiring continuously short positions in the cash account.
We enrich our analysis by inspecting the sensitivity of optimal shares w.r.t. the so-called wage replacement ratio, or simply replacement rate. The latter measures the pension fund's capacity to furnish a post-retirement income adequately commensurate to the pre-retirement one. At the retirement date, the wage replacement ratio is

$$wrr = \frac{b_T}{L(T)},$$

and typically it ranges in $(0, 1)$. Obviously, ratio values approaching 1 indicate the pension fund's ability to maintain pre-retirement living standards during retirement. Concerning our base scenario, we record $wrr = 0.26$, revealing the need to investigate how the optimal proportions invested in risk-free and risky assets change if the fund manager aims to increase such a ratio. Figure 3.6 represents how optimal shares vary according to increasing values of the replacement rate, namely, $wrr = 0.56$ and $wrr = 0.76$. Fixing the median value for $L(T)$, a widening wrr requires rising annuity installments, and consequently, a higher annuity value. Since the wrr is not linked to the fund member's longevity in our framework, the fund manager faces the need to guarantee a greater annuity value by accelerating the capital accumulation. Then, as emerges from Figure 3.6, when the wrr increases, the fund's manager allocates more wealth in the rolling bond during the aggressive phase of the accumulation period. Conversely, the optimal proportions in the rolling longevity bond decrease, leverage effect through short positions in the stock. Looking at the conservative phase, the optimal proportions invested in the rolling longevity bond continue to be relevant, meaning to say that such an asset is the reference to protect retirement capital against longevity risk.
We conclude our numerical application by examining how the mortality phenomenon affects the optimal proportions invested in risky and risk-free assets. In particular, we aim to isolate the mortality impact by adding to the base scenario two alternatives: the “mortality down" scenario, corresponding to the 0.5-quantile of the simulated mortality intensity process, and the “mortality up" scenario, representing the 95-quantile. The former represents a scenario with higher longevity, while the latter considers increasing mortality. All the other random sources are portrayed by their

base scenario. Figure 3.7 summarizes the numerical results. As expected, for both genders, the "mortality down" scenario is characterized by wider optimal shares in the rolling longevity bond w.r.t. both the base scenario and the "mortality up" one. Indeed, the scenario with prominent longevity involves the prevalence of the longevity risk upon the interest rate risk. Then, the fund's manager finances long positions in the rolling longevity bond, taking greater (in absolute value) short position in the rolling bond. Moreover, the duration of the aggressive investment approach reduces, implying lower (greater) short positions (in absolute value) in the stock (cash account). During the conservative phase, the cash account also has a greater weight in the fund's asset allocation. Looking at the "mortality down" scenario, the opposite considerations hold.

Chapter 4

On the optimal design of a new class of Proportional PI strategy in a jump-diffusion framework

PPI strategy can be seen as a generalized version of the standard CPPI strategy. Indeed, the former is characterized by a multiplier m, which is no longer constant but time-varying, that is, $m = \{m(t),\ t \in [0,T]\}$, to provide better adaptability of the strategy's exposure to market fluctuations. As argued in Kingston [1989], the optimality of PPI strategies is defined for investors endowed with a hyperbolic absolute risk aversion (HARA) utility function given by

$$U^{HARA}(x) = \frac{(x-G)^{1-\delta}}{1-\delta}, \tag{4.1}$$

for $\delta > 0$ and $\delta \neq 1$. PPI insurers aim to determine the multiplier to maximize the expected HARA utility of terminal portfolio value under the constraint that the terminal portfolio value exceeds or equals the guaranteed amount G. Equivalently, a PPI investor aims to determine the optimal multiplier that maximizes the expected constant relative risk aversion (CRRA) utility function, $U^{CRRA}(x) = \frac{x^{1-\delta}}{1-\delta}$, of the terminal cushion, under the constraint that the terminal cushion is non-negative. This optimization problem has been solved by Zieling et al. [2014], under the hypotheses that the market is frictionless and the dynamics of the underlying risky asset and the reserve asset follow geometric Brownian motions. As discussed in Section 1.2.1, in this setup, the portfolio protection is efficient $\mathbb{P}$-a.s., which means that the constraint for the terminal cushion is not binding, and the standard dynamic programming approach applies. The optimal time-varying multiplier is the sum of two components: the myopic demand expressed in terms of the Merton solution, and a strongly model-dependent intertemporal hedging demand. Moreover, if one considers a market model with constant drift and volatility of the underlying risky asset, and constant risk-free interest rate, then the optimal multiplier is also constant and given by the product of the instantaneous Sharpe ratio and the inverse of the risk aversion parameter. In this case, the PPI strategy collapses into CPPI strategy.

However, the above results only apply in frictionless market. Indeed, as soon as frictions are introduced, either in the form of unpredictable downward jumps appear in the dynamics of the underlying (Cont and Tankov [2009]) or trading restrictions (Balder et al. [2009]), PI strategies fail to meet one of their objectives, and may lead to the so-called gap risk with positive probability. Technically, as shown in Section 1.2.3, the occurrence of gap risk is equivalent to the existence of a critical time $\tau := \inf\left\{t \in [0,T] : W^{PPI}(t) \leq F(t)\right\}$. Hence, under a standard PPI strategy, from τ on, the residual wealth is entirely invested into the reserve asset, meaning that the corresponding cushion process satisfies

$$(C(t))^{+} = \begin{cases} C(t), & t \in [0,\tau), \\ 0, & t \in [\tau,T]. \end{cases} \tag{4.2}$$

To illustrate the effects of the gap risk on the portfolio insurers, we perform a historical simulation of the CPPI strategy applied on the Standard and Poor 500 index from 2006 to 2013 with a constant multiplier of 10. As shown in Figure 4.1, due to the sudden market collapse in 2008, the CPPI

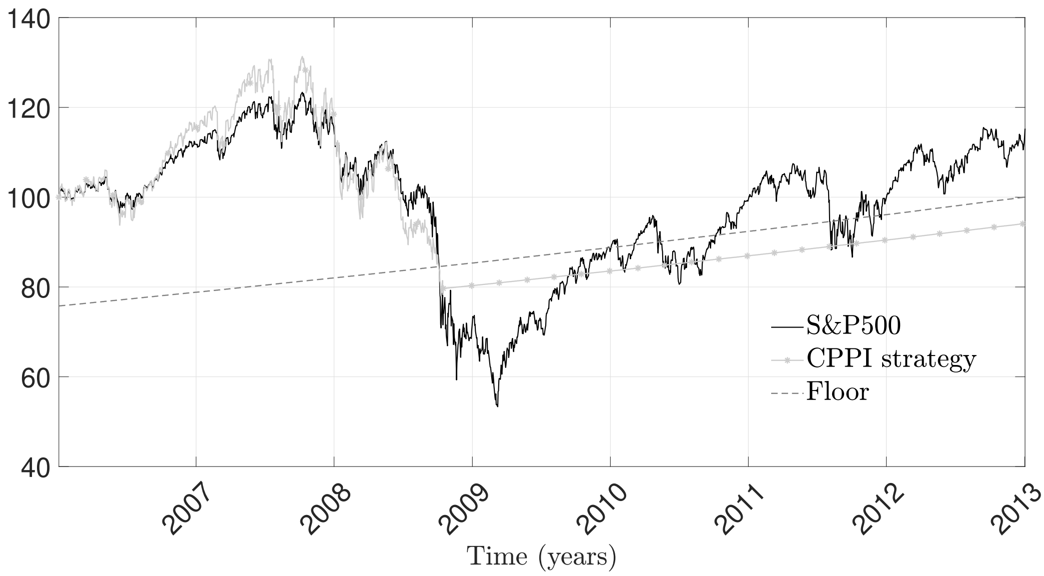


Figure 4.1 Historical simulations for the CPPI strategy.

strategy cannot be adequately rebalanced and thus falls below the floor. From that moment on, the remaining wealth is entirely allocated to the risk-free asset meaning that the strategy is unable to secure the guaranteed amount at maturity. Furthermore, although the Standard and Poor 500 nicely recovers from the beginning of 2009 onwards, the occurrence of gap risk makes the CPPI strategy unable to get any equity market participation.

Hence, the unbroken popularity of portfolio strategies cannot be explained based on standard utility theory. For this reason, a more recent strand of literature uses the positive framework of behavioural finance to explain the widespread popularity of PI strategies. In particular, Dichtl and Drobetz [2011] carried out an extensive empirical analysis showing that both CPPI and PPI strategies can also be explained using the *cumulative prospect theory* (CPT) developed by Tversky and Kahneman [1992]. The authors proved that PI investors are loss-averse and endowed with an S-shaped utility function, which is concave for gains and convex for losses. As a consequence, portfolio insurers evaluate the investment outcome by the corresponding deviation from a reference point, coinciding with the guaranteed amount at maturity, and assess potential gains and losses asymmetrically. In the CPT framework, Escobar-Anel et al. [2020] study the optimal design of a variant of the CPPI strategy, the so-called generalized behavioural portfolio insurance (GBPI) strategy. Unlike the standard CPPI, the GBPI works even when the portfolio falls below the floor due to the gap risk. After such an event, the GBPI maintains a positive exposure to the risky assets to limit potential loss and reach the funded area. The authors show the proposed GBPI strategy to be optimal in a CPT world, assuming a discrete-time rebalancing, a diffusive dynamics for the risky asset, and a constant risk-free rate.

In the present chapter, we provide a criterion for determining the optimal multiplier of a new PPI strategy to maximize the expected S-shaped utility of the terminal cushion at maturity. One of the novelties of our work consists in extending the idea of Escobar-Anel et al. [2020] to maintain a positive risky exposure even when the cushion becomes negative because of the gap risk. In addition, we consider a more realistic market setting, allowing for downward jumps in the underlying risky asset dynamics. We solve the optimization problem using a procedure based on the following two steps. First we show the equivalence between the optimization problem with an S-shape utility function with that of its concave envelop. Then we apply a modification of the so-called *martingale method*, based on duality theory, where the original dynamic problem is transformed into an equivalent static problem. Here the main difficulty arises to to market incompleteness, that does not allow for a univocal characterization of the state price density. To deal with that we rely on a technique developed by Michelbrink and Le [2012] which combines the martingale method and *worst case* probability, and allows to determine PPI strategy, and the state price density by solving a non-linear system of equations. Establishing sufficient conditions for both the existence and the uniqueness of solutions of this system is very challenging, as argued in Michelbrink and Le [2012]. We discuss some cases where the optimized multiplier is unique and given in closed or quasi-closed form. We also perform an accurate numerical analysis by using several specifications of jump size distributions, providing remarkable results. In particular, we prove that the optimized multiplier of the proposed PPI strategy is such that gap risk never occurs during the entire investment horizon.

The remainder of the chapter is organized as follows. In Section 4.1, we describe the market model with jumps. The optimization problem is introduced in Section 4.2, also containing a discussion about the concavification technique. Section 4.3 determines the optimal PPI strategy and discusses a few examples where a (semi-)closed form of the optimal multiplier can be established. Finally, we perform a numerical analysis in Section 4.4.

4.1 The financial market model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let T be a finite time horizon. Consider a one-dimensional Brownian motion $Z_S = \{Z_S(t),\ t \in [0,T]\}$, and a jump process described by an homogeneous Poisson random measure $N(\mathrm{d}t, \mathrm{d}y)$, independent of each other. We let $\nu(\mathrm{d}y)$ be the σ-finite compensator of $N(\mathrm{d}t, \mathrm{d}y)$, with support in a measurable set $E \subseteq \mathbb{R}$. Let $\mathbb{F} = \{\mathcal{F}(t),\ t \in [0,T]\}$ be a complete and right continuous filtration given by

$$\mathcal{F}(t) := \sigma\left\{\left(Z_S(s), N\left((0,s], \bar{A}\right)\right) \mid \bar{A} \in \mathcal{B}(E),\ s \in [0,t]\right\} \vee \bar{\mathcal{N}},$$

with $\mathcal{F}(T) = \mathcal{F}$ and $\mathcal{F}(0)$ being the trivial σ-algebra, where $\bar{\mathcal{N}}$ is the collection of all $\mathbb{P}$-null sets and $\mathcal{B}(E)$ is the Borel-σ-algebra on E. Next, we consider a financial market which consists of a zero-coupon bond (ZCB) with price process $P = \{P(t),\ t \in [0,T]\}$ maturing at time T, and a stock with price process $S = \{S(t),\ t \in [0,T]\}$. We assume that a fund manager can buy and sell continuously without restrictions or transaction costs. The ZCB price dynamics satisfies

$$\begin{cases} \dfrac{\mathrm{d}P(t)}{P(t)} = r\mathrm{d}t, \quad t \in [0,T], \\ P(T) = 1, \end{cases} \tag{4.3}$$

where $r > 0$ is the constant risk-free interest rate. The stock dynamics is assumed to follow a geometric jump-diffusion process, that is

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t-)} = \mu\mathrm{d}t + \sigma\mathrm{d}Z_S(t) + \displaystyle\int_E \gamma(t,y)N(\mathrm{d}t,\mathrm{d}y), \quad t \in [0,T], \\ S(0) = \tilde{s} > 0, \end{cases} \tag{4.4}$$

where $\mu \in \mathbb{R}$ is the drift of the risky asset, $\sigma \in \mathbb{R}_+$ is the volatility of the risky asset, and $\gamma(t,y)$ is an $\mathbb{F}$-adapted càdlàg σ-field accounting for the amplitude of jumps in the stock price. To ensure non-negativity of the risky asset's price, we assume that $1 + \int_E \gamma(t,y)N(\mathrm{d}t,\mathrm{d}y) \geq 0$. Note that that equation (4.4) admits a unique strong solution.
We consider a self-financing trading strategy $\boldsymbol{\pi} = \{(\pi(t), 1-\pi(t)),\ t \in [0,T]\}$ where $\pi(t)$ is the fraction of portfolio value invested in the risky asset at time t (consequently, $1-\pi(t)$ is the fraction of wealth invested in the bond at time t). We assume standard integrability conditions on $\boldsymbol{\pi}$,

$$\mathbb{E}\left[\int_0^T |S(s)||\pi(s)|\left(|\mu| + \int_E |\gamma(s,y)|\nu(\mathrm{d}y)\right) + S^2(s)\pi^2(s)\sigma^2\mathrm{d}s\right] < \infty. \tag{4.5}$$

Then, for all $t \in [0,T]$, the dynamics of the wealth process $\tilde{W}^{\boldsymbol{\pi},PPI} = \left\{\tilde{W}^{\boldsymbol{\pi},PPI}(t),\ t \geq 0\right\}$ is given by

$$\frac{\mathrm{d}\tilde{W}^{\boldsymbol{\pi},PPI}(t)}{\tilde{W}^{\boldsymbol{\pi},PPI}(t-)} = \left[r + \pi(t)\left(\mu - r\right)\right]\mathrm{d}t + \pi(t)\sigma\mathrm{d}Z_S(t) + \pi(t)\int_E \gamma(t,y)N(\mathrm{d}t,\mathrm{d}y), \quad t \in [0,T],$$

with $\tilde{W}^{\boldsymbol{\pi},PPI}(0) = \tilde{w}_0$ being the initial endowment. We assume that the exposure to the risky asset is proportional to the cushion, that is $m(t) \cdot \tilde{C}^m(t)$ for all $t \in [0,T]$, where $\tilde{C}^m(t) := \tilde{W}^{\boldsymbol{\pi},PPI}(t) - F(t)$, regardless of gap risk. Formally, we have the following definition.

Definition 4.1.1. *An admissible strategy is an* $\mathbb{F}$*-predictable process* $m = \{m(t),\ t \in [0,T]\}$ *such that*

(i) $m(t) \cdot \tilde{C}^m(t) = \pi(t)\tilde{W}^{\boldsymbol{\pi},PPI}(t), \quad t \in [0,T],$

(ii) the integrability condition in equation (4.5) *is satisfied,*

(iii) $\tilde{C}^m(t) \geq -F(t)$ *for all* $t \in [0,T]$*, and for every initial cushion* $\tilde{c}_0 = \tilde{w}_0 - G \cdot P(0) > 0$*. Equivalently, an admissible multiplier* m *satisfies* $\tilde{W}^{\pi,PPI}(t) \geq 0$*, for all* $t \in [0,T]$*, and for every initial wealth* $\tilde{w}_0$*.*

We denote by $\mathcal{M}(\tilde{c}_0)$ the set of all admissible multipliers m for a given initial cushion $\tilde{c}_0 > 0$. This set is non-empty, as an investor can always invest all his/her funds into the bond, that is $m(t) = 0$ for all $t \in [0,T]$. By stretching the definition slightly, we refer to the strategy mentioned above as PPI. In terms of the multiplier m, the PPI portfolio value process $\tilde{W}^{\pi,PPI}$ reads as

$$\begin{aligned} \mathrm{d}\tilde{W}^{\pi,PPI}(t) &= \mathrm{d}F(t) + \left(\tilde{W}^{\pi,PPI}(t-) - F(t)\right)[r + m(t)(\mu - r)]\,\mathrm{d}t \\ &\quad + \left(\tilde{W}^{\pi,PPI}(t-) - F(t)\right)\left[m(t)\sigma \mathrm{d}Z_S(t) + m(t)\int_E \gamma(t,y)N(\mathrm{d}t,\mathrm{d}y)\right], \end{aligned}$$

for every $t \in [0,T]$, with $\tilde{W}^{\pi,PPI}(0) = \tilde{w}_0$. As in Section 1.2, the floor process satisfies $\mathrm{d}F(t) = rF(t)r\mathrm{d}t$, with $F(T) = G$. Hence, the dynamics of the cushion process are

$$\begin{cases} \dfrac{\mathrm{d}\tilde{C}^m(t)}{\tilde{C}^m(t-)} = [r + m(t)(\mu - r)]\,\mathrm{d}t + m(t)\sigma \mathrm{d}Z_S(t) + m(t)\displaystyle\int_E \gamma(t,y)N(\mathrm{d}t,\mathrm{d}y), \quad t \in [0,T], \\ \tilde{C}^m(0) = \tilde{c}_0. \end{cases} \tag{4.6}$$

4.2 The optimization problem

The presence of jumps in the dynamics of the underlying risky asset impedes the determination of the optimal multiplier, even under broader definitions of PPI strategies. This is a mere consequence of gap risk which may lead to negative values for the cushion process and hence does not permit modelling the preferences of the insurer via a CRRA utility function. Therefore, in the present work, we adopt the prospect utility theory to model the preference of the PI insurer. This choice is justified by the findings of Dichtl and Drobetz [2011], who proved, through extensive numerical analysis, that PI insurers' behaviour is more appropriately described by the theory devised by Tversky and Kahneman [1992]. In particular, Dichtl and Drobetz [2011] argue that PI insurers evaluate their investment decisions regarding the potential losses and gains relative to a reference level that coincides with the guaranteed amount at maturity. They also found that PI insurers are loss-averse, perceiving losses and gains asymmetrically. More precisely, they evaluate the marginal utility of potential losses higher than that of potential gains. Mathematically, such features could be stylised via S-shaped utility functions. An S-shaped utility function on $[0,+\infty)$ defined as follows

$$U(x;G) = \begin{cases} U_1(x - G), & x \geq G, \\ U_2(G - x), & 0 \leq x < G, \end{cases} \tag{4.7}$$

where the guaranteed amount G is the reference point, $U_1(x)$ and $U_2(x)$ are strictly increasing, strictly concave, continuously differentiable on $[0,+\infty)$ satisfying $U_1'(x) < U_2'(x)$, $U_i(0) = 0$, $\lim_{x\to+\infty} U_i(x) = +\infty$, $\lim_{x\to 0^+} U_i'(x) = +\infty$, $\lim_{x\to+\infty} U_i'(x) = 0$, and $0 \leq U_i'(x) \leq M_i\left(1 + x^{\beta_i}\right)$, for all

$x \geq 0$, and for some constants $M_i > 0$, $\beta_i \in (0,1)$, $i = 1,2$, see Bian et al. [2011] for further details. As in Dong and Zheng [2019], we choose $U_1(x)$ and $U_2(x)$ such that

$$U(x;G) = \begin{cases} \dfrac{(x-G)^{1-\delta_1}}{1-\delta_1}, & x \geq G, \\ -\bar{\lambda}\dfrac{(G-x)^{1-\delta_2}}{1-\delta_2}, & 0 \leq x < G, \end{cases} \tag{4.8}$$

where $\delta_i \in (0,1)$, for $i = 1,2$ and $\bar{\lambda} > \frac{1-\delta_2}{1-\delta_1}$. Hence, the fund's manager aims to determine the optimal multiplier of our new version of the PPI strategy in order to maximize the expected S-shaped utility of the terminal wealth $\tilde{W}^{\pi,PPI}(t)$ over the guaranteed amount. His problem can be stated as follows.

Problem 4.2.1. *For the initial data* $(t,\tilde{w}) \in [0,T] \times (0,+\infty)$,

$$\textit{Maximize } J(t,\tilde{w};m(\cdot)) = \mathbb{E}^{t,\tilde{w}}\left[U\left(\tilde{W}^{\pi,PPI}(T);G\right)\right] \textit{ over all } m(\cdot) \in \mathcal{M}(\tilde{c}), \tag{4.9}$$

where $\mathbb{E}^{t,\tilde{w}}$ *denotes the conditional expectation given* $\tilde{W}^{\pi,PPI}(t) = \tilde{w}$.

Due to the special choice of the utility function, we do not restrict to consider strategies leading to positive cushion. Even if the gap occurs during the investment horizon, the fund manager maintains the exposure to the risky asset proportional to the negative cushion to minimize the difference between the underfunded portfolio value and the guaranteed amount at maturity. Following Zieling et al. [2014], we recast the optimization problem 4.2.1 in terms of the cushion as follows.

Problem 4.2.2. *For the initial data* $(t,\tilde{c}) \in [0,T] \times (-\infty,+\infty)$,

$$\textit{Maximize } J(t,\tilde{c};m(\cdot)) = \mathbb{E}^{t,\tilde{c}}\left[\tilde{U}\left(\tilde{C}^m(T);G\right)\right] \textit{ over all } m(\cdot) \in \mathcal{M}(\tilde{c}), \tag{4.10}$$

where $\mathbb{E}^{t,\tilde{c}}$ *denotes the conditional expectation given* $\tilde{C}^m(t) = \tilde{c}$, *and* $\tilde{U}(\cdot\,;G)$ *is an S-shaped utility on* $[-G,\infty)$ *with reference point equal to zero, defined as*

$$\tilde{U}(x;G) = \begin{cases} \dfrac{x^{1-\delta_1}}{1-\delta_1}, & x \geq 0, \\ -\bar{\lambda}\dfrac{(-x)^{1-\delta_2}}{1-\delta_2}, & -G \leq x < 0. \end{cases}$$

From now on, we omit the dependence of the multiplier m in the notation of the cushion process to help for readability. We define the value function corresponding to problem 4.2.2 as follows

$$\begin{cases} v(t,\tilde{c}) := \sup\limits_{m \in \mathcal{M}(\tilde{c})} J(t,\tilde{c};m(\cdot)), \\ v\left(t,\tilde{C}(T)\right) := \tilde{U}\left(\tilde{C}(T);G\right). \end{cases} \tag{4.11}$$

4.2.1 The concavification

We approach the optimization problem 4.2.2 via the martingale method. Here, a few difficulties arise due to the incompleteness of the market and the non-concavity of the utility function. We begin by addressing the problem that the objective function of the optimization problem depicted in equation (4.10) is not concave in $\tilde{C}$. To overcome this issue, we apply the concavification technique, see, e.g., Carpenter [2000] and Nicolosi et al. [2018]. Denote by a function $\bar{f}^c$ the concave envelope of a function $\bar{f}$ with domain D by

$$\bar{f}^c := \inf\left\{g : D \to \mathbb{R} \text{ s.t. } g \text{ is concave, } g(x) \geq \bar{f}(x),\ \forall x \in D\right\}.$$

Figure 4.2 represents $\tilde{U}$ as a function of the terminal cushion $\tilde{C}(T)$. This objective function is not concave with respect to the decision variable $\tilde{C}(T)$. However, for each $G > 0$, $\tilde{U}(\cdot\,; G)$ exhibits a concavification $\tilde{U}^c(\cdot\,; G)$ illustrated by the dotted line in Figure 4.2. In particular, $\tilde{U}^c$ replaces part of the original function with a straight line from $-G$ to another point, $\hat{c}(G) > 0$, at which the slope of such a straight line equals the slope of $\tilde{U}$. The following Proposition shows that for every $G > 0$, the point $\hat{c}(G) > 0$ exists and is unique.

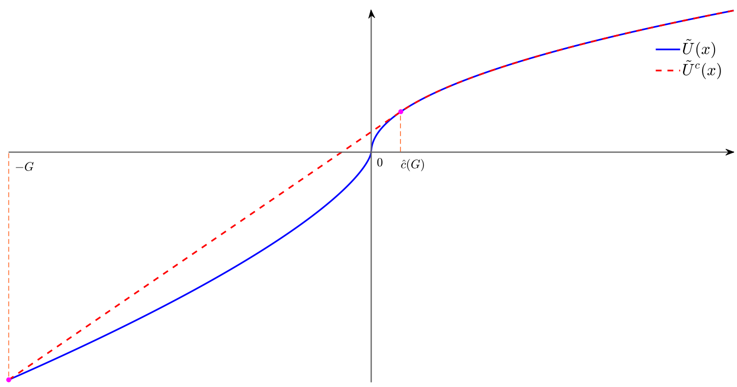


Figure 4.2 The solid line plots the utility $\tilde{U}(x)$, the dotted line is the concave envelope of $\tilde{U}(x)$. In this plot we have used the parameters: $\delta_1 = 0.3$, $\delta_2 = 0.5$, $\bar{\lambda} = 2.25$ and G=100.

Proposition 4.2.3. *Assume that* $\dfrac{\bar{\lambda}}{1-\delta_2} > \dfrac{1}{1-\delta_1} > 0$ *and* $\delta_i \in (0,1)$ *for* $i = 1, 2$. *Then, for every* $G > 0$, *there exists a unique point* $\hat{c}(G) > 0$ *which is the solution of the following equation*

$$\frac{\tilde{U}\left(\hat{c}(G); G\right) - \tilde{U}(-G; G)}{\hat{c}(G) + G} = \tilde{U}'\left(\hat{c}(G); G\right). \tag{4.12}$$

Proof. Recall that $\tilde{U}(x; G)$ is differentiable for every $x \neq 0$. Let us consider $x \in (-\infty, 0)$. Using the condition stated in equation (4.12), we define the function

$$f(x) := \begin{cases} -\bar{\lambda}\dfrac{\delta_2}{1-\delta_2}(-x)^{1-\delta_2} - \bar{\lambda}\dfrac{G}{(-x)^{\delta_2}} + \bar{\lambda}\dfrac{G^{1-\delta_2}}{1-\delta_2}, & x \in (-\infty, 0) \\ \dfrac{\delta_1}{1-\delta_1}x^{1-\delta_1} - \frac{G}{x^{\delta_1}} + \bar{\lambda}\dfrac{G^{1-\delta_2}}{1-\delta_2}, & x \in (0, \infty). \end{cases} \tag{4.13}$$

We see that for $x \in (-\infty, 0)$ the function $f(x)$ has a maximum at $x = -G$, with $f(-G) < 0$. Since $\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f(x) = -\infty$, there is no $\hat{c}(G) \in (-\infty, 0)$ such that $f(\hat{c}(G)) = 0$. If we consider

$x \in (0, +\infty)$, since $\lim_{x \to 0^+} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$, then there exists $\hat{c}(G) > 0$ such that $f(\hat{c}(G)) = 0$. Moreover, since $f'(x) > 0$ for all $x \in (0, +\infty)$, then $\hat{c}(G)$ is also unique. This concludes the proof. $\square$

The concavified function $\tilde{U}^c(x; G)$ is given by

$$\tilde{U}^c(x; G) = \begin{cases} \dfrac{x^{1-\delta_1}}{1-\delta_1}, & x \geq \hat{c}(G), \\ \hat{k}x + \dfrac{G}{\hat{c}(G) + G} \dfrac{\hat{c}(G)^{1-\delta_2}}{1-\delta_2} - \bar{\lambda} \dfrac{G^{1-\delta_1}}{\hat{c}(G) + G} \dfrac{\hat{c}(G)}{1-\delta_1}, & -G \leq x < \hat{c}(g), \end{cases}$$

where

$$\hat{k} = \frac{1}{\hat{c}(G) + G} \left(\frac{\hat{c}(G)^{1-\delta_2}}{1-\delta_2} + \frac{\bar{\lambda} G^{1-\delta_1}}{1-\delta_1} \right),$$

is the slope of the tangent line (see, Figure 4.2) and $\hat{c}(G) > 0$ is the unique solution of equation (4.12). Furthermore, $\tilde{U}^c(x; G) \geq \tilde{U}(x; G)$ for all $(x, G) \in [-G, +\infty) \times (0, +\infty)$ and $\tilde{U}^c(x; G) = \tilde{U}(x; G)$ for $x = -G$ and for all $x > \hat{c}(G)$.

Lemma 4.2.4. *The optimal terminal cushion is achieved at* $\tilde{C}^\star(T) \geq \hat{c}(G)$.

Proof. We first show that the optimal terminal cushion $\tilde{C}^\star(T)$ never takes values on $(-G, \hat{c}(G))$ where the true and the concavified objective functions are different. Suppose that there is a set $\bar{\Omega} \subseteq \Omega$ such that $\tilde{U}\left(\tilde{C}^\star(T, \omega); G\right) \neq \tilde{U}^c\left(C^\star(T, \omega); G\right)$, for all $\omega \in \Omega$, equivalently, $\tilde{C}^\star(T, \omega)$ takes values in $(-G, \hat{c}(G))$ for all $\omega \in \bar{\Omega}$. We also let $\mathbb{P}\left(\tilde{C}^\star(T) < \hat{c}(G)\right) = q$ for some $q \in (0, 1)$. We want to show that, in this case, such $\tilde{C}^\star(T)$ is not optimal. For all $\omega \in \bar{\Omega}$, there exists $\alpha(\omega) \in (0, 1)$ such that $\tilde{C}^\star(T, \omega) = \alpha(\omega) \cdot (-G) + (1 - \alpha(\omega)) \cdot \hat{c}(G)$. Now, define the random variable

$$\hat{C}(T)(\omega) = \begin{cases} -G, & \text{if } \omega \in \Omega_1, \\ \hat{c}(G), & \text{if } \omega \in \bar{\Omega} \setminus \Omega_1, \\ \tilde{C}^\star(T, \omega), & \text{if } \omega \in \Omega \setminus \bar{\Omega}. \end{cases}$$

We also assume that $\mathbb{P}\left(\hat{C}(T) = -G\right) = p_1$ for some $p_1 \in (0, q)$ and $\mathbb{P}\left(\hat{C}(T) = \hat{c}(G)\right) = q - p_1$. Note that $\tilde{U}\left(\hat{C}(T); G\right) = \tilde{U}^c\left(\hat{C}(T); G\right)$ everywhere and that $\hat{G}$ is an admissible cushion. Moreover

$$\mathbb{E}\left[\tilde{U}(\hat{C}(T); G)\right] = \tilde{U}(-G; G)p_1 + \tilde{U}(\hat{c}(G); G)(q - p_1) + \mathbb{E}\left[\tilde{U}(\hat{C}(T); G)\mathbb{1}_{\Omega \setminus \bar{\Omega}}\right].$$

On the other hand we also observe that

$$\mathbb{E}\left[\tilde{U}(\tilde{C}^\star(T); G)\right] = \mathbb{E}\left[\tilde{U}\left(-G\alpha + \hat{c}(G)(1 - \alpha); G\right) \mathbb{1}_{\bar{\Omega}}\right] + \mathbb{E}\left[\tilde{U}(\hat{C}(T); G)\mathbb{1}_{\Omega \setminus \bar{\Omega}}\right]. \tag{4.14}$$

Hence if we choose $p_1 = 0$ this implies that $\mathbb{E}\left[\tilde{U}(\hat{C}(T); G)\right] \geq \mathbb{E}\left[\tilde{U}(\tilde{C}^\star(T); G)\right]$. Next we also exclude that $\tilde{C}^\star(T) = -G$. Indeed, if one implements the trivial strategy, i.e. that corresponding to the multiplier $m = 0$, gets the terminal cushion $\tilde{C}^0(T) = \tilde{c}_0 \cdot e^{rT}$, which is non-negative since $\tilde{c}_0 \geq 0$ by construction. Comparing terminal cushions $\tilde{C}^\star(T) = -G$ $\mathbb{P}$−a.s. and $\tilde{C}^0(T)$ one would get that $\mathbb{E}\left[\tilde{U}(\tilde{C}^\star(T); G)\right] < \mathbb{E}\left[\tilde{U}(\tilde{C}^0(T); G)\right]$, which contradicts the optimality of $\tilde{C}^\star(T)$. $\square$

Since $\tilde{C}^{\star}(T) \geq \hat{c}(G)$, we can restrict to the case where $\tilde{U}^c(y,G) = \frac{y^{1-\delta_1}}{1-\delta_1}$. Hence, we can recast the optimization problem 4.2.2 into the following equivalent one.

Problem 4.2.5. *For the initial data $(t,\tilde{c}) \in [0,T] \times (-\infty,+\infty)$*

$$\text{Maximize } J\left(t,\tilde{c};m(\cdot)\right) = \mathbb{E}^{t,\tilde{c}}\left[\frac{\left(\tilde{C}(T)\right)^{1-\delta_1}}{1-\delta_1}\right] \text{ over all } m(\cdot) \in \mathcal{M}\left(\tilde{c}\right), \tag{4.15}$$

where $\mathbb{E}^{t,\tilde{c}}$ denotes the conditional expectation given $\tilde{C}(t) = \tilde{c}$.

The value function corresponding to the latter equivalent problem is given by

$$\begin{cases} v(t,\tilde{c}) = \sup\limits_{m \in \mathcal{M}(\tilde{c})} J(t,\tilde{c};m(\cdot)), \\ v\left(t,\tilde{C}(T)\right) = \dfrac{\left(\tilde{C}(T)\right)^{1-\delta_1}}{1-\delta_1}. \end{cases} \tag{4.16}$$

4.3 The martingale approach under market incompleteness

Now, we are ready to tackle the second issue, i.e. the incompleteness of the market, which impedes the resolution of optimization problem 4.2.5 via the standard martingale method. To overcome such a drawback, we use a generalization of the martingale method that applies to incomplete markets. In particular, we rely on the results of Michelbrink and Le [2012], suitably taylored to our setup.

Remark 4.3.1. *Alternatively one could apply the dynamic programming approach, and characterize the value of the optimization as the unique viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation. Moreover uniqueness in the classical sense holds under additional conditions on the parameters of the model (see, e.g. Pham [1998]). The optimal PPI strategy can only be expressed in feedback form and the value function must be computed numerically. Therefore we resort to the martingale method suitably combined with the worst case probability measure as described below, which permits the simultaneous characterization of the state price density and the optimal PPI strategy in terms of the solution of an implicit system.*

First, we have to rewrite the dynamic optimization problem 4.2.5 into an equivalent static one. Under the assumption of no arbitrage, there are infinitely many equivalent martingale measures (EMMs) that can be characterized by the family of Radon-Nikodym densities $\Pi^{\mathbf{\Xi}}$ so that

$$\left.\frac{\mathrm{d}\mathbb{Q}_{\mathbf{\Xi}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}(T)} = \Pi^{\mathbf{\Xi}}(t), \tag{4.17}$$

parameterized by $\mathbf{\Xi} = (\Xi_S(t), \Xi_J(t,y))$. In particular, $\Xi_S(t)$ and $\Xi_J(t,y)$ are the market prices of risk for the diffusion and for jumps of the stock price, respectively. For the predictable process $\Xi_S = \{\Xi_S(t),\ t \in [0,T]\}$, and predictable random field $\Xi_J = \{\Xi_J(t,y),\ t \in [0,T]\}$, we assume that the process $\Pi^{\mathbf{\Xi}}$ is a martingale and it is the solution of the SDE

$$\frac{\mathrm{d}\Pi^{\mathbf{\Xi}}(t)}{\Pi^{\mathbf{\Xi}}(t-)} = \Xi_S(t)\mathrm{d}Z_S(t) - \int_E \left(1 - \Xi_J(t,y)\right)\left(N(\mathrm{d}t,\mathrm{d}y) - \nu(\mathrm{d}t,\mathrm{d}y)\right), \tag{4.18}$$

for every $t \in [0,T]$, with $\Pi^{\Xi}(0)=1$. If $\Xi_J(t,y)>0$ for all $y \in E$ and $t \in [0,T]$, it is well known that $\Pi^{\Xi}(t)$ can be rewritten as

$$\Pi^{\Xi}(t) = \exp\left\{ \int_0^t \Xi_S(s)\mathrm{d}Z_S(s) - \frac{1}{2}\int_0^t \left(\Xi_S(s)\right)^2 \mathrm{d}s \right. \\ \left. + \int_0^t \int_E \log \Xi_J(s,y) N(\mathrm{d}s,\mathrm{d}y) + \int_0^t \int_E \left(1-\Xi_J(s,y)\right)\nu(\mathrm{d}y)\mathrm{d}s \right\}, \quad (4.19)$$

see, Theorem 4.61 of Jacod and Shiryaev [2013] for further details. By Girsanov's Theorem, if $\Pi^{\Xi}(t)$ is a $\mathbb{P}$-martingale with expectation equal to one, we get that

$$Z_S^{\mathbb{Q}_\Xi}(t) = Z_S(t) - \int_0^t \Xi_S(s,r(s))\mathrm{d}s, \quad t \in [0,T], \quad (4.20)$$

is a Brownian motion under $\mathbb{Q}_\Xi$ and the random measure defined by

$$N^{\mathbb{Q}_\Xi}(\mathrm{d}t,\mathrm{d}y) = N(\mathrm{d}t,\mathrm{d}y) - \int_0^t \Xi_J(s,y,r(s))\nu(\mathrm{d}y)\mathrm{d}s, \quad t \in [0,T], \quad (4.21)$$

is a compensated Poisson random measure under $\mathbb{Q}$. To guarantee that Π^{Ξ} is a martingale with $\mathbb{E}\left[\Pi^{\Xi}(t)\right]=1$, one could assume a generalization of the Novikov's condition for jump-diffusion processes, that is

$$\mathbb{E}\left\{ e^{\frac{1}{2}\int_0^T (\Xi_S(u))^2 \mathrm{d}u + \int_0^T (1-\Xi_J(u,y))^2 \mathrm{d}u} \right\} < \infty,$$

see, e.g. Theorem 9 of Protter et al. [2008]. To summarize, we consider the following definition.

Definition 4.3.2. *The vector process Ξ is an admissible market price of risk if it is predictable and the following conditions hold*

(i) $\Xi_J(t,y)>0$ for all $y \in E$ and $t \in [0,T]$,

(ii) $\Pi^{\Xi}(t)$ defined by (4.19) is a martingale with expected value equal to one,

(iii) for all $t \in [0,T]$ it holds that

$$\mu - r + \sigma \Xi_S(t) + \int_E \gamma(t,y)\Xi_J(t,y)\nu(\mathrm{d}y) = 0, \quad (4.22)$$

We let Θ be the set of all admissible market prices of risk.

Condition (iii) of Definition 4.3.2 ensures that the discounted stock price $S(t)e^{-\int_0^t r(s)\mathrm{d}s}$ is a local martingale under the corresponding new probability measure $\mathbb{Q}_\Xi$ defined by equation (4.17).

Assumption 4.3.1. *The set Θ is non-empty.*

For every $\Xi \in \Theta$, using the definition of the $\mathbb{Q}_\Xi$-Brownian motions, the $\mathbb{Q}_\Xi$-compensator of the Poisson random measure and the condition (4.22), it can be easily seen that the process $\tilde{C}^{\mathbb{Q}_\Xi}$ defined as

$$\tilde{C}^{\mathbb{Q}_\Xi}(t) := \tilde{C}(t)e^{-rt}, \quad t \in [0,T], \quad (4.23)$$

is a local $\mathbb{Q}_{\Xi}$-martingale. To obtain properties of $\tilde{C}^{\mathbb{Q}_{\Xi}}$ under the original probability $\mathbb{P}$, we introduce the state price density associated with $\Xi \in \Theta$.

Definition 4.3.3. *For $\Xi = (\Xi_S(t), \Xi_J(t,y))$, the process defined by*

$$H^{\Xi}(t) := \Pi^{\Xi}(t) e^{-rt}$$

is called the state price density associated with Ξ, where Π^{Ξ} is defined by (4.19)*. The dynamics of H^{Ξ} is given by*

$$\frac{\mathrm{d}H^{\Xi}(t)}{H^{\Xi}(t-)} = -r\mathrm{d}t + \Xi_S(t)\mathrm{d}Z_S(t) - \int_E (1 - \Xi_J(t,y))\, \tilde{N}(\mathrm{d}t, \mathrm{d}y),$$

for every $t \in [0,T]$. For $\Xi \in \Theta$, the process M defined by

$$M(t) := H^{\Xi}(t)\tilde{C}(t), \quad t \in [0,T], \tag{4.24}$$

is a local $\mathbb{P}$-martingale. Moreover, for all $m \in \mathcal{M}(\tilde{c}_0)$, the following budget constraint holds

$$\mathbb{E}^{\mathbb{Q}}\left[\tilde{C}(T)e^{-rT}\right] \le \tilde{c}_0 \ \text{ or, equivalently, } \ \mathbb{E}\left[\tilde{C}(T)H^{\Xi}(T)\right] \le \tilde{c}_0,$$

with $\tilde{c}_0 := \tilde{v}_0 - G \cdot P(0,T)$, see, e.g., Proposition 3.3 of Michelbrink and Le [2012]. Hence, the static optimization problem equivalent to the Problem 4.2.5 is

$$\max_{\tilde{C} \in \mathcal{F}(T)} \mathbb{E}\left[\frac{\left(\tilde{C}(T)\right)^{1-\delta_1}}{1-\delta_1}\right], \tag{4.25}$$

with budget constraint

$$\mathbb{E}\left[\tilde{C}(T)H^{\Xi}(T)\right] \le \tilde{c}_0. \tag{4.26}$$

For $\Xi = (\Xi_S(t), \Xi_J(t,y))$, we define

$$\mathcal{X}_{\Xi}(y) := y^{-\frac{1}{\delta_1}} \mathbb{E}\left[\left(H^{\Xi}(T)\right)^{1-\frac{1}{\delta_1}}\right], \quad y > 0, \tag{4.27}$$

and consider the set $\tilde{\Theta} \subseteq \Theta$ of all change of measures for which $\mathcal{X}_{\Xi}(y)$ is finite. For a fixed $\tilde{c}_0 > 0$ and $\Xi \in \tilde{\Theta}$, we define the non-negative random variable Y_{Ξ} as

$$Y_{\Xi} := \left(\mathcal{X}^{-1}(c) H^{\Xi}(T)\right)^{-\frac{1}{\delta_1}} \tag{4.28}$$

As shown in Michelbrink and Le [2012], Y_{Ξ} satisfies the property into the following Lemma.

Lemma 4.3.4. *For any $\Xi \in \tilde{\Theta}$, Y_{Ξ} defined in equation* (4.28) *satisfies*

(i) $\mathbb{E}\left[H^{\Xi}(T)Y_{\Xi}\right] = \tilde{c}_0$,

(ii) $\mathbb{E}\left[\min\left(\frac{Y_{\Xi}^{1-\delta_1}}{1-\delta_1}, 0\right)\right] > -\infty$,

(iii) $\mathbb{E}\left[\frac{\left(\tilde{C}(T)\right)^{1-\delta_1}}{1-\delta_1}\right] \leq \mathbb{E}\left[\frac{Y_{\Xi}^{1-\delta_1}}{1-\delta_1}\right]$ *for all* $m \in \mathcal{M}(\tilde{c}_0)$.

An immediate consequence of point (*iii*) of Lemma 4.3.4 is that

$$\sup_{\tilde{m}\in\mathcal{M}(\tilde{c}_0)} \mathbb{E}\left[\frac{\left(\tilde{C}(T)\right)^{1-\delta_1}}{1-\delta_1}\right] \leq \inf_{\Xi\in\tilde{\Theta}} \mathbb{E}\left[\frac{Y_{\tilde{\Xi}}^{1-\delta_1}}{1-\delta_1}\right], \tag{4.29}$$

or in other terms, the expected utility of the auxiliary process outperforms or is at least equal to the expected utility of any admissible multiplier.

Definition 4.3.5. *A martingale measure* $\mathbb{Q}_{\Xi}$ *obtained by equation* (4.17) *in terms of a* $\hat{\Xi} \in \tilde{\Theta}$ *is called optimal for the infimum problem defined in equation* (4.29) *if*

$$\mathbb{E}\left[\frac{Y_{\hat{\Xi}}^{1-\delta_1}}{1-\delta_1}\right] = \inf_{\Xi\in\tilde{\Theta}} \mathbb{E}\left[\frac{Y_{\Xi}^{1-\delta_1}}{1-\delta_1}\right], \tag{4.30}$$

where Y_{Ξ} *is defined by equation* (4.28).

In the following, we link the optimal martingale measure $\mathbb{Q}_{\Xi}$ defined by means of equation (4.30) with the optimal multiplier which solve (4.16). First, we consider for any $\Xi \in \tilde{\Theta}$ the martingale M^{Ξ} defined as follows

$$M^{\Xi}(t) := \mathbb{E}\left[H^{\Xi}(T)Y_{\Xi}|\mathcal{F}(t)\right], \tag{4.31}$$

such that $M^{\Xi}(0) = \tilde{c}_0$ $\mathbb{P}-$a.s. for every $\Xi \in \tilde{\Theta}$. Then, in terms of M^{Ξ}, we have

$$\tilde{C}_{\Xi}(t) := \frac{M^{\Xi}(t)}{H^{\Xi}(t)}, \quad t \in [0,T], \quad \Xi \in \tilde{\Theta}. \tag{4.32}$$

To derive the optimal multiplier, and optimal market risk premiums, the martingale representation coefficients of M^{Ξ} in equation (4.31) need to be computed. To do this, we first have to decompose the state price density H^{Ξ} as in the following Proposition.

Proposition 4.3.6. *For all* $\Xi \in \tilde{\Theta}$*, the following decomposition holds*

$$\left(H^{\Xi}(t)\right)^{-\frac{1-\delta_1}{\delta_1}} = \frac{G\left(t, H^{\Xi}(t)\right)}{\tilde{G}\left(t\right)}, \quad t \in [0,T], \tag{4.33}$$

where the process $\tilde{G}$ *is given by*

$$\begin{aligned}\tilde{G}(t) = \exp\Bigg\{ & \frac{1-\delta_1}{\delta_1} r(T-t) + \int_t^T \int_E \left[-\frac{1-\delta_1}{\delta_1}\left(1-\Xi_J(s,y)\right) + \left(\Xi_J(s,y)\right)^{-\frac{1-\delta_1}{\delta_1}} - 1\right] \nu(\mathrm{d}y)\mathrm{d}s \\ & + \frac{1}{2}\frac{1-\delta_1}{\delta_1^2} \int_t^T \left(\Xi_S(s)\right)^2 \mathrm{d}s \Bigg\}, \quad t \in [0,T],\end{aligned} \tag{4.34}$$

and the process G is a martingale given by

$$G\left(t, H^{\boldsymbol{\Xi}}(t)\right) = \exp\Bigg\{ -\frac{1-\delta_1}{\delta_1}\int_0^t \Xi_S(s)\mathrm{d}Z_S(s) - \frac{1}{2}\left(\frac{1-\delta_1}{\delta_1}\right)^2\int_0^t (\Xi_S(s))^2\,\mathrm{d}s$$
$$+\int_0^t\int_E \ln\left((\Xi_J(s,y))^{-\frac{1-\delta_1}{\delta_1}}\right)N(\mathrm{d}s,\mathrm{d}y) - \int_0^t\int_E\left[(\Xi_J(s,y))^{-\frac{1-\delta_1}{\delta_1}} - 1\right]\nu(\mathrm{d}y)\mathrm{d}s\Bigg\}, \tag{4.35}$$

for every $t \in [0,T]$.

Proof. Since the process $H^{\boldsymbol{\Xi}}$ is Markovian, for all $\boldsymbol{\Xi} \in \tilde{\Theta}$, we can define the process

$$G\left(t, H^{\boldsymbol{\Xi}}(t)\right) := \mathbb{E}\left[\left(H^{\boldsymbol{\Xi}}(T)\right)^{-\frac{1-\delta_1}{\delta_1}} |\mathcal{F}(t)\right]. \tag{4.36}$$

The process $G\left(t, H^{\boldsymbol{\Xi}}(t)\right)$ is a martingale under $\mathbb{P}$, therefore, it can be characterized as the solution of the following PIDE

$$0 = \frac{\partial G\,(t,h)}{\partial t} + \frac{\partial G(t,h)}{\partial h}h\left[-r + \int_E (1-\Xi_J(t,y))\,\nu(\mathrm{d}y)\right] + \frac{1}{2}\frac{\partial^2 G(t,h)}{\partial h^2}h^2\left(\Xi_S(t)\right)^2$$
$$+\int_E \left(G(t,h+y) - G(t,h)\right)\nu(\mathrm{d}y), \tag{4.37}$$

with boundary condition

$$G\,(T,h) = h^{-\frac{1-\delta_1}{\delta_1}}. \tag{4.38}$$

To solve the problem depicted in (4.37), we make the ansatz

$$G\,(t,h) = h^{-\frac{1-\delta_1}{\delta_1}}\tilde{G}\,(t)\,, \quad t \in [0,T]. \tag{4.39}$$

with $\tilde{G}\,(t) = e^{g_1(t)}$ for all $t \in [0,T]$ and $g_1(T) = 0$. Substituting this ansatz in equation (4.37), we obtain the following ODE

$$0 = \frac{\mathrm{d}g_1(t)}{\mathrm{d}t} + \frac{1-\delta_1}{\delta_1}r - \int_E\left[\frac{1-\delta_1}{\delta_1}\left(1-\Xi_J(t,y)\right) - \left(\Xi_J(t,y)\right)^{-\frac{1-\delta_1}{\delta_1}} + 1\right]\nu(\mathrm{d}y) + \frac{1}{2}\frac{1-\delta_1}{\delta_1^2}\left(\Xi_S(t)\right)^2,$$

whose well behaved solution up to time T, is given in equation (4.34). A sufficient condition for G to be a martingale is that

$$\mathbb{E}\left\{\int_0^T \frac{(1-\delta_1)^2}{\delta_1^2}\left(\Xi_S(s)\right)^2\mathrm{d}s + \int_0^T\int_E\left|\left(\Xi_J(s,y)\right)^{-\frac{1-\delta_1}{\delta_1}} - 1\right|\nu(\mathrm{d}y)\mathrm{d}s\right\} < +\infty. \tag{4.40}$$

Note that under the Novikov's condition, the condition on Ξ_S is satisfied. Hence the right integrability of Ξ_J provides an additional sufficient condition for having $\boldsymbol{\Xi} \in \tilde{\Theta}$. □

Proposition 4.3.7. *Let $a_S(t)$ and $a_J(t,y)$ be the unique martingale representation coefficients of $M^{\boldsymbol{\Xi}}$ depending on $\boldsymbol{\Xi} \in \tilde{\Theta}$. Then, the dynamics of $M^{\boldsymbol{\Xi}}$ is given by*

$$\mathrm{d}M^{\boldsymbol{\Xi}}(t) = a_S\,(t)\,\mathrm{d}Z_S(t) + \int_E a_J\,(t,y)\left(N\,(\mathrm{d}t,\mathrm{d}y) - \nu\,(\mathrm{d}y)\,\mathrm{d}t\right), \quad t \in [0,T], \tag{4.41}$$

where

$$a_S\,(t) = -M^{\boldsymbol{\Xi}}(t-)\frac{1-\delta_1}{\delta_1}\Xi_S(t),$$
$$a_J\,(t,y) = M^{\boldsymbol{\Xi}}(t-)\left[(\Xi_J(t,y))^{-\frac{1-\delta_1}{\delta_1}} - 1\right].$$

Proof. Thanks to the decomposition of the state price density $H^{\boldsymbol{\Xi}}$ given in Proposition 4.3.6, we can rewrite $\mathcal{X}_{\boldsymbol{\Xi}}(y)$ as follows

$$\mathcal{X}_{\boldsymbol{\Xi}}(y) = y^{-\frac{1}{\delta_1}}\mathbb{E}\left[\left(H^{\boldsymbol{\Xi}}(T)\right)^{-\frac{1-\delta_1}{\delta_1}}\right] = y^{-\frac{1}{\delta_1}}\cdot G\left(0, H^{\boldsymbol{\Xi}}(0)\right).$$

We denote by $\hat{y}_{\boldsymbol{\Xi}}$ the unique point such that $\mathcal{X}_{\boldsymbol{\Xi}}(\hat{y}_{\boldsymbol{\Xi}}) = \tilde{c}_0$ for given $\tilde{c}_0$. Then

$$\hat{y}_{\boldsymbol{\Xi}} = \left(\frac{\tilde{c}_0}{G\left(0, H^{\boldsymbol{\Xi}}(0)\right)}\right)^{-\delta_1}, \tag{4.42}$$

and

$$Y_{\boldsymbol{\Xi}} = \left(\tilde{c}_0^{-\delta_1}H^{\boldsymbol{\Xi}}(T)\right)^{-\frac{1}{\delta_1}} = \frac{\tilde{c}_0}{G\left(0, H^{\boldsymbol{\Xi}}(0)\right)}\cdot\left(H^{\boldsymbol{\Xi}}(T)\right)^{-\frac{1}{\delta_1}}. \tag{4.43}$$

Substituting equation (4.43) into equation (4.31), we obtain

$$\begin{aligned} M^{\boldsymbol{\Xi}}(t) &= \mathbb{E}\left[H^{\boldsymbol{\Xi}}(T)\cdot\frac{\tilde{c}_0}{G\left(0, H^{\boldsymbol{\Xi}}(0)\right)}\cdot\left(H^{\boldsymbol{\Xi}}(T)\right)^{-\frac{1}{\delta_1}}\middle|\mathcal{F}(t)\right] \\ &= \frac{\tilde{c}_0}{G\left(0, H^{\boldsymbol{\Xi}}(0)\right)}\cdot\mathbb{E}\left[\left(H^{\boldsymbol{\Xi}}(T)\right)^{-\frac{1-\delta_1}{\delta_1}}\middle|\mathcal{F}(t)\right] \\ &= \frac{\tilde{c}_0}{G\left(0, H^{\boldsymbol{\Xi}}(0)\right)}\cdot G\left(t, H^{\boldsymbol{\Xi}}(t)\right). \end{aligned} \tag{4.44}$$

By applying Itô's formula on equation (4.44), we obtain the dynamics of $M^{\boldsymbol{\Xi}}(t)$ which reads as follows

$$\frac{\mathrm{d}M^{\boldsymbol{\Xi}}(t)}{M^{\boldsymbol{\Xi}}(t-)} = -\frac{1-\delta_1}{\delta_1}\Xi_S(t)\mathrm{d}Z_S(t) + \int_E\left[(\Xi_J(t,y))^{-\frac{1-\delta_1}{\delta_1}} - 1\right](N(\mathrm{d}t,\mathrm{d}y) - \nu(\mathrm{d}y)\mathrm{d}t). \tag{4.45}$$

By comparing equation (4.41) with equation (4.45) we obtain the result. □

Then the following results characterizes the optimal multiplier and state price density, which in turn determines the worst case martingale measure. For the optimal market price of risk $\hat{\boldsymbol{\Xi}} \in \tilde{\Theta}$, the corresponding process $\tilde{C}^{\hat{\boldsymbol{\Xi}}}$ defined by equation (4.32) gives the optimal cushion.

Theorem 4.3.8. *Suppose that there exist a $\hat{\boldsymbol{\Xi}} \in \tilde{\Theta}$ and a multiplier $\hat{m}_{\hat{\boldsymbol{\Xi}}} \in \mathcal{M}$ that satisfy*

$$\begin{cases} m\,(t)\,\sigma = -\dfrac{\Xi_S(t)}{\delta_1}, \\ m\,(t)\,\gamma(t,y) = (\Xi_J(t,y))^{-\frac{1}{\delta_1}} - 1. \end{cases} \tag{4.46}$$

for all $y \in E$. Assume further that the SDE depicted in equation (4.6) *has a solution for $m = \hat{m}_{\hat{\boldsymbol{\Xi}}}$.*

Then $\hat{m}_{\hat{\Xi}}$ is a solution to the optimization problem 4.16. The corresponding cushion process is given by $\tilde{C}(t) = \tilde{C}^{\hat{\Xi}}(t)$ $\mathbb{P}-a.s.$, for all $t \in [0,T]$, where $\tilde{C}^{\hat{\Xi}}$ is defined in equation (4.32).

Proof. To prove that $\tilde{C}(t) = \tilde{C}^{\hat{\Xi}}(t)$ $\mathbb{P}$-a.s, we have to show that both processes evolves according to the same stochastic differential equation. Let $\hat{\Xi} \in \tilde{\Theta}$ and $\hat{m}_{\hat{\Xi}} \in \mathcal{M}$ satisfy the system of equations (4.46). It follows from equation (4.32) that the dynamics of $\tilde{C}^{\hat{\Xi}}(t)$ satisfies

$$\mathrm{d}\tilde{C}^{\hat{\Xi}}(t) = \mathrm{d}\left(\frac{M^{\hat{\Xi}}(t)}{H^{\hat{\Xi}}(t)}\right), \quad t \in [0,T]. \tag{4.47}$$

Applying Itô's formula for jump-diffusion processes, and using equation (4.41), we have that,

$$\begin{aligned}\mathrm{d}\left(\frac{M^{\hat{\Xi}}(t)}{H^{\hat{\Xi}}(t)}\right) = \frac{M^{\hat{\Xi}}(t-)}{H^{\hat{\Xi}}(t-)}\Bigg\{ & r\mathrm{d}t + \Xi_S(t)\frac{\Xi_S(t)}{\delta_1}\mathrm{d}t - \frac{\Xi_S(t)}{\delta_1}\mathrm{d}Z_S(t) \\ & + \int_E \left[(\Xi_J(t,y))^{-\frac{1}{\delta_1}} - 1\right]\left(N(\mathrm{d}t,\mathrm{d}y) - \Xi_J(t,y)\nu(\mathrm{d}y)\mathrm{d}t\right)\Bigg\},\end{aligned} \tag{4.48}$$

for every $t \in [0,T]$. It follows from equation (4.47), and equation (4.48) that

$$\begin{aligned}\frac{\mathrm{d}\tilde{C}^{\hat{\Xi}}(t)}{\tilde{C}^{\hat{\Xi}}(t-)} &= r\mathrm{d}t + \Xi_S(t)\frac{\Xi_S(t)}{\delta_1}\mathrm{d}t - \frac{\Xi_S(t)}{\delta_1}\mathrm{d}Z_S(t) \\ &+ \int_E \left[(\Xi_J(t,y))^{-\frac{1}{\delta_1}} - 1\right]\left(N(\mathrm{d}t,\mathrm{d}y) - \Xi_J(t,y)\,\nu(\mathrm{d}y)\mathrm{d}t\right),\end{aligned} \tag{4.49}$$

for $t \in [0,T]$. Substituting equation (4.46) into equation (4.49) and, recalling that $\hat{\Xi} \in \tilde{\Theta}$ satisfies equation (4.22), we obtain

$$\mathrm{d}\tilde{C}^{\hat{\Xi}}(t) = \tilde{C}^{\hat{\Xi}}(t-)\left[r + \hat{m}_{\hat{\Xi}}(t)(\mu - r)\right]\mathrm{d}t + \tilde{C}^{\hat{\Xi}}(t-)\hat{m}_{\hat{\Xi}}(t)\sigma\mathrm{d}Z_S(t) + \tilde{C}^{\hat{\Xi}}(t-)\hat{m}_{\hat{\Xi}}(t)\int_E \gamma(t,y)N(\mathrm{d}t,\mathrm{d}y)$$

which coincides with the SDE in equation (4.6) satisfied by the cushion process that follows the strategy $\hat{m}_{\hat{\Xi}}$. Moreover, since $\tilde{C}(0) = \tilde{C}^{\hat{\Xi}}(0) = \tilde{c}_0$, we conclude that $\tilde{C}(t) = \tilde{C}^{\hat{\Xi}}(t) = \tilde{c}_0$, $\mathbb{P}$-a.s., for all $t \in [0,T]$. Finally, we need to show that $\hat{m}_{\hat{\Xi}}$ is a solution to 4.16. Thanks to Lemma 4.3.4 and equation (4.29), we only need to show that

$$\mathbb{E}\left[\frac{\left(\tilde{C}(T)\right)^{1-\delta_1}}{1-\delta_1}\right] = \mathbb{E}\left[\frac{(Y_{\hat{\Xi}})^{1-\delta_1}}{1-\delta_1}\right], \tag{4.50}$$

for $m = \hat{m}_{\hat{\Xi}}$, which is true if $\tilde{C}(t) = Y_{\hat{\Xi}}$ $\mathbb{P}$-a.s. The latter equality comes from the facts that $\tilde{C}(t) = \tilde{C}^{\hat{\Xi}}(t)$, for all $t \in [0,T]$, $\mathbb{P}$-a.s., and that by construction of $\tilde{C}^{\hat{\Xi}}$. □

In view of the absence of arbitrage (see equation (4.22)) we can characterize the the optimal multiplier in terms of the market parameters only. That is, if the equation

$$\mu - r - \sigma^2\delta_1 m(t) + \int_E \gamma(t,y)\left[m(t)\gamma(t,y) + 1\right]^{-\delta_1}\nu(\mathrm{d}y) = 0, \tag{4.51}$$

has a unique solution $\hat{m}_{\hat{\Xi}}(t)$ for every $t \in [0,T]$, then $\hat{m}_{\hat{\Xi}}(t)$ is the optimal multiplier. In the sequel we discuss specific conditions under which the equation above has a unique solution and hence the optimal multiplier is well defined. For the sake of simplicity, we further assume that the jump size is time-independent, that is $\gamma(t,y) = \gamma(y)$ for every $t \in [0,T]$. In this case, the Girsanov kernel $\boldsymbol{\Xi}$ with $\Xi_S(t) := \Xi_S$ and $\Xi_J(t,y) := \Xi_J(y)$ satisfies a simplified version of the condition stated in equation (4.22) which is given by

$$\mu - r + \sigma \Xi_S + \int_{\mathbb{R}\setminus\{0\}} \gamma(y) \Xi_J(y) \nu\,(\mathrm{d}y) = 0. \tag{4.52}$$

The optimal multiplier is no longer time-varying but constant and satisfies the following conditions

$$\begin{cases} m\sigma = -\dfrac{\Xi_S}{\delta_1}, \\ m\gamma(y) = (\Xi_J(y))^{-\frac{1}{\delta_1}} - 1. \end{cases} \tag{4.53}$$

Taking condition (4.52) on $\hat{\boldsymbol{\Xi}}$ into account, we get that the optimal multiplier $\hat{m}_{\hat{\Xi}}$ can be characterized as the solution of the equations

$$\mu - r - \delta_1 \sigma^2 m + \int_{\mathbb{R}\setminus\{0\}} \frac{\gamma(y)}{(1+\gamma(y)m)^{\delta_1}} \nu\,(\mathrm{d}y) = 0, \tag{4.54}$$

Next we discuss the sufficient conditions for the existence and uniqueness of the solution of equation (4.54). Let $\Phi \subseteq [-1, +\infty)$ be the subset in which all jumps $\gamma(\cdot)$ lie. Furthermore, denote by $\phi_1 = \inf_{y \in \mathbb{R}\setminus\{0\}} \gamma(y)$ and $\phi_2 = \sup_{y \in \mathbb{R}\setminus\{0\}} \gamma(y)$, the infimum and the supremum of the set Φ, respectively. Since $\delta_1 \in (0,1)$, we observe that the optimal solution $\hat{m}_{\hat{\Xi}}$, if it exists, has to necessarily satisfy $1 + \gamma(y)\hat{m}_{\hat{\Xi}} > 0$ for all $y \in \mathbb{R} \setminus \{0\}$. The latter condition guarantees that the corresponding cushion is non-negative at any time $t \in [0,T]$ (see equation (4.6)).

Proposition 4.3.9. *Assume that the integrals*

$$\int_{\mathbb{R}\setminus\{0\}} \gamma(y) \left(1 - \frac{\gamma(y)}{\phi_1}\right)^{-\delta_1} \nu(\mathrm{d}y), \ \textit{and} \ \int_{\mathbb{R}\setminus\{0\}} \gamma(y) \left(1 - \frac{\gamma(y)}{\phi_2}\right)^{-\delta_1} \nu(\mathrm{d}y), \tag{4.55}$$

are well-defined or, at most, are $\pm\infty$. If the model parameters are such that

$$\mu - r + \frac{\delta_1 \sigma^2}{\phi_2} + \int_{\mathbb{R}\setminus\{0\}} \gamma(y) \left(1 - \frac{\gamma(y)}{\phi_2}\right)^{-\delta_1} \nu\,(\mathrm{d}y) \geq 0, \tag{4.56}$$

$$\mu - r + \frac{\delta_1 \sigma^2}{\phi_1} + \int_{\mathbb{R}\setminus\{0\}} \gamma(y) \left(1 - \frac{\gamma(y)}{\phi_1}\right)^{-\delta_1} \nu\,(\mathrm{d}y) \leq 0, \tag{4.57}$$

then there exists a unique $\hat{m}_{\hat{\Xi}}$ that solves equation (4.54)*. Moreover:*

(i) If $-1 \leq \phi_1, \phi_2 < 0$, then $\hat{m}_{\hat{\Xi}} \in (-\infty, -1/\phi_1)$;

(ii) If $0 < \phi_1, \phi_2 \leq +\infty$, then $\hat{m}_{\hat{\Xi}} \in (-1/\phi_2, +\infty)$;

(iii) If $-1 \leq \phi_1 < 0 < \phi_2 \leq +\infty$, then $\hat{m}_{\hat{\Xi}} \in (-1/\phi_2, -1/\phi_1)$.

Proof. Let $\left(\tilde{\phi}_1, \tilde{\phi}_2\right)$ be the range for the multiplier corresponding to each of the three cases (i)-(iii). Given the optimality condition, we define

$$g\left(m\right) := \mu - r - \delta_1\sigma^2 m + \int_{\mathbb{R}\setminus\{0\}} \frac{\gamma(y)}{\left(1+\gamma(y)m\right)^{\delta_1}}\nu\left(\mathrm{d}y\right), \tag{4.58}$$

whose first derivative $g'\left(m\right)$ is given by

$$g'\left(m\right) = -\delta_1\sigma^2 - \delta_1\int_{\mathbb{R}\setminus\{0\}} \frac{\gamma^2(y)}{\left(1+\gamma(y)m\right)^{1+\delta_1}}\nu\left(\mathrm{d}y\right). \tag{4.59}$$

Since $g'\left(m\right) < 0$, then there exists a unique point $\hat{m}_{\hat{\Xi}} \in \left(\tilde{\phi}_1, \tilde{\phi}_2\right)$ such that $g\left(\hat{m}_{\hat{\Xi}}\right) = 0$ only if $g\left(\tilde{\phi}_2\right) \geq 0$, and $g\left(\tilde{\phi}_1\right) \leq 0$. However, the latter relations are exactly the ones stated into the proposition, and this concludes the proof. □

Remark 4.3.10. *In all cases, the optimal multiplier is such that the optimal terminal cushion is non-negative in the whole time interval* $[0, T]$*, meaning that for our new version of PPI strategy gap risk does not occur, even when the dynamics of* S *jumps. It is also worth mentioning that the optimal multiplier may be negative, which implies that the optimal investment in the risky asset is negative, equivalently selling short the asset. From a financial point of view, if the stock price shows a downward trend, generated for instance by many negative jumps and a small or negative excess return, then short-selling the asset may turn losses into profits.*

4.4 Numerical analysis

In this Section, we propose a few numerical experiment where the results of Proposition 4.3.9 are applied to compute the optimal multiplier $\hat{m}_{\hat{\Xi}}$. We consider different specification of the jump size distributions and we also run a sensitivity analysis on model parameters. We compare three cases: (*i*) constant jump size; (*ii*) double exponential distribution of the jump sizes, namely, the Kou's model; (*iii*) Gaussian distribution of the jump sizes, i.e., the Merton's model.

4.4.1 Optimal multiplier under a jump-diffusion model with constant jump size

Assume that the risky asset price process follows the dynamic

$$\frac{\mathrm{d}S\left(t\right)}{S\left(t-\right)} = \mu\mathrm{d}t + \sigma\mathrm{d}Z_S(t) + \tilde{\gamma}\mathrm{d}N(t), \tag{4.60}$$

where N is an homogeneous Poisson process with intensity rate $\tilde{\lambda} > 0$, and $\tilde{\gamma} \in [-1, 0)$ is the constant negative jump size. The optimal multiplier $\hat{m}_{\hat{\Xi}}$ is determined in the corollary below.

Corollary 4.4.1. *In a market model where the risky asset price process is described by equation* (4.60)*, the optimal multiplier* $\hat{m}_{\hat{\Xi}} \in (-\infty, -1/\tilde{\gamma})$ *is the unique solution of*

$$\mu - r - \delta_1\sigma^2 m + \tilde{\lambda}\tilde{\gamma}\left(1+\tilde{\gamma}m\right)^{-\delta_1} = 0. \tag{4.61}$$

Proof. To prove the result we note that $\phi_1 = \tilde{\gamma} \in (-1, 0)$, and $\phi_2 = 0$, so the optimal multiplier $\hat{m}_{\hat{\Xi}}$ must take value in $(-\infty, -1/\tilde{\gamma})$. We easily see that, $g'(m) = -\delta_1\sigma^2 - \delta_1\tilde{\gamma}^2 (1 + \tilde{\gamma}m)^{-\delta_1} < 0$, and that

$$\lim_{m\to-\infty} \mu - r - \delta_1\sigma^2 m + \tilde{\lambda}\tilde{\gamma} (1 + \tilde{\gamma}m)^{-\delta_1} = +\infty, \qquad \lim_{m\to-\tilde{\gamma}^{-1}} \mu - r - \delta_1\sigma^2 y + \tilde{\lambda}\tilde{\gamma} (1 + \tilde{\gamma}m)^{-\delta_1} = -\infty,$$

which implies that $\hat{m}_{\hat{\Xi}}$ exists and that it is the unique solution of equation (4.61). □

We fix the model parameters according to Table 4.1. Then, we vary each of them (one by one) to run a sensitivity analysis on the optimal multiplier. The numerical analysis is relatively easy to obtain for the constant jump size due to the quasi-explicit form of the multiplier. Nevertheless, they provide a useful comparison case study. Figure 4.3 shows that the multiplier is decreasing with

$\mu - r$	σ	$\tilde{\lambda}$	$\tilde{\gamma}$	δ_1	$\hat{m}_{\hat{\Xi}}$
0.20	0.30	11	−0.03	0.60	−2.18

Table 4.1 Parameters of Jump diffusion model with negative constant jump size

respect to the jump intensity, the volatility and the risk aversion level, and it is increasing in the excess return of the underlying risky asset and the jump size. What is interesting and deviates from the classical results on PPI in the diffusion setting, is that the multiplier may be negative. Since the optimal multiplier does not allow for negative cushion (see Remark 4.3.10), then having $\hat{m}_{\hat{\Xi}}(t) < 0$ implies that the fund manager should short-sell the risky asset. Such behaviour occurs when the intensity of negative jumps increases, or the jump size (in absolute value) becomes prohibitive, or the excess return gets close to zero (or negative). Each of these effect pushes down the value of the risky asset, and hence short-selling the asset may contrast with such a downward trend. We have also represented the value function with respect to time to maturity and the initial cushion in left chart of Figure 4.4. In particular, it is increasing with respect to both time and the initial value of the cushion and concave. Most interestingly, we have numerically verified the theoretical result that the gap risk does not occur. Indeed, in right chart of Figure 4.4 we have plotted the dynamic portfolio values $q_0(t)$, $q_{99}(t)$ between $\mathbb{P}(\tilde{W}^{PPI}(t) \leq q_0(t)) = 0$ and $\mathbb{P}(\tilde{W}^{PPI}(t) \leq q_{99}(t)) = 0.99$ for every $t \leq T$. In particular, the zero quantile $q_0(t)$ is always above the floor value, as indicated in the red dashed line.

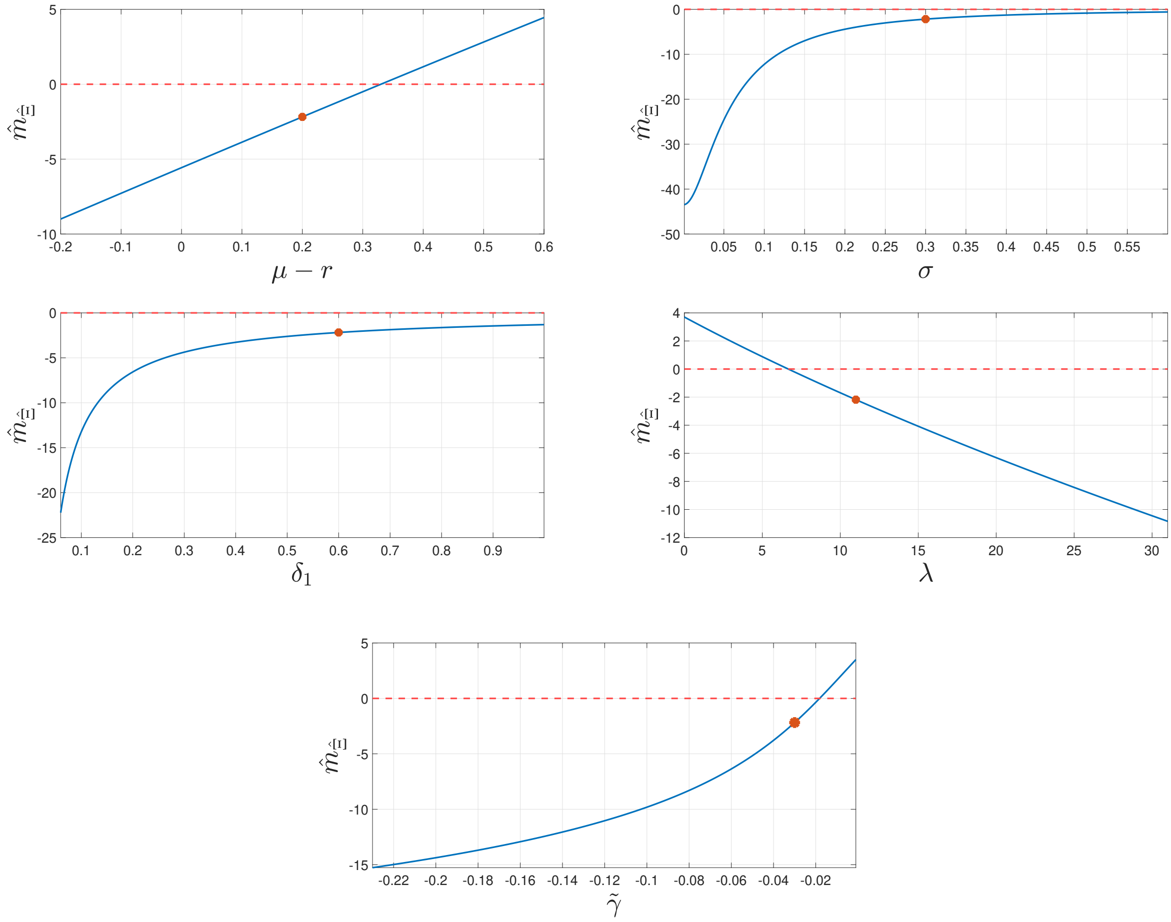


Figure 4.3 Sensitivity analysis for the optimal multiplier with respect to the jump-diffusion model with constant negative jump size. The blue line is the value of the multiplier for the different values of the parameter indicated under each panel and the red dot corresponds the value of the multiplier under the parameter configuration in Table 4.1.

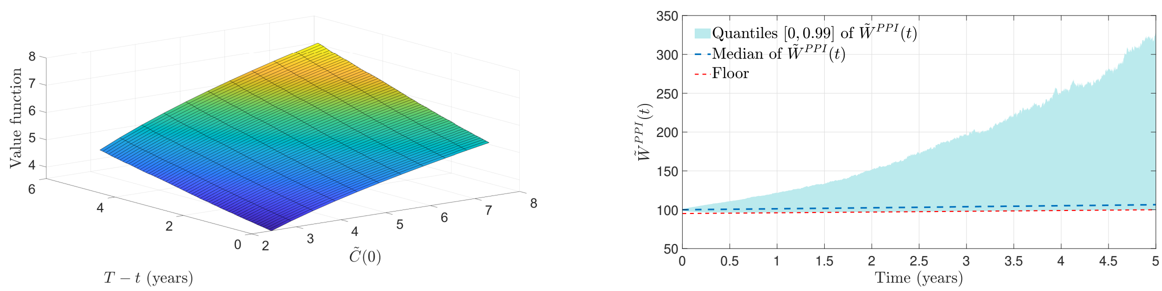


Figure 4.4 Value function for the constant jump size, with respect to time to maturity and initial cushion (left chart). Median (dotted line) and extreme scenarios for the portfolio value in case of constant jump size. The shaded area represents the portfolio values between the zero and the 99% quantiles. The red dashed line represents the level of the floor which is always below the zero quantile (right chart).

4.4.2 Optimal multiplier under Kou's and Merton's models

Kou's model assumes the following dynamics for the underlying risky asset S

$$\frac{\mathrm{d}S(t)}{S(t-)} = \mu \mathrm{d}t + \sigma \mathrm{d}Z_S(t) + \mathrm{d}\left[\sum_{j=1}^{N(t)} (V_j - 1)\right], \quad t \in [0, T], \tag{4.62}$$

where N is an homogeneous Poisson process with intensity rate $\tilde{\lambda} > 0$, and $\{V_1, V_2, \dots\}$ is a sequence of independent and identically distributed random variables such that, for every j, $\log(V_j)$ is subject to the following asymmetric double exponential density function

$$f(y) = p\eta_+ e^{-\eta_+ y} \mathbb{1}_{y \geq 0} + (1-p)\, \eta_- e^{\eta_- y} \mathbb{1}_{y<0}, \tag{4.63}$$

where $\eta_+ > 1$, is the parameter governing the severity of upward jumps, $\eta_- > 0$ is the parameter governing the amplitude of downward jumps, and $p \in (0,1)$ is the probability of an upward jump to occur. By applying the result in Proposition 4.3.9, we derive the following condition for the existence and uniqueness of the optimal multiplier that maximizes the expected utility of terminal cushion.

Corollary 4.4.2. *Assume that the stock price process follows the Kou's model stated in equation* (4.62)*. If the model parameters satisfy*

$$\mu - r - \frac{p\tilde{\lambda}}{1-\eta_+} - \frac{(1-p)\,\tilde{\lambda}}{1+\eta_-} \geq 0, \tag{4.64}$$

$$\mu - r - \delta_1 \sigma^2 - \frac{p\eta_+ \tilde{\lambda}}{(1-\delta_1-\eta_+)(\delta_1+\eta_+)} + \frac{(1-p)\eta_- \tilde{\lambda}}{(1-\delta_1+\eta_-)(\delta_1-\eta_-)} \leq 0, \tag{4.65}$$

then there exists a unique optimal multiplier $\hat{m}_{\hat{\Xi}} \in [0,1]$ *that can be found by solving*

$$\mu - r - \delta_1 \sigma^2 m + \tilde{\lambda} \int_{\mathbb{R}\setminus\{0\}} \frac{e^y - 1}{[1 + (e^y - 1)\, m]^{\delta_1}} \left[p\eta_+ e^{-\eta_+ y} \mathbb{1}_{y \geq 0} + (1-p)\, \eta_- e^{\eta_- y} \mathbb{1}_{y<0}\right] \mathrm{d}y = 0. \tag{4.66}$$

Proof. Since for Kou's model the jump sizes are such that $\phi_1 = -1$ and $\phi_2 = +\infty$, the optimal multiplier $\hat{m}_{\hat{\Xi}}$ if exists, lies in $[0,1]$. Moreover, since also in this case $g'(m) < 0$ for all $m \in [0,1]$, the condition for existence and uniqueness stated in Proposition 4.3.9 become $g(0) > 0$, and $g(1) < 0$. Hence, we have to compute $g(0)$ and $g(1)$, and find the conditions on Kou's model parameters such that the latter inequalities are satisfied. The expressions for $g(0)$ and $g(1)$ are given by

$$\begin{aligned} g(0) &= \mu - r + \tilde{\lambda} \int_{\mathbb{R}\setminus\{0\}} (e^y - 1) \left(p\eta_+ e^{-\eta_+ y} \mathbb{1}_{\{y \geq 0\}} + (1-p)\, \eta_- e^{\eta_- y} \mathbb{1}_{\{y<0\}}\right) \mathrm{d}y \\ &= \mu - r - \frac{p\tilde{\lambda}}{1-\eta_+} - \frac{(1-p)\,\tilde{\lambda}}{1+\eta_-}, \end{aligned}$$

$$
\begin{aligned}
g(1) &= \mu - r - \delta_1\sigma^2 + \tilde{\lambda}\int_{\mathbb{R}\setminus\{0\}} \left(e^y - 1\right) e^{-\delta_1 y}\left(p\eta_1 e^{-\eta_1 y}\mathbb{1}_{\{y\geq 0\}} + (1-p)\,\eta_2 e^{\eta_2 y}\mathbb{1}_{\{y<0\}}\right) \mathrm{d}y \\
&= \mu - r - \delta_1\sigma^2 - \frac{\tilde{\lambda}p\eta_+}{\left(1-\delta_1-\eta_+\right)\left(\delta_1+\eta_+\right)} + \frac{\tilde{\lambda}(1-p)\eta_-}{\left(1-\delta_1+\eta_-\right)\left(\delta_1-\eta_-\right)},
\end{aligned}
$$

respectively. Thus to guarantee that $g(0) > 0$, and $g(1) < 1$, the conditions stated in equation (4.64) and equation (4.65) need to be satisfied. This ensures the existence and the uniqueness for an optimal multiplier $\hat{m}_{\hat{\Xi}} \in [0, 1]$, and concludes the proof. □

As in the previous example we consider the set of parameters in table 4.2, and change their values one by one. Figure 4.5 shows results that are consistent with the case of constant jump size.

$\mu - r$	σ	$\tilde{\lambda}$	p	η_+	η_-	δ_1	$\hat{m}_{\hat{\Xi}}$
0.24	0.26	20	0.72	64.94	49.02	0.60	0.77

Table 4.2 Kou model parameters

We stress that, when η_- increases, the expected size of negative jumps decreases. That would mean smaller average downward moves of the risky asset price, suggesting a larger exposure to the risky asset may be advantageous. The situation reverts for η_+, that is, when the parameter governing severity of positive jumps increases, upward moves become less pronounced, leading to a smaller exposure. What is interesting in this case is that the optimal multiplier assumes value in $[0, 1]$, which, in virtue of the relation $\pi(t)\tilde{W}^{PPI}(T) = m(t)\tilde{C}(t)$, translates into an investment strategy where short-selling and borrowing from the bank account are not allowed (here we also used the fact that $\tilde{C}(t) < \tilde{W}^{PPI}(t)$). The value function for the optimization problem in Kou's specification shares the same features as the one depicted in Figure 4.4 for the constant jump size case. Hence, for the sake of brevity, we have omitted it here. Figure 4.6 shows the extreme scenarios for the portfolio values, and it confirms that the optimal investment strategy allows the portfolio value process to stay well above the floor, indicated with the dotted red line.

Under Merton's model the price process of the underlying risky asset is still defined as in equation (4.62), with a different assumption on the jump size distribution. In particular, jump amplitudes $\{V_1, V_2, \dots\}$ are independent identically distributed random variables such that, for all j, $\log(V_j)$ has normal density function

$$
f(y) = \frac{1}{\sqrt{2\pi\sigma_J^2}} \exp\left\{-\frac{\left(y-\mu_J\right)^2}{2\sigma_J^2}\right\}, \tag{4.67}
$$

for constant parameters $\mu_J \in \mathbb{R}$, and $\sigma_J > 0$ representing the mean and the standard deviation of jump size distribution, respectively. For this model, the parameters' conditions for the optimal multiplier to exist and be unique are as follows.

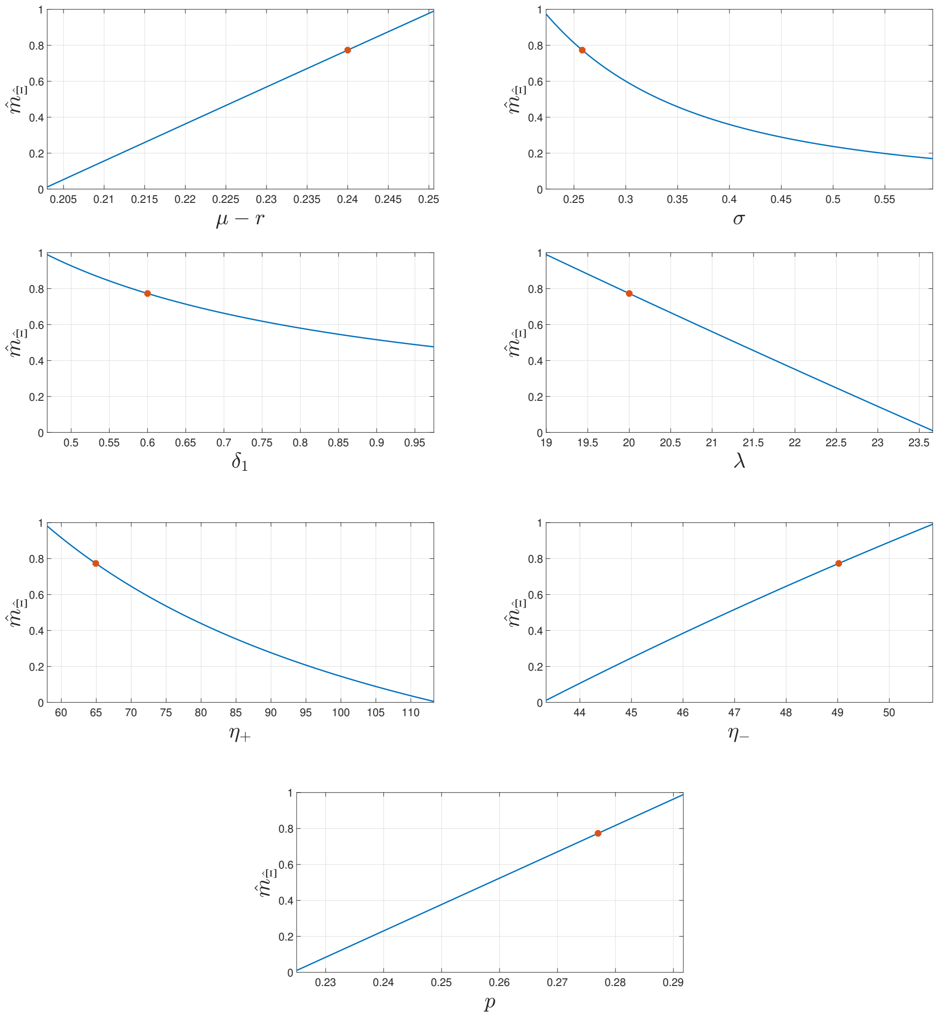


Figure 4.5 Sensitivity analysis for the optimal multiplier with respect to Kou's model parameters. The blue line is the value of the multiplier for the different values of the parameter indicated under each panel and the red dot corresponds the value of the multiplier under the parameter configuration in Table 4.2.

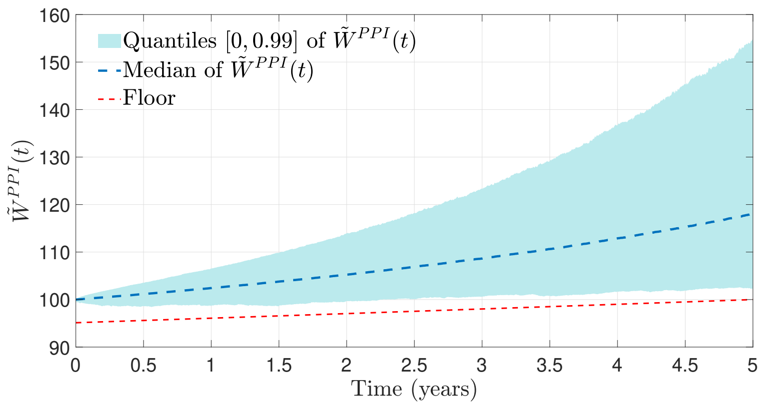


Figure 4.6 Median (dotted line) and extreme scenarios for the portfolio value in case of Kou's model. The light blue area represents the portfolio values between the zero and the 99% quantiles. The red dashed line represents the level of the floor which is always below the zero quantile.

Corollary 4.4.3. *Assume that the stock price process follows the Merton model. If the model parameters satisfy*

$$\mu + \tilde{\lambda}\left(\exp\left\{\mu_J + \frac{\sigma_J^2}{2}\right\} - 1\right) \geq 0, \tag{4.68}$$

$$\mu - \sigma^2\delta_1 + \tilde{\lambda}\left(\exp\left\{(1-\delta_1)\mu_J + \frac{(1-\delta_1)^2\sigma_J^2}{2}\right\} - \exp\left\{-\delta_1\mu_J + \frac{\delta_1^2\sigma_J^2}{2}\right\}\right) \leq 0, \tag{4.69}$$

then there exists a unique optimal multiplier $\hat{m}_{\hat{\Xi}} \in [0,1]$ *that solves*

$$\mu - r - \delta_1\sigma^2 m + \tilde{\lambda}\int_{\mathbb{R}\setminus\{0\}} \frac{e^y - 1}{\left[1 + (e^y - 1)\, m\right]^{\delta_1}} \frac{1}{\sqrt{2\pi\sigma_J^2}} \exp\left\{-\frac{(y-\mu_J)^2}{2\sigma_J^2}\right\} \mathrm{d}y = 0. \tag{4.70}$$

Proof. The proof replicates the lines of that of Corollary 4.4.2. □

We now fix the parameter according to the following values and perform a sensitivity analysis by varying each of them. In Figure 4.7 we analyze the behaviour of the optimal multiplier with respect to excess return, volatility, frequency of jumps in the underlying risky asset, and the investor's risk aversion. The results are comparable to those in the previous cases (i.e. constant jump sizes

$\mu - r$	σ	$\tilde{\lambda}$	μ_J	σ_J	δ_1	$\hat{m}_{\hat{\Xi}}$
0.09	0.35	20	−0.01	0.15	0.60	0.22

Table 4.3 Merton's model parameters

and the Kou model). The optimal multiplier increases with the parameter μ_J, which indicates higher exposure to the risky asset as the expected value of jumps sizes increases, for fixed volatility $\sigma_J = 0.15$. The optimal multiplier also increases with respect to standard deviation of the jump size distribution. Indeed, when σ_J is small, jump sizes tend to cluster around negative values. Conversely, larger values of σ_J lead to dispersion of jump sizes across a wider range, encompassing positive values. As for Kou's model, the optimal multiplier assumes value in $[0,1]$, implying that our new investment strategy does not allow for short-selling nor borrowing from the bank account. Since the value function has the same features as in the previous two specifications of jump size distributions, we have not reported it. Furthermore, for comprehensive analysis, we display the extreme scenarios of portfolio values in Figure 4.8. These scenarios highlight that the optimal investment strategy effectively maintains the portfolio value above the floor, thereby mitigating gap risk throughout the entire investment horizon $[0,T]$.

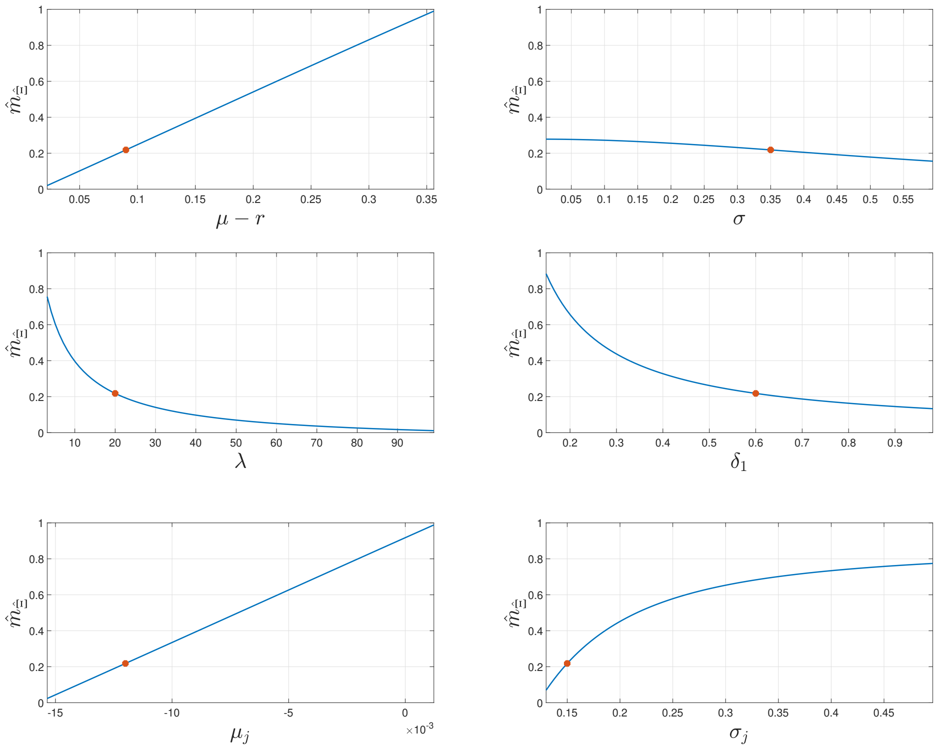


Figure 4.7 Sensitivity analysis for the optimal multiplier with respect to Merton's model parameters. The blue line is the value of the multiplier for the different values of the parameter indicated under each panel and the red dot corresponds the value of the multiplier under the parameter configuration in Table 4.3.

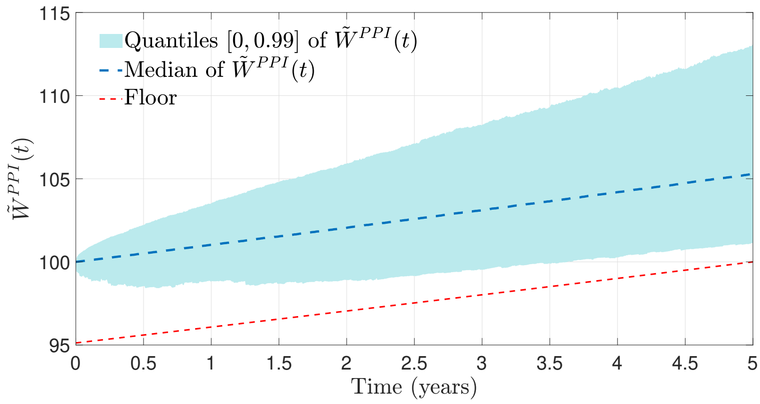


Figure 4.8 Median (dotted line) and extreme scenarios for the portfolio value in case of Merton's model (right chart). The light blue area represents the portfolio values between the zero and the 99% quantiles. The red dashed line represents the level of the floor which is always below the zero quantile.

Conclusions

The present research investigates modified versions of Portfolio Insurance strategies to improve the management of various types of risk. The studies confirmed our proposals' suitability in hedging against financial and longevity risks.
Looking at investigations in Chapter 2, we introduce a new type of portfolio insurance strategy, namely the G-TIPP as an underlying asset for options embedded in structured investment products with capital protection. Through detailed numerical simulations within the hybrid Heston-Vasicek model, we obtain that European call options linked to G-TIPP strategies exhibit better characteristics than options with pure risky securities and other PI strategies (such as G-CPPI) as underlying assets. In particular, we find that European call options linked to G-TIPP are cheaper than those linked to pure risky securities and G-CPPI, which leads to a significant increase in the participation rate. Furthermore, we show that options on G-TIPP are less sensitive to market volatility. For all these reasons, G-TIPP options in a structured investment product with capital protection make it possible to maintain a relatively high participation rate (capital protection being equal), regardless of market volatility and interest rate levels. Therefore, this new G-TIPP option represents an attractive and appropriate approach to structured products. Hence, we expect portfolio solutions of the latter types will spread widely over markets, especially in a market environment characterized by low-interest rate levels and high volatility. Moreover, it is worth noting that G-TIPP options are not traded on the market. Hence, the issuer can obtain the corresponding payoff by implementing suitable dynamic hedging strategies. The construction of such hedging strategies is subject of our ongoing research.
Looking at investigations in Chapter 3, we consider an optimal investment problem for a DC pension fund, guaranteeing a target annuity value at retirement. Considering investment and longevity risks, the fund manager implements a PO-PPI strategy during the accumulation period to maximize the expected utility of the terminal surplus between the fund final wealth and the target annuity value. We formulate the manager stochastic optimal control problem and solve it through an auxiliary formulation. Then, closed-form expressions for optimal controls are derived, enabling a numerical application to investigate the impact of both the financial and mortality parameters of PO-PPI parameters and optimal asset allocations. In summary, our findings show that a PO-PPI strategy allows for protecting the retirement saving accumulation, taking into account the longevity behaviour characterizing the decumulation period. As longevity grows, the optimal investment leverage increases to support guarantee protection. In addition, the risk-averse fund manager considers the rolling longevity bond as a crucial asset in building and protecting the retirement capital. Such an asset allows the hedge of longevity risk and, at the same time, diversifies the

financial portfolio of the pension plan. Looking at investigations in Chapter 4, we consider the problem of optimal design of a new version of PPI strategy in a mathematical setup where portfolio dynamics may present downward jumps generated by, e.g., sudden market drawdown. Incorporating such features in a market model brings gap risk with positive probability, as discussed, for instance, by Cont and Tankov [2009]. Hence, a relevant task of a manager or an insurer is to hedge against gap risk to be able to fulfil the liabilities at maturity. We attempted to provide a criterion for determining the optimal PPI strategy that maximizes expected utility at maturity and, at the same time, avoids or minimizes gap risk by keeping equity market participation during the trading horizon, even when the cushion becomes negative. In the problem formulation, we introduced two modifications to the standard setup. We proposed a generalized definition of the PPI strategy that does not exclude negative values for the multiplier, and we described the manager preferences via an S-shaped utility function. From a technical viewpoint, we have addressed the optimization using a worst-case martingale approach and concavification to account for the features of the utility function. Our main result provides a procedure for simultaneously computing the state price density for the worst-case probability measure and the optimized multiplier in terms of the solution of a system of equations. Unfortunately, there could be model parameter configurations for which the solution of the system may not exist or may not be unique. In this case, it is hard to identify the optimal PPI strategy. However, we discussed examples where the multiplier is given in closed or quasi-closed form, making our results readily applicable in a risk management context. Interestingly, in such examples, it can be explicitly proved that gap risk does not occur. Our next step in studying PPI strategies includes enriching the structure of jumps by considering a portfolio with several underlying assets subject to a common contagion factor.

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